Wild division algebras over Laurent series fields

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0.1 Introduction

In this paper we prove a decomposition theorem for wild division algebras over a Laurent series field with arbitrary residue field of characteristik greater than two. This theorem is a generalization of the decomposition theorems for tame division algebras given by Jacob and Wadsworth in [2]. An extensive analysis of the wild division algebras of degree p over a field F with complete discrete rank 1 valuation with $char(\bar{F})=p$ was given by Saltman in [9] (Tignol in [7] analyzed more general case of the defectless division algebras of degree p over a fild F with Henselian valuation). Here we treate a general case of the wild division algebras over a field F with complete discrete rank 1 valuation with $char(F)=p>2$.

The main result of this paper is Theorem 3; it is a corollary of Theorem 2 and propositions 4-6. Also we prove a generalization of Cohen's theorem [8], Thm.9 for any valued complete division algebra (Th. 1). Theorem 2 is a key tool in the proof of Theorem 3; also Theorem 2 is in some sence a generalization of Th. 8.3.2. in [13]. As a corollary we get the positive answer on the following conjecture: the exponent of A is equal to its index for any division algebra A over a C_2-field $F=F_0((t_1))((t_2))$ (see [13], corollary 4, §8.3.2.).

We use the notation of [2]. We always denote by D a division algebra finite dimensional over its center $F=Z(D)$. Recall that any Henselian valuation on F has a unique extension to a valuation on D.

Given a valuation v on D, we denote by Γ_D its value group, by V_D its valuation ring, by M_D its maximal ideal and by $\bar{D}=V_D/M_D$ its residue division ring.

By [5], p.21 one has the fundamental inequality

$$[D:F]\geq|\Gamma_D:\Gamma_E|\cdot[\bar{D}:\bar{F}].$$

D is called defectless over F if equality holds and defective otherwise. It is known that D is defectless if it has a discrete valuation of rank 1.

Jacob and Wadsworth in [2] introduced the basic homomorphism

$$\theta_D:\Gamma_D/\Gamma_F\rightarrow Gal(Z(\bar{D})/\bar{F})$$

induced by conjugation by elements of D. They showed that θ_D is surjective and $Z(\bar{D})$ is the compositum of an abelian Galois and a purely inseparable extension of $\bar{F}$.

We say D is tame division algebra if $char(\bar{F})=0$ or $char(\bar{F})=q\neq 0$, D is defectless over F, $Z(\bar{D})$ is separable over $\bar{F}$, and $q\not||ker(\theta_D)|$. We say D is wild division algebra if it is non tame.

We call a division algebra D *inertially split* if $Z(\bar{D})$ is separable over $\bar{F}$, the map θ_D is an isomorphism, and D is defectless over F.

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0.2 Cohen's theorem

The following result is a generalazation of Cohen's theorem [8], Thm.9 (see also [4], Th.1).

Theorem 1 *Let (F,v) be a valued field which is complete with respect to a discrete rank 1 valuation v. Suppose $char F = char \bar{F}$. Let D be a division algebra with $Z(D) = F$ and $[D:F] < \infty$.*

Then there exists a section $\bar{D} \hookrightarrow D$ of the residue homomorphism $D \to \bar{D}$.

Proof. Since F is a comlete field, F is a Henselian field and v extends uniquely to a valuation w on D. It is known ([2], prop. 1.7) that $Z(\bar{D})$ is the compositum of a purely inseparable extension of $\bar{F}$ and a cyclic Galois extension. Let $\tilde{Z}(\bar{D})$ be a separable part of the extension $Z(\bar{D})/\bar{F}$, i.e. $Z(\bar{D})/\tilde{Z}(\bar{D})$ is a purely inseparable extension and $\tilde{Z}(\bar{D})/\bar{F}$ is a cyclic Galois extension. There exists an inertial lift Z of $\tilde{Z}(\bar{D})$ over F, Z is Galois over F, and by classical Cohen's theorem there exists a section $\tilde{Z}(\bar{D}) \hookrightarrow Z$.

Consider the centralizer $C = C_D(Z)$ of Z in D. Then we have $\bar{C} = \bar{D}$.

Indeed, by Double Centralizer Theorem we have $[D:F] = [C:F][Z:F]$ and $[Z:F] = |Gal(\tilde{Z}(\bar{D})/\bar{F})|$. By [2], prop.1.7 a homomorphism $\theta_D : \Gamma_D/\Gamma_F \to Gal(Z(\bar{D})/\bar{F})$ is surjective, so for any parameter z we have $\theta_D(w(z)) = \sigma$, where $<\sigma>= Gal(\tilde{Z}(\bar{D})/\bar{F})$. It is clear that $z \notin C$. Now let $u_1, \ldots, u_{[C:F]}$ be a F-basis of C. It is easy to see that the elements $u_j, zu_j, \ldots, z^{n-1}u_j$, $j = 1, \ldots, [C:F]$, where $n = ord(\sigma)$, the order of σ, are linearly independent, so form a basis for D over F. Since

$$w(F < zu_j, \ldots, z^{n-1}u_j, j = 1, \ldots, [C:F] >) \cap \Gamma_C = 0,$$

this implies that for any element $x \in D$ with $w(x) = 0$ we can find elements $r_1, \ldots r_{[C:F]} \in F$ such that $x = r_1u_1 + \ldots + r_{[C:F]}u_{[C:F]} \mod M_D$. Hence $\bar{C} = \bar{D}$.

Now we have $Z = Z(C)$, and $Z(\bar{C})/\bar{Z}$ is a purely inseparable extension. Let W be a maximal separable subfield in $\bar{C}$. Let $\tilde{Z}(W)$ be a maximal separable subfield of the extension $W/\bar{Z}$. By Hensel's lemma (see prop.4 in [4]) there exists an inertial lift $Z(W)$ and a section $\tilde{Z}(W) \hookrightarrow Z(W)$. Consider the centralizer $D_1 = C_C(Z(W))$.

By Double Centralizer Theorem, $[D_1:Z][Z(W):Z] = [C:Z]$. We have $[D_1:Z] = [\bar{D_1}:\tilde{Z}(W)][\tilde{Z}(W):\bar{Z}][\Gamma_{D_1}:\Gamma_Z]$, $[Z(W):Z] = [\tilde{Z}(W):\bar{Z}] = ind\bar{C}$, $[C:Z] = [\bar{C}:\bar{Z}][\Gamma_C:\Gamma_Z] = (ind\bar{C})^2[Z(\bar{C}):\bar{Z}][\Gamma_C:\Gamma_Z]$. Therefore, $[\bar{D_1}:\tilde{Z}(W)][\Gamma_{D_1}:\Gamma_Z] = [Z(\bar{C}):\bar{Z}][\Gamma_C:\Gamma_Z]$. Note that if $a \in D_1$, then $\bar{a} \in W = \tilde{Z}(W)Z(\bar{C})$. Hence, $[\bar{D_1}:\tilde{Z}(W)] \leq [Z(\bar{C}):\bar{Z}]$ and $[\Gamma_{D_1}:\Gamma_Z]$ must be equal to $[\Gamma_C:\Gamma_Z]$, $\bar{D_1} = W$. So $\bar{D_1}$ is a field and $\bar{D_1}/\overline{Z(D_1)}$ is a purely inseparable extension. Therefore, $[\bar{D_1}:\overline{Z(D_1)}] \leq [\Gamma_{D_1}:\Gamma_Z]$. Indeed, one can consider a maximal separable subfield U in D_1. If $\bar{U} \neq \overline{Z(D_1)}$, then by Cohen's theorem U must contain a purely inseparable element, a contradiction. So, $U/Z(D_1)$ is a totally ramified extension.

We claim that $[\bar{D_1}:\overline{Z(D_1)}] = [\Gamma_{D_1}:\Gamma_Z]$. Suppose $[\bar{D_1}:\overline{Z(D_1)}] < [\Gamma_{D_1}:\Gamma_Z]$. Take an element $a \in D_1$ such that $\bar{a}^p \in \bar{Z}$. By [12], Thm. 8.2.9 the field $Z(D_1)(a)$ is a complete valued field as a finite extension of a complete valued field. By Cohen's theorem there exists a lift a' of $\bar{a}$ in $Z(D_1)(a)$, i.e. $a'^p \in Z$. Consider the centralizer $D_2 = C_{D_1}(Z(D_1)(a'))$. We have $[D_2:Z(D_1)]p = [D_1:Z(D_1)]$, $[D_2:Z(D_1)] = [\bar{D_2}:$

$\overline{Z(D_2)}]p[\Gamma_{D_2}:\Gamma_{Z(D_1)}] = [\bar{D}_2:\overline{Z(D_1)}][\Gamma_{D_2}:\Gamma_{Z(D_2)}]$. Since $[\bar{D}_2:\overline{Z(D_1)}] \leq [\bar{D}_1:\overline{Z(D_1)}]$ and $[\Gamma_{D_2}:\Gamma_{Z(D_1)}] = [\Gamma_{D_2}:\Gamma_{Z(D_2)}] \leq [\Gamma_{D_1}:\Gamma_{Z(D_1)}]$, there are two possibilities: $[\bar{D}_2:\overline{Z(D_1)}]p = [\bar{D}_1:\overline{Z(D_1)}]$ and $[\Gamma_{D_2}:\Gamma_{Z(D_1)}] = [\Gamma_{D_1}:\Gamma_{Z(D_1)}]$ or $[\bar{D}_2:\overline{Z(D_1)}] = [\bar{D}_1:\overline{Z(D_1)}]$ and $[\Gamma_{D_2}:\Gamma_{Z(D_1)}]p = [\Gamma_{D_1}:\Gamma_{Z(D_1)}]$.
In both cases we have $[\bar{D}_2:\overline{Z(D_2)}] < [\Gamma_{D_2}:\Gamma_{Z(D_2)}]$ and $[\bar{D}_2:\overline{Z(D_2)}] < [\bar{D}_1:\overline{Z(D_1)}]$. Continuing this line of reasoning, we get a division algebra D_n with $\bar{D}_n = \overline{Z(D_n)}$ and $\Gamma_{D_n} \neq \Gamma_{Z(D_n)}$, a contradiction.

Take an element $a \in D_1$ such that $\bar{a}^p \in \bar{Z}$. By Cohen's theorem there exists a lift a' of $\bar{a}$ in $Z(D_1)(a)$, i.e. $a^p \in Z$. Consider the centralizer $C_1 = C_C(\bar{Z}(a'))$. We claim that $\bar{C}_1 = \bar{D}$.

This follows from the fact that $\Gamma_{C_1} \subset \Gamma_C$, $\Gamma_{C_1} \neq \Gamma_C$, which one can prove using the same arguments as above. Namely, if $\Gamma_{C_1} \subset \Gamma_C$, then p must divide $[\Gamma_C:\Gamma_{C_1}]$ as $[C:Z] = [C_1:Z][Z(a):Z] = [C_1:Z]p = [\bar{C}_1:\bar{Z}][\Gamma_{C_1}:\Gamma_Z]p = [\bar{C}:\bar{Z}][\Gamma_C:\Gamma_Z] = [\bar{D}:\bar{Z}][\Gamma_C:\Gamma_{C_1}][\Gamma_{C_1}:\Gamma_Z]$ for some $k, k_1 \in \mathbb{N}$. Hence by Double Centralizer Theorem we have $[\bar{C}_1:\bar{Z}] = [\bar{C}:\bar{Z}]$, so $\bar{C}_1 = \bar{C} = \bar{D}$.

Continuing this line of reasoning, we see that there exists a subalgebra $C_n \subset D$ with $\bar{C}_n \subset \bar{D}$ and $Z(\bar{C}_n) = \overline{Z(C_n)}$. By [2], Th. 2.8(a), Th.2.9, C_n contains an inertial lift $\tilde{C}_n$ of $\bar{C}_n$, i.e. $\bar{\tilde{C}}_n = \bar{C}_n$ and $\Gamma_{\tilde{C}_n} = \Gamma_{Z(\tilde{C}_n)}$. By Cohen's theorem the field $F' = \overline{Z(C_n)}$ can be embedded into $Z(\tilde{C}_n)$. Consider the algebra $A = \bar{\tilde{C}}_n \otimes_{F'} Z(\tilde{C}_n)$. It is easy to see that A is an unramified division algebra with $\bar{A} = \bar{\tilde{C}}_n = \bar{D}$. Therefore by [6], Th.31, $A \cong \tilde{C}_n$; so there exists a section $\bar{D} \hookrightarrow \tilde{C}_n$.

The theorem is proved.

□

0.3 Decomposition theorem

In this part we prove a generalization of Jacob-Wadsworth's decomposition theorem ([2], Th.6.3., lemma 6.2) for finite dimensional division algebras over a Laurent series field $k((t))$, $char\, k > 2$.

First we recall some results from [4]. Let D be a finite dimensional division algebra over a complete valued field $F = k((t))$. Let w be a unique extension of the valuation v to D. We will denote by z any parameter of D, i.e. any element with $< w(z) >= \Gamma_D$. Consider the ring $\mathbb{Z} < \alpha, \delta >$ of noncommutative polinomials in two variables. Define the map

$$\sigma : \mathbb{Z} < \alpha, \sigma > \to \mathbb{Z} < \alpha, \delta, \delta_i; i \geq 1 >,$$
$$\sigma(\alpha^{a_1}\delta^{b_1}\ldots\alpha^{a_n}\delta^{b_n}) = \alpha^{a_1}\delta_{b_1}\ldots\delta_{b_{n-1}}\alpha^{a_n-1}\delta^{b_n},$$

where $a_1, b_n \geq 0$, $a_i, b_j \geq 1$, $i > 1$, $j < n$ for every word in $\mathbb{Z} < \alpha, \delta >$.

Let $S_i^k \in \mathbb{Z} < \alpha, \delta >$, $i \geq k$, $i \geq 1$ be polynomials given by the following formula:

$$S_i^k = \sum_{\tau \in S_i/G} \tau(\underbrace{\alpha\ldots\alpha}_{i-k}\underbrace{\delta\ldots\delta}_{k}),$$

where S_i is a permutation group and G is an isotropy subgroup.

Lemma 1 *([4], lemma 2) The polynomials S_i^k satisfy the following property:*

$$S_i^i=\delta^i,\quad S_i^0=\alpha^i,\quad S_{i+1}^{k+1}=\alpha S_i^{k+1}+\delta S_i^k$$

Proposition 1 *([4], prop. 1,2) D is isomorphic to a local skew-field $\bar{D}((z))$, where*

$$zaz^{-1}=\alpha(a)+\delta_1(a)z+\delta_2(a)z^2+\dots,\quad a\in\bar{D};$$

here $\alpha:\bar{D}\to\bar{D}$ is an automorphism and $\delta_i:\bar{D}\to\bar{D}$ are linear maps such that the map δ_i satisfy the identity

$$\delta_i(ab)=\sum_{k=0}^{i}\sigma(\delta^{i-k}\alpha)(a)\sigma(S_i^k\alpha)(b),\quad a,b\in\bar{D}$$

The proof is an easy combination of the prooves of [4], prop.1,2 and Cohen's theorem 1.

Remark Note that the values $\sigma(S_i^k\alpha)$ and $\sigma(\delta^{i-k}\alpha)$ belong to the subring $\mathbb{Z}<\alpha,\delta_i,i\geq 1>$, so the formula is well defined.

Corollary 1 *([4], corol. 1) Suppose $\alpha=Id$. Then*

$$\delta_i(ab)=\delta_i(a)b+\sum_{k=1}^{i}\delta_{i-k}(a)\sum_{(j_1,\dots,j_l)}C_{i-k+1}^l\delta_{j_1}\dots\delta_{j_l}(b),$$

where $\delta_0=\alpha$ and the second sum is taken over all the vectors $(j_1,\dots,j_l)$ such that $0<l\leq min\{i-k+1,k\}$, $j_m\geq 1$, $\sum j_m=k$.

Definition 1 *Let us define maps ${}_m\delta_i:\bar{D}\to\bar{D}$, $m\in\mathbb{Z}$, $i\in\mathbb{N}$ as follows.*

$$z^maz^{-m}=\alpha^m(a)+{}_m\delta_1(a)z+{}_m\delta_2(a)z^2+\dots,\quad a\in\bar{D}.$$

If $m=0$, put ${}_m\delta_i=0$.

Note that if $\alpha=id$, then ${}_m\delta_i=0$ for $m=p^k$, where k is sufficiently large, k depends on i. Moreover, ${}_m\delta_i={}_{m+p^k}\delta_i$ for k sufficiently large. We will use also the following notation:

$${}_m\tilde{\delta}_i={}_{-m}\delta_i.$$

Proposition 2 *(i) The maps ${}_m\delta_i$ satisfy the following identities:*

$${}_m\delta_i(ab)={}_m\delta_i(a)\alpha^{i+m}(b)+\alpha^m(a){}_m\delta_i(b)+\sum_{k=1}^{i-1}{}_m\delta_{i-k}(a){}_{i-k+m}\delta_k(b)$$

(ii) Suppose $\alpha=id$. Then the maps ${}_m\delta_i$ satisfy the following identities:

$${}_m\delta_i(ab)={}_m\delta_i(a)b+a{}_m\delta_i(b)+\sum_{k=1}^{i-1}{}_m\delta_{i-k}(a)\sum_{(j_1,\dots,j_l)}C_{i-k+m}^l\delta_{j_1}\dots\delta_{j_l}(b)$$

where the second sum is taken over all the vectors $(j_1,\dots,j_l)$ such that $0<l\leq min\{i-k+m,k\}$, $j_m\geq 1$, $\sum j_m=k$; $C_j^k=0$ if $j=0$, and $C_j^k=C_{j+p^q}^k$ for $q>>0$ if $j\leq 0$.

Proof. For any $a, b \in \bar{D}$ we have

$$\alpha^m(ab)z^m + {}_m\delta_1(ab)z^{m+1} + {}_m\delta_2(ab)z^{m+2} + \ldots = z^m(ab) =$$

$$(*) \qquad (\alpha^m(a)z^m + {}_m\delta_1(a)z^{m+1} + {}_m\delta_2(a)z^{m+2} + \ldots)b$$

If we represent the right-hand side of $(*)$ as a series with coeffitients shifted to the left and then compare the corresponding coeffitients on the left-hand side and right-hand side, we get some formulas for ${}_m\delta_i(ab)$. We have to prove that these formulas are the same as in our proposition.

Let

$$z^{i+m-k}b = \alpha^{i+m-k}(b)z^{i+m-k} + \ldots + x'_k z^{i+m} + \ldots$$

and

$$(\alpha^m(a)z^m + {}_m\delta_1(a)z^{m+1} + {}_m\delta_2(a)z^{m+2} + \ldots)b = \alpha^m(ab)z^m + y_{m+1}z^{m+1} + y_{m+2}z^{m+2} + \ldots$$

Then we have

$$y_{i+m} = \alpha^m(a)x'_i + \sum_{k=0}^{i-1} {}_m\delta_{i-k}(a)x'_k$$

In the proof of [4], prop.2 we have shown that

$$z^{i+1-k}b = \alpha^{i+1-k}(b)z^{i+1-k} + \ldots + \sigma(S_i^k\alpha)(b)z^{i+1} + \ldots$$

Hence $x'_k = \sigma(S_{i+m-1}^k\alpha)(b)$ for $k < i$. It is easy to see that $x'_i = {}_m\delta_i(b)$, $x'_0 = \alpha^{i+m}(b)$ and $\sigma(S_{i+m-1}^k\alpha) = {}_{i+m-k}\delta_k$, which proves (i).

For $\alpha = id$, by corollary 1,

$$\sigma(S_{i+m-1}^k\alpha)(b) = \sum_{(j_1,\ldots,j_l)} C_{i-k+m}^l \delta_{j_1}\ldots\delta_{j_l}(b),$$

where $l, j_1, \ldots, j_l$ were defined in our proposition. This proves (ii).
The proposition is proved.
□

Lemma 2 *([4], lemma 3)*

Suppose the following relation holds:

$$zaz^{-1} = a^\alpha + a^{\delta_j}z^j + a^{\delta_{j+1}}z^{j+1} + \ldots, \quad a \in \bar{D}$$

where $\delta_1 = \ldots = \delta_{j-1} = 0$, $\delta_j \neq 0$. *Then*

(i) for $z' = z + bz^{q+1}$ *we have*

$$z'az'^{-1} = a^\alpha + \ldots + a^{\delta_{q-1}}z'^{q-1} + (a^{\delta_q} + ba^{\alpha^{q+1}} - a^\alpha b)z'^q + \ldots$$

i.e. $a^{\delta'_q} = a^{\delta_q} + ba^{\alpha^{q+1}} - a^\alpha b$.

(ii) Suppose $\alpha^n = id$ *,* $n \geq 1$ *. Then for* $z' = z + bz^{q+1}$ *,* $n|q$ *we have*

$$z'az'^{-1} = a^{\alpha} + \ldots + a^{\delta_{q+j-1}} z'^{q+j-1} +$$

$$(a^{\delta_{q+j}} + b(a^{\delta_j})^{\alpha^q} - a^{\delta_j} b^{\alpha^j} + b\sum_{k=1}^{q}((a^{\alpha^k})^{\delta_j})^{\alpha^{q-k}} - a^{\delta_j}\sum_{k=0}^{j-1} b^{\alpha^k}) z'^{q+j} + \ldots$$

In particular, if $\alpha = id$ *, then*

$$z'az'^{-1} = a + \ldots + a^{\delta_{q+j-1}} z'^{q+j-1} + (a^{\delta_{q+j}} + (q-j)a^{\delta_j} b) z'^{q+j} + \ldots$$

(iii) for $z' = bz$ *,* $b \in \bar{D}$ *,* $b \neq 0$ *we have*

$$z'az'^{-1} = a^{\alpha} + a^{\delta_j}(b^{-1})^{\alpha} \ldots (b^{-1})^{\alpha^j} z'^j + \ldots$$

The following theorem is a main result of this paper.

Theorem 2 *Let* $u \in Z(\bar{D})$ *be a purely inseparable element and* $char F > 2$ *. Then* $u^p \in \bar{F}$ *.*

Proof. By proposition 1 we can assume that $u \in \bar{D} \subset D$. Without loss of generality we can assume that $Z(\bar{D})/\bar{F}$ is a purely inseparable extension. Moreover, it can be assumed that $Z(\bar{D}) = \bar{F}(u)$. Then by Scolem-Noether theorem we can choose a parameter z such that $\alpha = id$. Suppose

$$zuz^{-1} = u + \delta_i(u)z^i + \ldots,$$

i.e. $\delta_1|_{F(u)} = \ldots = \delta_{i-1}|_{F(u)} = 0$, $\delta_i(u) \neq 0$. Suppose $u^{p^k} \in F$.

Without loss of generality it can be assumed that the following property holds:

ψ) let δ_j, $j > i$ be the first map such that $\delta_j \neq 0$ if j is not divisible by i and $\delta_j \neq c_{j/i}\delta_i^{j/i}$ for some $c_{j/i} \in \bar{D}$ otherwise; then $j = 2i \mod p$.

Indeed, let δ_j be the first map such that $\delta_j \neq 0$ and $i \not| j$, $j \neq 2i \mod p$. Then by lemma 2, (ii) there exists a parameter z' such that $\delta_j'(u) = 0$. Therefore by corollary 1, $\delta_j'|_{F(u)} = 0$. By Scolem-Noether theorem, δ_j' is an inner derivation. Therefore by lemma 2 (ii) there exists a parameter z such that $\delta_j = 0$.

If δ_j is the first map such that $i|j$, $\delta_j|_{F(u)} \neq c_{j/i}\delta_i^{j/i}|_{F(u)}$ and $j \neq 2i \mod p$, then by lemma 2, (ii) there exists a parameter z' such that $\delta_j'(u) = c_{j/i}\delta_i'^{j/i}(u) = c_{j/i}\delta_i^{j/i}(u)$, where

$$c_{j/i} = \frac{(i+1)\ldots(i(j/i-1)+1)}{(j/i)!} \tag{1}$$

One can easily show that $\delta_j'|_{F(u)} = c_{j/i}\delta_i'^{j/i}|_{F(u)}$. By Scolem-Noether theorem, $(\delta_j' - c_{j/i}\delta_j'^{j/i})$ is an inner derivation. Therefore by lemma 2 (ii) there exists a parameter z such that $\delta_j = c_{j/i}\delta_i^{j/i}$.

Let's divide our proof in two steps.

Step 1. First we prove that $(i,p) = 1$.

Lemma 3 *Let j be the minimal positive integer such that $\delta_j|_{F(u^{p^l})} \neq 0$, $l \geq 0$. Then the maps ${}_n\delta_m$, $kj \leq m < (k+1)j$, $k \in \{1, \ldots, p-1\}$ satisfy the following properties:*

i) there exist elements $c_{m,k} \in \bar{D}$ such that

$$({}_n\delta_m - c_{m,1}\delta - \ldots - c_{m,k}\delta^k)|_{F(u^{p^l})} = 0,$$

where $\delta : \bar{D} \to \bar{D}$ is any F-linear map such that $\delta|_{F(u^{p^l})}$ is a derivation, $\delta(u^j) = 0$ for $j \notin p^l\mathbb{N}$, $\delta(u^{p^l}) = 1$, $c_{kj,k} = c(\delta_j(u^{p^l}))^k$, $c \in \mathbb{F}_p$.

ii) $c_{kj,k} \neq 0$ iff $n, n+j, \ldots, n+(k-1)j \neq 0 \mod p$.

Proof. i) The proof is by induction on k. Let $a, b \in F(u^{p^l})$. Put $t = u^{p^l}$. For $k = 1$, by proposition 2, (ii) we have

$${}_n\delta_m(ab) = {}_n\delta_m(a)b + a_n\delta_m(b)$$

because all the maps δ_q, $q < j$ are equal to zero on $F(u^{p^l})$. Hence, ${}_n\delta_m$ is a derivation on $F(u^{p^l})$ and $c_{j,1} = {}_n\delta_j(t) = n\delta_j(t)$.

For arbitrary k, by proposition 2, (i) and by the induction hypothesis we have (**)

$${}_n\delta_m(t^q) = q_n\delta_m(t)t^{q-1} + {}_n\delta_j(t)(\sum_{l=0}^{q-2}(c_1\delta + \ldots + c_{k-1}\delta^{k-1})(t^{q-1-l})t^l) + \ldots + {}_n\delta_s(t)(\sum_{l=0}^{q-2}(g_1\delta)(t^{q-1-l})t^l),$$

where $c_j, g_j \in \bar{D}$, $s > m - 2j$. Therefore, ${}_n\delta_m(t^p) = 0$, because $k \leq p-1$ and $\sum_{l=0}^{p-2} c_j\delta^j(t^{p-1-l})t^l = 0$ for $j \leq p-2$. Hence, ${}_n\delta_m|_{F(t)} = c_{m,1}\delta + \ldots + c_{m,p-1}\delta^{p-1}$ and we only have to show that $c_{m,q} = 0$ for $q > k$.

Using (**) we can calculate $c_{m,j}$. We have

$$c_{m,1} = {}_n\delta_m(t);$$

$$c_{m,2} = \frac{1}{2!}{}_n\delta_m(t^2) - 2c_{m,1}t = \frac{1}{2}{}_n\delta_j(t)(c_1\delta(t)) + \ldots + {}_n\delta_s(t)(g_1\delta(t))$$

$$\ldots$$

$$c_{m,q} = \frac{1}{q!}({}_n\delta_j(t)(\sum_{l=0}^{q-2} c_{q-1}\delta^{q-1}(t^{q-1-l})t^l) + \ldots + {}_n\delta_{m-q+1}(t)(\sum_{l=0}^{q-2} g_{q-1}\delta^{q-1}(t^{q-1-l})t^l))$$

$$= \frac{1}{q}({}_n\delta_j(t)c_{q-1} + \ldots + {}_n\delta_{m-q+1}(t)g_{q-1})$$

Hence, $c_{m,k+1} = \ldots = c_{m,p-1} = 0$ and $c_{kj,k} = c\delta_j(t)c_{k-1} = \tilde{c}(\delta_j(t))^k$, $c, \tilde{c} \in \mathbb{F}_p$. Note that $c_{(p-1)j,p} = ({}_j\delta_j(t))^{p-1}$.

ii) Suppose $n = 0 \mod p$. For $k = 1$ we have ${}_n\delta_j|_{F(t)} = n\delta_j|_{F(t)} = 0$. For arbitrary k we have

$${}_n\delta_{kj}(t^q) = q_n\delta_{kj}(t)t^{q-1} + {}_n\delta_j(t)\sum_{r=0}^{q-2} {}_{n+j}\delta_{(k-1)j}(t^{q-1-r})t^r + \ldots + \sum_{l=j+1}^{kj} {}_n\delta_l(t)\sum_{r=0}^{q-2} {}_{n+l}\delta_{kj-l}(t^{q-1-r})t^r$$

Since ${}_n\delta_j(t)=0$ and ${}_{m_1}\delta_h|_{F(t)}=c_1\delta+\ldots+c_{k-2}\delta^{k-2}$ for $h<(k-1)j$, the same arguments as in i) show that $c_{kj,k}=0$.

Suppose $n+rj\neq 0 \mod p$, $r<k-1$. The same arguments as above show that in this case $c_{kj,k}({}_n\delta_{kj})=0$ iff $c_{(k-1)j,k-1}({}_{n+j}\delta_{(k-1)j})=0$. So, by induction, $c_{kj,k}({}_n\delta_{kj})=0$ iff $c_{(k-r)j,k-r}({}_{n+rj}\delta_{(k-r)j})=0$, which proves ii).

The lemma is proved.

□

Lemma 4 *If $p|i$, then there exists a map δ_j such that $\delta_j(u^{p^k})\neq 0$.*

Proof. We claim that $\delta_{p^q i}$ is the first map such that $\delta_{p^q i}|_{F(u^{p^q})}\neq 0$. The proof is by induction on q. For $q=0$, there is nothing to prove. For arbitrary q, put $t=u^{p^{q-1}}$. By proposition 2 we have

$$\delta_{p^q i}(t^p)=\delta_{p^{q-1}i}(t)\sum_{r=0}^{p-2}{}_{1+p^{q-1}i}\delta_{p^{q-1}i(p-1)}(t^{p-1-r})t^r+\sum_{l=p^{q-1}i+1}^{p^q i-1}\delta_l(t)\sum_{r=0}^{p-2}{}_{1+l}\delta_{p^q i-l}(t^{p-1-r})t^r$$

By induction and lemma 3, ${}_{1+l}\delta_{p^q i-l}|_{F(t)}=c_1\delta+\ldots+c_{p-2}\delta^{p-2}$ for $l>p^{q-1}i$. Therefore, $\sum_{r=0}^{p-2}{}_{1+l}\delta_{p^q i-l}(t^{p-1-r})t^r=0$. By lemma 3, (ii), ${}_{1+p^{q-1}i}\delta_{p^{q-1}i(p-1)}|_{F(t)}=c_1\delta+\ldots+c_{p-1}\delta^{p-1}$ with $c_{p-1}\neq 0$. Hence, $\delta_{p^q i}(t^p)=-c_{p-1}\delta_{p^{q-1}i}(t)\neq 0$.

The same arguments show that $\delta_j(t^p)=0$ for $j<p^q i$. So, $\delta_{p^q i}$ is the first non-zero map on $F(u^{p^q})$.

□

Step 2.

From now on $(i,p)=1$. Note that $\delta_i(u)\in Z(\bar{D})$. Indeed, for any $a\in\bar{D}$ we have

$$\delta_i(au)=\delta_i(a)u+a\delta_i(u)=\delta_i(ua)=u\delta_i(a)+\delta_i(u)a$$

Therefore, $a\delta_i(u)=\delta_i(u)a$. Since $(i,p)=1$, there exists $k_1\in\mathbb{N}$ such that p^k divides $1+k_1 i$. Therefore by lemma 2 (iii) there exists a parameter z' such that $\delta_i'(u)=(\delta_i(u))^{1+k_1 i}\in Z(\bar{D})$.

So we can assume that $\delta_i(u)\in Z(\bar{D})$ and ψ) holds.

To prove our theorem it is sufficient to show that there exists a parameter z such that the maps δ_j satisfy the following property:

$(*)$ If j is not divisible by i, then $\delta_j|_{F(u)}=0$. If j is divisible by i, then $\delta_j|_{F(u)}=c_{j/i}\delta_i^{j/i}|_{F(u)}$ with some $c_{j/i}\in\bar{D}$.

To show it we prove that if property $(*)$ does not hold, then there exists a map δ_j such that $\delta_j(u^{p^k})\neq 0$. Suppose $(*)$ does not hold and δ_{2i+mp} is the first map which does not satisfy $(*)$. So, $\delta_{2i+mp}(u)\neq 0$. Note that $\delta_q(u)=0$ for $i<q<2i+mp$.

Note that $\delta_{2i+mp}(u),\tilde{\delta}_{2i+mp}(u)\in Z(\bar{D})$. Indeed, by proposition 2, ${}_i\tilde{\delta}_{2i+mp}|_{F(u)}$ is a derivation. Therefore, ${}_i\tilde{\delta}_{2i+mp}(u)\in Z(\bar{D})$. Since $\delta_i(u)\in Z(D)$ and $\delta_q(u)=0$ for $i<q<2i+mp$, ${}_i\tilde{\delta}_{2i+mp}(u)=i\tilde{\delta}_{2i+mp}(u)$ and $\tilde{\delta}_{2i+mp}(u)\in Z(\bar{D})$. Therefore, $\delta_{2i+mp}(u)\in Z(\bar{D})$.

First we prove that there exists a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \leq 2i+mp$ and ${}_{2i+mp+(p-1)i}\tilde{\bar{\delta}}_q(u) = 0$ for $q \neq 2i \mod p$, $q > 2i+mp$; here $\bar{\delta}_q$ are maps given by the parameter $\bar{z}$. Put $j(1) = 2i+mp+(p-1)i$.

Suppose ${}_{j(1)}\tilde{\delta}_q(u) \neq 0$, $q > 2i+mp$ and $q \neq 2i \mod p$. By definition,

$$ {}_{j(1)}\tilde{\delta}_q(u) = -j(1)\delta_q(u) + \sum \delta_{k_1}\dots\delta_{k_l}(u), $$

where $k_i < q$. By lemma 2, (ii) for any $a \in \bar{D}$ there exists a parameter $\bar{z}_q$ such that

$$ \bar{z}_q u \bar{z}_q^{-1} = u + \delta_i(u)\bar{z}_q^i + \dots + \delta_{q-1}(u)\bar{z}_q^{q-1} + a\bar{z}_q^q + \dots $$

Therefore, there exists an element $a \in \bar{D}$ such that ${}_{j(1)}\tilde{\bar{\delta}}_q(u) = 0$. It is easy to see that the sequence $\{\bar{z}_q\}$ converges in D. So, $\bar{z} = lim\bar{z}_q$.

Lemma 5 *Put $\kappa = j(1) = 2i+mp+(p-1)i$. Then there exists a parameter z such that the following properties hold:*

(i) ${}_{\kappa}\tilde{\delta}_{2i+mp+(p-1)i}$ is the first map such that ${}_{\kappa}\tilde{\delta}_{2i+mp+(p-1)i}|_{F(u^p)} \neq 0$.

(ii) ${}_{\kappa}\tilde{\delta}_{2i+mp+(p-1)i+i+mp}(u^p) \neq 0$ and ${}_{\kappa}\tilde{\delta}_r|_{F(u^p)} = 0$ for $j(1) < r < j(1)+i+mp$.

(iii) ${}_{\kappa}\tilde{\delta}_{2i+mp+(p-1)i}(u^p) \in Z(\bar{D})$, ${}_{\kappa}\tilde{\delta}_{2i+mp+(p-1)i+i+mp}(u^p) \in Z(\bar{D})$.

Proof.

i) Put $w := 2i+mp+(p-1)i$. By proposition 2 we have

$$ {}_{\kappa}\tilde{\delta}_w(u^p) = {}_{\kappa}\tilde{\delta}_i(u)\sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q + {}_{\kappa}\tilde{\delta}_{2i+mp}(u)\sum_{q=0}^{p-2} {}_{2i+mp-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q + $$

$$ \sum_{k=2i+mp+1}^{w-1} {}_{\kappa}\tilde{\delta}_k(u)\sum_{q=0}^{p-2} {}_{k-\kappa}\delta_{w-k}(u^{p-1-q})u^q $$

By lemma 3, ${}_{k-\kappa}\delta_{w-k}|_{F(u^p)} = c_1\delta + \dots + c_{p-2}\delta^{p-2}$ for $w-k < (p-1)i$ and ${}_{2i+mp-\kappa}\delta_{(p-1)i}|_{F(u^p)} = c_1\delta + \dots + c_{p-1}\delta^{p-1}$ with $c_{p-1} = ({}_i\delta_i(u))^{p-1} \neq 0$.

By proposition 2 we have

$$ {}_{i-\kappa}\delta_{w-i}(u^q) = q_{i-\kappa}\delta_{w-i}(u)u^{q-1} + {}_{i-\kappa}\delta_i(u)\sum_{r=0}^{q-2} {}_{m_1}\delta_{w-2i}(u^{q-1-r})u^r + $$

$$ \sum_{s=2i+mp}^{w-i-1} {}_{i-\kappa}\delta_s(u)\sum_{r=0}^{q-2} {}_{s+i-\kappa}\delta_{w-i-s}(u^{q-1-r})u^r $$

By lemma 3, ${}_{s+i-\kappa}\delta_{w-i-s}|_{F(u)} = c_1\delta + \dots + c_{p-3}\delta^{p-3}$ for $w-i-s < (p-2)i$. Since $i-\kappa = 0 \mod p$, ${}_{i-\kappa}\delta_i(u) = 0$ and ${}_{i-\kappa}\delta_{2i+mp}(u) = 0$. So, ${}_{i-\kappa}\delta_{w-i}|_{F(u)} = c_1\delta + \dots + c_{p-2}\delta^{p-2}$. Hence,

$$ {}_{\kappa}\tilde{\delta}_w(u^p) = -{}_{\kappa}\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1} = -{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1} \neq 0 $$

and ${}_{\kappa}\tilde{\delta}_w(u^p) \in Z(\bar{D})$.

The same arguments show that ${}_{\kappa}\tilde{\delta}_w$ is the first map such that ${}_{\kappa}\tilde{\delta}_w|_{F(u^p)} \neq 0$.

ii) Using the same arguments as above, we can find a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \leq j(1)$ and ${}_{j(1)}\tilde{\bar{\delta}}_q(u^p) = 0$ for $q > j(1)$, $q \neq 2i \mod p$. Since ${}_{j(1)}\tilde{\bar{\delta}}_r|_{F(u^p)}$, $j(1) < r < j(1) + i + mp$ are derivations (see lemma 3), ${}_{j(1)}\tilde{\bar{\delta}}_r|_{F(u^p)} = 0$ for $j(1) < r < j(1) + i + mp$.

Note that now ${}_{j(1)}\tilde{\bar{\delta}}_q(u)$, $q > j(1)$, $q \neq 2i \mod p$ may be not equal to zero.

For $j(1) < w \leq 2i + mp + (p-1)i + i + mp$, $w = 2i \mod p$, by proposition 2 we have

$$ {}_{\kappa}\tilde{\delta}_w(u^p) = {}_{\kappa}\tilde{\delta}_i(u) \sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q + {}_{\kappa}\tilde{\delta}_{2i+mp}(u) \sum_{q=0}^{p-2} {}_{2i+mp-\kappa}\delta_{w-2i-mp}(u^{p-1-q})u^q + \ldots + $$

$$ {}_{\kappa}\tilde{\delta}_{w-(p-1)i}(u) \sum_{q=0}^{p-2} {}_{w-(p-1)i-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q + \sum_{k=w-(p-1)i+1}^{w-1} {}_{\kappa}\tilde{\delta}_k(u) \sum_{q=0}^{p-2} {}_{k-\kappa}\delta_{w-k}(u^{p-1-q})u^q $$

By lemma 3, ${}_{k-\kappa}\delta_{w-k}|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $w - k < (p-1)i$.

Let us prove that ${}_{2i+mp-\kappa}\delta_\zeta|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)i < \zeta < (p-1)i + i + mp$.

If $(p-1)i < \zeta < 2i + mp$, then it is clear that ${}_{2i+mp-\kappa}\delta_\zeta|_{F(u)} = 0$. By proposition 2, for $\zeta \geq 2i + mp$ and $q < p$ we have

$$ {}_{2i+mp-\kappa}\delta_\zeta(u^q) = q_{2i+mp-\kappa}\delta_\zeta(u)u^{q-1} + $$

$$ {}_{2i+mp-\kappa}\delta_i(u) \sum_{r=0}^{q-2} {}_m\delta_{\zeta-i}(u^{q-1-r})u^r + {}_{2i+mp-\kappa}\delta_{2i+mp}(u) \sum_{r=0}^{q-2} {}_{m_1}\delta_{\zeta-2i-mp}(u^{q-1-r})u^r + \ldots $$

Since $\zeta - 2i - mp < (p-2)i$, ${}_{m_1}\delta_s|_{F(u)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ for $s \leq \zeta - 2i - mp$.

To show that ${}_m\delta_{\zeta-i}|_{F(u)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ we use induction on r, where $i + mp \leq \zeta - (p-2-r)i < 2i + mp + ri$. For arbitrary r we can use the same calculations, so we only have to prove that ${}_m\delta_{\zeta-(p-2-r)i}|_{F(u)} = 0$ for some $r \geq 0$.

There exists $r \geq 0$ such that $i + mp \leq \zeta - (p-2-r)i < 2i + mp$. If $2i + mp > 2i$, then ${}_m\delta_{\zeta-(p-2-r)i} = c\delta_i^k$, $k > p$ if $i|\zeta$ and ${}_m\delta_{\zeta-(p-2-r)i} = 0$ otherwise. So, ${}_m\delta_{\zeta-(p-2-r)i}|_{F(u)} = 0$. If $2i + mp \leq 2i$, then $(p-1)i < \zeta < pi$. So, $i < \zeta - (p-2)i < 2i + mp$ and ${}_m\delta_{\zeta-(p-2)i} = 0$.

Let us prove that ${}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp}|_{F(u)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$. Note that ${}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}|_{F(u)} = c_1\delta + \ldots + c_{r+1}\delta^{r+1}$ with $c_{r+1} \neq 0$ for any $0 \leq r \leq p-2$.

Indeed, by proposition 2 we have

$$ {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}(u^q) = q_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}(u)u^{q-1} + {}_{2i+mp-\kappa}\tilde{\delta}_i(u) \sum_{t=0}^{q-2} $$

$$ \sum_{(j_1,\ldots,j_l), l \geq p} C^l_{\kappa-i-mp}\delta_{j_1}\ldots\delta_{j_l}(u^{q-1-t})u^t + {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp}(u) \sum_{t=0}^{q-2} {}_{\kappa}\delta_{ri}(u^{q-1-t})u^t + \ldots $$

By lemma 3, ${}_{\kappa}\delta_{ri}|_{F(u)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r \neq 0$ and ${}_m\delta_s|_{F(u)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $s < ri$.

If there exists $j_k \geq 2i+mp$, then $j_1+\ldots+\hat{j_k}+\ldots+j_l \leq ri$; so there exists $j_t < i$ and $\delta_{j_1}\ldots\delta_{j_l} = 0$. If there are no $j_k \geq 2i+mp$, then $\delta_{j_k} = c\delta_i^k$, $c \in \mathbb{Q}$ and $\delta_{j_1}\ldots\delta_{j_l}|_{F(u)} = 0$, because $l \geq p$.

Hence by lemma 3, ${}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp+ri}|_{F(u)} = c_1\delta + \ldots + c_{r+1}\delta^{r+1}$ with $c_{r+1} = \frac{1}{r+1}{}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^r = \frac{1}{r+1}{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^r \neq 0$.

We have

$$ {}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp} + {}_{2i+mp-\kappa}\tilde{\delta}_{(p-2)i+2i+mp} + {}_{2i+mp-\kappa}\delta_i \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{(p-3)i+2i+mp} + \ldots + $$

$$ {}_{2i+mp-\kappa}\delta_{(p-2)i} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{2i+mp} + {}_{2i+mp-\kappa}\delta_{(p-3)i+2i+mp} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_i = 0 $$

We have ${}_{2i+mp-\kappa}\delta_{ri} \cdot {}_{2i+mp-\kappa}\tilde{\delta}_{(p-2-r)i+2i+mp}|_{F(u)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} = \frac{1}{p-1-r}{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2}$.

Since $({}_{2i+mp-\kappa}\tilde{\delta}_i)^p|_{F(u)} = 0$, and using induction, we get ${}_{2i+mp-\kappa}\delta_{(p-2)i+2i+mp}|_{F(u)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with

$$ c_{p-1} = -(1 + \ldots + \frac{1}{p-1}){}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} - $$

$$ (1+\ldots+\frac{1}{p-2}){}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} - \ldots - {}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} = -{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2} \neq 0 $$

Note that ${}_{\kappa}\tilde{\delta}_{w-(p-1)i}(u)\sum_{q=0}^{p-2} {}_{w-(p-1)i-\kappa}\delta_{(p-1)i}(u^{p-1-q})u^q = 0$.

Indeed, $w-(p-1)i-\kappa = 2i \mod p$. Therefore by lemma 3, (ii), ${}_{w-(p-1)i-\kappa}\delta_{(p-1)i}|_{F(u)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

Finally, let us prove that $\sum_{q=0}^{p-2} {}_{i-\kappa}\delta_{w-i}(u^{p-1-q})u^q = 0$. By proposition 2 we have

$$ {}_{i-\kappa}\delta_{w-i}(u^q) = q\,{}_{i-\kappa}\delta_{w-i}(u)u^{q-1} + {}_{i-\kappa}\delta_i(u)\sum_{r=0}^{q-2} {}_{2i-\kappa}\delta_{w-2i}(u^{q-1-r})u^r + $$

$$ \sum_{s=2i+mp}^{w-i-1} {}_{i-\kappa}\delta_s(u)\sum_{r=0}^{q-2} {}_{s+i-\kappa}\delta_{w-i-s}(u^{q-1-r})u^r $$

Since $i-\kappa = 0 \mod p$, ${}_{i-\kappa}\delta_s(u) = 0$ for $s < 2i+mp+(p-1)i$. For $s \geq 2i+mp+(p-1)i$ we have $w-i-s \leq mp$ and ${}_{s+i-\kappa}\delta_{w-i-s} = c\delta_i^k$, $c \in \mathbb{Q}$. Therefore, to prove our assertion it is sufficient to prove that ${}_{w-(p-2)i-\kappa}\delta_{(p-2)i}|_{F(u)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$. But $w-(p-2)i-\kappa = 3i \mod p$. Therefore by lemma 3, (ii) we get the proof of (ii).

Now we have ${}_{\kappa}\tilde{\delta}_w(u^p) = -{}_{\kappa}\tilde{\delta}_{2i+mp}(u)(-{}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-2}) \in Z(\bar{D})$, which proves (iii).

The lemma is proved.

□

Consider the following two cases.

Case 1. $\delta_i(\tilde{\delta}_{2i+mp}(u)) = 0$ or $i+mp < i$. In this case we have shown that $\delta_i({}_i\tilde{\delta}_{j(1)}(u^p)) = 0$ and $\delta_i({}_i\tilde{\delta}_{j(1)+i+mp}(u^p)) = 0$.

Lemma 6 *Let $\delta_{j(n+1)}$ be the first map such that $\delta_{j(n+1)}|_{F(u^{p^{n+1}})} \neq 0$. Suppose the following conditions hold:*

i') ${}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^p)|_{F(u^{p^{n+1}})} \neq 0$ and ${}_{j(n)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i+mp$;

ii') $\delta_i({}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})) = 0$ and $\delta_i({}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})) = 0$.

Then there exists a parameter z such that the following conditions hold:

i) ${}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)}$ is the first map such that ${}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)}|_{F(u^{p^{n+2}})} \neq 0$;

ii) ${}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp+(p-1)j(n+1)+i+mp}(u^{p^{n+2}}) \neq 0$ and ${}_{j(n+1)}\tilde{\delta}_r|_{F(u^{p^{n+2}})} = 0$ for $j(n+1)+i+mp+(p-1)j(n+1) < r < j(n+1)+i+mp+(p-1)j(n+1)+i+mp$;

iii) $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+2)}(u^{p^{n+2}})) = 0$ and $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+2)+i+mp}(u^{p^{n+2}})) = 0$, where $j(n+2) = j(n+1)+i+mp+(p-1)j(n+1)$.

Proof. First we prove that there exists a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \leq j(n+1)+i+mp$ and ${}_{j(n+1)}\tilde{\bar{\delta}}_q(u^{p^{n+1}}) \neq 0$ only if $q = 2i \mod p$ for $q > j(n+1)+i+mp$; here $\bar{\delta}_q$ are the maps given by the parameter $\bar{z}$.

Suppose ${}_{j(n+1)}\tilde{\bar{\delta}}_q(u^{p^{n+1}}) \neq 0$, $q > j(n+1)+i+mp$ and $q \neq 2i \mod p$. By definition, ${}_{j(n+1)}\tilde{\delta}_q(u^{p^{n+1}}) = -j(n+1)\delta_q(u^{p^{n+1}}) + \sum \delta_{k_1}\ldots\delta_{k_l}(u^{p^{n+1}})$, where $k_i < q$. By lemma 2, (ii), for any $a \in \bar{D}$ there exists a parameter $\bar{z}_q$ such that

$$\bar{z}_q u^{p^{n+1}} \bar{z}_q^{-1} = u^{p^{n+1}} + \delta_{j(n+1)}(u^{p^{n+1}})\bar{z}_q^{j(n+1)} + \ldots + \delta_{q-1}(u^{p^{n+1}})\bar{z}_q^{q-1} + a\bar{z}_q^q + \ldots$$

Therefore there exists an element $a \in \bar{D}$ such that ${}_{j(n+1)}\tilde{\delta}_q(u^{p^{n+1}}) = 0$. It is easy to see that the sequence $\{\bar{z}_q\}$ converges in D. So, $\bar{z} = lim\bar{z}_q$.

Now we prove that ${}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}}) \neq 0$ and ${}_{j(n+1)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i+mp$ and $\delta_i({}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})) = 0$.

We have $j(n+1) = j(n) \mod p$. Therefore,

$$z^{-j(n+1)}u^{p^{n+1}}z^{j(n+1)} = z^{-pk}(z^{-j(n)}u^{p^{n+1}}z^{j(n)})z^{pk} = z^{-pk}(u^{p^{n+1}} + {}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{j(n+1)} +$$

$${}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \ldots)z^{pk} =$$

$$z^{-pk}u^{p^{n+1}}z^{pk} + z^{-pk}{}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{pk}z^{j(n+1)} + {}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \ldots =$$

$$u^{p^{n+1}} + {}_{j(n)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}})z^{j(n+1)} + {}_{j(n)}\tilde{\delta}_{j(n+1)+i+mp}(u^{p^{n+1}})z^{j(n+1)+i+mp} + \ldots,$$

because ii') provide $\delta_i(\delta_r(u^{p^{n+1}})) = 0$ for $j(n+1) < r < j(n+1)+i+mp$. So, $z^{-pk}u^{p^{n+1}}z^{pk} = u^{p^{n+1}} \mod M_D^{j(n+1)+2i+mp}$.

i) Put $w = j(n+1) + i + mp + (p-1)j(n+1)$, $t = u^{p^{n+1}}$. By proposition 2 we have

$$ {}_{j(n+1)}\tilde{\delta}_w(t^p) = {}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t) \sum_{q=0}^{p-2} {}_{i+mp}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q + $$

$$ \sum_{k=j(n+1)+i+mp+1}^{w-1} {}_{j(n+1)}\tilde{\delta}_k \sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q $$

By lemma 3, ${}_{k-j(n+1)}\delta_{w-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $w-k < (p-1)j(n+1)$ and ${}_{i+mp}\delta_{(p-1)j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$. Therefore, ${}_{j(n+1)}\tilde{\delta}_w(t^p) = -{}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t)c_{p-1} \neq 0$.

The same arguments show that ${}_{j(n+1)}\tilde{\delta}_w$ is the first map such that ${}_{j(n+1)}\tilde{\delta}_w(t^p) \neq 0$.

ii) Put $t = u^{p^{n+1}}$. Using the same arguments as above, we can find a parameter $\bar{z}$ such that $\bar{\delta}_q = \delta_q$ for $q \leq j(n+1) + i + mp + (p-1)j(n+1)$ and ${}_{j(n+1)}\tilde{\bar{\delta}}_q(t^p) = 0$ for $q > j(n+1) + i + mp + (p-1)j(n+1)$, $q \neq 2i \mod p$. Since ${}_{j(n+1)}\tilde{\bar{\delta}}_r|_{F(t^p)}$, $j(n+1) + i + mp + (p-1)j(n+1) < r < j(n+1) + i + mp + (p-1)j(n+1) + i + mp$ are derivations (see lemma 3), ${}_{j(n+1)}\tilde{\bar{\delta}}_r|_{F(t^p)} = 0$ for $j(n+1) + i + mp + (p-1)j(n+1) < r < j(n+1) + i + mp + (p-1)j(n+1) + i + mp$.

For $j(n+1)+i+mp+(p-1)j(n+1) < w \leq j(n+1)+i+mp+(p-1)j(n+1)+i+mp$, $w = 2i \mod p$, by proposition 2 we have

$$ {}_{j(n+1)}\tilde{\delta}_w(t^p) = {}_{j(n+1)}\tilde{\delta}_{j(n+1)+i+mp}(t) \sum_{q=0}^{p-2} {}_{i+mp}\delta_{(p-1)j(n+1)+i+mp}(t^{p-1-q})t^q + \ldots + $$

$$ {}_{j(n+1)}\tilde{\delta}_{w-(p-1)j(n+1)}(t) \sum_{q=0}^{p-2} {}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q + $$

$$ \sum_{k=w-(p-1)j(n+1)+1}^{w-1} {}_{j(n+1)}\tilde{\delta}_k(t) \sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q $$

By lemma 3, ${}_{k-j(n+1)}\delta_{w-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $w-k < (p-1)j(n+1)$. Therefore, $\sum_{q=0}^{p-2} {}_{k-j(n+1)}\delta_{w-k}(t^{p-1-q})t^q = 0$.

Note that ${}_{j(n+1)}\tilde{\delta}_{w-(p-1)j(n+1)}(t) \sum_{q=0}^{p-2} {}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}(t^{p-1-q})t^q = 0$.

Indeed, $w - (p-1)j(n+1) - j(n+1) = 2i \mod p$. Therefore by lemma 3 (ii), ${}_{w-(p-1)j(n+1)-j(n+1)}\delta_{(p-1)j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

Let us prove that ${}_{k-j(n+1)}\delta_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $(r+1)j(n+1) < \zeta < (r+1)j(n+1) + i + mp$, $r \leq p-2$.

The proof is by induction on r. By ii') and i'), $\delta_s(t) = 0$ for $j(n+1) < s < j(n+1) + i + mp$. Therefore for $r = 0$, ${}_{k-j(n+1)}\delta_\zeta|_{F(t)} = 0$.

For arbitrary r we have

$$ {}_{k-j(n+1)}\delta_\zeta(t^q) = q_{k-j(n+1)}\delta_\zeta(t) + {}_{k-j(n+1)}\delta_{j(n+1)}(t) \sum_{r=0}^{q-2} {}_k\delta_{\zeta-j(n+1)}(t^{q-1-r})t^r + $$

$$\sum_{s=j(n+1)+i+mp}^{\zeta-1} {}_{k-j(n+1)}\delta_s(t) \sum_{r=0}^{q-2} {}_{s+(k-j(n+1))}\delta_{\zeta-s}(t^{q-1-r})t^r$$

By lemma 3, ${}_{s+(k-j(n+1))}\delta_{\zeta-s}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta - s < rj(n+1)$.

For any m ${}_m\delta_{\zeta-j(n+1)}|_{F(t)} = 0$ if $r = 1$, and ${}_m\delta_{\zeta-j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ by induction and lemma 3, because

$${}_m\delta_{\zeta-j(n+1)}(t^q) = q\,{}_m\delta_{\zeta-j(n+1)}(t)t^{q-1} + {}_m\delta_{j(n+1)}(t) \sum_{l=0}^{q-2} {}_{m_1}\delta_{\zeta-2j(n+1)}(t^{q-1-l})t^l +$$

$$\sum_{s=j(n+1)+i+mp}^{\zeta-j(n+1)-1} {}_m\delta_s \sum_{l=0}^{q-2} {}_{m_s}\delta_{\zeta-j(n+1)-s}(t^{q-1-l})t^l,$$

and by lemma 3, ${}_{m_s}\delta_{\zeta-j(n+1)-s}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $s \geq j(n+1) + i + mp$.

The same arguments show that ${}_{k-j(n+1)}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $(r+1)j(n+1) < \zeta < (r+1)j(n+1) + i + mp$.

Let us show that ${}_{i+mp}\delta_{(p-1)j(n+1)+i+mp}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$, $c_{p-1} \neq 0$. Put $\zeta = (p-1)j(n+1) + i + mp$. We have

$${}_{i+mp}\delta_\zeta + {}_{i+mp}\tilde{\delta}_\zeta + \sum_{s=1}^{w-k-1} {}_{i+mp}\delta_{\zeta-s} \cdot {}_{i+mp}\tilde{\delta}_s = 0$$

First we prove that ${}_{i+mp}\tilde{\delta}_{rj(n+1)+i+mp}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r = \frac{1}{r}{}_{i+mp}\tilde{\delta}_{j(n+1)+i+mp}(t)\left({}_{j(n+1)}\delta_{j(n+1)}(t)\right)^{r-1} \neq 0$. We use the same arguments as above. The proof is by induction on r. For $r = 0$, since $i + mp < j(n+1)$, ${}_{i+mp}\tilde{\delta}_{i+mp}|_{F(t)} = 0$.

Put $w = rj(n+1) + i + mp$. For arbitrary r we have

$${}_{i+mp}\tilde{\delta}_w(t^q) = q\,{}_{i+mp}\tilde{\delta}_w(t)t^{q-1} + {}_{i+mp}\tilde{\delta}_{j(n+1)}(t) \sum_{r=0}^{q-2} {}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t^{q-1-r})t^r +$$

$$\sum_{s=j(n+1)+i+mp}^{w-1} {}_{i+mp}\tilde{\delta}_s(t) \sum_{r=0}^{q-2} {}_{s-i-mp}\delta_{w-s}(t^{q-1-r})t^r$$

By lemma 3, ${}_{s-i-mp}\delta_{w-s}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $w - s < (r-1)j(n+1)$ and ${}_{j(n+1)}\delta_{(r-1)j(n+1)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ with $c_{r-1} = \left({}_{j(n+1)}\delta_{j(n+1)}(t)\right)^{r-1} \neq 0$.

By proposition 2 we have

$${}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t^q) = q\,{}_{j(n+1)-i-mp}\delta_{w-j(n+1)}(t)t^{q-1} +$$

$${}_{j(n+1)-i-mp}\delta_{j(n+1)}(t) \sum_{r=0}^{q-2} {}_{m_1}\delta_{w-2j(n+1)}(t^{q-1-r})t^r +$$

$$\sum_{s=j(n+1)+i+mp}^{w-j(n+1)-1} {}_{j(n+1)-i-mp}\delta_s(t) \sum_{r=0}^{q-2} {}_{s+j(n+1)-i-mp}\delta_{w-j(n+1)-s}(t^{q-1-r})t^r$$

By lemma 3, ${}_{s+j(n+1)-i-mp}\delta_{w-j(n+1)-s}|_{F(t)}=c_1\delta+\ldots+c_{r-3}\delta^{r-3}$ for $w-j(n+1)-s<(r-2)j(n+1)$. Since $j(n+1)-i-mp=0 \mod p$, ${}_{j(n+1)-i-mp}\delta_{j(n+1)}(t)=0$ and ${}_{j(n+1)-i-mp}\delta_{j(n+1)+i+mp}(t)=0$ (here we use also i') and ii')). Therefore,
${}_{j(n+1)-i-mp}\delta_{w-j(n+1)}|_{F(t)}=c_1\delta+\ldots+c_{r-2}\delta^{r-2}$ and lemma 3 completes the proof.

The same arguments show that ${}_{i+mp}\tilde{\delta}_{rj(n+1)}|_{F(t)}=c_1\delta+\ldots+c_{r-1}\delta^{r-1}$. Therefore, ${}_{i+mp}\delta_{\zeta-s}\cdot{}_{i+mp}\tilde{\delta}_s|_{F(t)}=c_1\delta+\ldots+c_{p-1}\delta^{p-1}$ only if $s=j(n+1)$ or $s>j(n+1)$ and $s=rj(n+1)+i+mp$. Note that $c_{p-1}=\frac{1}{r}{}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}$ if $s=rj(n+1)+i+mp$.

Using the same arguments for the maps ${}_{i+mp}\delta_{\zeta-rj(n+1)}$, we get ${}_{i+mp}\delta_\zeta|_{F(t)}=c_1\delta+\ldots+c_{p-1}\delta^{p-1}$, where

$$c_{p-1}=-(1+\frac{1}{2}+\ldots+\frac{1}{p-1}){}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}-$$

$$(1+\ldots+\frac{1}{p-2}){}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}-\ldots-{}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}=$$

$$-{}_i\tilde{\delta}_{j(n+1)+i+mp}(t)({}_i\delta_{j(n+1)}(t))^{p-2}\neq 0.$$

This completes the proof of ii) and iii).
The lemma is proved.
□

Case 2. $\delta_i(\tilde{\delta}_{2i+mp}(u))\neq 0$.

Lemma 7 *Suppose $i+mp>i$. Put $t=u^p$. Then there exists a parameter z such that the following properties hold:*

i) ${}_{j(1)}\tilde{\delta}_{j(1)+i+mp+(p-1)j(1)}$ is the first map such that ${}_{j(1)}\tilde{\delta}_{j(1)+i+mp+(p-1)j(1)}|_{F(t^p)}\neq 0$.

ii) ${}_{j(1)}\tilde{\delta}_r|_{F(t^p)}=0$ for $j(2)<r<j(2)+i$ and ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p)\neq 0$, where $j(2)=j(1)+i+mp+(p-1)j(1)$.

iii) ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p)\in Z(\bar{D})$, ${}_{j(1)}\tilde{\delta}_{j(2)}(t^p)\in Z(\bar{D})$.

Proof. To prove i) one can repeat the proof of i) in lemma 6. Note that ${}_{j(1)}\tilde{\delta}_{j(2)}(t^p)=-{}_{j(1)}\tilde{\delta}_{j(1)+i}({}_i\delta_{j(1)}(t))^{p-2}\in Z(\bar{D})$.

ii) As in the case 1 we can find a parameter $\bar{z}$ such that $\bar{\delta}_q=\delta_q$ for $q\leq j(1)+i+mp$, $\tilde{\bar{\delta}}_q(t^p)=0$ for $q\neq 2i \mod p$, $q>j(2)$.

For $r=2i \mod p$, by proposition 2 we have

$${}_{j(1)}\tilde{\delta}_r(t^p)={}_{j(1)}\tilde{\delta}_{j(1)+i+mp}(t)\sum_{q=0}^{p-2}{}_{i+mp}\delta_{r-j(1)-i-mp}(t^{p-1-q})t^q+$$

$$\sum_{k=j(1)+i+mp+1}^{r-1}{}_{j(1)}\tilde{\delta}_k(t)\sum_{q=0}^{p-2}{}_{k-j(1)}\delta_{r-k}(t^{p-1-q})t^q$$

By lemma 3, ${}_{k-j(1)}\delta_{r-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $r-k < (p-1)j(1)$.

Note that ${}_{j(1)}\tilde{\delta}_{r-(p-1)j(1)}(t)\sum_{q=0}^{p-2} {}_{r-pj(1)}\delta_{(p-1)j(1)}(t^{p-1-q})t^q = 0$.

Indeed, $r - pj(1) = 2i \mod p$. Therefore by lemma 3 (ii), ${}_{r-pj(1)}\delta_{(p-1)j(1)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

The same arguments as in lemma 6 (ii) show that ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)j(1) < s < (p-1)j(1)+i$.

Let us prove that ${}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$.

First let us show that ${}_{i+mp}\tilde{\delta}_{\zeta}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ for $\zeta = (r+1)j(1)+i$, $r \le p-2$. For $r=0$, ${}_{i+mp}\tilde{\delta}_{\zeta}|_{F(t)}$ is a derivation. Since ${}_{j(1)}\tilde{\delta}_{\zeta}(t) = 0$ for $\zeta = j(1)+i$ and $j(1) = i+mp \mod p$ and $p > 2$, we have ${}_{i+mp}\tilde{\delta}_{\zeta}(t) = 0$ and ${}_{i+mp}\tilde{\delta}_{\zeta}|_{F(t)} = 0$.

For arbitrary r we have

$$ {}_{i+mp}\tilde{\delta}_{\zeta}(t^q) = q\, {}_{i+mp}\tilde{\delta}_{\zeta}(t)t^{q-1} + {}_{i+mp}\tilde{\delta}_{j(1)}(t)\sum_{r=0}^{q-2} {}_{j(1)-i-mp}\delta_{\zeta-j(1)}(t^{q-1-r})t^r + $$

$$ \sum_{k=j(1)+i+1}^{\zeta-1} {}_{i+mp}\tilde{\delta}_k(t)\sum_{r=0}^{q-2} {}_{k-i-mp}\delta_{\zeta-k}(t^{q-1-r})t^r $$

By lemma 3, ${}_{k-i-mp}\delta_{\zeta-k}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta - k < rj(1)$, i.e. $k \ge j(1)+i+1$.

By definition,

$$ {}_{j(1)-i-mp}\delta_{\zeta-j(1)} = \sum_{(j_1,\ldots,j_l),l\ge p} C^l_{j(1)-i-mp}\delta_{j_1}\ldots\delta_{j_l}, $$

because $j(1)-i-mp = 0 \mod p$. Since $l \ge p$, there exist j_k, j_{k_1} such that $j_k < j(1)$ and $j_{k_1} < j(1)$. Thus, $j_1 + \ldots + \hat{j_k} + \ldots + \hat{j}_{k_1} + \ldots + j_l < rj(1)$ and $\delta_{j_1}\ldots\delta_{j_l}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$. Hence by lemma 3, ${}_{i+mp}\tilde{\delta}_{\zeta}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$.

Now we have

$$ {}_{i+mp}\delta_{(p-1)j(1)+i} + {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i} + \sum_{k=i}^{(p-1)j(1)} {}_{i+mp}\delta_k \cdot {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i-k} = 0 $$

with ${}_{i+mp}\delta_k \cdot {}_{i+mp}\tilde{\delta}_{(p-1)j(1)+i-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $k \neq (p-2)j(1)+i$.

Therefore,

$$ {}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2} - {}_{i+mp}\delta_{(p-2)j(1)+i} \cdot {}_{i+mp}\tilde{\delta}_{j(1)}|_{F(t)} $$

So by induction,

$$ {}_{i+mp}\delta_{(p-1)j(1)+i}|_{F(t)} = \tilde{c}_1\delta + \ldots + \tilde{c}_{p-2}\delta^{p-2} + {}_{i+mp}\delta_i({}_{i+mp}\delta_{j(1)})^{p-1}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2} + $$

$$ {}_{i+mp}\delta_i(({}_i\delta_{j(1)}(t))^{p-1})\delta^{p-1} $$

Since ${}_i\delta_{j(1)}(t) = -{}_{j(1)}\tilde{\delta}_{j(1)}(t) = {}_i\tilde{\delta}_{2i+mp}(u)({}_i\delta_i(u))^{p-1}$, we have ${}_{i+mp}\delta_i({}_i\delta_{j(1)}(t)) \neq 0$, which completes the proof of ii).

Finally, ${}_{j(1)}\tilde{\delta}_{j(2)+i}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(1)+i+mp}(t)_{i+mp}\delta_i({}_i\delta_{j(1)}(t))({}_i\delta_{j(1)}(t))^{p-2} \in Z(\bar{D})$, because ${}_{i+mp}\delta_i$ is a derivation and ${}_i\delta_{j(1)}(t) \in Z(\bar{D})$. This proves iii).
□

The following lemma completes the proof of Case 2 and of Theorem.

Lemma 8 *Suppose the following conditions hold:*

i') ${}_{j(1)}\tilde{\delta}_{j(n)+i}|_{F(u^{p^n})} \neq 0$ and ${}_{j(1)}\tilde{\delta}_r|_{F(u^{p^n})} = 0$ for $j(n) < r < j(n)+i$, $n \geq 1$;

ii') ${}_{j(1)}\tilde{\delta}_{j(n)}(u^{p^n}), {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n}) \in Z(\bar{D})$.

Then there exists a parameter z such that

i) ${}_{j(1)}\tilde{\delta}_{j(n)+i+(p-1)j(n)}$ is the first map such that ${}_{j(1)}\tilde{\delta}_{j(n)+i+(p-1)j(n)}|_{F(u^{p^{n+1}})} \neq 0$;

ii) ${}_{j(1)}\tilde{\delta}_{j(n+1)+i}(u^{p^{n+1}}) \neq 0$ and ${}_{j(1)}\tilde{\delta}_r|_{F(u^{p^{n+1}})} = 0$ for $j(n+1) < r < j(n+1)+i$, where $j(n+1) = j(n)+i+(p-1)j(n)$;

iii) ${}_{j(1)}\tilde{\delta}_{j(n+1)}(u^{p^{n+1}}), {}_{j(1)}\tilde{\delta}_{j(n+1)+i}(u^{p^{n+1}}) \in Z(\bar{D})$.

Proof. By induction, $j(n) = i \mod p$. Put $a = {}_{j(1)}\tilde{\delta}_{j(n)}(u^{p^n})$. We have $a^{kj(n)+1} = a^{p^l} \in Z(D)$ for some $k \in \mathbb{Z}$. Put $z' = (a^{-kj(1)}z^{j(1)})^{1/j(1)}$. We claim that i'), ii') hold for ${}_{j(1)}\tilde{\delta'}_{j(n)+i}$, i.e. ${}_{j(1)}\tilde{\delta'}_{j(n)+i}|_{F(u^{p^n})} \neq 0$, and ${}_{j(1)}\tilde{\delta'}_r|_{F(u^{p^n})} = 0$ for $j(n) < r < j(n)+i$, and ${}_{j(1)}\tilde{\delta'}_{j(n)+i}(u^{p^n}), {}_{j(1)}\tilde{\delta'}_{j(n)}(u^{p^n}) \in Z(\bar{D})$. Moreover, ${}_{j(1)}\tilde{\delta'}_{j(n)}(u^{p^n}) \in Z(D)$.
Note that $\alpha' = id$, because $a \in Z(\bar{D})$. We have

$$z'^{-j(1)}u^{p^n}z'^{j(1)} = z^{-j(1)}u^{p^n}z^{j(1)} = u^{p^n} + az^{j(n)} + {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})z^{j(n)+i} + \dots$$

Let us show that

$$z'^{j(n)} = a^{-kj(n)}z^{j(n)} \mod \wp^{j(n)+i+1}$$

We have

$$z'^{j(n)} = z'^{j(n)-j(1)}a^{-kj(1)}z^{j(1)}.$$

It is easy to see that $z' = a^{-k}z + xz^{i+1} + \dots,\ x \in \bar{D}$. Since $j(n) - j(1) = 0 \mod p$, $z'^{j(n)-j(1)} = (a^{-k}z)^{j(n)-j(1)} \mod \wp^{j(n)-j(1)+i+1}$. Now we have

$$(a^{-k}z)^{j(n)-j(1)}a^{-kj(1)} = a^{-kj(n)}z^{j(n)-j(1)} + xz^{j(n)-j(1)+i} + \dots,$$

where $x = [-kj(1) - k(j(1)+1) - \dots - k(j(n)-1)]a^{-kj(n)-1}\delta_i(a) = 0$. Therefore,

$$z'^{-j(1)}u^{p^n}z'^{j(1)} = u^{p^n} + a^{p^l}z'^{j(n)} + {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})a^{k(j(n)+i)}z'^{j(n)+i} + \dots$$

and ${}_{j(1)}\tilde{\delta'}_{j(n)+i}(u^{p^n}) = {}_{j(1)}\tilde{\delta}_{j(n)+i}(u^{p^n})a^{k(j(n)+i)} \in Z(\bar{D})$, ${}_{j(1)}\tilde{\delta'}_{j(n)}(u^{p^n}) = a^{p^l} \in Z(D)$. So i'), ii') hold.

Now to prove i) one can repeat the proof of i) in lemma 6. Note that ${}_{j(1)}\tilde{\delta}_{j(n+1)}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(n)+i}({}_i\delta_{j(n)}(t))^{p-2} \in Z(\bar{D})$.

ii) We use the same arguments as in ii) of lemma 7. Put $t = u^{p^n}$.

For $r = 2i \mod p$, by proposition 2 we have

$$ {}_{j(1)}\tilde{\delta}_r(t^p) = {}_{j(1)}\tilde{\delta}_{j(n)}(t) \sum_{q=0}^{p-2} {}_{j(n)-j(1)}\delta_{r-j(n)}(t^{p-1-q})t^q + {}_{j(1)}\tilde{\delta}_{j(n)+i}(t) \sum_{q=0}^{p-2} {}_{j(n)-j(1)+i}\delta_{r-j(n)-i}(t^{p-1-q})t^q + $$

$$ \sum_{k=j(n)+i+1}^{r-1} {}_{j(1)}\tilde{\delta}_k(t) \sum_{q=0}^{p-2} {}_{k-j(1)}\delta_{r-k}(t^{p-1-q})t^q $$

By lemma 3, ${}_{k-j(1)}\delta_{r-k}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ if $r-k < (p-1)j(n)$.

Note that ${}_{j(n)-j(1)}\delta_{r-j(n)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$. Indeed, by proposition 2 we have

$$ {}_{j(n)-j(1)}\delta_{r-j(n)}(t^q) = q\,{}_{j(n)-j(1)}\delta_{r-j(n)}(t)t^{q-1} + {}_{j(n)-j(1)}\delta_{j(n)}(t) \sum_{s=0}^{q-2} {}_{2j(n)-j(1)}\delta_{r-2j(n)}(t^{q-1-s})t^s + $$

$$ {}_{j(n)-j(1)}\delta_{j(n)+i}(t) \sum_{s=0}^{q-2} {}_{2j(n)-j(1)+i}\delta_{r-2j(n)-i}(t^{q-1-s})t^s + \ldots + $$

$$ {}_{j(n)-j(1)}\delta_{j(n)+2i}(t) \sum_{s=0}^{q-2} {}_{2j(n)-j(1)+2i}\delta_{r-2j(n)-2i}(t^{q-1-s})t^s + \ldots $$

Recall that $r \le pj(n) + 2i$. By lemma 3, ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-3}\delta^{p-3}$ if $s < (p-2)j(n)$. Since ${}_{j(1)}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have $\delta_{j(n)}(t) \in Z(D)$. Since $j(n) - j(1) = 0 \mod p$ and $\delta_{j(n)}(t) \in Z(D)$ and $charF > 2$, we have ${}_{j(n)-j(1)}\delta_{j(n)+ei}(t) = 0$ for $e \le p-1$, which completes the proof.

Note that ${}_{j(1)}\tilde{\delta}_{r-(p-1)j(n)}(t) \sum_{q=0}^{p-2} {}_{r-(p-1)j(n)-j(1)}\delta_{(p-1)j(n)}(t^{p-1-q})t^q = 0$.

Indeed, $r - (p-1)j(n) - j(1) = 2i \mod p$. Therefore by lemma 3 (ii), ${}_{r-(p-1)j(n)-j(1)}\delta_{(p-1)j(n)}|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$.

The same arguments as in lemma 6 (ii) show that ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{p-2}\delta^{p-2}$ for $(p-1)j(n) < s < (p-1)j(n) + i$.

Let us prove that ${}_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with $c_{p-1} \neq 0$.

Note that ${}_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r \neq 0$ for any $1 \le r \le p-1$.

Indeed, by proposition 2 we have

$$ {}_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}(t^q) = q\,{}_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}(t)t^{q-1} + {}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)}(t) \sum_{s=0}^{q-2} $$

$$ {}_{j(1)-i}\delta_{(r-1)j(n)+i}(t^{q-1-s})t^s + {}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)+i}(t) \sum_{s=0}^{q-2} {}_{j(1)}\delta_{(r-1)j(n)}(t^{q-1-s})t^s + \ldots $$

By lemma 3, ${}_{j(1)}\delta_{(r-1)j(n)}|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ with $c_{r-1} \neq 0$ and ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$ for $s < (r-1)j(n)$.

Let us prove that ${}_{j(1)-i}\delta_{(r-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$. By proposition 2 we have

$$ {}_{j(1)-i}\delta_{(r-1)j(n)+i}(t^q) = q_{j(1)-i}\delta_{(r-1)j(n)+i}(t)t^{q-1} + {}_{j(1)-i}\delta_{j(n)}(t)\sum_{s=0}^{q-2} {}_{j(n)+j(1)-i}\delta_{(r-2)j(n)+i}(t^{q-1-s})t^s + $$

$$ {}_{j(1)-i}\delta_{j(n)+i}(t)\sum_{s=0}^{q-2} {}_{j(n)+j(1)}\delta_{(r-2)j(n)}(t^{q-1-s})t^s + \ldots $$

Since ${}_{j(1)}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have $\delta_{j(n)}(t) \in Z(D)$. Since $j(1) - i = 0 \mod p$, ${}_{j(1)-i}\delta_{j(n)}(t) = 0$ and ${}_{j(1)-i}\delta_{j(n)+i}(t) = 0$. By lemma 3, ${}_m\delta_s|_{F(t)} = c_1\delta + \ldots + c_{r-3}\delta^{r-3}$ for $s < (r-2)j(n)$. So, ${}_{j(1)-i}\delta_{(r-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{r-2}\delta^{r-2}$.

Hence by lemma 3, ${}_{j(n)-j(1)+i}\tilde{\delta}_{rj(n)+i}|_{F(t)} = c_1\delta + \ldots + c_r\delta^r$ with $c_r = \frac{1}{r}{}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{r-1} = \frac{1}{r}{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{r-1} \neq 0$.

The same arguments show that ${}_{j(n)-j(1)+i}\tilde{\delta}_\zeta|_{F(t)} = c_1\delta + \ldots + c_{r-1}\delta^{r-1}$ for $\zeta < rj(n)+i$.

Therefore we have

$$ {}_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i} + {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i} + $$

$$ \sum_{w=1}^{(p-1)j(n)+i-1} {}_{j(n)-j(1)+i}\delta_w \cdot {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i-w} = 0, $$

where ${}_{j(n)-j(1)+i}\delta_w \cdot {}_{j(n)-j(1)+i}\tilde{\delta}_{(p-1)j(n)+i-w}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ only if $w = rj(n)$. In this case $c_{p-1} = \frac{1}{p-1-r}{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2}$.

Since ${}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)}(t) \in Z(D)$, we have ${}_{j(n)-j(1)+i}\delta_i({}_{j(n)-j(1)+i}\tilde{\delta}_{j(n)})^{p-1} = 0$. So using induction, we get ${}_{j(n)-j(1)+i}\delta_{(p-1)j(n)+i}|_{F(t)} = c_1\delta + \ldots + c_{p-1}\delta^{p-1}$ with

$$ c_{p-1} = -(1 + \ldots + \frac{1}{p-1}){}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2} - $$

$$ (1 + \ldots + \frac{1}{p-2}){}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2} - \ldots - {}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2} = $$

$$ -{}_i\tilde{\delta}_{j(n)+i}(t)({}_i\delta_{j(n)}(t))^{p-2} \neq 0. $$

Therefore, ${}_{j(1)}\tilde{\delta}_{j(n+1)+i}(t^p) = -{}_{j(1)}\tilde{\delta}_{j(n)+i}(t)c_{p-1} \in Z(\bar{D})$. This proves ii) and iii). The lemma is proved.

□

The theorem is proved.

□

Lemma 9 *Let* $n = |Gal(Z(\bar{D})/\bar{F})|$.

There exists a parameter z *such that*

$$ zaz^{-1} = \alpha(a) + \delta_n(a)z^n + \delta_{2n}(a)z^{2n} + \ldots, \quad a \in \bar{D} $$

So, $\delta_j = 0$ *if* $n \not| j$.

One can repeat the proof of lemma 10 in [4] to prove the lemma.

Proposition 3 *Suppose* $Z(\bar{D})/\bar{F}$ *is not a separable extension. Then* p *does not divide* $|Gal(Z(\bar{D})/\bar{F})|$.

Proof. Suppose p divides $|Gal(Z(\bar{D})/\bar{F})|$. By lemma 9 there exists a parameter z such that

$$zaz^{-1} = \alpha(a) + \delta_n(a)z^n + \delta_{2n}(a)z^{2n} + \dots, \quad a \in \bar{D},$$

where $n = |Gal(Z(\bar{D})/\bar{F})|$.

Since $Z(\bar{D})/\bar{F}$ is a compositum of a purely inseparable extension and abelian Galois extension, there exists an element $u \in Z(\bar{D})$ such that $\alpha(u) = u$, i.e. u is a purely inseparable element; so by theorem 2 $u^p \in Z(D)$.

In this case lemma 3 holds for $l = 0$ and we can repeat the arguments in the proof of lemma 4 to show that $\delta_{pi}(u^p) \neq 0$, which is a contradiction.
□

Proposition 4 *We have* $D \cong D_1 \otimes_F D_2$, *where* D_1, D_2 *are division algebras such that* D_1 *is an inertially split algebra, and* $Z(\bar{D}_2)/\bar{F}$ *is a purely inseparable extension.*

So, $D \sim A \otimes_F B \otimes_F D_2$, *where* A *is a cyclic division algebra and* B *is an unramified division algebra.*

Proof. By [10], p.261, $D \cong D_1 \otimes_F \dots \otimes_F D_k$, where $[D : F] = p_1^{r_1} \dots p_k^{r_k}$ and $[D_i : F] = p_i^{r_i}$. Let $p_2 = p$. By proposition 3, $Z(\bar{D}_2)/\bar{F}$ is a purely inseparable extension. Since D_i are defectless over F, $D_1, D_3, \dots D_k$ are inertially split. Decomposition theorems [2], Thm. 5.6-5.15 complete the proof.
□

Proposition 5 *Let* D_2 *be a division algebra such that* $Z(\bar{D}_2)/\overline{Z(D_2)}$ *is a purely inseparable extension. Then* $D_2 \cong D_3 \otimes_{Z(D_2)} D_4$, *where* D_3 *is an unramified division algebra and* D_4 *is a division algebra such that* $\bar{D}_4$ *is a field,* $\bar{D}_4/\overline{Z(D_2)}$ *is a purely inseparable extension,* $[\bar{D}_4 : \overline{Z(D_2)}] = [\Gamma_{D_4} : \Gamma_{Z(D_2)}]$.

Proof. By theorem 2 there exists a subfield $\overline{Z(D_2)} \subset K \subset Z(\bar{D}_2)$ such that the extension $Z(\bar{D}_2)/K$ has degree p. Then by theorem 1 and 2 there exists a lift $\tilde{K}$ of K in D_2, i.e. $\bar{\tilde{K}} = K$, $\Gamma_{\tilde{K}} = \Gamma_{Z(D_2)}$, $K \subset \tilde{K}$.

Consider the centralizer $C_1 = C_{D_2}(\tilde{K})$. By theorem 1 we have $\bar{C_1} = \bar{D}_2$, $Z(\bar{C_1})/\overline{Z(C_1)}$ is a purely inseparable extension of degree p, $Z(\bar{C_1}) = \overline{Z(C_1)}(u)$. By theorem 2 there exists a parameter z such that $C_1 \cong \bar{C_1}((z))$ as a vector space with the relation

$$zaz^{-1} = a + \delta_i(a)z^i + c_{2i}\delta_i^2(a)z^{2i} + \dots, \quad c_{ki} \in \mathbb{F}_p$$

and $zuz^{-1} = u + xz^i$, where $x \in Z(C_1)$. Therefore, δ_i^p is a derivation trivial on the centre $Z(\bar{C_1})$, hence by Scolem-Noether theorem it is an inner derivation.

We claim that $z^p \in Z(C_1)$. To prove it, consider a subalgebra $W = \bar{C_1}((z^i)) \subset C_1$ (note that $Z(W) \neq Z(C_1)$). It exists because of the type of the relation in C_1.

We have

$$z^{-i} a z^{i} = a - i\delta_i(a) z^{i}, \quad a \in \bar{C}_1$$

in W. Therefore,

$$z^{-pi} a z^{pi} = a - i^p \delta_i^p(a) z^{pi}, \quad a \in \bar{C}_1$$

and

$$z^{pi} a z^{-pi} = a + \delta_1'(a) z^{pi} + {\delta_1'}^2(a) z^{2pi} + \dots,$$

where $\delta_1' = i^p \delta_i^p$. So,

$$z^p a z^{-p} = a + \frac{1}{i}\delta_1'(a) z^{pi} + c_2 \frac{1}{i^2}{\delta_1'}^2(a) z^{2pi} + \dots,$$

where c_k are given by (1) in theorem 2. So, $z^p \in Z(C_1)$ iff $\delta_i^p = 0$. Suppose $\delta_i^p \neq 0$. Consider an element $Y \in Z(C_1)$, $w(Y) > 0$. Let

$$Y = a_1 z^p + \dots$$

First note that

$$Y = a_1 z^p + a_2 z^{2p} + a_3 z^{3p} + \dots, \quad a_i \in \bar{C}_1$$

Indeed, Y must satisfy $[Y, u] = 0$. Since $u \in Z(\bar{C}_1)$, we then have $[z^{i_k}, u] = 0$ for every k, where

$$Y = \sum_{k=1}^{\infty} a_k z^{i_k}$$

Therefore, $p | i_k$.

Then, Y must satisfy $Ya = aY$ for any $a \in \bar{C}_1$. Therefore, $a_1, \dots a_i \in Z(\bar{C}_1)$ and we must have

$$a a_{i+1} - a_{i+1} a = a_1 \delta_1'(a)/i$$

and

$$a a_{2i+1} - a_{2i+1} a = a_i \delta_1'(a) + a_1 c_2 {\delta_1'}^2(a).$$

So ${\delta_1'}^2 = \delta$, where δ is a derivation, which is a contradiction. Therefore, ${\delta_1'}^2 = \delta = 0$ and $\delta_1' = 0$, and $z^p \in Z(C_1)$.

Consider the centralizer $C_2 = C_{C_1}(\tilde{K}(z))$. It's clear that $[\bar{C}_2 : Z(\bar{C}_2)] = [\bar{C}_2 : \overline{Z(C_2)}] = ind\bar{C}_1$. By theorem 2 there exists a subalgebra $\bar{C}_2 \subset C_2$. Consider now the centralizer $C_3 = C_{D_2}(Z(D_2)(z))$. We have $\bar{C}_2 \subset C_3$, $\bar{C}_3 \cong \bar{C}_2$, because $[Z(\bar{C}_3) : \overline{Z(D_2)}] = [\Gamma_{C_3} : \Gamma_{Z(D_2)}] = [Z(\bar{C}_2) : \overline{Z(D_2)}] = [K : \overline{Z(D_2)}]$. By induction on dimension of $\bar{D}_2$ we get the existence of a subalgebra $\bar{C}_4$ such that $[\bar{C}_4 : Z(\bar{C}_4)] = [\bar{C}_2 : Z(\bar{C}_2)]$, $Z(\bar{C}_4) = \overline{Z(D_2)}$. Therefore there exists an unramified subalgebra $C_4 \subset D_2$ such that $[C_4 : Z(D_2)] = [C_4 : Z(D_2)] = [C_4 : Z(C_4)] = [\bar{D}_2 : Z(\bar{D}_2)]$. By Double Centralizer Theorem, $D_2 \cong C_4 \otimes_{Z(D_2)} D_4$ with $[\bar{D}_4 : \overline{Z(D_2)}] = [\Gamma_{D_4} : \Gamma_{Z(D_2)}]$, so $\bar{D}_4$ is a field.
The proposition is proved.
□

Proposition 6 *Let* D_2 *be a division algebra such that* $\bar{D}_2$ *is a field,* $\bar{D}_2/\overline{Z(D_2)}$ *is a purely inseparable extension.*

Then $D_2 \cong A_1 \otimes_{Z(D_2)} \ldots \otimes_{Z(D_2)} A_m$ *, where* A_i *are cyclic division algebras of degree* p *,* $[\bar{A}_i : \overline{Z(D_2)}] = [\Gamma_{A_i} : \Gamma_{Z(D_2)}]$ *.*

Proof. The proof immediately follows from theorem 2 and [1], Thm.3, §2.8.(see [11] for the proof of this theorem).
□

So, we get the following decomposition theorem.

Theorem 3 *Let* D *be a finite dimensional central division algebra over a field* F *with a discrete complete rank 1 valuation,* $char(F) = p > 2$ *.*

Then $D \cong D_1 \otimes_F D_2 \otimes_F A_1 \otimes_F \ldots \otimes_F A_m$ *, where* A_i *are cyclic division algebras of degree* p *,* $[\bar{A}_i : \overline{Z(D)}] = [\Gamma_{A_i} : \Gamma_{Z(D)}]$ *,* D_1 *is an inertially split division algebra,* $(ind(D_2), p) = 1$ *,* D_2 *is an unramified division algebra.*

Corollary 2 *The following conjecture: the exponent of* A *is equal to its index for any division algebra* A *over a* C_2 *-field* F *(see for example [1], 3.4.5.) has the positive answer for* $F = F_0((t_1))((t_2))$ *,* F_0 *is an algebraically closed field.*

Acknowledgements

I am very grateful to Professor E.-W. Zink, Professor A. N. Parshin and M. Grabitz for useful discussions and attention to my work.

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