r-Maximal Sets

Eberhard Herrmann

Abstract

The r-maximal sets and their properties were investigated in several papers. Here it will be given a systematical presentation of all these results including also few which are still not published. In the centre of interest are the atomless, r-maximal sets and the different methods of constructing them. In particular in the paper are treated the already known lattice properties of the r-maximal sets. But also degree properties of them and more general of the r-cohesive sets are given. Further the paper includes also index set estimations of special classes of r-maximal sets and considers the relationship between classes of r-maximal sets and other classes of c.e. sets.

Introduction

The notion of maximal set can be generalized in respect to many different point of views. Thus if we take the c.e. superset structure of a maximal set (factored by finite differences between sets) we get the two-element Boolean algebra. If we require for a coinfinite c.e. set that its c.e. superset structure (modulo finite differences) is a Boolean algebra we get the more general notion of hyperhypersimple set.
An other generalization we get by the following consideration: The c.e. sets disjoint to an c.e. set form an ideal. For a maximal set this ideal consists only of finite sets. If we factor the c.e. superset structure of a maximal set by this ideal we get again the two-element Boolean algebra. If we require for an arbitrary c.e. set that this factorization (by this ideal, which for arbitrary c.e. sets can include also infinite c.e. sets) gives the two-element Boolean algebra then we get the notion of $\mathcal{D}$-maximal set. The known results about the $\mathcal{D}$-maximal sets show that the class of $\mathcal{D}$-maximal sets is much greater than the class of maximal sets.
A third generalization of the notion of maximal set follows if we require for the c.e. superset structure of an coinfinite c.e. set that only the complemented elements in this sublattice (mod.fin.dif.) form the two-element Boolean algebra. Coinfinite c.e. sets with such a property are called r-maximal.

This third generalization, i.e. the r-maximal sets, is the topic of investigations of this paper. While the existence of r-maximal sets which are not maximal follows easily from Lachlan's major subset theorem, the general description of all possibilities of r-maximal sets e.g. in respect to their c.e. superset structures is complicated and still not completely done. The richness of the class of r-maximal sets follows from the so-called atomless, r-maximal sets. The existence of such r-maximal sets was firstly shown by Robinson and later again by different construction methods.

We give in subpoint 1 the precise definition of the r-maximal sets and describe roughly the position of the r-maximal sets inside the lattice of c.e. sets. In subpoint 2 we give three different constructions of atomless, r-maximal sets. Each of them will be used in later subpoints for showing further properties of the r-maximal sets.
The description of the lattice-theoretic properties of the r-maximal sets, i.e. the current situation of investigation of the description of the c.e. superset lattices of the r-maximal sets will be done in subpoint 3. Throughout some about these structures is already known, a complete description of these is still not done. About the automorphism properties of the r-maximal sets almost nothing is known.
Basic facts on the T-degrees of r-maximal sets can be easily concluded from other results. The T-degrees of the r-cohesive sets in general are much sigher and outside the class of high sets even is equal to the T-degrees of the cohesive sets. This degree class was intensively investigated by Jockusch and others. This will be given in subpoint 4.
Subpoint 5 includes the index set estimations of the r-maximal sets and important subclasses of them. Here the new results about the index set estimations of the atomless, r-maximal sets and the major subsets are given.
In subpoint 6 the notion of monotonic set and of $1-1$ set analysed by Madan and Robinson is considered. We shall see that these notions are closely connected with the r-maximal sets, more precisely are subnotions of the notion of r-maximal set.
In the last subpoint 7 we investigate the appearence of r-cohesive sets in the lattice of c.e. sets in the form as d.-c.e. sets. Here the new notion of r-maximal major subsets is introduced and the results from Lerman, Shore and Soare concerning these special subsets are given.

Nevertheless already some about the r-maximal sets is known, there are still many open problems and questions concerning these sets. There are given in many places through the whole paper and show that the theory of the r-maximal sets is rather in a beginning stage than in a final one.

Our symbols and notions are almost identical with those in [So87].
In difference we use $I\!N$ as symbol for the numbers and not ω. With $\mathcal{E}$ we denote the lattice of c.e. sets under inclusion. Instead of "modulo finite differences between sets" we write shortly "(mod $=^*$)". If X is a set with $X^{<\omega}$ we denote as usual the set of finite sequences with members from X. Usually we work with $2^{<\omega}$, i.e. $\{0,1\}^{<\omega}$. In some theorems we shall need $I\!N^{<\omega}$. $\langle\ \rangle$ is the symbol for the empty sequence. For $\sigma \in X^{<\omega}$ with $[\sigma]$ we denote the set $\{\nu \in X^{<\omega} : \sigma \preceq \nu\}$. For nodes α and β with $\alpha \cap \beta$ we denote the common initial part of both. Let $\alpha \in X^{<\omega}$ with $\alpha \neq \langle\ \rangle$. Then α^- means the immediate predecessor of α. For α from $X^{<\omega}$ $|\sigma|$ means the length of σ. If $\sigma, \nu \in 2^{<\omega}$ then $\sigma <_1 \nu$ means that σ is lesser than ν respectively to the lexicographical order in $2^{<\omega}$.

Contents

1 The definition of the r-maximal sets and the main classes of r-maximal sets

We start in this subpoint with the definition of the basic notions for the paper, namely of the r-cohesive sets and by using this of the r-maximal sets. Directly from the definition follows that the notion of r-maximal set generalizes that of the maximal set. Further we compare the class of r-maximal sets with other classes of c.e. sets and after this with analogously defined objects in another lattice as $\mathcal{E}$ which also is of interest for the Computability Theory.

Definition 1.1 A subset X of $\mathbb{N}$ is called *computable cohesive* (shortly: *r-cohesive*) if X is infinite and for every computable set R

$$(1.1) \qquad R \cap X =^* \emptyset \qquad \text{or} \qquad \overline{R} \cap X =^* \emptyset.$$

Thus an r-cohesive set cannot be splitted by a computable set into two infinite parts. Obviously every cohesive set is r-cohesive and every r-cohesive set is immune.

1.1 Existence of r-cohesive sets, which are not cohesive

That the notion of r-cohesive set properly extends that of cohesive set can be shown quite easily. For showing this of course the existence of a noncomputable c.e. set is necessary.

Let Y be a noncomputable c.e. set. We define a sequence $(R_n)_{n\geq 0}$ of computable sets as follows:

$$\begin{aligned} R_0 &= \mathbb{N} \\ R_{n+1} &= R_n \cap W_n : \ \ W_n \text{ is computable} \wedge R_n \cap W_n \cap Y \text{ is not computable} \\ &= R_n \cap \overline{W_n} : \ \ W_n \text{ is computable} \wedge R_n \cap \overline{W_n} \cap Y \text{ is not computable} \\ &= R_n \ ; \quad \text{otherwise.} \end{aligned}$$

Let X be a set which has an element from $R_n \cap Y$ and one from $R_n \cap \overline{Y}$ for every $n \geq 0$.

We see that for every n R_n is computable and $R_n \cap Y$ is not computable (inductive over n). Thus for every n the sets $R_n \cap Y$ and $R_n \cap \overline{Y}$ are both infinite. Further if W_e is cofinite then obviously $R_{e+1} = R_e \cap W_e$ by the definition of R_{e+1}. Hence a number belongs only to finitely many R_n, i.e. $X \cap Y$ and $X \cap \bar{Y}$ are both infinite. This gives that X is infinite, but not cohesive.

X is r-cohesive. Suppose not. Let R be a computable set such that $X \cap R$ and $X \cap \bar{R}$ are both infinite. Let $R = W_e$. If $R_{e+1} = R_e \cap W_e$ then $\bar{R} \cap X =^* \emptyset$, since $R_{e+1} \cap \bar{R} = \emptyset$ and $X \subseteq^* R_n$ for every n.

Thus $R_e \cap W_e \cap Y$ is computable. Hence $R_{e+1} = R_e \cap \bar{W}_e = R_e \cap \bar{R}$. But then $R_{e+1} \cap R$ and thus $R \cap X =^* \emptyset$. Thus both possibilities lead to a contradiction, i.e. such an R does not exist.

Definition 1.2 A subset A of $I\!N$ is called *computable maximal* (shortly: *r-maximal*) if A is c.e. and $\bar{A}$ is r-cohesive.

Let R-Max be the symbol for the class of r-maximal sets. Thus we have $A \in R-\mathrm{Max}$ iff $\mathcal{L}_r^*(A)$ is two-element.

1.2 Main classes of r-maximal sets

The class $R-\mathrm{Max}$ can be divided into three main subclasses by considering the c.e. superset structures of them. One obvious subclass is Max – the class of maximal sets. The second one follows from the fact

$$X \in R-\mathrm{Max} \wedge Y \subset_m X \Rightarrow Y \in R-\mathrm{Max}.$$

Thus the class MS_{Max} – the class of major subsets of the maximal sets, forms another subclass of $R-\mathrm{Max}$. Easy to see is that if $A \in MS_{\mathrm{Max}}$ then the maximal set M s.th. $A \subset_m M$ is a maximal element (mod $=^*$) in $\mathcal{L}^*(A)$ and converse if for $A \in R-\mathrm{Max}$ Y^* is maximal in $\mathcal{L}^*(A)$ with $Y^* \neq A^*$ then $A \subset_m Y$ and Y is a maximal set. By using the Reduction principle we see that for $A \in R-\mathrm{Max}$ $\mathcal{L}^*(A)$ can have at most one maximal element.

For the union of Max and MS_{Max} we also write $R-\mathrm{Max}_{\mathbf{atm}}$ (the class of atomic r-maximal sets).
The definition of the r-maximal sets does not imply that in the c.e. superset structure is a maximal element (mod $=^*$).
Denote with $R-\mathrm{Max}_{\mathbf{atl}}$ the class of atomless[1)], r-maximal sets. Not obvious also by allowing to use other known facts is the existence of atomless, r-maximal sets.
Since just these r-maximal sets are the most ones inside the class $R-\mathrm{Max}$, in the following subpoints above all these sets are investigated starting in subpoint 2 with different proofs of the existence of such sets.

1.3 Relationship between $R-$Max to other classes of c.e. sets

Directly from the definition of the r-maximal sets we can compare them with other well-known classes of c.e. sets. From this we get a roughly description of the class $R-$Max in $\mathcal{E}$.

We use beside already known symbols the following ones of classes of c.e. sets considered here:

$$\begin{aligned}
Q-\mathrm{Max} &- \text{the class of } q\text{-maximal sets}\\
\mathcal{HH} &- \text{the class of } hh\text{-simple sets}\\
\mathcal{MS}^1_{\mathrm{Max}} &= \mathrm{Max} \cup MS_{\mathrm{Max}} \quad (= R-\mathrm{Max}_{\mathrm{atm}})\\
\mathcal{MS}^1_{HH} &= \mathcal{HH} \cup MS_{HH}
\end{aligned}$$

1) More precisely would be to say "co-atomless". But since in $\mathcal{E}^*$ are no atoms the notion of "atomless" instead of "co-atomless" does not lead to misunderstandings.

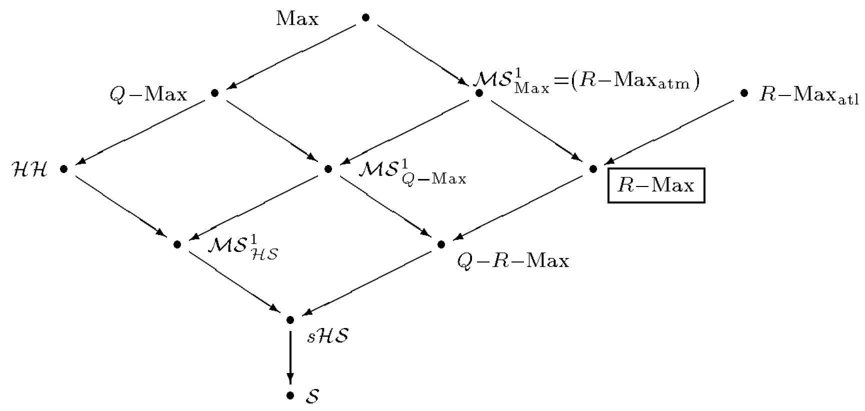
$Q - R - \mathrm{Max}$ – the class of $q - r$-maximal sets. (This are the coinfinite c.e. sets A with $\mathcal{L}_r^*(A)$-finite, what is the same with that A is equal to a finite, nonempty intersection of r-maximal sets.)

$s\mathcal{HS}$ – the class of strongly hypersimple sets

$\mathcal{S}$ – the class of simple sets

(For the definition of the classes Q-Max, $\mathcal{HH}$, $s\mathcal{HS}$ and $\mathcal{S}$ see e.g. [So87]).

We have the following schema about the relationship between the classes given above: (For two classes $\mathcal{X}$ and $\mathcal{I}$ of sets we write $\mathcal{X} \to \mathcal{I}$ if $\mathcal{X}$ is a subclass of $\mathcal{I}$.)



All inclusions in the schema are properly and no further inclusion hold between the classes considered there.

Remark. In subpoint 6 there are still considered further classes of c.e. sets and their relationship to $R-$Max. Among these classes will be also the well-known class of dense simple sets. But the relationships given there are not so obvious as here in the above schema and need proofs.

1.4 The lattice of c.e. ideals, Π^0_1-classes

A structure with many similarities to $\mathcal{E}$ is the lattice of c.e. ideals of $2^{<\omega}$. A subset Δ of $2^{<\omega}$ is an *c.e. ideal* if Δ is an c.e. set (by using an effective coding of the elements of $2^{<\omega}$ by numbers) and Δ is an ideal, i.e.

$$\sigma \in \Delta \wedge \sigma \preceq \tau \Rightarrow \tau \in \Delta \qquad \text{and}$$
$$\sigma * (0) \in \Delta \wedge \sigma * (1) \in \Delta \Rightarrow \sigma \in \Delta.$$

The well-known notion of Π^0_1-class correspondenses closely with the notion of c.e. ideal. Namely Δ is a c.e. ideal iff $\overline{\Delta}$ is a Π^0_1-class.

The family of all c.e. ideals with the usual inclusion (between subsets of $2^{<\omega}$) form a lattice which we denote with $\mathcal{E}_I$.

With $\cap$ we denote the intersection of two ideals (who is equal to the usual intersection of two sets) and with $\oplus$ the union of two ideals, i.e. the smallest ideal including both. (If Δ_1 and Δ_2 are both c.e. ideals then $\Delta_1 \oplus \Delta_2$ also is c.e.)

The lattices $\mathcal{E}$ and $\mathcal{E}_I$, or more precisely $\mathcal{E}^*$ and $\mathcal{E}_I$ similar each to the other. But that even their elementary theories differ follows easily from the facts that in $\mathcal{E}$ Max$\neq R-$ Max, but in $\mathcal{E}_I$ the both corresponding notions coincide as we show.

An c.e. ideal Δ is called *maximal* (*r-maximal*) if $\Delta \neq 2^{<\omega}$ and

$$(\forall \Delta' \in \mathcal{E}_I)(\Delta \subseteq \Delta',\ \Delta \neq \Delta' \Rightarrow \Delta' = 2^{<\omega})$$
$$\Big((\forall \Delta', \Delta'' \in \mathcal{E}_I)(\Delta' \cap \Delta'' = \emptyset \wedge \Delta' \oplus \Delta'' = 2^{<\omega} \Rightarrow$$
$$(\Delta \oplus \Delta' = 2^{<\omega} \vee \Delta \oplus \Delta'' = 2^{<\omega}))\Big).$$

Let Δ be an c.e. ideal with $\Delta \neq 2^{<\omega}$ and not maximal. Then $\overline{\Delta}$ includes at least two infinite branches. Let $\sigma \in 2^{<\omega}$ such that $\sigma * (0) \notin \Delta$ and $\sigma * (1) \notin \Delta$. Then for $\Delta' = [\sigma * (0)]$ and Δ'' – the smallest ideal including all $[v]$ with $|v| = |\sigma| + 1$ and $v \neq \sigma * (0)$ we have $\Delta' \cap \Delta'' = \emptyset$. $\Delta' \oplus \Delta'' = 2^{<\omega}$, but $\Delta \oplus \Delta' \neq 2^{<\omega}$ and $\Delta \oplus \Delta'' \neq 2^{<\omega}$. Hence Δ is not r-maximal.

Similar as for $\mathcal{E}$ when $\mathcal{E}$ is factorized by finite differences between the sets we get $\mathcal{E}^*$ also $\mathcal{E}_I$ can be factorized by a similar congruence relation. This is that which is generated by the filter (in $\mathcal{E}_I$) of the cofinite ideals. This means: Let Ω be a subset of $2^{<\omega}$ with $[\Omega]$ we denote the set of infinite branches in Ω. Let $\mathbb{F}$ be

$$\{\Delta - \text{c.e. ideal in } 2^{<\omega} : [\bar{\Delta}] \text{ is finite}\}.$$

Then the factor lattice $\mathcal{E}_{I/\mathbb{F}}$ is the analogy for $\mathcal{E}_I$ as $\mathcal{E}^*$ for $\mathcal{E}$. But also $\mathcal{E}_{I/\mathbb{F}}$ and $\mathcal{E}^*$ have different elementary theories as Cenzer and Nies showed.

2 Basic constructions of atomless, r-maximal sets

In this subpoint we will give three constructions of atomless, r-maximal sets. All three nevertheless differ from the construction of a maximal set similar this. But every of them generalizes the maximal set construction in a different way.

These different generalizations are not the onliest reasons for giving them. An other motivation is that every of them again becomes generalized for showing further facts on r-maximal sets. This will be done in the subpoints 3, 5 and 6. But then the constructions are still more complicated, and thus from the point of view of understanding them it seems to be reasonable to give at first the basic constructions. But we believe that the basic constructions for themselves are also of interest.

2.1 Tower construction

This construction was given by Robinson [Rob66]. A modification of it is given in [So87, p. 196] which we give here. To give it a special notion in particularly suitable for atomless, r-maximal sets will be introduced.

Call a sequence $(H_n)_{n\geq 0}$ (not necessarily c.e.) of c.e. sets *tower* if $H_n \subset_\infty H_{n+1}$ for all n and $\bigcup\limits_{n\geq 0} H_n = I\!N$.

Lemma 2.1 *Let A be a coinfinite c.e. set. If there is a tower $(H_n)_{n\geq 0}$ with $A \subseteq H_0$ and*

(2.1) $\qquad (\forall\, e)(\exists\, n)(W_e \subseteq^* H_n \vee \bar{A} \subseteq^* W_e)$ [2)]

then A is an atomless, r-maximal set.

Proof. Let W be an c.e. set with $A \subseteq W \neq^* I\!N$. Then by (2.1) there is an n such that $W \subseteq^* H_n$. But then $W \subset^*_\infty H_{n+1}$. Hence W is not maximal, what means that A is atomless.
Let R be a computable set with $A \cup R \neq^* I\!N$. Then by (2.1) there is an n such that $A \cup R \subseteq^* H_n$. But then $\bar{H}_n \subseteq^* A \cup \bar{R}$ and thus $A \cup \bar{R} \not\subseteq^* H_e$ for every e. This gives $\bar{A} \subseteq^* \bar{R}$. Thus R does not split $\bar{A}$ nontrivially.

Theorem 2.2 (Robinson) *There is a coinfinite c.e. set A and a tower $(H_n)_{n\geq 0}$ with $A \subseteq H_0$ such that* (2.1) *is satisfied.*

Proof. The (stepwise) construction of A similars the maximal set construction, but with a different state measure and in the construction steps more elements are taken to A_{s+1} as in the maximal set construction.
We will use here a simultaneous computable enumeration of $(W_e)_{e\geq 0}$ with the property:

(2.2) $\qquad x \in W_{e,s} \Rightarrow x < s.$

Let $A_0 = \emptyset$ and $T_0(n) = n$ for all $n \geq 0$. Suppose A_s and T_s are given, where T_s is strictly increasing with $\mathrm{rg}(T_s) = \bar{A}_s$.
Let $\mathrm{st}^0(T_s(n), e, s)$ be the following 01-sequence of length $e+1$:

$$\begin{aligned} \mathrm{st}^0(T_s(n), e, s)(j) &= 1 \ \text{ if } (\exists\, t \leq s)(\exists\, m)[T_s(n) = T_t(m) \wedge j < (m)_0 \wedge T_t(m) \in W_{j,t}] \\ &= 0 \ \text{ otherwise}, \qquad\quad j \leq e. \end{aligned}$$

Thus only for $j \leq e$ "$x \in W_j$" has influence to $\mathrm{st}^0(x, e, s)$ when in some step $t \leq s$ $x \in W_{j,t}$ and x was in that step on some place m with $j < (m)_0$. [3)]
Look if there are numbers e and i with $e < i$ such that

(2.3) $\qquad \mathrm{st}^0(T_s(e), e, s) <_e \mathrm{st}^0(T_s(i), e, s).$

[2)] In [So87] instead of (2.1) the stronger requirement with $n = e$ is given, but this strong condition is not necessary.

[3)] For a better understanding of st^0 it is useful to assign every number m the pair $((m)_0, (m)_1)$ from $I\!N^2$. Then $\mathrm{st}^0(T_s(m), e, s)(j) = 1$ for all $e \geq j$ and all $s \geq s'$ if $T_s(m) = T_{s'}(k)$, $j < (k)_0$ and $T_{s'}(k) \in W_{j,s'}$. Observe that $n < m$ does not imply $(n)_0 < (m)_0$, what is important for the construction.

If yes choose the smallest such e and for this the smallest i and define:

$$A_{s+1} = A_s \cup \{T_s(k) : e \leq k \leq s,\ k \neq i\}$$

$$\begin{aligned} &T_{s+1}(j) = T_s(j): \ \ j < e \\ &T_{s+1}(e+k) = T_s(s+k): \ \ k \geq 1 \\ &T_{s+1}(e) = T_s(i). \end{aligned}$$

(That all number $T_s(k)$, $e \leq k \leq s$, $k \neq i$ are placed into A is important. Observe that by (2.2) $s \leq T_s(s)$ and thus for all $k \geq s$ $\ T_s(k) \notin W_{e,s}$ for all $e \geq 0$.)

If such numbers e and i satisfying (2.3) do not exist do nothing in step $s+1$ (i.e. $A_{s+1} = A_s$ and $T_{s+1} = T_s$).
Let $A = \bigcup_{s \geq 0} A_s$.

1. $\lim_s T_s(k)$ exists for all k.

The function $\mathrm{st}^0(x, e, s)$ is increasing in s and bounded by $\mathrm{st}_W(x, e, s)$. In (2.3) we require $>_e$ and $e < i$. Thus if for all $e' < e$ and s_0 $\ T_{s_0}(e') = \lim_s T_s(e')$ this also must hold for $T_{s_1}(e)$ for some $s_1 \geq s_0$.
Let T be the limit function. Since $\bar{A} = \mathrm{rg}(T)$ and T is strictly increasing, A is coinfinite.

2. Let $(X_n)_{n \geq 0}$ be an c.e. sequence and C a coinfinite c.e. set. We say that C *is* $(X_n)_{n\geq 0}$*-maximal* if

$$(\forall n)(X_n \subseteq^* C \vee \bar{C} \subseteq^* X_n).$$

(Thus C is maximal iff C is $(W_e)_{e \geq 0}$-maximal.)
Let $U_{n,s}$ be the set

$$\{x : (\exists\, m)(\exists\, t \leq s)(x \in W_{n,t} \wedge x = T_t(n) \wedge n < (m)_0\}.$$

We see that $x \in U_{j,s}$ iff $\mathrm{st}^0(x, e, s)(j) = 1$, $j \leq e$. By the maximalization of the e-state for $T_s(e)$ respectively to st^0 in the construction, A is $(U_n)_{n \geq 0}$-maximal. Thus

$$(2.4) \qquad U_n \cap \bar{A} \ \text{ is finite or } \ \bar{A} \subseteq^* U_n.$$

Let H_n be

$$A \cup \{T(m) : (m)_0 \leq n,\ m \geq 0\}, \quad n \geq 0.$$

From (2.4) it follows

$$W_n \subseteq^* H_n \vee \bar{A} \subseteq^* W_n.$$

Since $U_n \subseteq W_n$, $\bar{A} \subseteq^* U_n$ implies $\bar{A} \subseteq^* W_n$. Suppose $U_n \cap \bar{A}$ is finite.
If for some m with $n < (m)_0$ $\ T(m) \in W_n$ then $T(m) \in U_n$, by the definition of U_n.
Thus $U_n \cap \bar{A}$ is finite implies that there are only finitely many such m's.

3. Let for $i \geq 0$ the set C_n be equal to

$$A \cup \{T(\langle n, e\rangle) : e \geq 0\}.$$

We show that all C_n's are c.e. Given n let σ_n be the 01-sequence of length $n+1$ defined by

$$\begin{aligned}\sigma_n(j) &= 1 \ : \ \bar{A} \subseteq^* U_j \\ &= 0 \ : \ \text{otherwise}, \quad j \leq n.\end{aligned}$$

Let m' be such that $\mathrm{st}^0(T(m), n) = \sigma_n$ for all $m > m'$. (By the $(U_n)_{n\geq 0}$-maximality of A, σ_n and m' exists.)
Let s_0 be s.th. $T(k) = T_{s_0}(k)$ for all $k \leq m'$. Now we claim that C_n is equal (mod $=^*$) to

$$\begin{aligned}A \cup \{x : (\exists\, s)(s_0 < s \wedge (\exists\, k)(m' < k \leq s \wedge (k)_0 = n \wedge T_s(k) = x \wedge \\ (\forall \ell)(m' < \ell \leq k)(\mathrm{st}^0(T_s(\ell), n, s) = \sigma_n \wedge T_s(\ell) = T_{s+1}(\ell)))\}.\end{aligned} \tag{2.5}$$

The set (2.5) obviously includes C_n (mod $=^*$). We show the converse.
Since in (2.5) $k \leq s$, if for some $t \geq s$ in step $t+1$ $\quad T_{t+1}(\ell) \neq T_t(\ell)$ for $\ell < k$ then $T_{t+1}(\ell) = x$ or x cones to A. But $T_{t+1}(\ell) = x$ means $\mathrm{st}(T_t(\ell), n', t) <_\ell \mathrm{st}^0(x, n', t+1)$ for some n'
But $n' \leq n$ is not possible ba the choice of s and $n' < \ell$ in (2.5) and $n < n'$ is not possible, since $T_s(k) = x$ and $(k)_0 < n$. Then by the definition of st^0 $\quad T_s(\ell)$, $\ell \leq k$ cannot come into $U_{n'',s'}$ in some step $s' \geq s$ for some $n < n'' \leq n'$.
Thus $T_s(\ell) = T_{s+1}(\ell)$ for $m' < \ell \leq k$ implies that if $x = T_s(k) = T_{s+1}(k)$ then $x = T(k)$ or x cones to A.

4. Let H_n be $C_0 \cup \ldots \cup C_n$. The H_n is c.e. and $(H_n)_{n\geq 0}$ is a tower for A with the property (2.1). $\square$

Corollary 2.3 *There are atomless, r-maximal sets.*

Proof. The set A constructed in Theorem 2.2 has the properties mentioned in Lemma 2.1. Hence A is an atomless, r-maximal set.

2.2 Strongly finite c.e. sequence

Independently Lachlan gave in [La68a] a different method for the construction of an atomless, r-maximal set. This construction works with strongly finite c.e. sequence and the priority is measured respectively to how many numbers of W_e are into the members of this sequence.

Theorem 2.4 (Lachlan) *There exists an atomless, r-maximal set A.*

Proof. Let $(E_i)_{i\geq 0}$ be a strongly finite c.e. sequence with $|E_i| = 2^i \cdot f(i)$, where f is a computable function with $f(i) \geq 2$, $i \geq 0$. Further we claim $\bigcup\limits_{n\geq 0} E_n = I\!N$.
In every step s we construct a strongly finite c.e. sequence $(F_{n,s})_{n\geq 0}$ starting with $F_{n,0} = E_n$ and with $F_{n,s+1} \subseteq F_{n,s}$ and a computable function p_s, starting with $p_0(n) =$

n, $n \geq 0$. By using $(F_{n,s})_{n,s\geq 0}$ and p_s, $s \geq 0$ we define later the set A.
Having $(F_{n,s})_{n\geq 0}$ we define the following e-state:

$$\begin{aligned} \mathrm{st}_F(x,e,s)(j) &= 1 \qquad \text{if } \ F_{x,s} \subseteq W_{j,s} \\ &= 0 \qquad \text{otherwise}, \quad j \leq \ell. \end{aligned}$$

Step $s+1$: a) (Definition of $(F_{n,s+1})_{n\geq 0}$). Look if

$$(2.6) \qquad (\exists\, m)(\exists\, n)\Big[n \in \mathrm{rg}(p_s) \wedge m < n \wedge F_{n,s} \not\subseteq W_{m,s} \wedge |F_{n,s}| \leq 2 \cdot |F_{n,s} \cap W_{m,s}|\Big].$$

If (2.6) holds for some numbers m and n, choose first the smallest such n and for this n the smallest such m. Let n_0 and m_0 be these numbers and define

$$\begin{aligned} F_{n_0,s+1} &= F_{n_0,s} \cap W_{m_0,s} \\ F_{n,s+1} &= F_{n,s} \qquad \text{for all } \ n \neq n_0. \end{aligned}$$

If such numbers do not exist let $F_{n,s+1} = F_{n,s}$ for all n.
b) (Definition of p_{s+1}). Having p_s we define p_{s+1}.

$$\begin{aligned} p_{s+1}(0) &= (\mu z)(z \in \mathrm{rg}(p_s) \wedge \mathrm{st}_F(z,0,s) = \max\{\mathrm{st}_F(y,0,s) : y \in \mathrm{rg}(p_s)\}) \\ p_{s+1}(i+1) &= (\mu z)(z \in \mathrm{rg}(p_s) \wedge p_{s+1}(i) < z \wedge \\ &\qquad \mathrm{st}_F(z,i+1,s) = \max\{\mathrm{st}_F(y,i+1,s) : y \in \mathrm{rg}(p_s) \wedge p_{s+1}(i) < y\}). \end{aligned}$$

Obviously p_s is a computable function, since $(W_{e,s})_{e\geq 0}$ and $(F_{n,s})_{n\geq 0}$ are both strongly finite c.e. sequences.

By definition p_s is increasing. Further, since $F_{n,s+1} \subseteq F_{n,s}$ and $W_{e,s} \subseteq W_{e,s+1}$ for all n, e and s $\mathrm{st}_F(x,e,s) \leq \mathrm{st}_F(x,e,s+1) \leq \underbrace{1\ldots1}_{e+1-\text{times}}$, $\lim_s F_{n,s}$ and $\lim_s p_s(i)$ exist for all n and all i.
Let be F_n and $p(i)$ be the limit values respectively. We see that $2 \cdot |F_{n,s+1}| \geq |F_{n,s}|$ and $F_{n,s+1} \subsetneq F_{n,s}$ can happen only n times, since in (2.6) $m < n$ is required. Thus $|F_n| \geq f(n)$. Now let A be the set

$$I\!N - \bigcup\{F_n : n \in \mathrm{rg}(p)\}.$$

1. A is an c.e. set, since $(F_{n,s} \setminus F_{n,s+1})_{n\geq 0}$ is a strongly finite c.e. sequence (even more at most one n $F_{n,s} \setminus F_{n,s+1} \neq \emptyset$) and $\mathrm{rg}(p_{s+1}) \subseteq \mathrm{rg}(p_s)$. Further A is coinfinite, since $\mathrm{rg}(p)$ is ∞ and every $F_n \neq \emptyset$, $n \geq 0$.

2. Let $\mathrm{st}_F(x,e) = \lim_s \mathrm{st}_F(x,e,s)$. By the construction of $p_s(i)$ (i.e. it is the number $> p_s(i-1)$ will the greatest $\mathrm{st}_F(\ ,i,s))$ for every m and for almost all $x \in \mathrm{rg}(p)$ $\mathrm{st}_F(x,m)$ are the same.
From this we get that for every $e \geq 0$ either for almost all $n \in \mathrm{rg}(p)$ $F_n \subseteq W_e$ or for almost all $n \in \mathrm{rg}(p)$ $2 \cdot |F_n \cap W_e| < |F_n|$. The first case obviously means that $A \cup W_e =^* I\!N$.

3. $\bar{A}$ is r-cohesive. Let R be a computable set. Then by 2. if not $\bar{A} \subseteq^* R$ and not $\bar{A} \subseteq^* \bar{R}$ for almost all $n \in \mathrm{rg}(p)$

$$2 \cdot |F_n \cap R| < |F_n| \quad \text{ and } \quad 2 \cdot |F_n \cap \bar{R}| < |F_n|.$$

But this means $R \cup \bar{R} \neq^* I\!N$. Hence $\bar{A} \subseteq^* R$ or $\bar{A} \subseteq^* \bar{R}$ must hold.

4. A is atomless. Suppose $A \cup W_e$ is coinfinite. Hence for a.a. $n \in \mathrm{rg}(p)$

$$2 \cdot |F_n \cap W_e| < |F_n|.$$

Thus for a.a. $n \in \mathrm{rg}(p)$ $\;|F_n \setminus W_e| \geq 2$. But this means that $A \cup W_e$ is not maximal, see [Ro67, p. 304]. □

Remarks 2.5 1) We see that the set Ω equal to

$$(2.7) \qquad \{n : E_n \subseteq A\}$$

is c.e., since $(E_n)_{n\geq 0}$ is strongly finite. But it holds even that the set (2.7) is maximal. Suppose not. Then there is an c.e. set B with $\Omega \subset_\infty B \subset_\infty I\!N$. Since $E_n \setminus F_n \subseteq A$ for all n by definition of A, $E_n \subseteq A$ is equivalent with $F_n \subseteq A$. Thus for infinitly may n with $F_n \cap A = \emptyset$, $F_n \subseteq B$, but also for ∞ may n with $F_n \cap A = \emptyset$ also $F_n \cap B = \emptyset$. But this contradicts 2. above.

2) The set constructed in Theorem 2.4 has a further, beside the atomless r-maximality property. Namely, if we take for f the function $f(i) = i + 2$, $i \geq 0$, then this set is not dense simple. The maximal sets and the major subsets of the maximal sets are dense simple. In general, since the r-maximal lays very high inside the lattice $\mathcal{E}$, it seems to be that all r-maximal sets are dense simple. Thus the existence of r-maximal, not dense simple sets is rather surprising.
We get that the set constructed in Theorem 2.4 with $f(i) = i + 2$ is not dense simple, since for the sequence $(E_n)_{n\geq 0}$ holds

$$(2.8) \qquad (\exists^\infty n)(|E_n \cap \bar{A}| \geq n),$$

by the choice of f and $|F_n| \cdot 2^n > |E_n| = 2^n \cdot (n + 2)$. But an infinite set with the property (2.7) is not dense immune, see [So87, p. 212, 1.10]. Hence A is not dense simple.
Observe that in (2.8) we have not: $(\forall\, n)(\exists\, m)(|E_m \cap \bar{A}| \geq n)$, where m can be much greater than n. Since in this case $\bar{A}$ must not be dense immune. This will be important later in Lemma 6.9, 4).

2.3 The common state of numbers

A third method for constructing atomless r-maximal sets is given in [He.ta2]. There the notion "e-state of a number x after s enumeration steps of $(W_e)_{e\geq 0}$" is generalized to the "common e-state of finitely many numbers $x_1, \ldots, x_n$ after s enumeration steps".

Let $x_1, \ldots, x_n$, e and s be numbers. With $\mathrm{st}_W(x_1, \ldots, x_n; e, s)$ we denote the "*common e-state of the number $x_1, \ldots, x_n$ at step s*" what is defined by

$$\begin{aligned} \mathrm{st}_W(x_1, \ldots, x_n; e, s)(j) &= 1 \quad \text{if } \; x_1 \in W_{j,s} \wedge \ldots, \wedge x_n \in W_{j,s} \\ &= 0 \quad \text{if for one } \; i \;\; x_i \notin W_{j,s},\; 1 \leq i \leq n,\; j \leq e. \end{aligned}$$

(The common e-state of one number of course coincide with the usual e-state of this number.)

In the following construction in opposite to the maximal set construction we move not only one number, but finitely many numbers by reason of a greater common e-state (than other ones).

Theorem 2.6 *There exists an atomless, r-maximal set A.*

Proof. Let $\Gamma \subseteq I\!N^2$ be the set $\{(n,m) \in I\!N^2 : n \leq m,\, n \geq 0\}$ and $\Gamma_i = \{(i,m) : i \leq m\}$, $i \geq 0$. $\left(\text{Thus } \Gamma = \bigcup_{i\geq 0} \Gamma_i\right)$.
Further we assume that we have a computable, well-ordering of order type ω between all finite sequences of pairs $(n,m) \in \Gamma$. (Thus knowing that there is a finite sequence satisfying a computable condition we can find the smallest sequence w.r.t. to this ordering having this property.)
We shall construct stepwise the set A together with a mapping $T : \Gamma \mapsto I\!N$.

Construction.

Step 0: Let $A_0 = \emptyset$ and T_0 be a computable bijection between Γ and $I\!N$.

Step $s+1$: Look if

$$\begin{aligned}
&(\exists\, n)(\exists\, e \leq n)(\exists\, (k_0,m_0),\ldots,(k_n,m_n) \in \Gamma, \text{all different})\\
&\quad [n < k_i,\; i = 0,1,\ldots,n \wedge \mathrm{st}_W(T_s((0,n)),\ldots,T_s((n,n)); e,s) <_\ell \\
&\qquad \mathrm{st}_W(T_s((k_0,m_0)),\ldots,T_s((k_n,m_n)); e,s)].
\end{aligned} \tag{2.9}$$

If yes choose the smallest n (let n_0 be this) and the smallest $n+1$-tuple for this n_0. Let be this $(k_0^0,m_0^0),\ldots,(k_n^0,m_n^0)$.
Now define

$$T_{s+1}((i,n_0)) = T_s((k_i^0,m_i^0)) : i = 0,1,\ldots,n_0.$$

Let $\ell = \max\{m_i^0 : i = 0,1,\ldots,n_0\} + 1$.

$$\begin{aligned}
&T_{s+1}((k_i^0,m_i^0) = T_s((\ell+i,\ell+i)) : i = 0,\ldots,n_0\\
&T_{s+1}((\ell+k,\ell+k)) = T_s((\ell+n_0+k+1,\ell+n_0+k+1)),\; k \geq 0\\
&T_{s+1}((i,j)) = T_s((i,j)) \quad \text{for all other pairs}
\end{aligned}$$

$$A_{s+1} = A_s \cup \{T_s((i,n_0)); i = 0,\ldots,n_0\}.$$

Result.

Let A be $\bigcup_{s\geq 0} A_s$. By the effectivity of the construction, A is c.e.

1) For every pair $(n,m) \in \Gamma$ $\lim_s T_s((n,m))$ exists (proof by induction). Suppose $T_{s+1}((0,0)) \neq T_s((0,0))$. Then $T_{s+1}((0,0)) \in W_{0,s}$. But this can happen at most one time. Hence $\lim_s T_s((0,0))$ exists.
Suppose the limit exists for all pairs (n,m) with $m < i$. Let s_0 be such that for $s \geq s_0$

$T_s((n,m)) = T_{s_0}((n,m))$ for all (n,m) with $m < i$. If for an $s > s_0$ $T_{s+1}((k,i)) \neq T_s((k,i))$ for a k with $0 \leq k \leq i$ then the common i-state of the $T_{s+1}((k,i))$, $0 \leq k \leq i$ is greater than that of the $T_s((k,i))$'s. But this can happen only finitely often. Thus also $\lim_s T_s((k,i))$ exists.
Let T be the limit function of $(T_s)_{s\geq 0}$. Since all T_s are bijections between Γ and $\bar{A}_s$, T is a bijection between Γ and $\bar{A}$. From this we get that A is coinfinite.

2) For every: $A \cup T(\Gamma_i)$ is c.e.
This follows easily from the facts: $T_0(\Gamma_i)$ is c.e. and only finitely many elements placed while the construction on pairs from Γ_i are moved to pairs from Γ_j, $j < i$, since in (2.9) $n < k_i$ is required. (By the construction no number from a place in Γ_i can be moved to a pair from Γ_k with $i < k$).
Thus also $A \cup T(\Gamma_0) \cup \ldots \cup T(\Gamma_i)$ is c.e. for every $i \geq 0$.

3) For every e

$$(2.10) \qquad (\exists^\infty i)(W_e \cap T(\Gamma_i) \neq \emptyset \Rightarrow T(\Gamma) \subseteq^* W_e).$$

(Proof by induction over e). Let e be a number and suppose for all $j < e$ (2.10) holds where e is replaced by j. Then for an $j < e$:

(i) there is an k_j such that $W_j \subseteq A \cup T(\Gamma_0) \cup \ldots \cup T(\Gamma_{k_j})$ or

(ii) $T(\Gamma) \subseteq^* W_j$.

From this it follows that there is a number m_0 such that for all $m \geq m_0$ in case (i) for all n, $m_0 \leq n \leq m$ $T((n,m)) \notin W_j$ and in case (ii) for all n $T((n,m)) \in W_j$.
Let s_0 be such that $T((\ell,k)) = T_{s_0}((\ell,k))$ for all $k < m_0$.
Suppose $(\exists^\infty i)(W_e \cap T(\Gamma_i) \neq \emptyset)$. Then we can find an $s \geq s_0$ and $m_0 + 1$ numbers from $\bigcup_{i > m_0} \Gamma_i$ such that their common m_0-state at step s is 1 for all j from (ii) or $j = e$. Thus this $m_0 + 1$-tuple (or a smaller $m_0 + 1$-tuple with the same m_0-state is moved to the pairs (k, m_0), $0 \leq k \leq m_0$. This shows that $T(\Gamma) \subseteq^* W_e$.

A is c.e. and coinfinite by 1). A has no maximal superset. If $A \subseteq W_e \neq^* I\!N$ then by 3) there is an i s.th. $W_e \subseteq A \cup T(\Gamma_0) \cup \ldots \cup T(\Gamma_i)$. But then $T(\Gamma_{i+1}) \cap W_e = \emptyset$. Hence $W_e \subset_\infty W_e \cup T(\Gamma_{i+1}) \subset_\infty I\!N$.
$\bar{A}$ is r-cohesive by 3). If R is a computable set with $A \cup R \neq^* I\!N$ then $R \subseteq A \cup T(\Gamma_0 \cup \ldots \cup \Gamma_i)$ for some i. Hence $\bar{A} \subseteq^* A \cup \bar{R}$, by (2.10) (and thus $\bar{A} \cap R =^* \emptyset$). □

Later, e.g. in Lemma 3.3 we need a special property of the construction given in Theorem 2.6 (as also in Theorem 2.2). For $i \geq 0$ let A_i be the set

$$\{x : (\exists\, s)(x \in T_s(\Gamma_i) \wedge x \in A_{s+1})\}.$$

This means that x comes to A at step $s+1$ from a pair (i,m) for some m. Important are two properties of the sets A_i, $i \geq 0$.

- The sequence $(A_i)_{i\geq 0}$ is an c.e. sequence and
- $R_i = A_i \cup T(\Gamma_i)$ are c.e. sets for every $i \geq 0$.

Remark 2.7

a) We see that if X is an infinite subset of Γ with

$$(\forall\, i)(\Gamma_i \cap X - \text{ finite})$$

then $T(X)$ is a cohesive set. This follows easily from property 3).

b) In Theorem 5.3 we shall modify the above construction in such a way that the final function T will be defined not for all pairs from Γ. But the properties 1) to 3) true for all $T((n,m))$, $(n,m) \in \Gamma$ for which $T((n,m))$ remains. Then we get

$$(\exists^{<\infty} i, \text{ but not zero many})(T(\Gamma_i) \text{ is infinite}) \wedge$$
$$(\exists^{\infty} i)(T(\Gamma_i) \text{ is not empty}) \Rightarrow A \in \mathrm{MS}_{\mathrm{Max}}$$
$$(\exists^{\infty} i)(T(\Gamma_i) \text{ is infinite}) \Rightarrow A \in R - \mathrm{Max}_{\mathrm{atl}}.$$

This fact will be used there.

3 On the structures $\mathcal{L}^*(A)$ for A atomless, r-maximal

One of the main topics, not only for the theory of the r-maximal sets, but for the general lattice analysis of the c.e. sets, is the description of the possibilities of the c.e. superset structures of the atomless, r-maximal sets. Already from the basic constructions in subpoint 2 it can be concluded quite easily that there are more than only one isomorphism type of such lattices as we show. Let A and B be c.e. sets with $A \subseteq B$ and $B - A$ infinite. A set $C \in \mathcal{L}(A, B)$ is called simple in $\mathcal{L}(A, B)$ if $C \neq^* B$ and

$$(3.1) \qquad (\forall\, D \in \mathcal{L}(A, B))(D \neq^* A \Rightarrow D \cap C \neq^* A).$$

(We see that a set S is simple in the usual meaning if S is simple in $\mathcal{L}(\emptyset, I\!N)$.)
The r-maximal set constructed in Theorem 2.4 is not dense simple, see Remark 2.5, 2). But every coinfinite c.e. set A, not dense simple has in $\mathcal{L}(A)$ an element which is simple in $\mathcal{L}(A)$, see [He.ta2]. On the other side the sets constructed in Theorem 2.2, as also in Theorem 2.6 do not have such special elements in their c.e. superset structures. In Remark 3.2 the isomorphism type of these structures is given, from what the nonexistence of such relatively simple sets at once follows. [4)]
Thus there are at least two isomorphism types of c.e. superset structures of atomless, r-maximal sets. In the following theorem we shall show that there are infinitely many nonisomorphic such sublattices. But a complete description of all isomorphism types $\mathcal{L}^*(A)$ for $A \in R - \mathrm{Max}_{\mathrm{atl}}$ is still unknown.

[4)] In [So87] it is remarked that by modifying the construction in Theorem 2.2 it can be constructed and atomless, r-maximal set A such that $\mathcal{L}(A)$ has an simple set in $\mathcal{L}(A)$.

3.1 Basic calculation of $\mathcal{L}^*(A)$, the number of nonisomorphic intervals $\mathcal{L}^*(A)$'s, $A \in R - \mathrm{Max}_{\mathrm{atl}}$

Probably the best method for characterizing the isomorphism types of $\mathcal{L}^*(A)$ for $A \in R - \mathrm{Max}_{\mathrm{atl}}$ seems to be the form as it was done by Lachlan in [La68a] for the c.e. superset structures of the hh-simple sets. This was what we will call "Basic calculation" of $\mathcal{L}^*(A)$. In this style the structures for the atomless, r-maximal sets A would be characterized by the following four conditions:

1° $\mathcal{L}^*(A)$ is an infinite $\exists\forall\exists$-lattice with 0 and 1,

2° $\mathcal{L}^*_r(A)$ consists of two elements (A^* and $\mathbb{N}^*$),

3° $(\forall\, B \in \mathcal{L}(A))(B \neq^* \mathbb{N} \Rightarrow (\exists\, C \in \mathcal{L}(A))(B \subset_\infty C \subset_\infty \mathbb{N}))$.

4° $(\forall\, B \in \mathcal{L}(A))(A \subset_\infty B \subset_\infty \mathbb{N} \Rightarrow \mathcal{L}^*(A, C) \cong \mathcal{L}_{\mathrm{ms}})$ ($\mathcal{L}_{\mathrm{ms}}$ – the major subset interval).

We see easily that the conditions 1° to 4° are satisfied for every $A \in R - \mathrm{Max}_{\mathrm{atl}}$. The main still open question is if also the converse implication holds, i.e. if for every distributive lattice $\mathcal{L}$ which has the properties 1° to 4° there is an $A \in R - \mathrm{Max}_{\mathrm{atl}}$ s.th. $\mathcal{L} \cong \mathcal{L}^*(A)$?

*

Another possibility to classify $\mathcal{L}^*(A)$, $A \in R - \mathrm{Max}_{\mathrm{atl}}$ could be the "ideal characterization". For $A \in R - \mathrm{Max}_{\mathrm{atl}}$ let $\mathcal{L}^-(A)$ be

$$\{B \in \mathcal{L}(A) : B \neq^* \mathbb{N}\}.$$

Obviously $\mathcal{L}^-(A)$ is a nonprincipal ideal (in $\mathcal{L}(A)$).

> Let X and Y be c.e. sets with $X \subset_m Y$. Exists a characterization of nonprincipal ideals in $\mathcal{L}(X, Y)$, e.g. in the form
>
> $$\{Z \in \mathcal{L}(X, Y) : Z \subset^* U\}$$
>
> for sets U with $X \subseteq U \subseteq Y$ such that the above ideals have the same isomorphism types as the $\mathcal{L}^-(A)'s$ and U is from some arithmetical class?

*

A weaker fact was shown by Cholak and Nies in [ChN.ta]. The following theorem improves the consideration in the introduction of this subpoint.

Theorem 3.1 (Cholak/Nies) *There are infinitely many atomless, r-maximal sets with pairwise non-isomorphic c.e. superset structures.*

Proof. The construction of the atomless, r-maximal sets given here is a generalization of that from Theorem 2.2. We shall work with infinite computable trees $\Gamma \subseteq I\!N^{<\omega}$ and assume that an effective coding $\alpha \in \Gamma \leftrightarrow i_\alpha \in I\!N$ for all notes from Γ, except $\langle\ \rangle$, onto $I\!N$ with the property

$$\alpha \prec \beta \Rightarrow i_\alpha < i_\beta,\ \ \alpha, \beta \in \Gamma \setminus \{\langle\ \rangle\}$$

is given.
It will be constructred stepwise an $1-1$ function $T_s : I\!N \to I\!N$, the set A_s (with $\bar{A}_s = \mathrm{rg}(T_s)$), both with a similar meaning as in Theorem 2.2, and a function $d_s : I\!N \to I\!N$ with $d_s(m) \leq m$ (more precisely $d_s(m)$ can have only at most two values for $s \geq 0$, m and a special number lesser than m). The meaning of d_s is: While the construction we shall not try to maximalize the m-state of $T_s(m)$ resp. to st^0 as in Theorem 2.2, but the $d_s(m)$-state. Since for every n for a.a. m and a.a. s $\ d_s(m) > n$, the constructed set A also will be $(U_n)_{n\geq 0}$-maximal (what is necessary for the r-maximality of A).
The set A will have the following properties:

(i) For ever $\alpha \in \Gamma$, $\alpha \neq \langle\ \rangle$ the set A_α equal to

$$A \cup \{T(\langle i_\beta, \ell\rangle) : \ell \geq 0,\ \beta \leq \alpha\}$$

will be c.e.

(ii) For $\alpha \in \Gamma$ and every $j \in I\!N$ such that $\alpha * (j) \in \Gamma$ $\ A_\alpha$ will be simple in $\mathcal{L}(A_{\alpha^-}, A_{\alpha*(j)})$.

(iii) The "tower sets" H_n for $n \geq 0$ will be

$$A \cup \bigcup \{A_\alpha : i_\alpha \leq n\},$$

(i.e. $A \cup W_n \subseteq^* H_n$ or $A \cup W_n =^* I\!N$ for $n \geq 0$).

(For ensuring the property (ii) we need the function d_S. The sets A_α with $i_\alpha = n$ and H_n in general are different sets and have different meanings for A.)
We see that from (i) it follows for $\alpha, \beta \in \Gamma : A_\alpha \cap A_\beta = A_{\alpha\cap\beta}$. We remember that every number m is equal to a number of the form $\langle i, k\rangle$ for some unique numbers i and k and m is equal to a number of the form $\langle x, \langle y, z, u\rangle\rangle$ for some unique numbers x, y, z and u. We assume that both correspondences are bijective (the first between $I\!N$ and $I\!N^2$ and the second one between $I\!N$ and $I\!N^4$).

Construction

Step 0: Define $T_0(m) = m$, $m \geq 0$, $A_0 = \emptyset$ and $d_0(m) = m$, $m \geq 0$.

Step $s+1$: This step consists of two parts. First a) which gives T_s^+, A_s^+ and d_s^+ (by using T_s, A_s and d_s) and then b) giving T_{s+1}, A_{s+1} and d_{s+1} (by using T_s^+, A_s^+ and d_s^+).

Part a). Look if

$$(\exists\, e)(\exists\, i)(e < i \le s \wedge \mathrm{st}^0(T_s(e), d_s(e), s) < \mathrm{st}^0(T_s(i), d_s(e), s)). \tag{3.2}$$

If yes let e_0 be the smallest such e and for e_0 let i_0 be the smallest i s.th. (e_0, i_0) satisfies (3.2). Now define

$$\begin{aligned}
A_s^+ &= A_s \cup \{T_s(k) : e_0 \le k \le s,\ k \ne i_0\} \\
T_s^+(k) &= T_s(k) : k < e_0 \\
T_s^+(e_0) &= T_s(i_0) \\
T_s^+(e_0 + k) &= T_s(s + k) : k > 0 \\
d_s^+(m) &= d_s(m) \quad \text{for all} \quad m \ge 0.
\end{aligned}$$

If (3.2) does not hold let $T_s^+ = T_s$, $A_s^+ = A_s$ and $d_s^+ = d_s$.

Part b). Look if

$$\begin{aligned}
&(\exists\, e)(\exists\, i)(e < i \le s \wedge\ e = \langle i_\alpha, \langle j, n, \ell\rangle\rangle \text{ for some } \alpha, j, n, \ell \wedge \\
&\qquad\qquad i = \langle i_{\alpha * (j)}, k\rangle \text{ for some } k \wedge \\
&\mathrm{st}^0(T_s^+(e), \langle i_\alpha, j, n\rangle, s) \le \mathrm{st}^0(T_s^+(i), \langle i_\alpha, j, n\rangle, s) \wedge \\
&(\forall\, \ell' \le \ell)(T_s^+(\langle i_\alpha, \langle j, n, \ell'\rangle\rangle) \notin W_{n,s} \wedge T_s^+(i) \in W_{n,s}).
\end{aligned} \tag{3.3}$$

If yes find the last e and i (first e than i) s.th. (3.3) holds and define

$$\begin{aligned}
A_{s+1} &= A_s^+ \cup \{T_s^+(k) : e \le k \le s,\ k \ne i\} \\
T_{s+1}(k) &= T_s^+(k) : k < e \\
T_{s+1}(e) &= T_s^+(i) \\
T_{s+1}(e + k) &= T_s^+(s + k) : k > 0 \\
d_{s+1}(e) &= \langle i_\alpha, j, n\rangle \\
d_{s+1}(m) &= d_s^+(m) : (m)_0 \ne i_\alpha \\
d_{s+1}(m) &= m \text{ for all } m = \langle i_\alpha, \langle j, n, \ell''\rangle\rangle \text{ with } \ell'' \ne \ell.
\end{aligned}$$

If (3.3) is not satisfied let $T_{s+1} = T_s^+$, $A_{s+1} = A_s^+$ and $d_{s+1} = d_s^+$.

Result. Let $A = \bigcup_{s \ge 0} A_s$.

1. For every m $\lim_s T_s(m)$ exists. (By induction over m.) We assume that this holds for all $i < m$. Let s_0 be such that $T_{s+1}(i) = T_s(i)$ for $s \ge s_0$ and $i < m$. If $T_{s+1}(m) \ne T_s(m)$ for $s \ge s_0$ happens two times by part b) then in the second case the $\langle i_\alpha, j, n\rangle$-state of $T_s(m)$ is greater than in the first case. Since at least in the first time after part b) in which part a) happens, $d_t(m) = \langle i_\alpha, j, n\rangle$. Thus, this can happen only finitely often. But case a) also can happen at most finitely often.

2. For all n for almost all m and almost all s $d_s(m) > n$. By definition of d_{s+1} in part b) last line we see that among the numbers $\langle i_\alpha, \langle j, n, \ell\rangle\rangle$, $e \ge 0$ at most for one

$d_{s+1}(m) = \langle i_\alpha, j, n\rangle$ and for all others $d_{s+1}(m) = m$. But numbers $\langle x, y, z\rangle$ lesser than a given one are only finitely many.

3. A is $(U_n)_{n\geq 0}$-maximal.

In both construction parts we maximalize the $d_s(m)$-state respectivley to $(U_n)_{n\geq 0}$ of $T_s(m)$. By 2. we get the $(U_n)_{n\geq 0}$-maximality of A.

4. We have

$$\begin{aligned}(\forall\, n)(\forall\, i_\alpha)(\forall\, j)\ [&W_n \cap \{T((\langle i_{\alpha*(j)}, \ell\rangle) : \ell \geq 0\}) \text{ is infinite} \Rightarrow \\ &W_n \cap \{T((\langle i_\alpha, \ell\rangle) : \ell \geq 0) \text{ is not empty}].\end{aligned} \tag{3.4}$$

Assume not. Choose some n i_α and j such that (3.4) does not hold. Let σ be the $\langle i_\alpha, j, n\rangle$-state of $\bar{A}$, see 2. Then there is an m_0 s.th. for $m \geq m_0$ all $T(m)$ hve the $\langle i_\alpha, j, n\rangle$-state σ.

Choose some $m_1 \geq m_0$ with $m_1 = \langle i_\alpha, \langle j, n, \ell\rangle\rangle$ and some $m_2 > m_1$ with $m_2 = \langle i_{\alpha*(j)}, k\rangle$ and $T(m_2) \in W_n$. Such numbers m_1 and m_2 exist by our assumption. Let s_0 be such that for all $i \leq m_2$ $\;T_{s_0}(i) = T(i)$. Then for all $s \geq s_0$ $\;(m_1, m_2)$ satisfies (3.3), but never get attention, since $s \geq s_0$ and the choice of s_0. But this is not possible.

5. The sets A_α defined in (i) are c.e. sets. Given α. Let σ be the $i_\alpha + 1$-state of $\bar{A}$. Let m_0 be s.th. for all $m \geq m_0$ $\;T(m)$ has the $i_\alpha + 1$-state σ. Let s_0 be s.th. $T(k) = T_{s_0}(k)$ for $k \leq m_0$ and part b) does not hold in a step $s+1$, $s \geq s_0$ for a number $\langle i_\beta, j, n\rangle < i_\alpha$ and a state $\sigma' \supseteq \sigma$.

Then A_α is equal (mod $=^*$) to

$$\begin{aligned}[y : (\exists\, t \geq s_0)(y \in A_{t+1} \vee \\ &(\exists\, k)(m_0 < k \leq t,\ T_t(k) = y,\ k = \langle i_\beta, j\rangle \text{ for some } \beta \preceq \alpha \wedge \\ &(\forall\, m)(m_0 < m \leq k \Rightarrow \mathrm{st}^0(T_t(m), i_\alpha, t) = \sigma) \wedge \\ &\wedge(\forall\, m \leq k)(T_{t+1}(m) = T_t(m)].\end{aligned}$$

6. The sets $H_n = \bigcup\{A_\alpha : i_\alpha \leq n\}$ form a tower with $W_n \subseteq^* H_n$ or $A \cup W_n =^* A$.

Observe we have assumed $\beta \prec \alpha \Rightarrow i_\beta < i_\alpha$. Thus the sets H_n are c.e. The property (2.1) can be shown similar as in Theorem 2.2.

$*$

Let $\Gamma_n = \{\sigma_n \in I\!N^{<\omega} : |\sigma| \leq n\}$. For all $n \geq 1$ $\;\Gamma_n$ are infinite, computable trees. Thus there are sets A_n constructed by means of Γ_n. We show that $\mathcal{L}^*(A_n) \not\cong \mathcal{L}^*(A_m)$ for $n \neq m$, $n, m \geq 1$.

With A^n_α, $\alpha \in \Gamma_n$ we denote the "basic sets" of A_n, $n \geq 1$, and for $\alpha \neq \langle\,\rangle$ with X^n_α the "atom" $A^n_\alpha - A^n_{\alpha^-}$. Suppose for $n > m \geq 1$ $\;\Phi$ is an isomorphism between $\mathcal{L}^*(A_n)$ and $\mathcal{L}^*(A_m)$. Denote with Ω_α, $\alpha \in \Gamma_n$ the set

$$\{\sigma \in \Gamma_m : \Phi(X^n_\alpha) \cap X^m_\sigma - \infty\}.$$

1) Ω_α is a finite, nonempty set.
Since $X^n_\alpha \subseteq A^n_\alpha$ and $\Phi(A^n_\alpha) \subseteq \bigcup_{\alpha\in\theta} A^m_\alpha$ and θ is finite, see above, the set Ω must be finite too.
2) For finite, nonempty sets $\theta_1, \theta_2 \subseteq I\!N^{<\omega}$ we write $\theta_1 \preceq \theta_2$ if

$$(\forall\, \sigma \in \theta_2)(\exists\, v \in \theta_1)(v \preceq \sigma).$$

For $\alpha \in \Gamma_n$, $\alpha \neq \langle\ \rangle$ and $\alpha * j \in \Gamma_n$ it holds

$$\Omega_\alpha \preceq \Omega_{\alpha * j}.$$

Suppose not. Let $\sigma \in \Omega_{\alpha*j}$ s.th. for all $\sigma' \preceq \sigma \Rightarrow \sigma' \notin \Omega_\alpha$. Take the set A^m_σ. This set is c.e., hence $\Phi^{-1}(A^m_\sigma)$ also. We have $\Phi^{-1}(A^n_\sigma) \cap (A^n_{\alpha*j} - A^n_\alpha) - \infty$, since $\sigma \in \Omega_{\alpha*j}$. Thus $\Phi^{-1}(A^m_\sigma)\} \cap (A^n_\alpha - A^n_{\alpha^-}) - \infty$. Hence $A^m_\sigma \cap (\Phi(A^n_\alpha) - \Phi(A^n_{\alpha^-}))$ is infinite. This means that for some $\sigma' \preceq \sigma \quad X^m_{\sigma'} \cap \Phi(X^n_\alpha) - \infty$. Hence $\sigma' \in \Omega_\alpha$.
3) For every atom X^m_σ only for finitely many $\alpha \in \Gamma_n$

$$X^m_\sigma \cap \Phi(X^n_\alpha) \quad \text{is infinite.}$$

If not, since every A^n_α consists only of finitely many atoms (united with A_n) and $X^m_\sigma \subseteq A^m_\sigma$, $\Phi^{-1}(A^m_\sigma)$ would be not included into finitely many sets A^n_α. But this contradicts 6. in the theorem.
4) For every $\alpha \in \Gamma_n$, $\alpha \neq \langle\ \rangle$, $|\alpha| < n$ there is an j with $\alpha * j \in \Gamma_n$ s.th.

$$\Omega_\alpha \prec \Omega_{\alpha*j},$$

i.e. $\Omega_\alpha \preceq \Omega_{\alpha*j} \wedge (\forall\, v \in \Omega_\alpha)(\forall\, \sigma \in \Omega_{\alpha*j})(v \preceq \sigma \Rightarrow v \prec \sigma)$. Take the set Ω_α (this set is finite, by 1)) and consider $\{j : \Omega_{\alpha*j} \cap \Omega_\alpha \neq \emptyset\}$. By 4), this set is finite. Thus even for almost all j we have

$$\Omega_\alpha \preceq \Omega_{\alpha*j} \Rightarrow \Omega_\alpha \prec \Omega_{\alpha*j}.$$

But by 3) we have $\Omega_\alpha \preceq \Omega_{\alpha*j}$. Hence $\Omega_\alpha \prec \Omega_{\alpha*j}$ for almost all j.
5) For Ω finite, not empty let $\mathrm{ord}(\Omega)$ be equal to

$$\min\{|\sigma| : \sigma \in \Omega\}.$$

We have for at least one α with $\alpha \in \Gamma_n$, $\alpha \neq \langle\ \rangle \quad \mathrm{ord}(\Omega_\alpha) \geq |\alpha|$. (By induction over the length of $\alpha \in \Gamma_n$). For $\alpha \in \Gamma_n$ with $|\alpha| = 1 \quad \mathrm{ord}(\Omega_\alpha) \geq 1$, since X^n_α is an infinite set and thus $\Omega_\alpha \neq \emptyset$. Having $\mathrm{ord}(\Omega_\alpha) \geq |\alpha|$, by 4) we find an j s.th. $\Omega_\alpha \prec \Omega_{\alpha*j}$. Hence $\mathrm{ord}(\Omega_{\alpha*j}) \geq |\alpha| + 1$.
Since for some $\alpha \in \Gamma_n$ with $|\alpha| = n \quad \mathrm{ord}(\Omega_\alpha) \geq n$, but $m < n$ and thus Γ_m has no element of order n, such mapping Φ cannot exist. □

Remark 3.2 The set A_1 from Theorem 3.1 is of particulary interest. In [ChN.ta] the atomless, r-maximal sets A having a sequence $(C_i)_{i\geq 0}$ of c.e. supersets with $C_i - A - \infty$ for every i, $C_i \cap C_j = A$ for $i \neq j$ and

$$(\forall\, D \in \mathcal{L}(A))(D \neq^* I\!N \Rightarrow (\exists\, n)(D \subseteq C_0 \cup C_1 \cup \ldots \cup C_n)) \tag{3.5}$$

are called *triangles*. The set A_1 is a triangle, since the sets $A_{(j)}$, $j \geq 0$ have the properties mentioned above for $(C_i)_{i\geq 0}$. We use $\mathcal{T}r$ as symbol for the class of triangles.

- A sequence $(C_i)_{i\geq 0}$ as above is called *basis for* $\mathcal{L}^-(A)$. A basis for $\mathcal{L}^-(B)$ with B triangle is not uniquely determined, but two basises are closely related. For $D \in \mathcal{L}^-(B)$ with $D - B - \infty$ we say that $D' \in \mathcal{L}(B)$ is a relatively splitting half of D if

  $$(\exists\, D'' \in \mathcal{L}(B))(D' \cap D'' = B \wedge D' \cup D'' = D).$$

  If $(C_i)_{i\geq 0}$ and $(E_i)_{i\geq 0}$ are two basises for $\mathcal{L}^-(B)$ then every E_i is a finite union of splitting halfs of sets C_i and converse.

- The class of triangles even is elementary definable in $\mathcal{E}$. It holds: "A is a triangle" if

  $$\begin{aligned} &A \text{ is an atomless, } r\text{-maximal set } \wedge \\ (3.6) \qquad &(\forall\, V \in \mathcal{L}^-(A))(\exists\, W \in \mathcal{L}^-(A))[V \subseteq^* W \wedge \\ &\quad (\forall\, S \in \mathcal{L}^-(A))(\exists\, T \in \mathcal{L}^-(A))(W \cap T = A \wedge S \subseteq^* W \cup T)] \end{aligned}$$

  $\Rightarrow$ Let $(C_i)_{i\geq 0}$ be a basis for $\mathcal{L}^-(A)$. Given $V \in \mathcal{L}^-(A)$, by (3.5) $V \subseteq C_0 \cup \ldots \cup C_n$ for some n. Let $W = C_0 \cup \ldots \cup C_n$. Hence $V \subseteq^* W$. If $S \in \mathcal{L}^-(A)$ then $S \subseteq C_0 \cup \ldots \cup C_m$ for some m again by (3.5). Now let $T = \bigcup\{C_i : n < i \leq m\}$ if $n < m$ and $T = A$ otherwise. Then $W \cap T = A$ by definition and $S \subseteq^* W \cup T$.

  $\Leftarrow$ By using (3.6) we can construct a basis for $\mathcal{L}^-(A)$. Take some $V \in \mathcal{L}^-(A)$ with $v - A - \infty$. For this V exists W s.th. (3.6) is satisfied (for all S). Let $C_0 = W$. Now take a $V \in \mathcal{L}^-(A)$ with $V - C_0 - \infty$ and choose W for $C_0 \cup V$. Then we define $C_1 = W - C_0$. Repeating this method infinitely often, where every $W \in \mathcal{L}^-(A)$ is considered, we get a basis $(C_i)_{i\geq 0}$ for $\mathcal{L}^-(A)$.

  By a similar condition as (3.6) also every set for $\mathcal{L}^-(A)$ which is a member of a basis for $\mathcal{L}^-(A)$ can be defined elementarily in $\mathcal{E}$.

- From the definition of triangles it follows easily that if A is a triangle then

  $$\mathcal{L}^*(A) \cong \bigoplus_W \mathcal{L}_{\mathrm{ms}},$$

  where $\bigoplus_W \mathcal{L}_{\mathrm{ms}}$ denotes the isomorphism type of the weak product of $\mathcal{L}_{\mathrm{ms}}$ – the major subset interval (mod $=^*$).

Thus all triangles have isomorphic c.e. superset structures. Still open for triangles is their automorphism characterization, see the end of this subpoint.
That triangles do not have in their superset structures a set simple in them is obvious. Since the atomless, r-maximal set constructed in Theorem 2.2 and also in Theorem 2.6 are triangles we showed here that what was mentioned in the introduction of subpoint 3.

$*$

After the triangles the next greater class of atomless, r-maximal sets which seems to be possible to characterize from the point of view of the possibilities of their c.e. superset structures is the class of major subsets of triangles.
Obviously every triangle is a major subset of a triangle, since the class $\mathcal{T}r$ is closed upwards (inside the class of coinfinite c.e. sets), but not converse.
Let A be a triangle and $X \subset_{\mathrm{sm}} A$. Then X is not a triangle. To see this take a coinfinite c.e. superset Y of X. Then $(Y \cap A) - X$ is ∞, what contradicts (3.6). For the elements from $\mathcal{MS}_{\mathcal{T}r}$ (the major subsets of triangles) $-\mathcal{T}r$ we do not have a characterization of the c.e. superset structures as for triangles (by one isomorphism type).

Lemma 3.3 *There are elements X and Y from $\mathcal{MS}_{\mathcal{T}r} - \mathcal{T}r$ such that $\mathcal{L}(X) \ncong \mathcal{L}(Y)$.*

Proof. Let A be the set from Theorem 2.6 which is a triangle, $(A_i)_{i\geq 0}$ the c.e. sequence and $(R_i)_{i\geq 0}$ the sequence defined there. (The sequence $(A \cup R_i)_{i\geq 0}$ is a basis for $\mathcal{L}^-(A)$.) Now let $X \subset_{\mathrm{sm}} A$ and Ω any infinite and coinfinite c.e. set. Let Y be equal to

$$X \cup \bigcup \{A_i : i \in \Omega\}.$$

Since $(A_i)_{i\geq 0}$ is an c.e. sequence and Ω is c.e., Y also is c.e.

- $X \notin \mathcal{T}r$, see above.
- $Y \in \mathcal{MS}_{\mathcal{T}r} - \mathcal{T}r$. $X \subseteq Y$ by definition. $A - Y - \infty$, since $A_i - X - \infty$ and Ω is coinfinite. Thus $Y \subset_{\mathrm{m}} A$, $Y \notin \mathcal{T}r$. Let B be coinfinite with $A \subseteq B$. If B would be a basis set for $\mathcal{L}^-(Y)$ then every other set $C \in \mathcal{L}^-(Y)$ has the form

  $$(C \cap B) \cup C'$$

  with $C' \cap B = Y$. But for an i with $i \notin \Omega$ and $R_i \cap B = A$ (such an i exists, since $A \in \mathcal{T}r$ and $\bar{\Omega} - \infty$) has the property $R_i \cap B \neq^* Y$, since $A_i - X \subseteq R_i \cap B$ and $(A_i - x) \cap Y = \emptyset$. Thus B is no basis set. But if Y would be from $\mathcal{T}r$ then every set from $\mathcal{L}^-(Y)$ is included into a basis set for $\mathcal{L}^-(Y)$.
- $\mathcal{L}(X) \ncong \mathcal{L}(Y)$. The set A is a set simple in $\mathcal{L}(X)$, but in $\mathcal{L}(Y)$ there are no such elements. If $z \in \mathcal{L}^-(Y)$ would be simple in $\mathcal{L}(Y)$ then $A \cup Z$ much more would it be. But $A \cup Z \subseteq A \cup R_0 \cup \ldots \cup R_n$ for some n. Take $i \in \Omega$ and $i > n$. Then $R_i - Y - \infty$, but $R_i \cap (A \cup Z) = Y$. □

Interesting would be to have a complete description of all possible isomorphism types of $\mathcal{L}^*(X)$ for $X \in \mathcal{MS}_{\mathcal{T}r}$ (mod $\mathcal{L}_{\mathrm{ms}}$), i.e. by using $\mathcal{L}_{\mathrm{ms}}$ as parameter to characterize all $\mathcal{L}^*(X)$'s.

$\ast$

Question. In [HarN98] it is shown that in the major subset intervals Σ^0_3-embedding is equal to computable embedding (mod $=^*$), i.e. let $A \subset_{\mathrm{m}} B$. Then

$$(\forall\, g \leq_T \emptyset'')(\exists\, f \text{ rec.fct})(\forall\, n)[(B - A) \cap W_{g(n)} =^* (B - A) \cap W_{f(n)}].$$

Holds this property also inside $\mathcal{L}(C)$ for $C \in R - \mathrm{Max}$?

A partial classification of $R - \mathrm{Max}$ is of particularly interest. This is the description of all nonempty subclasses of $R - \mathrm{Max}$ which can be defined by an $\exists\forall\exists$-formula in the style as in [La68b]. The basic structure is $(\mathbb{B}A(\mathcal{E}^*), \mathcal{E}^*)$, i.e. the Boolean closure of $\mathcal{E}^*$ with the unary relation $\mathcal{E}^*$ satisfied by the c.e. sets (mod $=^*$) in the language for a Boolean algebra with a unary relation for $\mathcal{E}^*$ and all formulas have quantifiers restricted to this predicate. What by using such formulas can defined in $R - \mathrm{Max}$? A first view on this problem shows that there are more than ten different subclasses of $R - \mathrm{Max}$.

3.2 r-Maximal sets and the congruence relation $\approx_{\mathrm{ms}}$

Closely connected with the relation $\subset_{\mathrm{m}}$ is a congruence relation in $\mathcal{E}$ which is defined below. In respect to this congruence relation the r-maximal sets have some special properties satisfied only by these c.e. sets.

Definition 3.4 For c.e. sets A and B we write $A \approx_{\mathrm{ms}} B$ if

$$A \cap B \subset_{\mathrm{m}} A \cup B \quad \text{or} \quad A \cap B =^* A \cup B.$$

We are here interested in the relationship between $\approx_{\mathrm{ms}}$ and $R - \mathrm{Max}$. We see easily that for A and B from $R - \mathrm{Max}$ $\;A \approx_{\mathrm{ms}} B$ iff $A \cup B \neq^* \mathbb{N}$. Further that the equivalence classes respectively to $\approx_{\mathrm{ms}}$ including an r-maximal set are exactly the maximal elements in the factor lattice $\mathcal{E}/_{\approx_{\mathrm{ms}}}$.
The maximal sets are automorphic in $\mathcal{E}$ each to the other, see [So74]. Holds an analogously property also for the maximal elements in $\mathcal{E}/_{\approx_{\mathrm{ms}}}$? The results in [So74] give beside the automorphism of the maximal sets also an answer to this question. The answer is yes, also the maximal elements in $\mathcal{E}/_{\approx_{\mathrm{ms}}}$ are automorphic each to the other, what will be shown in the following. For doing it we need the following consideration:
Let $\approx_R$ be the relation: X, Y c.e., $X \approx_R Y$ if

(3.7) $\qquad (\forall\, R \text{ computable set})(R \subseteq^* X \Leftrightarrow R \subseteq^* Y).$

By using the Reduction principle (for the definition of it, see e.g. [So87, p. 30]) we see that for c.e. sets A and B

(3.8) $\qquad A \approx_{\mathrm{ms}} B \quad \text{iff} \quad A \approx_R B,$

i.e. both relations coincide in $\mathcal{E}$. In the following we use the equivalence (3.8).
Let X be an infinite c.e. set. With $\mathcal{E} \cap X$ we denote the sublattice of all c.e. sets which are included into X and with R_X a unary relation satisfied by all computable sets in $\mathcal{E} \cap X$. From the results in [So74] the following can be concluded:

(3.9) $\qquad$ "Let A and B be noncomputable c.e. sets. Then the structures $(\mathcal{E} \cap A, R_A)$ and $(\mathcal{E} \cap B, R_B)$ are isomorphic."

(That $\mathcal{E} \cap A$ and $\mathcal{E} \cap B$ are isomorphic is easy to see, but in (3.9) we require from the isomorphism between both to preserve the computable subsets. This additionally condition makes the proof of the existence of the isomorphism very complicated).

Let φ be an isomorphism between $(\mathcal{E}\cap A, R_A)$ and $(\mathcal{E}\cap B, R_B)$. Then for $X, Y \in \mathcal{E}\cap A$

$$X \subset_{\mathrm{m}} Y \quad \text{iff} \quad \varphi(X) \subset_{\mathrm{m}} \varphi(Y),$$

i.e. φ preserves $\subset_{\mathrm{m}}$.
Suppose $X \subset_{\mathrm{m}} Y$. Let R be a computable set with $R \subseteq \varphi(Y)$. Then $\varphi^{-1}(R)$ is a computable set with $\varphi^{-1}(R) \subseteq Y$. Thus $\varphi^{-1}(R) \subseteq^* X$, what gives $R \subseteq^* \varphi(X)$. This shows that $\varphi(X) \subset_{\mathrm{m}} \varphi(Y)$.
The converse implication is shown in the same way.

Theorem 3.5 [He85] *Let A and B be r-maximal sets. Then $[A]/\approx_{\mathrm{ms}}$ and $[B]/\approx_{\mathrm{ms}}$ are automorphic in $\mathcal{E}/\approx_{\mathrm{ms}}$.*

Proof. Since A and B are r-maximal, both are not computable, Thus there is an isomorphism φ between $(\mathcal{E}\cap A, R_A)$ and $(\mathcal{E}\cap B, R_B)$, see (3.9). We show that φ generates an automorphism Φ of $\mathcal{E}/\approx_{\mathrm{ms}}$ which maps $[A]/\approx_{ms}$ to $[B]/\approx_{\mathrm{ms}}$. Let $X \in \mathcal{E}$. We define $\Phi([X]/\approx_{\mathrm{ms}})$ as follows:

$$\begin{array}{ll} \text{(i)} & \\ & \\ \text{(ii)} & \Phi([X]/\approx_{\mathrm{ms}}) = \\ \text{(iii)} & \end{array} \begin{cases} [\varphi(X\cap \bar{R}) \cup \overline{\varphi(\bar{R})}]/\approx_{\mathrm{ms}} : \text{ if } \bar{A} \subseteq^* X, \\ (\text{where } R \text{ is a computable set with } \bar{A} \subseteq^* R \subseteq X) \\ [\varphi(X\cap A)]/\approx_{\mathrm{ms}} : \text{ if } \bar{A} \not\subseteq^* X, \text{but } X - A \text{ is infinite} \\ [\varphi(X)]/\approx_{\mathrm{ms}} : \text{ if } X \subseteq^* A. \end{cases}$$

At first we see that Φ is well-defined, since the choice of R in (i) is unessentional.
Suppose $\bar{A} \subseteq^* X$ and R_1, R_2 are computable sets with $\bar{A} \subseteq^* e_i \subseteq X$, $i = 1, 2$. Let $R_0 = R_1 \cap R_2$. We show that

$$(3.10) \qquad \varphi(X\cap \bar{R}_0) \cup \overline{\varphi(\bar{R}_0)} = \varphi(X\cap \bar{R}_1) \cup \overline{\varphi(\bar{R}_1)}.$$

By chancing the role of R_1 with R_2 we get the equality of the sets. Hence the equality of the equivalence classes resp. to $\approx_{\mathrm{ms}}$.
We have

$$X = (X\cap \bar{R}_1) \cup (\bar{R}_0 \cap R_1) \cup R_0,$$

where $\bar{R}_0 \cap R_1$ and R_0 are computable sets, the sets are pairwise disjoint and $\bar{R}_1, \bar{R}_0 \cap R_1 \subseteq A$. Further

$$(3.11) \qquad \overline{\varphi(\bar{R}_1)} = \varphi(\bar{R}_0 \cap R_1) \cup \overline{\varphi(\bar{R}_0)}.$$

Since all sets $\bar{R}_1$, $\bar{R}_0 \cap R_1$ and $\bar{R}_0$ are computable, also their image by φ are computable. Further

$$(3.12) \qquad \varphi(X\cap \bar{R}_0) = \varphi(X\cap \bar{R}_1) \cup \varphi(\bar{R}_0 \cap R_1).$$

The combination of (3.11) and (3.12) gives the equality (3.10).
In case (ii) we see that by the assumption of X we have

$$X \approx_{\mathrm{ms}} X \cap A.$$

Since φ preserves $\approx_{\text{ms}}$, we get

$$\varphi(X) \approx_{\text{ms}} \varphi(X \cap A),$$

hence $[\varphi(X)]/\approx_{\text{ms}} = [\varphi(X \cap A)]/\approx_{\text{ms}}$. □

Let $\mathcal{L}$ be a lattice and $\approx$ a congruence relation in $\mathcal{L}$. Then there is a canonical homomorphism from the group $\text{Aut}(\mathcal{L})$ into the group $\text{Aut}(\mathcal{L}/\approx)$ by the assignment:

$$\begin{array}{ll} \Phi \in \text{Aut}(\mathcal{L}) \longrightarrow \Phi/\approx \ \in \text{Aut}(\mathcal{L}/\approx) & \quad \text{with} \\ \Phi/\approx([X]/\approx) = [Y]/\approx \quad \text{if} \ \ \Phi(X) = Y. & \end{array}$$

In [La68a] it is shown that this canonical homomorphism for the congruence relation $=^*$ is surjective. For $\approx_{\text{ms}}$ we have

Corollary 3.6 *The mapping* $\text{Aut}(\mathcal{E}) \underset{\text{can.}}{\longrightarrow} \text{Aut}(\mathcal{E}/\approx_{\text{ms}})$ *is not surjective.*

Proof. Let X and Y be r-maximal, not automorphic in $\mathcal{E}$, e.g. $X \in \text{Max}$ and $Y \in \mathcal{MS}_{\text{Max}}$. Then

$$[X]/\approx_{\text{ms}} \ \cong_{\mathcal{E}/\approx_{\text{ms}}} [Y]/\approx_{ms}, \text{ by Theorem 3.5, but } \ X \not\cong_{\mathcal{E}} Y. \qquad □$$

Questions 3.7

1) Exhause the equivalence classes of the r-maximal sets respectively to $\approx_{\text{ms}}$ (considered as lattices) all isomorphism types of equivalence classes respectively to $\approx_{\text{ms}}$ (as lattices); symbolically:

$$(\forall\, X \text{ not computable c.e.})(\exists\, Y \ r\text{-max.})([X]/\approx_{\text{ms}} \ \cong_{\Phi} [Y]/\approx_{\text{ms}})?$$

2) Let A and B be r-maximal sets. Holds $A \cong_{\mathcal{E}} B$ iff $[A]/\approx_{\text{ms}}$ and $[B]/\approx_{\text{ms}}$ are isomorphic, where an isomorphism between both maps A to B. (Here both classes $[A]/\approx_{\text{ms}}$ and $[B]/\approx_{\text{ms}}$ are considered as sublattices of $\mathcal{E}$.) symbolically:

$$A, B \in R - \text{Max},\ [A]/\approx_{\text{ms}} \ \cong_{\Phi} [B]/\approx_{\text{ms}} \text{ with } \Phi(A) = B \text{ iff } A \cong_{\mathcal{E}} B?$$

This question says that the automorphism properties of the r-maximal sets are completely determined by the structure of their equivalence classes respectively to $\approx_{\text{ms}}$ considered as lattices and the place of the r-maximal sets inside their equivalence classes.

3.3 On the orbits of the r-maximal sets

On the orbits of the r-maximal, not maximal sets not many is known. In [Ch91], Theorem 3.21 it is shown that for every not q-maximal, noncomputable c.e. set A $\mathcal{O}(A)$ (the orbit of A) is not equal to

$$\{B \text{ r.e} : \mathcal{L}(B) \cong \mathcal{L}(A)\}, \tag{3.13}$$

hence in particulary also for all not maximal, r-maximal sets.
To conjecture is that the class (3.13) decomposes into infinitely many orbits. This is a special case of the general conjecture in [So78]. Good candidates for showing this general fact are the classes $\mathcal{MS}_{\text{Max}}$ and $\mathcal{T}r$.

Conjecture 3.8 *The class $\mathcal{MS}_{\text{Max}}$ as also $\mathcal{T}r$ consists of infinitely many orbits.*

One of the orbits included into $\mathcal{MS}_{\text{Max}}$, but also unknown seems to be the family

$$\{X \text{ r.e} : X \subset_{\text{sm}} M,\ M - \text{maximal}\}.$$

4 Degrees of r-maximal and r-cohesive sets

The relationship between the r-maximal sets and more general of the r-cohesive sets and the reducibilities was also investigated in some papers. Some facts follows also from results shown for other classes of c.e. sets.

4.1 The degrees of the r-maximal sets

The class of T-degrees of the r-maximal sets can be easily characterized by using well-known facts.

a) It holds: $\deg_T(\text{Max}) = \mathbb{H}^1$ (Martin's Theorem, see e.g. [So87]).

b) $\deg_T(fs\mathcal{HS}) \subseteq \mathbb{H}^1$ (see [So87]).

Thus, since $\text{Max} \subseteq R - \text{Max} \subseteq fs\mathcal{HS}$, $\deg_T(R - \text{Max}) = \mathbb{H}^1$. We get this also by using

c) $\deg_T(\mathcal{MS}(A)) = \mathbb{H}^1$ (Jockusch showed "$\subseteq$", see [Jo71] and Lerman "$\supseteq$", see [Le71]).

Thus for $A \in R - \text{Max}$ either $A \in \text{Max}$ or $A \subset_m B$ for some $B \in \mathcal{L}(A)$. Hence from a) and b) we also get $\deg_T(R - \text{Max}) = \mathbb{H}^1$.
By similar argumentations we get the same characterization also for the main subclasses of $R - \text{Max}$.

- $\deg_T(\text{Max}) = \mathbb{H}^1$, see a)
- $\deg_T(\mathcal{MS}_{\text{Max}}) = \mathbb{H}^1$, see c)
- $\deg_T(R - \text{Max}_{\text{atl}}) = \mathbb{H}^1$, see c).

The properties given above can be still essentionally improved.

Theorem 4.1 *Let A be an r-maximal set. Then*

$$(4.1) \qquad \deg_T(\{B \text{ c.e.} : \mathcal{L}^*(B) \cong \mathcal{L}^*(A)\})$$

is equal to $\mathbb{H}^1$.

Proof. In [Ch91], Theorem 3.20 it is proven:

$$(\forall\, A \text{ noncomputable, c.e. set})(\forall\, \underline{h} \in \mathbb{H}^1)(\exists\, C \text{ c.e.})[C \in \underline{h} \wedge \mathcal{L}^*(A) \cong \mathcal{L}^*(B)].$$

From this we get that $\mathbb{H}^1$ is included into (4.1).
The other inclusion is obvious. □

4.2 The degrees of orbits of r-maximal sets

Different is the situation for the orbits of r-maximal sets.
For special r-maximal sets A we know that $\deg_T(\mathcal{O}(A)) \neq \mathbb{H}^1$. If A is a not maximal, r-maximal set with the splitting property (see for this [MaShSt81]) then $\deg_T(A)$ is not a half of a minimal pair. The existence e.g. of major subsets of maximal sets which are promptly simple and then having the splitting property we get by a small modification of the usual major subset construction of a promptly simple, maximal set. In the construction it is useful to use the modified witness function (for the promptly simplicy of the maximal set) given in [So87, p. 287, point 1.11].
Since the splitting property even is elementary definable in the lattice $\mathcal{E}$ and there are high degrees which are halfs of a minimal pair, not every high degree is realized by the members of this orbit.
But there are also r-maximal sets not satisfying even a greater elementary property, namely they are not d-simple, see [LeSo80]. We get this easily, since $X \subset_{\mathrm{sm}} Y$, $Y \in R - \mathrm{Max}$ implies $X \in R - \mathrm{Max}$ and X is not d-simple.
Since $\subset_{\mathrm{sm}}$ is closed downwards (inside $\subset_{\mathrm{m}}$), by Conjecture 3.8 and the result of Lerman in [Le71] the class

$$\{X \text{ c.e.} : X \subset_{\mathrm{sm}} M,\ M\text{-maximal}\}$$

would be an orbit of an r-maximal set realising every high degree.

4.3 The degrees of $(R - \mathrm{Max})^{\#}$

Let $\mathcal{X}$ be a class of coinfinite c.e. sets. With $\mathcal{X}^{\#}$ we denote the class

$$\{Y \text{ c.e.} : \mathcal{L}(Y) \cap \mathcal{X} = \emptyset\}.$$

there is a well-known result from Lachlan/Shoenfield, see [So87] that

$$\deg_T(\mathrm{Max}^{\#}) = \overline{\mathbb{L}_2}.$$

Shore asked, see [Od.ta] what is with

$$\deg_T((R - \mathrm{Max})^{\#})\,?$$

Since $(R-\text{Max})^{\#} \subseteq \text{Max}^{\#}$ and e.g. for $\mathcal{HH}_{\text{atl}}$ – the class of atomless hh-simple sets – we have $\mathcal{HH}_{\text{atl}} \subseteq R-\text{Max}^{\#}$,

$$I\!\!H^1 \subseteq \deg_T((R-\text{Max})^{\#}) \subseteq \overline{I\!\!L_2}.$$

A more precise description of $\deg_T((R-\text{Max})^{\#})$ is still unknown, holds also the equality between this degree class and $\overline{I\!\!L_2}$?

4.4 On the degrees of r-cohesive sets

The characterization of the T-degrees of all r-cohesive sets is more complicated as that of those which are complements of c.e. sets. The degrees of this greater family of sets were investigated in several papers, e.g. in [Jo73] and [JoSt93]. Nevertheless many properties of this degree class are known a complete characterization still is not given.

For a characterization of the T-degrees of the r-cohesive sets we define the following class of degrees $\mathcal{C}$:

Let $\mathcal{C}$ be the set of degrees $\underline{c}$ such that

$$(\forall\, \psi\text{-part., 01-function } \leq_T \underline{0}')(\exists\, f\text{-total fct. } \leq_T \underline{c})[\psi \subseteq f].$$

Obviously $\mathcal{C}$ is closed upwards and for any $\underline{c} \in \varphi \quad \underline{0}' \leq \underline{c}$. (To the last take, C_k the characteristic function of the creative set K. $C_k \leq_T \underline{0}'$ and is even total. Thus $C_k \leq \underline{c}$. Hence $\underline{0}' \leq \underline{c}$.)

Similar easy is to see that even $\underline{0}' < \underline{c}$ is true. By Post's Theorem a Σ_2 set is c.e. in $\underline{0}'$. Take a Σ_2-set which is not Δ_2. Apply to X the relativized Friedberg splitting of Σ_2-sets. Then we get the pair (X_0, X_1). Let φ be the function: $\varphi(x) = 0$ if $x \in X_0$ and $\varphi(x) = 1$ if $x \in X_1$. $\varphi \leq_T \underline{0}'$, since X_0 and X_1 are c.e. in $\underline{0}'$. But there is no Δ_2-set Y with $X_0 \subseteq Y$ and $X_1 \cap Y = \emptyset$. Hence any f with $\varphi \subseteq f$ cannot be $\leq_T \underline{0}'$.

Theorem 4.2 ([JoSt93]) *$\mathcal{C}$ is equal to*

$$\{\underline{c}' : \underline{c} \text{ is an } r\text{-cohesive degree}\}.$$

Remark. The converse of Theorem 4.2 does not hold, i.e. there is a degree $\underline{c}$ with $\underline{c}' \in \mathcal{C}$ and c is not r-cohesive, see later Theorem 4.8.

Corollary 4.3 *The classes of* low_1 *sets and of r-cohesive sets are disjoint.*

In the following we show that the cohesive sets coincide with the r-cohesive sets outside of the class of high sets.

Theorem 4.4 ([JoSt93]) *If A is an r-cohesive, not high set then A is cohesive.*

Proof. Let A be r-cohesive and W_e be an c.e. set with $W_e \cap A$ infinite. We define a function f. Let $f(x) = \min\{s : (\exists\, a \geq x)(a \in W_{e,s} \cap A)\}$. We see that f is total and A-computable, since $W_e \cap A$ is infinite. Since A is not high, there is an increasing computable function h not dominated by f. If $f(x) < h(x)$ for an x then there is some $a \geq x$ with $a \in A \cap W_{e,h(a)}$, since $f(x) < h(x) \leq h(a)$. Thus the set R equal to $\{a : a \in W_{e,h(a)}\}$ is a computable subset of W_e which has infinite many elements common with A. Therefore $A \subseteq^* R$, since A is r-cohesive and thus $A \subseteq^* W_e$. Hence A is cohesive. $\square$

Corollary 4.5 *The T-degrees of the r-cohesive sets coincide with the T-degrees of the cohesive sets.*

Proof. Both degree classes include $\mathbb{H}^1$, see a) in 4.1. Outside of $\mathbb{H}^1$ even the class of sets are equal each to the other, see Theorem 4.4, hence much more the corresponding degree classes. $\square$

Theorem 4.6 ([JoSt93]) *There is an r-cohesive set which is* low$_2$.

More interesting is the combination of Theorem 4.4 with 4.6. From this we get that there is a cohesive set which is low$_2$, hence not high. This gives an answer of a long time open problem of Jockusch.

From Cooper's result that every cohesive set X with $X \leq_T \emptyset'$ is high and Theorem 4.4 we get the same for every r-cohesive set.

Another property of not high, r-cohesive sets is given in the following theorem:

Theorem 4.7 ([JoSt93]) *If A is r-cohesive, but not high, then A is included into a complement of an effective simple set.*

Proof. We can assume that A is cohesive, by Theorem 4.4. Let p_A be the principal function of A. Since A is not high there is a strictly increasing computable fct. f s.th. $p_A(2n) < f(n)$ for ∞ many n and $2n \leq f(n)$ for all n.

By Post's simple set construction relatively to f there is a simple set B with $p_{\bar{B}}(n) \geq f(n)$ for all n. $\bar{B}$ is infinite, since $2n \leq f(n)$.

$\bar{B}$ is effectively immune by f. If $W_e \subseteq \bar{B}$ then $|W_e| < f(e)$, otherwise an element from W_e comes to B. Since A is cohesive, it must be almost contained in B or almost contained in $\bar{B}$. If $A \subseteq^* B$ (i.e. for some k $\ (k,\infty) \cap A \subseteq B$). Thus $p_B(n) \leq p_A(n+k)$ for all n. Hence $f(n) \leq p_A(n)$ for all sufficiently large n, what is not possible. Thus $A \cap B =^* \emptyset$. Let D be the finite set $A \cap B$. Then $B' = B - D$ is disjunct with A, but also effectively simple. $\square$

Theorem 4.8 ([JoSt93]) *There is a cohesive degree a and a noncohesive degree b with $a' = b'$.*

From Theorem 4.8 we get that the converse of Theorem 4.2 does not hold.

In [JoSt93] there are given still further properties of the T-degrees of the r-cohesive sets. Thus there r-cohesiveness is compared with special subclasses of immune sets and in particular of the corresponding degree classes.

4.5 m-reducibility and r-maximal sets

The relationship between the r-maximal sets and the m-reducibility was investigated by Degtev in [De72], see also [Od81].

Theorem 4.9 (Degtev) *Let A be r-maximal. Then for every nontrivial c.e. superset B of A B is m-incomparable with A.*

Proof. B is nontrivial means $A \neq^* B$ and $B \neq^* I\!N$. We show a more general fact. let B_1 and B_2 be infinite subsets of $\bar{A}$ with $B_1 =^* B_2$. Then B_1 and B_2 are m-incomparable. For contrary suppose $B_1 \leq_m B_2$ by f, i.e.

$$x \in B_1 \iff f(x) \in B_2.$$

– At first we see that the computable set $\{x : x = f(x)\}$ includes only finitely many elements from $\bar{A}$. If not then, since A is r-maximal, it includes $\bar{A}$ (mod $=^*$). But since $B_1 \neq^* B_2$, this is not possible.

– Next we see that the set

$$(4.2) \qquad \{x \in \bar{A} : f^{-1}(\{x\}) \in \bar{A} \wedge \{x\} \neq f^{-1}(\{x\}) \neq \emptyset\}$$

is infinite.
For $x \in B_1 \Rightarrow f(x) \in B_2 \Rightarrow f^{-1}(\{f(x)\}) \subseteq B_1 \subseteq \bar{A}$ and further $f^{-1}(\{f(x)\})$ is finite, since A is a simple set.
Further $x \neq f(x)$ for almost all $x \in B_1$. $x \in f^{-1}(\{f(x)\})$, hence $\{x\} \neq \{f^{-1}(x)\}$. This gives that the set (4.2) even intersected with B_2 is infinite.

– Now we define (inductively in respect to the magnitude of the numbers) a computable set R. Let R be the set of numbers x such that

$$(4.3) \qquad \neg(\exists\, y)(y < x,\ f(y) = x,\ y \in R) \wedge \neg(f(x) < x,\ f(x) \in R).$$

The set R has following properties:

$$(4.4) \qquad (\forall\, x)(\forall\, y)(x \neq y,\ f(y) = x \Rightarrow \{y, x\} \cap \bar{R} \neq \emptyset).$$

If $x \in R$ then if $y < x$, $y \notin R$ and thus $\{y, x\} \cap \bar{R} \neq \emptyset$ or $x < y$ and thus $x = f(y) < y$ and $f(y) \in R$. But then $y \notin R$, by the second condition in (4.3).
From (4.2) and $\operatorname{rg} f \cap B_2 - \infty$ it follows that $\bar{A} \cap \bar{R}$ is ∞, hence $\bar{A} \subseteq^* \bar{R}$. Hence $\bar{A} \cap R$ is finite. Further

$$(4.5) \qquad (\forall\, x)((f^{-1}(\{x\}) \cup \{x, f(x)\}) \cap R \neq \emptyset).$$

If $x \notin R$ then for some $y \neq x$, $f(y) = x$, $y \in R$, hence $f^{-1}(\{x\}) \cap R \neq \emptyset$ or $f(x) \in R$, by (4.3).
Now consider the computable set S equal to $\{x : f(x) \in R\}$. Then $S \cap B_1 =^* \emptyset$, since $x \in B_1 \Rightarrow f(x) \in B_2 \subseteq \bar{A}$. But $\bar{A} \cap R$ is finite and $f^{-1}(\{z\})$ is finite for $z \in B_2$. Hence $\bar{S} \cap \bar{A}$ is infinite.
But for $x \in B_1 \to f(x) \in B_2 \subseteq \bar{A}$ and $f^{-1}(\{f(x)\}) \subseteq B_1 \subseteq \bar{A}$. Thus except for finitely many $f(f(x)) \in R$ by (4). Hence $f(x) \in S$. But $\operatorname{rg} f \cap B_2 - \infty$. Hence $S \cap \bar{A} - \infty$. Both $\bar{S} \cap \bar{A} - \infty$ and $S \cap \bar{A} - \infty$ contradicts that A is r-maximal. Thus our assumption $B_1 \subseteq_m B_2$ was false. Now let $B_1 = \bar{A}$ and $B_2 = \bar{B}$. Hence $\bar{A}$ and $\bar{B}$ are m-incomparable. □

5 Index sets of the class of r-maximal sets and subclasses of r-maximal sets

On the index sets of the whole class of r-maximal sets as also on subclasses the estimations are known. We will give here the known results on the index sets, but also open problems concerning this topic.

5.1 The index set of the r-maximal sets

By using the fact: $(\Sigma_4^0, \Pi_4^0) \leq_1 (\mathbb{I}S(\neg fs\mathcal{HS}), \mathbb{I}S(\mathrm{Max}))$, ($fs\mathcal{HS}$-class of finitely strongly hypersimple sets) see [So87, p. 268] or even the stronger fact with $\mathbb{L}$ow instead of $fs\mathcal{HS}$, see [So87, p. 263], we get:

Theorem 5.1 *$\mathbb{I}S(R - \mathrm{Max})$ is Π_4^0-complete.*

Proof. Obviously $\mathrm{Max} \subseteq R - \mathrm{Max} \subseteq fs\mathcal{HS}$. Further it is easy to see that $R - \mathrm{Max}$ has a Π_4^0-definition. Thus from the facts above we get the Π_4^0-completeness of $\mathbb{I}S(R - \mathrm{Max})$.

Corollary 5.2 *$\mathbb{I}S(R - \mathrm{Max} \cap \mathcal{DS})$ is Π_4^0-complete.*

Proof. The proof is identical to that of Theorem 5.1 with $R - \mathrm{Max} \cap \mathcal{DS}$ in place of $R - \mathrm{Max}$.

5.2 The index sets of the main subclasses of r-maximal sets

Also for the main subclasses of $R - \mathrm{Max}$ we know the estimations of their index sets. We get those by using the following theorem:

Theorem 5.3 ([He.ta1]) *Let Q be a (unary) Σ_5^0-relation. Then there is a c.e. sequence $(U_i)_{i \geq 0}$ such that*

$$\begin{aligned} Q(i) &\Longrightarrow U_i \in \mathcal{MS}_{\max}\,, \\ \neg Q(i) &\Longrightarrow U_i \in R - \mathrm{Max}_{\mathrm{atl}}. \end{aligned}$$

Proof. Let $Q(i)$ be from Σ_5^0. Then there is a computable function $\lambda(i,x,y,s)$ with $\lambda(i,x,y,s) \leq \lambda(i,x,y,s+1)$ such that

$$Q(i) \Longrightarrow (\exists\, x_0)(\forall\, x \geq x_0)(\exists\, y_0)(\forall\, y \geq y_0)(\forall\, k)(\exists\, s)(k \leq \lambda(i,x,y,s))$$

and

$$\neg Q(i) \Longrightarrow (\exists^\infty x)(\forall\, y)(\exists\, k)(\forall\, s)(\lambda(i,x,y,s) \leq k)\,.$$

We write $\lambda(i,x,y)$ for $\lim_s \lambda(i,x,y,s)$ (and take also ω as limit value).
Let $p(i,n)$ be a computable function having as image pairs (x,y) with $0 \leq x \leq y$ with the property:

$$\mathrm{card}(\{n : p(i,n) = (x,y)\}) = \begin{cases} \lambda(i,x,y) & : \; 0 < x < y \\ 0 & : \; \text{else (i.e. } x = 0 \vee x = y). \end{cases}$$

Construction. Fix an i. We describe the construction of U_i. In the following we omit this i. (Thus we shall write U_s instead of $U_{i,s}$, $p(n)$ for $p(i,n)$ and so on.)

Similar as in Theorem 2.6 we construct stepwise U, a (move) function T and here still two other objects. We need still a function C defined for all pairs $(n,m) \in \Gamma$ and m-states κ into $I\!N$. C counts how often a number is placed on (n,m) by reason of a state κ in the meaning described below. Further we need another state function $\mathrm{st}_M(x,i,s)$ (which differs from $\mathrm{st}_W(x,i,s)$). When a number is moved by the i-state κ, then $\mathrm{st}_M(x,i,s) = \kappa$.
(Thus, e.g., if in step s x comes to W_e then $\mathrm{st}_W(x,e,s+1) \neq \mathrm{st}_W(x,e,s)$, but so long x is on the same place $\mathrm{st}_M(x,e,t)$ for $t \geq s$ does not change its value.)

Step 0. Let $U_0 = \emptyset$. T_0 be a computable bijection between Γ and $I\!N$. $C_0((n,m),\kappa) = m$ for all pairs (n,m) and m-states κ and $\mathrm{st}_M(x,i,0) = (0\dots0)$, $i+1$-times for all x and i.

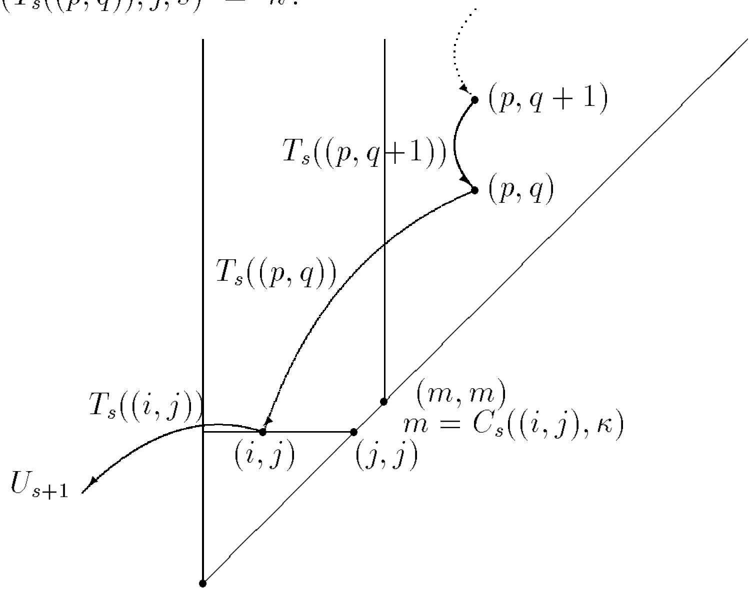
Step $\boldsymbol{s+1}$. We assume that U_s, T_s, C_s and $\mathrm{st}_M(x,i,s)$ are given.

$\boldsymbol{(s+1=2n)}$. Take the pair $p(n)$. Suppose $p(n) = (k,\ell)$. Then define

$$\begin{aligned}
&U_{s+1} = U_s \cup \{T_s(p(n))\}\\
&T_{s+1}((k,\ell+\varepsilon)) = T_s((k,\ell+\varepsilon+1)) \;:\; \varepsilon \geq 0\\
&T_{s+1}((k',\ell')) = T_s((k',\ell')) \;:\; \text{else}\\
&\mathrm{st}_M(T_{s+1}((k,\ell+\varepsilon)),i,s+1) = (0\dots0),\ \varepsilon, i \geq 0\\
&C_{s+1} = C_s.
\end{aligned}$$

$\boldsymbol{(s+1=2n+1)}$. Look if there are pairs (i,j), (p,q) and a j-state κ with the properties:

$$\begin{aligned}
C_s((i,j),\kappa) &< p\\
\mathrm{st}_M(T_s((i,j)),j,s) &<_\ell \kappa\\
\mathrm{st}_W(T_s((p,q)),j,s) &= \kappa\,.
\end{aligned}$$



If not do nothing. If yes choose the smallest pair (i,j) (first j then i) for this the greatest j-state κ and for these the smallest pair (p,q).
Let (i_0,j_0), κ_0, (p_0,q_0) these objects and define

$$T_{s+1}((i_0,j_0)) = T_s((p_0,q_0))$$
$$T_{s+1}((p_0,q_0+\varepsilon)) = T_s((p_0,q_0+\varepsilon+1))\,, \quad \varepsilon \geq 0$$

unchanced for all other pairs.

$$C_{s+1}((i_0,j_0)),\kappa_0) = C_s((i_0,j_0),\kappa_0)+1$$
$$\mathrm{st}_M(T_{s+1}((i_0,j_0)),j_0,s+1) = \kappa_0$$
$$\mathrm{st}_M(T_{s+1}((p_0,q_0+\varepsilon)),q_0+\varepsilon,s+1) = (0\ldots0)\,, \quad \varepsilon \geq 0$$
$$U_{s+1} = U_s \cup \{T_s((i_0,j_0))\}\,.$$

Result. Let $U = \bigcup_{s\geq 0} U_s$. By construction of U_s, U is c.e.

1. $\lim_s T_s((n,m))$ exists iff $\{s : p(s) = (n,m)\}$ is finite.

$\Rightarrow$ Obvious by step $s+1 = 2n$.

$\Leftarrow$ At first we see that $x = T_s((n,m)) = T_{s+1}((n',m'))$ implies that $(n,m) = (n',m')$ or $m' < m$ (we have $C_0((n,m),\kappa) = m$). Further, if $\{s : p(s) = (m,n)\}$ is finite, also all pairs (n,m') with $m' \leq m$ are enumerated only finitely often by p and at least the function C_s is weakly increasing in s.
Thus there is a step s_0 such that for all $s \geq s_0$, for all pairs (n',m') with $m' \leq m$ and for all m'-states v we have that $C_s((n',m'),v) > m$ or is constant, $p(s) \neq (n,m')$ for all m' with $m' \leq m$ and $T_{s+1}((k,\ell)) = T_s((k,\ell))$ for all pairs (k,ℓ), $\ell \leq m$ which are only finitely often enumerated by p. But then for a $s \geq s_0$ $\;T_{s+1}((n,m)) \neq T_s((n,m))$ is possible only if $s+1$ is odd and in step $s+1$ $\;(n,m) = (i_0,j_0)$.
But then $\mathrm{st}_M(T_{s+1}((n,m)),m,s+1) > \mathrm{st}_M(T_s((n,m),m,s)$. Since this can happen only finitely often $\lim_s T_s((n,m))$ must exist.

Let T be the limit function for the pairs (n,m) for which $\lim_s T_s((n,m))$ exists. T is injective, further $T((0,m))\downarrow$ and $T((m,m))\downarrow$. Since domain $(T) = \bar{U}$, U is coinfinite.

2. Let $T(\Gamma_i)$ be the set

$$\{T((i,m)) : T((i,m))\downarrow,\, m \geq i\}\,.$$

For every i the set $U \cup T(\Gamma_0) \cup \ldots \cup T(\Gamma_i)$ is a c.e. set. This follows easily from the fact

$$x = T_s((p,q)) \Longrightarrow x \in U_{s+1} \vee x = T_{s+1}((p',q')) \wedge p' \leq p\,.$$

3. For every e

$$(*) \qquad (\exists^\infty i)(T(\Gamma_i) \cap W_e \neq \emptyset) \Longrightarrow \bar{U} \subseteq^* W_e\,.$$

Given a number e and suppose for all $f < e$ $(*)$ is satisfied (with f in place of e in $(*)$).
For a pair (n,m) with $T((n,m))\downarrow$ define $\mathrm{st}_M(T((n,m)),m) = \lim_s \mathrm{st}_M(T_s((n,m)),m,s)$.
We assume that for every $f < e$ for almost all pairs (n,m) with $f \leq m$ at $T((n,m))\downarrow$

either $\mathrm{st}_M(T((n,m)),m)(\ell)=1$ or for almost all such pairs $\mathrm{st}_M(T((n,m)),m)(f)=0$. Thus there is a number $m\geq\ell$ such that for all pairs (p,q) with $m\leq q$ and $T((p,q))\downarrow$ $\mathrm{st}_M(T((p,q)),\ell-1)$ is the same.
If $(\exists^\infty i)(T(\Gamma_i)\cap W_e\neq\emptyset)$ then by the assumption on e, the fact that $\mathrm{st}_M(T_s((p,q)),e,s)\leq \mathrm{st}_W(T_s((p,q)),e,s)$ and by the assumption on the st_M-state of length $e-1$ there cannot be a pair (i,j) with $m\leq j$, $T((i,j))\downarrow$ at $\mathrm{st}_M(T((i,j)),e)(e)=0$.

4. Suppose $Q(i)$. Then

$$(\exists\, x_0)(\forall\, x\geq x_0)(\exists\, y_0)(\forall\, y\geq y_0)(\lambda(i,x,y)=\omega)\,.$$

Thus there is a k such that for all $k'>k$ we have that $T(\Gamma_{k'})$ is finite (but not empty, $T((k',k')\downarrow)$). But then $U\cup T(\Gamma_0\cup\ldots\cup\Gamma_k)$ is maximal, by 3. $T(\Gamma_0\cup\ldots\cup\Gamma_k)$ is infinite, since $T((0,y))\downarrow$ for all $y\geq 0$. By 3. we have $U\subset_m U\cup T(\Gamma_0\cup\ldots\cup\Gamma_k)$. Both facts give that $U\in\mathcal{MS}_{\max}$.

5. Suppose $\neg Q(i)$. Then

$$(\exists^\infty x)(\forall\, y)(\lambda(i,x,y)<\omega)\,.$$

Then, since $\lambda(i,x,y)<\omega$ implies $T((x,y))\downarrow$ by 1., we have as infinite tower for U, which by 3. satisfies (2.1). But this means that $U\in R-\mathrm{Max}_{\mathrm{atl}}$.

Corollary 5.4

(i) $I\!I S(\mathcal{MS}_{\mathrm{Max}})-\Sigma^0_5$*-complete.*

(ii) $I\!I S(R-\mathrm{Max}_{\mathrm{atm}})-\Sigma^0_5$*-complete.*

(iii) $I\!I S(R-\mathrm{Max}_{\mathrm{atl}})-\Pi^0_5$*-complete.*

Remark. From Theorem 5.3 we can conclude also that $I\!I S(\mathrm{Atl})$ [5)] is Π^0_5-complete, since this class has a Π^0_5-definition. But this was already shown before by Jockusch, see [So87, p. 262] and also by Lempp, see [Lem87].

Questions 5.5 1) Let $\mathcal{MS}_{\mathrm{QM}}$ be the class of the major subsets of the q-maximal sets. An easy corollary from results in [Lem87] is the Σ^0_5-completeness of its index set. But holds the stronger fact:

$$(\Pi^0_5,\Sigma^0_5)\leq_{\mathrm{m}}(I\!I S(\neg\mathcal{MSS}),I\!I S(\mathcal{MS}_{\mathrm{QM}})),$$

where $\mathcal{MSS}=\{A \text{ c.e. }:(\exists B)(A\subset_{\mathrm{m}}B\}$? The Π^0_5-completeness of $I\!I S(\neg\mathcal{MSS})$ is shown in [Ja96].
2) Is the index set of the triangles Π^0_6-complete?

In subpoint 6 further classes of r-maximal sets will be defined and by using the results given here their index sets can be easily characterized.

5) Atl – the class of coinfinite, atomless c.e. sets.

5.3 Index sets of c.e. superset classes of r-maximal sets

Let $A \in R-\mathrm{Max}$. We will consider here the index sets of $\mathrm{FIN}_{\bar{A}} = \{W_e : W_e \cap \bar{A} \text{ is finite}\}$ and of $\mathrm{COF}_{\bar{A}} = \{W_e : A \cup W_e =^* I\!N\}$.

The index sets $\mathrm{FIN}_{\bar{A}}$ are Σ^0_3-complete for any r-maximal, not maximal set A. This follows at once from the fact shown by Maass in [Ma85]. He proved that $\mathrm{FIN}_{\bar{A}}$ is Σ^0_3-complete for any $A \in \mathcal{MSS}$. But $R - \mathrm{Max} \setminus \mathrm{Max}$ is a subclass of $\mathcal{MSS}$. Thus we get the equivalence

$$A \in R - \mathrm{Max} \wedge \bar{A} \in S I\!L_2 \Leftrightarrow A \text{ maximal.}$$

For the class $\mathrm{COF}_{\bar{A}}$ the situation is more complicated, i.e. is not still cleared for the r-maximal sets in general.
Easy to see is that for $A \in R - \mathrm{Max}_{\mathrm{atm}}$ $\mathrm{COF}_{\bar{A}} \leq_T \emptyset''$. Since for A and B with $A \subset_{\mathrm{m}} B \vee A =^* B$

$$W_e \cup A =^* I\!N \Leftrightarrow W_e \cup B =^* I\!N.$$

If B is maximal (then obviously $\mathrm{COF}_{\bar{B}} \leq_T \emptyset''$) we get $\mathrm{COF}_{\bar{A}} \leq_T \emptyset''$.
But also some atomless, r-maximal sets have this property.

Lemma 5.6 *Let $A \in \mathcal{T}r$. Then $\mathrm{COF}_{\bar{A}} \in \Delta^0_3$.*

Proof. Given $A \in \mathcal{T}r$ let $B \in \mathcal{L}(A)$ be a basis element for $\mathcal{L}^-(A)$, i.e. for every $C \in \mathcal{L}^-(A)$ there are sets $C_1, C_2 \in \mathcal{L}(A)$ with $C_1 \subseteq B$, $C_2 \cap B = A$ s.th. $C = C_1 \cup C_2$. Then

$$W_e \cap \bar{A} \neq^* I\!N \Leftrightarrow (\exists f)(\exists k)(W_f \subseteq B \wedge W_k \cap B \subseteq A \wedge W_e = W_f \cup W_k). \quad \square$$

For the r-maximal set A constructed in Theorem 2.4 we also have $\mathrm{COF}_{\bar{A}} \in \Delta_3$. It holds

$$A \cup W_e =^* I\!N \text{ iff } (\exists n)(\forall n \geq n_0)(F_n \subseteq A \vee F_n \setminus W_e \neq \emptyset).$$

"$F_n \setminus W_e$" is Π_1, since $(F_n)_{n\geq 0}$ is strongly c.e.
Easy to see is also that for $A \in R-\mathrm{Max}$ for all B and C from $[A]/\approx_{\mathrm{ms}}$ $\mathrm{COF}_{\bar{B}} = \mathrm{COF}_{\bar{C}}$. We have

$$B \approx_{\mathrm{ms}} C \Rightarrow (B \cup W_e =^* I\!N \Leftrightarrow C \cup W_e =^* I\!N).$$

The r-maximal sets A with $\mathrm{COF}_{\bar{A}} \in \Delta_3$ have a property which seems to be useful for the characterization of $\mathcal{L}(A)$, what in the following we discuss. We consider a weaken of the notion of tower from subpoint 2.

> Call a sequence $(H_n)_{n\geq 0}$ (not necessarily c.e.) of c.e. sets *weak tower* if for all $n \geq 0$ $H_n \subseteq H_{n+1}$ and $H_n \neq^* I\!N$.

Obviously if for a coinfinite c.e. set A there is a weak tower $(H_n)_{n\geq 0}$ with $A \subseteq H_0$ such that (2.1) is satisfied then A is r-maximal (not necessarily atomless). Interesting is the case when for A there is a weak tower which is c.e., i.e. $(H_n)_{n\geq 0}$ – the weak tower – is an c.e. sequence.

Lemma 5.7 *Let A be a r-maximal set. Then A has a weak c.e. tower $(H_n)_{n\geq 0}$ with $A \subseteq H_0$ satisfying* (2.1) *iff* $\mathrm{COF}_{\bar{A}} \leq_T \emptyset''$.

Proof. $\Rightarrow$ Suppose A has an c.e. weak tower $(H_n)_{n\geq 0}$ with $A \subseteq H_0$ and (2.1) is satisfied. Then we have

$$A \cup W_e \neq^* I\!N \quad \text{iff} \quad (\exists\, n)(W_e \subseteq^* H_n),\ e \geq 0.$$

Thus "$A \cup W_e \neq^* I\!N$" belongs to Σ_3^0 and thus $\mathrm{COF}_{\bar{A}}$ to Δ_3^0.
$\Rightarrow$ Suppose $\mathrm{COF}_{\bar{A}} \in \Delta_3^0$. Let λ be a computable function with

$$e \in \mathrm{COF}_{\bar{A}} \quad \text{iff} \quad (\forall\, i)(\lambda e(i) < \omega),\ e \geq 0.$$

Let $U_{e,i}$ be the set

$$\bigcup\{W_{e,s} : \lambda_e(i, s+1) \neq \lambda_e(i,s),\ s \geq 0\}.$$

We see that for $e \notin \mathrm{COF}_{\bar{A}}$ there is an i with $\lambda_e(i) = \omega$, hence $U_{e,i} = W_e$ and for $e \in \mathrm{COF}_{\bar{A}}$ then for all e and i $U_{e,i}$ is finite. Let for $x \geq 0$ V_x be equal to $\bigcup\limits_{\langle e,i\rangle \leq x} U_{e,i} \cup A$. Then $(V_x)_{x\geq 0}$ is an c.e. sequence and is a tower for A with $A \subseteq V_0$ satisfying (2.1). □

Thus if for $A \in R - \mathrm{Max}\,\mathrm{COF}_{\bar{A}}$ then A has a c.e. weak tower. But the existence of a weak c.e. tower for A implies the existence of a computable function f such that $(W_{f(e)})_{e\geq 0}$ "bounds" every coinfinite c.e. superset of A by condition (2.1), what together with the effectivity presented isomorphism between major subset intervals, shown by Maass and Stob, would gives a good tool for charactering $\mathcal{L}^-(A)$.

Sorely not every r-maximal set has a c.e. weak tower what from the next theorem in connection with Lemma 5.7 follows.

Theorem 5.8 (Nies, Lempp, Solomon) *There is an r-maximal set A with* $\mathrm{COF}_{\bar{A}}$*–Σ_3-complete.*

Proof. Let Q be a Σ_3-predicate. We construct an r-maximal set A and a c.e. sequence of sets $(C_e)_{e\geq 0}$ such that

$$Q(e) \Longleftrightarrow A \cup C_e =^* I\!N.$$

In the first part of the proof we describe the construction of the set A. For doing this at first we give the basic idea of this construction. After it A will be constructed. This will be an infinitely many simultaneous application of the slightly modified basic method. This approach seems to be good for the understanding of the general construction. In the second part of the proof the description of $(C_e)_{e\geq 0}$ is given.
Let $(R_e^0, R_e^1)_{e\geq 0}$ be a duple c.e. sequence with

- $R_e^0 \cup R_e^1 = I\!N$ or $R_e^0 \cup R_e^1$ is finite,
- $R_e^0 \cap R_e^1 = \emptyset,\quad e \geq 0,$

– $(\forall e)(W_e - \text{computable} \Rightarrow (\exists f)(W_e = R^0_f))$.

(Basic construction). In the following we construct a coinfinite set C and infinitely many pairwise disjoint subsets M_i of $I\!N$, $i \geq 0$, with $\bigcup\{M_i : i \geq 0\} = I\!N$ and for some e all sets M_i, $i < e$, are finite, $M_e - C$ is infinite and for all $j > e$ $M_j - C$ are empty. (In the general construction every M_i becomes decomposed in the same way into sets M_{ij}, $j > i$, every set M_{ij} into sets M_{ijk}, $k > j$, and so on.)
We construct C stepwise and together with C a function T such that T_s gives a finite list of elements of $M_{i,s}$ at step s. The number of these lists also is finite. Thus

$$\{(i,j) : T_s(i,j)\downarrow\} - \text{ finite}, \quad T_s(i,j)\downarrow \wedge j' \leq j \Rightarrow T_s(i,j')\downarrow .$$

Hence $\{j : T_s(i,j)\downarrow\}$ is a finite initial part of $I\!N$. The greatest j such that $T_s(i,j)\downarrow$ (if $T_s(i,0)\downarrow$) is of particulary interest. We denote it with $x_{i,s}$.

Step 0. Define $C_0 = \emptyset$, $x_{0,0} = 0$, $T_0(0, x_{0,0}) = 0$. (For all other pairs (i,j) $T_0(i,j)\uparrow$.)

Step $s+1$. We have C_s, numbers $x_{0,s}, x_{1,s}, \ldots, x_{s,s}$ and T_s injective with $T_s(i,j) \Leftrightarrow i \leq s \wedge j \leq x_{i,s}$.
Look if there is an $e \leq s$ such that

$$[0, T_s(e, x_{e,s})] \subseteq R^0_{e,s} \cup R^1_{e,s}. \tag{5.1}$$

- If not let all unchanced and define $x_{s+1,s+1} = 0$ and $T_{s+1}(s+1, 0)$-equal to a fresh element (i.e. equal to a number which is not in C_{s+1} and is not one of the numbers $T_s(i,j)$, $(i,j) \in I\!N$, for which $T_s(i,j)\downarrow$.
- If yes let e be the smallest such number and do the following:
  (i) Put all numbers $z \leq \max\{T_s(i,j) : T_s(i,j)\downarrow\}$ which are not of the form: $T_s(i,k)$, $i \leq e$, $k \leq x_{i,s}$ to C_{s+1}.
  (ii) If $T_s(e, x_{e,s}) \in R^0_{e,s}$ than put additionally all numbers $T_s(e,k) \in R^1_{e,s}$ $(k < x_{e,s})$ to C_{s+1}.

Now define

$$\begin{aligned}
x_{i,s+1} &= x_{i,s} : i < e \\
x_{i,s+1} &= 0 : e < i \leq s+1 \\
x_{i,s+1}\uparrow & \quad \text{for } i > s+1 \\
x_{e,s+1} &= x_{e,s} + 1 \quad \text{if } T_s(e, x_{e,s}) \in R^1_{e,s} \\
&= |\{k : T_s(e,k) \in R^0_{e,s},\ k \leq x_{e,s+1}\}| + 1 \ \text{ if } T_s(e, x_{e,s}) \in R^0_{e,s}.
\end{aligned}$$

Further

$$\begin{aligned}
T_{s+1}(i,k) &= T_s(i,k), \quad i < e,\ k \leq x_{i,s} \\
T_{s+1}(e,i) &= T_s(e,i), \quad i < x_{e,s+1} - 1 \\
&= T_s(e, x_{e,s}),\ i = x_{e,s+1} - 1 \\
T_{s+1}(i, x_{i,s+1}) &- \text{ a fresh element for every } i \text{ with } e \leq i \leq s+1.
\end{aligned}$$

(Of course different pairs become assigned different fresh numbers; thus T_{s+1} is injective.)

Result. Let $C = \bigcup\{C_s : s \geq 0\}$. We see that C is a c.e. set.

1. Let $f = (\mu z)(R_z^0 \cup R_z^i = I\!N)$. Then for $e < f$ (1) can happen only finitely often. Let s_0 be such that for all $s \geq s_0$ (1) does not hold for an $e < f$.
Let $x_i =_{\text{df}} \lim_s x_{i,s}$, $i < f$. Then for $i < f$ $T(i,j) =_{\text{df}} \lim_s T_s(i,j)$ exists iff $j \leq x_i$. But this are only finitely many pairs. For $s > s_0$ $T_s(f,l)$, $l < x_{f,s}$ cannot come to C_{s+1} by (i), but only by (ii). Hence $T_s(f,l) \neq T_{s+1}(f,l)$ can be at most one time. Further by definition of f (1) holds infinitely often for $e = f$ and $x_{f,s}\downarrow$ for all $s \geq f$. Hence $\lim_s x_{f,s} = \infty$. Let $T(f,l) =_{\text{df}} \lim_s T_s(f,l)$. Then $T(f,l) \notin C$ for all $l \geq 0$. Hence C is coinfinite. For $e > f$ $x_{e,s} = 0$ for infinitely many s and $\lim_s T_s(e, x_{e,s})\uparrow$.

2. From construction case (ii) we get

$$\bar{C} \subseteq^* R_f^0 \quad \text{or} \quad \bar{C} \subseteq^* R_f^1.$$

3. We see that the numbers $T_s(i, x_{i,s})$ for $i \leq s$ $(x_{i,s}\downarrow)$ are the onliest among $T_s(i,j)$ with $T_s(i,j)\downarrow$ which can chance their places, i.e. that can be $T_s(i, x_{i,s}) = T(i,l)$, $l < x_{i,s}$ for some $t > s$. For all other numbers $T_s(i,j)$, $j < x_{i,s}$ either they are putted into C or for all $t > s$ $T_s(i,j) = T_t(i,j)$.
(The places of the $T_s(i, x_{i,s})$'s are still not determined.)

4. The sets M_i about which was spoken in the beginning of the construction are equal to

$$\bigcup_{s \geq i}\{T_s(i,k) : 0 \leq k \leq x_{i,s}\}.$$

(The sets M_i "measure" the growth of the set s $R_{i,s}^0 \cup R_{i,s}^1$.)

More interesting are other sets which we denote with X_j. Let X_j be equal to

$$\bigcup_{s \geq 0}\bigcup_{i \geq 0}\{T_s(i,j) : T_s(i,j)\downarrow \wedge T_s(i,j) \neq T_s(i, x_{i,s})\}.$$

$(X_j)_{j \geq 0}$ is a c.e. sequence of disjoint sets with $X_j \cap \bar{C}$-finite, $X_0 \cap \bar{C} \neq \emptyset$ ($X_i \cap \bar{C} \neq \emptyset$ for every $i \geq 0$, but we do not need it) and $\bar{C} \subseteq^* \bigcup\{X_i : i \geq 0\}$. (We have only $\subseteq^*$, since the numbers $T(i, x_i)$, $i < f$ are not in the union.)

(General construction, construction of A**)**. Let Γ be the set

$$\{\sigma \in I\!N^{<\omega} : \sigma \neq \langle\ \rangle,\ \sigma = (\sigma_0\sigma_1 \ldots \sigma_{n-1}),\ \sigma_0 < \sigma_1 < \ldots < \sigma_{n-1}\}.$$

Further let $\theta_0, \theta_1, \ldots$ be a computable sequence of the elements of Γ such that every element of Γ appears infinitely often in this sequence.
We construct a function T_s, set A_s^n, $n \geq 0$ with $A_n^0 \subseteq A_n^1 \subseteq \ldots$ (A_s^0 has the same meaning as C_s from the basic construction and A_s^n for the similar construction on level n) and numbers $x_{\sigma,s}$, $\sigma \in \Gamma$. (In every step s only finitely many of them are defined.)

Step 0. Define $A_0^n = \emptyset$ for all $n \geq 0$ and all other objects, i.e. $x_{\sigma,0}$ and $T_0(\sigma, i)$ are undefined.

Step $s+1$. We have A^n_s, $n \geq 0$, $x_{\sigma,s}$ for finitely many σ's from Γ and

$$x_{\sigma,s}\downarrow \Rightarrow T_s(\sigma, i)\downarrow \quad \text{for} \quad i \leq x_{\sigma,s}.$$

(The converse, i.e. for some $i \leq x_{\sigma,s}$ $T_s(\sigma, i)\downarrow \Rightarrow x_{\sigma,s}\downarrow$ in general is not true.) Further we assume that $T_s(\sigma, \ )$ is injective in σ.

Take θ_s. (In the following we write only θ for θ_s.) Look if $x_{\theta,s}\downarrow$.

– If not, look if there is a fresh element on

$$(5.2) \qquad \{T_s(\theta^-, 1), T_s(\theta^-, 2), \ldots, T_s(\theta^-, l)\} - A_s^{|\theta|-1},$$

where l is the greatest number for which $T_s(\theta^-, l)\downarrow$ and lesser than $x_{\theta^-,s}$ (if $x_{\theta^-,s}\downarrow$) [6].
"Fresh" means not equal to $T_s(\theta^- * i, j)$ for no i and j.
If there is a fresh element let $x_{\theta,s+1}$ be equal to

$$\max\{l : T_s(\theta, l)\downarrow\} + 1$$

and $T_{s+1}(\theta, x_{\theta,s+1})$ be this fresh element. If there is no fresh element go to the next step.

– If $x_{\theta,s}\downarrow$ then look if

$$(5.3) \qquad [0, T_s(\theta, x_{\theta,s})] \subseteq R^0_{e,s} \cup R^1_{e,s},$$

where $\theta = \theta^- * e$. If not go to the next step. If yes then

(iii) Take all z with

$$z \leq \max\{T_s(\theta^- * i, j) : T_s(\theta^- * i, j)\downarrow\}$$

which are not of the form $T_s(\theta^- * i, k)$, $i \leq e$, $\theta^- * i \in \Gamma$ to A^n_{s+1} for all $n \geq |\theta^-|$.

(iv) If $T_s(\theta, x_{\theta,s}) \in R^0_{e,s}$ then take additionally all $T_s(\theta, k) \in R^l_{e,s}$ $(k \neq 0)$ to A^n_{s+1} $(n \geq |\theta^-|)$. ($k \neq 0$ above will be ensure that A will be coinfinite.)

Now define

$$x_{\theta,s+1} = x_{\theta,s} + 1 \qquad \text{if } T_s(\theta, x_{\theta,s}) \in R^l_{e,s} \text{ and there is a fresh element in } \{T_s(\theta^-, i) : 0 < i < x_{\theta^-,s}\} \text{ for } T_{s+1}(\theta, x_{\theta,s+1}).$$

If by (iii) a number $T_s(\sigma, i)$ comes to A^n_{s+1} then $T_{s+1}(\sigma, i)\uparrow$ (except that $\sigma = \theta$ and l becomes $x_{\theta,s+1}$. We see that if this holds for $T_s(\sigma, i)$ then also for all $T_s(\sigma, j)$ with $i < j$ for which $T_s(\sigma, j)\uparrow$. Thus after every step for every $\nu \in \Gamma$ $T_{s+1}(\nu, \cdot)$ is a initial part of $\mathbb{N}$.

Result. Let $A^n = \bigcup_{s \geq 0} A^n_s$ and $A = \bigcup_{n \geq 0} A^n$.

1. We have $A^0 \subseteq A^1 \subseteq \ldots \subseteq$ and all sets and A are c.e. By the construction $(A^n_s)_{n,s \geq 0}$ is a strong c.e. sequence of finite sets.

2. Let $e_0 < e_1 < \ldots$ be all indices in order of magnitude such that $R^0_i \cup R^1_i = \mathbb{N}$ iff i is one of the e_j's.

[6] θ^- means the immediate predecessor of θ. If $\theta^- = \langle\ \rangle$ instead of the set (5.2) take the set $\mathbb{N} \setminus A^0$, and without the restriction for l.

Then for $\sigma = (e_0 \dots e_k e)$ with $e < e_{k+1}$ the requirement (5.3) can happen only finitely often, but for every $m \geq 0$ and $(e_0 \dots e_m)$ infinitely often. For $\sigma \in \Gamma$ let M_σ be

$$\{T_s(\sigma, k) : T_s(\sigma, k)\downarrow,\ s \geq 0,\ k \geq 0\}.$$

Then for σ and $\sigma * e \in \Gamma$ we have $M_{(\sigma e)} \subseteq M_\sigma$. Thus only the sets $M_{(e^0)}, M_{(e^0 e^1)}, \dots$ are infinite inside $\bar{A}$ (even more include $\bar{A}$ (mod=*)). Further for every $k \geq 0$ and $m \geq 0$ $\lim_s T_s((e_0 \dots e_m), k)$ and belongs to $\bar{A}$.

Construction of C_e, $e \geq 0$.
Having the Σ_3-predicate Q we can find a c.e. sequence $(U_{e,i,s})_{e,i,s \geq 0}$ with $U_{e,i,s}$ a finite initial part of $I\!N$ (weakly) increasing in s (let $U_{e,i} =_{\text{df}} \lim_s U_{e,i,s}$), such that

$$Q(e) \iff (\exists\, i)(Q_{e,i} = I\!N).$$

The generalization of the sets X_j from the basic construction are the sets $X_k^{(n)}$, $n \geq 1$, $k \geq 0$. Let $X_k^{(n)}$ be the sets

$$\bigcup_{s \geq 0} \bigcup_{i_1 \geq 0} \bigcup_{i_2 > i_1} \bigcup_{i_n > i_{n-1}} \{T_s((i_1 \dots i_n), k) : \text{ for } \sigma = (i_1 i_2 \dots i_n), \\ T_s(\sigma, k)\downarrow \wedge T_s(\sigma, k) \neq T_s(\sigma, x_{\sigma,s}) \text{ if } T_s(\sigma, x_{\sigma,s})\downarrow\}.$$

(Thus $X_j = X_j^{(l)}$, $j \geq 0$).
The sets $X_k^{(n)}$ have following properties:

1° $X_0^{(n)} \cap \bar{A} \neq \emptyset$.

2° $X_j^{(n)} \cap \bar{A}$-finite.

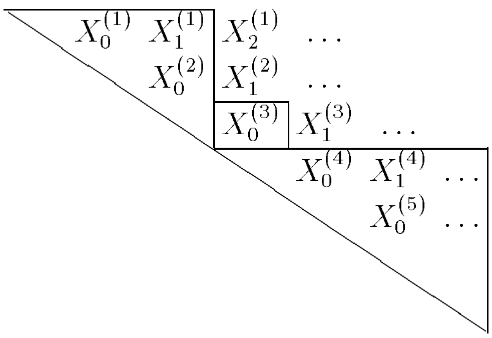
3° $X_j^{(n)} \cap X_i^{(n)} = \emptyset \quad (i \neq j)$ for all $n \geq 1$ and $i, j \geq 0$.

4° $\bar{A} \subseteq^* \bigcup \{X_j^{(n)} : j \geq 0\}$.

5° $X_j^{(n)} \cap \bigcup \{X_j^{(m)} : j + m < n\} \cap \bar{A} = \emptyset$.

6° $X_0^{(n)} \cap X_j^{(m)} \cap \bar{A} = \emptyset$ for all $j \geq 0$ and $m > n$.

We have the schema:



The sets in the triangles and $X_0^{(3)}$ have no common elements in $\bar{A}$. (But $X_0^{(3)} \cap \bar{A} \neq \emptyset$.)

Fix e. In the following we write C_s and $U_{i,s}$ for $C_{e,s}$ and $U_{e,i,s}$ respectively. In step $s+1$ we have a sequence of numbers $0 < a_{0,s} < a_{1,s} < \ldots < a_{s,s}$ with $|U_{i,s}| < a_{i,s}$, $i \leq s$, $a_{i,s} \leq a_{i,s+1}$ and $|U_i| < \infty \Leftrightarrow \lim_s a_{i,s}$. Let a_i be the limit if it exists. Let C_s be

$$(5.4) \qquad \bigcup_{i+1=0,\ldots,s} \bigcup_{k,j} \{X_k^{(j)} : a_{i,s} < k+j < a_{i+1,s}\} \quad (a_{-1} =_{\mathrm{df}} 0)$$

($C_0 =_{\mathrm{df}} \emptyset$). $C = \bigcup\{C_s : \ s \geq 0\}$.
If $a_{i,s}\uparrow \infty$, but $a_{i-1}\downarrow$ then $\bigcup\{X_{i-1}^{(a+1)} : k \geq 0\} \subseteq C$, see (5.4). Hence $\bar{A} \subseteq^* C$. If for all $i \quad a_i\downarrow$ then $(\bigcup_{i\geq 0} X_{i_0}^{(a)} \setminus A) \cap C = \emptyset$ and $\bigcup_{i\geq 0} X_{i_0}^{(a)} \setminus A$ is infinite. Hence $A \cup C_e \neq^* I\!\!N$. $\square$

Question 5.9 *Exists for every r-maximal set A an r-maximal set B with a c.e. weak tower such that $\mathcal{L}(A) \cong \mathcal{L}(B)$ (or even more $A \cong_{\mathcal{E}} B$)?*

6 r-Maximal sets, co-Monotone and co-1-1 sets

In [MadRob82] there were investigated notions which are closely related with the class $R-\mathrm{Max}$; more precisely which form a subclass of $R-\mathrm{Max}$. These notions are:

Definition 6.1 Let $X \subseteq I\!\!N$ be infinite. X is called *monotonic* (1–1) if

$$(6.1) \qquad \begin{aligned} &(\forall\, f \text{ rec. fct})\Big[(\exists\, n)(\forall\, m, m' \geq n)(m, m' \in X,\ m \leq m' \\ &\quad \Rightarrow f(m) \leq f(m')) \vee f \text{ is constant on } X \ (\mathrm{mod} =^*)\Big]. \end{aligned}$$

$$(6.2) \qquad \begin{aligned} &\Big((\forall\, f \text{ rec. fct})[(\exists\, n)(\forall\, m, m' \geq n)(m, m' \in X,\ m \neq m' \\ &\quad \Rightarrow f(m) \neq f(n)) \vee f \text{ is constant on } X \ (\mathrm{mod} =^*)]\Big). \end{aligned}$$

We are interested here in coinfinite c.e. sets having monotonic or 1–1 complements. The following facts hold, see [MadRob82]:

1) A coinfinite c.e. set has a monotonic complement iff it has a 1–1 complement.

2) Every maximal set has a monotonic (and thus by 1)) also a 1–1 complement (Owings).

3) Every infinite set which is monotonic or 1–1 is dense immune. Thus together with 1) and 2) we get that the T-degrees of the monotonic sets as also of the 1–1 sets is equal to $I\!\!H^1$.

6.1 The position of the co-monotonic sets inside $R-\mathrm{Max}$

Denote with Co-Mon (Co–1–1) the class of coinfinite c.e. sets with monotonic (1–1) complements. From 1) we get Co–Mon=Co–1–1, from 2) Max $\subseteq$ Co–Mon and from 3) Co–Mon$\subseteq \mathcal{DS}$ (the class of dense simple sets).

Lemma 6.2 *Every A from* Co–Mon *belongs to* $R-\mathrm{Max}$.

Proof. Let X be an infinite set not r-cohesive. Then there is a computable set R such that $R \cap X$ and $\bar{R} \cap X$ are both infinite. Let $f(x) = 0$ for $x \in R$ and $f(x) = 1$ for $x \in \bar{R}$. Then f is neither monotonic (mod $=^*$) on X, nor 1–1 (mod $=^*$) on X. Thus for a coinfinite c.e. set A if $\bar{A}$ is not r-cohesive then $\bar{A}$ is not monotonic and not 1–1.
□

In the following Lemma 6.3 we need a fact also shown in [MadRob82]:

4) Let W be an c.e. set and $(S_n)_{n\geq 0}$ be an c.e. sequence of disjoint sets. Then

- a) There is a disjoint c.e. sequence $(T_n)_{n\geq 0}$ with $\bigcup\limits_{n\geq 0} T_n = \bigcup\limits_{n\geq 0} S_n$ and $T_n \cap \bar{W} \neq \emptyset$ for all n or
- b) there is a computable function g such that for almost all n

$$(\forall\, x)(x \in S_n \cap \bar{W} \Rightarrow x \leq g(n)).$$

Lemma 6.3 (Madan, Robinson) *If $Y \in$ Co–Mon and $X \subset_{\mathrm{m}} Y$ then $X \in$ Co–Mon.*

Proof. Let f be a computable function not constant (mod $=^*$) on $\bar{Y}$ and $S_n = f^{-1}(\{n\})$, $n \geq 0$. Since $\bar{Y}$ is monotonic, $|S_n \cap \bar{Y}| \leq 1$ for almost all n and $S_n \cap \bar{Y}$ is finite for all n. If for an n_0 $S_{n_0} \cap \bar{Y}$ would be infinite, since f is not constant (mod $=^*$) on $\bar{Y}$, $f(x) = 0$ for $x \in S_{n_0}$ and $f(x) = 1$ for all other x would contradict that $\bar{Y}$ is monotonic.
Let n_0 be such that for $n \geq n_0$ $|S_n \cap \bar{Y}| \leq 1$. Then we have

$$\bar{Y} \subseteq^* \bigcup_{n\geq 0} S_n \quad \text{and} \quad \bar{Y} \quad \text{is } r\text{-cohesive, by Lemma 6.2.}$$

Thus by fact 4) there is a computable function g s.th.

$$(\forall\, n)(\forall\, x)(x \in S_n \cap \bar{Y} \Rightarrow x \leq g(n)).$$

Thus for all $n \geq n_0$

$$(6.3) \qquad |S_n \cap \bar{Y}| \leq 1 \land x \in S_n \cap \bar{Y} \Rightarrow x \leq g(n).$$

Let x_n be the last enumerated element from S_n with $x \leq g(n)$, not earlier enumerated into Y, for $n \geq n_0$. Let be Ω the set of all these x_n's. Then Ω is c.e. and $\bar{Y} \subseteq^* \Omega$. Thus $\bar{X} \subseteq^* \Omega$ and f is 1–1 on Ω, since $f(x_n) = n$. Hence f is 1–1 (mod $=^*$) on $\bar{X}$.
If f is constant (mod $=^*$) on $\bar{Y}$, i.e. $\bar{Y} \subseteq^* f^{-1}(\{k\})$ for some k then also $\bar{X} \subseteq^* f^{-1}(\{k\})$, since $f^{-1}(\{k\})$ is c.e. Both together give that $\bar{X}$ is an 1–1 set. □

Corollary 6.4 *If $A \in \mathcal{MS}_{\text{Max}}$ then $A \in$ Co–Mon.*

Proof. Since every maximal set is co-monotonic, see fact 2), from Lemma 6.3 we get it also for its major subsets. □

Remark. From Lemma 6.3 we can conclude that all elements in an equivalence class resp. to $\approx_{\text{ms}}$ are co-monotonic or none is it.
Suppose $X \in$ Co–Mon and $Y \in [X]/\approx_{\text{ms}}$. Then $Y \approx_{\text{ms}} X \cup Y$, hence $\bar{Y} \subseteq^* \bar{X}$, what implies that $\bar{Y}$ is monotonic or $Y \subset_{\text{m}} X \cup Y$. But $\overline{X \cup Y} \subseteq \bar{X}$, hence is monotonic and thus $Y \in$ Co–Mon, by Lemma 6.3.

$*$

The next question concerns the relationship between Co–Mon and $R-$ Max: Holds Co–Mon $= R -$ Max$_{\text{atm}}$ or not? We get an answer by comparing both index sets.

Lemma 6.5 *$\{e : W_e \in$ Co–Mon$\}$ is Π^0_4-complete.*

Proof. Since Max $\subseteq$ Co–Mon $\subseteq R -$ Max and obviously $R -$ Max $\subseteq fs\mathcal{HS}$, by using the fact in 5.1 we need only to show that "$W_e \in$ Co–Mon" has a Π^0_4-definition. Considering (6.1) we have

$$\bar{W}_e - \infty \wedge \forall f[(\varphi_f - \text{total} \Rightarrow \exists n \forall m, n'(m, m' \notin W_e,\, m < m' \Rightarrow \varphi_f(m) \leq \varphi_f(m')) \\ \vee \exists a \exists n \forall m > n(m \notin W_e \Rightarrow \varphi_f(m) = a)].$$

This is $\Pi^0_3 \wedge \forall[\Pi^0_2 \to \exists\forall(\forall \to \exists) \vee \exists\forall\exists] \equiv \Pi^0_4$. □

Corollary 6.6 *$R -$ Max$_{\text{atm}} \neq$ Co–Mon.*

Proof. The index set of $R -$ Max$_{\text{atm}}$ is Σ^0_5-complete, see Theorem 5.3 and that of Co–Mon Π^0_4-complete, see Lemma 6.5. Hence both classes cannot coincide. □

Corollary 6.7 *There are atomless, r-maximal sets which are co-monotonic.*

Proof. From Lemma 6.2 and Corollary 6.6. □

Fact 3) and Lemma 6.2 together give the inclusion Co–Mon $\subseteq R -$ Max $\cap\, \mathcal{DS}$. This inclusion was more intensively investigated in [MadRob82] as the following theorem shows:

Theorem 6.8 (Madan, Robinson) *There is an r-maximal and dense simple set W, which is not co-monotonic.*

Proof. The construction of W will be a generalization of that in Theorem 2.4. Now we do not fix at the beginning a finitely strong c.e. sequence for the whole construction as $(E_i)_{i\geq 0}$ in Theorem 2.2 and in the steps we construct $F_{i,s}$ with $F_{i,s+1} \subseteq F_{i,s} \subseteq E_i$, but here it can happen that for some s $F_{i,s+1}$ is defined new in such a way that $F_{i,s+1} \cap F_{i,s} = \emptyset$. Since the elements of $\bar{W}$ are included into $F_{i,s}$, $i \geq 0$, if $F_{i,s}$ consists

only of sufficiently large elements, $\bar{W}$ will be dense immune. The constuction method from Theorem 2.2 and this "$F_{i,s}$-chance principle" together give that W is atomless, r-maximal and dense simple.
Since further the sets $F_{i,s}$, $i, s \geq 0$ are either included one in the other or are disjoint, the sequences are strongly finite and the sets $F_{i,s}$ have at least two elements, it can be defined a recursicve function decreasing inside every $F_{i,s}$. This negates that $\bar{W}$ is monotonic.
Let $r_s(n)$ be the function:

$$\begin{array}{ll} \max\{\varphi_{e,s}(n): & \text{For all } e < n \text{ for which } (\forall\, x \leq n)(\varphi_{e,s}(n)\downarrow) \text{ is satisfied}\} \\ 0 & \text{If such } e \text{ does not exist.} \end{array}$$

Further let $(A_n)_{n\geq 0}$ be a computable decompositon of $\mathbb{N}$ into infinite, pairwise disjoint computable sets with $\bigcup\limits_{n\geq 0} A_n = \mathbb{N}$. At every step s we define an (increasing) function $y_s(n)$ with the meaning: $F_{n,s} \subseteq A_{y_s(n)}$.
We say that n *requires attention at step* $s+1$ *by* e if

$$e < n \wedge F_{n,s} \not\subseteq W_{e,s+1} \wedge |F_{n,s}| \leq 2 \cdot |F_{n,s} \cap W_{e,s+1}|.$$

Let $\mathrm{st}_R(x, e, s)$ be the special $e+1$ state of x at step $s+1$ defined by

$$\begin{aligned} \mathrm{st}_R(x,e,s)(j) &= 1 \quad \text{if } F_{x,s} \subseteq W_{e,s+1} \\ &= 0 \quad \text{otherwise}, \quad j \leq e. \end{aligned}$$

For two finite, nonempty sets $X, Y \subseteq \mathbb{N}$ we write $X < Y$, if $\forall x \in X\ \forall y \in Y$ $(x < y)$.

Construction.

Step 0:

- Let $y_0(n) = n$ for all $n \geq 0$.
- $F_0(0)$ – the set of the first two elements of $A_{y_0(0)}$ in order of magnitude.
  $F_0(n+1)$ the first 2^{n+2} elements of $A_{y_0(n+1)}$ in order of magnitude such that $F_0(n) < F_0(n+1)$.
- g will be defined on $\bigcup\limits_{n\geq 0} F_0(n)$ in such a way that for
  $F_0(n) = \{a_1 < a_2 < \ldots < a_{2n+1}\}$ $\quad g(a_i) = 2^{n+1} - i$, $i = 1, \ldots, 2^{n+1}$.
- Let $T_0(n)$ be the increasing enumeration of $\bigcup\limits_{n\geq 0} F_0(n)$ and $W_0 = \emptyset$.

Step $s+1$: If there is no n for which $T_s(n) < r_{s+1}(n)$ do nothing. Otherwise let $n_0 = (\mu n)(T_s(n) < r_{s+1}(n))$.
Now we do the following (in order (i) to (iv)):

(i) Suppose $T_s(n_0) \in F_s(m_0) \subseteq A_{y_s(m_0)}$ for some m_0. For every $m \geq m_0$ define new sets $\widehat{F}_s(m)$ inside $A_{y_s(m)}$ such that

– $x \in \widehat{F}_s(m_0) \Rightarrow r_{s+1}(n_0) < x$

(6.4) $\quad -|\widehat{F}_s(m)| = 2^{m+1}$

– $\widehat{F}_s(m) < \widehat{F}_s(m+1)$

– $\widehat{F}_s(m)$ consists of elements which are "new", i.e. are not in dom(g), not in any $W_{e,s}$, not in W_s, not in $F_s(m)$.

(ii) If $(\exists\, n < m_0)(\exists\, e < n)$ (n gets attention by e at step $s+1$) then take the smallest such n (let be this n_1) and for this the smallest e (let be this e_1) and define

$$\begin{aligned} \widehat{F}_s(k) &= F_s(k) \cap W_{e_1,s} \ : \ k = n_1 \\ &= F_s(k) \qquad\quad : \ 0 \le k < m_0,\ k \ne n_1. \end{aligned}$$

(iii) If $(\exists\, m)(\exists\, n)(\exists\, e)[e \le m < n < m_0$ [7] $\wedge\ \mathrm{st}_R(m,e,s) < \mathrm{st}_R(n,e,s)]$ (where st_R is defined by means of $\widehat{F}_s(k)$ instead of $F_s(k)$) then m_1 be the smallest such m and n_2 be the smallest n for m_1. Define

$$\begin{aligned} y_{s+1}(m_1+k) &= y_s(n_2+k), \quad k \ge 0 \\ y_{s+1}(k) &= y_s(k) : \quad k < m_1 \\ F_{s+1}(k) &= \widehat{F}_s(k) : \quad k < m_1 \\ F_{s+1}(m_1+k) &- \text{ the first } 2^{m_1+k+1} \text{ elements} \qquad (6.5) \\ &\quad\ \text{of } \widehat{F}_s(n_2+k) \ : \ \text{if } |\widehat{F}_s(n_2+k)| > 2^{m_1+k+1} \\ &= \widehat{F}_s(n_2+k) \ : \ \text{otherwise, } k \ge 0. \qquad (6.6) \end{aligned}$$

(iv) – Put into W_{s+1} all $x \le s$ with $x \notin \bigcup_{n=1}^{\infty} F_{s+1}(n)$ and all x from W_s.

– Let T_{s+1} be an increasing enumeration of $\bigcup_{n \ge 0} F_{s+1}(n)$.

– Define $g(x) = 0$ for $x \in W_{s+1}$ for which g was not still defined and if $F_{s+1}(n)$ if not still defined define g similar as in step 0.

Result. At first we see that step $s+1$ infinitely often happens (i.e. there is an n with $T_s(n) < r_{s+1}(n)$). After every step the function T_s is computable. Hence for some $s' \ge s$ and some n $T_{s'}(n) < r_{s'+1}(n)$ and thus the construction (i) to (iv) will be taken place.

In the construction step $s+1$ we start with y_s, F_s, g_s, T_s, W_s and after

(i) we have y_s, $\widehat{F}_s(m)$, $m \ge m_0$, g_s, T_s, W_s

(ii) we have y_s, $\widehat{F}$ for all m, g_s, T_s, W_s

[7] The requirement "$n < m_0$" is not necessary, since for $n \ge m_0$ $\mathrm{st}_R(n,e,s) = 0$ by (i) and (ii) and thus $\mathrm{st}_R(m,e,s) < \mathrm{st}_R(n,e,s)$ cannot hold.

(iii) we have y_{s+1}, F_{s+1}, g_s, T_s, W_s

(iv) we have y_{s+1}, F_{s+1}, g_{s+1}, T_{s+1}, W_{s+1}.

1. $(\forall\, n)(\exists\, s)(\forall\, s' > s)(y_{s'}(n) = y_s(n) \wedge F_{s'}(n) = F_s(n))$.
(By induction). Given n let s_0 be such that for $k < n$ and $s \geq s_0 \quad y_s(k) = y_{s_0}(k)$ and $F_s(k) = F_{s_0}(k)$.
Let $s_1 \geq s_0$ be such that for all $s \geq s_1 \quad r_s(k) = r_{s+1}(k)$ for all $k < 2^{n+2} - 1$. After step $s_1 \quad n$ maybe some $m \geq m_0$ in (i) on behalf of some $k < 2^{n+2} - 1$. But for every k at most once, since by (6.4) and (6.6) $\quad |\bigcup\limits_{i \leq n} F_s(i)| < 2^{n+2} - 1$ for all s.
Hence there is a step $s_2 \geq s_1$ such that for $s \geq s_2$ in step $s+1$ in part (i) always $n < m_0$.
Further we see that the parts (ii) and (iii) imply $T_s(n) < T_{s+1}(n)$, hence do not disturb that (i) is not satisfied for $k < 2^{n+1} - 1$.
Part (iii) can happen only by a greater n-state of which there are only finitely many. Thus $y_s(n) = y_{s_3}(n)$ for some $s_3 \geq s_2$ and all $s \geq s_3$. But $(F_s(n))_{s \geq s_3}$ is decreasing. Thus for some $s_4 \geq s_3 \quad F_s(n) = F_{s_4}(n)$ for all $s \geq s_4$.
Let $F(n) = \lim_s F_s(n)$ and $y(n) = \lim_s y_s(n)$, $n \geq 0$.

2. $(\forall\, n)(\forall\, s)(|F_s(n)| \geq 2)$.
Given n let s_0 be such that $F(k) = F_{s_0}(k)$ and $y(k) = y_{s_0}(k)$ for $k \leq n$. Thus $F(n) = A_{y(k)}$. For $s \geq s_0$ let $h(s)$ be such that $F_s(h(s)) \subseteq A_{y(n)}$. Then $h(0) = y(n)$ and $h(s_0) = n$ and h is decreasing.
(This means $F_{t_i}(h(t_i)) \subseteq A_{y(n)}$, $i = 0, 1, \ldots, k$).
Let $s_1 < s_0$ be the greatest number such that (i) holds for some $m \leq h(s_1)$ in step s_1. (If s_1 does not exist, let it be 0.)
Then at the end of step $s_1 \quad |F_{s_1}(h(s_1))| = 2^{h(s_1)+1}$, by (1) and for all s with $s_1 < s \leq s_0$ we have $F(n) \subseteq F_s(h(s)) \subseteq F_{s_1}(h(s_1))$.
The sets $F_s(h(s))$ can get attention by a number e at most one time. Then also $e < h(s)$ must be and $F_s(h(s))$ becomes divided at mostly by $\frac{1}{2}$.
Let s' be the greatest number with $s_1 \leq s' < s_0$ such that $h(s') < h(s'-1)$ and (3) holds for $F_{s'}(h(s'))$. Is such s' does not exists let $s' = s_1$. Then $|F_{s'}(h(s'))| = 2^{h(s')+1}$ and for all $s'' > s'$ only (ii) holds for $F_{s''}(h(s''))$ by some $e < h(s'') \leq h(s')$ (and at most once). Thus $|F_{s_0}(h(s_0))| \geq \dfrac{1}{2^{n(s')}} \cdot 2^{h(s')+1} = 2$. hence $|F(n)| \geq 2$.

3. W ist r-maximal.

Suppose R is computable and W_{e_0}, W_{e_1} are R and $\bar{R}$ respectively. By the maximalization of the e-states for n with $e < n$ either for almost all $n > e$ and all $e \quad F(n) \subseteq W_e$ or $|W_e \cap F(n)| < \frac{1}{2} \cdot |F(n)|$.
If for both W_{e_0} and W_{e_1} the second case hold then $|W_{e_0} \cap F_n| < \frac{1}{2}|F_n|$ and $|W_{e_1} \cap F_n| < \frac{1}{2}|F_n|$ for almost all n. But this gives $(W_{e_0} \cup W_{e_1}) \cap \bar{A} \neq^* \bar{A}$. Thus for one set the first case must hold, i.e. $F(n) \subseteq W_{e_0}$ for almost all n. Hence $\bar{A} \subseteq^* W_{e_0}$. Similar if $F(n) \subseteq W_{e_1}$ for almost all n. This gives that $\bar{A}$ is r-cohesive. $\square$

6.2 Classification of the class $R-\mathrm{Max}$

In [Sh85] a special property of subsets of $I\!N$ was investigated which is comparable with the notions considered here. This is:

Let $\mathcal{D}$ be the class of computable functions f such that $(I\!D_{f(n)})_{n\geq 0}$ is a sequence of pairwise disjoint sets with $\bigcup\limits_{n\geq 0} I\!D_{f(n)} = I\!N$.

For $X \subseteq I\!N$ let $\overline{\sup}(X)$ be

(6.7) $$\sup_{f\in\mathcal{D}}(\lim_{n\to\infty}(\sup_{m>n}(|I\!D_{f(m)}\cap X|))).$$

Lemma 6.9 1) *(Schmidt) If $A \in R-\mathrm{Max}$ then $\overline{\sup}(\bar{A})$ is equal to 1 or to ∞.*

2) *If $A \in$ Co–Mon then $\overline{\sup}(\bar{A}) = 1$. (Thus in particulary for $A \in \mathrm{Max}$ or $A \in \mathcal{MS}_{\mathrm{Max}}$ $\overline{\sup}(\bar{A}) = 1$.)*

3) *If A is a coinfinite c.e. set and $A \notin \mathcal{DS}$ then $\overline{\sup}(\bar{A}) - \infty$.*

4) *There is $A \in R-\mathrm{Max}$, $A \in \mathcal{DS}$ and $\overline{\sup}(\bar{A}) - \infty$.*

Proof. 1) Suppose $\overline{\sup}(\bar{A}) = n > 1$ for an c.e. set A. Let $(I\!D_{f(k)})_{k\geq 0}$ be a sequence s.th. (6.7) is equal to n. Then we can find a pair of computable sets R_0, R_1 with $R_1 = \bar{R}_0$ s.th. $R_0 \cap \bar{A}$, $R_1 \cap \bar{A}$ are both infinite. Hence A is not r-maximal.

2) Suppose for $A \in R-\mathrm{Max}$ $\overline{\sup}(\bar{A}) - \infty$. Let $I\!D_{f(n)}$ be a sequence s.th. (6.7) is ∞. Then g defined by $g(x) > g(y)$ for $x, y \in I\!D_{f(n)}$ with $x < y$ is a computable function, but not monotone (mod $=^*$).

3) Let $(E_n)_{n\geq 0}$ be a finitely strong c.e. sequence s.th.

(6.8) $$(\exists^\infty n)(|E_n \cap \bar{A}| \geq n).$$

Then $I\!D_{f(n)} = E_n$ shows that (6.7) is ∞.

4) This follows from Theorem 6.8 by a small modification of the construction there. In Theorem 6.8 we started with $|F_0(n)| = 2^{n+1}$ and when case (i) happens then again $|F_s(n)| = 2^{n+1}$ is claimed. But if we replace 2^{n+1} by $2^{n+1}\cdot n$ already in step 0 and all later we get in 2) of the result $(\forall\, n)(|F(n)| \geq 2\cdot n)$. Now let $(E_n)_{n\geq 0}$ be the sequence with $E_n = \bigcup F_s(n)$ s.th. for $s' < s$ $F_s(n) \cap F_s(n) = \emptyset$ together with $F_0(n)$. (Every set $F_s(n)$ is a member of E_n.) $(E_n)_{n\geq 0}$ is a strong finite c.e. sequence for which (6.7) gives ∞. Further the set is dense simple and r-maximal, but not co-monotonic. $\square$

Remark. The set constructed in 4) of Lemma 6.9 is dense simple, since (6.8) does not hold. We have only $|E_m \cap \bar{A}| \geq n$ for sufficiently large m, but not for $m = n$.

Thus we have the following classification of $R-\mathrm{Max}$ in respect to the notions considered here:

Denote with $\overline{\sup}_1$ the class of r-maximal sets A with $\overline{\sup}(\bar{A}) = 1$.

$$\mathrm{Max} \underset{\neq}{\rightarrow} \mathrm{Max} \cup MS_{\mathrm{Max}} \underset{\neq}{\rightarrow} \text{Co–Mon} \rightarrow \overline{\sup}_1 \underset{\neq}{\rightarrow} R-\mathrm{Max} \cap \mathcal{DS} \underset{\neq}{\rightarrow} R-\mathrm{Max}.$$

Question. Is the inclusion between Co–Mon and $\overline{\text{sup}}_1$ proper or not? Since $I\!IS(\overline{\text{sup}}_1)$ is Π^0_4-complete, see below, so as $I\!IS(\text{Co–Mon})$, see Lemma 6.5 the comparasion of the index sets of these classes gives no answer to this question.

We have "$W_e \in \overline{\text{sup}}_1$" if

$$(W_e \in R - \text{Max}) \wedge (\forall\, f)(\varphi_f \in \mathcal{D} \Rightarrow (\exists\, n)(\forall\, m > n)(|I\!D_{\varphi_f(m)} \cap \bar{W}_e| \leq 1)).$$

$$"\varphi_f \in \mathcal{D} \Leftrightarrow \varphi_f - \text{total} \wedge (\forall\, n)(\forall\, m)(I\!D_{f(n)} \cap I\!D_{f(m)} = \emptyset \wedge (\forall\, x)(\exists\, m)(x \in I\!D_{f(m)})$$

$$"|I\!D_{\varphi_f(m)} \cap \bar{W}_e| \leq 1 \Leftrightarrow \neg(\exists\, x)(\exists\, y)[x \neq y \wedge (\exists\, z)(\varphi_f(m) = z \wedge x, y \in I\!D_z \cap \bar{W}_e)$$

All together give: $\Pi^0_4 \wedge \forall(\Pi_2 \Rightarrow \exists\forall\, \Pi_2) \equiv \Pi^0_4$.

$*$

Similar as for other notions, e.g. dense immune, also the notions of monotone and of 1–1 sets can be refined by taking partial computable functions instead of computable function.

An infinite set $X \subseteq I\!N$ is called *strongly monotonic* (*strongly* 1–1) if

$$(\forall\, \varphi - \text{partial computable function})(\exists\, n)(\forall\, m_1, m_2 > n)$$
$$[m_1, m_2 \in X \wedge m_1 \leq m_2 \wedge \varphi(m_1)\downarrow \wedge \varphi(m_2)\downarrow \Rightarrow \varphi(m_1) \leq \varphi(m_2)]$$

$$\Big((\forall\, \varphi - \text{partial computable function})(\exists\, n)(\forall\, m_1, m_2 > n)$$
$$[m_1 \neq m_2 \wedge \varphi(m_1)\downarrow \wedge \varphi(m_2)\downarrow \Rightarrow \varphi(m_1) \neq \varphi(m_2)]\Big).$$

We call an c.e. set A *strongly co-monotonic* (*strongly* co–1–1) if $\bar{A}$ is infinite and is strongly monotonic (strongly 1–1).

Lemma 6.10 ([He85]) *Let A be a coinfinite c.e. set. Then the following conditions are equivalent:*

a) *A is maximal.*
b) *A is strongly co-monotonic.*
c) *A is strongly* co–1–1.

Proof. Since strongly co-monotonic is included into Co–Mon, similar for strongly Co–1-1 $\subseteq$ Co–1–1, by definition, both classes are subclasses of $R - \text{Max}$.
Let $A \in R - \text{Max}$, but not maximal. Hence $A \subset_{\text{m}} B$ for some c.e. superset B. Let $(S_i)_{i\geq 0}$ be a disjoint c.e. sequence of finite sets with $S_i \cap \bar{A} \neq \emptyset$ for every i. Then

$$g(S_{2n}) = 0 \;:\; n \geq 0$$
$$g(S_{2n+1}) = 1 \;:\; n \geq 0$$

is partial computable and not 1–1 (mod $=^*$) on $\bar{A}$. Further, since every S_i is finite

$$\forall\, m\, \exists\, n > m \quad S_{2m+1} < S_{2n}.$$

But $g(S_{2m+1}) = 1$ and $g(S_{2n}) = 0$. Hence g is not monotonic (mod $=^*$) on $\bar{A}$.
If A is maximal then $\bar{A}$ is cohesive. Hence $\bar{A} \subseteq^* \text{dom}(g)$ or $\bar{A} \cap \text{dom}(g) =^* \emptyset$ for every partial computable function g. Thus from 2) we get it also for partial computable functions.

7 r-Maximal major subsets

In the lattice of c.e. sets the notion of r-cohesive set does not only appear in connection with the notion of r-maximal set (as the complement of such sets), but still in another form, namely as difference of (special) computable enumerable sets. Suppose A and B are c.e. sets with $A \subset_\infty B$ such that $B - A$ is r-cohesive. Then there are possible two cases. The first is that $B - A$ is co-c.e. But then $A \cup \bar{B}$ is r-maximal and thus does not give nothing new. But the second one, namely that $B - A$ is not co-c.e., as we shall see leads to new points of view inside the investigations on $\mathcal{E}$. This second appearance of r-cohesive sets in $\mathcal{E}$ was analyzed in [LeShSo78]. In [He89] and [HeKu94] special aspects on $\mathcal{E}$ were considered which also have consequences to the d.-c.e., not co-c.e., r-cohesive sets.

The following easy to see observation restricts the appearance of the d.-c.e., not co-c.e., r-cohesive sets inside $\mathcal{E}$ essentional. If A and B are c.e. sets such that $B - A$ is r-cohesive, not co-c.e. then every computable set R does not split nontrivially $B - A$, hence in particular also all with $\bar{B} \subseteq R$. Thus in this case $B - A$ not co-c.e. gives that $\bar{B} \subseteq R$ implies $\bar{A} \subseteq^* R$. This means $A \subset_m B$. Thus the analyse of r-cohesive d.-c.e., not co-c.e. sets is an analyse of c.e. sets and their major subsets. But on the other side for arbitrary c.e. sets A and B with $A \subset_m B$ in general $B - A$ must not be r-cohesive, since computable sets which not inlcude $\bar{B}$ can split $B - A$ nontrivially, not negating with this that $A \subset_m B$. This leads to the definition:

Definition 7.1 (Lerman, Shore, Soare) Let A and B be c.e. sets with $A \subset_m B$. If $B - A$ is r-cohesive we say that A *is an r-maximal major subset of B.*

If A is an r-maximal major subset of B then we write shortly: A is an r.m. major subset of B and use the abbreviation: $A \subset_{\mathrm{rm}} B$.

Obviously $\subset_{\mathrm{rm}}$ is a lattice-theoretic relation. Here we will analyse the c.e. sets and their degrees from the point of view when they have such special subsets and when not. We shall see that this problem has a nontrivial solution. The main results shown in [LeShSo78] about the r.m. major subsets were used there for giving an answer to a question of Post on the general structure of $\mathcal{E}$, see later subpoint 7.2.

7.1 The computable enumerable sets with r.m. major subsets

We start our considerations with the analysis of the c.e. sets which have r.m. major subsets. Inside this we investigate at first in A) the c.e. sets with the property that every major subset is r-maximal and then in B) the c.e. sets with r.m. major subsets in general.

A) The relation "$\subset_{\mathrm{rm}}$" in general is not equal to "$\subset_{\mathrm{m}}$" as it can be seen easily and will also follows in particular from that what is shown later. But for special c.e. sets both relations coincide. These c.e. sets can be even described quite easily. For doing this we have to generalize the notion of r-maximal set in the same way as in the Introduction the notion of maximal set was generalized to that of $\mathcal{D}$-maximal set.

Let A be a coinfinite c.e. set. With $\mathcal{D}(A)$[8] we denote the family $\{X \text{ c.e.} : A \cap X = \emptyset\}$. Obviously $\mathcal{D}(A)$ is an ideal in $\mathcal{E}$, hence $\mathcal{D}_{\mathcal{L}}(A) = \{A \cup X : X \in \mathcal{D}(A)\}$ is an ideal in $\mathcal{L}(A)$. Thus $\mathcal{L}(A)$ can be factored by $\mathcal{D}_{\mathcal{L}}(A)$. The factor lattice we denote with $\mathcal{L}(A)/\mathcal{D}_{\mathcal{L}}(A)$. Let $\mathcal{D}_r(A) = \{R \text{ computable} : R \cap A = \emptyset\}$ and $\mathcal{D}_{\mathcal{L},r}(A)$ be $\{A \cup R : R \in \mathcal{D}_r(A)\}$. It holds

$$(\mathcal{L}(A)/_{\mathcal{D}(a)})_r \cong \mathcal{L}_r(A)/\mathcal{D}_{\mathcal{L},r}(A).$$

For $\mathcal{L}_r(A)/\mathcal{D}_{\mathcal{L},r}(A)$ we write shortly $\mathcal{L}_r/\mathcal{D}(A)$[9].
A c.e. set A is called $\mathcal{D} - r$*-maximal* if $\mathcal{L}_r/\mathcal{D}(A)$ consists of exactly two elements, i.e. the factor lattice $\mathcal{L}(A)/\mathcal{D}_{\mathcal{L}}(A)$ has only two complemented elements[10].
Obviously c.e. sets which are simple and $\mathcal{D} - r$-maximal are exactly the r-maximal sets. But outside the class of simple sets there are still many further $\mathcal{D} - r$-maximal sets.

Lemma 7.2 *Let A be a noncomputable c.e. set. Then*

$$(\forall Z)(Z \subset_m A \Rightarrow Z \subset_{\text{rm}} A) \quad \textit{iff} \quad A \textit{ is } \mathcal{D} - r\textit{-maximal.}$$

Proof. $\Rightarrow$ Suppose $Z \subset_m A$ and let R be a computable set. Then if $\bar{A} \subseteq^* R$, $A - Z \subseteq^* R$. If $R \subseteq^* A$ then $R \subseteq^* Z$. Both, since $Z \subset_m A$. Let R be split nontrivially $\bar{A}$. We have $R \cap A$ and $\bar{R} \cap A$. If both are noncomputable then there are Z_1, Z_2 with $Z_1 \subset_m R \cap A$, $Z_2 \subset_m \bar{R} \cap A$. Thus $Z_1 \cup Z_2 \subset_m A$, but is not r-maximal. Hence A is not $\mathcal{D} - r$-maximal.
$\Leftarrow$ If R splits $A - Z$ nontrivially then $R \cap A$ and $\bar{R} \cap A$ are both not computable, since $Z \subset_m A$. Hence A is not $\mathcal{D} - r$-maximal. □

A criterion for the case that all major subsets of a c.e. set are r-maximal is given in the following lemma:

Lemma 7.3 *Let A be a noncomputable c.e. set. Then every major subset of A is r-maximal iff there is an r.m. major subset which is small in A.*

Proof. $\Rightarrow$ Obvious, since $\subset_{\text{sm}} \Rightarrow \subset_{\text{m}}$.
$\Leftarrow$ $A \notin \mathcal{D} - r$-maximal, see Lemma 7.2. Then there is R computable such that $R \cap A$, $\bar{R} \cap A$ are both not computable. Let $X \subset_m A$. Thus $R \cap A \subseteq^* X$ or $\bar{R} \cap A \subseteq^* X$. Suppose $R \cap A \subseteq^* X$. Let $U = A \cup R$ and V equal to $\bar{R} \cap A$. Then $U \cap (A - X) = \bar{R} \cap (A - X)$. Thus $(R - A) \cup V$ is c.e. and equal to $((R - A) \cup V) \cap R$, what is equal to $R - A$. This means that $A \cap R$ is c.e., hence $A \cap R$ is computable. Thus $X \subset_m A$ does not exist. □

---

[8] The letter $\mathcal{D}$ is used for symbolizing this family of c.e. sets, since these c.e. sets are disjoint with A.

[9] We see that for simple sets A $\mathcal{L}(A)/\mathcal{D}_{\mathcal{L}}(A) = \mathcal{L}^*(A)$ and $\mathcal{L}_r/\mathcal{D}(A) = \mathcal{L}_r^*(A)$.

[10] In [LeShSo78] the notion "almost recursive" is introduced. A c.e. set A is called *almost recursive* if A is not computable and

$$(\forall R \text{ computable})(A \cap R \text{ is computable } \vee A \cap \bar{R} \text{ is computable}).$$

It is easy to see that both notions "$\mathcal{D} - r$-maximal" and "almost recursive" coincide.

Still not cleared is the question which degrees include c.e. sets of the above kind, i.e. having only major subsets which are r-maximal. By using Lemma 7.2 this degree class is equal to $\deg_T(\mathcal{D} - r - \mathrm{Max})$. Inside the simple sets $\mathcal{D} - r - \mathrm{Max} = r - \mathrm{Max}$, hence only high degrees appear. But in the whole lattice $\mathcal{E}$ $\mathcal{D} - r - \mathrm{Max}$ includes many other well-known classes, e.g. the so-called hemimaximal sets. Hence to $\deg_T(\mathcal{D} - r - \mathrm{Max})$ belong also low degrees. It is to conjecture that all c.e. degrees unequal 0 belong to this class.

B) More interesting is the general case, namely to characterize the c.e. sets having r.m. major subsets, but not necessarily all major subsets must be of this kind. We shall see that already inside the class of simple sets this is a greater class then that considered in A).

From A) it follows that there are two different types of r-maximal major subsets. If $\mathcal{L}_r/\mathcal{D}(A)$ has atoms then for some computable set R $A \cup R$ is $\mathcal{D} - r$-maximal. Thus $X \subset_m A$ with $\bar{R} \cap A \subseteq X$ is r-maximal. We call such r-maximal major subsets *r-separable*.

Preference function

A necessary and sufficent criterion for a c.e. set to have r.m. major subsets was given in [LeShSo78].

Definition 7.4 Let A be a c.e. set. Suppose $(R_i)_{i \geq 0}$ is a c.e. sequence consisting of exactly all computable sets, e.g. $R_i = \{x : (\forall\, y \leq x)(\varphi_i(y) \downarrow \wedge \varphi_i(x) = 0)\}$, $i \geq 0$. For $\sigma = (\sigma_0 \ldots \sigma_{k-1}) \in 2^{<\omega}$ with $\bigcap_\sigma R_i$ we denote the intersection

$$R_0^{\sigma_0} \cap \ldots \cap R_{k-1}^{\sigma_{k-1}}$$

where $R_i^0 = R_i$ and $R_i^1 = \bar{R}_i$. (For $\sigma = < > \bigcap_\sigma R_i =_{\mathrm{df}} I\!N$).

A function $h : I\!N \to \{0, 1\}$ is called *preference function for* A (*with respect to* $(R_i)_{i \geq 0}$) if for every $n \geq 0$ for $\sigma = (h(0) \ldots h(n-1))$ the set

$$(7.1) \qquad \bigcap_\sigma R_i \cap A$$

is not computable[11)].

[11)]In [LeShSo78] instead of the condition (7.1) there are given the two requirements:

$$(7.2) \qquad \bigcap_\sigma R_i \cap \bar{A} - \infty \qquad \text{and} \qquad \bigcap_\sigma R_i \cap A - \infty.$$

It is easy to see that these two conditions are equivalent with (7.1), since $(R_i)_{i \geq 0}$ includes all computable sets. But for defining a preference function for nonsimple c.e. sets this other definition seems to be not suitable, since we cannot define $h(n)$ knowing only $h(i)$, $i < n$. If for $\sigma = (h(0) \ldots h(n-1))$ both sets $\bigcap_\sigma R_i \cap A \cap R_n$ and $\bigcap_\sigma R_i \cap \bar{A} \cap R_n$ are infinite, we cannot define $h(n) = 0$, since $\bigcap_\sigma R_i \cap \bar{A} \cap R_n$ can be computable and then for some R_j, $j > n$ we get a contradiction. If A is simple then $\bigcap_\sigma R_i \cap \bar{A} \cap R_n - \infty$ is sufficient (since this implies the second condition). Further if $\bigcap_\sigma R_i \cap \bar{A} \cap R_n$ is infinite then it cannot be computable.

Remarks 7.5 – We see easily that a c.e. set A has a preference function iff A is not computable and such a function can be found at least from Δ_4.
– There is no preference function from Δ_2.
Suppose h is a preference function for A w.r.t. $(R_i)_{i\geq 0}$ from Δ_2. Let $h(i,s) \Rightarrow h(i)$, $i \geq 0$ and $h(i,s)$ computable. We define a sequence $(T_i)_{i\geq 0}$ with T_i equal to

$$\bigvee\{[0,s] : h(i,s) = 1,\ s \geq 0\} \cup \bigvee\{R_{i,s} : h(i,s) = 0,\ s \geq 0\}.$$

We see that $(T_i)_{i\geq 0}$ is an c.e. sequence with

$$\begin{array}{lll} T_i =^* R_i & \text{if} & h(i) = 0, \\ T_i = I\!N & \text{if} & h(i) = 1. \end{array}$$

Let R be a computable set. Then $R = R_i$ for some i. If $h(i) = 1$ then for $R_j = \bar{R}$ $h(j) = 0$. Thus for every computable set R $\bigcap_{i\geq 0} T_i \subseteq^* R$ or $\bigcap_{i\geq 0} T_i \cap R =^* \emptyset$. But for all j $\bigcap_{j\leq i} T_i - \infty$. This implies that there is a computable set T with $T \subseteq^* \bigcap_{j\leq i} T_i$ for every j, what cannot be.
– If h is a preference function for A w.r.t. $(R_i)_{i\geq 0}$ and $h \in \Delta_3$ and $(R_i^0)_{i\geq 0}$ is another c.e. sequence of exactly all computable sets then there is a preference function h^0 for A w.r.t $(R_i^0)_{i\geq 0}$ and h^0 also is from Δ_3. Thus for a c.e. set the sequence $(R_i)_{i\geq 0}$ is unimportant for the existence of a Δ_3-preference function.

Theorem 7.6 ([LeShSo78]) *A c.e. set A has a preference function h from Δ_3 iff A has an r.m. major subset.*

Proof. $\Leftarrow$ Suppose $X \subset_{\mathrm{rm}} A$. Let h be defined by

$$\begin{array}{lllr} h(i) = 0 & \text{if} & A - X \subseteq^* R_i & \text{and} \\ h(i) = 1 & \text{if} & (A - X) \cap R_i =^* \emptyset. & \end{array}$$

We see that for every i one of the above cases must hold, since $A - X$ is r-cohesive. Further h belongs to Δ_3. We have $h(i) = 0$ if

$$(\exists\, x)(\forall\, y > x)(y \in A \Rightarrow y \in X \cup R_i)$$

and $h(i) = 1$ if

$$(\exists\, x)(\forall\, y > x)(y \in A \cap R_i \Rightarrow y \in X).$$

$\Rightarrow$ This implication follows from a more general fact which we give below.

$*$

An ideal $\mathcal{I}$ in $\mathcal{L}_r/\mathcal{D}(A)$ is called Σ_3-ideal if

$$\{e : W_e \text{ is computable } \wedge [A \cup W_e]/\mathcal{D}_r(A) \in \mathcal{I}\}$$

is Σ_3.
Let X be a c.e. subset of A and $\mathcal{I}_{A,r}(X)$ be the class

$$\{[A \cup R]/\mathcal{D}_r(A) : R - \text{computable} \wedge A \cap R \subseteq^* X\}.$$

Then $\mathcal{I}_{A,r}(X)$ is a Σ_3-ideal in $\mathcal{L}_r/\mathcal{D}(A)$.

– If $[A \cup R]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$ and $[A \cup R]/\mathcal{D}_r(A) = [A \cup R^0]/\mathcal{D}_r(A)$ then $R^0 \cap A \subseteq^* X$.

– If $[A \cup R]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$ and $[A \cup S]/\mathcal{D}_r(A) \leq [A \cup R]/\mathcal{D}_r(A)$ then $S \subseteq R \cup R^0$ with $R^0 \in \mathcal{D}_r(A)$. Thus $R \cap A \subseteq^* X$ gives $S \cap A \subseteq^* X$.

– $[A \cup R_0]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$ and $[A \cup R_1]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$ imply $R_0 \cap A \subseteq^* X$ and $R_1 \cap A \subseteq^* X$. Thus $(R_0 \cup R_1) \cap A \subseteq^* X$. Hence $[A \cup R_0 \cup R_1]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$. But $[A \cup R_0 \cup R_1]/\mathcal{D}_r(A) = [A \cup R_0]/\mathcal{D}_r(A) \vee [A \cup R_1]/\mathcal{D}_r(A)$.

– $[A \cup W_e]/\mathcal{D}_r(A) \in \mathcal{I}_{A,r}(X)$ if

$$W_e \quad \text{is computable } \wedge A \cap W_e \subseteq^* X.$$

But this is Σ_3.
An ideal $\mathcal{I}$ in $\mathcal{L}_r/\mathcal{D}(A)$ is called *maximal* if $\mathcal{I} \neq \mathcal{L}_r/\mathcal{D}(A)$ and for every ideal $\mathcal{I}^0$ in $\mathcal{L}_r/\mathcal{D}(A)$

$$\mathcal{I} \subseteq \mathcal{I}^0,\ \mathcal{I} \neq \mathcal{I}^0 \Rightarrow I^0 = \mathcal{L}_r/\mathcal{D}(A).$$

Thus $\mathcal{I}$ is maximal iff for every computable set R

$$\begin{aligned}
&R - A \in \mathcal{D}_r(A) \Rightarrow [A \cup R]/\mathcal{D}_r(A) \in \mathcal{I} \qquad \text{and}\\
&\bar{R} - A \in \mathcal{D}_r(A) \Rightarrow [A \cup R]/\mathcal{D}_r(A) \notin \mathcal{I} \qquad \text{and}\\
&R - A, \bar{R} - A \notin \mathcal{D}_r(A) \Rightarrow ([A \cup R]/\mathcal{D}_r(A) \in \mathcal{I} \wedge [A \cup \bar{R}]/\mathcal{D}_r(A) \notin \mathcal{I}) \text{ or}\\
&\qquad\qquad\qquad\qquad ([A \cup R]/\mathcal{D}_r(A) \notin \mathcal{I} \wedge [A \cup \bar{R}]/\mathcal{D}_r(A) \in \mathcal{I}).
\end{aligned}$$

The connection between r.m. major subsets of A and Σ_3-ideals in $\mathcal{L}_r/\mathcal{D}(A)$ shows the following lemma:

Lemma 7.7 *Let $X \subset_{\mathrm{m}} A$. Then $X \subset_{\mathrm{rm}} A$ iff $\mathcal{I}_{A,r}(A)$ is maximal in $\mathcal{L}_r/\mathcal{D}(A)$.*

Proof. If $A - X$ is r-cohesive then for R computable $R - A \in \mathcal{D}_r(A)$ implies $R \cap A$ is computable, hence $R \cap A \subseteq^* X$, $\bar{R} - A \in \mathcal{D}_r(A)$ implies $A - X \subseteq^* R$, hence not $A \cap R \subseteq^* X$ and in the third case $(A - X) \cap R =^* \emptyset$ or $(A - X) \cap \bar{R} =^* \emptyset$. Similar can be shown that if $\mathcal{I}_{A,r}(A)$ is maximal then $A - X$ is r-cohesive. □

Remark 7.8 An easy Corollary of the ideal definability Lemma of Harrington is the fact that for every Σ_3-ideal $\mathcal{I}$ in $\mathcal{L}_r/\mathcal{D}(A)$ there is $X \subset_m A$ such that $\mathcal{I} = \mathcal{I}_{A,r}(X)$.

If $\mathcal{I}$ is a Σ_3-ideal in $\mathcal{L}_r/\mathcal{D}(A)$ then $\{A \cap R : [A \cup R]/\mathcal{D}_r(A) \in \mathcal{I}\}$ is a Σ_3-ideal in Spl(A) (the set of splitting halfs of A) including all computable sets R with $R \subseteq A$. Thus there is a c.e. set C with the property that for every computable set R

$$[A \cup R]/\mathcal{D}_r(A) \in \mathcal{I} \text{ iff } (\exists\, T \text{ computable})(A \cap R \subseteq^* C \cup T).$$

Let $X = C \cup Y$, where $Y \subset_{\mathrm{sm}} A$. Then $\mathcal{I}_{A,r}(X) = \mathcal{I}$.

Thus Lemma 7.7 and Remark 7.8 together give:

"A has an r.m. major subset iff $\mathcal{L}_r/\mathcal{D}(A)$ has a maximal Σ_3-ideal".

Lemma 7.9 *Suppose h is a Δ_3-preference function for A. Then $\mathcal{L}_r/\mathcal{D}(A)$ has a Σ_3-maximal ideal.*

Proof. Let $\mathcal{I}$ be the family

$$\{[A \cup R_i]/\mathcal{D}_r(A) : h(i) = 1,\, i \geq 0\}.$$

We show that $\mathcal{I}$ is a Σ_3-maximal ideal in $\mathcal{L}_r/\mathcal{D}(A)$.
– $\mathcal{I}$ has a Σ_3-definition. "$[A \cup W_e]/\mathcal{D}_r(A) \in \mathcal{I}$" if

$$W_e \text{ is computable} \wedge (\exists\, i)(W_e = R_i \wedge h(i) = 1).$$

– $\mathcal{I}$ is an ideal. $[A \cup R_i]/\mathcal{D}_r(a) \in \mathcal{I}$ and $[A \cup R]/\mathcal{D}_r(A) \leq [A \cup R_i]/\mathcal{D}_r(A)$, R-computable then $R \subseteq A \cup R_i \cup T$, $T \in \mathcal{D}_r(A)$. Let $R = R_j$. If $R - A$ is c.e. then $h(j) = 1$. If $R - A$ is not c.e., but there is a $S \in \mathcal{D}_r(A)$ with $R \cup A \cup S = I\!N$ then $h(j) = 0$. But thus $h(i) = 0$, what is a contradiction. If both $R - A$ and $\bar{R} - A$ are not c.e., but $h(j) = 0$ then by our assumption $[A \cup R]/\mathcal{D}_r(A) \leq [A \cup R_i]/\mathcal{D}_r(A)$ this gives $h(i) = 0$, what is not true.
Suppose $[A \cup R_1]/\mathcal{D}_r(A), [A \cup R_2]/\mathcal{D}_r(A) \in \mathcal{I}$. If $h(i) = 0$ ($R_i = \bar{R}_1$) and $h(j) = 0$ ($R_j = \bar{R}_2$) then for $R_k = R_i \cap R_j \quad h(k) = 0$. Hence for $R_t = R_i \cup R_j \quad h(t) = 1$.
– $\mathcal{I}$ is maximal. This follows from the fact that for every computable set R there is an i with

$$R = R_i \wedge (h(i) = 1 \text{ or } \bar{R} = R_j \wedge h(j) = 1).$$

Thus the sequence $(R_i)_{h(i)=0, i \geq 0}$ is cofinal, i.e. no computable set splits nontrivially all R_i from the above sequence. □

Remark 7.8 and Lemma 7.9 give the second implication in Theorem 7.6.

Existence of c.e. sets with r.m. major subsets

By the consideration in A), see the remark in the beginning of B) only the not r-separable r.m. major subsets are of interest. In respect to this we have

Theorem 7.10 ([LeShSo78]) *There is an atomless, hh-simple set with r.m. major subsets.*

This result was still slightly improved in [HeKu94]. There it is shown

Theorem. *For every hh-simple, not q-maximal set A there is a hh-simple set B such that*

$$\mathcal{L}(A) \cong \mathcal{L}(B) \wedge B \textit{ has a not r-separable, r.m. major subset.}$$

We see that for atomless, hh-simple sets both theorems say the same.

$*$

Having a simple set S with r.m. major subsets by using Theorem 7.6 we get at once that also every simple subset S^0 of S has r.m. major subsets, since $\bigcap_\sigma R_i \cap \bar{S} - \infty$ implies $\bigcap_\sigma R_i \cap \bar{S}^0 - \infty$ and in the case of simple sets this is sufficient for to have r.m. major subsets. Later in Theorem 7.11 this observation will be still improved by the degree description of S^0.

Conjecture. For every noncomputable c.e. set A there is a c.e. set B with

$$\mathcal{L}_r/\mathcal{D}(A) \cong \mathcal{L}_r/\mathcal{D}(B) \wedge B \text{ has not } r\text{-separable r.m. major subsets.}$$

The relation RM( )

Let $\mathrm{RM}(X)$ hold for a c.e. set X if X has r.m. major subsets and $\mathrm{RM}_{\mathrm{ns}}(X)$ if X has not r-separable r.m. major subsets. We have

(i) If $\mathcal{L}_r/\mathcal{D}(X)$ is atomless then $\mathrm{RM}(X)$ iff $\mathrm{RM}_{\mathrm{ns}}(X)$.

(ii) $\mathrm{RM}(A) \wedge B$ is simple subset of A then $\mathrm{RM}(B)$. (The same for $\mathrm{RM}_{\mathrm{ns}}$).

(iii) RM is $\approx_{\mathrm{ns}}$-closed.

(iv) Inside the hh-simple, not q-maximal set RM_{ns} is $\mathcal{L}$-closed, i.e.

$$(\forall\, A)(\exists\, B)(\mathcal{L}(A) \cong \mathcal{L}(B) \wedge \mathrm{RM}_{\mathrm{ns}}(B)).$$

T − degrees of sets with r.m. major subsets

That there are many c.e. sets with r.m. major subsets is shown in the following theorem even in a strong form.

Theorem 7.11 ([LeShSo78]) *Let $\underline{a}$ be a noncomputable c.e. T-degree and M be a simple set. Then there is a simple set A with $A \in \underline{a}$ and $A \subseteq M$. (Thus every noncomputable c.e. T-degree includes simple sets with r.m. major subsets).*

Proof. Given $\underline{a} > 0$. Let M be a simple set. Further let B be a c.e. set with $B \in \underline{a}$ and $b = (b_s)_{s\geq 0}$ an effective enumeration of B such that the computation function associated with b E^b_B fails to dominate some computable function f. (See for this fact [Rob68]). We can find easily a computable enumeration m of M such that E^m_M dominates f. Now let A be the elements of M permitted by B, i.e. A is equal to

$$\{m_s : (\exists\, t > s)(b_t \leq m_s)\}.$$

We show that A is the wanted set.
– $A \leq_T B$. "$a \in A$" if $(\exists\, b \leq a)(b \in B,\, b = b_t$ for some t and $a \in \{m_0, \ldots, m_{t-1}\})$.
– $B \leq_T A$. We see that $M - A$ is infinite. We have

f is not dominated by E^b_B and f is dominated by E^b_B.
Thus $(\exists^\infty z)(\exists^\infty s)(B_s|z = B|z \wedge M_s|z \neq M|z)$.

Hence infinitely many x from M does not come into A.
$M - A$ infinite gives $B \leq_T A$. "$b \in B$". Find an m with $b < m$, $m \in M$, $m \notin A$. Suppose m_s is it. Then $m_s \notin A$ means $B_s|m_s = B|m_s$. Thus $b \in B$ iff $b \in B_s$.
– A is simple. Let W_e be infinite. Thus $W_e \cap M$ is infinite, since M is simple. If $W_e \cap A$ is finite then there is an effective list of infinite many elements from $M - A$. But this implies that B is computable, what is not true.
– If we assume RM(M) then see property (ii) of the relation RM( ) also RM(A). Such M exists by Theorem 7.10. □

Theorem 7.12 ([LeShSo78]) *If A is simple and $A \in I\!\!L_2$ (i.e. A is* low$_2$*) then* RM(A).

Proof. $A \in I\!\!L_2$, i.e. $A'' \equiv_T \emptyset''$. Thus the set $\mathrm{INF}_{\bar{A}}$ equal to

$$\{e : W_e \cap A^c - \infty\}$$

is computable in $\emptyset''$.
Given $(R_i)_{i \geq 0}$ a c.e. sequence of exactly all computable sets let f be a computable function with $R_i = W_{f(i)}$, $i \geq 0$. We construct a preference function h for A (w.r.t. $(R_i)_{i \geq 0}$) by recursion computable in $\emptyset''$. Hence h will be computable in $\emptyset''$, i.e. $h \in \Delta_3$.

$$\begin{aligned} h(0) &= 0 \qquad \text{if } f(0) \in \mathrm{INF}_{\bar{A}} \\ &= 1 \qquad \text{else.} \end{aligned}$$

Suppose $h(0), \ldots, h(n)$ are already defined. Let e be an index of

$$\bigwedge\{R_i : h(i) = 0,\ i = 0, \ldots, n\}.$$

Knowing $h(i)$, $i \leq n$ we can find e effectively. Define

$$\begin{aligned} h(n+1) &= 0 \qquad \text{if } U(e, f(n+1)) \in \mathrm{INF}_{\bar{A}} \\ &= 1 \qquad \text{else,} \end{aligned}$$

where U is a computable function with $W_e \cup W_f = W_{U(e,f)}$. Since A is simple, h such defined is a preference function for A. □

Remark. The assumption that A was simple is essentional, see footnote 11) of this point. Indeed later in Theorem 7.14 we shall see that this restriction is necessary, i.e. for all low$_2$ c.e. sets it is not true.

7.2 The computable enumerable sets without r.m. major subsets

In the second paragraph we deal with the c.e. sets without r.m. major subsets. We shall see that there are many c.e. sets of this kind, nevertheless inside the simple sets from the point of view of the T-degrees of such sets is not so easy to characterize these sets.

The existence of in particular atomless, hh-simple sets without r.m. major subsets was shown in [LeShSo78]. This together with the result shown in Theorem 7.10 answers the question of Post if for c.e. sets A and B the implication

$$\mathcal{L}(A) \cong \mathcal{L}(B) \Rightarrow A \cong_{\mathcal{E}} B$$

holds? The two kinds of atomless, hh-simple sets show that in $\mathcal{E}$ this implication in general is not true.

A some stronger property for all hh-simple sets was shown in [He89]. There it is proven

Theorem 7.13 *For every hh-simple set A there is a hh-simple set B with*

$$\mathcal{L}(A) \cong \mathcal{L}(B) \wedge \mathcal{L}_r^*(B) \quad \textit{has only principal } \Sigma_3\textit{-ideals.}$$

An atomless, hh-simple set cannot have a Σ_3-maximal ideal which is principal and thus by using Lemma 7.7 it has also no r.m. major subset.

That in the lattice $\mathcal{E}$ there are many sets without r.m. major subsets is shown in the following theorem:

Theorem 7.14 ([LeShSo78]) *For every noncomputable c.e. T-degree $\underline{a}$ there is a c.e. set A with $A \in \underline{a}$ and A has no r.m. major subset.*

Proof. Let B be a c.e. set with $B \in \underline{a}$ and $(b_s)_{s \geq 0}$ be an effective listing of B. We shall construct an c.e. set A with $A \equiv_T B$ and A will not have a Δ_3-preference function by diagonalizing over all Δ_3 01-valued functions. For doing it we need the following two sequences:

– Let $(Q_i)_{i \geq 0}$ be a computable sequence of computable sets with

$$\bigcap_{\sigma} Q_i - \infty \qquad \text{for every } \sigma \in 2^{<\omega} \qquad \text{and}$$
$$\bigcup_{i \geq 0} Q_i = I\!N\ .$$

– Further let $(C_e)_{e \geq 0}$ be a strong sequence of finite and disjoint sets with

$$\bigcup_{e \geq 0} C_e = I\!N \qquad \text{and}$$
$$\bigcap_{\sigma} Q_i \cap C_e \neq \emptyset \qquad \text{for every } \sigma \in 2^{<\omega}, \text{ with } |\sigma| \leq e.$$

The set A will satisfies two groups of requirements, a group of positives and one of negatives. The positives are:

P_e: For exactly the e's with $e \in B$ exactly one number from C_e comes to A (in the step s with $b_s = e$).

We see that if all P_e are satisfied then obviously $A \equiv_T B$.
"$e \in B$" if $C_e \cap A \neq \emptyset$. "$x \in A$". For x we can find effectively e with $x \in C_e$. If $e \in B$ then $e = b_s$ and one number y from C_e comes to A. If $y \neq x$ or $e \notin B$ then $x \notin A$.
The negative requirements are designed to pairs of Σ_3 sets to ensure that there is no

Δ_3-preference function for A. For doing this we consider all pairs (S_i, T_i), $i \geq 0$ of Σ_3-sets. Such a pair defines a Δ_3 01-valued function h if $\bar{S}_i = T_i$ by

$$\begin{array}{lll} h(i) = 0 & \text{if } x \in S_i & \text{and} \\ h(i) = 1 & \text{if } x \in T_i. & \end{array}$$

Let f be a computable function with $Q_i = R_{f(i)}$. We will ensure:

N_i: If $\bar{S}_i = T_i$ then $Q_i^{h(f(i))} \cap \bar{A}$ is computable.

Observe that $Q_i^{h(f(i))} \cap \bar{A} = R_{f(i)}^{h(f(i))} \cap \bar{A}$. But if h is a preference function then for every x $R_{f(i)}^{h(f(i))} \cap \bar{A}$ must be noncomputable, see (7.1).
Let $(S_i^n)_{n,i\geq 0}$ and $(T_i^n)_{n,i\geq 0}$ be c.e. sequences such that

$$\begin{array}{ll} f(i) \in S_i \Leftrightarrow (\exists\, n)(S_i^n - \infty) & \text{and} \\ f(i) \in T_i \Leftrightarrow (\exists\, n)(T_i^n - \infty) & \end{array}$$

Construction. Stage s. Suppose $e = b_s$. Define $\sigma \in 2^{<\omega}$ with $|\sigma| = e$. Let φ and ψ be the partial functions

$$\begin{array}{ll} \varphi(i,s) \;=\; (\mu n)(e \leq \max(S_{i,s}^n)) & \text{and} \\ \psi(i,s) \;=\; (\mu n)(e \leq \max(T_{i,s}^n)). & \end{array}$$

If

$$\varphi(i,s)\downarrow \wedge (\psi(i,s)\downarrow \Rightarrow \varphi(i,s) \leq \psi(i,s)) \tag{7.3}$$

then define $\sigma_i = 1$. If

$$\psi(i,s)\downarrow \wedge (\varphi(i,s)\downarrow \wedge \psi(i,s) \leq \varphi(i,s)) \tag{7.4}$$

then define $\sigma_i = 0$. In all other cases let $\sigma_i = 0$. We put an element from $C_e \cap \bigcap_{\sigma} Q_i$ into A.

Result. The requirements P_e are satisfied.
Fix i and suppose $\bar{S}_i = T_i$. Suppose $h(f(i)) = 0$. (The case $h(f(i)) = 1$ goes similar). Thus $f(i) \in S_i$. Let n_0 be the smallest n such that S_i^n is infinite. All $T_{i,s}^n$ are finite by assumption.
Let $m = \max(\bigcup_{l \leq n_0} T_i^l)$. The set $Q_i \cap A$ is computable. The set R equal to

$$\{b \in B : b = b_s \wedge b_s > \max(S_{i,s}^{n_0})\}$$

is computable. Hence $\{a \in A : a \in C_b \wedge b \in R\} \cap Q_i$ is computable. But this is equal (mod $=^*$) to $A \cap Q_i$. Hence also $Q_i - A$ is computable. □

From Theorem7.10 we know that not every c.e. T-degree includes simple sets without r.m. major subsets. In [LeShSo78] it was tried to characterize this degree class.
Let $I\!\!F$ be the class

$$\{\underline{a}\text{-c.e. } T\text{-degree} : \underline{a} \text{ has a simple set without r.m. major subset}\}.$$

Lemma 7.15 ([LeShSo78]) *For the class $\mathbb{F}$ we have the inclusions*

$$\mathbb{H}^1 \subseteq \mathbb{F} \subseteq \overline{\mathbb{L}}_2. \tag{7.5}$$

Proof. The second inclusion above was shown as contraposition in Theorem 7.10. The first one follows from the following:
Take a high simple set without r.m. major subset. For an example of it, the atomless, hh-simple set A mentioned before. By [Le71] for every high c.e. T-degree d there is a major subset B of A and $B \in d$. But RM( ) is $\approx_{\mathrm{ms}}$-closed. Thus also for B we have RM(B). B as major subset of a simple set obviously also is simple. $\square$

The inclusions (7.5) do not characterize $\mathbb{F}$ completely. A more precise description of the class $\mathbb{F}$ is still unknown.

References

[Ch91] P.A. Cholak, Automorphisms of the lattice of c.e. sets. Dissertation (Dept. Math.) University Wisconsin, Madison 1991.

[ChN.ta] P.A. Cholak, A. Nies, Atomless, r-maximal set (to appear).

[De72] A.N. Degtev, m-powers of simple sets. Algebra and Logic 11 (1972), 74-80.

[HarN98] L. Harrington, A. Nies, Coding in the partial ordering of enumerable sets. Adv. Math. 133 (1998), 133-162.

[He85] E. Herrmann, Maximal sets (A survey on the maximal set inside the theory of the c.e. sets and their degrees) (typewritten manuscript) (1985).

[He89] E. Herrmann, Automorphisms of the lattice of recursively enumerable sets and hyperhypersimple sets. Prof. of the 8th Int. Congr. of Logic, Phil. Science, Moscow (1987), North-Holland, Studies in Logic Vol. 126 (1989), 179-190.

[He.ta1] E. Herrmann, The index set of the major subsets of the maximal sets (to appear).

[He.ta2] E. Herrmann, Recursively enumerable sets (The lattice structure under inclusion and general properties) (to appear).

[HeKu94] E. Herrmann, M. Kummer, Diagonals and $\mathcal{D}$-maximal sets. Journ. Symb. Logic 59 (1994), 60-72.

[Ja96] M.A. Jahn, Σ^0_5-completeness of index sets arising from the lattice of r.e. sets. Ann. Pure Applied Logic 80 (1996), 55-67.

[Jo71] C.G. Jockusch, Review of Lerman (1971). Mathematical Reviews 45 (1973), No. 3200.

[Jo72] C.G. Jockusch, Degrees in which the recursive sets are uniformly recursive. Canad. J. Math. 24 (1972), 1092-1099.

[Jo73] C.G. Jockusch, Upward closure and cohesive degrees. Israel J. Math. 15 (1973), 332-335.

[JoSo72] C.G. Jockusch, R.I. Soare, Π^0_1-classes and degrees of theories. Trans. Amer. Soc. 173 (1972), 33-56.

[JoSt93] C.G. Jockusch, F. Stephan, A cohesive set which is not high. Math. Logic Quarterly 39 (1993), 515-530.

[La68a] A.H. Lachlan, On the lattice of recursively enumerable sets. Trans. Amer. Math. Soc. 130 (1968), 1-37.

[La68b] A.H. Lachlan, The elementary theory of recursively enumerable sets. Duke Math. J. 35 (1968), 123-146.

[Lem87] S. Lempp, Hyperarithmetical Index Sets in Recursion Theory. Trans. Amer. Math. Soc. 303 (1987), 559-583.

[LemNSo.ta] S. Lempp, A. Nies, R. Solomon, On the filter of computable enumerable supersets of an r-maximal set (to appear).

[Le71] M. Lerman, Some theorems on r-maximal sets and major subsets of recursively enumerable sets. Journ. Symb. Logic 36 (1971), 193-215.

[LeSo80] M. Lerman, R.I. Soare, d-simple sets, small sets, and degree classes. Pacific J. Math. 87 (1980). 135-155.

[LeShSo78] M. Lerman, R.A. Shore, R.I. Soare, r-Maximal Major Subsets. Israel Journ. Math. 31 (1978), 1-18.

[Ma85] W. Maass, Major subsets and automorphisms of recursively enumerable sets. In: Recursion Theory, Proc. Symp. Pure Math. 42, Amer. Math. Soc. (1985), 21-32.

[MaSt83] W. Maass, M. Stob, The intervals of the lattice of recursively enumerable sets determined by major subsets. Ann. Pure Appl. Logic 24 (1983), 189-212.

[MaShSt81] M. Maass, R.A. Shore, M. Stob, Splitting properties and jump classes. Israel Journ. Math. 39 (1981), 210-224.

[MadRob82] D.B. Madan, R.W. Robinson, Monotone and $1-1$ sets. J. Austr. Math. Soc. (Series A) 33 (1982), 62-75.

[N.ta] A. Nies, Intervals of the lattice of computably enumerable sets and effective Boolean algebras (to appear).

[Od81] P. Odifreddi, Strong Reducibilities. Bull. Amer. Math. Soc. 4 (1981), 37-86.

[Od89] P. Odifreddi, Classical Recursion Theory, Vol. I. North Holland (1989), Studies in Logic Vol. 125.

[Od.ta] P. Odifreddi, Classical Recursion Theory, Vol. II (to appear).

[Rob66] R.W. Robinson, Recursively enumerable sets not contained in any maximal set. Nat. Amer. Math. Soc. 13 (1966), 325.

[Rob68] R.W. Robinson, A dichotomy of the recursively enumerable sets. Z. Math. Logik Grundl. Math. 14 (1968), 339-356.

[Ro67] H. Rogers, Theory of Recursive Functions and Effective Computability. McGraw-Hill, New York 1967.

[Sh85] J.D. Schmidt, On the class of r-maximal sets. Mathematical Reviews 88 f: 03039.

[So74] R.I. Soare, Automorphisms of the lattice of recursively enumerable sets, Part I: Maximal sets. Ann. of Math. 100 (1974), 80–120.

[So78] R.I. Soare, Recursively enumerable sets and degrees. Bull. Amer. Math. Soc. 84 (1978), 1149–1181.

[So87] R.I. Soare, Recursively enumerable sets and degrees. Perspectives Math. Logic, Springer Verlag, Berlin 1987.