Jacobi theta embedding of a hyperbolic 4-space with cusps

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Abstract

Starting from a fixed elliptic curve with complex multiplication we compose lifted quotients of elliptic Jacobi theta functions to abelian functions in higher dimension. In some cases, where complete Picard-Einstein metrics have been discovered on the underlying abelian surface (outside of cusp points), we are able to transform them to Picard modular forms. Basic algebraic relations of basic forms come from different multiplicative decompositions of these abelian functions in simple ones of the same lifted type. In the case of Gauß numbers the constructed basic modular forms define a Baily-Borel embedding in $\mathbf{P}^{22}$. The relations yield explicit homogeneous equations for the Picard modular image surface.

Acknowledgement. For the celebration of 200 years since the appearence of Gauß' *Disquisitiones Arithmeticae*.

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Contents

1 Introduction: Basic problems, motivations

1.) Rational Cuboid Problem.
Find a cuboid with rational edges and (face) diagonals. There is no solution until now. By some work of H.Shiga and others, see [SN], the rational cuboids define (and are defined up to similarity by) rational points on a $\mathbf{Q}$-model of the $K3$-surface $E \times E / < -1 >$, E an elliptic curve with Gauß number multiplication. This K3 surface is Picard modular.

2.) Hilbert's 12-th Problem.
Explicit construction of "nice" number fields via special values of transcendental functions with more than one variable. There is no completely understood example until now.

3.) Uniformization theory for systems of partial differential equations of Picard-Fuchs type, see [Yos].

Example 1.1 *(closely related with this paper). Set*

$$I := \int \frac{dx}{\sqrt[4]{x(x+1)(x-t_0)^2(x-t_1)^2(x-t_2)^2}}.$$

For $t_0 = 1$ this multivalued function $I(t_1, t_2)$, defined outside of the line configuration

$$t_1 t_2 (t_2 - t_1)(t_2 - 1)(t_2 + 1)(t_1 - 1)(t_1 + 1) = 0,$$

satisfies special Picard-Fuchs equations, see [Ho86], ch. II.

4.) String theory.
There is a question of Stieberger (Princeton, 2000, private communication): Understand Picard modular forms of the field of Gauß numbers, and present them in most explicit manner.

5.) Construction of almost compact real 4-spaces with Kähler-Einstein metric of negative constant curvature.
The metric should be complete outside of finitely many "cusp points" compactifying the space. Find explicit complex quasiprojective models.

There are respectable lists of candidates for attacking these problems simultaneously. The most interesting cases are those which appear in all these tables, and some of these are related to all of the above problems. We refer to the lists of

(i) Le Vavasseur [LeV], 1893 (thesis adviced by Picard)

(ii) Mostow-Deligne [MD], 1986

(iii) Hizebruch (and others) [BHH], 1987

(iv) Thurston [Th], 1998

We pick out a real complex 2-dimensional case appearing in all the lists, which is connected with Gauß numbers. Most important for these problems is to find generating modular forms together with basic relations, both as explicit as possible. This is the aim of this paper.

For necessary preparations we give in the sections 2 - 6 a summary of basic notions and results. The most recent are only available in HU-preprints at the moment of writing. One can find them on my homepage at Humboldt-University. After defining the remarkable η-form via euclidean-hyperbolic coordinate change, the transfer of abelian functions on special ball quotient surfaces to modular ones is explained (section 7). We remember to elliptic Jacobi theta functions and quotients of them in section 8. In section 9 we pull them back along elliptic fibrations and find multi-periodic compositions of them, which we call abelian Σ-quotients. We discover an imaginary quadratic version of Heisenberg groups in section 10. Then we look at divisors of abelian Σ-quotient and discover relations via different multiplicative decompositions in simple ones (section 11). Sections 12 - 14 are dedicated to normalizing constants in order to get the most simple and clean relations. In section 15 we present for each weight ≥ 2 explicit modular forms of the following Eisenstein quality: they are non-cuspidal at precisely one cusp point. For any given cusp we construct such a form.

Our lifting construction of "abelian modular forms" yields explicit basic Picard modular forms of weights ≤ 3 and relations in the Gauß number case as explained in the final sections 16 and 17. The projective Baily-Borel embedding of the underlying Picard modular surface with all basic forms shows that we are very near to generate the whole ring of modular forms and to generate explicitly the ideal of all relations between generators.

This should be checked by computer algebra for a forthcoming paper. It needs also more place, but seems to be not difficult with help of [He] (adviced by Langlands), to present Jacobi-Fourier series of our modular forms of abelian type at a cusp. I believe that our abelian approach to modular forms works also in higher dimension, at least in Picard modular cases appearing in the above lists (i) - (iv).

Acknowledgement: I have to thank N. Vladov for alluding me to the neat Gauß number model with six elliptic cusp curves investigated in the last sections.

2 Modular approach

The example of problem 3.) is connected with the family of plane curves

$$C_{\mathfrak{t}}:\ y^4=(x-1)(x+1)(x-t_0)^2(x-t_1)^2(x-t_2)^2,\qquad \mathfrak{t}=(t_0,t_1,t_2)\in\mathbf{C}^3\setminus\{\mathfrak{o}\}. \tag{1}$$

Via plane projective closure and normalization one gets smooth compact models $\widetilde{C}_{\mathfrak{t}}$. The general members of the family have genus 3. We correspond to $\mathfrak{t}$ the point

$$t = \mathbf{P}\mathfrak{t} = (t_0 : t_1 : t_2) \in \mathbf{P}^2(\mathbf{C}).$$

It is easy to see that the isomorphy class of $\widetilde{C}_{\mathfrak{t}}$ is well defined by t. So the projective plane appears as parameter space of isomorphy classes of our family. The symmetric group S_3 acts by permutation of t_0, t_1, t_2. In [HV] we proved

Proposition 2.1 *The quotient surface $\mathbf{P}^2/S_3$ is a compactified moduli space of the family (1).*

This means that a Zariski open part of $\mathbf{P}^2/S_3$ parametrizes precisely the genus 3 curves of the family (1). The three projective lines $C_k :\ t_i = t_j,\ \{i,j,k\} = \{1,2,3\}$, on $\mathbf{P}^2$ are tangent lines of an S_3-invariant quadric C_0. For a first understanding of C_0 we move the zeros ± 1 in (1) to a common one fixing the other zeros. Then the curve C_0 parametrizes the limit points on $\mathbf{P}^2$. The *Apollonius curve*

$$A\ := C_0 + C_1 + C_2 + C_3 \in Div\,\mathbf{P}^2$$

supports the *orbital Apollonius cycle*

$$\mathbf{A}\ := 4C_0 + 4C_1 + 4C_2 + 4C_3 + P_1 + P_2 + P_3 + K_1 + K_2 + K_3,$$

with intersection points

$$P_i = C_j \cap C_k,\ \{i,j,k\} = \{1,2,3\},\ K_i = C_0 \cap C_i,\ i = 1,2,3.$$

Now we want to uniformize the *orbital surface* $(\mathbf{P}^2,\ \mathbf{A})$. Let

$$\mathbf{B}\ := \{(z_1, z_2) \in \mathbf{C}^2;\quad |z_1|^2 + |z_2|^2 < 1\} \subset \mathbf{C}^2 \subset \mathbf{P}^2$$

be the 2-dimensional complex unit ball,

$$\Gamma_1\ := \mathbf{SU}((2,1), \mathbf{Z}[\mathrm{i}]) \subset \mathbf{Gl}_3(\mathbf{C})$$

the full *Picard modular group* acting effectively on $\mathbf{B}$ and $\Gamma(1+\mathrm{i})$ the congruence subgroup defined by the exact sequence

$$1 \longrightarrow \Gamma(1+\mathrm{i}) \longrightarrow \Gamma_1 \longrightarrow \mathbf{SU}((2,1), \mathbf{Z}[\mathrm{i}]/(1+\mathrm{i})) \longrightarrow 1,$$

Notice that

$$\mathbf{SU}((2,1), \mathbf{Z}[\mathrm{i}]/(1+\mathrm{i})) \cong \mathbf{O}(3, \mathbf{F}_2) \cong S_3.$$

with the field $\mathbf{F}_2$ consisting of two elements.

Uniformization Theorem 2.2 *[HV]. The Baily-Borel compactification of the Picard modular surface* $\mathbf{B}/\Gamma(1+\mathrm{i})$ *is equal to the projective plane. There are precisely* 3 *compactifying cusp points, which we identify with* K_1, K_2, K_3. *The quotient morphism extended to the Baily-Borel compactification*

$$\mathbf{B} \longrightarrow \mathbf{B}/\Gamma(1+\mathrm{i}) \subset \widehat{\mathbf{B}/\Gamma(1+\mathrm{i})} = \mathbf{P}^2$$

is a locally finite covering branched along the Apollonius curve A. *The orbital cycle (with ramification indices, cusp points and singularities) of this covering is the Apollonius cycle* $\mathbf{A}$. *The ramification locus on* $\mathbf{B}$ *is the* $\Gamma(1+\mathrm{i})$*-orbit of* 4 *discs* $\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$.

□

The proof of the theorem needs rather new rational invariants of orbital surfaces, orbital curves and orbital points. Especially, we dispose on orbital Euler numbers, orbital signatures of orbital surfaces, orbital Euler numbers and orbital selfintersections of orbital curves. The theory of these $\mathbf{Q}$-invariants (originally called *orbital heights*) has been developed in [Ho98a]. With a general proportionality theorem, see [Ho00], one can decide whether an orbital surface has ball uniformization or not. On the other hand, starting from Picard modular groups, one can calculate the orbital invariants in terms of fine arithmetic hermitian lattice theory described also in [Ho98a]. Bringing both sides together we proved the above Uniformization Theorem in [HV].

In general we define in this paper a *Picard modular group* Γ as a subgroup of finite index in the (special) *full Picard modular group*

$$\Gamma_1 = \Gamma_1^K = \mathbf{SU}((2,1), \mathfrak{O}_K),$$

where $K = \mathbf{Q}(\sqrt{-d})$, $d \in \mathbf{N}_+$, is an imaginary quadratic number field and $\mathfrak{O}_K$ the ring of all algebraic integers in K. Of main interest are the *Picard modular congruence subgroups* defined as subgroups of Γ_1, containing a *principal congruence subgroup* $\Gamma_1(\mathfrak{a})$. The latter is defined as kernel of the reduction homomorphism of Γ_1 *mod* $\mathfrak{a}$, $\mathfrak{a}$ an ideal of $\mathfrak{O}_K$ invariant under complex conjugation:

$$1 \longrightarrow \Gamma(\mathfrak{a}) \longrightarrow \Gamma_1 \longrightarrow \mathbf{SU}((2,1), \mathfrak{O}_K/\mathfrak{a}).$$

For the rest of the paper we use the shorter notation *Picard modular group* as synonyme for Picard modular congruence subgroup.

3 Hyperbolic metric on neat ball quotients

For the construction of the *hyperbolic metric* on $\mathbf{B}$ one starts with the boundary distance function

$$N(z) := 1 - z_1\bar{z}_1 - z_2\bar{z}_2, \ z = (z_1, z_2) \in \mathbf{B}.$$

The corresponding Kähler form is defined by

$$\omega = -\frac{i}{2\pi} \cdot \sum g_{j\bar{k}} dz^j d\bar{z}^k = -\frac{i}{2\pi} \cdot \partial\bar{\partial} \log N.$$

This $(1,1)$-form is invariant under the action of

$$Aut_{hol}\mathbf{B} = \mathbf{PU}((2,1),\mathbf{C}).$$

The Ricci form of ω is defined as

$$\rho = \frac{i}{2\pi} \cdot \sum R_{j\bar{k}} dz^j d\bar{z}^k$$

with coefficients

$$R_{j\bar{k}} = -\partial^2(\log\det g)/\partial z^j \partial \bar{z}^k.$$

It holds that $\rho = -3 \cdot \omega$. If the Ricci and Kähler form coincide up to a real constant factor, then the underlying metric is called *Kähler-Einstein*. For a short proof in the ball case we refer to [BHH], Appendix B. It turns out that the holomorphic sectional curvature is negative constant. Such metrics are called (complex) *hyperbolic*.

Definition 3.1 . *A* ball lattice *is a discrete subgroup* Γ *of* $G = \mathbf{PU}((2,1),\mathbf{C})$ *(or of* $\mathbf{U}((2,1),\mathbf{C})$*) such that for the volume of a* Γ*-fundamental domain* $\mathfrak{F}_\Gamma$ *of* Γ *on* $\mathbf{B}$ *with respect to any* G*-invariant volume form it holds that*

(i) $vol(\mathfrak{f}_\Gamma) < \infty$, $\quad \mathfrak{f}_\Gamma$ *a* Γ*-fundamental domain.*

The group $\Gamma \subset G$ *is a* neat *ball lattice iff it satisfies (i) and for each element* γ *of* Γ *the eigenvalues of* γ *generate a torsion free subgroup of* $\mathbf{C}^*$*. If* Γ *is neat then*

(ii) Γ *acts freely on* $\mathbf{B}$*, that means each* $id \neq \gamma \in \Gamma$ *acts without fixed points.*

Theorem 3.2 *For neat ball lattices* Γ *it holds that*

(iii) The quotient morphism $\mathbf{B} \longrightarrow \mathbf{B}/\Gamma$ *is a universal covering.*

(iv) $\mathbf{B}/\Gamma$ *supports a complete hyperbolic (Kähler-Einstein) metric.*

(v) The Baily-Borel compactification $\widehat{\mathbf{B}/\Gamma}$ *is a normal projective algebraic surface.*

(vi) $\widehat{\mathbf{B}/\Gamma} = \mathbf{B}/\Gamma \cup \{\hat{\kappa}_1, ..., \hat{\kappa}_h\}$*, with elliptic cusp singularities* $\hat{\kappa}_i$*.*

$\square$

(vi) means that the minimal resolution of each of these cusp singularities $\hat{\kappa}_j$ is an ellipic curve T'_j. For explicit local construction via cusp bundles we refer to [Ho98a], ch. IV. The property (v) holds for any quotient of a symmetric domain by an arithmetic group. This is a theorem of Baily-Borel [BB].

Fixing notations, keep in mind the following diagram in the category of complex surfaces with neat ball lattice Γ:

$$A \xleftarrow{\sigma} A' = X'_\Gamma \xrightarrow{\hat{p}} \hat{X}_\Gamma \longleftarrow X_\Gamma = \mathbf{B}/\Gamma \tag{2}$$

with minimal (elliptic) singularity resolution $\hat{p}$ and (smooth) minimal model A of A'.

4 Picard modular forms

Let Γ be a Picard modular group. It acts via argument shifting on the field $\mathfrak{Mer}(\mathbf{B})$ of meromorphic functions on the ball.

Definition 4.1 . *The invariant field*

$$\mathfrak{F}_\Gamma = \mathfrak{Mer}(\mathbf{B})^\Gamma \subset \mathfrak{Mer}(\mathbf{B})$$

is called the field of Γ-modular functions.

This is an algebraic function field, namely the function field of the quasiprojective surfaces appearing in (2):

$$\mathfrak{F}_\Gamma = \mathbf{C}(X_\Gamma) = \mathbf{C}(\hat{X}_\Gamma) = \mathbf{C}(X'_\Gamma).$$

For each $n \in \mathbf{N}$ we have a representation ρ_n of Γ in $\mathfrak{Mer}(\mathbf{B})$, defined by

$$\rho_n(\gamma):\ f \mapsto j_\gamma^{-n} \cdot \gamma^*(f),\ \gamma \in \Gamma,\ f \in \mathfrak{Mer}(\mathbf{B}),$$

where $j_\gamma(z)$ denotes the Jacobi determinant function of γ on $\mathbf{B}$. The ring of holomorphic functions on $\mathbf{B}$ is denoted by $\mathfrak{Hol}(\mathbf{B})$.

Definition 4.2 . *A holomorphic function f on $\mathbf{B}$ is called a* Γ-modular form *of (algebraic)* weight n *(of Haupttypus) iff it is an eigenfunction of $\rho_n(\Gamma) \subset Aut\,\mathfrak{Hol}(\mathbf{B})$ with eigenvalue* 1. *More explicitly, this means that*

$$\gamma^*(f)(z) = f(\gamma(z)) = j_\gamma^n \cdot f,\ \forall z \in \mathbf{B},\ \gamma \in \Gamma.$$

The $\mathbf{C}$-vector space of all Γ-modular forms of weight n is denoted by $[\Gamma, n]$. The product of forms of weight m or n is a modular form of weight $m+n$. Therefore the direct sum

$$R[\Gamma] := \bigoplus_{n=0}^{\infty} [\Gamma, n]$$

has the structure of a graded ring.

Theorem 4.3 *(Baily-Borel [BB]). Let Γ be a Picard modular group. Then:*

(o) $[\Gamma, 0] = \mathbf{C}$;

(i) $[\Gamma, n]$ *is finite dimensional for all* $n \in \mathbf{N}$;

(ii) $R[\Gamma]$ *is a normal finitely generated* $\mathbf{C}$*-algebra of dimension* 3;

(iii) $\hat{X}_\Gamma \cong Proj\ R[\Gamma]$.

□

This was the way of Baily-Borel to recognize the existence and normal projective structure of the compactification of $X_\Gamma = \mathbf{B}/\Gamma$.

From now on we assume that Γ is a **neat** ball lattice. In this case we have

Proposition 4.4 *(see e.g. [Ho98b]). There is a natural isomorphism of graded rings*

$$R[\Gamma] \cong \bigoplus_{n=0}^{\infty} H^0(X'_\Gamma, (\Omega^2_{X'}(\log T'))^n) \tag{3}$$

onto the ring of logarithmic pluricanonical forms of the (smoothly compactified) quotient surface X'_Γ with (disjoint) elliptic compactification divisor

$$T' = \sum_{j=1}^{h} T'_j. \tag{4}$$

With $X' := X'_\Gamma$ and a canonical divisor $K_{X'}$ on X' we change from bundle sections to rational functions. The divisor $K_{X'} + T'$ is called *logarithmic canonical*, and its multiples $n(K_{X'} + T')$, are *logarithmic pluricanonical divisors* on X' with respect to T'. For $n \in \mathbf{N}$ we have $\mathbf{C}$-vector space isomorphisms

$$[\Gamma, n] \cong H^0(X'_\Gamma, (\Omega^2_{X'}(\log T'))^n) \cong H^0(X', n(K_{X'} + T')). \tag{5}$$

The *cusp points* of a Picard modular group of the imaginary quadratic field K are precisely the K-points of $\partial\mathbf{B}$, that means the boundary points of $\mathbf{B}$ with coordinates in K. Let $\hat{\mathbf{B}}$ be the join of $\mathbf{B}$ and the set of Γ-cusps. The quotient map $\mathbf{B} \longrightarrow \mathbf{B}/\Gamma$ extends surjectively to $\hat{\mathbf{B}} \longrightarrow \hat{X}_\Gamma$. Around cusps κ and their image cusp points $\hat{\kappa} = \Gamma\kappa \subset \partial\mathbf{B}$ it is a well-understood locally analytic map, see [Ho98a], ch. IV.

Definition 4.5 . *A Γ-modular form φ is called a* cusp form *iff it vanishes at each Γ-cusp $\kappa \in \partial\mathbf{B}$.*

Algebraic criterion: $\varphi \in [\Gamma, n]$ is a cusp form iff the corresponding plurilogarithmic canonical form vanishes along T'.

The $\mathbf{C}$-vector space of Γ-cusp forms of weight n is denoted by $[\Gamma, n]_{cusp}$. The isomorphisms (5) restrict to

(6) $$[\Gamma, n]_{cusp} \cong H^0(X', nK_{X'} + (n-1)T') \cong H^0(X'_\Gamma, (\Omega^2_{X'}(\log T'))^n \otimes \mathfrak{T}'^{-1}),$$

where $\mathfrak{T}' = \mathcal{O}_{X'}(T')$.

5 Neat coabelian ball lattices

Definition 5.1 . *A neat ball lattice* Γ *is called* coabelian *if the smooth minimal model of the surface* X'_Γ *is an abelian surface* A*. Such a lattice is called* coabelian of first kind *iff the only curves on* X'_Γ *contractible to a regular point are irreducible.*

In this paper all coabelian ball lattices are assumed to be of first kind. So the exceptional curve L of the birational morphism $A \longleftarrow X'$ is a disjoint sum

(7) $$L = L_1 + ... + L_s,\ \mathbf{P}^1 \cong L_k \subset X',\ (L_k^2) = -1.$$

Each elliptic cusp curve T'_j intersects L properly. Namely, the selfintersection index of T'_j on X' is negative (it resolves the cusp singularity $\hat{\kappa}_j$), and its image curve of T'_j on A must be an elliptic curve $T_j \cong T'_j$ because there is no rational curve on abelian surfaces. Since O is a canonical divisor on A, it follows from the adjunction formula that $(T_j^2) = 0$ on A. Now it is clear that all components of L intersect T'_j at most simply at one point and

$$-(T_j'^2) = (L \cdot T'_j) = \#\{k \in \{1, ..., s\};\ L_k \cap T'_j \neq \emptyset\}.$$

In this case we extend Diagram (2) to the following *Euclidean-Hyperbolic Change Diagram* (EHC)

(8) $$\begin{array}{ccccccccc} & \mathbf{C}^2 & & & & & \mathbf{B} & \\ & \downarrow & & & & & \downarrow & \\ \mathbf{C}^2/\Lambda = & A & \longleftarrow & A' & \longrightarrow & \hat{X}_\Gamma & \hookleftarrow & X_\Gamma & = \mathbf{B}/\Gamma \end{array}$$

with vertical universal coverings and horizontal birational morphisms and $\Lambda \cong \mathbf{Z}^4$ a lattice in $\mathbf{C}^2$.

Now we start from an abelian surface A with (reduced) "elliptic divisor"

$$T = T_1 + ... + T_h \in Div\, A,\ T_j \text{ elliptic curve}.$$

As in the above diagram we blow up all intersection points of the components of T. This is the set $Sing\, T$ of all singularities of the curve T. The inverse image of T is the logarithmic canonical divisor

$$L + T' = L_1 + ... + L_s + T'_1 + ... + T'_h = K_{A'} + T'$$

with exceptional (canonical) divisor $L = K_{A'}$ on A' as described in (7) and elliptic proper transforms T'_j on A'. We look for a criterion which allows to decide whether A' is a neat ball quotient surface X'_Γ with smooth compactification divisor T'. Then we are again in the situation described in the EHC-diagram (8). In this case we call T a *hyperbolic elliptic divisor* on A. We set

$$S_j := T_j \cap Sing\, T, \;\; s := \# Sing\, T, \; s_j := \# S_j.$$

Since all components T'_j of T' are contractable (to a cusp point), the selfintersections $({T'}^2)$ are negative. So we notice

$$1 \leq s_j \leq s, \; j = 1, ..., h$$

as first necessary criterion for T to be hyperbolic. It is equivalent to claim that $s > 0$, because each elliptic curve intersects properly each pair of non-parallel elliptic curves on an abelian surface.

Definition 5.2 . *The* singularity rate *of the (intersecting) divisor* $T = \sum_{i=1}^{h} T_i$ *is denoted and defined by*

$$\mathfrak{r}(T) := \frac{1}{s} \sum_{i=1}^{h} s_i$$

Theorem 5.3 *([Ho00])*

(i) For each elliptic divisor T on an abelian surface A the singularity rate $\mathfrak{r}(T)$ is not greater than 4.

(ii) T is hyperbolic if and only if $\mathfrak{r}(T) = 4$.

□

For the proof in [Ho00] we needed the precise characterization of ball quotient surfaces ([Ho98a]) in general, working with (rational) orbital heights, combined with the Cyclic Covering Theorem for construction of suitable cyclic covers of A of general type.

Example 5.4 *[Ho00]. Let A be the biproduct $E \times E$ of the elliptic curve*

$$E: \, Y^2 = X^3 + X, \; A = A(\mathbf{C}) = \mathbf{C}^2/\Lambda, \; \Lambda \cong \mathbf{Z}[\mathrm{i}]^2.$$

E has complex multiplication with the field $\mathbf{Q}(\mathrm{i})$ of Gauß numbers. On $\mathbf{C}^2$ we define eight lines by equations

$$\begin{array}{llll} u+v = 0, & u-v = 0, & u+iv = 0, & u-iv = 0, \\ u+\omega_1 = 0, & u+\omega_2 = 0, & v+\omega_1 = 0, & v+\omega_2 = 0, \end{array} \tag{9}$$

with Λ-incongruent half periods $\omega_1, \omega_2 \in \frac{1}{2}\Lambda \setminus \frac{1+\mathrm{i}}{2}\Lambda$. Along the universal covering $\mathbf{C}^2 \longrightarrow A$ their images are elliptic curves $T_1, ..., T_8$ on A. For $T = T_1 + ... + T_8$ it is not difficult to determine the pairwise intersection points, see [Ho00]. The hyperbolic condition (ii) is satisfied again:

$$\begin{aligned} s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 &= 4+4+4+4+2+2+2+2 \\ &= 24 = 4 \cdot 6 = 4 \cdot s. \end{aligned} \tag{10}$$

Remark 5.5 . *The corresponding ball lattice Γ is Picard modular. This has been proved also in [Ho00] showing that A' is a finite covering of the projective plane with orbital Apollonius cycle blown up at K_1, K_2, K_3 such that Γ is a normal subgroup of finite index in $\mathbf{SU}((2,1), \mathbf{Z}[\mathrm{i}])$. We refer back to section 2.*

6 Dimension formulas for modular forms of neat coabelian lattices and the modular η-form

In the happy case of a neat coabelian lattice Γ we dispose on dimension formulas for $[\Gamma, n]_{cusp}$ and almost all $[\Gamma, n]$.

Theorem 6.1 *([Ho01]). For neat coabelian lattices Γ with elliptic compactification curve $T' = T_1' + ... + T_h'$ it holds that*

$$\dim[\Gamma, n]_{cusp} = \begin{cases} 1, & \text{if } n = 1; \\ 3\binom{n}{2}s, & \text{if } n > 1. \end{cases} \tag{11}$$

$$\dim[\Gamma, n] = \begin{cases} 1, & \text{if } n = 0; \\ 3\binom{n}{2}s + h, & \text{if } n > 1. \end{cases} \tag{12}$$

□

The proof was a combination of Riemann-Roch, Kodaira vanishing, Baily-Borel embedding theorems, (long) exact sequence and local compactification techniques. The dimension formulas recover the number h of cusp points:

Corollary 6.2 *For $n \geq 2$ and coabelian neat ball lattices Γ it holds that*

$$\#(\hat{X}_\Gamma \setminus X_\Gamma) = h = \dim([\Gamma, n]/[\Gamma, n]_{cusp}),$$

□

Using the EHC-Diagram (8) we want to transfer certain Λ-periodic meromorphic functions on $\mathbf{C}^2$ to Γ-modular forms. We work with global coordinates u, v on $\mathbf{C}^2$ and z_1, z_2 on $\mathbf{B}$. By abuse of language we call the former *euclidean* and the latter *hyperbolic* coordinates. So we dispose around each point

$$P \in A \setminus supp\, T = A' \setminus supp\,(L + T') = X_\Gamma \setminus supp\, L \tag{13}$$

simultaneously on local euclidean coordinates u, v and local hyperbolic coordinates z_1, z_2.

Via holomorphic sections of logarithmic pluricanonical bundles the **C**-linear isomorphisms

$$\iota_n : H^0(A', n(L+T')) \longrightarrow [\Gamma, n], \ f \mapsto \varphi^{(n)} =: \iota_n(f)$$

are realized in two steps. First by local euclidean-hyperbolic coordinate transfer

$$f \cdot (\mathrm{d}u \wedge \mathrm{d}v)^{\otimes n} = \varphi^{(n)} (\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes n} \tag{14}$$

around any point P as in (13). Then we consider $\varphi^{(n)}$ as local analytic function on the ball $\mathbf{B}$. We use the same notation $\varphi^{(n)}$ for its (unique) analytic extension to $\mathbf{B}$, which belongs to $[\Gamma, n]$. Since f belongs also to $H^0(A', N(L+T'))$ for all $N \geq n$, it produces a series of Γ-modular functions $\varphi^{(N)} = \iota_N(f)$. Fortunately, we are able to understand the quotients $\varphi^{(N)}/\varphi^{(n)}$ as powers of a remarkable cusp form, one and the same for all f.

Proposition 6.3 *There is a cusp form $\eta = \eta_\Gamma$ generating the vector space $[\Gamma, 1]_{cusp}$ such that the diagram*

$$\begin{tikzcd} H^0(A', n(L+T')) \arrow[r] \arrow[d, "\iota_n"'] & H^0(A', N(L+T')) \arrow[d, "\iota_N"] \\ {[\Gamma, n]} \arrow[r, "\cdot \eta^{N-n}"] & {[\Gamma, N]} \end{tikzcd}$$

are commutative for all $0 \leq n \leq N$. This means that

$$\varphi^{(N)} = \iota_N(f) = \eta^{N-n} \cdot \iota_n(f) = \eta^{N-n} \cdot \varphi^{(n)}$$

Proof. With the constant function $1 = 1_{A'} \in H^0(A', O)$ on A' we define

$$\eta = \eta_\Gamma \ := \iota_1(1_{A'}) \tag{15}$$

or, by local equation (14),

$$1_{A'} \cdot (\mathrm{d}u \wedge \mathrm{d}v) = \mathrm{d}u \wedge \mathrm{d}v = \eta \cdot \mathrm{d}z_1 \wedge \mathrm{d}z_2. \tag{16}$$

For products $f \cdot h$, $f \in H^0(A', n(L+T'))$, $h \in H^0(A', m(L+T'))$ the corresponding local equations yield

$$\iota_{n+m}(f \cdot h) = \iota_n(f) \cdot \iota_m(h) \in H^0(A', (n+m)(L+T')).$$

Especially, for $h = 1_{A'} \in H^0(A', m(L+T'))$, $m = N - n$, we get

(17)
$$\iota_N(f) = \iota_{n+m}(f \cdot 1_{A'}) = \iota_n(f) \cdot \iota_m(1_{A'}) = \iota_n(f) \cdot (\iota_1(1_{A'}))^m = \iota_n(f) \cdot \eta^{N-n}.$$

From the definitions (16) and (15) it is clear that $0 \neq \eta \in [\Gamma, 1]$. In Lemma 7.3 below we will see that η must be a cusp form. The dimension formula (11) for $n = 1$ yields

$$[\Gamma, 1]_{cusp} = \mathbf{C}\eta = \mathbf{C}\eta_\Gamma.$$

□

Convention. For each $f \in H^0(A', N(L + T'))$ there is a minimal $n \in \mathbf{N}$ such that $f \in H^0(A', n(L + T'))$. We omit the index n, if n is choosen for f in this minimal manner setting

$$\varphi = \iota(f) = \iota_n(f) = \iota_N(f)/\eta^{N-n}.$$

Clearly, φ is not divisible by η. Starting from a Γ-modular form $\varphi^{(N)} \in [\Gamma, N]$ we see that n is defined as smallest natural number such that $\varphi^{(N)}$ is decomposable into $\varphi^{(n)}\eta^{N-n}$ in the ring of holomorphic functions on $\mathbf{B}$.

It may happen that

$$f \cdot g = h,\ f \in H^0(A', k(L + T'),\ g \in H^0(A', m(L + T'),\ h \in H^0(A', n(L + T'),$$

but $n < k + m$, even if k and m are minimal choosen. Setting $\varphi = \iota_k(f)$, $\gamma = \iota_m(g)$ and $\chi = \iota_n(h)$ it is easy to see from the definition of ι_{k+m} that the above *inhomogeneous relation* transferes by means of the η-form to the *homogenized relation*

$$\varphi \cdot \gamma = \eta^{k+m-n} \cdot \chi.$$

So the abstractly defined η plays algebraically the role of a homogenizing element. It gives also the chance to determine η as a root of the quotient of $\varphi{\cdot}\gamma/\chi$ of explicitly known modular forms $\varphi \cdot \gamma$,χ . More generally, each (inhomogeneous) polynomial relation

$$0 = \sum_{\mathfrak{i} \in \mathbf{N}^r} c_{\mathfrak{i}} f_1^{i_1} \cdot \ldots \cdot f_r^{i_r},\ c_{\mathfrak{i}} \in \mathbf{C}, \quad \mathfrak{i} = (i_1, ..., i_r),$$

in the ring $H^0(A', *(L + T')) := \bigcup_{n=0}^{\infty} H^0(A', n(L + T'))$ transfers to a homogeneous relation

(18)
$$0 = \sum_{\mathfrak{i} \in \mathbf{N}^r} c_{\mathfrak{i}} \varphi_1^{i_1} \cdot ... \cdot \varphi_r^{i_r} \eta^{N-n_{\mathfrak{i}}},$$

in $[\Gamma, N]$ with functions $f_j \in H^0(A', k_j(L + T'))$, $\varphi_j = \iota_{k_j}(f_j)$, $f_1^{i_1} \cdot ... \cdot f_r^{i_r} \in H^0(A', n_{\mathfrak{i}}(L+T'))$, $N = max\{n_{\mathfrak{i}}\}$. One has only to multiply the inhomogeneous polynomial with $(\mathrm{d}u \wedge \mathrm{d}v)^N$, to apply the transformations (17) together with (14) to each summand and to cancel the common factor $(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^N$.

Remark 6.4 . *Via quotients of functions one gets the well-defined and well-known isomorphism of function fields*

$$\begin{array}{l} \Upsilon : \mathfrak{Mer}(A) \xrightarrow{\sim} \mathfrak{F}_\Gamma = \mathfrak{Mer}(\mathbf{B})^\Gamma, \\ \qquad f/g \quad \mapsto \quad \Upsilon(f/g) := \iota_n(f)/\iota_n(g), \ f, g \in H^0(A', n(L+T')). \end{array}$$

Especially, $f = f/1 \in H^0(A', n(L+T)$ *corresponds to* $\iota_n(f)/\eta^n$.

7 Modular transfer of abelian functions

We write $(f)_A$ or $(f)_{A'}$ for the (principal) divisor on A or A' of an (abelian) function $f \in \mathfrak{Mer}(A) = \mathfrak{Mer}(A')$. For the Λ-periodic meromorphic function F on $\mathbf{C}^2$ corresponding to f we define

$$(F)_A := (f)_A \in Div\ A, \quad (F)_{A'} := (f)_{A'} \in Div\ A'.$$

Similarly, for a Γ-modular function $\varphi \in \mathfrak{Mer}(\mathbf{B}/\Gamma) = \mathfrak{Mer}(A')$ we use the notation $(\varphi)_{A'}$ for the corresponding divisor on A'. The divisor $(\omega)_A$ of higher differential forms $\omega \in H^0(A, (\Omega_A^2)^{\otimes n})$ on A is globally defined by patching local divisors of the functions f appearing in local presentations $\omega = f(\mathrm{d}u \wedge \mathrm{d}v)^{\otimes n}$. Comparing after blowing up local coordinates around points we get

$$(\mathrm{d}u \wedge \mathrm{d}v)_{A'} = \sigma^*((\mathrm{d}u \wedge \mathrm{d}v)_A) = L,$$

hence

$$\sigma^*((\omega)_A) = \sigma^*((f)_A) + nL = (f)_{A'} + nL.$$

The geometry around cusps (Jacobi-Fourier series, see [He] or [Ho98a]) forces us to set $(\mathrm{d}z_1 \wedge \mathrm{d}z_2)_{A'} = -T'$ and extend it to

$$(\varphi(\mathrm{d}z_1 \wedge \mathrm{d}z_2)^{\otimes n})_{A'} = (\varphi)_{A'} - nT'$$

definigng $(\varphi)_{A'} \in Div\, A'$ for modular forms $\varphi \in [\Gamma, n]$. The coefficients coincide with the zero orders of φ along irreducible (local) curves on $\mathbf{B}$ lifted from A' and, by definition, at cusps related with the elliptic compactification curves T'_j. With the definition (14) of ι_n we are able to compare divisors, namely

$$(f)_{A'} + nL = (\varphi)_{A'} - nT', \ \varphi = \iota_n(f),$$

hence

$$(\varphi)_{A'} = (f)_{A'} + n(L+T'), \tag{19}$$

which can be also used as short natural definition. For instance,

$$(\eta)_{A'} = (1)_{A'} + (L+T') = L+T', \ \eta = \iota_1(1).$$

We want to define for φ a cycle (of zeros) on $\hat{\mathbf{B}}$. For this purpose we choose a representative set $\kappa = \{\kappa_1, ..., \kappa_h\}$ of Γ-cusps on the boundary of $\mathbf{B}$ corresponding elementwise to the elliptic cusp curves $T'_1, ..., T'_h$ on A'. Let

$$Div_\kappa \mathbf{B} = \mathbf{Z}\kappa_1 \oplus ..... \oplus \mathbf{Z}\kappa_h$$

be the free abelian group generated by the points $\kappa_1, ..., \kappa_h$. For φ we denote the (zero) order at κ_j by $v_{\kappa_j}(\varphi)$ coinciding with the T'_j-coefficient of $(\varphi)_{A'}$. The cusp cycle of φ on κ is defined as

(20)

$$\begin{aligned}(\varphi)_\kappa \ := v_{\kappa_1}(\varphi)\kappa_1 + ... + v_{\kappa_h}(\varphi)\kappa_h &= v_{T'_1}(\varphi)\kappa_1 + ... + v_{T'_h}(\varphi)\kappa_h \\ &= (n + v_{T_1}(f))\kappa_1 + ... + (n + v_{T_h}(f))\kappa_h\end{aligned}$$

The value of the modular function

$$\frac{\varphi}{\eta^n} = \frac{\iota_n(f)}{\eta^n} = \Upsilon(f) at \kappa_j$$

is well-defined as

$$\Upsilon(f)(\kappa_j) = \frac{\varphi}{\eta^n}(\kappa_j) \ := f(\hat{\kappa}_j) = f(T'_j) = f(T_j) \in \mathbf{C} \cup \{\infty\}$$

via Hartogs' theorem, if the value is finite.

Remark 7.1 . *Also φ itself has a well-defined values $\varphi(\kappa_j)$ at κ_j. Each is defined as the constant term of the Fourier-Jacobi (power) series of φ at κ_j (after fixed normalized transfer to a standard cusp, see [He]). There is no pole. In correspondence with (20) $\varphi = \iota_n(f)$ does not vanish at κ_j iff $v_{T_j}(f) = -n$. In this case we call φ* not cuspidal *at κ_j. In the opposite case we say that φ is* cuspidal *there.*

Lemma 7.2 *(disc criterion) Assume that stabilizers G_P at $P \in S = Sing T$ of the group*

$$G = G(A, T) = \{g \in Aut A;\ g(T) = T\}$$

are bigger than the cyclic (T_k-centralizer) groups

$$Z_{T_k}(G) \ := \{g \in G;\ g|_{T_k} = id_{T_k}\}, \quad \forall T_k \ni P.$$

Then the smooth rational curves $\hat{p}(L_j)$ on $\widehat{\mathbf{B}/\Gamma}$ are compactified disc quotients

$$L_j \cong \hat{p}(L_j) = \widehat{\mathbf{D}_j/\Gamma_{\mathbf{D}_j}},\ j = 1, ..., h,$$

where the $\mathbf{D}'_j s$ are linear subdiscs of the ball $\mathbf{B}$ and

$$\Gamma_{\mathbf{D}_j} = N_\Gamma(\mathbf{D}_j)/Z_G(\mathbf{D}_j),\ N_\Gamma(\mathbf{D}_j) = \{\gamma \in \Gamma;\ \gamma(\mathbf{D}_j) = \mathbf{D}_j\}.$$

Proof. The stabilizer condition implies the existence of an element $g \in G_P$ with P as isolated fixed point. This element acts on A' as reflection with $L_P = \sigma^{-1}(P)$ as reflection curve. The elements of G act also on $\mathbf{B}/\Gamma \subset \hat{X}_\Gamma$ and lift to $\mathbf{B}$ along the universal covering $\mathbf{B} \longrightarrow \mathbf{B}/\Gamma$. Especially, g is lifted to a reflection on $\mathbf{B}$ because of local isomorphy. The reflection curve is a linear disc $\mathbf{D}_j$ because $Aut\,\mathbf{B}$ consists of linear transformations.

□

Through the paper we will assume that the disc criterion of the lemma is always satisfied. We choose representative discs $\mathbf{D}_j$ lifting L_j, $j = 1, ..., h$, set $\mathbf{D} = \{\mathbf{D}_1, ..., \mathbf{D}_h\}$ and define the free abelian group

$$Div_{\mathbf{D}}\mathbf{B} = \mathbf{Z}\mathbf{D}_1 \oplus ... \oplus \mathbf{Z}\mathbf{D}_h.$$

For each Γ-modular function $\varphi = \iota_n(f)$ we denote the zero orders along $\mathbf{D}_j$ by $v_{\mathbf{D}_j}(\varphi)$. They coincide with $v_{L_j}(f)$, which is equal to the L_j-coeffitient of $(\varphi)_{A'} = (f)_{A'} + n(L + T') \in Div\ A'$. We write

$$\begin{aligned}(\varphi)_{\mathbf{D}} &:= v_{\mathbf{D}_1}(\varphi)\mathbf{D}_1 + ... + v_{\mathbf{D}_h}(\varphi)\mathbf{D}_h, \\ &= (n + v_{L_1}(f))\mathbf{D}_1 + ... + (n + v_{L_h}(f))\mathbf{D}_h \in Div_{\mathbf{D}}\mathbf{B},\end{aligned}$$

which defines also the $\hat{\mathbf{D}}$-cycle

$$(\varphi)_{\hat{\mathbf{D}}} := (\varphi)_{\mathbf{D}} + (\varphi)_\kappa \tag{21}$$

on

$$Div_{\hat{\mathbf{D}}}\mathbf{B} := Div_{\mathbf{D}}\mathbf{B} \oplus Div_\kappa\mathbf{B} = \mathbf{Z}\mathbf{D}_1 \oplus ... \oplus \mathbf{Z}\mathbf{D}_h \oplus \mathbf{Z}\kappa_1 \oplus ..... \oplus \mathbf{Z}\kappa_h.$$

For instance,

$$(\eta)_{\hat{\mathbf{D}}} = \mathbf{D}_1 + ... + \mathbf{D}_h + \kappa_1 + ... + \kappa_s.$$

By the way we proved

Lemma 7.3 *The η-function $\iota_1(1)$ has simple zeros at the cusps and also along the discs lifted from the lines L_j. It has no other zeros on $\hat{\mathbf{B}}$.*

□

For $\varphi = \iota_n(f)$ we want to relate the T_j-coefficients of $(f)_A$ with the coefficients of $(\varphi)_{\hat{\mathbf{D}}}$. Let D be a divisor on A, $D' = \sigma'(D)$ its proper transform and $D^* = \sigma^*(D)$ its inverse image on A'. The embeddings

$$\sigma',\, \sigma^* :\ Div\ A \longrightarrow Div\ A' \cong \sigma'(Div\ A) \oplus \mathbf{Z}L_1 \oplus ... \oplus \mathbf{Z}L_s$$

are connected by the formula

$$\sigma^*(D) = \sigma'(D) + \mu(D) \cdot L,$$

where

$$\mu(D)\cdot L := \mu_1(D)L_1 + ... + \mu_s(D)L_s$$

with the s-tuple

$$\mu(D) = (\mu_1(D), ..., \mu_s(D)) \in \mathbf{Z}^s$$

of multiplicities $\mu_j(D) = \mu_{P_j}(D)$ of D at P_j. Notice that the multiplicity map $\mu : \; Div\ A \longrightarrow \mathbf{Z}^s$ is additive. Important is the multiplicity tupel of T

$$\mu(T) =: (t_1, ..., t_s).$$

We set for $f \in \mathfrak{Mer}(A)$

$$\begin{aligned}\mu(f) &:= \mu((f)_A),\ v_T(f) := (v_{T_1}(f), ..., v_{T_h}(f)),\\ v_T(f)\cdot T' &:= v_{T_1}(f)T_1' + ... + v_{T_h}(f)T_h'\end{aligned}$$

Restricting to functions $f \in H^0(A, kT)$, $k \geq 0$, we get

$$(f)_{A'} = \sigma'((f)_A) + \mu(f)\cdot L = \sigma'((f)_A^0) + v_T(f)\cdot T' + \mu(f)\cdot L \tag{22}$$

where $(f)_A^0$ is the positive part (zero divisor). Restricting to the part of the divisor supported by $T' + L$ we write

$$(f)_{L+T'} = \mu(f)\cdot L + v_T(f)\cdot T'. \tag{23}$$

We proved

Lemma 7.4 *The abelian function f with poles only on T belongs to $H^0(A', nL + kT')$ if and only if*

(i) $\mu(f) \geq -(n, ..., n) \in \mathbf{Z}^s$.

(ii) $v_T(f) \geq -(k, ..., k) \in \mathbf{Z}^h$,

□

Assume that $k = n$ and the conditions (i) and (ii) are satisfied. Then the Γ-modular form $\varphi = \iota_n(f)$ of weight n is well-defined. From (19), (21), (22) and (23) it follows that

$$\begin{aligned}(\varphi)_{A'} &= \sigma'((f)_A^0) + (\mu(f) + (n, ..., n))\cdot L + (v_T(f) + (n, ..., n))\cdot T',\\ (\varphi)_{L+T'} &= (\mu(f) + (n, ..., n))\cdot L + (v_T(f) + (n, ..., n))\cdot T',\\ (\varphi)_{\hat{\mathbf{D}}} &= (\mu(f) + (n, ..., n))\cdot \mathbf{D} + (v_T(f) + (n, ..., n))\cdot \kappa,\\ &= (v_L(f) + (n, ..., n))\cdot \mathbf{D} + (v_T(f) + (n, ..., n))\cdot \kappa\end{aligned} \tag{24}$$

where $\mathbf{D}$, $v_L(f)$, κ stand for $(\mathbf{D}_1, ..., \mathbf{D}_s)$, $(v_{L_1}(f), ..., v_{L_s}(f))$ respectively $\kappa = (\kappa_1, ..., \kappa_h)$ and the $\cdot$'s are understood as sum of componentwise products again. We see also more immediately now that

Corollary 7.5 *An abelian function $f \in \mathfrak{Mer}(A)$ belongs to $H^0(A', n(L+T'))$ for a suitable $n \geq 0$ if and only if the support of its pole divisor on A is supported by T. The smallest possible n coincides with $-min\{0, \mu_1(f), ..., \mu_s(f)\}$. The Γ-modular form $\varphi = \iota_n(f)$, $f \in H^0(A', n(L+T'))$, is a cusp form iff $v_T(f) \geq -(n-1, ..., n-1)$.*

8 Some classical elliptic theta functions and their quotients

First we remember to some classical quasi-periodic functions on $\mathbf{C}$ with respect to a lattice $\mathbf{Z}^2 \cong \Lambda \subset \mathbf{C}$. Fixing notations we work with half periods

$$\omega_1,\, \omega_2,\, \omega_3 = \omega_1 + \omega_2 \in \frac{1}{2}\Lambda$$

such that

$$\Lambda = \mathbf{Z} \cdot 2\omega_1 + \mathbf{Z} \cdot 2\omega_2,\ \tau\ := \omega_2/\omega_1 \in \mathbf{H}:\ Im\,\tau > 0.$$

The classical σ-function on $\mathbf{C}$ is defined by

$$\sigma(z) = \sigma_\Lambda(z) = z \cdot \prod_{\lambda \in \Lambda^{\cdot}} ((1 - \frac{z}{\lambda}) \mathrm{e}^{\frac{z}{\lambda} + \frac{z^2}{\lambda^2}})$$

with $\Lambda^{\cdot} = \Lambda \setminus \{0\}$. This is an odd function. The Weierstraß ζ-function is defined as

$$\zeta(z) = \zeta_\Lambda(z) = \frac{d}{dz} \log \sigma(z) = \sigma'(z)/\sigma(z).$$

The following basic properties can be found in several textbooks of function theory, e.g. in [Pr] and [Rue].

1.) $\sigma(z + 2\omega_k) = -\mathrm{e}^{2\eta_k(z+\omega_k)} \cdot \sigma(z)$ with constants $\eta_k := \zeta(\omega_k)$, $k = 1, 2$.

2.) The quotient $\prod_{i=1}^r \sigma(z - a_i)/\prod_{i=1}^r \sigma(z - b_i)$, $a_i, b_i \in \mathbf{C}$, is an elliptic function if and only if $\sum_{i=1}^r (a_i - b_i) = 0$.

3.) $\sigma(z) = 2\omega_1 \mathrm{e}^{\eta_1 \frac{z^2}{2\omega_1}} \cdot \vartheta_1(\frac{z}{2\omega_1})/\vartheta_1'(0)$, with classical theta series

$$\vartheta_1(z) = \vartheta_1(z, \tau) = -\mathrm{i}q^{1/4}\mathrm{e}^{\mathrm{i}\pi z} \cdot \vartheta_0(z + \frac{\tau}{2}, \tau),$$

where

$$\begin{aligned} \vartheta_0(z) = \vartheta_0(z,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \mathrm{e}^{2\pi \mathrm{i} n z} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n^2}{2}} (\mathrm{e}^{-2\pi \mathrm{i} n z} + \mathrm{e}^{2\pi \mathrm{i} n z}) \\ &= 1 + 2 \cdot \sum_{n=1}^{\infty} (-1)^n q^{\frac{n^2}{2}} \cos(2\pi n z), \quad q = \mathrm{e}^{2\pi \mathrm{i} \tau}. \end{aligned}$$

We choosed the notations and detailled presentation of [Rue] in order to recognize the series expansion of the σ-function in the most convenient manner.

4.) The derivative of the Weierstraß ζ-function is the negative of the Weierstraß $\wp$-function of the lattice Λ: $\zeta'(z) = -\wp(z)$.

The quasi-periodic property 1.) indicates that $\sigma(z)$ is a theta function in a general sense. It extends really to

1'.) $\sigma(z + 2\omega) = \psi(2\omega)\mathrm{e}^{\eta(2\omega)(z+\omega)}$, $\omega \in \frac{1}{2}\Lambda$,

Thereby $\eta : \Lambda \longrightarrow \mathbf{C}$ is the additive homomorphism defined by

$$\zeta(z + \lambda) = \zeta(z) + \eta(\lambda),\ \lambda \in \Lambda.$$

This is correct because ζ' is a Λ-periodic function by 4.). The homorphism extends linearly to $\eta : \mathbf{C} \longrightarrow \mathbf{C}$. Moreover, ψ is the character on $(\Lambda, +)$ with exact sequence

$$0 \longrightarrow 2\Lambda \longrightarrow \Lambda \xrightarrow{\psi} \{\pm 1\} \longrightarrow 1.$$

We refer to [La], Appendix 2.

Altogether one gets a group homomorphism

$$\frac{1}{2}\Lambda \longrightarrow \mathcal{O}^*_{hol}(\mathbf{C}),\ \omega \mapsto \epsilon^{\omega}(z) := \psi(2\omega)\mathrm{e}^{\eta(\omega)z}.$$

We need it for the functional relations

$$\sigma(z + \omega) = \epsilon^{\omega}(2z)\sigma(z - \omega) = \begin{cases} \mathrm{e}^{2\eta(\omega)z}\sigma(z - \omega), & \text{if } \omega \in \Lambda; \\ -\mathrm{e}^{2\eta(\omega)z}\sigma(z - \omega), & \text{if } \omega \in \frac{1}{2}\Lambda \setminus \Lambda. \end{cases} \tag{25}$$

9 Σ-quotients in dimension 2

Let $K = \mathbf{Q}(\sqrt{-d}$, $d \in \mathbf{N}_+$, be an imaginary quadratic number number field, $\mathfrak{O} = \mathfrak{O}_K$ its ring of integers, $c \in \mathbf{C}^*$ a constant specified later. For the $\mathbf{C}^2$-lattice

$$\Lambda = \Lambda_1 \times \Lambda_1,\ \Lambda_1 = c\mathfrak{O}$$

We want to construct Λ-periodic functions on $\mathbf{C}^2$ in explicit manner. We will pick them out from the image of the following map:

$$Mat_2(K) \times (\mathbf{C}^{2m} \times \mathbf{C}^{2l} \times \mathbf{C}^{2n}) \longrightarrow \mathfrak{Mer}(\mathbf{C}^2),\ m, l, n \in \mathbf{N}$$

sending $(M; \begin{pmatrix} \mu & \lambda & \nu \\ \mu' & \lambda' & \nu' \end{pmatrix})$ to

$$\Sigma_M^{\begin{smallmatrix} \mu & \lambda & \nu \\ \mu' & \lambda' & \nu' \end{smallmatrix}} := \frac{\prod_{i=1}^m \sigma_{\mathfrak{m}_1\mu_i} \cdot \prod_{j=1}^l \sigma_{\mathfrak{m}_3\lambda_j} \cdot \prod_{k=1}^n \sigma_{\mathfrak{m}_2\nu_k}}{\prod_{i=1}^m \sigma_{\mathfrak{m}_1\mu'_i} \cdot \prod_{j=1}^l \sigma_{\mathfrak{m}_3\lambda'_j} \cdot \prod_{k=1}^n \sigma_{\mathfrak{m}_2\nu'_k}}, \tag{26}$$

where

$$M \in Mat_2(K),\ \mu, \mu' \in \mathbf{C}^m,\ \lambda, \lambda' \in \mathbf{C}^l,\ \nu, \nu' \in \mathbf{C}^n,$$

$\mathfrak{m}_1$, $\mathfrak{m}_2$ are the rows of M, $\mathfrak{m}_3 = \mathfrak{m}_1 + \mathfrak{m}_2$, and

$$\sigma_{\mathfrak{m},\rho}(u, v) \ := \sigma(\mathfrak{m} \cdot \mathfrak{u} + \rho),$$

with

$$\mathfrak{m} = (m_1, m_2) \in K^2,\ \mathfrak{u} = (u, v) \in \mathbf{C}^2,\ \rho \in \mathbf{C},\ \mathfrak{m} \cdot \mathfrak{u} = m_1 u + m_2 v.$$

Theorem 9.1 *For $M \in \mathbf{Gl}_2(\mathfrak{O})$ the meromorphic function*

$$\Sigma(u, v) = \Sigma_{M}^{\begin{smallmatrix} \mu & \lambda & \nu \\ \mu' & \lambda' & \nu' \end{smallmatrix}}(u, v)$$

is Λ-periodic if

$$\sum_{i=1}^{m}(\mu_i - \mu'_i) = -\sum_{j=1}^{l}(\lambda_i - \lambda'_i) = \sum_{k=1}^{n}(\nu_i - \nu'_i). \tag{27}$$

Proof. Since $\Lambda = (\Lambda_1 \times \mathfrak{o}) \oplus (\mathfrak{o} \times \Lambda_1)$, it suffices to show that our Σ-quotients are Λ_1-periodic for fixed u and for fixed v under the assumption (27). Fix v, for instance. We are in a similar situation as in 2.), section 8, which is a consequence of 1.). For simplicity we work with the constant $c = 1$.

First case: $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. With $\mathfrak{m}_1 = (1, 0)$, $\mathfrak{m}_2 = (0, 1)$, $\mathfrak{m}_3 = (1, 1)$ the factors of $\Sigma(u, v)$ are:

$$\prod_{i=1}^{m} \sigma(u + \mu_i),\ \prod_{j=1}^{l} \sigma(u + v + \lambda_j),\ \prod_{k=1}^{n} \sigma(v + \nu_k);$$
$$\prod_{i=1}^{m} \sigma(u + \mu'_i), \prod_{j=1}^{l} \sigma(u + v + \lambda'_j), \prod_{k=1}^{n} \sigma(v + \nu'_k).$$

According to 1.), section 8, in the denominator and numerator of the function $\Sigma(u+2\omega_p, v)$, $p=1,2$, appear six products

$$(-1)^m \prod_{i=1}^{m} \sigma(u+\mu_i)\cdot \exp(2\eta_p \sum_{i=1}^{m}(u+\mu_i+\omega_p)),$$
$$(-1)^l \prod_{j=1}^{l} \sigma(u+v+\lambda_j)\cdot \exp(2\eta_p \sum_{j=1}^{l}(u+v+\lambda_j+\omega_p)),$$
$$(-1)^m \prod_{i=1}^{m} \sigma(u+\mu_i')\cdot \exp(2\eta_p \sum_{i=1}^{m}(u+\mu_i'+\omega_p)),$$
$$(-1)^l \prod_{j=1}^{l} \sigma(u+v+\lambda_j')\cdot \exp(2\eta_p \sum_{j=1}^{l}(u+v+\lambda_j'+\omega_p)),$$
$$\prod_{k=1}^{n} \sigma_(v+\nu_k)\ ,\ \prod_{k=1}^{n} \sigma_(v+\nu_k').$$

Comparing products in numerator and denominator we see that $\sum(u+2\omega_p, v) = \Sigma(u,v)$ holds, if

$$\sum_{i=1}^{m} \mu_i + \sum_{j=1}^{l} \lambda_i = \sum_{i=1}^{m} \mu_i' + \sum_{j=1}^{l} \lambda_i'$$

Fixing u instead of v we get periodicity in the second argument, if the relations

$$\sum_{j=1}^{l} \lambda_i + \sum_{k=1}^{n} \nu_k = \sum_{j=1}^{l} \lambda_i' + \sum_{k=1}^{n} \nu_k'$$

are satisfied. Both relations together are equivalent to (27).

We let the group $\mathbf{Gl}_2(K)$ act on $Mat_2(K)$ by multiplication from the right side. It transfers to an action on the space of our σ-functions $\sigma_M = \sigma_M^{\substack{\mu\ \lambda\ \nu \\ \mu'\ \lambda'\ \nu'}}$. Fixing the upper translation indices and writing the variables as column we have

$$\Sigma_{MG}(u,v) = \Sigma_M(G\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)) = G^*(\Sigma_M),\ G \in \mathbf{Gl}_2(K).$$

Lemma 9.2 *The function σ_M is Λ-periodic iff σ_{MG} is $G^{-1}\Lambda$-periodic.*

Proof. For $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in \Lambda$ we get

$$\Sigma_{MG}(\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) + G^{-1}\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)) = \Sigma_M(G\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)) = \Sigma_M(G\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)) = \Sigma_{MG}(u,v).$$

In order to finnish the proof of the theorem take $G \in \mathbf{Gl}_2(\mathfrak{O})$. We know already by our first step that Σ_E, E the unit matrix of order 2, is Λ-periodic, if the relations (27) are satisfied. Therefore, by the lemma, $\Sigma_G = \Sigma_{EG}$ is a $G^{-1}\Lambda = \Lambda$-periodic function. The relations have not been changed.

$\square$

Remark 9.3 . *It may happen that some of the six products in our Σ-quotient do not really occur. This means they are equal to* 1. *Indeed, we allowed* m, l *or* n *in (26) to be equal* 0. *In this case the corresponding difference sum in the relations (27) has to be substituted by* 0, *too. A new proof is not nessessary because it is the same to assume that for* m, l *or* n *the corresponding two products in numerator and denominator coincide, hence cancel. For instance* $l = 0$ *is equivalent to* $l > 0$ *and* $\lambda = \lambda' \in \mathbf{C}^l$.

Remark 9.4 . *Applications of affine transformations of* $\mathbf{C}^2$ *with* $\mathbf{Gl}_2(\mathfrak{O})$*-linear part also do not change the linear relations (27). Besides of the* $\mathbf{Gl}_2(\mathfrak{O})$*-transforma-tion one has only to change the* μ_i, μ'_i *by* $\mu_i + c$, $\mu'_i + c$, *the* ν_k, ν'_k *by* $\nu_k + d$, $\nu'_k + d$, *and the* λ_j, λ'_j *by* $\lambda_j + c + d$, $\lambda'_j + c + d$, *respectively, where* $(c, d) \in \mathbf{C}^2$ *is the translation vector of the affine transformation (image of* $(0, 0)$*).*

10 Group actions

The group action of $\mathbf{Gl}_2(K)$ on Σ-quotients is accomponied with the $(m+l+n)$-th power of the translation group $\mathfrak{T}_2(\mathbf{C}) = (\mathbf{C}^2, +)$ via the upper index vectors:

$$\begin{pmatrix} \lambda_i \\ \lambda'_i \end{pmatrix} \mapsto \begin{pmatrix} \lambda_i \\ \lambda'_i \end{pmatrix} + \begin{pmatrix} a_i \\ a'_i \end{pmatrix}, \begin{pmatrix} \mu_j \\ \mu'_j \end{pmatrix} \mapsto \begin{pmatrix} \mu_j \\ \mu'_j \end{pmatrix} + \begin{pmatrix} b_j \\ b'_j \end{pmatrix} \begin{pmatrix} \nu_k \\ \nu'_k \end{pmatrix} \mapsto \begin{pmatrix} \nu_k \\ \nu'_k \end{pmatrix} + \begin{pmatrix} c_k \\ c'_k \end{pmatrix}$$

The relations (27) are preserved, if we restrict the action to the multi-diagonal translation subgroup $\mathfrak{T}_1(\mathbf{C})^{m+l+n}$ of $\mathfrak{T}_2(\mathbf{C})^{m+l+n}$ working with $a_i = a'_i$, $b_j = b'_j$, $c_k = c'_k$. Combined with the linear group we get an action of $\mathbf{Gl}_2(K) \times \mathfrak{T}_1(\mathbf{C})^{m+l+n}$ on the set of Σ-quotients, especially on the subset of Σ-quotients invariant with respect to a lattice commensurable with Λ, see Lemma 9.2.

Moreover, the affine group $\mathbf{Gl}_2(K) \cdot \mathfrak{T}_2(\mathbf{C})$ acts as subgroup of $\mathbf{Gl}_2(K) \times \mathfrak{T}_1(\mathbf{C})^{m+l+n}$ on the set of Σ-quotients. The elements are identified with $\mathbf{Gl}_3(\mathbf{C})$-elements of form

$$\tilde{G} = \{ \begin{pmatrix} G & \mathfrak{t} \\ \mathfrak{o} & 1 \end{pmatrix}, \; G \in \mathbf{Gl}_2(K), \; \mathfrak{t} \in \mathbf{C}^2 \}.$$

Identifying the argument vectors u, v of our functions with $\begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$ the affine action of $\tilde{G}$ is explicitly described by

$$\tilde{G}^* F(u, v) = \tilde{G}^* F \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = F(\tilde{G} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}).$$

Applied to each factor of a Σ-quotient we get

$$\tilde{G}^* \sigma(\mathfrak{m} \cdot \mathfrak{u} + \rho) = \tilde{G}^* \sigma((\mathfrak{m}, 0) \cdot \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} + \rho) = \sigma((\mathfrak{m}, 0) \tilde{G} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} + \rho) = \sigma(\mathfrak{m} G \begin{pmatrix} u \\ v \end{pmatrix} + \mathfrak{m} \cdot \mathfrak{t} + \rho).$$

Further restriction yields an action of an imaginary quadratic version of the *Heisenberg group*

$$\mathfrak{H}(K^2, \mathbf{Q}\Lambda) := \mathbf{Gl}_2(K) \cdot \mathfrak{T}(\mathbf{Q}\Lambda) \cong \mathbf{Gl}_2(K) \cdot \mathfrak{T}_2(K)$$

(with obvious notations) on the function field generated by Σ-quotients invariant under lattice translations commensurable with Λ. Restricting further we get with Theorem 9.1 the following

Corollary 10.1 *The subgroup $\mathfrak{H}(\mathfrak{O}^2, \mathbf{Q}\Lambda) := \mathbf{Gl}_2(\mathfrak{O}) \cdot \mathfrak{T}(\mathbf{Q}\Lambda)$ acts on the subfield $\Sigma(\mathbf{C}^2, \Lambda)$, generated by all Λ-periodic Σ-quotients of the field $\mathfrak{Mer}(\mathbf{C}^2, \Lambda)$ $\cong \mathfrak{Mer}(\mathbf{C}^2/\Lambda)$ of abelian functions on the surface $A = \mathbf{C}^2/\Lambda$.*

□

Explicitly, the action of

$$\tilde{G} = \begin{pmatrix} G & \mathfrak{t} \\ \mathfrak{o} & 1 \end{pmatrix} \in \mathfrak{H}(\mathfrak{O}^2, \mathbf{Q}\Lambda)$$

on Σ-quotients can be written as

$$\tilde{G}^* \Sigma_M^{\begin{pmatrix} \mu & \lambda & \nu \\ \mu' & \lambda' & \nu' \end{pmatrix}} = \Sigma_{GM}^{\tilde{G}\begin{pmatrix} \mu & \lambda & \nu \\ \mu' & \lambda' & \nu' \end{pmatrix}}$$

with obvious action on pairs in the upper indices, e.g.

$$\begin{pmatrix} \mu_i \\ \mu_i' \end{pmatrix} \mapsto \tilde{G} \cdot \begin{pmatrix} \mu_i \\ \mu_i' \\ 1 \end{pmatrix} \mapsto G \cdot \begin{pmatrix} \mu_i \\ \mu_i' \end{pmatrix} + \mathfrak{t}$$

More simply, $\tilde{G}$ acts on each σ-factor $\sigma_{\mathfrak{m},\mu}$ by index multiplication

$$\sigma_{\mathfrak{m},\mu} \mapsto \tilde{G}^* \sigma_{\mathfrak{m},\mu} = \sigma_{(\mathfrak{m},\mu)\tilde{G}}$$

11 Elliptic divisors of Σ-quotients and relations

The inverse image of the canonical projection $p_\Lambda : \mathbf{C}^2 \longrightarrow A = \mathbf{C}^2/\Lambda$ identifies the abelian function field $\mathbf{C}(A) = \mathfrak{Mer}(A)$ with the field $\mathfrak{Mer}(\mathbf{C}^2, \Lambda)$ of meromorphic Λ-periodic functions on $\mathbf{C}^2$. So each $F \in \mathfrak{Mer}(\mathbf{C}^2, \Lambda)$ corresponds uniquely to an abelian function f such that $F = p_\Lambda^*(f)$. We are mainly interested on abelian functions f with elliptic divisors on A, or at least with elliptic pole divisors $(f)_\infty$. The elliptic curves on A through O correspond via p_Λ^* to $\mathbf{Q}\Lambda$-lines on $\mathbf{C}^2$, which are defined as affine complex lines on $\mathbf{C}^2$ going throug two different points of $\mathbf{Q}\Lambda$. We have bijective correspondences

$$\begin{aligned} \{elliptic\, curves\, through\, O \in A\} &\Longleftrightarrow \{\mathbf{Q}\Lambda - lines\, through\, O \in \mathbf{C}^2\}/\Lambda \\ \{elliptic\, curves\, on\, A\} &\Longleftrightarrow (\{\mathbf{Q}\Lambda - lines\, on\, \mathbf{C}^2\} + \mathbf{C}^2)/\Lambda, \end{aligned}$$

where the action of $\mathbf{C}^2$ (and Λ) is defined by additive shifting. Intermediately we have also the correspondence

$$\{elliptic\,curves\;with\;several A - torsion\;points\} \Longleftrightarrow \{\mathbf{Q}\Lambda - lines\}/\Lambda.$$

Here "with several A-torsion points" means: more than one or, equivalently, with a dense subset of A-torsion points. On this way each elliptic curve C on A is defined by a linear equation (remember $\Lambda = c\cdot\mathfrak{O}^2$)

$$C:\ \alpha u + \beta v + \rho = 0,\ \alpha,\beta\in K,\ \rho\in\mathbf{C}.$$

We can and will assume that the coefficient pair (α,β) is *primitive*, which means that α and β are integral and have no common $\mathfrak{O}$-factor except for units. If $\mathfrak{O}_K$ is a principal domain, the primitive coefficient pair (α,β) is the row of a $\mathbf{Gl}_2(\mathfrak{O})$-matrix. It is uniquely determined by C up to $\mathfrak{O}^*$-mutiplication and ρ up to Λ-shifts after fixing α and β. Moreover, C has A-torsion points iff $\rho\in\Lambda_1$.

For our Σ-quotients $\Sigma_M^{\substack{\mu\ \lambda\ \nu\\ \mu'\ \lambda'\ \nu'}}$ we assume that we cannot cancel factors of the denominator and the numerator, this means

$$\lambda_i\neq\lambda_i',\ \mu_j\neq\mu_j',\ \nu_k\neq\nu_k'$$

for all possible numerations. We restrict ourselves also to Λ-periodic Σ-quotients with upper indices lying in $\mathbf{Q}\Lambda_1$ and $M\in\mathbf{Gl}_2(\mathfrak{O})$. The set of all of them is denoted by $\Sigma(\mathfrak{O}^2,\mathbf{Q}\Lambda_1)$. The divisor of $\Sigma_M^{\substack{\mu\ \lambda\ \nu\\ \mu'\ \lambda'\ \nu'}}\in\Sigma(\mathfrak{O}^2,\mathbf{Q}\Lambda_1)$ on A is an elliptic divisor whose components are universally covered by the following $\mathbf{C}^2$-lines

$$\begin{aligned} zeros: \alpha u+\beta v+\mu_i=0,\ (\alpha+\beta)u+(\gamma+\delta)v+\lambda_j=0,\ \gamma u+\delta v+\nu_k=0;\\ poles: \alpha u+\beta v+\mu_i'=0,\ (\alpha+\beta)u+(\gamma+\delta)v+\lambda_j'=0,\ \gamma u+\delta v+\nu_k'=0,\end{aligned}$$

$$M=\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\in\mathbf{Gl}_2(\mathfrak{O}),\ \lambda_i,\lambda_i',\mu_j,\mu_j',\nu_k,\nu_k'\in\mathbf{Q}\Lambda_1=cK.$$

with relations

$$\sum_{i=1}^{m}(\mu_i-\mu_i') = -\sum_{j=1}^{l}(\lambda_i-\lambda_i') = \sum_{k=1}^{n}(\nu_i-\nu_i'). \tag{28}$$

We see that each elliptic component of the divisor $div_A(\Sigma_M^{\substack{\mu\ \lambda\ \nu\\ \mu'\ \lambda'\ \nu'}})$ has (several) A-torsion points.

Two elliptic curves on A are called *parallel* if and only if they have no common point or they coincide. If all components of an elliptic divisor are parallel, then we call it a *parallel divisor*. Each elliptic divisor D on A splits uniquely into the sum of maximal parallel subdivisors. They are called the *parallel components* of D. The divisor of $\Sigma_M\begin{smallmatrix}\mu & \lambda & \nu\\ \mu' & \lambda' & \nu'\end{smallmatrix}$ has in general three parallel components described by

$$\begin{gathered}\alpha u+\beta v+\mu_i=0,\ \alpha u+\beta v+\mu_i'=0,\ i=1,...,m,\\ (\alpha+\beta)u+(\gamma+\delta)v+\lambda_j=0,\ (\alpha+\beta)u+(\gamma+\delta)v+\lambda_j'=0,\ j=1,...,l,\\ \gamma u+\delta v+\nu_k=0,\ \gamma u+\delta v+\nu_k'=0,\ k=1,...,n\end{gathered}$$

with coefficient relations (28).

The case of two parallel components happens iff precisely one of the upper index bounds l,m,n vanishes. We loose nothing if we assume that $m=0$. The two parallel components are described by the elliptic curves with linear equations

$$\alpha u+\beta v+\mu_i=0,\ \alpha u+\beta v+\mu_i'=0,\ i=1,...,m,$$

respectively

$$\gamma u+\delta v+\nu_k=0,\ \gamma u+\delta v+\nu_k'=0,\ k=1,...,n.$$

with coefficient relations

$$\sum_{i=1}^{m}\mu_i=\sum_{i=1}^{m}\mu_i',\ \sum_{k=1}^{n}\nu_k=\sum_{k=1}^{n}\nu_k.$$

The cases with only one parallel component are exhausted by $l=n=0$, Then the zero and pole equations are

$$\alpha u+\beta v+\mu_i=0,\ \alpha u+\beta v+\mu_i'=0,\ i=1,...,m\neq 0,$$

with only one relation

$$\sum_{i=1}^{m}\mu_i=\sum_{i=1}^{m}\mu_i'.$$

The elliptic divisors on A of all possible abelian Σ-quotients are called Σ-*divisors*. Their pole divisors are called Σ-*pole divisors*. The minimal number of components of a Σ-pole divisor is

$$\begin{cases}3, & \text{if it has precisely 3 parallel components;}\\ 4, & \text{if it has precisely 2 parallel components;}\\ 2, & \text{if it has precisely 1 parallel component.}\end{cases}$$

The parallel components of each Σ-divisor with precicely two of them are (parallel) Σ-divisors themselves corresponding to the product of two Σ-quotients. These are *reducible* Σ-divisors, where a Σ-divisor is called *irreducible* or *simple*

iff it is not the sum of two smaller Σ-divisors. So the most simple Σ-divisors are those with two parallel pole components (*binary*), with three parallel pole components (*ternary parallel*) and with three (pairwise) non-parallel pole components (*triangular*).
binary:

$$\begin{aligned} \alpha u + \beta v + \mu = 0,\ \alpha u + \beta v + \mu' = 0,\\ \alpha u + \beta v + \nu = 0,\ \alpha u + \beta v + \nu' = 0, \end{aligned}$$

with relation

$$\mu + \nu = \mu' + \nu'.$$

ternary parallel:

$$\begin{aligned} \alpha u + \beta v + \mu = 0,\ \alpha u + \beta v + \mu' = 0,\\ \alpha u + \beta v + \nu = 0,\ \alpha u + \beta v + \nu' = 0,\\ \alpha u + \beta v + \lambda = 0,\ \alpha u + \beta v + \lambda' = 0, \end{aligned}$$

with relation

$$\mu + \nu + \lambda = \mu' + \nu' + \lambda'.$$

triangular:

$$\begin{gathered} \alpha u + \beta v + \mu = 0, \alpha u + \beta v + \mu' = 0,\\ (\alpha+\beta)u + (\gamma+\delta)v + \lambda = 0, (\alpha+\beta)u + (\gamma+\delta)v + \lambda' = 0,\\ \gamma u + \delta v + \nu = 0,\ \gamma u + \delta v + \nu' = 0, \end{gathered}$$

with relations

$$\mu - \mu' = \lambda - \lambda' = \nu - \nu'.$$

The Σ-divisors generate an additive subgroup $Div_\Sigma A$ of the principal divisor subgroup of $Div\ A$. The additive decomposition of a divisor $D \in Div_\Sigma A$ into irreducible Σ-divisors is not unique as we will see below. That's the source of algebraic relations between different products of simple abelian Σ-quotients. Namely, if

$$D = D_1 + ... + D_k = C_1 + ... + C_m \tag{29}$$

are two different decompositions of $D = div_A(F)$ into Σ-divisors $D_i = div_A(F_i)$, $C_j = div_A(G_j)$ with Λ-periodic Σ-quotients F_i, G_j, then we have a relation

$$f_1 \cdot ... \cdot f_k = const \cdot g_1 \cdot ... \cdot g_m \tag{30}$$

for the corresponding abelian Σ-quotients $f_i, g_j \in \mathfrak{Mer}(A)$ because the quotient of the two products in (30) has no zeros and poles on A. This relation pulls back along p_Λ to

$$(31) \qquad F_1 \cdot \ldots \cdot F_k = const \cdot G_1 \cdot \ldots \cdot G_m$$

We have the following problems:

I. Find divisors in $Div_\Sigma A$ with different Σ-*decompositions* (29).

II. Determine the constants in the corresponding relations (31).

III. Transfer such relations to relations between modular forms.

Basically, we look for relations between products of special binary, ternary and triangular Σ-quotients. Let us start with

Σ-quotients of P-type

These are special abelian binary Σ-quotients

$$P^\omega_{\alpha\beta\kappa} := \frac{\sigma(\alpha u + \beta v + \kappa + \omega) \cdot \sigma(\alpha u + \beta v + \kappa - \omega)}{\sigma(\alpha u + \beta v + \kappa) \cdot \sigma(\alpha u + \beta v + \kappa)},$$
$$\mathfrak{O}\alpha + \mathfrak{O}\beta = \mathfrak{O},\ \omega \in \frac{1}{2}\Lambda_1 \setminus \Lambda_1,\ \kappa \in \mathbf{C}.$$

zeros : $\alpha u + \beta v + (\kappa + \omega) = 0$, $\alpha u + \beta v + (\kappa - \omega) = 0$;

pole : $\alpha u + \beta v + \kappa = 0$ (double)

This means that the pole divisor of P on A consists of a double elliptic curve described in the second row, and the zero divisor is also a double elliptic curve described in the first row. The zero curve is a shift of the pole curve by a honest 2-torsion point of A.

Triangular Σ-quotients of Δ-type

These are special triangular Σ-quotients

$$\Delta^\omega_{M,\mu} := \frac{\sigma(\alpha u + \beta v + \mu_1 + \omega) \cdot \sigma((\alpha+\gamma)u + (\beta+\delta)v + \mu_3 - \omega) \cdot \sigma(\beta u + \delta v + \mu_2 + \omega)}{\sigma(\alpha u + \beta v + \mu_1) \cdot \sigma((\alpha+\gamma)u + (\beta+\delta)v + \mu_3) \cdot \sigma(\beta u + \delta v + \mu_2)},$$

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{Gl}_2(\mathfrak{O}),\ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \in \mathbf{C}^3,\ \omega \in \frac{1}{2}\Lambda_1.$$

zeros : $\alpha u + \beta v + \mu_1 + \omega = 0$,
$(\alpha+\gamma)u + (\beta+\delta)v + \mu_3 - \omega = 0$,
$\beta u + \delta v + \mu_2 + \omega = 0$;

poles : $\alpha u + \beta v + \mu_1 = 0$,
$(\alpha+\gamma)u + (\beta+\delta)v + \mu_3 = 0$,
$\beta u + \delta v + \mu_2 = 0$.

$\Delta - P$-relations

Now we get the first relations. To the above Δ-function $\Delta = \Delta^{\omega}_{M,\mu}$ we correspond Σ-quotients of P-type

$$P_1 := P^{\omega}_{\alpha\beta\mu_1},\ P_2 := P^{\omega}_{\gamma\delta\mu_2},\ P_3 := P^{-\omega}_{\alpha+\gamma,\beta+\delta,\mu_3} = P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_3}.$$

The products D^2 and $P_1 \cdot P_2 \cdot P_3$ have obviously the same divisor on A. Therefore we get as special case of (31) a relation written as

$$\Delta^2 = const \cdot P_1 \cdot P_2 \cdot P_3. \tag{32}$$

Σ-quotients of Q-type

These are special ternary Σ-quotients of parallel type.

$$Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa} := \frac{\sigma(\alpha u + \beta v + \kappa + \omega_1) \cdot \sigma(\alpha u + \beta v + \kappa + \omega_2) \cdot \sigma(\alpha u + \beta v + \kappa + \omega_3)}{\sigma(\alpha u + \beta v + \kappa) \cdot \sigma(\alpha u + \beta v + \kappa) \cdot \sigma(\alpha u + \beta v + \kappa)},$$

$\mathfrak{O}\alpha + \mathfrak{O}\beta = \mathfrak{O}$, $\kappa \in \mathbf{C}$, with (honest) half periods ω_i, $i = 1, 2, 3$, satisfying $\omega_1 + \omega_2 + \omega_3 = 0$, and with

$$\text{zeros}: \begin{array}{l} \alpha u + \beta v + (\kappa + \omega_1) = 0, \\ \alpha u + \beta v + (\kappa + \omega_2) = 0, \\ \alpha u + \beta v + (\kappa + \omega_3) = 0; \end{array} \qquad \text{pole}: \ \alpha u + \beta v + \kappa = 0 (\text{triple}),$$

The pole divisor is a triple elliptic curve. The zero divisor is the simple sum of the elliptic curves, which are shifts of the zero curve by the three different honest 2-torsion points of A.

Parallel $P - Q$-relations

Take $Q = Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa}$ as above. We correspond to Q the Σ-quotients of P-type

$$P_1 := P^{\omega_1}_{\alpha\beta\kappa},\ P_2 := P^{\omega_2}_{\alpha\beta\kappa},\ P_3 := P^{\omega_3}_{\alpha\beta\kappa}.$$

The square Q^2 and $P_1 \cdot P_2 \cdot P_3$ have the same divisor on A. Therefore we get the special relations

$$Q^2 = C \cdot P_1 \cdot P_2 \cdot P_3,\ C \in \mathbf{C}^*. \tag{33}$$

as described in (31).

Triangular $P - Q$-relations

These are triangular relations between functions of P-type and Q-type. First we need precise transformation laws for these functions. For this purpose we introduce for half periods ω the notation

$$\sigma^{\omega}_{pqr} := \sigma(pu + qv + r + \omega),\ (p, q, r) \in \mathbf{C}^3.$$

The lower index is understood as covector representing the linear function

$$f = pu + qv + r : \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \mapsto (p, q, r) \cdot \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

The affine transformations (M, μ) with $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{Gl}_2(\mathfrak{O})$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbf{C}^2$, act on the space of linear functions via

$$(M\mu)^* : \ f \mapsto (p, q, r) \begin{pmatrix} \alpha & \beta & \mu_1 \\ \gamma & \delta & \mu_2 \\ 0 & 0 & 1 \end{pmatrix} = (p\alpha + q\gamma, p\beta + q\delta, p\mu_1 + q\mu_1 + r).$$

It holds that

$$(M\mu)^* \sigma^{\omega}_{pqr} = \sigma^{\omega}_{(M\mu)^*(p,q,r)},$$

especially:

$$(M\mu)^* \sigma^{\omega}_{100} = \sigma^{\omega}_{\alpha\beta,\mu_1}, \ (M\mu)^* \sigma^{\omega}_{010} = \sigma^{\omega}_{\gamma\delta,\mu_2}, \ (M\mu)^* \sigma^{\omega}_{110} = \sigma^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2}.$$

Observe that the transformations work for arbitrary $\omega \in \mathbf{C}$. The actions extends to Σ-quotients, especially we get for P- and Q-type functions the elementary transformation laws

(34)
$$(M\mu)^* P^{\omega}_{100} = P^{\omega}_{\alpha\beta\mu_1}, \ (M\mu)^* P^{\omega}_{010} = P^{\omega}_{\gamma\delta\mu_2}, \ (M\mu)^* P^{\omega}_{110} = P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2};$$

(35)
$$\begin{aligned} (M,\mu)^* Q^{\omega_1\omega_2\omega_3}_{100} = Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\mu_1}, \ (M,\mu)^* Q^{\omega_1\omega_2\omega_3}_{010} &= Q^{\omega_1\omega_2\omega_3}_{\gamma\delta\mu_2}, \\ (M,\mu)^* Q^{\omega_1\omega_2\omega_3}_{110} &= Q^{\omega_1\omega_2\omega_3}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2}. \end{aligned}$$

We consider for half periods $\omega_1, \omega_2, \omega_3$ with $\omega_1 + \omega_2 + \omega_3 = 0$ the determinant functions in z (with constants u, v, at the moment, $\notin \Lambda_1$)

$$D(z) = \det \begin{pmatrix} 1 & \frac{\sigma(z+\omega_1)\sigma(z-\omega_1)}{\sigma(z)^2} & \frac{\sigma(z+\omega_1)\sigma(z+\omega_2)\sigma(z+\omega_3)}{\sigma(z)^3} \\ 1 & \frac{\sigma(u+\omega_1)\sigma(u-\omega_1)}{\sigma(u)^2} & \frac{\sigma(u+\omega_1)\sigma(u+\omega_2)\sigma(u+\omega_3)}{\sigma(u)^3} \\ 1 & \frac{\sigma(v+\omega_1)\sigma(v-\omega_1)}{\sigma(v)^2} & \frac{\sigma(v+\omega_1)\sigma(v+\omega_2)\sigma(v+\omega_3)}{\sigma(v)^3} \end{pmatrix}.$$

This is a Λ_1-periodic function by 2.) of section 8. It has obviously zeros at $z = u$ and $z = v$ and only (triple) poles on Λ_1. By Abel's theorem the third pole sits at $-(u + v)$. At $z = -(u + v)$ we get

$$0 = \det \begin{pmatrix} 1 & P^{\omega}_{110} & -Q^{-\omega_1,-\omega_2,-\omega_3}_{110} \\ 1 & P^{\omega}_{100} & Q^{\omega_1\omega_2\omega_3}_{100} \\ 1 & P^{\omega}_{010} & Q^{\omega_1\omega_2\omega_3}_{010} \end{pmatrix}.$$

Applying $(M, \mu)^*$ we receive with (34) and (35)

Proposition 11.1 *For half periods*

$$\omega = \omega_1,\ \omega_2,\ \omega_3,\ \omega_1 + \omega_2 + \omega_3 = 0$$

of Λ_1, $(\mu_1, \mu_2, \mu_1 + \mu_2) \in \mathbf{C}^3$ *and* $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathbf{Gl}_2(\mathfrak{O})$ *it holds that*

$$0 = \det \begin{pmatrix} (P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2})^2 & P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} & -Q^{-\omega_1,-\omega_2,-\omega_3}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} \\ 1 & P^{\omega}_{\alpha\beta\mu_1} & Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\mu_1} \\ 1 & P^{\omega}_{\gamma\delta\mu_2} & Q^{\omega_1\omega_2\omega_3}_{\gamma\delta\mu_2} \end{pmatrix}.$$

□

12 The Ramachandra-Kronecker constant

First we look for most simple equations of elliptic curves E with complex multiplication. We restrict us to the principal case $End\, E \cong \mathfrak{O} = \mathfrak{O}_K$, $E(\mathbf{C}) = \mathbf{C}/\Lambda$, $\Lambda = c \cdot \mathfrak{O}$, $c \in \mathbf{C}^*$. Different constants c do not change the isomorphy class but the Weierstraß equation of elliptic curves. The problem is to find c with Weierstraß equation of E with simplest integral coefficients. It has been solved in 1964 by Ramachandra in [Ra] employing Kronecker's limit formula.

Proposition 12.1 *Let* $\Gamma(z)$ *be the* Γ*-function on* $\mathbf{C}$ *extending* $n \mapsto n!$ *on* $\mathbf{N}$*. For*

$$\mathfrak{O} = \mathbf{Z} + \mathbf{Z}\mathrm{i}, \quad c = \frac{\zeta_8}{2}\sqrt{\pi}\Gamma(1/4)/\Gamma(3/4),\ \zeta_8\, \textit{a primitive } 8\textit{-th unit root}$$

the elliptic curve $\mathbf{C}/c(\mathbf{Z} + \mathbf{Z}\mathrm{i})$ *has Weierstraß equation*

$$Y^2 = 4(X^3 + X) = 4(X + \mathrm{i})(X - \mathrm{i})X. \tag{36}$$

Proof. We remember to the following ingredients. The Eisenstein series of a $\mathbf{C}$-lattice Λ are defined as

$$G_k = G_k^{\Lambda} := \sum_{\lambda \in \Lambda^{\cdot}} \lambda^{-k},\ 2 < k \in 2\mathbf{Z}.$$

With

$$g_2 = g_2^{\Lambda} = 60G_4,\ g_3 = g_3^{\Lambda} = 140G_6$$

we get the Weierstraß equation

$$E:\ 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$$

with

$$e_1 = \wp(\omega_1),\ e_2 = \wp(\omega_2),\ e_3 = \wp(\omega_3).$$

From the definitions follows

$$G_k^{c\Lambda} = c^{-k},\ g_2^{c\Lambda} = c^{-4} g_2^{\Lambda},\ g_3^{c\Lambda} = c^{-6} g_3^{\Lambda}.$$

The j-invariant of the isomorphy class of E is

$$j_E = g_2^3/\Delta,\ \Delta = g_2^3 - 27g_3^2 \text{ (discriminant)}.$$

For the standard lattices $\mathbf{Z} + \mathbf{Z}\tau$, $\tau \in \mathbf{H}$ (upper half plane) the Eisenstein series have the explicit Fourier expansions

$$G_k(\tau) = G_k^{\mathbf{Z}+\mathbf{Z}\tau} = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with the Riemann ζ-function $\zeta(z)$, $q = q(\tau) = \mathrm{e}^{2\pi \mathrm{i} \tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. By a formula of Euler (see e.g. [BF], III, Prop. 7.14) the zeta-values $\zeta(k)$ belong to $\mathbf{Q}B_k\pi^k = \mathbf{Q}\pi^k$. For $\tau = \mathrm{i}$ it is clear now from the Fourier expansion that $G_4(i)$, hence also $g_2(\mathrm{i})$, must be a real number. Since (36) is obviously an elliptic curve with $\mathbf{Q}(\mathrm{i})$-multiplication, we have $g_3(\mathrm{i}) = 0$. Therefore $g_2(\mathrm{i})$ is the real root of $\Delta(\mathrm{i})$. By Ramachandra [Ra] the latter value is known:

$$\Delta(\mathrm{i}) = (\sqrt{\pi/2}\Gamma(1/4)/\Gamma(3/4))^{12},$$

hence

$$g_2(\mathrm{i}) = \frac{\pi^2}{4} (\Gamma(1/4)/\Gamma(3/4))^4 \in \mathbf{R}.$$

The constant we look for is

$$c = \zeta_8 \sqrt{\pi/2}\Gamma(1/4)/\Gamma(3/4),$$

because

$$g_2^{c(\mathbf{Z}+\mathbf{Z}\mathrm{i})} = c^{-4} g_2^{\mathbf{Z}+\mathbf{Z}\mathrm{i}} = c^{-4} g_2(\mathrm{i}) = -4.$$

□

13 Abelian prime functions and constants of relations

For the $\mathbf{C}$-lattice $\Lambda_1 = \mathbf{Z} \cdot 2\omega_1 + \mathbf{Z} \cdot 2\omega_2$ the simplest and most interesting Σ-quotients are

$$\begin{aligned} w_{\alpha\beta\kappa}^{+\omega} &:= \frac{\sigma(\alpha u + \beta v + \kappa + \omega)}{\sigma(\alpha u + \beta v + \kappa)} = \sigma^{+\omega}/\sigma, \\ w_{\alpha\beta\kappa}^{-\omega} &:= \frac{\sigma(\alpha u + \beta v + \kappa - \omega)}{\sigma(\alpha u + \beta v + \kappa)} = \sigma^{-\omega}/\sigma \end{aligned}$$

with

$$\sigma^{\pm\omega} = \sigma^{\pm\omega}_{\alpha\beta\kappa} = \sigma(\alpha u + \beta v + \kappa \pm \omega).$$

These meromorphic functions are transcendental over $\mathfrak{Mer}(A)$. For half periods ω we have a decomposition of our P-type functions

$$P^{\omega}_{\alpha\beta\kappa} = w^{\omega}_{\alpha\beta\kappa} \cdot w^{-\omega}_{\alpha\beta\kappa} = \frac{\sigma^{+\omega} \cdot \sigma^{-\omega}}{\sigma^2_{\alpha\beta\kappa}}.$$

Following classical ideas of Jacobi in one variable (amplitudinus functions, elliptic integrals) we modify the simple Σ-quotients in order to get an algebraic decomposition of $P^{\omega}_{\alpha\beta\kappa}$. For this purpose we introduce for $\omega \in \frac{1}{2}\Lambda_1$ the functions

$$\epsilon^{\omega}_{\alpha\beta\kappa} := \epsilon^{\omega}(\alpha u + \beta v + \kappa) = \psi(2\omega)\mathrm{e}^{\eta(\omega)\cdot(\alpha u+\beta v+\kappa)} = \pm\mathrm{e}^{\eta(\omega)\cdot(\alpha u+\beta v+\kappa)}.$$

and set for half periods ω

$$\begin{aligned} W^{\omega,+}_{\alpha\beta\kappa} &:= -\epsilon^{\omega}_{-\alpha,-\beta,-\kappa} \cdot w^{\omega}_{\alpha\beta\kappa} = \mathrm{e}^{-\eta(\omega)(\alpha u+\beta v+\kappa)}\frac{\sigma(\alpha u + \beta v + \kappa + \omega)}{\sigma(\alpha u + \beta v + \kappa)}, \\ W^{\omega,-}_{\alpha\beta\kappa} &:= \epsilon^{\omega}_{\alpha\beta\kappa} \cdot w^{-w}_{\alpha\beta\kappa} = -\mathrm{e}^{\eta(\omega)(\alpha u+\beta v+\kappa)}\frac{\sigma(\alpha u + \beta v + \kappa - \omega)}{\sigma(\alpha u + \beta v + \kappa)}. \end{aligned}$$

By definitions we decomposed P-type functions

$$P^{\omega}_{\alpha\beta\kappa} = -W^{\omega,+}_{\alpha\beta\kappa} \cdot W^{\omega,-}_{\alpha\beta\kappa}. \tag{37}$$

On the other hand we deduce with

$$z = \alpha u + \beta v + \kappa,\ \epsilon^{\omega} = \epsilon^{\omega}_{\alpha\beta\kappa} = \epsilon^{\omega}(z)$$

from (25)

$$\begin{aligned} -W^{\omega,+}_{\alpha\beta\kappa} &= \epsilon^{\omega}(-z)\frac{\sigma(z+\omega)}{\sigma(z)} = \epsilon^{\omega}(-z)\epsilon^{\omega}(2z)\frac{\sigma(z-\omega)}{\sigma(z)} \\ &= \epsilon^{\omega}(z)\frac{\sigma(z-\omega)}{\sigma(z)} = W^{\omega,-}_{\alpha\beta\kappa}, \end{aligned}$$

hence (37) can be written as

$$P^{\omega}_{\alpha\beta\kappa} = (W^{\omega,+}_{\alpha\beta\kappa})^2 = (W^{\omega,-}_{\alpha\beta\kappa})^2.$$

Proposition 13.1 *For*

$$\alpha, \beta \in \mathfrak{O},\ \mathfrak{O}\alpha + \mathfrak{O}\beta = \mathfrak{O},\ \kappa \in \mathbf{C},\ \omega \in \frac{1}{2}\Lambda_1 \setminus \Lambda_1$$

the functions $W^{\omega,+}_{\alpha\beta\kappa}$ and $W^{\omega,-}_{\alpha\beta\kappa}$ are algebraic of degree 2 over $\mathfrak{Mer}(A)$. These are precisely the two different roots of the P-type function $P^{\omega}_{\alpha\beta\kappa}$. They are Λ'-periodic for a sublattice Λ' of Λ of index 2. This means that they belong to the function field $\mathfrak{Mer}(B)$ of unramified abelian double cover $B = \mathbf{C}^2/\Lambda'$ of A.

Proof. It only remains to prove the last statement. Omitting indices we deduce from the Λ-periodicity of P

$$W^2(\mathfrak{u}+\mu) = P(\mathfrak{u}+\mu) = P(\mathfrak{u}) = W^2(\mathfrak{u}),\ \mathfrak{u} = (u,v) \in \mathbf{C}^2,\ \mu \in \Lambda.$$

Therefore

$$W(\mathfrak{u}+\mu) = \delta(\mu)W(\mathfrak{u}),\ \delta : \Lambda \longrightarrow \{\pm 1\}$$

a surjective quadratic character on Λ. The kernel of δ is the sublattice Λ' we look for.

$\square$

Definition 13.2 . *The W-type functions appearing in Proposition 13.1 are called* abelian prime functions for Λ.

Now we are able to determine the constant C in our $P-Q$-relations (33).

Proposition 13.3 *The constant in relation (33) is equal to -1; this means*

$$Q^2 = -P_1P_2P_3 \tag{38}$$

with abelian functions

$$Q = Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa},\ P_i = P^{\omega_i}_{\alpha\beta\kappa},\ i = 1,2,3.$$

Proof. Instead of number triples $\alpha\beta\kappa$ we will write sometimes the corresponding linear forms $\alpha u + \beta v + \kappa$ as lower index, for instance $w^\omega_z = w^\omega_{\alpha\beta\kappa}$ with $z = \alpha u + \beta v + \kappa$. Then

$$\begin{aligned} Q^2 &= (w^{\omega_1}_z)^2(w^{\omega_2}_z)^2(w^{\omega_3}_z)^2 \\ &= w^{\omega_1}_z w^{\omega_2}_z w^{\omega_3}_z w^{-\omega_1}_z w^{-\omega_2}_z w^{-\omega_3}_z (-\mathrm{e}^{2\eta(\omega_1)z})(-\mathrm{e}^{2\eta(\omega_2)z})(-\mathrm{e}^{2\eta(\omega_3)z}) \\ &= -P_1P_2P_3 \end{aligned}$$

because η is a homomorphism, see $1'$.) in section 8, and $\omega_1+\omega_2+\omega_3 = 0$.

$\square$

Now consider the Δ-type function $\Delta^\omega_{M,\mu}$ as in (32) and the related P-type functions

$$P_1 := P^\omega_{\alpha\beta\mu_1},\ P_2 := P^\omega_{\gamma\delta\mu_2},\ P_3 := P^{-\omega}_{\alpha+\gamma,\beta+\delta,\mu_3},$$

Proposition 13.4 *The constant in relation (32) is equal to $-\mathrm{e}^{2\eta(\omega)(\mu_1+\mu_2-\mu_3)}$. This means that*

$$(\Delta^\omega_{M,\mu})^2 = -\mathrm{e}^{2\eta(\omega)(\mu_1+\mu_2-\mu_3)}P_1P_2P_3.$$

Proof. We set

$$\begin{aligned} z_1 &:= \alpha u + \beta v + \mu_1,\ z_2 := \beta u + \delta v + \mu_2, \\ z_3 &:= (\alpha+\gamma)u + (\beta+\delta)v + \mu_3 = z_1 + z_2 + (\mu_3 - \mu_1 - \mu_2). \end{aligned}$$

By definitions we can write

$$\Delta^{\omega}_{M,\mu} = w^{\omega}_{z_1} w^{\omega}_{z_2} w^{-\omega}_{z_3},$$

hence

$$\begin{aligned} \Delta^2 &= (w^{\omega}_{z_1})^2 (w^{\omega}_{z_2})^2 (w^{-\omega}_{z_3})^2 \\ &= w^{\omega}_{z_1} w^{\omega}_{z_2} w^{-\omega}_{z_3} w^{-\omega}_{z_1} w^{-\omega}_{z_2} w^{\omega}_{z_3} \cdot \varepsilon \\ &= \varepsilon P_1 P_2 P_3 \end{aligned}$$

with factor

$$\varepsilon = (-\mathrm{e}^{2\eta(\omega)z_1})(-\mathrm{e}^{2\eta(\omega)z_2})(-\mathrm{e}^{2\eta(-\omega)z_3}) = -\mathrm{e}^{2\eta(\omega)(\mu_1+\mu_2-\mu_3)}.$$

□

Corollary 13.5 *The normalized triangular abelian functions*

$$D^{\omega}_{M,\mu} := \mathrm{e}^{\eta(\omega)(\mu_3-\mu_1-\mu_2)} \Delta^{\omega}_{M,\mu}$$

satisfy

$$(D^{\omega}_{M,\mu})^2 = -P_1 P_2 P_3.$$

□

At the end of the section we modify the triangular $P-Q$-relation of Proposition 11.1.

Proposition 13.6 *With the notations there it holds that*

$$0 = \det \begin{pmatrix} (P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2})^2 & P^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} & -Q^{\omega_1,\omega_2,\omega_3}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} \\ 1 & P^{\omega}_{\alpha\beta\mu_1} & Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\mu_1} \\ 1 & P^{\omega}_{\gamma\delta\mu_2} & Q^{\omega_1\omega_2\omega_3}_{\gamma\delta\mu_2} \end{pmatrix}.$$

Proof. With $\mathfrak{z} = (\alpha+\gamma, \beta+\delta, \mu_1+\mu_2)$ we have by (25)

$$\begin{aligned} Q^{\omega_1,\omega_2,\omega_3}_{\mathfrak{z}} &= w^{\omega_1}_{\mathfrak{z}} \cdot w^{\omega_2}_{\mathfrak{z}} \cdot w^{\omega_3}_{\mathfrak{z}} \\ &= \epsilon^{\omega_1}(2z)\epsilon^{\omega_2}(2z)\epsilon^{\omega_3}(2z) \cdot w^{-\omega_1}_{\mathfrak{z}} \cdot w^{-\omega_2}_{\mathfrak{z}} \cdot w^{-\omega_3}_{\mathfrak{z}} \\ &= Q^{-\omega_1,-\omega_2,-\omega_3}_{\mathfrak{z}} \end{aligned}$$

because

$$\epsilon^{\omega_1}(2z)\epsilon^{\omega_2}(2z)\epsilon^{\omega_3}(2z) = \epsilon^{\omega_1+\omega_2+\omega_3}(2z) = \epsilon^0(2z) = 1.$$

Now we have only to substitute one coefficient in the matrix of Proposition 11.1.

□

14 Further normalizations of functions and relations

We divide each simple Σ-factor

$$w^{\omega}_{\alpha\beta\mu} = \frac{\sigma(\alpha u + \beta v + \mu + \omega)}{\sigma(\alpha u + \beta v + \mu)}$$

of abelian Σ-quotients by the constant $\sigma(-\omega)$ and extend it to the Σ-quotients. So we define (with admissible indices as usual)

$$\mathfrak{W}^{\omega}_{\alpha\beta\kappa} := W^{\omega}_{\alpha\beta\kappa}/\sigma(\omega)$$

$$\begin{aligned} \mathfrak{P}^{\omega}_{\alpha\beta\kappa} &:= P^{\omega}_{\alpha\beta\kappa}/\sigma(\omega)\sigma(-\omega) \\ \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa} &:= Q^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa}/\sigma(\omega_1)\sigma(\omega_2)\sigma(\omega_3) \\ \mathfrak{D}^{\omega}_{M,\mu} &:= D^{\omega}_{M,\mu}/\sigma(\omega)^3. \end{aligned} \tag{39}$$

Using $\sigma(-\omega) = -\sigma(\omega)$ it is easy to prove by $\sigma(\omega)$-divisions in Propositions 11.1, 13.6 and Corollary 13.5 the following

Theorem 14.1 *For admissible indices as in the three propositions of the previous sections, $\omega_1 + \omega_w + \omega_3 = 0$, it holds that*

(i) $(\mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa})^2 = \mathfrak{P}^{\omega_1}_{\alpha\beta\kappa}\mathfrak{P}^{\omega_2}_{\alpha\beta\kappa}\mathfrak{P}^{\omega_3}_{\alpha\beta\kappa}$,

(ii) $0 = \det\begin{pmatrix} (\mathfrak{P}^{\omega_i}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2})^2 & \mathfrak{P}^{\omega_i}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} & -\mathfrak{Q}^{\omega_1,\omega_2,\omega_3}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} \\ 1 & \mathfrak{P}^{\omega_i}_{\alpha\beta\mu_1} & \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\alpha\beta\mu_1} \\ 1 & \mathfrak{P}^{\omega_i}_{\gamma\delta\mu_2} & \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\gamma\delta\mu_2} \end{pmatrix}$,
i = 1,2,3,

(iii) $(\mathfrak{D}^{\omega}_{M,\mu})^2 = \mathfrak{P}^{\omega}_{\alpha\beta\mu_1}\mathfrak{P}^{\omega}_{\gamma\delta\mu_2}\mathfrak{P}^{\omega}_{\alpha+\gamma,\beta+\delta,\mu_3}$.

□

Corollary 14.2 *Let $Y^2 = 4(X-e_1)(X-e_2)(X-e_3)$ be the Weierstraß equation of the elliptic curve E with $E(\mathbf{C}) = \mathbf{C}/\Lambda_1$. The abelian function*

$$\mathfrak{P}_{\alpha\beta\kappa} := \mathfrak{P}^{\omega_i}_{\alpha\beta\kappa} + e_i,\ \alpha\mathfrak{O} + \beta\mathfrak{O} = \mathfrak{O},\ \kappa \in \mathbf{C},\ i = 1,2,3,$$

is correctly defined (independently on index i). It has a double pole divisor on $L^0_{\alpha\beta\kappa}$: $\alpha u + \beta v + \kappa = 0$. Moreover, the following relations are satisfied:

(i ') $(\mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\alpha\beta\kappa})^2 = (\mathfrak{P}^{\omega_1}_{\alpha\beta\kappa} - e_1)(\mathfrak{P}^{\omega_2}_{\alpha\beta\kappa} - e_2)(\mathfrak{P}^{\omega_3}_{\alpha\beta\kappa} - e_3)$,

(ii ') $0 = \det\begin{pmatrix} 1 & \mathfrak{P}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} & -\mathfrak{Q}^{\omega_1,\omega_2,\omega_3}_{\alpha+\gamma,\beta+\delta,\mu_1+\mu_2} \\ 1 & \mathfrak{P}_{\alpha\beta\mu_1} & \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\alpha\beta\mu_1} \\ 1 & \mathfrak{P}_{\gamma\delta\mu_2} & \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\gamma\delta\mu_2} \end{pmatrix}$.

Proof. We use the following well-known relations for the Weierstraß $\wp$-function, see e.g. [Pr], III, Par. 4:

$$\wp(z) - e_i = \frac{\sigma(z+\omega_i)}{\sigma(\omega_i)\sigma(z)} \cdot \frac{\sigma(z-\omega_i)}{\sigma(-\omega_i)\sigma(z)},$$

With $z = \alpha u + \beta v + \kappa$ and definition (39) we have

$$\wp(z) - e_i = \mathfrak{P}^{\omega_i}_{\alpha\beta\kappa}.$$

Therefore

$$\mathfrak{P}_{\alpha\beta\kappa}(u,v) := \mathfrak{P}^{\omega_i}_{\alpha\beta\kappa} + e_i$$

is correctly defined. So (i′) follows immediately from relation (i) in the theorem. For (ii′) one has in (ii) only to add e_i-times the first column in the determinant to the second.

□

15 Basic non-cuspidal abelian modular forms

As in section 5 we consider the minimal compact model $A = \mathbf{C}^2/\Lambda$ of a neat coabelian ball quotient $\mathbf{B}/\Gamma$ with elliptic compactification divisor
$T = T_1 + ... + T_h$. We use the notations of diagram (8). Without much loss of generality we assume as usual that

$$\Lambda = \Lambda_1 \times \Lambda_1,\ \Lambda_1 = c\mathfrak{O},\ \mathfrak{O} = \mathfrak{O}_K,\ c \in \mathbf{C}^*,$$

K an imaginary quadratic number field,

$$\begin{gathered} T_j : \alpha_j u + \beta_j v + \kappa_j,\ j = 1,...,h, \\ \alpha_j\mathfrak{O} + \beta_j\mathfrak{O} = \mathfrak{O},\ \kappa_j \in \mathbf{C}. \end{gathered}$$

The abelian functions

$$\mathfrak{P}_j := \mathfrak{P}_{\alpha_j\beta_j\kappa_j},\ \mathfrak{Q}_j := Q^{\omega_1\omega_2\omega_3}_{\alpha_j\beta_j\kappa_j},$$

with half periods ω_k, $\omega_1 + \omega_2 = \omega_3 = 0$, have poles $2T_j$ or $3T_j$, respectively, on A. Therefore $\mathfrak{P}_j^p\mathfrak{Q}_j^q$ is an abelian function with pole divisor $(2p+3q)T_j$. Looking at multiplicities of divisors of these functions at points of $Sing\,T$, see (23), we see that the pole divisors on A' are

$$\begin{aligned} (\mathfrak{P}_j)^\infty_{A'} &= -(\mathfrak{P}_j)_{L+T'} = 2T'_j + \sum_{P_i\in T_j} 2L_i, \\ (\mathfrak{Q}_j)^\infty_{A'} &= -(\mathfrak{Q}_j)_{L+T'} = 3T'_j + \sum_{P_i\in T_j} 3L_i, \\ (\mathfrak{P}_j^k)^\infty_{A'} &= -(\mathfrak{P}_j^k)_{L+T'} = 2kT'_j + \sum_{P_i\in T_j} 2kL_i, \\ (\mathfrak{P}_j^k\mathfrak{Q}_j)^\infty_{A'} &= -(\mathfrak{P}_j^k\mathfrak{Q}_j)_{L+T'} = (2k+3)T'_j + \sum_{P_i\in T_j} (2k+3)L_i, \end{aligned} \tag{40}$$

Therefore the abelian functions $\mathfrak{P}_1^p, ..., \mathfrak{P}_h^p$ belong to $H^0(A', 2p(L+T'))$. For $p > 0$ they are linearly independent because of independent T'-parts of the pole divisors. The same is true for

$$\mathfrak{P}_1^p\mathfrak{Q}, ..., \mathfrak{P}_h^p\mathfrak{Q} \in H^0(A', (2p+3)(L+T')),\ p \geq 0.$$

Theorem 15.1 *The modular transfers of the above products of $\mathfrak{P}$- and $\mathfrak{Q}$-functions on A*

$$\pi_j^{(n)} = \begin{cases} \iota_n(\mathfrak{P}_j^{n/2}), & \text{if } 2 \leq n \text{ even}, \\ \iota_n(\mathfrak{P}_j^{(n-3)/2}\mathfrak{Q}), & \text{if } 3 \leq n \text{ odd}, \end{cases}$$

fill exactly the non-cusp parts of $[\Gamma, n]$ for each $n \geq 2$. More precisely, $\pi_j^{(n)}$ has a zero (of order n) at the at all cusps except for κ_j (and its Γ-transforms), where it actually does not vanish, and

$$[\Gamma, n] = [\Gamma, n]_{cusp} \oplus \mathbf{C}\pi_1^{(n)} \oplus ... \oplus \mathbf{C}\pi_h^{(n)}. \tag{41}$$

Proof. Knowing the pole divisors of the abelian $\mathfrak{P}$- and $\mathfrak{Q}$-functions on A', see (40), we get the $\hat{\mathbf{D}}$-cycles of their modular transforms by (24). Namely, setting

$$\pi_j = \pi_j^{(2)} = \iota(\mathfrak{P}_j),\ \pi_j' = \pi_j^{(3)} = \iota(\mathfrak{Q}_j),\ j = 1, ..., h, \tag{42}$$

we get

$$(\pi_j)_{\hat{\mathbf{D}}} = \sum_{P_i \in T_j} 2\mathbf{D}_i + \sum_{i \neq j} 2\kappa_i,$$

$$(\pi_j')_{\hat{\mathbf{D}}} = \sum_{P_i \in T_j} 3\mathbf{D}_i + \sum_{i \neq j} 3\kappa_i, \tag{43}$$

hence

$$(\pi_j^{(n)})_{\hat{\mathbf{D}}} = \sum_{P_i \in T_j} n\mathbf{D}_i + \sum_{i \neq j} n\kappa_i.$$

Looking at the cusp contributions one recognizes that the sums on the right-hand side of (41) must be direct. On the other hand we dispose the dimension formula in Corollary 6.2 yields

$$h = \dim([\Gamma, n]/[\Gamma, n]_{cusp}),$$

which proves the theorem.

□

A Γ-cusp form of weight $n \geq 1$ is called *new* iff it is not divisible by η in the graded ring $R[\Gamma]$ of Γ-modular forms. Each subspace of $[\Gamma, n]_{cusp}$ complementary to $\eta[\Gamma, n-1]$ is called a *space of new cusp forms* of weight n. Such a space will be denoted by $[\Gamma, n]_{cusp}^{new}$

Remark 15.2 . *Imagine an euclidean metric on $[\Gamma, n]_{cusp}$ like Peterssons scalar product for modular forms. Then the space $[\Gamma, n]_{cusp}^{new}$ can and will be uniquely defined as orthogonal complement of $\eta[\Gamma, n-1]$ in $[\Gamma, n]_{cusp}$.*

Corollary 15.3 *For $n \geq 2$ the application of the modular transfer ι_n changes the filtration*

$$\begin{aligned} \mathbf{C} = H^0(A', O) \quad &\subset H^0(A', (L+T') \subset ..... \subset H^0(A', (n-1)(L+T')) \\ &\subset H^0(A', nL + (n-1)T') \subset H^0(A', n(L+T')) \end{aligned}$$

to the η-filtration

$$\mathbf{C}\eta^n \subset \eta^{n-1}[\Gamma, 1] \subset ..... \subset \eta[\Gamma, n-1] \subset [\Gamma, n]_{cusp} \subset [\Gamma, n]$$

of $[\Gamma, n]$ with splitting

$$[\Gamma, n] = \eta[\Gamma, n-1] \oplus [\Gamma, n]_{cusp}^{new} \oplus (\mathbf{C}\pi_1^{(n)} \oplus ... \oplus \pi_h^{(n)})$$

and dimensions

$$\dim[\Gamma, n]_{cusp}^{new} = \begin{cases} 0, & n = 1; \\ 3s - \dim[\Gamma, 1], & n = 2; \\ 3s(n-1) - h, & n \geq 3. \end{cases} \tag{44}$$

Proof. We have only to calculate the dimensions. For $n \geq 3$ the dimensions of the spaces

$$[\Gamma, n]_{cusp}^{new} \cong [\Gamma, n]_{cusp}/\eta[\Gamma, n-1]$$

are

$$3\binom{n}{2}s - 3\binom{n-1}{2}s - h = 3s(n-1) - h.$$

by the dimension formulas in Theorem 6.1. The cases $n = 1, 2$ can be checked in the same way.

□

Remark 15.4 . *The strategy of finding explicit bases of $[\Gamma, n]$ for all n in concrete situations is reduced to solve this problem first for $n = 1$, then determine successively bases of $[\Gamma, n]_{cusp}^{new}$ for $n = 2, 3, 4, ...$ Since there exists a natural number k such that $R[\Gamma] = \mathbf{C}[[\Gamma, 1], ..., [\Gamma, k]]$, the procedure stops theoretically after finitely many steps. This happens iff the dimensions of the n-th homogeneous parts of the graded algebra $\mathbf{C}[[\Gamma, 1], ..., [\Gamma, k]]$, which should be calculable with the knowledge of relations, coincide with $\dim[\Gamma, n] = 3\binom{n}{2}s + h$ for all $n \geq 2$.*

Corollary 15.5 *The η-form is algebraic over the function fields $\mathbf{C}(\pi_j, \pi_j')$, $j = 1, ..., h$. More precisely, if $A = E \times E$, E the elliptic CM-curve with Weierstraß equation*

$$E: \; Y^2 = 4X^3 - g_2 X - g_3 = 4(X - e_1)(X - e_2)(X - e_3),$$

then the algebraic relations

$$\pi_j'^2 = \pi^3 - \frac{g_2}{4}\pi\eta^4 - \frac{g_3}{4}\eta^6$$

are satisfied. So, η is integral (algebraic) over $\mathbf{C}[\pi_j, \pi_j']$ of degree 6, if $g_3 \neq 0$, and

$$\eta^4 = \frac{4(\pi^3 - \pi_j'^2)}{g_2 \pi}$$

in the CM-case of Gauß numbers.

Proof. With our notations the relation (i$'$) of Corollary 14.2 can be written as

$$\mathfrak{Q}_j'^2 = P_j^3 - \frac{g_2}{4}\mathfrak{P}_j - \frac{g_3}{4}$$

We have only to apply η -homogenization (18) to this relation.

□

16 Generators of low cohomology groups in the case of Gauß numbers

The following model is closely connected with the earlier known Example 5.4. As there the underlying lattice is

$$\Lambda = \Lambda_1 \times \Lambda_1, \; \Lambda_1 = c\mathbf{Z}[\mathrm{i}],$$

with Ramachandra-Kronecker constant c defined in Proposition (12.1), section 12. The Weierstraß equation for the elliptic curve E with respect to Λ_1 is

$$(45) \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3) = 4(X - \mathrm{i})(X + \mathrm{i})X = 4(X^3 + X),$$

satisfied by the $\wp$- and $\wp'$-function of this lattice. As basic half periods of Λ_1 we fix

$$\omega_1 := c \cdot \frac{1}{2}, \; \omega_2 := c \cdot \frac{\mathrm{i}}{2}, \; \omega_3 := c \cdot \frac{1 + \mathrm{i}}{2},$$

From the first of the obvious relations

$$(46) \qquad \wp(\mathrm{i}z) = -\wp(z),\ \wp'(\mathrm{i}z) = \mathrm{i}\wp(z),$$

follows

$$\wp(\omega_1) = -\wp(\omega_2),\ \wp(\omega_3) = \wp(\mathrm{i}\omega_3) = -\wp(w_3).$$

Therefore ω_3 is a zero of $\wp(z)$. Being a half period it is a double zero and the only one by Abel's theorem (up to Λ_1-shifts). Comparing with (36) we have without loss of generality (changing c by $\mathrm{i}c$ if necessary)

$$e_1 = \wp(\omega_1) = \mathrm{i},\ e_2 = \wp(\omega_2) = -\mathrm{i},\ e_3 = \wp(\omega_3) = 0.$$

So $\wp(z)$ has a double zero at ω_3. Therefore $\wp(z+\omega_3)$ has a double pole at ω_3 and a double zero at 0. The same holds for $1/\wp(z)$, hence it is a $\mathbf{C}^*$-multiple of $\wp(z+\omega_3)$. Comparing the values at $z = \omega_1$ we see that

$$(47) \qquad \wp(z+\omega_3) = 1/\wp(z),\ \wp(z+\omega_2) = 1/\wp(z+\omega_1),$$

for later use.

We define the elliptic divisor $T = T_1 + \ldots + T_6$ on A by following (covering) linear equations:

$$(48) \qquad \begin{aligned} &T_1 : \mathfrak{l}_1 := u = 0,\ T_2 : \mathfrak{l}_2 := v = 0,\ T_3 :\ \mathfrak{l}_3 + \omega_3 := u + v + \omega_3 = 0,\\ &T_4 : \mathfrak{l}_4 := u + (1+\mathrm{i})v = 0,\ T_5 :\ \mathfrak{l}_5 := (1-\mathrm{i})u + v = 0,\\ &T_6 : \mathfrak{l}_6 + \omega_3 := u + iv + \omega_3 = 0. \end{aligned}$$

By calculations as explained in [BHH], 1.4, one finds precisely three intersection points P_1, P_2, P_3 of pairs of T-components represented by

$$P_1 :\ (\omega_3, 0),\ P_2 :\ (0, \omega_3),\ P_3 :\ (0,0) \in \frac{1}{2}\Lambda.$$

The indices are choosed such that

$$(49) \qquad P_i \notin T_j \Longleftrightarrow i \equiv j \mathop{mod} 3.$$

With $s = 3$, $s_1 = \ldots = s_6 = 2$ the hyperbolicity condition $4s = \sum s_i$ is satisfied.

Proposition 16.1 *Blowing up the three intersection points P_1, P_2, P_3 on $A = E \times E$ one gets a Picard modular surface $A' = X'_\Gamma$.*

Proof. Let B be the abelian surface of Example 5.4. It is an unramified double covering of A. The eight elliptic curves on $B = E \times E$ defined by (9) are the preimages of the six curves defined in (48). More explicitly, the Galois group of this covering is generated by additive shifting (t_3, t_3) on B, t_3 the only non-trivial $(1+\mathrm{i})$-torsion point on E. Blowing up intersection points we get an unramified covering B' over A'. The restriction to the finite parts (omitting both elliptic T'-divisors) yields commensurability of the corresponding ball lattices. Since the B-lattice is Picard modular by Remark 5.5, this also true for the ball lattice of (A, T).

$\square$

We recommend the reader to draw a picture of the six curves T_j through the three points P_i with our numerations. There are eight triangular subdivisors of T, namely

$$\begin{array}{llll} T_1+T_2+T_3, & T_1+T_2+T_6, & T_1+T_5+T_3, & T_1+T_5+T_6, \\ T_2+T_4+T_3, & T_2+T_4+T_6, & T_4+T_5+T_3, & T_4+T_5+T_6. \end{array}$$

Each of them is a pole divisor of an abelian function of Δ-type. Namely, consider the eight $\mathbf{Z}[\mathrm{i}]$-unimodular matrices

$$\begin{array}{llllllll} M_{123} & = & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), & M_{126} & = & \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{smallmatrix}\right), & M_{153} & = & \left(\begin{smallmatrix} \mathrm{i} & 0 \\ 1-\mathrm{i} & 1 \end{smallmatrix}\right), \\ M_{156} & = & \left(\begin{smallmatrix} -1 & 0 \\ 1-\mathrm{i} & 1 \end{smallmatrix}\right), & M_{243} & = & \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ 1 & 1+\mathrm{i} \end{smallmatrix}\right), & M_{246} & = & \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1+\mathrm{i} \end{smallmatrix}\right), \\ M_{453} & = & \left(\begin{smallmatrix} 1 & 1+\mathrm{i} \\ -1+\mathrm{i} & -1 \end{smallmatrix}\right), & M_{456} & = & \left(\begin{smallmatrix} i & -1+\mathrm{i} \\ 1-\mathrm{i} & 1 \end{smallmatrix}\right) \end{array} \tag{50}$$

Up to $\mathfrak{O}^*$-multiples, the rows consists of u, v-coefficient pairs of two of the linear equations (48) for the T_1, T_2, T_4, T_5 corresponding to the first two indices of the matrix. The $\mathfrak{O}^*$-factors $\xi_m = \xi_m(ijk)$, $m = 1, 2, 3$, with

$$\begin{pmatrix} \xi_1 \mathfrak{l}_i \\ \xi_2 \mathfrak{l}_j \\ \xi_3 \mathfrak{l}_k \end{pmatrix} \equiv \begin{pmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \\ \mathfrak{m}_1 + \mathfrak{m}_2 \end{pmatrix}, \ \left(\begin{smallmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_2 \end{smallmatrix}\right) = M_{ijk}$$

for the eight above matrices are written as columns in the following *factor matrix*

$$\Xi = (\xi_m(ijk)) = \begin{pmatrix} 1 & 1 & \mathrm{i} & -1 & -\mathrm{i} & -1 & 1 & \mathrm{i} \\ 1 & \mathrm{i} & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -\mathrm{i} & 1 & 1 & \mathrm{i} & 1 \end{pmatrix}. \tag{51}$$

The corresponding normalized Δ-type functions

$$\mathfrak{D}_{ijk} := \mathfrak{D}^{\omega_3}_{M,\mu} \tag{52}$$

with the above matrices $M = M_{ijk}$ and $\mu = (0, 0, \omega_3)$. For instance,

$$\mathfrak{D}_{123} \sim \Delta^{\omega_3}_{M_{123},\mu} = \frac{\sigma(u+\omega_3)\sigma(v+\omega_3)\sigma(u+v)}{\sigma(u)\sigma(v)\sigma(u+v+\omega_3)}.$$

The (triangular) pole divisor on A of this function is $T_i + T_j + T_k$. The zero divisor is the sum of the pole components shifted by P_1 (or P_2 with the same effort). Altogether we get

$$(\mathfrak{D}_{ijk})_A = (P_{33} + T_i) + (P_{33} + T_j) + (P_{33} + T_k) - (T_i + T_j + T_k),$$

where P_{33} is one of the sixteen 2-torsion points

$$P_{ij} = (\omega_i, \omega_j)\, mod\, \Lambda$$

of A. The multiplicity triples are same: $\mu(\mathfrak{D}_{ijk}) = (-1, -1, -1)$. Therefore

$$(\mathfrak{D}_{ijk})_{L+T'} = -(L_1 + L_2 + L_3) - (T_i + T_j + T_k)$$

implying

$$\mathfrak{D}_{ijk} \in H^0(A', (L + T')),\ \delta_{ijk} := \iota(\Delta_{ijk}) \in [\Gamma, 1]. \tag{53}$$

It is not difficult to recognize that the disc criterion 7.2 for (A, T) is satisfied. This can be verified by the following

Proposition 16.2 *The automorphism group $G(A, T)$ can be identified with the (half) integral part $\mathfrak{H}(\mathfrak{O}^2, \frac{1}{2}\Lambda)$ of the Heisenberg group $\mathfrak{H}(K^2, \mathbf{Q}\Lambda)$. Via linear parts it is isomorphic to the subgroup of $\mathbf{Gl}_2(\mathfrak{O})$ generated by the three elements*

$$\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$$

This is a central extension of the rotation group $\mathbf{O}_+ \cong S_4$ of the octahedron with exact sequence

$$1 \longrightarrow I = < \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} > \longrightarrow G(A, T) \longrightarrow \mathbf{O}_+ \longrightarrow 1.$$

$G(A, T)$ acts transitively on the set $\{T_1, ..., T_6\}$ of compononents of T with ineffective kernel I.

Proof (idea). Send pairs of T-components to other pairs and check, which of these maps extend to (half) integral Heisenberg group elements. This has been done by MAPLE.

□

We recommend the reader to draw a real 3-dimensional graph: Represent the boundarys of three discs $\mathbf{D}_i$ over L_i, $i = 1, 2, 3$ by three mutually perpendicular big circles on the 2-sphere in $\mathbf{R}^3$. Correspond the six intersection points to six cusps and to the six T'-components contracted to cusp points on $\widehat{\mathbf{B}/\Gamma}$. The intersection points span an octahedron. On this way the action of $G(A, T)/I$ is via T-components visible as original octahedron rotation group.

By (24) the $\hat{\mathbf{D}}$-cycles of the modular forms (53) on $\hat{\mathbf{B}}$ are

$$(\delta_{ijk})_{\hat{\mathbf{D}}} = \kappa_p + \kappa_q + \kappa_r,\ \{i, j, k, p, q, r\} = \{1, ..., 6\}. \tag{54}$$

Definition 16.3 . *The algebraic functions $f_1, ..., f_r$ (on any smooth compact complex algebraic manifold V) satisfies the* strong descending divisor condition *iff there are linear combinations $g_1, ..., g_r$ of them with strongly descending* minimum chain *of divisors*

$$(g_1) > \min\{(g_1), (g_2)\} > ... > \min\{(g_1), ..., (g_r)\}. \tag{55}$$

Thereby the minimum of several divisors is defined componentwise.

Lemma 16.4 *If $f_1, ..., f_r$ satify the strong descending divisor condition, then they are linearly independent.*

Proof. It suffices to prove that the linear combinations $g_1, ..., g_r$ in 16.3 are linearly independent. By assumption, the principal divisor (g_j) has a component $n_j C_j$, C_j irreducible, which is lower than the C_j-components of all linear combinations of $g_1, ..., g_{j-1}$, $j = 2, ..., r$. Therefore g_j cannot be a linear combination of its predecessors.

□

With these preparation we are able to determine the dimension of $[\Gamma, 1]$.

Proposition 16.5 *With notations of this and the previous section it holds that*

$$\dim H^0(A', L + T') = 5. \tag{56}$$

Together with the constant function 1 *the quadruple $\mathfrak{D}_{i_m j_m k_m}$, $m = 1, ..., 4$, yields a basis of $H^0(A', L + T')$, if the cardinality of $\bigcup_{m=1}^{n} \{i_m, j_m, k_m\}$ is equal to $2 + n$ for $n = 1, ..., 4$.*

Example 16.6 . $1, \mathfrak{D}_{123}, \mathfrak{D}_{126}, \mathfrak{D}_{153}, \mathfrak{D}_{243}$ *are linearly independent.*

Proof. Without loss of generality we restrict us to the example. A minimum chain of pole divisors of these functions is

$$\begin{aligned} O > -(T_1 + T_2 + T_3) > -(T_1 + T_2 + T_3 + T_6) > -(T_1 + T_2 + T_3 + T_5 + T_6) \\ > -(T_1 + T_2 + T_3 + T_4 + T_5 + T_6). \end{aligned}$$

Taking successively general linear combinations of first subsequences of the functions in 16.6 we get a sequence of functions with the above pole chain. But then we are in the situation of (55) with $r = 5$. Therefore $h^0(A', L + T') \geq 5$.

By (54) and Remark 7.1 the modular forms $\delta_{ijk} \in [\Gamma, 1]$ are non-cuspidal at $\kappa_i, \kappa_j, \kappa_k$ and cuspidal at the other three cusps. Let $\delta = \iota_1(\mathfrak{D})$ be an element of $[\Gamma, 1]$. We substract successively suitable $\mathbf{C}$-multiples of forms δ_{ijk} till we get a constant function. We do it for the general and worst case $(\mathfrak{D})_T = -(T_1 + ... + T_6)$. This means that δ is non-cuspidal at all cusps $\kappa_1, ..., \kappa_6$. So we can find $c_1 \in \mathbf{C}^*$ such that $\delta - c_1\delta_{126}$ is cuspidal at κ_1. Successively we find constants c_j, $j \leq 4$, such that

$$\varphi := \delta - c_1\delta_{126} - c_2\delta_{153} - c_3\delta_{243} - c_4\delta_{123} \in [\Gamma, 1]$$

is cuspidal at $\kappa_6, \kappa_5, \kappa_4, \kappa_3$. For the abelian function $f \in H^0(A', L + T')$ with $\varphi = \iota_1(f)$ this means that $(f)_A \geq -T_1 - T_2$. But $H^0(A, -T_1 - T_2) = \mathbf{C}$, because T_1 and T_2 intersect each other at precisely one point, namely O. Therefore the restrictions of $f \in H^0(A, -T_1 - T_2)$ on the elliptic fibres of the fibrations

$$A \longrightarrow T_j \cong A/T_k \cong E,\ \{j, k\} = \{1, 2\},$$

with kernel T_k must be constants because they have at most one pole. So f is a constant, hence $\varphi \in \mathbf{C}\eta$ and

$$\delta \in \mathbf{C}\eta + \mathbf{C}\delta_{123} + \mathbf{C}\delta_{126} + \mathbf{C}\delta_{153} + \mathbf{C}\delta_{243}.$$

□

Corollary 16.7 *The space of Γ-modular forms of weight 1 is generated by four forms of (triangular) abelian Σ-quotients and η. For instance, the sequence η, δ_{123}, δ_{126}, δ_{153}, δ_{243} is a basis of $[\Gamma, 1]$.*

□

Theorem 16.8 *We dispose now on the complete list of dimensions:*

$$\dim[\Gamma, n]_{cusp} = \begin{cases} 1, & n = 1; \\ 9\binom{n}{2}, & n > 1. \end{cases} \tag{57}$$

$$\dim[\Gamma, n] = \begin{cases} 1, & n = 0; \\ 5, & n = 1; \\ 9\binom{n}{2} + 6, & n > 1. \end{cases} \tag{58}$$

Proof. Knowing $s = 3$ we have only to fill the $[\Gamma, 1]$-gap in Theorem 6.1 with (56).

□

Together with 44 we get

Corollary 16.9 *For low dimensions we have*

$$\begin{array}{llllll} \dim[\Gamma,1] & =5, & \dim[\Gamma,1]_{cusp} & =1, & \dim[\Gamma,1]_{cusp}^{new} & =0, \\ \dim[\Gamma,2] & =15, & \dim[\Gamma,2]_{cusp} & =9, & \dim[\Gamma,2]_{cusp}^{new} & =4, \\ \dim[\Gamma,3] & =33 & \dim[\Gamma,3]_{cusp} & =27, & \dim[\Gamma,3]_{cusp}^{new} & =12. \end{array}$$

□

We look for new cusp forms of weight 2. Instead of ω_3-shifts we work with

$$\mathfrak{D}'_{ijk} := \mathfrak{D}^{\omega_1}_{M,\mu} \tag{59}$$

with the same matrices $M = M_{ijk}$ and fixed $\mu = (0,0,\omega_3)$ as above. For instance,

$$\mathfrak{D}'_{123} \sim \frac{\sigma(u+\omega_1)\sigma(v+\omega_1)\sigma(u+v+\omega_2)}{\sigma(u)\sigma(v\sigma(u+v+\omega_3)}.$$

We have the same triangular pole divisors $T_i+T_j+T_k$ as for $\mathfrak{D}'_{ijk}$ but the zero divisors do not contain any of the points P_1, P_2, P_3. Therefore the multiplicity triple at these points is equal to $(-2,-2,-2)$, thus

$$(\mathfrak{D}'_{ijk})_{L+T'} = -2(L_1+L_2+L_3) - (T_i+T_j+T_k),$$

implying

$$\mathfrak{D}'_{ijk} \in H^0(A', 2L+T')),\ \delta'_{ijk} := \iota(\mathfrak{D}'_{ijk}) \in [\Gamma,2]_{cusp}.$$

By the above descending pole divisor method it is easy to find four linearly independent functions $\mathfrak{D}'_{ijk}$. Taking in account also the zero divisors and the 16 torsion points (four on each elliptic component) it is not difficult to generate a space $[\Gamma,2]^{new}_{cusp}$ with four of the functions δ'_{ijk}. The check of details is left to the reader. We pick out

$$\mathfrak{D}'_1 := \mathfrak{D}'_{456},\ \mathfrak{D}'_2 := \mathfrak{D}'_{453},\ \mathfrak{D}'_3 := \mathfrak{D}'_{246},\ \mathfrak{D}'_4 := \mathfrak{D}'_{156}, \tag{60}$$

as basis of a subspace of $H^0(A', 2L+T')$ complementary to $H^0(A', L+T')$. With Corollary 16.9 and Theorem 15.1 we get

Proposition 16.10 *It holds that*

$$\begin{aligned} [\Gamma,2]_{cusp} &= \eta[\Gamma,1] \oplus \mathbf{C}\delta'_{456} \oplus \mathbf{C}\delta'_{453} \oplus \mathbf{C}\delta'_{246} \oplus \mathbf{C}\delta'_{156} \\ [\Gamma,2] &= [\Gamma,2]_{cusp} \oplus \mathbf{C}\pi_1 \oplus \ldots \oplus \mathbf{C}\pi_6. \end{aligned}$$

□

With index abbreviation as in (60) and

$$\delta_1 := \delta_{123},\ \delta_2 := \delta_{126},\ \delta_3 := \delta_{153},\ \delta_4 := \delta_{243} \tag{61}$$

we find also explicit cusp forms of weight 3

$$\delta_1\delta_2',\ \delta_1\delta_3',\ \delta_1\delta_4',\ \delta_2\delta_1',\ \delta_3\delta_1',\ \delta_4\delta_1'. \tag{62}$$

The strong descending pole divisor condition for the corresponding products $\mathfrak{D}_i\mathfrak{D}_j'$ is satisfied. Therefore they are linearly independent. We find further explicit new cusp forms by multiplication with π_m, $m = 1, ..., 6$.

Lemma 16.11 *The product $\pi_m\delta_{ijk} \in [\Gamma, 3]$ is a cusp form iff $m \notin \{i, j, k\}$.*

Proof. The abelian functions $\mathfrak{P}_m$ defined in section 15 with double zero along $P_m + T_m$, where the point index must be taken modulo 3. Counting multiplicities at P_1, P_2, P_3 we get

$$(\mathfrak{P}_m)_{A'} = 4L_m - 2L - 2T_m'$$

with L-index modulo 3. Therefore the divisor of $\mathfrak{P}_m\mathfrak{D}_{ijk}$ is not smaller than $-2T'$ iff $m \notin \{i, j, k\}$. Now we get the statement via modular transfer.

□

Going back to abelian functions it is easy to see that

Example 16.12 . *The modular forms $\delta_1\pi_5, \delta_2\pi_5, \delta_4\pi_5,\ \delta_1\pi_6, \delta_3\pi_6, \delta_4\pi_6$ are linearly independent.*

Together with the six cusp forms (62) we get a 12-dimensional space generated by new cusp forms of weight 3. More is not possible by Corollary 16.9. On this way one gets with the notations of (42), Corollary 16.9 and Theorem 15.1 the

Theorem 16.13 *The homogeneous parts of degree lower than 4 of the graded subring*

$$\mathbf{C}[\eta, \delta_1, ..., \delta_4, \delta_1', ..., \delta_4', \pi_1, ..., \pi_6, \pi_1', ..., \pi_6']$$

of the ring of Γ-modular forms $R[\Gamma]$ coincide with $[\Gamma, k]$, $k = 0, 1, 2, 3$, respectively.

□

17 Basic relations in the case of Gauß numbers

We set $\mathfrak{P}_i := \mathfrak{P}_{\mathfrak{l}_i,\kappa}$ with

$$\kappa = \kappa(i) = \begin{cases} \omega_3, & \text{for } i = 3, 6, \\ 0, & \text{for } i = 1, 2, 4, 5, \end{cases}$$

such that $-2T_i$ is the pole divisor of $\mathfrak{P}_i$. With the same κ's we define

$$\mathfrak{Q}_i := \mathfrak{Q}^{\omega_1\omega_2\omega_3}_{\mathfrak{l}_i,\kappa(i)}$$

Theorem 17.1 . *The abelian functions $\mathfrak{P}_i, \mathfrak{Q}_i$ and $\mathfrak{D}_{ijk}$, $\mathfrak{D}'_{ijk}$ defined in (52) or (59), respectively, satisfy the following algebraic equations*

(I) $\mathfrak{Q}_i^2 = \mathfrak{P}_i^3 + \mathfrak{P}_i,\ i = 1, ..., 6,$

(II) $\det \begin{pmatrix} 1 & (\xi_1(ijk)^2\mathfrak{P}_i & \xi_1(ijk)\mathfrak{Q}_i \\ 1 & (\xi_2(ijk)^2\mathfrak{P}_j & \xi_2(ijk)\mathfrak{Q}_j \\ \mathfrak{P}_k^2 & \xi_3(ijk)^2\mathfrak{P}_k & \xi_3(ijk)\mathfrak{Q}_k \end{pmatrix} = 0,$

(III) $\mathfrak{D}^2_{ijk} = \xi_1(ijk)^2\xi_2(ijk)^2\xi_3(ijk)^2\mathfrak{P}_i\mathfrak{P}_j\mathfrak{P}_k,$

(III ') $\mathfrak{D}'^2_{ijk} = (\xi_1(ijk)^2\mathfrak{P}_i + \mathrm{i})(\xi_2(ijk)^2\mathfrak{P}_j + \mathrm{i})(\xi_3(ijk)^2\mathfrak{P}_k + \mathrm{i}).$

for $(i, j, k) = (1, 2, 3), (1, 2, 6), (1, 5, 3), (1, 5, 6),$
$(2, 4, 3), (2, 4, 6), (4, 5, 3), (4, 5, 6).$

Proof. The six equations (I) come from (i$'$) of Corollary 14.2 specialized to our case of Weierstraß equation (45) with $e_1 = \mathrm{i}$, $e_2 = -\mathrm{i}$, $e_3 = 0$.

The determinant relations are specialisations of (ii$'$) in the same corollary. In the first two rows of the determinants in (ii$'$) we have to take into consideration the multiplication factors listed in the columns of Ξ in (51) assigning the the difference of linear forms in the T_i-equations (48) and the rows used in the eight matrices (50). The multiplication of the arguments by i in the functions $\wp(z)$ and $\wp'(z)$ has the representation matrix $\left(\begin{smallmatrix} \mathrm{i}^2 & 0 \\ 0 & \mathrm{i} \end{smallmatrix}\right)$, see (46). A comparision with the definitions of $\mathfrak{P}_j$, $\mathfrak{Q}_j$ via σ-function shows that this is also the representation matrix of this pair with respect to argument multiplication by i. The reader should check it first for $\mathfrak{P}_1 = \wp(u)$, $\mathfrak{Q}_1 = 2\wp'(u)$ and then for the other $\mathfrak{P}_j$, $\mathfrak{Q}_j$ via substitutions $z = \mathfrak{l}_j \cdot \mathfrak{u} + \kappa_j$. As the last row one has to take $(1, \mathfrak{P}_{\mathfrak{l}_k,0}, \mathfrak{Q}_{\mathfrak{l}_k,0})$, $k = 3$ or 6 in accordance with (ii$'$). Observe that

$$\mathfrak{P}_{\mathfrak{l}_k,0}(u, v) = \mathfrak{P}_k(u + \omega_3, v),\ \mathfrak{Q}_{\mathfrak{l}_k,0}(u, v) = \mathfrak{Q}_k(u + \omega_3, v).$$

With the classical determinant formula (addition theorem) for $\wp$ and $\wp'$ one receives

$$0 = \det \begin{pmatrix} 1 & \wp(\omega_3) & -\wp'(\omega_3) \\ 1 & \wp(z) & -\wp'(z) \\ 1 & \wp(z + \omega_3) & \wp'(z + \omega_3) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & \wp(z) & -\wp'(z) \\ 1 & \wp(z + \omega_3) & \wp'(z + \omega_3)) \end{pmatrix}$$
$$= \wp'(z + \omega_3)\wp(z) + \wp(z + \omega_3)\wp'(z).$$

Together with (47), we get

$$\wp(z+\omega_3) = \frac{1}{p(z)},\ \wp'(z+\omega_3) = -\frac{\wp'(z)}{\wp(z)^2}$$

and after linear u, v-substitutions for z

$$\mathfrak{P}_{\mathfrak{l}_k,0} = \mathfrak{P}_k(u+\omega_3, v) = 1/\mathfrak{P}_k,\ \mathfrak{Q}_{\mathfrak{l}_k,0} = \mathfrak{Q}_k(u+\omega_3, v) = -\frac{\mathfrak{Q}_k}{\mathfrak{P}_k^2}.$$

Substitution in

$$0 = \det\begin{pmatrix} 1 & (\xi_1(ijk)^2\mathfrak{P}_i & \xi_1(ijk)\mathfrak{Q}_i \\ 1 & (\xi_2(ijk)^2\mathfrak{P}_j & \xi_2(ijk)\mathfrak{Q}_j \\ 1 & \xi_3(ijk)^2\mathfrak{P}_{\mathfrak{l}_k,0} & \xi_3(ijk)\mathfrak{Q}_{\mathfrak{l}_k,0} \end{pmatrix}$$

yields (II) after multiplying the third row by $\mathfrak{P}_k^2$.

The last 16 relations (III) and (III$'$) follow immediately from (iii) in Theorem 14.1.

□

We look for the simplest algebraic relations of the generators of the graded ring of modular forms

$$\mathbf{C}[\eta, \delta_1, ..., \delta_4, \delta'_1, ..., \delta'_4, \pi_1, ..., \pi_6, \pi'_1, ..., \pi'_6]$$

and for a projective embedding of $\widehat{\mathbf{B}/\Gamma}$ using all of them. For this purpose we set

$$\begin{aligned} \mathfrak{P}_0 &:= \mathfrak{P}_1(u+\omega_3, v) = \wp(u+\omega_3) = 1/\mathfrak{P}_1, \\ \mathfrak{Q}_0 &:= \mathfrak{Q}_1(u+\omega_3, v) = -\frac{\mathfrak{Q}_1}{\mathfrak{P}_1^2} = -\mathfrak{Q}_1\mathfrak{P}_0^2 \end{aligned}$$

and

$$\pi_0 := 1/\pi_1,\ \pi'_0 := -\pi'_1/\pi_1^2$$

connected with the modular functions

$$\Upsilon(\mathfrak{P}_0) = \eta^2\pi_0,\ \Upsilon(\mathfrak{Q}_0) = \eta\pi'_0$$

With the additional notations of (60) and similar index abbreviations $\mathfrak{D}_m$ for some $\mathfrak{D}_{ijk}$ used in (61) we correspond to the 22 variables

(63) $$X_0, Y_0, U_1, ..., U_4, V_1, ..., V_4, X_1, ..., X_6, Y_1, ...Y_6$$

the 22 abelian functions

$$(64) \qquad \mathfrak{P}_0, \mathfrak{Q}_1, \mathfrak{D}_1, ..., \mathfrak{D}_4, \mathfrak{D}'_1, ..., \mathfrak{D}'_4, \mathfrak{P}_1, ..., \mathfrak{P}_6, \mathfrak{Q}_1, ..., \mathfrak{Q}_6,$$

and the 22 abelian modular functions being their Υ-images by Remark 6.4.

$$(65) \qquad \eta^2\pi_0, \eta\pi'_0, \frac{\delta_1}{\eta}, ..., \frac{\delta_4}{\eta}, \frac{\delta'_1}{\eta}, ..., \frac{\delta'_4}{\eta}, \frac{\pi_1}{\eta^2}, ..., \frac{\pi_6}{\eta^2}, \frac{\pi'_1}{\eta^3}, ..., \frac{\pi'_6}{\eta^3},$$

respectively.

Theorem 17.2 . *As well the* 22 *abelian functions (64) as the* 22 *modular functions (65) satisfy the affine relations corresponding to the following* 20 *homogeneous equations after setting* $Z = 1$*:*

(O) $X_0X_1 = Z^2, \ X_0Y_1 + X_1Y_0 = 0$;

$$(I) \quad \begin{array}{l} Y_1^2 = X_1^3 + X_1Z^2, \ Y_2^2 = X_2^3 + X_2Z^2, \ Y_3^2 = X_3^3 + X_3Z^2, \\ Y_4^2 = X_4^3 + X_4Z^2, \ Y_5^2 = X_5^3 + X_5Z^2, \ Y_6^2 = X_6^3 + X_6Z^2; \end{array}$$

$$(II) \quad \begin{array}{r} (Y_0 - Y_2)X_3 + (-X_2 + X_0)Y_3 + X_0Y_2 - Y_0X_2 = 0, \\ (Y_0 - \mathrm{i}Y_2)X_6 + (X_2 + X_0)Y_6 + \mathrm{i}X_0Y_2 + Y_0X_2 = 0, \\ (-Y_6 + Y_0)X_4 + (X_0 - X_6)Y_4 + X_0Y_6 - Y_0X_6 = 0, \\ (-Y_3 - \mathrm{i}Y_0)X_5 + (-X_0 - X_3)Y_5 - X_0Y_3 + \mathrm{i}Y_0X_3 = 0; \end{array}$$

$$(III) \quad \begin{array}{l} ZU_1^2 - X_1X_2X_3 = 0, \ ZU_2^2 + X_1X_2X_6 = 0, \\ ZU_3^2 + X_1X_5X_3 = 0, \ ZU_4^2 + X_2X_4X_3 = 0; \end{array}$$

$$(III') \quad \begin{array}{l} ZV_1^2 = (-X_4 + \mathrm{i}Z)(X_5 + \mathrm{i}Z)(X_6 + \mathrm{i}Z), \\ ZV_2^2 = (X_4 + \mathrm{i}Z)(X_5 + \mathrm{i}Z)(-X_3 + \mathrm{i}Z), \\ ZV_3^2 = (X_2 + \mathrm{i}Z)(X_4 + \mathrm{i}Z)(X_6 + \mathrm{i}Z), \\ ZV_4^2 = (-X_1 + \mathrm{i}Z)(X_5 + \mathrm{i}Z)(X_6 + \mathrm{i}Z). \end{array}$$

Proof. The relations (O) are homogenized defining relations for $\mathfrak{P}_0$ and $\mathfrak{Q}_0$. Relations (I), (III) and (III$'$) come from the corresponding affine relations in Theorem 17.1. For (II) we change from the determinant relations of Theorem 17.1 to simpler determinant relations working with $\mathfrak{P}_0$, $\mathfrak{Q}_0$ instead of $\mathfrak{P}_1$, $\mathfrak{Q}_1$. We pick out the four index triples 123, 126, 146 and 153. Then we apply immediately (ii$'$) of Corollary 14.2 with $\mu_2 = 0$, $\mu_1 = \mu_1 + \mu_2 = \omega_3$.

□

Remark 17.3 . *The auxiliary functions* $\mathfrak{P}_0$*,* $\mathfrak{Q}_0$ *do not belong to* $H^0(A', *(L + T'))$*. They have the advantage to diminish the degrees of the algebraic determinant equations. Moreover, starting with* $\mathfrak{P}_1$*,* $\mathfrak{P}_2$*, from the* *above equations one can derive an explicit and shortest "triangular" system of algebraic equations for our basic functions in the following order*

$$\mathfrak{P}_0, \mathfrak{Q}_0, \mathfrak{P}_3, \mathfrak{Q}_3, \mathfrak{P}_6, \mathfrak{Q}_6, \mathfrak{P}_4, \mathfrak{Q}_4, \mathfrak{P}_5, \mathfrak{Q}_5, \mathfrak{D}_1, ..., \mathfrak{D}_4, \mathfrak{D}'_1, ..., \mathfrak{D}'_4,$$

where "triangular" means that each of these functions satisfies an polynomial equation in one variable with coefficitions in the polynomial ring generated by its predesessors. For instance, the first determinant equation of (II) yields after substitution $\mathfrak{Q}_3 = \sqrt{\mathfrak{P}_3^3 + \mathfrak{P}_3}$ *such a polynomial equation with coefficients in* $\mathbf{C}[\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_0, \mathfrak{Q}_0]$.

Definition 17.4 . *A* Baily-Borel morphism $\Phi: \mathbf{B} \longrightarrow \mathbf{P}^N$ *is an everywhere defined projectivized analytic map from* $\mathbf{B}$ *defined by* Γ*- modular forms* $\varphi_0, ..., \varphi_N$ *of the same weight applied to ball points:*

$$\mathbf{B} \ni \beta \mapsto (\varphi_0(\beta) : ... : \varphi_N(\beta)) \in \mathbf{P}^N.$$

It is also assumed, that there is no cusp, where all φ_i *vanish simultaneously.*

Because of same automorphy factors and Hartog's theorem Φ factors through a well-defined analytic morphism $\varphi: \mathbf{B}/\Gamma \longrightarrow \mathbf{P}^N$, which can be extended uniquely to an algebraic morphism $\hat{\varphi}$. If it is an embedding, that means an isomorphy onto its image surface, then we say that $\hat{\varphi}$ is a *Baily-Borel embedding*. Such embeddings exist for each arithmetic group Γ acting on $\mathbf{B}$ by a celebrated theorem of Baily-Borel in [BB]. It works with a basis of $[\Gamma, N]$ for sufficiently high $N > 0$.

Theorem 17.5 *The Baily-Borel map* Φ *of the* 23*-tuple of* Γ*-modular forms*

$$\begin{aligned}&(\hat{\varphi}_0, \hat{\varphi}_1, ..., \hat{\varphi}_{22}) = \\ &(\eta^3, \eta^5\pi_0, \eta^4\pi'_0, \eta^2\delta_1, ..., \eta^2\delta_4, \eta\delta'_1, ..., \eta\delta'_4, \eta\pi_1, ..., \eta\pi_6, \pi'_1, ..., \pi'_6)\end{aligned} \tag{66}$$

of weight 3 *induces a Baily-Borel embedding*

$$\hat{\varphi}: \widehat{\mathbf{B}/\Gamma} \longrightarrow \mathbf{P}^{22}.$$

Working with homogeneous coordinate functions (63), the points of the image surface satisfy the homogeneous equations (O), (I), (II), (III) and (III') of Theorem 17.2.

Proof. At the cusp κ_i the modular form π'_i does not vanish by (43). For instance, using also (49), we have

$$(\pi'_1)_{\hat{\mathbf{D}}} = 3\mathbf{D}_2 + 3\mathbf{D}_3 + 3\kappa_2 + ... + 3\kappa_5.$$

It is clear that $\pi'_1, ..., \pi'_6$ distuinguish the cusps $\kappa_1, ..., \kappa_6$. We see also that π'_1 does not vanish identically on $\mathbf{D}_1$. Taking into account the other forms π'_i it follows that Φ is well-defined on $\mathbf{D}_1 \cup \mathbf{D}_2 \cup \mathbf{D}_3$ outside of a discrete subset.

Therefore $\hat{\varphi}$ is well-defined on $L_1 \cup L_2 \cup L_3 \subset \widehat{\mathbf{B}/\Gamma}$ outside of a finite set of points sitting inside of $\mathbf{B}/\Gamma$. But $\hat{\varphi}$ is also well-defined outside of L. Namely, with identifications

$$A^* := \widehat{\mathbf{B}/\Gamma} \setminus supp\,(L) = A' \setminus supp\,(L + T') = A \setminus supp\,(T)$$

the map $\hat{\varphi}$ coincides on this open subset with the map to the affine space $\mathbf{A}^{22}$ induced by the abelian functions (64). Namely, since η has no zeros over A^*, we can divide our embedding modular forms (66) by η^3. Forgetting the 0-coordinate we get the affine part of the map from A^* into $\mathbf{A}^{22}$ by modular functions whose pullbacks along Υ, defined in 6.4, coincide with (64). It is elementary to check that the functions (64) have no common zero on A^*: The torsion point P_{33} (or P_{13}) is the only intersection point of of the zero divisors of $\mathfrak{D}_1, ..., \mathfrak{D}_4$ (or of $\mathfrak{D}_1', ..., \mathfrak{D}_4'$, respectively). So $\hat{\varphi}$ is well-defined on A^*, thus Φ is well-defined outside a discrete subset of $\mathbf{B}$. Via Hartogs' theorem we get Φ on $\mathbf{B}$ by analytic extension.

Next we prove that $\hat{\varphi}$ is injective. The elliptic functions $\wp$ and $\wp'$ embed the elliptic curve into $\mathbf{P}^2$. On the affine part we get an embedding of $E \setminus \{O\}$ into $\mathbf{A}^2$. Taking biproducts, the abelian functions $\mathfrak{P}_1$, $\mathfrak{Q}_1$, $\mathfrak{P}_2$, $\mathfrak{Q}_2$ embed A into $\mathbf{P}^2 \times \mathbf{P}^2$ and $A \setminus supp\,(T_1 + T_2)$ into $\mathbf{A}^4$. They distuinguish also the tangent lines through O, which are the points of the space $L_3 \cong \mathbf{P}^1$. Altogether $\hat{\varphi}$ is injective on $A' \setminus supp\,(T') = \mathbf{B}/\Gamma$.

As last step we prove that $\hat{\varphi}$ is an isomorphism onto its image surface. For this purpose we pull back $\hat{\varphi}$ to $\varphi' : A' \longrightarrow Im\,\hat{\varphi} \subset \mathbf{P}^{22}$. This is a birational morphism contracting some connected exceptional curves. It contracts T' to six elliptic cusp points. More is not possible because φ' coincides with $\hat{\varphi}$ outside of T', where it is injective. Therefore $\hat{\varphi}$ is an embedding.

□

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