Remarks on the Prehistory of Sobolev Spaces

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Contents

1. Introduction

From the very beginning of the development of the classical calculus of variations it was considered as evident that minimum problems for variational integrals [1)] which have a finite lower bound, admit a minimizing function. In particular, the validity of the following so-called “Dirichlet Principle” was widely accepted: the variational integral

$$\mathcal{D}(u) := \int_{\Omega} |Du|^2 \, \mathrm{d}x \qquad \text{(Dirichlet integral)}$$

admits a minimizing function on every appropriate subset of functions of $C^1(\Omega)$ (Ω a bounded domain in $\mathbb{R}^N$). This principle played an important role in the development of the theory of analytic functions. It was used by B. Riemann without mathematically satisfactory justification [(1) 2)].

In 1870, K. Weierstrass [Wei 1] observed that the existence of minimizing functions for variational integrals is, however, by no means guaranted. To illustrate this, he considered the following minimum problem. Minimize the variational integral

$$\mathcal{I}(u) := \int_{-1}^{1} t^2 \big(u'(t)\big)^2 \, \mathrm{d}t$$

over the set

$$\mathcal{K} := \big\{ u \in C^1\big([-1,1]\big) \;\big|\; u(-1) = -1, u(1) = 1 \big\} \,.$$

Clearly, $\mathcal{I}(u) \geq 0$ for all $u \in \mathcal{K}$. To determine the infimum of $\mathcal{I}$ on $\mathcal{K}$, define

$$u_\varepsilon(t) := \frac{\arctan \frac{t}{\varepsilon}}{\arctan \frac{1}{\varepsilon}} \,, \qquad \varepsilon > 0 \,, \; t \in [-1,1] \,.$$

Then $u_\varepsilon \in \mathcal{K}$ and $\mathcal{I}(u_\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that

$$\inf \big\{ \mathcal{I}(u) \;\big|\; u \in \mathcal{K} \big\} = 0 \,,$$

however there does not exist a function $u_0 \in \mathcal{K}$ such that $\mathcal{I}(u_0) = 0$. Indeed, $\mathcal{I}(u_0) = 0$ implies $u_0(t) = \text{const}$ for all $t \in [-1,1]$, hence $u_0(-1) = u_0(1)$, i.e. $u_0 \notin \mathcal{K}$.

We consider an example of a minimum problem for $\mathcal{D}(u)$ for functions of several variables. Define

$$\Omega := B_1(0) \setminus \{0\} = \big\{ x \in \mathbb{R}^N \;\big|\; 0 < |x| < 1 \big\} \,,$$

$$\mathcal{M} := \left\{ u \in C(\bar{\Omega}) \cap C^1(\Omega) \;\middle|\; u(0) = 1, \; u(x) = 0 \; \forall \, |x| = 1, \; \int_{\Omega} |Du|^2 \, \mathrm{d}x < +\infty \right\} .$$

[1)] That is, integrals of the type $\displaystyle\int_{\Omega} f(x, u, Du) \, \mathrm{d}x$.

[2)] The numbers in brackets ${}^{(\,)}$ refer to the Comments at the end of Sect. 2.

The minimum problem for the Dirichlet integral $\mathcal{D}(u)$ is now as follows:

$$\textit{minimize } \mathcal{D}(u) = \int_\Omega |Du|^2 \, \mathrm{d}x \textit{ over } \mathcal{M}.$$

Clearly, $\mathcal{D}(u) \geq 0$ for all $u \in \mathcal{M}$. To determine the infimum of $\mathcal{D}(u)$ when u runs through $\mathcal{M}$, fix $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $t \leq 1$, $0 \leq \eta(t) \leq 1$ for $1 < t < 2$ and $\eta(t) = 0$ for $t \geq 2$. For $0 < \varepsilon < 1$, define

$$u_\varepsilon(x) := \begin{cases} 1 - \eta\left(\dfrac{\log|x|}{\log\varepsilon}\right) & \text{if } \ 0 < |x| \leq 1\,, \\ 1 & \text{if } \ x = 0\,. \end{cases}$$

Then

$$u_\varepsilon(x) = \begin{cases} 1 & \text{if } \quad |x| \leq \varepsilon^2\,, \\ 0 & \text{if } \ \varepsilon \leq |x| \leq 1\,. \end{cases}$$

Hence $u_\varepsilon \in C_c^\infty\big(B_1(0)\big)$; in particular,

$$\frac{\partial u_\varepsilon}{\partial x_i}(x) = \begin{cases} 0 & \text{if } \quad |x| < \varepsilon^2\,, \\ -\eta'\left(\dfrac{\log|x|}{\log\varepsilon}\right)\dfrac{x_i}{(\log\varepsilon)|x|^2} & \text{if } \ \varepsilon^2 \leq |x| \leq \varepsilon\,, \\ 0 & \text{if } \ \varepsilon < |x| \leq 1 \end{cases}$$

$(i = 1, \ldots, n)$. Using spherical coordinates we find

$$\begin{aligned} \int_{B_1(0)} \big|Du_\varepsilon(x)\big|^2 \, \mathrm{d}x &\leq \max_{\mathbb{R}} |\eta'| \frac{n|B_1|}{(\log\varepsilon)^2} \int_{\varepsilon^2}^{\varepsilon} r^{n-3} \, \mathrm{d}r \\ &= \max_{\mathbb{R}} |\eta'| \frac{n|B_1|}{(\log\varepsilon)^2} \begin{cases} -\log\varepsilon & \text{if } \ N = 2, \\ \dfrac{1}{N-2}\left(\varepsilon^{N-2} - \varepsilon^{2(N-2)}\right) & \text{if } \ N \geq 3 \end{cases} \\ &\longrightarrow 0 \ \text{ as } \ \varepsilon \to 0\,. \end{aligned}$$

We obtain

$$\inf\big\{\mathcal{D}(u) \,\big|\, u \in \mathcal{M}\big\} = 0\,.$$

Analogously as in Weierstrass' example, there does not exist $u_0 \in \mathcal{M}$ such that $\mathcal{D}(u_0) = 0$ (indeed, $\mathcal{D}(u_0) = 0$ would imply $u_0(x) = \text{const}$ for all $|x| \leq 1$, but this u_0 does not obey the boundary conditions both in $x = 0$ and on $\big\{x \,\big|\, |x| = 1\big\}$).

■

Weierstrass' example indicated the necessity to establish with complete rigor the existence of minimizing functions for variational integrals within appropriate function classes. The first important progress in justifying the "Dirichlet Principle" has been made by using the class of continuous functions of several variables which are absolutely continuous in each variable for almost all of the others. These works were initiated by Italian mathematicians during the first two decades of the last century (cf. Sect. 2.1). Subsequently various existence theorems for minimizing functions within this class of functions have been proved for rather general types of variational integrals.

Since the 1920's, another impulse for the use of essentially the same function classes came from the Göttingen school and was motivated by the rapidly growing interaction between functional analysis and the theory of partial differential equations (cf. Sect. 2.2). Here the functional analytic framework for boundary and eigenvalue problems for partial differential equations led to the consideration of the class of those L^2-functions which have weak derivatives in L^2. These function classes turned out to be an appropriate frame for the study of spectral properties of differential operators.

Finally, various different approaches to the concept of generalized solution to partial differential equations also contributed to the invention of function classes of the types mentioned above (cf. Sect. 2.3 and 2.4).

From the mid 1930's on, S. L. Sobolev studied weak solutions to hyperbolic equations and slightly later the minimization of certain variational integrals. These studies led him to the use of the class of those L^p-functions whose generalized derivatives of order m are in L^p. This class of functions has been later called "Sobolev space" and was denoted by $W_p^{(m)}$ (cf. Sect. 2.5).

The aim of the present paper is to sketch some aspects of these developments up to the appearence of S. L. Sobolev's book [So 8] in 1950. In the Appendix we give a brief discussion of the historical development of the concept of approximation of integrable functions by smooth functions.

2. Aspects of the Prehistory of Sobolev Spaces

After Weierstrass' critique of Riemann's use of the "Dirichlet Principle" several investigations have been devoted to the justification of this principle.

A first rigorous existence proof for a minimizing function for the Dirichlet integral $\mathcal{D}(u)$ ($u \in C(\bar{\Omega}) \cap C^1$, $u = g$ on $\partial\Omega$) has been given by D. Hilbert [H 1] in 1900 (cf. Sect. 2.2). Although written in the realm of "classical techniques", this paper marked the beginning of the so-called direct methods of the calculus of variations[3)].

From the historical point of view, the justification of the "Dirichlet Principle" [4)]

[3)]The role of Hilbert's work for the development of the direct methods of the calculus of variations is discussed in Giaquinta, M.: *Hilbert e il calcolo delle variazioni.* Le Matematiche 55 (2000), suppl. no. 1, 47-58. See also the "Comments" (2) below.

[4)]A detailed discussion of the history of the "Dirichlet Principle" is given in M. Giaquinta / S. Hildebrandt [GH], S. Hildebrandt [Hid 2] and A. S. Monna [Mon].

The counterexamples by F. Prym and J. Hadamard to the "Dirichlet Principle" are mentioned in [Hid 2] (see also: Leis, R.: *Zur Entwicklung der angewandten Analysis und mathematischen Physik in den letzen hundert Jahren.* In: Ein Jahrhundert Mathematik 1890-1990. Hrsg.: G. Fischer, F. Hirzebruch, W. Scharlau, W. Törning. Dtsch. Math.-Ver., Friedr. Vieweg, Braunschweig 1990; pp. 497-500.).

(and, more generally, the proof of the existence of minimizing functions for variational integrals) gave the first impulse for the invention of classes of functions which later on turned out to coincide with the Sobolev spaces $W_2^{(1)}$ resp. $W_1^{(1)}$.

2.1 The Work of B. Levi, G. Fubini and L. Tonelli.

An important step towards a mathematically satisfactory justification of the "Dirichlet Principle" has been made by B. Levi [Lev]. He considered the Dirichlet integral $\mathcal{D}(u)$ for the following class of functions $u = u(x, y)$ in a bounded domain $\Omega \subset \mathbb{R}^2$:

1) *u is continuous in $\overline{\Omega}$;*

2) *u has partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ satisfying*

$$\int\limits_{x_0}^{x} \frac{\partial u(\xi, y)}{\partial \xi} d\xi = u(x, y) - u(x_0, y)$$

for almost all lines $y = const$, and analogously for x and y interchanged;

3) *u takes prescribed values on the boundary $\partial\Omega$;*

4) $\mathcal{D}(u) < +\infty$.

B. Levi proved the existence of a minimizing sequence for $\mathcal{D}(u)$ belonging to this set of functions and converging uniformly in $\overline{\Omega}$ to a limit function u^*; moreover, he showed that u^* has partial derivatives $\dfrac{\partial u^*}{\partial x}$, $\dfrac{\partial u^*}{\partial y}$ in L^2 (in fact in L^1) and that u^* renders a minimum to $\mathcal{D}(u)$ over the set of u's define above (cf. [Lev ; pp. 338-347]).

Thus, in contrast to Hilberts' work [H 1] (and R. Courant's works [Co 1]-[Co 5]), B. Levi used extensively the Lebesgue integral in his approach to the "Dirichlet Principle". This allowed him to work with a larger class of functions in which the minimizing function for the Dirichlet integral $\mathcal{D}(u)$ could easier be found.

G. Fubini [Fu 1] used the same class of functions $u = u(x, y)$ in $\Omega \subset \mathbb{R}^2$ for his investigation of the "Dirichlet Principle" as B. Levi (cf. [Fu 1; p. 65]). [5)] He proved the following results:

1. Let (u_i) be a minimizing sequence for $\mathcal{D}(u)$ in this class, i.e.

$$\mathcal{D}(u_i) = d + \varepsilon; \qquad \lim_{i\to\infty} \varepsilon_i = 0,$$

where

$$d = inf\ \mathcal{D}(u).$$

[5)] G. Fubini has been familar with the works of H. Weber, D. Hilbert and B. Levi on the "Dirichlet Principle".

Then a subsequence (u_{i_k}) can be extracted such that the series $\sum_{k=1}^{\infty} \varepsilon_{i_k}^{1/3}$ converges. At each point of Ω (except for a set of measure zero), there holds

$$\lim_{k\to\infty} u_{i_k} = v; \qquad\qquad \mathcal{D}(v) = d$$

2. There holds

$$\lim_{k\to\infty} \int\limits_{\Lambda} u_{i_k} ds = \int\limits_{\Lambda} v ds$$

for each line segment or curve Λ in Ω. If ω is a subdomain of Ω, then there holds

$$\lim_{k\to\infty} \int\limits_{\omega} \frac{\partial u_{i_k}}{\partial x_j} dx_1 dx_2 = \int\limits_{\omega} \frac{\partial v}{\partial x_j} dx_1 dx_2 \qquad (j = 1, 2).$$

In addition, the function v is continuous in $\overline{\Omega}$ and harmonic in Ω.

The methods of proof of these results make extensive use of arguments from Lebesgue measure and integration. Later G. Fubini completed and extended these results in [Fu 2] and [Fu 3].

In [T 1], L. Tonelli developed the concept of lower semicontinuity for variational integrals

$$\int\limits_a^b f(t, u(t), u'(t)) dt$$

where u is an absolutely continuous function on the interval $[a, b]$ [6)]. A systematic presentation of the calculus of variations for this class of variational integrals is given in his monograph [T 2].

Later L. Tonelli studied the problem of characterization of the class of surfaces given by

$$z = f(x, y), \qquad (x, y) \in R = [0, 1] \times [0, 1],$$

for which the area $\mathcal{S}$ defined by Lebesgue, can be expressed by

$$\mathcal{S} = \int\limits_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy.$$

In [T 3] he proved that these functions have to be continuous in R, absolutely continuous in each variable for almost all values of the other, and to have the first order

[6)] The investigation of real functions with discontinuous derivatives can be traced back to the works of U. Dini. The concept of absolutely continuous functions of one variable is due to G. Vitali. The work of B. Levi, L. Tonelli and G. Fubini is rooted in the Italian school of analysis. This school largely contributed to Lebesgue measure and integration.

partial derivatives in $L^1(R)$. This class of functions has been further investigated in [T 4]. The same class of functions was then used in [T 5] as frame for proving theorems on the lower semicontinuity of variational integrals.

The paper [T 6] contains a large part of L. Tonelli's fundamental contributions to the direct methods of the calculus of variations for functions of two variables. After defining appropriate classes of bounded domains $D \subset \mathbb{R}^2$, L. Tonelli introduced *absolutely continuous functions u in any bounded domain $D \subset \mathbb{R}^2$* (cf. also [T 5]):

1) u is continuous in D;

2) for almost all y_0 and x_0, $u(x, y_0)$ and $u(x_0, y)$ are absolutely continuous on each intersection of the straight lines $y = y_0$ and $x = x_0$, respectively with D;

3) the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are integrable on D in the sense of Lebesgue.

For this class of functions, he then proved the following results:

- the uniform continuity in the interior of D or in $\overline{D}$, respectively, of the family of those absolutely continuous functions u satisfying

$$\int_D \left(\left| \frac{\partial u}{\partial x} \right|^{2+\alpha} + \left| \frac{\partial u}{\partial y} \right|^{2+\alpha} \right) dxdy \leq Const, \quad (\alpha > 0) \tag{2.1}$$

  (cf.[T 6; pp. 97, 104]) [7];

- the existence of a "trace" on $\partial\mathcal{D}$ for functions u satisfying (2.1) (cf. p. 100).

L. Tonelli next introduced the concept of a "*complete class of functions with respect to $F\left(x, y, u(x, y), \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$*" (cf. p. 118) and proved then several existence theorems for minimizing functions for the variational integral

$$\int_D F\left(x, y, u(x, y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dxdy.$$

These existence results are completed in [T 7].

The paper [T 6] (and O. Nikodym's paper [N 1]) seem to be the first works where the above class of absolutely continuous functions are studied as a mathematical object of independent interest.

2.2 The Göttingen School.

In 1900, D. Hilbert [H 1] presented a proof of the existence of a minimizing function for the Dirchlet integral $\mathcal{D}(u)$ in an appropriate function class. His method

[7] L. Tonelli thus proved the imbeddings $W_p^{(1)}(D) \subset C(D)$ resp. $W_p^{(1)}(D) \subset C(\overline{D})$ $(p > 2)$.

of proof consists in constructing a suitable sequence of functions and proving its convergence to the minimum of $\mathcal{D}(u)$ via a compactness argument. The detailed proof appeared in [H 3].

D. Hilbert's work [H 1] gave a fundamental programmatic impulse for the development of an existence theory for minimizing functions for variational integrals. (2) From the 1920's on, a large part of the work of R. Courant, D. Hilbert, K. Friedrichs and others in Göttingen was concerned with applying methods of functional analysis for proving the existence of minimizing functions for variational integrals, and for solving boundary value and eigenvalue problems for partial differential equations. In particular, broad interest was devoted to the study of spectral properties of differential operators.

These studies led to the necessity to extend the notion of classical partial derivative to a concept of generalized derivative in order to obtain spaces of differentiable functions which are complete with respect to the L^2-norm of the function and their generalized derivatives. These function spaces have been introduced as the completion of the vector space

$$\left\{ u \in C^1(\Omega) \Big| \int_\Omega \left(u^2 + \sum_{j=1}^N \left(\frac{\partial u}{\partial x_j} \right)^2 \right) dx < +\infty \right\} \tag{2.2}$$

with respect to the metric

$$d(u,v) := \left\{ \int_\Omega \left((u-v)^2 + \sum_{j=1}^N \left(\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j} \right)^2 \right) dx \right\}^{1/2} .$$

Later on it turned out that this completion is linearly isometric to the Sobolev space $W_2^{(1)}(\Omega)$.

The completion of (2.2) with respect to the metric $d(\cdot,\cdot)$ (i. e. the space $W_2^{(1)}$) as a mathematical object of its own right has, however, not attracted independent research interest by the Göttingen school. A specific feature of many works of this school in the period 1912 - 1930, was to dispense with techniques of Lebesgue measure and integration and in place of it to work with improper Riemann integrals. A basic idea behind these works on minimum problems and eigenvalue problems for partial differential equations consisted in introducing the bilinear form

$$[u,v] := \int_\Omega \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + buv \right) dx$$

for u, v in the space (2.2), extending this form from (2.2) onto its completion (" ... to the ideal elements") and applying then methods from functional analysis. With regard to eigenvalue problems, K. Friedrich wrote:

> *"Die erste Aufgabe ist, die Räume der zulässigen Funktionen anzugeben. Diese sind nun zunächst keine Hilbertschen Räume; sie werden aber*

durch Adjunktion idealer Elemente zu Hilbertschen Räumen fortgesetzt; wir verzichten darauf, diese idealen Elemente durch nach Lebesgue quadratisch integrierbare Funktionen zu realisieren; insbesondere deshalb, weil gezeigt werden kann, daß die "Eigenelemente", die vor allem interessieren, doch schon den Ausgangsfunktionsräumen angehören."

(cf. [F 2; part I, pp. 467-468]).

From the numerous works of the Göttingen mathematicians which develop these ideas, we only mention the following ones.

- In 1919, R. Courant [Co 1] outlined a new justification of the "Dirichlet Principle" which was simpler than D. Hilbert's method on the one hand, and which could simpler be applied to problems in conformal mapping theory on the other one. This new method was then fully elaborated in [Co 2].

  Subsequently R. Courant [Co 3] - [Co 6] developed the foundations of the variational methods for eigenvalue problems for partial differential equations within the frame of the space (2.2).

  Most of the results known up to the end of the 1940's, are presented in the monograph [Co 7].

- The paper K. Friedrichs [Fr 1] was motivated by the investigations of R. Courant [Co 2] - [Co 5].

  The papers [Fr 2] - [Fr 5] represent an important part of K. Friedrichs' contributions to spectral theory of partial differential operators. These investigations seem to be stimulated by the rapidly developing interaction between quantum mechanics and linear unbounded operators in Hilbert spaces.

- The paper F. Rellich [Rel] played a fundamental role in a number of works of the Göttingen school on eigenvalue problems (cf. e.g. [CH; pp. 489 - 495]).

- In Chap. 7 of part II of the famous monograph [CH], R. Courant and D. Hilbert develop a systematic theory of boundary value and eigenvalue problems for partial differential equations by using variational methods in the frame of the completion of (2.2) with respect to the metric d.

Finally, in his paper [Wey], H. Weyl presented an elegant solution of the "Dirichlet Principle" in terms of the orthogonal projection of a vector of a Hilbert space onto a closed subspace, and proved a regularity theorem for the minimizing function of the Dirichlet integral (the "Weyl's lemma"). This paper which appeared after the period of the Göttingen school, makes extensive use of Lebesgue integration theory.

2.3 The Contributions of O. Nikodym and J. Leray.

The works of B. Levi [Lev] and G. Fubini [Fu 1] were the motivation for O. Nikodym [N 1] to investigate the class (BL) (*"fonction de M. Beppo Levi"*) of those functions u in a bounded domain $D \subset \mathbb{R}^3$ which are defined a. e. in D, absolutely continuous on almost all line segments parallel to the axis intersecting D, and have partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}$ which are square integrable in D. He introduced the semi-norm

$$[u]_D := \left\{ \int\limits_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dx\, dy\, dz \right\}^{1/2}$$

for functions u of the class (BL) and proved the following results:

- the functions of the class (BL) are square integrable in D;
- the function class (BL) is complete with respect to the semi-norm $[\cdot]_D$;
- for u, u_k ($k \in \mathbb{N}$) in the function class (BL) satisfying $[u_k - u]_D \to 0$ as $k \to \infty$, there exist $a_k \in \mathbb{R}$ such that $\|u_k + a_k - u\|_{L^2(D)} \to 0$.

Later it became clear that the functions of the class (BL) can be identified with functions of the Sobolev space $W_2^{(1)}$.

O. Nikodym's work marked a significant step of the development of the theory of Sobolev spaces. Although the paper [N 1] arose from O. Nikodym's investigations on the "Dirichlet Principle", in contrast to his predecessors B. Levi and G. Fubini, he studied the class of functions (BL) as a mathematical object of its own right:

> *"Je mi propose ici de développer quelques propriétés des fonctions (BL) non seulement à cause de leur importance, mais surtout parce qu'elles sont intéressantes en elles mèmes."*

(cf. [N 1; p. 129]). In the paper [N 2] which appeared some years later, O. Nikodym developed the basics of the space (BL) within the context of the Hilbert space terminology [(3)].

An important approach to the investigation of the non-stationary Navier-Stokes equations in $\mathbb{R}^3$ has been developed by J. Leray [Ler] in 1934. He multiplied these equations by smooth divergence-free vector functions and performed an integration by parts in t over the interval $[0, T]$, and in x over $\mathbb{R}^3$. This test function method gives an integral identity which represents the "weak formulation" of the non-stationary Navier-Stokes equations in $\mathbb{R}^3 \times [0, T]$ (including an initial datum). To prepare this, J. Leray introduced the notion of "quasi-derivative":

> ***"Définition des quasi-dérivées:***
>
> *Soient deux fonctions de carrés sommables sur $\mathbb{R}^3$, u et $u_{,i}$; nous dirons que $u_{,i}$ est la quasi-dérivée de u par rapport à x_i quand la relation*
>
> $$\int\limits_{\mathbb{R}^3} \left(u(x) \frac{\partial \varphi}{\partial x_i}(x) + u_{,i}(x) \varphi(x) \right) dx = 0$$

> *sera vérifiée; rappelons que dans cette relation φ représente une quelconque des fonctions admettant des dérivées premières continues qui sont, comme ces fonctions elles-mémes, de carrés sommables sur $\mathbb{R}^3$."*

(cf. [Ler; p. 205]) (4).

Besides this notion, he introduced the "approximation d'une fonction mesurable par une suite de fonctions régulières", i. e. (for a locally integrable function u in $\mathbb{R}^3$)

$$u_\varepsilon(x) := \frac{1}{\varepsilon^3} \int\limits_{\mathbb{R}^3} \lambda\left(\frac{|x-y|^2}{\varepsilon^2}\right) u(y)\, dy\,, \qquad \varepsilon > 0\,,\ x \in \mathbb{R}^3\,,$$

where $\lambda \in C^\infty([0,+\infty[)$, $\lambda \geq 0$ in $[0,+\infty[$, $\lambda(s) = 0$ for all $s \geq 1$ and

$$4\pi \int\limits_0^1 \lambda(s^2) s^2\, ds = 1$$

(cf. [Ler; p. 206]).

J. Leray then established some properties of functions having "quasi-derivatives" and studied the convergence of u_ε as $\varepsilon \to 0$ for $u \in L^2(\mathbb{R}^3)$, and for $u \in L^2(\mathbb{R}^3)$ which have "quasi-derivatives" $u_{,i} \in L^2(\mathbb{R}^3)$ $(i = 1, 2, 3)$. He used these results for proving the existence of a solution to the above integral identity (representing the weak formulation of the non-stationary Navier-Stokes equations).

Thus J. Leray used the Sobolev space $W_2^{(1)}(\mathbb{R}^3)$ as frame for his existence theory for weak solutions to the non-stationary Navier-Stokes equations in $\mathbb{R}^3$. .

2.4 The Work of G. C. Evans, C. W. Calkin and Ch. B. Morrey.

G. C. Evans used the Lebesgue-Stieltjes integral for studies in potential theory. The motivation behind the work [E 1] (1920) he described as follows:

> *"These studies originated in 1907, when it first became apparent to me that the theory was unnecessarily complicated by the for of the Laplacian operator, but I did no work on the subject until 1913, when it occurred to me to use instead of the operator*
>
> $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u\,,$$
>
> *the operator*
>
> $$\lim_{h\to 0} \frac{u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)}{h^2}\,,$$
>
> *or the operator*
>
> $$\lim_{\sigma\to 0} \int_s \frac{\partial u}{\partial n}\, ds\,,$$

where s is a closed contour containing an area σ which is allowed to approach zero. The first of these operators had been used by H. Petrini, and conditions for the existence of the limit discussed [8)]*. The second idea, under the form of the equation*

$$\int_s \frac{\partial u}{\partial n}\, ds = 0\,,$$

had, it turned out, been discussed by Bôcher [9)]*, with relation to Laplace's equation; of the two it is obviously the concept which is more closely allied with the physical interpretation."*

(cf. [E 1]; pp. 253-254]).

G. C. Evans then studied in details Stieltjes potentials. With these results at hand he first generalized the gradient vector in $\mathbb{R}^2$ directly (cf. [E 1; p. 274]), and introduced next the notion of generalized derivative of a function u:

"We say that $D_\alpha u$, the generalized derivative in the direction α of $u(M)$, is the limit, if such limit exists, of the expression

$$D_\alpha u := \lim_{\sigma \to 0} \frac{1}{\sigma} \int\limits_s u\, d\alpha'$$

where the fixed direction α' makes an angle $\dfrac{\pi}{2}$ with the fixed direction α, and σ denotes the area enclosed by s; it is understood that σ tends towards 0 in such a way that the ratio $\dfrac{\sigma}{d^2}$, where d is the diameter of σ, remains different from 0 by some positive quantity."

(cf. [E 1; p. 275]).

After this G. C. Evans studied the relation between his notion of generalized derivative and the differentiability a. e.. He proved: If $u(x, y)$ is a potential function in the open rectangle $\Omega \subset \mathbb{R}^2$ of its generalized derivatives then there exists a point $(x_0, y_0) \in \Omega$ such that the function

$$\bar{u}(x, y) := u(x_0, y_0) + \int\limits_{y_0}^{y} D_y u(x_0, \eta)\, d\eta + \int\limits_{x_0}^{x} D_x u(\xi, y)\, d\xi$$

differs from $u(x, y)$ only on a set of measure zero; "... moreover, *almost everywhere in the rectangle* the derivative in the usual sense $\dfrac{\partial \bar{u}}{\partial x}$ exists and is identical with $D_x u$." (cf. [E 1; p. 278]).

8) H. Petrini, "Les derivées premiers et secondes du potentiel", Acta Mathematica, vol. 31 (1908), pp. 127-332.

9) M. Bôcher, "On harmonic functions in two dimensions", Proceedings of the American Academy of Science, vol. 41 (1905-06).

Later G. C. Evans became familiar with the works of L. Tonelli [T 3], [T 4]. In the paper [E 2] he established the following:

> *"... L. Tonelli formulated the definition of "funzione di due variabili assolutamente continua", in a study of general problems relating to areas of surfaces. In this appendix the two notions are compared. It happens that if in the definition of potential function of generalized derivatives the function is assumed to be continuous, as a point function, the specialized concept thus obtained is identical with the one, just mentioned, formulated by Tonelli."*

(cf. [E 2; pp. 43-45]).

Thus, the motivation behind the work of G. C. Evans to generalize the concept of partial derivative, was to solve Poisson's equation with more singular right hand sides. This was done by using Green's formula and the "test curves method". In [E 2], G. C. Evans has schown that the two definitions of generalized derivatives which arose from completely different sources, namely minimization of variational integrals and potential theory, essentially concide.

The work of G. C. Evans and L. Tonelli inspired C. W. Calkin [Ca] and Ch. B. Morrey [Mor 1] [10)] to investigate in detail the space of those continuous functions $u = u(x_1, \cdots, x_N)$ which are absolutely continous in each variable x_j for almost all of the other variables $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N$ and have their derivatives $\dfrac{\partial u}{\partial x_i}$ in $L^p (i = 1, \cdots, N; 1 \leq p < +\infty)$. These functions may be viewed as a natural generalization of G. C. Evan's functions which are potential functions of their generalized derivatives.

Among other results, C. W. Calkin and Ch. B . Morrey prove that these fucntions form a Banach space with respect to the norm

$$\left\{ \int\limits_{\Omega} \left(|u|^p + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p \right) dx \right\}^{1/p},$$

and that for each subdomain $\Omega' \subset \overline{\Omega'} \subset \Omega$ there exists a sequence of uniformly Lipschitz functions (u_k) $(k \in \mathbb{N})$ in Ω' such that

$$\int\limits_{\Omega'} \left(|u_k - u|^p + \sum_{i=1}^{N} \left| \frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|^p \right) dx \to 0 \qquad \text{as } k \to \infty.$$

Moreover, these authors make clear the notion of boundary values for these functions and establish the general form of linear continuous functionals on the space of these fucntions.

Thus, C. W. Calkin and Ch. B. Morrey introduced and studied independently of S. L. Sobolev, a space of functions which was later denoted by $W_p^{(1)}(\Omega)$. A detailed

[10)] The two papers [Ca] and [Mo1] are in fact a joint work.

discussion of the relations between the function spaces introduced by C. W. Calkin and Ch. B. Morrey, and the Sobolev spaces $W_p^{(1)}$ is given by S. Hildebrandt [Hid 1].

In [Mor 2], Ch. B. Morrey proved results on the Hölder continuity of functions in $W_p^{(1)}$ which are nowadays well-known as "Morrey growth lemma" resp. "Morrey estimate".

2.5 The Work of S. L. Sobolev.

The work of S. L. Sobolev is grown in the St. Peterburg school of partial differential equations. During many decades, the scientific activities of this school were connected with the work of V. A. Steklov, V. I. Smirnov and N. M. Gjunter.

After finishing his studies in 1929, S. L. Sobolev was employed at the Seismological Institute of the Academy of Sciences in St. Peterburg up to 1932. During this time his mathematical research was mainly concerned with wave propagation in inhomogeneous media.

In 1935, S. L. Sobolev [So 1] presented a theory of generalized solutions to the wave equation. He sketched the influence of N. M. Gjunter's work concerning this concept of solution, as follows:

> *"As we shall see later, very closely to this field of ideas are the investigations of N. M. Gjunter which are concerned with the potential equation and the heat equation. N. M. Gjunter showed that for these problems of mathematical physics it is proven to be useful to pass from the differential equation in its classical form to the investigation of certain integral identities which contain derivatives of orders smaller than those of the differential equation we started from."* (Russian)

(cf. [So 1; p. 39]) (cf. also the miscellaneous remarks below).

In this paper, a generalized solution to the wave equation is defined as the L^1-limit of C^2-solutions of this equation. These investigations made extensive use of the mean function (=mollifier) of an integrable function.

The paper [So 2] also appeared in 1935. In this paper, S. L. Sobolev introduced a concept of continuous linear functionals on spaces of continuously differentiable functions (later on called "distibutions of finite order") and announced an existence theorem for a (later on called "distributional") solution to a large class of hyperbolic equations. The proofs of these results are presented in [So 3] (5).

S. L. Sobolev did not continue the study of this new concept of solution to hyperbolic equations, but turned to the investigation of continuously differentiable functions which are square integrable in an open set of $\mathbb{R}^N$, and to the study of the polyharmonic equation. In the paper [So 4] he announced the following result:

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying a cone condition. Let $L_s(A)$ denote the set of all functions $u \in C^s(\Omega)$ such that

$$\int_\Omega \left(\frac{\partial^j u}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_N^{\alpha_N}}\right)^2 dx \le A = \text{const}$$

for all $\alpha_1 + \ldots + \alpha_N = j \le s$. *Define* $k := [\frac{N}{2}] + 1$. *Then:*

1. *The functions of* $L_k(A)$ *are uniformly bounded in* Ω.
2. *The functions of* $L_k(A)$ *are uniformly Hölder continuous in* Ω *with Hölder exponent* $\mu < 1$ *for* N *even, and with Hölder exponent* $\mu = \frac{1}{2}$ *for* N *odd.*

In this paper, S. L. Sobolev presented an integral representation for continuously differentiable functions which is called nowadays *"Sobolev integral representation"*. This integral representation is then used in [So 6] and [So 7].

The results of [So 4] were generalized by V. I. Kondrašov [K] from L^2 to L^p $(1 < p < +\infty)$.

The proofs of the results which were announced in [So 4], appeared in [So 5]. In that paper, S. L. Sobolev also proved the existence of a generalized solution to the Dirichlet boundary value problem for the polyharmonic equation by establishing the existence of a minimizing function to the associated variational integral. This variational method seems to be inspired by works of the Göttingen school (in particular, by K. Friedrichs' paper [F 1] [(6)].

Slightly later in 1938, S. L. Sobolev [So 6] introduced the class of those L^1-functions which have all generalized (=weak) derivatives of a fixed order ν in L^p. For this function class he stated results which were later called "imbedding theorems". The proof of these results appeared in [So 7]; its summary is:

"'Appelons espace $L_p^{(\nu)}$ *l'espace fonctionnel linéaire qui est formé de toutes les fonctions de* n *variables réelles* $\varphi(x_1, \ldots, x_N)$ *dont les dérivées partielles jusqu'à l'ordre* l *existent et sont sommables à la puissance* $p > 1$ *dans chaque partie bornée de l'espace* $x_1, \ldots, x_N$. *La dérivée*

$$\frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$$

est définie comme une fonction qui satisfait à l'équation

$$\int_\infty \cdots \int \psi \frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \, dx_1 \ldots dx_N = \int_\infty \cdots \int (-1)^\alpha \varphi \frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \, dx_1 \ldots dx_N \,,$$

quelle que soit la fonction ψ *continue ayant des dérivées jusqu'à l'ordre* l *et s'annulant en dehors d'un domaine borné* D.

On démontre le théorème suivant:

Théorème. L'espace $L_p^{(\nu)}$ est une partie de l'espace $L^{(\nu-l)}_{\frac{1}{\frac{1}{p}-\frac{l}{N}}}$.

Ce résultat est un complément des résultats de l'auteur et de V. I. Kondrachov, qui ont démontré que l'espace $L_2^{(\nu)}$ est une partie de l'espace $C^{\nu-[\frac{N}{2}]-1}$[1], et l'espace $L_p^{(\nu)}$ est une partie de l'espace $C^{\nu-[\frac{N}{p}]-1}$ constitué des fonctions ayant des dérivées continues jusqu'à l'ordre $\nu-[\frac{N}{p}]-1$.

La démonstration est basée sur l'inégalité intégrale

$$\int \ldots \int \frac{g(x_1,\ldots,x_N)h(y_1,\ldots,y_N)}{\left[\sum\limits_{i=1}^{N}(x_i-y_i)^2\right]^{\frac{\lambda}{2}}}\,dx_1\ldots dx_N\,dy_1\ldots dy_N \le$$

$$\le K\left[\int\ldots\int |g|^p\,dx_1\ldots dx_N\right]^{\frac{1}{p}}\left[\int\ldots\int |h|^q\,dy_1\ldots dy_N\right]^{\frac{1}{q}} \tag{2.3}$$

où

$$\lambda = N\left(2-\frac{1}{p}-\frac{1}{q}\right),$$

qui est une généralisation de l'inégalité de F. Riesz [3]. L'exposé des résultats principaux de cet article se trouve dans la note de l'auteur [5]."

(cf. [So 7 ; pp. 496 - 497.]) In that paper, S. L. Sobolev proved inequality (2.3) (7), made use of his integral representation (presented in [So 4]), studied the mean functions (=mollifiers) of L^p-functions), and proved the imbedding theorems (outlined in the summary above) for domains which are starshaped with respect to a ball.

Later on S. L. Sobolev replaced the notation $L_p^{(\nu)}$ by $W_p^{(m)}$. In his famous monograph [So 8], which appeared in 1950, he studied systematically the spaces

$$W_p^{(m)}(\Omega) := \{u \in L^1(\Omega) \mid \forall\, |\alpha| = m\ \exists \text{ weak derivative } D^\alpha u \in L^p(\Omega)\}$$

and used them for the investigation of hyperbolic and elliptic equations.

Spaces of absolutely continuous functions whose partial derivatives are in L^p, have been used by B. Levi [Lev] (1906)($p=2$) and from the 1920's on by L. Tonelli [T 3] - [T 7] ($p=1$ and $p \ge 2$) as a setting for existence theories for minimum problems for variational integrals. In 1933, O. Nikodym studied the function class (BL) as a mathematical object of its own right. J. Leray [Ler] (1934) used the space $W_2^{(1)}(\mathbb{R}^3)$ as a frame for his investigations on the nonstationary Navier-Stokes equations. Then C. W. Calkin [Ca] and Ch. B. Morrey [Mor 1] (1940) introduced the space $W_p^{(1)}$ and proved a number of important properties of its elements.

From 1936 on, S.L. Sobolev began to develop the basics of the theory of the spaces $W_p^{(m)}$. Based on his integral representation for smooth functions and the estimate (2.3), he proved the embedding $W_p^{(m)} \subset L^q$ in 1938.

After the 1950's, the spaces $W_p^{(m)}$ became a rapidly increasing field of research. Later on these spaces have been named "Sobolev spaces".(8)

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Comments

1. Introduction

(1) B. Riemann made extensive use of the "Dirichlet Principle" in his Inauguraldissertation [Rim] (cf. sections 16-18, pp. 30-35). A discussion of B. Riemann's use of this principle in his approach to complex function theory can be found in **"Mathematics of the 19th Century"** [M ; pp. 212-215] and in S. S. Petrova [Pet]. More details are presented in S. Hildebrandt [Hid 2] and A. F. Monna [Mon].

2.2 *The Göttingen school.*

(2) In [Hib 1; pp. 185-186], D. Hilbert wrote:

"Das folgende ist ein Versuch der Wiederbelebung des Dirichlet'schen Princips. Indem wir bedenken, daß die Dirichletsche Aufgabe nur eine besondere Aufgabe der Variationsrechnung ist, gelangen wir dazu, das Dirichlet'sche Princip in folgender allgemeinerer Form auszusprechen: `Eine jede Aufgabe der Variationsrechnung besitzt eine Lösung, sobald hinsichtlich der Natur der gegebenen Grenzbedingungen geeignete einschränkende Annahmen erfüllt sind und nötigenfalls der Begriff der Lösung eine sinngemäße Erweiterung erfährt."`

[paper submitted: Göttingen, den 11. Oktober 1899.]

This programmatic idea became part of the 20st problem of Hilbert's famous speech at the International Congress of Mathematicians in 1900 (Paris):

"...

20. Allgemeines Randwertproblem.

...

Ich bin überzeugt, daß es möglich sein wird, diese Existenzbeweise durch einen allgemeinen Grundgedanken zu führen, auf den das `Dirichletsche` Prinzip hinweist, und der uns dann vielleicht in den Stand setzen wird, der Frage näher zu treten, *ob nicht jedes reguläre Variationsproblem eine Lösung besitzt, sobald hinsichtlich der gegebenen Grenzbedingungen gewisse Annahmen* - etwa die Stetigkeit und stückweise öftere Differentierbarkeit der für die Randbedingungen maßgebenden Funktionen - *erfüllt sind und nötigenfalls der Begriff der Lösung eine sinngemäße Erweiterung erfährt. ...* "

(cf. [Hib 2]; p. 119).

2.3 *The Contributions of O. Nikodym and J. Leray.*

(3) The results in [N 1] are part of O. Nikodym's investigations on the "Dirichlet Principle" for elliptic differential equations with symmetric coefficients. These results were presented on the II Congrès de Mathématiciens Roumains à Turnu Severin (1932) (cf. [N 1]; p. 129).

In [N 2], O. Nikodym published the details of his conference speech in 1932. In that paper he presented basic notions of the abstract Hilbert space, summarized the results on the function class (BL) obtained in [N 1], and studied then the differential equation

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(p\frac{\partial u}{\partial z}\right)+qu=0 \qquad (E)$$

in a bounded domain $D\subset R^3$, where p and q are measurable functions in D such that

$$0<\alpha\le p(P)\le\beta\,, \qquad\qquad q(P)\ge 0$$

$(\alpha,\beta=\text{const})$. He continued:

> *"Considèrons l'ensemble W de toutes le fonctions $u(P)$ qui sont du type (BL) dans D et pour laquelles l'intégrale de M. Lebesgue*
>
> $$\iiint\limits_{D} qu^2\,d\tau$$
>
> *existe. On voit que, si $f,\,g\in W$ et si λ est un nombre réel, les fonctions λf et $f+g$ appartiennet aussi à W. Il existe aussi l'intégrale*
>
> $$[u,v]_D:=\iiint\limits_{D}\left\{p\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right)+quv\right\}d\tau$$
>
> *que nous appellerons le produit scalaire de u et v."*

(cf. [N 2]; p. 119).

Then O. Nikodym established an existence theorem for (E) by minimizing $[u,u]_D$ in an appropriate subclass of W.

(4) The following result is easily proved.

Let u, v_i $(i=1,...,N)$ be functions in $L^2(\mathbb{R}^N)$. Then the following two statements are equivalent:

1° $$\int\limits_{\mathbb{R}^N}\left(u\frac{\partial\zeta}{\partial x_i}+v_i\zeta\right)dx=0 \qquad \forall\,\zeta\in C_c^\infty(\mathbb{R}^N);$$

2° $$\int\limits_{\mathbb{R}^N}\left(u\frac{\partial\varphi}{\partial x_i}+v_i\varphi\right)dx=0 \qquad \forall\,\varphi\in C^1(\mathbb{R}^N)$$

such that

$$\int\limits_{\mathbb{R}^N}\left(\varphi^2+\sum_{j=1}^{N}\left(\frac{\partial\varphi}{\partial x_j}\right)^2\right)dx<+\infty$$

$(i=1,...,N)$.

2.5 *The Work of S.L. Sobolev.*

(5) In [So 2], [So 3], S. L. Sobolev generalized the notion of a real function to the concept of a continuous linear functional on certain spaces of continuously differentiable functions. He also generalized the concept of classical operations with real functions (in the first line, the operation of differentiation) to operations on these spaces of functions. This led him to the result that the Cauchy problem for certain hyperbolic equations could be solved more easily.

Thus, in 1935-1936 S. L. Sobolev invented the concept of distribution (of finite order) and distributional solution to a partial differential equation, however, he did not pursue to develop a new mathematical theory starting from this concept.

L. Schwartz received important impulses from Heaviside's symbolic calculus and Dirac's δ-function (as well as from Bochner's formal functions, Bochner's generalized solutions and Leray's weak solutions) for generalizing the notion of classical derivative. The invention of the concept of distribution by L. Schwartz is closely connected with the work of French mathematicians on modern analysis during the period 1930-1945: " ... *These reflections date back to 1935, and in 1944, nine years later, I discovered distributions"* [11].

The historical development of the basics of the theory of distributions is described in great detail in the book J. Lützen [Lü]. The role of S. L. Sobolev's work with respect to development of this theory is also discussed in Appendix 3 of the third edition of [So 8].

(6) S. L. Sobolev has been familar with results of the Göttingen school. In [So 3; p. 268] he wrote:

> *"Our results represent in a more precise form known estimates which are due to the Göttingen school and are frequently encountered in various problems of the theory of partial differential equations.*
>
> *In certain special cases, for instance in the theory of quasilinear hyperbolic equations which were considered by Schauder, these more precise estimates allow to determine exactly the necessary number of continuous derivatives of the initial conditions for these equations."* (Russian)

In [So 5], S. L. Sobolev used a variational method to solve the Dirichlet problem for the polyharmonic equation. This method is a modification of an idea which has been developed in K. Friedrichs [Fr 1].

On the other hand, K. Friedrichs [Fr 3] referred to S. L. Sobolev's paper [So 4] and proved the following version of the main result of [So 4]:

> **Theorem.** *Let u be a function on which the operations $\nabla_1 = D$, $\nabla_2 = D^*D$, $\nabla_3 = D\,D^*D$... up to ∇_r can be applied such that $u, \nabla_1 u, ..., \nabla_r u$ are L^2-integrable. Then u is continuous and has continuous derivatives up to the order $r - m$ provided* $r \geq m := \left[\dfrac{N}{2}\right] + 1.$

(cf. [Fr 3; pp. 525, 540-543]).

(7) In [So 7], S. L. Sobolev proved the following inequality:

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{u(x)v(x)}{|x-y|^\lambda}\,dx\,dy \leq c\,\|\,u\,\|_{L^p(\mathbb{R}^N)}\,\|\,v\,\|_{L^q(\mathbb{R}^N)} \tag{+}$$

[11] See p. 218 of the book L. Schwartz: *A mathematician grappling with his century.* Birkhäuser Verlag, Basel 2001. - Chapter VI of this book is concerned whith the invention of distributions. The motivation for using the notion "distribution" is described on pp. 238-239.

where

$$\lambda = N\left(2 - \frac{1}{p} - \frac{1}{q}\right) \quad \left(N > 1,\, p > 1,\, q > 1,\, \frac{1}{p} + \frac{1}{q} < 1\right)$$

(cf. [So 7; pp. 477-481]). He referred to the paper F. Riesz [Ris]. In this paper, F. Riesz presented a new proof of the one-dimensional version of (+) which has been proved for the first time by G. H. Hardy and J. E. Littlewood [HL] (cf. also: Hardy, G. H.; Littlewood, J. L.; Pólya, G.: **Inequalities**. Cambridge Univ. Press 1934, 1952, pp. 288-289).

S. L. Sobolev extended the approach of F. Riesz from the one-dimensional case of (+) to the case of several variables. Inequality (+) is called nowadays "Hardy-Littlewood-Sobolev inequality".

---

(8) O. Nikodym [N 1] called the functions of $W_2^{(1)}$ "*fonctions de M. Beppo Levi*" (cf. Sect. 2.3 above).

The notion *"Beppo Levi space"* for $W_p^{(1)}$ has been used over some years by French and Italian mathematicians after 1950.

J. Deny/J. L. Lions [12)] use the notation "*espace de Beppo Levi attaché à E*" for the space of distributions

$$BL(E) := \left\{T \in \mathcal{D}'(\Omega) \,\middle|\, \frac{\partial T}{\partial x_i} \in E \quad (i = 1, ..., N)\right\}$$

(E a locally convex topological vector space, $\Omega \subset \mathbb{R}^N$ open) [if Ω is a bounded domain with smooth boundary $\partial\Omega$ and if $E = L^p(\Omega)$ $(1 \le p < +\infty)$ then $BL(L^p(\Omega)) = W_p^{(1)}(\Omega)$].

G. Prodi [13)] studied traces of functions of the space $W_2^{(1)}(\Omega)$. He wrote (p. 36):

"Indicheremo con $W(\Omega)$ la classe delle funzioni misurabili e localmente integrabili, dotate di derivate prime (in senso generalizzato) [1)] *a quadrato integrabile in Ω."*

[1)] *Cioè nel senso della teoria delle distribuzioni.*

The role of the "Beppo Levi space" in the work of the Italian school is also reflected in the obituary to G. Stampacchia [14)].

Concerning the use of B. Levi's name, G. Fichera [15)] wrote:

> *" These spaces, at least in the particular case p=2 , were known since the very beginning of this century, to the Italian mathematicians Beppo Levi [Lev] and Guido Fubini [Fu 1] who investigated the Dirichlet minimum principle for elliptic equations. Later on many mathematicians have used these spaces in their work. Some French mathematicians, at the beginning of the fifties, decided to invent a name for such spaces as, very often, French mathematicians like to do. They proposed the name Beppo Levi spaces. Although this name is not very exciting*

[12)] Deny, J.; Lions, J. L.: *Les espaces du type de Beppo Levi.* Ann. Inst. Fourier Grenoble 5 (1953-1954), 305-370.

[13)] Prodi, G.: *Tracce sulla frontiera delle funzioni di Beppo Levi.* Rend. Sem. Mat. Univ. Padova 26 (1956), 36-60.

[14)] *Guido Stampacchia (1922-1978).* Boll. Un. Mat. Ital. ser. 5, vol. 15-A (1978), 715-756.

[15)] Fichera, G.: *Analytic problems of hereditary phenomena.* Centro Intern. Mat. Estivo, ciclo I (1977), Bressanone 1977, pp. 111-169; footnote on pp. 146-147.

in the Italian language and it sounds because of the name "Beppo", somewhat peasant, the outcome in French must be gorgeous since the special French pronunciation of the names makes it to sound very impressive. Unfortunately this choice was deeply disliked by Beppo Levi, who at that time was still alive, and - as many elderly people - was strongly against the modern way of viewing mathematics. In a review of a paper of an Italian mathematician, who, imitating the Frenchman, had written something on "Beppo Levi spaces", he practically said that he did not want to leave his name mixed up with this kind of things. Thus the name had to be changed. A good choice was to name the spaces after S. L. Sobolev. Sobolev did not object and the name Sobolev spaces is nowdays universally accepted. "

Miscellaneous Remarks

1. Integral Inequalities. In 1894, H. Poincaré proved the following integral inequality:

$$\int\limits_{\Omega} u^2 dxdydz \leq c_0 \int\limits_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dxdydz$$

$$\textit{for all } u \textit{ such that } \int\limits_{\Omega} udxdydz = 0 \quad (\Omega \subset \mathbb{R}^3, c_0 = const)$$

(cf. [Poi 1; p. 76], [Poi 2; pp. 98-104]) . This inequality and its numerous generalizations are called nowadays *"Poincaré inequalities"*. These inequalities are an important tool for the study of (weak) solutions to partial differential equations and for Sobolev space functions.

V. A. Steklov was familiar with H. Poincaré's paper [Poi 1] . In [St 1; pp. 500-503] (1896-97) he presented a new proof of the "Poincaré inequality" which completely differs from H. Poincaré's original proof. On p. 566 of that paper, V. A. Steklov established the inequality

$$\int\limits_{\Omega} u^2 dxdydz \leq c_1 \int\limits_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dxdydz \tag{$*$}$$

$$\textit{for all } u \textit{ such that } u = 0 \textit{ on } \partial\Omega (\Omega \subset \mathbb{R}^3, c_1 = const).$$

Without being familar with V. A. Steklov's work [St 1], K. Friedrichs proved the inequality

$$\int\limits_{\Omega} u^2 dxdy \leq c \left\{ \int\limits_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dxdy + \int\limits_{\partial\Omega} u^2 ds \right\}$$

$(\Omega \subset \mathbb{R}^2, c = const)$ (cf. [Fr 1; p. 211, 229-233]). The N-dimensional generalization of this inequality (resp. its special case $(*)$) are usually called *"Friedrichs' inequality"*.

This inequality expressed in terms of quadratic forms, occurs in [Fr 2; I, p. 486]. It is used for proving compactness of linear operators.

The "Poincaré inequality" and the "Friedrichs inequality" are extensively used by D. Hilbert and R. Courant in [CH; pp. 486, 489, 511, 519].

2. The work of N. M. Gjunter. From the 1920's on, N. M. Gjunter began to work on solving partial differential equations by functions having discontinuous derivatives. He replaced the pointwise validity of a differential equation in a given domain by its integrated form over an arbitrary subdomain (the "test curve method"). First results of these studies are presented in [G 1].

Then N. M. Gjunter studied the problem of finding a function $\varphi = \varphi(x, y, z)$ such that

$$\frac{\partial \varphi}{\partial x} = u, \qquad \frac{\partial \varphi}{\partial y} = v \qquad \frac{\partial \varphi}{\partial z} = w$$

where u, v, w are given Lipschitz continuous functions satisfying

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

in a generalized form (cf. [G 2 ; pp. 366-372]). In part II of [G 2] he extended this method to the case of bounded integrable functions u, v, w and used the integrated form of the above compatibility conditions with respect to any subdomain. Both in [G 1] and [G 2] an extensive use of the Steklov mean

$$\phi(x, y, z) = \frac{1}{hkl} \int_x^{x+h} dx_1 \int_y^{y+k} dy_1 \int_z^{z+l} \varphi(x_1, y_1, z_1) dz_1$$

has been made (cf. Appendix 2.2). These methods are then systematically applied to the Poisson equation and to the equation

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + W\frac{\partial v}{\partial z} = f.$$

From the numerous works on these topics we only refer to the early papers [G 3] und [G 4]. It seems that these ideas of N. M. Gjunter have inspired S. L. Sobolev to introduce the concept of weak solution of a partial differential equation in terms of an integral identity involving test functions (cf. [So 1; 39]).

Later N. M. Gjunter [G 5], [G 6] studied in great detail the notion of the (Stieltjes) integral mean from the point of view of functions defined on families of subsets of $\mathbb{R}^N$ and the application of these results for solving boundary value problems for the Poisson equation. These works culminated in N. M. Gjunter's fundamental work on potential theory [16].

[16] Gjunter, N. M.: *La théorie du potentiel et ses applications aux problèmes fondamentaux de la physique mathématique.* Gauthier-Villars, Paris 1934. Russian Transl.: Gos. Izdat. Tech.-teor. Lit., Moskva 1953. German Transl.: [Günter] Teubner Verlagsgesellsch., Leipzig 1957.

Appendix

Approximation of Integrable Functions by Smooth Functions

1. Convolution with a smooth kernel

1.1. In 1885, K. Weierstrass [Wei 2] published his famous result on the uniform approximation of any continuous real function on an interval $[a, b]$ by polynominals. He began these investigations by the following observations:

"Ist $f(x)$ eine für jeden reellen Werth der Veränderlichen x eindeutig definierte, relle und stetige Function, deren absoluter Betrag eine endliche obere Grenze hat, so gilt bekanntlich die nachstehnende Gleichung, in der u ein zweite reelle Veränderliche bedeutet und unter k eine von x und u unabhängige positive Grösse zu verstehen ist:

$$\lim_{k\to 0} \frac{1}{k\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} f(u) e^{-\left(\frac{u-x}{k}\right)^2} du = f(x).$$

Der in dieser Gleichung ausgesprochene Satz lässt sich leicht verallgemeinern.

Es werde irgend eine Function $\psi(x)$ von derselben Beschaffenheit wie $f(x)$ angenommen, welche ihr Zeichen nicht ändert, der Gleichung $\psi(-x) = \psi(x)$ genügt und überdies der Bedingung entspricht, dass das Integral

$$\int\limits_{0}^{+\infty} \psi(x) dx$$

einen endlichen Werth haben muss, der mit ω bezeichnet werden möge . Setzt man dann

$$F(x, k) = \frac{1}{2k\omega} \int\limits_{-\infty}^{+\infty} f(u) \psi\left(\frac{u-x}{k}\right) du,$$

so ist

$$\lim_{k\to 0} F(x, k) = f(x). \text{ ''}$$

(cf [Wei 2; pp. 1-2]).

The result just mentioned, was the point of departure for proving the uniform approximation of a continuous function by polynomials [17]. Later on K. Weierstrass' technique has been developed into a tool of great importance in analysis: the convolution of integrable functions.

2. K. Ogura [O] (1919) developed a theory of approximation of a Riemann integrable function f on an interval by the following sequence of functions:

$$F_n[f(x)] = \frac{\int_a^b \varphi_n(x,t) f(t) dt}{\int_a^b \varphi_n(x,s) ds}, \; n \in \mathbb{N}, \; x \in [a,b].$$

The basic result in [O] on the convergence of these approximations is as follows:

Let $\varphi_n = \varphi_n(x,t)$ be bounded, non-negative and integrable with respect to t in the domain $a \le x,\ t \le b$, and let both

$$\frac{\int_a^{a_1} \varphi_n(x,t) dt}{\int_a^b \varphi_n(x,s) ds} \longrightarrow 0 \qquad \text{and} \qquad \frac{\int_{b_1}^{b} \varphi_n(x,t) dt}{\int_a^b \varphi_n(x,s) ds} \longrightarrow 0 \qquad \text{as } n \to \infty$$

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[17] E. Landau (*Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion.* Rend. Circ. Mat. Palermo 25 (1908), 337-345) gave a new and simpler proof of Weierstrass' result:

Let f be a continuous real function defined on the interval $[a,b]$. Then

$$\lim_{n\to\infty} \frac{\int_0^1 f(z)(1-(z-x)^2)^n dz}{2\int_0^1 (1-u^2)^n du} = f(x)$$

uniformly for all $x \in [a,b]$.

This result is widely used in the literature (e.g. in R. Courant/D. Hilbert: *Methoden der mathematischen Physik*, I. Springer-Verlag Berlin, Heidelberg 1924, 1931, 1968; pp. 55-57.)

S. Bernstein gave an independent proof of Weierstrass result by approximating a continuous function f by the sequence of polynomials

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

uniformly for x, where $a \le a_1 < a_2 \le x \le b_2 < b_1 \le b$.

Let f be bounded and integrable on $[a,b]$. Then:

1. If f is continuous at $x = x_0$ $(a < x_0 < b)$, then $F_n[f]$ converges uniformly to f in a neighbourhood of x_0.

2. For every $p > 0$,

$$\lim_{n\to\infty} \int_a^b \left| f(x) - F_n\Big[f(x)\Big] \right|^p dx = 0.$$

Then K. Ogura considered bounded, non-negative integrable functions $\varphi_n(t)$ defined for $t \in [a-b, b-a]$ such that

$$\lim_{n\to\infty} \int_{c_1}^{c_2} \varphi_n(t)dt = 0$$

for $a - b < c_1 < 0 < c_2 < b - a$, and applied the above results to the sequence of approximations

$$\int_a^b \varphi_n(t-x)f(t)dt, \qquad x \in [a,b].$$

With φ_n suitably choosen, these approximations include many of the known classical approximating functions, for instance

1) $$\frac{\int_a^b \left[1-(t-x)^2\right]^n f(t)dt}{2\int_0^1 (1-s^2)^n ds}$$

(L a n d a u ' s polynominal in the case where $a = 0$, $b = 1$),

2) trigonometric polynominals:

$$\frac{1}{2n\pi} \int_a^b \left[\frac{\sin \frac{n}{2}(t-x)}{\sin \frac{1}{2}} \right]^2 f(t)dt, \qquad (b - a \le 2\pi)$$

(F e j é r ' s integral in the case where $a = 0, b = 2\pi$);

3) $$\frac{n}{\sqrt{\pi}} \int_a^b e^{-n^2(t-x)^2} f(t)dt, \qquad (a < b)$$

(W e i e r s t r a s s ' transcendental integral function in the case where $a = -\infty, b = +\infty$).

K. Ogura next studied the differentiability of the function

$$x \longmapsto \int_a^b \varphi_n(t-x)f(t)dt$$

and their convergence to the derivative f'.

In 1919, K. Ogura thus developed many basic results on the convergence of the convolution

$$(\varphi_n * f)(x) = \int_a^b \varphi_n(t-x)f(t)dt$$

for integrable (in particular, smooth) functions φ_n and integrable (resp. differentiable) functions f.

1.3 The approximation of a locally Lebesgue integrable function in $\mathbb{R}^3$ by a sequence of convolutions with a C^∞-kernel has been invented by J. Leray in 1934 (cf. p. 10 above). As an example for the kernel he used the function

$$\lambda(s) = Ae^{\frac{1}{s-1}}, \qquad 0 < s < 1$$

(" ... A étant une constante convenable ... "; cf. [Ler; p. 206]).

This approximation method is used by J. Leray for studying functions of the Sobolev space $W_2^{(1)}(\mathbb{R}^3)$.

1.4 In 1935, S. L. Sobolev [So 1] studied weak solutions to the wave equations in two space variables. For these investigations he used the following approximation scheme for a Lebesgue integrable function f defined in a domain $D \subset \mathbb{R}^3$.

Let D_1 be a subdomain of D such that $dis\,t(D_1, \partial D) > \eta_1 > 0$. Let $(\eta_1, \eta_2, \cdots)$ be a sequence of reals satisfying $\eta_1 > \eta_2 > \cdots > 0$ and $\lim_{n\to\infty} \eta_n = 0$. In each ball with radius η_n and centre at $(x_0, y_0, t_0) \in D_1$ functions

$$(x, y, t) \longmapsto \omega_n(x, y, t; x_0, y_0, t_0)$$

are defined such that:

1) $\omega_n(x, y, t; x_0, y_0, t_0)$ is uniformly bounded for all n and all $(x_0, y_0, t_0) \in D_1$;

2) the functions $(x, y, t; x_0, y_0, t_0) \longmapsto \omega_n(x, y, t; x_0, y_0, t_0)$ is measurable;

3) $I_n(x_0, y_0, t_0) = \displaystyle\iiint_{(x-x_0)^2+(y-y_0)^2+(t-t_0)^2=r^2\le\eta_n^2} \omega_n(x, y, t; x_0, y_0, t_0)d\tau > \gamma V_n$

for all n and all $(x_0, y_0, t_0) \in D_1$, whereγ is a fixed positive constant and

$$V_n = \frac{4\pi}{3}\eta_n^3,$$

4) $\omega_n(x, y, t; x_0, y_0, t_0) = 0$ if $r^2 > \eta_n^2$.

Then S. L. Soboloev defined the functions

$$f_n(x_0, y_0, t_0) = \frac{\displaystyle\iiint\limits_{r^2 \le \eta_n^2} \omega_n(x, y, t; x_0, y_0, t_0) f(x, y, t) dxdydt}{I_n(x_0, y_0, t_0)}$$

and called them "*mean functions*". For these functions he proved a number of convergence properties needed for the further discussion in [So 1].

On p. 55 of [So 1] the functions ω_n are specialized as follows:

$$\omega_n(x, y, t; x_0, y_0, t_0) = \begin{cases} e^{\dfrac{r^2}{r^2 - \eta_n^2}} & if \quad r < \eta_n, \\ \\ 0 & if \quad r \ge \eta_n \end{cases}$$

$(r^2 = (x - x_0)^2 + (y - y_0)^2 + (r - t_0)^2)$.

Thus, J. Leray (1934) and S. L. Sobolev (1935) introduced (apparently independently from each other) the approximation of a Lebesgue integrable function by convolution with the kernel

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon^3}\omega\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0,\ x \in \mathbb{R}^3$$

where

$$\omega(x) \quad = \quad \begin{cases} a\, exp\left(-\dfrac{1}{1 - |x|^2}\right) & if \quad |x| < 1, \\ \\ 0 & if \quad |x| \ge 1, \end{cases}$$

$$a \quad = \quad \left(\int\limits_{|x|<1} exp\left(-\frac{1}{1 - |x|^2}\right) dx \right)^{-1}.$$

2. Approximation of an integrable function by its integral mean

The technique of "smoothing" an integrable function by its mean value (average) over a ball (resp. a cube) appeared independently over many years in several works.

2.1 In 1906, B. Levi [Lev; pp. 310-312] used " ... l'operazione di media

$$U(x,y|\theta) = \frac{1}{\pi\rho^2}\int\int_R u(x+\theta(x-\xi), y+\theta(y-\eta))d\xi d\eta ...\text{"}$$

(R = disc with radius ρ and centre at 0) in his studies on the justification of the "Dirichlet Principle".

2.2 In a series of papers, V.A. Steklov studied the representation of a continuous function f in terms of a series

$$f(x) = \sum_{n=1}^{\infty} a_n f_n(x).$$

In connection with this, he investigated the completeness of the system $\{f_1, f_2, \cdots\}$. As a tool for these investigations, V. A. Steklov introduced for continuous functions ψ the integral mean

$$\psi_h(x) = \frac{1}{h}\int_x^{x+h} \psi(s)ds, \qquad h > 0,$$

and more generally

$$\frac{1}{h_1 h_2 \cdot \ ... \ \cdot h_m} \int_x^{x+h_1} dx_1 \int_{x_1}^{x_1+h_2} dx_2 \cdots \int_{x_{m-1}}^{x_{m-1}+h_m} \psi(s)ds$$

(cf. [St 2; pp. 6,9], [St 4; p. 270]). Clearly, ψ_h is continuously differentiable, and $\psi_h(x) \to \psi(x)$ for all x, as $h \to 0$.

The function ψ_h is usually called the *Steklov mean of* ψ. The use of this approximation has been further developed in [St 3], [St 4] [18)].

[We note that the function ψ_h can be viewed as the convolution of the characteristic function $\chi_{]0,h[}$ of the interval $]0,h[$ and the function ψ:

$$\psi_h(x) = (\omega_h * \psi)(x) = \int_{\mathbb{R}} \omega_h(x-s)\psi(s)ds, \quad h > 0,$$

where

$$\begin{aligned} \omega(t) &= \chi_{]0,1[}(t) \\ \omega_h(t) &= \frac{1}{h}\omega\left(\frac{1}{h}\right) = \frac{1}{h}\chi_{]0,h[}(t), \quad t \in \mathbb{R}] \end{aligned}$$

[18)] The scientific work of V. A. Steklov is described in the article Kneser, A.: *Wladimir Stekloff zum Gedächtnis.* Jahresber. Dtsch. Math.-Ver. 38 (1929), 206-231, and in the book Vladimirov, V. S.; Markush, I. I.: *Academician STEKLOV. Mathematician Extraordinary.* Mir Publishers, Moscow 1983 (first published: Izd. Nauka, Moskva 1981).

2.3 C. W. Calkin [Ca] and Ch. B. Morrey [Mor 1] made an extensive use of the mean value (average) function

$$\frac{1}{(2h)^N}\int\limits_{x_1-h}^{x_1+h}\cdots\int\limits_{x_N-h}^{x_N+h} u(\xi_1,\cdots,\xi_N)d\xi_1\cdots d\xi_N$$

in their studies on absolutely continuous functions of several variables (cf. [Ca; pp. 172-173]). The integral mean of a function over a ball has been used in [E 3; pp. 235-236].

3. *The passage from "averaging" to "mollifying" in K. Friedrich's work*

The smoothing effect of the integral mean value (average)

$$\frac{1}{(2r)^N}\int\limits_{Q_r(x)} u(y)dy \quad {}^{19)},$$

($Q_r(x)$ = cube with side length $2r$ and centre at x) has been used by K. Friedrichs [Fr 2; II, p. 692] as a technical tool in his investigations on spectral properties of differential operators.

In [Fr 4; p. 527], K. Friedrichs then considered functions $e\in C^\infty(\mathbb{R})$ satisfying

$$e(t)\geq 0 \quad \text{if}\,|t|<1, \quad e(t)=0 \quad \text{if} \quad |t|\geq 1, \quad \int\limits_{-\infty}^{+\infty} e(t)dt=1,$$

and defined the kernels

19) Define $Q_r(x)=\left\{y\in\mathbb{R}^N \,\middle|\, |x_i-y_i|<r \quad (i=1,\cdots,N)\right\}$. For $\xi\in\mathbb{R}^N$ and $r>0$, set

$$\omega(\xi)=\frac{1}{2^N}\chi_{Q_1(0)}(\xi).$$

$$\omega_r(\xi)=\frac{1}{r^N}\omega\left(\frac{\xi}{r}\right)=\frac{1}{(2r)^N}\chi_{Q_r(0)}(\xi).$$

The convolution of ω_r and $u\in L^1(\mathbb{R}^N)$ is

$$(\omega_r*u)(x)=\int\limits_{\mathbb{R}^N}\omega_r(x-y)u(y)\,dy=\frac{1}{(2r)^N}\int\limits_{Q_r(x)} u(y)dy, \qquad x\in\mathbb{R}^N.$$

Then:

1. For fixed $r>0$, the function $x\longmapsto \int\limits_{Q_r(x)} u(y)dy$ is uniformly continuous in $\mathbb{R}^N$;
2. $\omega_r*u\to u$ in $L^1(\mathbb{R}^N)$ as $r\to 0$.

$$j_h(y-x) = \frac{1}{h^N} e\left(\frac{y_1 - x_1}{h}\right) \cdot ... \cdot e\left(\frac{y_N - x_N}{h}\right), \quad h > 0.$$

The integral operators

$$J_h u(x) = \int\limits_{\mathbb{R}^N} j_h(y-x)u(y)dy, \qquad u \in L^2(\mathbb{R}^N)$$

are the convolution of j_h and u [20]. K. Friedrichs then proved that $J_h u \to u$ in L^2 as $h \to 0$ (more results on $J_h u$ are presented on pp. 530-531).

The approximation of a (square) integrable function u by the sequence of C^∞-functions $J_h u$ is then extensively used in further works by K. Friedrichs.

In [Fr 5; p. 138] the operators J_h are called "*mollifiers*". The origin of this name is explained by P. Lax as follows:

> *" The name for the smoothing operators in this paper has an unusual origin. On English usage Friedrichs liked to consult his friend and colleague, Donald Flanders, a descendant of puritans and a puritan himself, with the highest standards of his own conduct, non-censorious toward others. In recognition of his moral qualities he was called Moll by his friends. When asked by Friedrichs what to name the smoothing operator, Flanders remarked that they could be named mollifiers after himself; Friedrichs was delighted, as on others occasions, to carry this joke into print. "*

(cf.: *Kurt Otto Friedrichs. Selecta.* Ed.: Cathleen S. Morawetz. Birkhäuser; Boston, Basel, Stuttgart 1986; vol. 1, p. 117: Commentary by P. Lax on [Fr 5]) .

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[20] K. Friedrichs has been familar with the paper by K. Ogura [O] (cf. footnote 12 on p. 527 in [Fr 4]).

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[21)] The third enlarged edition contains the following three appendices:

1. Remarks on the literature up to the mid of 1980's relevant for Sobolev spaces (the bibliography contains 339 references);
2. Translation of the paper [So 3] into Russian;
3. Remarks on the historical significance of [So 3] concerning the development of the theory of distributions.

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