Characterizing differential algebraic equations without the use of derivative arrays

R. März
Humboldt-University Berlin,
Institute of Mathematics,
Unter den Linden 6,
D-10099 Berlin, Germany

Abstract

For a large class of differential algebraic equations with properly stated leading terms and nonsmooth data, an index characterization in terms of the original coefficients and their only first partial derivatives is given. No higher derivatives are used, but several constant rank conditions appear to be essential.

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1 Introduction

Differential algebraic equations (DAEs) play an increasingly important role in the modelling, simulation and optimization of practical problems. In circuit simulation and in further fields of application, large DAE systems are composed via automatic generation. Although a wealth of physical knowledge has been implemented in the corresponding programs, the generated model may have higher index and, consequently, a direct numerical simulation without index reduction will not be possible or will provide a bad or even wrong result, respectively. *Hence, a prophylactic or simulation attendant index monitoring is desirable.*

Application problems are typically nonsmooth in the coefficients. For this reason, but also due to the large dimensions, an index determination via the formation of a derivative array (cf. [BrCaPe], [RaRh]) or a prolongation [Ve] is often no longer realizable.
Unfortunately, the so-called structural index ([ReMaBa]), which can be easily computed by means of the Pantelides algorithm and, thus, is widely used in practice, turned out to be inappropriate even for the characterization of linear DAEs with constant coefficients. This structural index does not cover the Kronecker index appropriately, it might be considerably larger as well as smaller. Not even the index 1 property is reliably reflected.
In [Mae1] an index criterion for linear standard form DAEs is formulated by means of a

matrix function sequence and special projectors. This tractability index does not require any derivative arrays. In [Mae1] it is further suggested to give an index notion for nonlinear DAEs via linearization, however, without putting it into practice. Recently, the concept of the tractability index was applied to develop an innovative index-monitor for the circuit simulation ([Es et all]).
Linear and nonlinear DAEs with properly stated leading terms are introduced in [BaMae], [Mae2], [HiMae]. In a linear DAE with properly stated leading term

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t),$$

with rectangular matrices $A(t), D(t)$, more information is built in than in a linear standard form DAE

$$G(t)x'(t) + F(t)x(t) = q(t).$$

Namely, it is precisely figured out which solution components are involved with their derivatives in fact. This additional information seems to be quite easily available in applications like circuit simulation (cf. [HiMaeTi]). Naturally, a more precise model should be beneficial in theory and computations.
After the tractability index for linear DAEs with properly stated leading terms was introduced in [Mae3], we now take up again the idea to define an index for nonlinear DAEs

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = 0$$

in such a way that all admissible linearizations have the same index. This opens up a way to index criteria and index monitors without derivative arrays. In particular, we thus obtain necessary criteria that can be realized in a relatively simple way.

The paper is organized as follows:
In Section 2, we compile fundamentals and conventions, particularly on properly stated leading terms and linearizations. This is followed by short discussions on nonlinear DAEs with tractability index 1 in Section 3 as well as discussions on linear DAEs with arbitrary index in Section 4. We take [HiMae] and [Mae3] as sources, which are completed by Theorem 3.4 on linearizations of index-1 DAEs and Theorem 4.4. on local matrix pencils of linear variable coefficient DAEs.
Finally, Section 5 provides new index criteria for nonlinear DAEs. As intended, Theorem 5.4 confirms all admissible linearizations of a nonlinear DAE with tractability index μ to have tractability index μ too, and, moreover, certain further uniform characteristics. These uniform characteristics are closely related to constant rank conditions that are important here.
Further necessary index μ criteria are given, in particular those concerning the local matrix pencils.

2 Fundamentals and conventions

In this paper we deal with DAEs

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = 0, \tag{2.1}$$

where $A(x,t) \in L(\mathbb{R}^n, \mathbb{R}^m)$, $D(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$, $b(x,t) \in \mathbb{R}^m$, for $x \in \mathcal{D}, t \in \mathcal{I}$, are given coefficients, $\mathcal{D} \subseteq \mathbb{R}^m$ is open and $\mathcal{I} \subseteq \mathbb{R}$ is an interval. The coefficients $A(x,t), b(x,t)$ depend continuously on their arguments and there are continuous partial derivatives $A_x(x,t), b_x(x,t)$. The coefficient $D(t)$ is continuously differentiable. The leading term in (2.1) is supposed to be properly stated (e.g. [HiMae], [BaMae], [Mae2]), i.e., the decomposition

$$ker\, A(x,t) \oplus im\, D(t) = \mathbb{R}^n, \quad x \in \mathcal{D},\ t \in \mathcal{I}, \tag{2.2}$$

is valid and there is a continuously differentiable projector function $R : \mathcal{I} \to L(\mathbb{R}^n)$ such that, for all $x \in \mathcal{D}, t \in \mathcal{I}$, the relations $R(t) = R(t)^2$, $im\, R(t) = im\, D(t)$, $ker\, A(x,t) = ker\, R(t)$ become true. Notice that, due to the smoothness of R, both subspaces in (2.2) must have constant dimensions and the matrices $A(x,t)$ and $D(t)$ have a common constant rank. Moreover, $ker\, A(x,t)$ is independent of x, it holds that $A(x,t) = A(x,t)R(t)$ for all $x \in \mathcal{D},\ t \in \mathcal{I}$.

Let $D(t)^- \in L(\mathbb{R}^n, \mathbb{R}^m)$ denote such a generalized inverse of $D(t)$ that, for $t \in \mathcal{I}$, the conditions (cf. e.g. [Na], [Zi])

$$D(t)^- D(t) D(t)^- = D(t)^-, \quad D(t)D(t)^- D(t) = D(t), \quad D(t)D(t)^- = R(t)$$

are satisfied. $D(t)^-$ is assumed to be continuously differentiable as $D(t)$ is so. At this place let us mention that adding a fourth condition, e.g. $D(t)^- D(t) = (D(t)^- D(t))^*$ (the superscript $*$ indicates the transpose), we would obtain a uniquely determined generalized inverse $D(t)^-$. However, we do not take this possibility for the sake of more computational flexibility.

Below, the matrices

$$\begin{aligned} G_0(x,t) &:= A(x,t)D(t), & (2.3)\\ B_0(y,x,t) &:= b_x(x,t) + (A(x,t)y)_x, & (2.4) \end{aligned}$$

constructed pointwise for $x \in \mathcal{D},\ t \in \mathcal{I},\ y \in \mathbb{R}^n$, shall serve as the basic data for creating index criteria. The identity

$$B_0(y,x,t) \equiv B_0(R(t)y,x,t)$$

is given a priori. Due to (2.2) it holds that (cf. [BaMae])

$$ker\, G_0(x,t) = ker\, D(t) =: N_0(t), \quad t \in \mathcal{I}.$$

By means of

$$P_0(t) := D(t)^- D(t), \quad Q_0(t) := I - P_0(t), \quad t \in \mathcal{I},$$

we introduce two further continuously differentiable projector functions $P_0, Q_0 : \mathcal{I} \to L(\mathbb{R}^m)$. Clearly, it results that $im\, Q_0(t) = N_0(t),\ t \in \mathcal{I}$.

Obviously, the matrices $G_0(x,t)$ and $B_0(y,x,t)$ depend continuously on their arguments and so does the following matrix

$$G_1(y,x,t) := G_0(x,t) + B_0(y,x,t)Q_0(t), \tag{2.5}$$

which is constructed from (2.4), (2.4) only, without any use of a higher derivative, and which shall play its role in the index criteria below.
Notice that $G_1(y,x,t) \equiv G_1(R(t)y,x,t)$ by construction.

The following simple example shall provide insight into the relationship between DAEs (2.1) with properly stated leading terms considered here and DAEs in standard form (e.g. [BrCaPe], [RaRh]).

Example 2.1 For given continuous functions $\alpha, \beta : \mathcal{I} \to \mathbb{R}$, $\alpha(t)$ positive on $\mathcal{I}$, the special DAE (2.1)

$$\begin{aligned} (x_1(t)+x_2(t))' - \alpha(t) &= 0, \\ x_1(t)(x_1(t)+x_2(t))' - \beta(t) &= 0, \end{aligned} \tag{2.6}$$

has the solution

$$x_1(t)+x_2(t) = x_1(t_0)+x_2(t_0) + \int_{t_0}^{t} \alpha(s)ds,$$

$$x_1(t) = \beta(t)/\alpha(t), \quad t \in \mathcal{I},$$

where $t_0 \in \mathcal{I}$. Here we have $m=2, n=1, \mathcal{D} = \mathbb{R}^2, D(t) = (1\ 1), R=1$,

$$A(x,t) = \begin{pmatrix} 1 \\ x_1 \end{pmatrix},\ D(t)^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ P_0(t) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ G_0(x,t) = \begin{pmatrix} 1 & 1 \\ x_1 & x_1 \end{pmatrix},$$

$$b(x,t) = -\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix},\ B_0(y,x,t) = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix},\ G_1(y,x,t) = \begin{pmatrix} 1 & 1 \\ x_1 & x_1 - y \end{pmatrix}.$$

Since the special matrix $G_1(y,x,t)$ is nonsingular on $\mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{I}$, $\mathbb{R}^+ := \{y \in \mathbb{R} : y > 0\}$, this DAE has tractability index 1 on $\mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{I}$ ([HiMae] and Section 3 below). Let

$$\mathcal{M}_0(t) := \{x \in \mathcal{D} : b(x,t) \in im\, A(x,t)\} = \{x \in \mathbb{R}^2 : \alpha(t)x_1 = \beta(t)\}$$

denote the constraint set which the solution values at time t have to belong to. To each given pair $t_0 \in \mathcal{I}, x_0 \in \mathcal{M}(t_0)$ there is a uniquely determined $y_0 \in \mathbb{R}$ such that $A(x_0,t_0)y_0 + b(x_0,t_0) = 0$, $y_0 = R(t_0)y_0$, i.e., $y_0 = \alpha(t_0)$. According to the solvability statements in [HiMae] there is exactly one solution $x(.)$ passing through $x(t_0) = x_0$, $(Dx)'(t_0) = y_0$. Naturally, in the above reasoning, the solutions that have a continuously differentiable sum $x_1(t)+x_2(t) = D(t)x(t)$ are continuous functions. The single components $x_1(t)$ and $x_2(t)$ are not necessarily continuously differentiable. In contrast to that, the standard DAE theory is bound to more smoothness. Solutions should be at least continuously differentiable (in all components). The standard form DAE corresponding to (2.6) reads $G_0(x(t),t)x'(t) + b(x(t),t) = 0$, i.e.,

$$\begin{pmatrix} 1 & 1 \\ x_1(t) & x_1(t) \end{pmatrix} x'(t) - \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = 0. \tag{2.7}$$

Clearly, to form the derivative array system for realizing differentiation index 1 one has to assume that $\alpha, \beta \in C^1(\mathcal{I}, \mathbb{R})$, additionally. Then the above solutions are continuously differentiable in fact.

In this paper, a continuous function $x_* : \mathcal{I}_* \to I\!R^m$ with values in $\mathcal{D}$, $\mathcal{I}_* \subseteq \mathcal{I}, \mathcal{I}_* \neq \phi$, is said to be a solution of the DAE (2.1) if $D(t)x_*(t)$ is continuously differentiable in t on $\mathcal{I}_*$ and equation (2.1) is satisfied pointwise for $t \in \mathcal{I}_*$. Hence, we are led to the function space

$$C_D^1(\mathcal{I}_*) := \{x \in C(\mathcal{I}_*, I\!R^m) : Dx \in C^1(\mathcal{I}_*, I\!R^n)\}. \tag{2.8}$$

Our main aim is finding index criteria for the DAE (2.1) in terms of the original data $G_0(x,t), B_0(y,x,t)$ without using higher derivatives. This shall be done in such a way that all linear DAEs arising from (2.1) by linearization have the same index as the DAE (2.1) itself. For this purpose let us take a closer look at linearizations. Let $x_* \in C_D^1(\mathcal{I}_*)$ with $x_*(t) \in \mathcal{D},\ t \in \mathcal{I}_*$ be fixed. Using variations $z := x - x_* \in C_D^1(\mathcal{I}_*)$ we expand, for $t \in \mathcal{I}_*$,

$$\begin{aligned} A(x(t),t) &= A(x_*(t),t) + A_x(x_*(t),t)z(t) + o(z(t)), \\ b(x(t),t) &= b(x_*(t),t) + b_x(x_*(t),t)z(t) + o(z(t)), \\ D(t)x(t) &= D(t)x_*(t) + D(t)z(t). \end{aligned}$$

Inserting into (2.1) and neglecting the terms $o(z(t))$ leads to

$$A_*(t)(D(t)z(t))' + B_*(t)z(t) = -A(x_*(t),t)(D(t)x_*(t))' - b(x_*(t),t), \tag{2.9}$$

with

$$A_*(t) \ := \ A(x_*(t),t), \tag{2.10}$$

$$\begin{aligned} B_*(t) \ &:= \ b_x(x_*(t),t) + A_x(x_*(t),t)z(t)(D(t)x_*(t))' \\ &= \ B_0((D(t)x_*(t))', x_*(t), t),\ t \in \mathcal{I}_*. \end{aligned} \tag{2.11}$$

Equation (2.9) is a linear DAE with respect to the variation z. The leading term of the DAE (2.9) is stated properly as a consequence of the corresponding property of the original DAE (2.1). Both, equation (2.9) but also

$$A_*(t)(D(t)z(t))' + B_*(t)z(t) = q(t) \tag{2.12}$$

are said to be *linearizations of the DAE (2.1) along* x_*. In general, for different functions x_* the linearizations (2.12) may behave quite differently. They may have different index, lack regularity, and suffer from index changes (cf. Examples 2.2 and 3.5 below).

Throughout this paper, $\mathcal{G} := \mathcal{D}_y \times \mathcal{D}_x \subseteq I\!R^n \times I\!R^m$ denotes an open set such that $\mathcal{D}_x \subseteq \mathcal{D},\ \mathcal{D}_y \cap im\, D(t) \neq \phi$, for $t \in \mathcal{I}$. We say that the function $x_* \in C_D^1(\mathcal{I}_*),\ \mathcal{I}_* \subseteq \mathcal{I}$ *has values in* $\mathcal{G}$ if $x_*(t) \in \mathcal{D}_x, R(t)(Dx_*)'(t) \in \mathcal{D}_y$, for $t \in \mathcal{I}_*$.

Example 2.2 Continue considering the DAE (2.6) from Example 2.1, which has tractability index 1 on $\mathcal{G} \times \mathcal{I}, \mathcal{D}_y = I\!R^+, \mathcal{D}_x = I\!R^2$. For $x_{*1}(t) := c_1, x_{*2}(t) := c_2 + dt, t \in \mathcal{I}$, with fixed $c \in I\!R^2, d \in I\!R$ it results that

$$A_*(t) = \begin{pmatrix} 1 \\ c_1 \end{pmatrix}, \quad B_*(t) = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix},$$

and the DAE (2.12) linearized along x_* is

$$\begin{aligned}(z_1(t)+z_2(t))' &= q_1(t),\\ c_1(z_1(t)+z_2(t))' + dz_1(t) &= q_2(t).\end{aligned} \tag{2.13}$$

This linear DAE has index 1 for all $(c,d) \in \mathcal{G}$ (that is, for $x_*(t) \in I\!R^2, (D(t)x_*(t))' \in I\!R^+, t \in \mathcal{I}$) as expected. If we choose $d = 0$, the linear DAE (2.13) loses its regularity.

As we have seen in the Examples 2.1 and 2.2, the matrix $G_1(y,x,t)$ may serve as an index-1 indicator. Further matrix functions are hoped to serve as criteria for higher index.

Remark 2.3 With the map $F : \mathcal{D}_F \subseteq C^1_D(\mathcal{I}_*) \to C(\mathcal{I}_*, I\!R^m)$, $\mathcal{D}_F := \{x \in C^1_D(\mathcal{I}_*) : x(t) \in \mathcal{D}, t \in (\mathcal{I}_*)\}$, $\mathcal{I}_* \subseteq \mathcal{I}$ compact, $Fx := A(x(.),.)(Dx)'(.) + b(x(.),.), x \in \mathcal{D}_F$, the DAE (2.1) may be represented as an operator equation $Fx = 0$.
Equipped with a natural norm, $C^1_D(\mathcal{I}_*)$ becomes a Banach space. In the usual way, one can realize that F is Fréchet differentiable and the DAE (2.12) linearized along x_* is nothing else but the equation $F'(x_*)z = q$.
In this context, defining properties of nonlinear operators via linearizations is common. For instance, a nonlinear C^1 map is said to be Fredholm if its Fréchet derivative is so at each element of the definition domain.

We finish this section by mentioning that the DAE (2.1) may be rewritten as a standard form DAE

$$G_0(x(t),t)x'(t) + b(x(t),t) + A(x(t),t)D'(t)x(t) = 0 \tag{2.14}$$

supposed the solutions are smooth enough (cf. Example 2.2).

3 On nonlinear index-1 DAEs

Nonlinear DAEs (2.1) with tractability index 1 are studied in [HiMae] in a more general setting. In particular, solvability assertions are given.

Definition 3.1 *The DAE (2.1) has tractability index 1 on $\mathcal{G} \times \mathcal{I}$ if $G_1(y,x,t)$ is nonsingular for $(y,x,t) \in \mathcal{G} \times \mathcal{I}$.*

Remark 3.2 Definition 3.1 is equivalent to [HiMae], Definition 2.2. Namely, $G_1(y,x,t)$ is a nonsingular matrix if the subspaces $N_0(t)$ and

$$S_0(y,x,t) := \{z \in I\!R^m : B_0(y,x,t)z \in im\, G_0(x,t)\}$$

intersect transversally, and vice versa.

Remark 3.3 If $G_1(y,x,t)$ is nonsingular, then the matrix pencil $\lambda G_0(x,t) + B_0(y,x,t) + G_0(x,t)W$ is regular with Kronecker index 1 independently of the chosen matrix $W \in L(I\!R^m)$ (cf. [GrMae1], Theorem A.13). Notice that the so-called local matrix pencil of the standard form DAE (2.14) is

$$\lambda G_0(x,t) + B_0(y,x,t) + G_0(x,t)D(t)^- D'(t).$$

Hence, if the DAE (2.1) has tractability index 1, then the standard form version (2.14) is *uniform index 1* in the sense of [BrCaPe], Definition 3.2.1, supposed the higher derivatives demanded in [BrCaPe] do exist.

Theorem 3.4 *If the nonlinear DAE (2.1) has tractability index 1 on $\mathcal{G} \times \mathcal{I}$, then, for each fixed function $x_* \in C^1_D(\mathcal{I}_*)$ with values in $\mathcal{G}, \mathcal{I}_* \subseteq \mathcal{I}$, the linearization along x_* has tractability index 1, and vice versa.*

Proof: Let $G_1(y,x,t)$ be nonsingular for $(y,x,t) \in \mathcal{G}\times\mathcal{I}$. Take $x_* \in C^1_D(\mathcal{I}_*)$ with values in $\mathcal{G}, \mathcal{I}_* \subseteq \mathcal{I}$, and form $G_{*0}(t) := A_*(t)D(t), B_{*0}(t) := B_*(t), G_{*1}(t) := G_{*0}(t)+B_{*0}(t)Q_0(t) = G_1(R(t)(Dx_*)'(t), x_*(t), t), t \in \mathcal{I}_*$. Since $G_{*1}(t)$ is nonsingular on $\mathcal{I}_*$, the linearization has index 1.
To show the contrary, we assume all linearizations to have tractability index 1. Take $t_0 \in \mathcal{I}, y_0 \in \mathcal{D}_y \cap im\, D(t_0), x_0 \in \mathcal{D}_x$ and consider the function

$$x_*(t) := x_0 + (t-t_0)D(t_0)^-\{y_0 - D'(t_0)x_0\}, t \in \mathcal{I}.$$

Compute $x_*(t_0) = x_0, (Dx_*)'(t_0) = D'(t_0)x_0 + R(t_0)\{y_0 - D'(t_0)x_0\},\ R(t_0)(Dx_*)'(t_0) = R(t_0)y_0 = y_0$, thus there is an interval $\mathcal{I}_* \subseteq \mathcal{I}$ so that $x_* \in C^1_D(\mathcal{I}_*)$ has values in $\mathcal{G}$. But then $G_1(y_0, x_0, t_0) = G_{*1}(t_0)$ is nonsingular. $\square$

As we have seen in Example 2.1/2.2, a rank change of $G_1(y,x,t)$ may indicate a singularity.

Example 3.5 Consider the DAE

$$\begin{aligned} x_1'(t) - x_3(t) &= 0 \\ x_2(t)(1-x_2(t)) - \gamma(t) &= 0 \\ x_1(t)x_2(t) + x_3(t)(1-x_2(t)) - t &= 0, \quad t \in \mathcal{I} := I\!R, \end{aligned} \tag{3.1}$$

which can be put into the form (2.1) with $n=1, m=3, D(t) = (1\ 0\ 0), R=1$,

$$A(x,t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad G_0(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(t)^- = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_0(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b(x,t) = \begin{pmatrix} -x_3 \\ x_2(1-x_2) - \gamma(t) \\ x_1x_2 + x_3(1-x_2) - t \end{pmatrix}, \quad G_1(y,x,t) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1-2x_2 & 0 \\ 0 & x_1 - x_3 & 1-x_2 \end{pmatrix}.$$

$\gamma : I\!R \to I\!R$ is a small continuous function. Notice that, for $\gamma(t) \equiv 0$, the DAE (3.1) is given in [AschPe], Example 9.2.
By inspection of $det\, G_1(y,x,t) = (1-2x_2)(1-x_2)$ we realize this DAE to have tractability index 1 on $\mathcal{G} \times \mathcal{I} = I\!R \times \mathcal{D}_x \times I\!R$, $\mathcal{D}_x = \{x \in I\!R^3 : x_2 \neq 1, x_2 \neq \frac{1}{2}\}$. $\mathcal{D}_x$ is split into three parts. The constraint set of the DAE (3.1)

$$\begin{aligned} \mathcal{M}_0(t) = \{x \in I\!R^3 : x_2 = \tfrac{1}{2}(1 \pm \sqrt{1-4\gamma(t)}), \\ x_1(1 \pm \sqrt{1-4\gamma(t)}) + x_3(1 \mp \sqrt{1-4\gamma(t)}) = 2t\ \} \end{aligned}$$

decomposes into the two parts, $\mathcal{M}_0^+(t), \mathcal{M}_0^-(t)$. If $\gamma(t) \equiv \frac{1}{4}$, which corresponds to $x_2 = \frac{1}{2}$, then $\mathcal{M}_0^+(t)$ and $\mathcal{M}_0^-(t)$ coincide. Then, there are only the solutions $x_{*2}(t) \equiv \frac{1}{2}, x_{*3}(t) \equiv -x_{*1}(t)+2t, x'_{*1}(t) = x_{*3}(t)$. The solvability statements proved in [HiMae] apply to $\mathcal{M}_0(t) \cap \mathcal{D}_x, t \in \mathcal{I}$.
If $\gamma(t)$ vanishes identically, then the DAE (3.1) has the only solutions

$$\begin{aligned} &x_{*1}(t) \equiv \tfrac{1}{2}t^2 + x_{*1}(0),\ x_{*2}(t) \equiv 0,\ x_{*3}(t) \equiv t, \\ \text{and}\quad &x_{*1}(t) \equiv t,\ x_{*2}(t) \equiv 1,\ x_{*3}(t) \equiv 1. \end{aligned}$$

The first solution lies in the region where the DAE has index 1. The linearization along the latter solution has the coefficients

$$A_*(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad D(t) = (1\ 0\ 0),\ B_*(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & t-1 & 0 \end{pmatrix}.$$

The matrix $G_{*1}(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & t-1 & 0 \end{pmatrix}$ is singular, i.e., the DAE linearized along x_* is no longer index-1 tractable. One can easily check the subspaces $ker\, G_{*1}(t)$ and $S_{*1}(t) := \{z \in I\!R^m : B_*(t)P_0(t)z \in im\, G_{*1}(t)\} = \{z \in I\!R^3 : z_1 = 0\}$ to intersect transversally, hence, the linearization along x_* has tractability index 2 (cf. [BaMae]).

If $\gamma(t)$ varies with t, $\gamma(\bar{t}) = \frac{1}{4}$ leads to $x_2(\bar{t}) = \frac{1}{2}$ and $\gamma(\bar{t}) = 0$ yields $x_2(\bar{t}) = 0$ or $x_2(\bar{t}) = 1$. In the particular case of $\gamma(t) = \frac{1}{4}\sin^2 t$ two different solutions corresponding to

$$x_2(t) = \frac{1}{2}(1 + |\cos t|) \quad \text{and} \quad x_2(t) = \frac{1}{2}(1 - |\cos t|)$$

pass through each point of

$$\mathcal{M}_0\left(k\frac{\pi}{2}\right) = \{x \in I\!R^3 : x_2 = \frac{1}{2},\ x_1 + x_3 = k\pi\},$$

that is, the rank change in the matrix $G_1(y,x,t)$ at $x_2 = \frac{1}{2}$ indicates bifurcations.
On this background we interpret the DAE (3.1) as an index-1 system that has different kinds of singularities at points where $G_1(y,x,t)$ undergoes rank drops. The matrix $G_1(y,x,t)$ reflects the peculiarities of the DAE well.

4 Characterizing linear DAEs

The index of a linear DAE

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t),\ t \in \mathcal{I}, \tag{4.1}$$

with properly stated leading term is characterized by means of a matrix function sequence built up from the coefficients A, D and B ([Mae3]).
As in Section 2, we use

$$G_0 := AD,\ B_0 := B,\ G_1 := G_0 + B_0Q_0,$$

as well as the projector functions R, P_0, Q_0 and the generalized inverse D^-. Here, all matrix functions depend on time t only. We drop the argument in this section. All definitions and relations are meant pointwise.

For $i \geq 1$, we successively determine projector functions Q_i, P_i such that $Q_i^2 = Q_i, im\, Q_i = ker\, G_i, P_i = I - Q_i$, and further, matrix functions

$$B_i := B_{i-1}P_{i-1} - G_i D^-(DP_0 \cdots P_i D^-)' DP_0 \cdots P_{i-1},$$

$$G_{i+1} := G_i + B_i Q_i. \tag{4.2}$$

By construction it holds that $rank\, G_i \leq rank\, G_{i+1}$.

Definition 4.1 *([Mae3]): The DAE (4.1) with properly stated leading term is said to be regular with tractability index μ if there is a matrix function sequence (4.2) such that*

(i) G_i *has constant rank* r_i *on* $\mathcal{I}, i \geq 0$,

(ii) $Q_i Q_j = 0,$ *for* $j = 0, 1, \ldots, i-1, i \geq 1$ (4.3)

(iii) $Q_i \in C(\mathcal{I}, I\!R^m), DP_0 \cdots P_i D^- \in C^1(\mathcal{I}, I\!R^n), i \geq 0,$

(iv) $r_0 \leq r_1 \leq \cdots \leq r_{\mu-1} < m,\ r_\mu = m.$

The tractability index is shown to be independent of the particular choice of the projectors Q_i. It remains invariant under regular transformations of the unknown function as well as under refactorizations of the leading term ([Mae3]).
Recall ([Mae3]) that rank changes of the matrix functions G_i and points where the condition (4.3) fails indicate singularities.

Remark 4.2 A numerical algorithm to realize the matrix sequence (4.2) subject to condition (4.3) is proposed in [La].

One might think that the derivatives $(DP_0 \cdots P_i D^-)'$ involved in (4.2) are higher derivatives of certain previous terms, but this is not the case. Behind the sequence (4.2) with condition (4.3) are advanced decompositions of the continuously differentiable subspace $im\, D$ into further such subspaces, namely

$$\begin{aligned} imD &= imDP_0 = imDP_0P_1 \oplus imDP_0Q_1 = \ldots \\ &= imDP_0 \cdots P_{i-1}P_i \oplus imDP_0 \cdots P_{i-1}Q_i \oplus imDP_0 \cdots P_{i-2}Q_{i-1} \oplus \cdots \oplus imDP_0Q_1. \end{aligned} \tag{4.4}$$

The terms $DP_0P_1D^-, DP_0Q_1D^-$ are projector functions, too. $DP_0P_1D^- \in C^1(\mathcal{I}, I\!R^n)$ implies $DP_0Q_1D^- = R - DP_0P_1D^- \in C^1(\mathcal{I}, I\!R^n)$ and so on (cf. [Mae3]).
The following assertion provides a compact recursion formula for G_{i+1} without using B_i.

Lemma 4.3 *Let a matrix function sequence (4.2) be given such that the properties (i), (ii) and (iii) in Definition 4.1 are satisfied. Let $\Omega_i \in C^1(\mathcal{I}, I\!R^n)$ be additional projector functions onto $im\, DP_0 \cdots P_{i-1}Q_i$, i.e., $\Omega_i^2 = \Omega_i$, $im\, \Omega_i = im\, DP_0 \cdots P_{i-1}Q_i, i \geq 1$.*
Then it holds that

$$G_{i+1} = G_i + B_0P_0 \cdots P_{i-1}Q_i + \left(\sum_{j=1}^{i} G_jP_0 \cdots P_j - \sum_{j=1}^{i-1} G_j \right) D^-\Omega_i' DP_0 \cdots P_{i-1}Q_i, i \geq 1. \tag{4.5}$$

Proof: First we compute

$$B_i = B_0P_0 \cdots P_{i-1} - \sum_{j=1}^{i} G_jD^-(DP_0 \cdots P_jD^-)'DP_0 \cdots P_{i-1}.$$

Taking into account the relations $DP_0 \cdots P_{i-1}Q_i = \Omega_i DP_0 \cdots P_{i-1}Q_i$, $im\, \Omega_i \subset ker\, DP_0 \cdots \cdots P_iD^-$ we obtain

$$\begin{aligned} &G_iD^-(DP_0 \cdots P_iD^-)'DP_0 \cdots P_{i-1}Q_i = G_iD^-(DP_0 \cdots P_iD^-)'\Omega_i DP_0 \cdots P_{i-1}Q_i \\ &= -G_iD^-DP_0 \cdots P_iD^-\Omega_i' DP_0 \cdots P_{i-1}Q_i. \end{aligned}$$

Further, for $j = 1, \ldots, i-1$, we find $im\, \Omega_i = im\, DP_0 \cdots P_{i-1}Q_iD^- \subseteq im\, DP_0 \cdots P_{i-1}D^- \subseteq im\, DP_0 \cdots P_jD^-$, hence $DP_0 \cdots P_jD^-\Omega_i = \Omega_i$.
This yields, for $j = 1, \ldots, i-1$,

$$\begin{aligned} &G_jD^-(DP_0 \cdots P_jD^-)'DP_0 \cdots P_{i-1}Q_i = G_jD^-(DP_0 \cdots P_jD^-)'\Omega_i DP_0 \cdots P_{i-1}Q_i \\ &= G_jD^-(DP_0 \cdots P_jD^-\Omega_i)'DP_0 \cdots P_{i-1}Q_i - G_jP_0 \cdots P_jD^-\Omega_i' DP_0 \cdots P_{i-1}Q_i \\ &= G_j(I - P_0 \cdots P_j)D^-\Omega_i' DP_0 \cdots P_{i-1}Q_i. \qquad \Box \end{aligned}$$

Theorem 4.4 *If the DAE (4.1) is regular with tractability index μ and the subspaces $im\, DP_0 \cdots P_{i-1}Q_i$ are time-invariant, then*

$$G_{i+1} = G_i + B_0P_0 \cdots P_{i-1}Q_i, \; i \geq 1, \tag{4.6}$$

is valid, and the local matrix pencil $\lambda A(t)D(t) + B(t)$ is regular with Kronecker index μ uniformly for all $t \in \mathcal{I}$.

Proof: The representation (4.6) results from Lemma 4.3 if we choose time-invariant projector functions Ω_i, hence, $\Omega_i' = 0$.
From (4.6), taking into account that $r_0 \leq \cdots \leq r_{\mu-1} < m, r_\mu = m$, we obtain the claimed properties of the pencil $\lambda G_0(t) + B_0(t)$ by means of [GrMae2], Theorem 3. $\Box$

In [Mae3] it is mentioned that the local matrix pencil of an index μ DAE (4.1) becomes regular with Kronecker index μ supposed that there is a sequence with time-invariant projectors $DP_0 \cdots P_iD^-, i \geq 1$. Now we observe, due to Theorem 4.4, that, more generally, if there is a sequence just with time-invariant intersections $(ker\, DP_0 \cdots P_iD^-) \cap im\, DD^-$, then the DAE (4.1) has tractability index μ if the local matrix pencil is regular with

Kronecker index μ uniformly on $\mathcal{I}$.
It has been well known for a long time that the local matrix pencil of a regular DAE may become singular (e.g. [BrCaPe]). The following example confirms this once more. At the same time it makes clear that time-invariant subspaces $DP_1 \cdots P_i D^-, i = 1, \cdots, \mu - 1$ (instead of $DP_1 \cdots P_{i-1} Q_i D^-$) do not lead to regular local matrix pencils.

Example 4.5 ([Mae3]): The DAE (4.1) given by the coefficients

$$A(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \\ 0 & 0 \end{pmatrix}, \ D(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 0 \end{pmatrix}, \ t \in \mathcal{I} = I\!R,$$

which reads in detail

$$x_2' + x_1 = q_1, \quad -tx_2' + x_3' = q_2, \quad -tx_2 + x_3 = q_3,$$

is regular with index 3. However, $det(\lambda G_0(t) + B_0(t)) = 0$ for all t and λ, i.e., the local matrix pencil is all overall singular. With

$$D(t)^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \ Q_0(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ G_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ Q_1(t) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{pmatrix},$$

$$G_2(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ Q_2(t) = \begin{pmatrix} 0 & -t & 1 \\ 0 & t & -1 \\ 0 & -t(t-1) & 1-t \end{pmatrix}, \ G_3(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & -t & 1 \end{pmatrix},$$

we find $detG_3(t) \equiv 1, \ r_0 = 2, \ r_1 = 2, \ r_2 = 2, \ r_3 = 3, \ (DP_1D^-)(t) = \begin{pmatrix} 0 & 0 \\ -t & 1 \end{pmatrix}$, $(DQ_1D^-)(t) = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix}$, $DP_1P_2D^- = 0, \ DP_1Q_2D^- = DP_1D^-$. While both $im\,(DP_1D^-)(t)$ and $im\,(DP_1P_2D)(t)$ are time-invariant, the subspace $(DQ_1D^-)(t)$ definitely varies with time, but this causes the local matrix pencil to become singular.

5 Nonlinear higher index DAEs

In this section we propose an index criterion for nonlinear DAEs (2.1). As basic data we shall use the original coefficients and their only first partial derivatives involved in $B_0(y, x, t)$. The index notion shall cover the Definitions 3.1 and 4.1 for nonlinear index-1 DAEs and linear higher index DAEs, respectively.
We continue working with the matrices

$$G_0(x, t), \ B_0(y, x, t), \ G_1(y, x, t) \tag{5.1}$$

as well as the projectors $P_0(t), Q_0(t), R(t)$ and the generalized inverse $D(t)^-$ introduced in Section 2.

As in Section 4 we pointwise determine a matrix function sequence, but now we have to consider all variables t, x, and y. This shall be done in such a way that for the matrix function sequences corresponding to DAEs linearized along $x_* \in C^1(\mathcal{I}_*)$ the relation $G_i((Dx_*)'(t),\ x_*(t), t) = G_{*i}(t),\ t \in \mathcal{I}_*$, becomes true.

Denote by $Q_1(y,x,t) \in L(\mathbb{R}^m)$ a projector onto $ker\, G_1(y,x,t),\ P_1(y,x,t) := I - Q_1(y,x,t)$, $y \in \mathbb{R}^n, x \in \mathcal{D}, t \in \mathcal{I}$. Then, assuming the product $D(t)P_1(y,x,t)D(t)^- \in L(\mathbb{R}^n)$ to be independent of y – we write $(DP_1D^-)(x,t)$ – we determine the next matrices

$$\begin{aligned} B_1(x', y, x, t) := B_0(y,x,t)P_0(t) - G_1(y,x,t)D(t)^-\{(DP_1D^-)_x(x,t)x' + \\ + (DP_1D^-)_t(x,t)\}D(t)P_0(t), \end{aligned} \tag{5.2}$$

$$G_2(x', y, x, t) := G_1(y,x,t) + B_1(x', y, x, t)Q_1(y,x,t), \tag{5.3}$$

pointwise for $x' \in \mathbb{R}^m,\ y \in \mathbb{R}^n,\ x \in \mathcal{D},\ t \in \mathcal{I}$.

Considering a linearization (2.12) along $x_* \in C^1(\mathcal{I}_*)$ we derive (cf. Section 3)
$G_1((Dx_*)'(t), x_*(t), t) = G_{*1}(t),\ (DP_1D^-)(x_*(t), t) = (DP_{*1}D^-)(t)$,
$\{(DP_1D^-)_x(x_*(t), t)x_*'(t) + (DP_1D^-)_t(x_*(t), t)\} = \frac{d}{dt}(DP_1D^-)(x_*(t), t) = (DP_{*1}D^-)'(t)$,
thus $B_1(x_*'(t), (Dx_*)'(t), x_*(t), t) = B_{*0}(t)P_0(t) - G_{*1}(t)D(t)^-(DP_{*1}D^-)'(t)D(t)P_0(t)$, $t \in \mathcal{I}_*$.
Of course, the values $x_*'(t)$ and $(Dx_*)'(t)$ are somehow related. Namely, it holds that

$$x_*'(t) = D(t)^-(y_*(t) - D'(t)x_*(t)) + z_*(t),\quad t \in \mathcal{I}_*, \tag{5.4}$$

with $y_*(t) := R(t)(Dx_*)'(t),\ z_*(t) := Q_0(t)x_*'(t),\ t \in \mathcal{I}_*$. Hence, the variables x' and y used in the formulas (5.1) and (5.2) are not independent of each other. By setting

$$x' = D(t)^-\{y - D'(t)x\} + z,\ y \in im\, D(t),\quad z \in N_0(t) \tag{5.5}$$

below, we reflect this relation. x' is fully determined by y and z.

In an analogous way as we have formed the matrices (5.3) and (5.2) we determine the following ones for $i > 1$:

$$G_{i+1}(x', y, x, t) := G_i(x', y, x, t) + B_i(x', y, x, t)Q_i(x', y, x, t), \tag{5.6}$$

with

$$\begin{aligned} B_i(x', y, x, t) \ &:= \ B_{i-1}(x', y, x, t)P_{i-1}(x', y, x, t) \\ &- \ G_i(x', y, x, t)D(t)^-\{(DP_0 \cdots P_iD^-)_x(x,t)x' \\ &\quad + (DP_0 \cdots P_iD^-)_t(x,t)\}(DP_0 \cdots P_{i-1})(x,t), \end{aligned} \tag{5.7}$$

pointwise for $x' \in \mathbb{R}^m,\ y \in \mathbb{R}^n,\ x \in \mathcal{D},\ t \in \mathcal{I}$. By construction, it holds that $rank\, G_{i+1}(x', y, x, t) \geq rank\, G_i(x', y, x, t)$. If $G_i(x', y, x, t)$ is nonsingular, the sequence becomes stationary, i.e., $G_{i+k} = G_i,\ k \geq 1$.
As we have seen in Example 3.5, points where $G_1(y,x,t)$ suffers from a rank drop indicate singularities for index-1 problems. Hence, the general index notion we want to realize is expected to contain e.g. a constant rank condition for $G_1(y,x,t)$. The next theorem confirms this conjecture.

Theorem 5.1 *Let $\mathcal{G} \times \mathcal{I}$ be connected. If all linearizations (2.12) along functions $x_* \in C^1(\mathcal{I}_*)$ with values in $\mathcal{G}, \mathcal{I}_* \subseteq \mathcal{I}$, are regular with common tractability index μ, then the matrix $G_1(y,x,t)$ has constant rank on $\mathcal{G} \times \mathcal{I}$.*

Proof: Assume that $G_1(y,x,t)$ changes its rank, i.e., there are two points $(y_i, x_i, t_i) \in \mathcal{G} \times \mathcal{I}, \;\; y_i = R(t_i)y_i, \;\; i = 1,2,$ such that $G_1(y_1,x_1,t_1)$ and $G_1(y_2,x_2,t_2)$ have different rank. These two points can be connected by a continuous curve lying in $\mathcal{G} \times \mathcal{I}$. Moving from (y_1,x_1,t_1) along the curve to (y_2,x_2,t_2) one necessarily meets a point $(y_0,x_0,t_0) \in \mathcal{G} \times \mathcal{I}, \;\; y_0 = R(t_0)y_0$, in each neighbourhood of which there are further points $(\bar{y},\bar{x},\bar{t}) \in \mathcal{G} \times \mathcal{I}, \bar{y} = R(\bar{t})\bar{y}$, with

$$rank G_1(\bar{y},\bar{x},\bar{t}) \neq rank G_1(y_0,x_0,t_0). \tag{5.8}$$

Consider the interpolation polynomials $x_*(.)$ determined by the conditions

$$\begin{aligned} x_*(t_0) &= x_0, \; x_*(\bar{t}) = \bar{x}, \\ x'_*(t_0) &= D(t_0)^-\{y_0 - D'(t_0)x_0\}, \quad x'_*(\bar{t}) = D(\bar{t})^-\{\bar{y} - D'(\bar{t})\bar{x}\}. \end{aligned}$$

Because of

$$R(t_0)(Dx_*)'(t_0) = y_0, \;\; R(\bar{t})(Dx_*)'(\bar{t}) = \bar{y},$$

we have $R(t)(Dx_*)'(t) \in \mathcal{D}_y, \; x_*(t) \in \mathcal{D}_x$, for $t \in \mathcal{I}_*$, supposed $\mathcal{I}_* \subseteq \mathcal{I}$ is small enough and $(\bar{y},\bar{x},\bar{t})$ is close to $(y_0,x_0,t_0), t_0, \bar{t} \in \mathcal{I}_*$.
Since $x_* \in C^1(\mathcal{I}_*)$ has now values in $\mathcal{G}$, we are allowed to consider the linearization along x_*. It holds that

$$G_{*1}(t) = G_1(R(t)(Dx_*)'(t), \; x_*(t), t), \;\; t \in \mathcal{I}_*,$$

and, in particular,

$$G_{*1}(t_0) = G_1(y_0,x_0,t_0), \;\; G_{*1}(\bar{t}) = G_1(\bar{y},\bar{x},\bar{t}).$$

By Definition 4.1, the matrix function $G_{*1}(t)$ does not change its rank, but this contradicts the relation (5.8).
In the consequence, those two points (y_1,x_1,t_1) and (y_2,x_2,t_2) we started with do not exist, that is, $G_1(y,x,t)$ has constant rank. □

Definition 5.2 *The DAE (2.1) has tractability index μ on $\mathcal{G} \times \mathcal{I}$ if there is a matrix function sequence (5.1), (5.2), (5.3), (5.6), (5.7) such that*

(i) $rank\, G_i(x', y, x, t) = r_i$
for $t \in \mathcal{I}, x \in \mathcal{D}_x, y \in \mathcal{D}_y, z \in N_0(t)$ *and* x' *defined by (5.5),* $i \geq 0$,

(ii) $Q_i(x',y,x,t)Q_j(x',y,x,t) = 0, \;\;$ *for* $j = 0, \ldots, i-1, \hfill (5.9)$

$t \in \mathcal{I}, x \in \mathcal{D}_x, y \in \mathcal{D}_y, z \in N_0(t)$ *and* x' *defined by (5.5),* $i \geq 0$, *(iii)* Q_i *is continuous,* $DP_0 \cdots P_i D^-$ *is continuously differentiable and does not depend on* x' *and* $y, i \geq 1$,

(iv) and $r_0 \leq r_1 \leq \ldots \leq r_{\mu-1} < m,\ r_\mu = m$.

Remark 5.3 As we have intended, Definition 5.2 covers Definition 3.1 as well as Definition 4.1 promptly.

Theorem 5.4 *If the nonlinear DAE (2.1) has tractability index* μ *on* $\mathcal{G}\times\mathcal{I}$*, then, for each function* $x_* \in C^1(\mathcal{I}_*)$ *with values in* $\mathcal{G}$*,* $\mathcal{I}_* \subseteq \mathcal{I}$*, the linearization along* x_* *has tractability index* μ *as well as the further characteristics* $r_{*i} = r_i,\ i \geq 0$.

Proof: The assertion is an immediate consequence of the above construction. □

Corollary 5.5 *Let the nonlinear DAE (2.1) have tractability index* μ *on* $\mathcal{G}\times\mathcal{I}$*. Then, for each fixed* $c \in \mathcal{D}_x$*, supposed there are intervals* $\mathcal{I}_c \subseteq \mathcal{I}$ *such that* $R(t)D'(t)c \in \mathcal{D}_y,\ t \in \mathcal{I}_c$*, the linear DAEs*

$$A(c,t)(D(t)x(t))' + B_0(R(t)D'(t)c, c, t)x(t) = q(t),\ t \in \mathcal{I}_c \tag{5.10}$$

have tractability index μ *and the uniform characteristics* $r_{ci} = r_i, i \geq 0$.

Proof: The constant function $x_*(t) := c,\ t \in \mathcal{I}_c$, has values in $\mathcal{G}$ so that Theorem 5.4 applies. □

If $0 \in \mathcal{D}_y$ and $D'(t) \equiv 0$, one can choose $\mathcal{I}_c = \mathcal{I}$ for each $c \in \mathcal{D}_x$. However, it may happen that $0 \notin \mathcal{D}_y$, as we have seen in Example 2.1.
In case of autonomous DAEs with $0 \in \mathcal{D}_y$, tractability with index μ requires that the linear constant coefficient DAEs

$$A(c)(Dx(t))' + B_0(0,c)x(t) = q(t) \tag{5.11}$$

have tractability index μ, or, equivalently, that $\lambda A(c)D + B_0(0,c)$ is a regular matrix pencil with Kronecker index μ, uniformly for all $c \in \mathcal{D}_x$.

In the necessary condition for tractability with index μ given by Corollary 5.5, the terms concerning the partial derivatives $(DP_0 \cdots P_{i-1}D^-)_x(x,t)$ disappear due to $x' = 0$. Therefore, this criterion is much easier to realize in practice.

Example 5.6 The semi-explicit DAE

$$\begin{aligned} x_1'(t) + b_1(x_1(t), x_2(t), t) &= 0 \\ b_2(x_1(t), x_2(t), t) &= 0 \end{aligned} \tag{5.12}$$

fits into the form (2.1) with $m = m_1 + m_2,\ n = m_1$,

$$A(x,t) = \begin{pmatrix} I \\ 0 \end{pmatrix},\ D(t) = (I\ \ 0),\ R(t) = I,\ P_0(t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},\ D(t)^- = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

The basic matrices

$$G_0(x,t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \; B_0(y,x,t) = \begin{pmatrix} B_{11}(x,t) & B_{12}(x,t) \\ B_{21}(x,t) & B_{22}(x,t) \end{pmatrix},$$

$B_{ik}(x,t) := b_{i_{x_k}}(x,t)$ do not at all depend on y. We drop this argument.
Compute

$$G_1(x,t) = \begin{pmatrix} I & B_{12}(x,t) \\ 0 & B_{22}(x,t) \end{pmatrix},$$

$N_1(x,t) = ker\, G_1(x,t) = \{x \in I\!R^m : z_1 + B_{12}(x,t)z_2 = 0, B_{22}(x,t)z_2 = 0\} = \{z \in I\!R^m : z_2 = Q_{22}(x,t)z_2, z_1 = -B_{21}(x,t)Q_{22}(x,t)z_2\}$, where $Q_{22}(x,t) \in L(I\!R^{m_2})$ denotes a projector of $I\!R^{m_2}$ onto $ker\, B_{22}(x,t)$.
With $H := B_{12}Q_{22}(B_{12}Q_{22})^-$, where $(B_{12}Q_{22})^-$ is pointwise a generalized inverse of $B_{12}Q_{22}$, we determine

$$Q_1(x,t) = \begin{pmatrix} H(x,t) & 0 \\ -Q_{22}(x,t)(B_{12}Q_{22})^-(x,t) & 0 \end{pmatrix}$$

to be a projector onto $N_1(x,t)$ and to satisfy $Q_1(x,t)Q_0(t) = 0$. Then we derive $DP_1D^- = H$,

$$G_2(x',x,t) = \begin{pmatrix} I + \mathfrak{A}(x',x,t)H(x,t) & B_{12}(x,t) \\ B_{21}(x,t)H(x,t) & B_{22}(x,t) \end{pmatrix},$$

$\mathfrak{A}(x',x,t) := B_{11}(x,t) + H_x(x,t)x' + H_t(x,t)$.

As it is well known, the DAE (5.12) has index 1 on $\mathcal{G} \times \mathcal{I}, \mathcal{G} = I\!R^n \times \mathcal{D}_x$, if the block $B_{22}(x,t)$ or, equivalently, $G_1(x,t)$ remains nonsingular for $x \in \mathcal{D}_x,\ t \in \mathcal{I}$.
$G_1(x,t)$ has constant rank if $B_{22}(x,t)$ has. In the constant rank case, $Q_{22}(x,t)$ can be chosen to be continuous.
If $B_{12}(x,t)Q_{22}(x,t)$ has constant rank, the generalized inverse $(B_{12}(x,t)Q_{22}(x,t))^-$ and, hence, the projectors $H(x,t)$ and $Q_1(x,t)$ are continuous. The only additional smoothness assumption is the assumption on H to be continuously differentiable.
A careful inspection of $G_2(x',x,t)$ shows this matrix to be nonsingular if the block

$$B_{22}(x,t) + B_{21}(x,t)B_{12}(x,t)Q_{22}(x,t) \in L(I\!R^{m_2}) \tag{5.13}$$

is so or, equivalently, if the pencil $\lambda B_{22}(x,t) + B_{21}(x,t)B_{12}(x,t)$ is regular with Kronecker index 1. Notice that this is independent of the choice of Q_{22}.

In the well-known special case of Hessenberg form size two DAEs one has $B_{22} = 0, Q_{22} = I, H = B_{12}B_{12}^-$. The block (5.13) is simply the product $B_{21}(x,t)B_{12}(x,t)$, whose nonsingularity is the well-known index-2 criterion for Hessenberg systems. Let us mention that $B_{21}(x,t)$ has constant rank m_2 then. $\square$

Remark 5.7 As in semi-explicit DAEs (cf. Example 5.6), in a large class of DAEs (2.1) the basic matrix $B_0(y,x,t) = (A(x,t)y)_x + b_x(x,t)$ does not depend on y. Namely, if $A(x,t)$ is independent of x, B_0 simplifies to $B_0(y,x,t) = b_x(x,t)$, i.e., we may drop this argument. Furthermore, the variable x' disappears in certain classes of problems (cf. Examples 5.8 and 5.9 below).
However, in general condition (iii) in Definition 5.2 plays the role of a structural condition on the admissible DAEs.

Example 5.8 Consider the DAE

$$\begin{aligned} x_1'(t) - x_2(t) - q_1(t) &= 0, \\ x_1(t) + \tfrac{1}{2}x_3(t)^2 - q_2(t) &= 0, \\ x_3(t) - q_3(t) &= 0, \end{aligned} \tag{5.14}$$

with $m=3,\ n=1,\ A = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $D = (1\ \ 0\ \ 0), D^- = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $R=1,\ P_0 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$,
$G_0 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$, $B_0(x) = \begin{pmatrix}0&-1&0\\1&0&x_3\\0&0&1\end{pmatrix}$.
Compute $G_1(x) = \begin{pmatrix}1&-1&0\\0&0&x_3\\0&0&1\end{pmatrix}$, $Q_1 = \begin{pmatrix}1&0&0\\1&0&0\\0&0&0\end{pmatrix}$, $Q_1Q_0 = 0,\ DP_1D^- = 0,\ DQ_1D^- = 1,\ G_2(x) = \begin{pmatrix}1&-1&0\\1&0&x_3\\0&0&1\end{pmatrix}$, $det\, G_2(x) = 1$, which characterizes the DAE (5.14) to have tractability index 2 on $\mathcal{G}\times\mathcal{I},\ \mathcal{G} = I\!R\times I\!R^3$. □

Example 5.9 For the special DAE in Hessenberg form

$$\begin{aligned} x_1'(t) - \alpha x_1(t) - q_1(t) &= 0, \\ x_2'(t) - \tfrac{x_3(t)}{x_2(t)} - q_2(t) &= 0, \\ x_1(t)^2 + x_2(t)^2 - 1 &= 0, \end{aligned} \tag{5.15}$$

with $m=3,\ n=2,\ \mathcal{I} = I\!R,\ \alpha\in I\!R$ being a constant, $\mathcal{D} = \{x\in I\!R^3 : x_2 > 0\}$,
$A = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$, $D = \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$, $D^- = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$, $P_0 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$, $R = \begin{pmatrix}1&0\\0&1\end{pmatrix}$,
$B_0(x) = \begin{pmatrix}-\alpha&0&0\\0&\frac{x_3}{x_2^2}&-\frac{1}{x_2}\\2x_1&2x_2&0\end{pmatrix}$, one obtains $G_1(x) = \begin{pmatrix}1&0&0\\0&1&-\frac{1}{x_2}\\0&0&0\end{pmatrix}$, $Q_1(x) = \begin{pmatrix}0&0&0\\0&1&0\\0&\frac{1}{x_2}&0\end{pmatrix}$,
$Q_1(x)Q_0 = 0$, $DQ_1(x)D^- = \begin{pmatrix}0&0\\0&1\end{pmatrix}$, $DP_1(x)D^- = \begin{pmatrix}1&0\\0&0\end{pmatrix}$, $G_2(x) = \begin{pmatrix}1&0&0\\0&1+\frac{x_3}{x_2^2}&-\frac{1}{x_2}\\0&2x_2&0\end{pmatrix}$,
$detG_2(x) = 2$.
The DAE (5.15) has tractability index 2 on $\mathcal{G}\times\mathcal{I},\ \mathcal{G} = I\!R^2\times\mathcal{D}$.

In the Examples 5.8 and 5.9 the subspaces $im\, DQ_1$ are constant. Further, the local matrix pencils $\lambda G_0 + B_0(x)$ are regular with Kronecker index 2. A similar situation results in a more general context too, as we will see below. Special information on the problem structure may simplify the matrix function sequence essentially.

Lemma 5.10 *Let a matrix function sequence (5.1), (5.2), (5.3), (5.6), (5.7) be given on $\mathcal{G} \times \mathcal{I}$ such that the properties (i), (ii) and (iii) of Definition 5.2 are valid. Let $\Omega_i \in C^1(\mathcal{D}_x \times \mathcal{I}, \mathbb{R}^n)$ be an additional projector function that pointwise projects $\mathbb{R}^n$ onto $im\, DP_0 \cdots P_{i-1}Q_i$, $i \geq 1$. Then it holds for $i \geq 1$ that*

$$G_{i+1} = G_i + B_0 P_0 \cdots P_{i-1} Q_i + \left(\sum_{j=1}^{i} G_j P_0 \cdots P_j - \sum_{j=1}^{i-1} G_j \right) D^- \{\Omega_{ix} x' + \Omega_{it}\} DP_0 \cdots P_{i-1} Q_i, \tag{5.16}$$

Proof: We apply the same arguments as used in Lemma 4.3 and take into account the rule

$$\begin{aligned}(DP_0 \cdots P_i D^- \Omega_i)_x(x,t)x' \;=\;& (DP_0 \cdots P_i D^-)_x(x,t)x'\Omega_i(x,t) \\ &+ (DP_0 \cdots P_i D^-)(x,t)\Omega_{ix}(x,t)x'.\end{aligned}$$

□

Corollary 5.11 *If the projectors Ω_i in Lemma 5.10 satisfy the condition*

$$\Omega_i(x,t) = \Omega_i(P_0(t)x,t), \quad x \in \mathcal{D}_x,\ t \in \mathcal{I},\ i \geq 1, \tag{5.17}$$

i.e., if the subspaces $im\, DP_0 \cdots P_{i-1}Q_i$, $i \geq 1$, do not depend on the nullspace component $Q_0(t)x$, then it holds that

$$\begin{aligned} G_{i+1}(x', y, x, t) &\equiv G_{i+1}(P_0(t)x', y, x, t) \\ &\equiv G_{i+1}(D(t)^- \{y - D'(t)x\}, y, x, t). \end{aligned}$$

Proof: Condition (5.17) implies $\Omega_{ix}(x,t) = \Omega_{ix}(P_0(t)x,t) = \Omega_{ix}(P_0(t)x,t)P_0(t)$, hence $\Omega_{ix}(x,t) = \Omega_{ix}(x,t)P_0(t)$, i.e., whenever x' occurs in (5.16), the projector $P_0(t)$ will be in front of it. Considering formula (5.5) gives $P_0(t)x' = D(t)^-\{y - D'(t)x\}$ and we are done.
□

Corollary 5.11 allows to verify the assertion of Theorem 5.4 for all $x_* \in C^1_D(\mathcal{I}_*)$ instead of $x_* \in C^1(\mathcal{I}_*)$.

Theorem 5.12 *If the DAE (2.1) has tractability index μ on $\mathcal{G} \times \mathcal{I}$ and if the subspaces $im\, D(t), im\,(DP_0 \cdots P_{i-1}Q_i)(x,t)$, $i \geq 1$, do not at all depend on x and t, then the matrix function sequence simplifies to*

$$G_{i+1}(0,y,x,t) = G_i(0,y,x,t) + B_0(y,x,t)(P_0 P_{i-1} Q_i)(x,t) \tag{5.18}$$

and the local matrix pencil

$$\lambda G_0(x,t) + B_0(y,x,t) \tag{5.19}$$

is regular with Kronecker index μ for $(y,x,t) \in \mathcal{G} \times \mathcal{I}$.

Proof: This assertion is a consequence of Lemma 5.10. Since there are constant projectors Ω_i, the term $\{\Omega_{ix}x' + \Omega_{it}\}$ vanishes identically so that the simpler matrix functions (5.18) result. Then, by [?], Theorem 3, the matrix pencil (5.19) is regular with Kronecker index μ. □

6 Final Remarks

The index notion for nonlinear DAEs with properly stated leading terms introduced here allows for index criteria for equations with low smoothness. We do not form any higher partial derivatives of the coefficients. Only the projectors that sequentially partition the subspace $im\, D(t)$ have to be continuously differentiable. Here, several constant rank conditions play an essential role. If they are violated, singularities will have to be expected. The paper does not aim at providing solvability statements. In case of linear differential equations, canonical projectors ([BaMae], [Schu]) seem to be helpful for describing the necessary smoothness requirements for solvability precisely.
A first numerical algorithm realizing an index monitoring for linear DAEs and, via linearization, for nonlinear DAEs, is suggested in [La]. This algorithm is based on linear algebra tools (e.g. [Zi]). For special applications it is hoped to make use of special structural properties as e.g. constant subspaces and to combine graph-theoretical determinations with numerical calculations. In particular, completing the index monitor discussed in [Es et all] by an appropriate index-3 criterion has been put on the agenda.
Concerning the basic structural condition on $DP_0 \cdots P_i D^-$ to depend on x and t exclusively (cf. Definition 5.2, (iii)), a further investigation will be necessary.

References

[AschPe] U.M. Ascher, L.R. Petzold: Computer methods for ordinary differential equations and differential-algebraic equations. SIAM Philadelphia, 1998.

[BaMae] K. Balla und R. Maerz: A unified approach to linear differential algebraic equations and their adjoint equations. Humboldt-Universitaet Berlin, Institut für Mathematik, Preprint 2000-18, to appear in *Zeitschrift für Analysis und ihre Anwendungen* 21, 2002.

[BrCaPe] K.E. Brenan, S.L. Campbell, L.R. Petzold: Numerical solution of initial-value problems in differential-algebraic equations. Elsevier Science Publ. Co, Inc. 1989.

[Es et all] D. Estévez Schwarz, U. Feldmann, R. Maerz, S. Sturtzel, C. Tischendorf: Finding beneficial DAE structures in circuit simulation. To appear in a special volume at Springer concerning mathematics for solving problems in industry and economy.

[GrMae1] E. Griepentrog, R. Maerz: Differential-algebraic equations and their numerical treatment. Teubner, Leipzig, 1986.

[GrMae2] E. Griepentrog, R. Maerz: Basic properties of some differential-algebraic equations. Zeitschrift für Analysis und ihre Anwendungen 8, 1989, 25-40.

[HiMae] I. Higueras, R. Maerz: Differential algebraic equations with properly stated leading terms. Computers Math. Applic. (to appear)

[HiMaeTi] I. Higueras, R. Maerz, C. Tischendorf: Numerically well formulated index-1 DAEs. Humboldt-Universitaet Berlin, Institut für Mathematik, Preprint 2001-5.

[La] R. Lamour: Index determination and calculation of consistent initial values for DAEs. In preparation.

[Mae1] R. Maerz: Numerical methods for differential-algebraic equations. Acta Numerica 1992, 141-198.

[Mae2] R. Maerz: Differential algebraic systems anew. To appear in Applied Numerical Mathematics.

[Mae3] R. Maerz: The index of linear differential algebraic equations with properly stated leading terms. *Preprint 2001–07, Humboldt-Universitaet Berlin, Institut für Mathematik,* to appear in: Results of Mathematics

[Na] M.Z. Nashed: Inner, outer, and generalized inverses in Banach and Hilbert spaces. Num. Func. Anal. and Optimiz. 9, 261-325, 1987.

[RaRh] P.J. Rabier, W.C. Rheinboldt: Theoretical and Numerical Analysis of Differential-Algebraic Equations. Handbook of Numerical Analysis Vol. VIII, Elsevier Publ. Amsterdam 2002.

[ReMaBa] G. Reißig, W.S. Martinson, P.J. Barton: Differential-algebraic equations of index 1 may have an arbitrarily high structural index. SIAM J. Scie. Comp. 21(6)2000, 1987-1990.

[Schu] I. Schumilina: Index-3 DAEs with properly stated leading term. Humboldt-Universitaet Berlin, Institut für Mathematik, Preprint 2001-20, 2001.

[Ve] G. Le Vey: Differential algebraic equations. A new look at the index. Research Report RR-2239, INRIA, 1994, ftp://ftp.inria.fr/INRIA/publication/RR/RR-2239.ps.gz

[Zi] G. Zielke: Motivation und Darstellung von verallgemeinerten Matrixinversen. Beitr. Num. Math. 7(1979), 177-218.