Stochastic DAEs in Circuit Simulation *

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Abstract

Stochastic differential-algebraic equations (SDAEs) arise as a mathematical model for electrical network equations that are influenced by additional sources of Gaussian white noise. We sketch the underlying analytical theory for the existence and uniqueness of strong solutions, provided that the systems have noise-free constraints and are uniformly of DAE-index 1. In the main part we analyze discretization methods. Due to the differential-algebraic structure, implicit methods will be necessary. We start with a general p-th mean stability result for drift-implicit one-step methods applied to stochastic differential equations (SDEs). We discuss its application to drift-implicit Euler, trapezoidal and Milstein schemes and show how drift-implicit schemes for SDEs can be adapted to become directly applicable to stochastic DAEs. Test results of a drift-implicit Euler scheme with a mean-square step size control are presented for an oscillator circuit.

1 Introduction

Electrical noise limits the performance of electronic circuits and, hence, requires the analysis or simulation of its effects. Due to decreasing signal to noise ratios in special applications linear noise analysis around the deterministic solution is no longer satisfactory. The noise influences such systems in an essentially nonlinear way. We deal with two sources of electrical noise, namely, thermal noise of resistors and shot noise of pn-junctions. They are modelled as external Gaussian white noise sources in parallel to the original element (see Figures 1 and 2). Nyquist's theorem (see e.g. [2, 4, 21]) states that the current through an arbitrary linear resistor having a resistance R, maintained in thermal equilibrium at a temperature T, can be described as the sum of the noiseless, deterministic current and a current due to a Gaussian white noise process with spectral density $S_{th} := \frac{2kT}{R}$, where k is Boltzmann's constant. Hence, the additional current is modelled as

$$I_{th} = \sigma_{th} \cdot \xi(t) = \sqrt{\tfrac{2kT}{R}} \cdot \xi(t),$$

where $\xi(t)$ is a standard Gaussian white noise process. In [20, 21] a thermo-dynamical foundation to apply this model to mildly nonlinear resistors and reciprocal networks is

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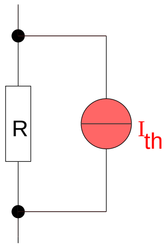
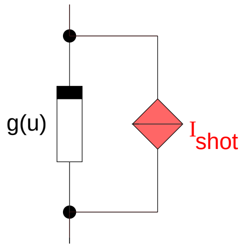
given.



thermal
noise
of a
resistor

Figure 1



shot
noise
of a
pn-junction

Figure 2

Shot noise of pn-junctions, caused by the discrete nature of current due to the elementary charge, is also modelled by a Gaussian white noise process, where the spectral density is proportional to the current I through the pn-junction: $S_{shot} := q|I|$, where q is the elementary charge. If the current through the pn-junction is described by a characteristic $I = g(u)$, where u is some voltage, the additional current is modelled by

$$I_{shot} = \sigma_{shot}(u) \cdot \xi(t) = \sqrt{q|g(u)|} \cdot \xi(t),$$

where $\xi(t)$ is a standard Gaussian white noise process. For a discussion of the model assumptions we refer to [2, 4, 20, 21].
The charge-oriented Modified Nodal Analysis (MNA) represents a standard tool in circuit simulation. The equations are generated automatically by combining the network topology, Kirchhoff's Current Law, and the characteristic equations describing the physical behaviour of the network elements. This results in large systems of DAEs, whose special structure was analyzed in a number of papers, e.g. [6, 8, 19]. We represent the topology of a network by means of the incidence matrix $(A_C, A_R, A_L, A_V, A_I, A_N)$, with indices referring to branches of capacitances, resistances, inductances, possibly controlled voltage and current sources, and n_N additional noise sources, respectively. Then the charge-oriented MNA system has the following structure (see [6, 8] for the deterministic case):

$$A_C q' + f_1(e, j_L, j_V, t) + A_N \mathrm{diag}\,(\sigma(A_N^T e, t))\xi(t) = 0 \tag{1.1}$$

$$\phi' - A_L^T e = 0 \tag{1.2}$$

$$A_V^T e - v_s(e, j_L, t) = 0 \tag{1.3}$$

$$q - q_C(A_C^T e, t) = 0 \tag{1.4}$$

$$\phi - \phi_L(j_L, t) = 0, \tag{1.5}$$

where $f_1(e, j_L, j_V, t) := A_R g(A_R^T e, t) + A_L j_L + A_V j_V + A_I i_s(e, j_L, j_I, t)$, and $q_C, g, \phi_L, v_s, i_s, \sigma$ are given, noiseless functions. The vector of unknowns describing the system behaviour consists of all node potentials e, the branch currents of current-controlled elements (inductances and voltage sources) j_L, j_V, and the charges q of capacitances, and the fluxes ϕ of inductances. ξ denotes an n_N-dimensional vector of independent standard Gaussian white noise processes. In industry-relevant applications one has to deal with a large number of unknowns and noise sources. The first block of equations (1.1) means a stochastic integral equation :

$$A_C q(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f_1(x(s), s)ds + \int_{t_0}^{t} A_N \mathrm{diag}\,(\sigma)(x(s), s)dw(s) = 0,$$

where the second integral is an Itô integral, and w denotes an n_N-dimensional Wiener process (or Brownian motion) given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq t_0}$ (see e.g. [3, 9] for the stochastic background). A solution $x = x(t,\omega)$ is a stochastic process depending on the time t and the chance element $\omega \in \Omega$. The parameter ω is omitted in the notations above. The solution $x(t) = x(t,\cdot)$ for fixed time t is a vector-valued random variable in $L^p(\Omega)$, $p \geq 1$, a realization $x(\cdot,\omega)$ is called a path.
The equations (1.1)-(1.5) form a specially structured *Stochastic Differential Algebraic Equation* (SDAE) of the type

$$Ax(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} G(x(s),s)dw(s) = 0, \tag{1.6}$$

where A is a constant singular matrix, t varies over a compact interval $\mathcal{J}$. The short-hand notation

$$Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) = 0 \tag{1.7}$$

emphasizes the relations of (1.6) to its deterministic counterpart but may be misleading for readers who are less familiar with the stochastic background. Though the notation $x'(t)$ is used in (1.7), a typical realization $x(\cdot,\omega)$ of the solution is nowhere differentiable.
A process $x(\cdot) = (x(t))_{t\in\mathcal{J}}$ is called a *strong solution* of (1.6) if it is adapted to the filtration (i.e., it does not depend on future information), and if, with probability 1, its sample paths are continuous, the integrals in (1.6) exist and (1.6) is satisfied.
In Section 2 we discuss some basics of an existence and uniqueness theory of strong solutions for SDAEs where we restrict to DAE-systems that have uniformly index 1 and noise-free constraints. In particular, we introduce the notion of an inherent regular SDE. The latter motivates to study discretization schemes first for SDEs. Hence, we provide in Section 3 a short introduction to p-th mean stability and convergence of general drift-implicit schemes. For the convenience of the reader the proof of the main stability result is shifted to the Appendix. In Section 4 we discuss several variants of drift-implicit schemes for SDAEs, namely, the drift-implicit Euler, trapezoidal and Milstein schemes. Special attention is paid to their convergence properties and to implementation issues. Finally, we report in Section 5 on numerical experience with the drift-implicit Euler scheme applied to the transient noise simulation in a ring-oscillator model.

2 Index 1 SDAEs

Due to the singularity of the matrix A the deterministic part of (1.6)

$$Ax'(t) + f(x(t),t) = 0, \tag{2.1}$$

where the solution x is now a deterministic function of t, forms a DAE. Solutions have to fulfil the constraints of the equation. The solution components belonging to $\ker A$ (we call them the algebraic components) do not occur under the differential operator d/dt, and the inherent dynamics live only in a lower-dimensional subspace. The DAE (2.1) is characterized as an index 1 DAE iff the constraints are locally solvable for the algebraic

components. Solving an index 1 DAE involves a coupling of an integration task and a nonlinear equation solving task. If a DAE is of higher index, the constraints are not locally solvable for the algebraic components, and there exist solution components that are determined only by a hidden differentiation step, which may cause serious difficulties in the numerical solution of such problems (see e.g. [1, 10]).
We assume here that the deterministic part (2.1) is globally an index 1 DAE in the sense that the constraints are regularly and globally uniquely solvable for the algebraic variables. The globally unique solvability is stronger than the deterministic index 1 condition, which requires only the non-singularity of the corresponding Jacobian and guarantees only local solvability of the constraints for the algebraic variables. The globally unique solvability holds for the MNA-system (1.1)-(1.5) if (see[22]) there are no loops of capacitances and voltage sources and no cut-sets of inductances or current sources, if the capacity, conductance, and inductance matrices are symmetric and uniformly positive definite, and if the controlled sources satisfy certain conditions described in [6] (see [6, 22]).
In [16, 17] it is shown that special conditions are needed to ensure solution processes that are not directly affected by white noise. Then the SDAEs are called SDAEs without direct noise, otherwise with direct noise. To avoid direct noise we have to assume that the noise sources do not appear in the constraints. This means that

$$\operatorname{im} G(x,t) \subseteq \operatorname{im} A \quad \forall (x,t) \in I\!R^n \times \mathcal{J}.$$

This is true for (1.7) if and only if there are always capacitances in parallel to a noise source. This is quite restrictive in the actual noise modelling (see also the example in Section 4). Nevertheless, one can also handle many situations where this condition is violated. Often noisy constraints are only needed for the determination of algebraic solution components that do not interact with the dynamical ones. Future work should be directed to a classification of such situations.
Under these conditions the constraints of the SDAE can be described by the deterministic equation

$$Rf(x(t),t) = 0,$$

where R is a projector along im$A$, i.e., $R^2 = R,\ \ker R = \operatorname{im} A$. Solving the constraints for the algebraic components

$$Rf(u+v,t) = 0,\ Av = 0 \Longleftrightarrow v = \hat{v}(u,t),$$

inserting the result into the differential equations, and scaling the system by a pseudo-inverse A^- (with $AA^- = I - R, A^-A$ a projector along $\ker A$) leads to a so-called *inherent* regular SDE in the differential components u:

$$u' + A^- f(u + \hat{v}(u,t),t) + A^- G(u + v(u,t),t)\xi(t) = 0 \tag{2.2}$$

It can be shown that (2.2), together with $x(t) = u(t) + \hat{v}(u(t),t)$, is equivalent to (1.6). Based on this, the following theorem on the existence and uniqueness of strong solutions of (1.6) is proved in [22]:

Theorem 2.1 *Let the above conditions be satisfied for (1.6), and assume that f and G are globally Lipschitz continuous with respect to x, continuous with respect to t, and that Ax^0 is $\mathcal{F}_{t_0}$ -measurable, independent of the Wiener process w, and has finite p-th mean for some $p \geq 1$.*
Then there exists a strong solution $x(\cdot)$ of the initial value problem

$$Ax(t) - Ax^0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} G(x(s), s)dw(s) = 0, \tag{2.3}$$

which is pathwise unique. Moreover, the solution $x(\cdot)$ has finite p-th mean.

Similarly, convergence properties of suitable drift-implicit discretization schemes for SDEs carry over to SDAEs. In the next section we therefore give some basic results for the discretization of SDEs.

3 Time discretization of stochastic differential equations

We consider the initial value problem for the SDE

$$x(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} G(x(s), s)dw(s) = 0,\ t \in \mathcal{J},\ x(t_0) = x_0, \tag{3.1}$$

where $\mathcal{J} = [t_0, T]$, $f : I\!R^n \times \mathcal{J} \to I\!R^n$, $G : I\!R^n \times \mathcal{J} \to I\!R^{n\times m}$, w is an m-dimensional Wiener process on a given probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\in\mathcal{J}}$, and x_0 is a given $\mathcal{F}_{t_0}$-measurable initial value, which is independent of the Wiener process w. We assume that there exists a pathwise unique strong solution $x(\cdot)$.
Let us consider a generally drift-implicit discretization scheme of the form

$$x_\ell = x_{\ell-1} + \varphi(x_{\ell-1}, x_\ell; t_{\ell-1}, h_\ell) + \psi(x_{\ell-1}; t_{\ell-1}, h_\ell, I_{t_{\ell-1},h_\ell}),\ \ell = 1, \ldots, N, \tag{3.2}$$

on the deterministic grid $t_0 < t_1 < \ldots < t_N = T$ with stepsizes $h_\ell := t_\ell - t_{\ell-1}$, $\ell = 1, \ldots, N$. Here, φ and ψ are functions defined on $I\!R^n \times I\!R^n \times \mathcal{T}$ and $I\!R^n \times \mathcal{T} \times I\!R^M$ with $\mathcal{T} := \{(t, h) : t, t + h \in \mathcal{J}, h \in I\!R_+\}$, respectively, and mapping to $I\!R^n$. By $I_{t,h}$ we denote a vector of M multiple stochastic integrals having the form

$$I_{i_1,\ldots,i_k;t,h} = \int_t^{t+h} \int_t^{s_1} \cdots \int_t^{s_{k-1}} dw_{i_1}(s_1)dw_{i_2}(s_2) \cdots dw_{i_k}(s_k)$$

where the indices $i_1, \ldots, i_k$ are in $\{0, 1, \ldots, m\}$, k is bounded by certain finite order k_{max} and $dw_0(s)$ corresponds to ds.
For example, for the family of drift-implicit Euler schemes

$$x_\ell := x_{\ell-1} + h_\ell(\alpha f(x_\ell, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell, \quad \ell = 1, \ldots, N,$$

where $\alpha \in [0,1]$, and $\Delta w_\ell := (w(t_\ell) - w(t_{\ell-1})) = (I_{i;t_{\ell-1},h_\ell})_{i=1}^m$, one has $k_{max} = 1,\ M = m$, and

$$\begin{aligned}\varphi(z,x;t,h) &:= h(\alpha f(x,t+h) + (1-\alpha) f(z,t)),\\ \psi(z;t,h,I_{t,h}) &:= G(z,t)(w(t+h)-w(t)) = \sum_{i=1}^m g_i(z,t)\int_t^{t+h} dw_i(s),\end{aligned}$$

where $g_i(z,t)$, $i = 1,\dots,m$, are the columns of the matrix $G(z,t)$.
The family of drift-implicit Milstein schemes differs from the Euler schemes by an additional correction term for the stochastic part. The Milstein schemes are described by the same function φ, and $k_{max} = 2,\ M = m + m^2$, and

$$\psi(z;t,h,I_{t,h}) := G(z,t)\Delta w_{t,h} + \sum_{j=1}^m (g_{j_x} G)(z,t) I_{(j);t,h},$$

where $\Delta w_{t,h} := w(t+h) - w(t) = (I_{i;t,h})_{i=1}^m$, and $I_{(j);t,h} := (I_{j,i;t,h})_{i=1}^m$.
In [22], a result on numerical stability of drift-implicit schemes (3.2) in the mean-square sense has been derived which allows to study the behaviour of (3.2) under perturbations. Next we present a variant of such a stability result which supplements and extends Theorem 5 in [22].

Theorem 3.1 *Let $p \ge 1$ and x_0 have finite p-th mean. Assume that the scheme (3.2) satisfies the following properties:*

- *for all $z, \tilde z, x, \tilde x \in I\!R^n$, $(t,h) \in \mathcal{T}$, $h \le h^1$ we have*

  (A1) $\quad |\varphi(z,x;t,h) - \varphi(\tilde z,\tilde x;t,h)| \le h(L_1|z-\tilde z| + L_2|x-\tilde x|)$

  for some positive constants h^1, L_1, L_2.

- *for all $(t,h) \in \mathcal{T}$, $h \le h^1$, and $\mathcal{F}_t$-measurable random vectors y, $\tilde y$ we have*

  (A2) $\quad I\!E(\psi(y;t,h,I_{t,h}) - \psi(\tilde y;t,h,I_{t,h})|\mathcal{F}_t) = 0,$

  (A3) $\quad I\!E(|\psi(y;t,h,I_{t,h}) - \psi(\tilde y;t,h,I_{t,h})|^p|\mathcal{F}_t) \le h^{\frac{p}{2}} L_3^p |y-\tilde y|^p,$

  (A4) $\quad I\!E|\psi(0;t,h,I_{t,h})|^p < \infty,$

  for some constant $L_3 > 0$.

Then there exists constants $a \ge 1$, $h^0 > 0$ and a stability constant $S > 0$ such that the following holds true for each grid $\{t_0, t_1, \dots, t_N\}$ having the property $h := \max_{\ell=1,\dots,N} h_\ell \le h^0$ and $h \cdot N \le a \cdot (T - t_0)$:
For all $\mathcal{F}_{t_0}$-measurable random vectors x_0^*, $\tilde x_0$ having finite p-th mean, for all $\ell \in \{1,\dots,N\}$ and $\mathcal{F}_{t_\ell}$-measurable perturbations d_ℓ^*, $\tilde d_\ell$ having finite p-th order moments the perturbed discrete system

$$\tilde x_\ell = \tilde x_{\ell-1} + \varphi(\tilde x_{\ell-1}, \tilde x_\ell; t_{\ell-1}, h_\ell) + \psi(\tilde x_{\ell-1}; t_{\ell-1}, h_\ell, I_{t_{\ell-1},h_\ell}) + \tilde d_\ell, \tag{3.3}$$

$\ell = 1,\dots,N$, has a unique solution $\{\tilde x_\ell\}_{\ell=0}^N$, and the following estimates are valid for any two solutions $\{x_\ell^*\}_{\ell=1}^N$ and $\{\tilde x_\ell\}_{\ell=0}^N$ of the perturbed discrete systems with perturbations

$\{d^*_\ell\}_{\ell=1}^N$ *and* $\{\tilde{d}_\ell\}_{\ell=1}^N$*:*

$$\mathbb{E} \max_{\ell=1,\dots,N} |x^*_\ell - \tilde{x}_\ell|^p \le S^p \Big(\mathbb{E}|x^*_0 - \tilde{x}_0|^p + \frac{\max_{\ell=1,\dots,N} \mathbb{E}|s_\ell|^p}{h^{\frac{p}{2}}} + \frac{\mathbb{E} \max_{\ell=1,\dots,N} |r_\ell|^p}{h^p} \Big), \quad (3.4)$$

$$\max_{\ell=1,\dots,N} \mathbb{E}|x^*_\ell - \tilde{x}_\ell|^p \le S^p \Big(\mathbb{E}|x^*_0 - \tilde{x}_0|^p + \frac{\max_{\ell=1,\dots,N} \mathbb{E}|s_\ell|^p}{h^{\frac{p}{2}}} + \frac{\max_{\ell=1,\dots,N} \mathbb{E}|r_\ell|^p}{h^p} \Big), \quad (3.5)$$

where $d_\ell := d^*_\ell - \tilde{d}_\ell$ *is splitted such that* $d_\ell = r_\ell + s_\ell$ *with* $\mathbb{E}(s_\ell|\mathcal{F}_{t_{\ell-1}}) = 0$.

The proof of Theorem 3.1 is given in the appendix.
Theorem 3.1 applies immediately to well-known schemes for SDEs. Here, we check the assumptions of Theorem 3.1 for the families of drift-implicit Euler and Milstein schemes. Condition (A1) follows from the Lipschitz continuity of the drift coefficient f, (A2) holds due to the explicit, non-anticipative discretization of the diffusion term, and the technical condition (A4) is satisfied since the function $G(0,\cdot)$ (and the functions $g_{j_x}G(0,\cdot)$) are bounded on the compact interval $\mathcal{J}$. Condition (A3) is a consequence of standard properties of moments of stochastic integrals and the Lipschitz continuity of the diffusion coefficient G (and in case of the Milstein scheme of the functions $g_{j_x}G$). For example, for the drift-implicit Euler scheme we obtain for any pair $(t,h) \in \mathcal{T}$ and any $\mathcal{F}_t$-measurable y, $\tilde{y}$

$$\begin{aligned} \mathbb{E}(|\psi(y;t,h,I_{t,h}) - \psi(\tilde{y};t,h,I_{t,h})|^p|\mathcal{F}_t) &= |G(y,t) - G(\tilde{y},t)|^p \mathbb{E}(|\Delta w_{t,h}|^p|\mathcal{F}_t) \\ &\le L_G^p |y - \tilde{y}|^p C_p h^{\frac{p}{2}} \end{aligned}$$

where L_G is a Lipschitz constant of G and C_p a universal constant.
In the special case $x^*_\ell = x(t_\ell)$, the perturbations d^*_ℓ form the local discretization errors. We split them into

$$d^*_\ell = (d^*_\ell - \bar{d}^*_\ell) + \bar{d}^*_\ell, \quad \text{where } \bar{d}^*_\ell := \mathbb{E}(d^*_\ell|\mathcal{F}_{t_{\ell-1}}),$$

and obtain, in comparison with the exact solution of the numerical scheme x_ℓ,

$$\begin{aligned} \| \max_{\ell=1,\dots,N} |x(t_\ell) - x_\ell| \, \|_{L_p} &\le S_p \Big(\max_{\ell=1,\dots,N} \|d^*_\ell - \bar{d}^*_\ell\|_{L_p}/h^{\frac{1}{2}} + \| \max_{\ell=1,\dots,N} |\bar{d}^*_\ell| \, \|_{L_p}/h \Big), \\ \max_{\ell=1,\dots,N} \|x(t_\ell) - x_\ell\|_{L_p} &\le S_p \Big(\max_{\ell=1,\dots,N} \|d^*_\ell - \bar{d}^*_\ell\|_{L_p}/h^{\frac{1}{2}} + \max_{\ell=1,\dots,N} \|\bar{d}^*_\ell\|_{L_p}/h \Big), \end{aligned}$$

where $\|x\|_{L_p} := (\mathbb{E}|x|^p)^{1/p}$. If, by consistency arguments, the local error terms on the right-hand side are of order $\mathcal{O}(h^\gamma)$, we have global convergence of order γ.

4 Discretization Schemes for Index 1 SDAEs

Nowadays, a wide spectrum of discretization schemes for SDEs is available (cf. [3, 9, 11, 14]). However, SDAEs require special schemes. First decoupling the SDAE numerically and then applying a scheme to the resulting inherent SDE would be an inefficient procedure in general. We aim at numerical methods for SDAEs that work directly on the given

implicit structure, as in the case of deterministic DAEs. Only little previous work has been done in this direction. In [16, 17] linear SDAEs are analyzed and the convergence of the drift-implicit Euler scheme is proved. In [13] a scheme with strong order 1 is developed for the specially structured SDAEs that arise in transient noise simulation for electronic circuits. Later we will point out its relation to the drift-implicit Milstein scheme.
Our approach also applies to nonlinear SDAEs. We present adaptations of known schemes for SDEs that are implicit in the deterministic and explicit in the stochastic part to the SDAE (1.6). Designing the methods such that the iterates x_ℓ have to satisfy the constraints of the SDAE at the current time-point t_ℓ

$$Rf(x_\ell, t_\ell) = 0,$$

is the key idea to adapt known methods for SDEs to (1.6).
The noise densities given in Section 1 contain small parameters. To exploit this in the analysis of the discretization errors we express the diffusion coefficient in the form

$$G(x,t) := \epsilon \tilde{G}(x,t), \quad \epsilon \ll 1. \tag{4.1}$$

4.1 Drift-Implicit Euler Scheme

On the deterministic grid $0 = t_0 < t_1 < \ldots < t_N = T$ the drift-implicit Euler scheme for (1.6) is given by

$$A\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell = 0, \tag{4.2}$$

where $h_\ell = t_\ell - t_{\ell-1}$, $\Delta w_\ell = w(t_\ell) - w(t_{\ell-1})$. Realizations of Δw_ℓ can be simulated as $N(0, h_\ell I)$-distributed random variables. The Jacobian of 4.2 is the same as in the deterministic setting.
The scheme (4.2) for the SDAE (1.6) possesses the same convergence properties as the drift-implicit Euler scheme for SDEs. In general, its order of strong convergence is 1/2, i.e.,

$$\|x(t_\ell) - x_\ell\|_{L_p} = (E|x(t_\ell) - x_\ell|^p)^{1/p} \le c \cdot h^{1/2}, \quad h := \max_{\ell=1,\ldots,N} h_\ell,$$

holds for the p-th mean norm of the global errors for $p \ge 1$. For additive noise, i.e., $G(x,t) = G(t)$, the order of strong convergence is 1. For small noise, i.e., $G(x,t) = \epsilon\tilde{G}(x,t)$, the error is bounded by $O(h + \epsilon^2 h^{1/2})$ (see [15], or [12] for related results).
The smallness of the noise also allows special estimates of local error terms, which can be used to control the stepsize. The local error for the Euler scheme applied to SDEs with small noise is analyzed in [15]. As long as stepsizes with

$$h_\ell \gg \epsilon^2$$

are used, the dominating local error term of (4.2) is

$$\begin{aligned} \|\bar{d}_\ell^*\|_{L_p}/h_\ell = \mathcal{O}(h_\ell) &= \frac{1}{2}\|A^-\big(f(x_\ell,t_\ell) - f(x_{\ell-1},t_{\ell-1})\big)\|_{L_p} + \mathcal{O}(\epsilon h_\ell^{1/2}) \\ &=: \eta_\ell + \mathcal{O}(\epsilon h_\ell^{1/2}), \end{aligned}$$

where A^- denotes a suitable pseudo-inverse of A. For $\epsilon \to 0$ it approaches the known error estimate in the deterministic setting. If an ensemble of solution paths is computed simultaneously, the estimate η_ℓ can be computed approximately and may be used to control the local error corresponding to a given tolerance. This results in an adaptive stepsize sequence that is uniform for all solution paths.

4.2 Drift-Implicit Milstein Scheme

We intend to design this method in such a way that it realizes the drift-implicit Milstein scheme for the inherent SDE $u' + \hat{f}(u,t) + \hat{G}(u,t)\xi(t) = 0$, i.e.,

$$\frac{u_\ell - u_{\ell-1}}{h_\ell} + \hat{f}(u_\ell, t_\ell) + \hat{G}(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell - \sum_{j=1}^{m}(\hat{g}_{j\,u}\hat{G})(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}I_j^\ell = 0,$$

where $I_j^\ell = (I_{j,i}^\ell)_{i=1}^m$, $I_{j,i}^\ell = \int_{t_{\ell-1}}^{t_\ell}\int_{t_\ell}^{s} dw_i(\tau)dw_j(s)$, $\hat{G} = (\hat{g}_1, \ldots, \hat{g}_m)$ and
$\hat{f}(u,t) := A^- f(u + \hat{v}(u,t), t)$, $\hat{G}(u,t) := A^- G(u + \hat{v}(u,t), t)$.
The Milstein scheme is strongly convergent of order $\gamma = 1$. It differs from the Euler scheme by an additional correction term for the stochastic part, which includes double stochastic integrals. For additive noise the additional term vanishes and both schemes coincide.
The Milstein scheme for the inherent SDE is realized by

$$A\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{\Delta w_\ell}{h_\ell} - \sum_{j=1}^{m}(g_{j\,x} x_u A^- G)(x_{\ell-1}, t_{\ell-1})\frac{I_j^\ell}{h_\ell} = 0,$$

where $G = (g_1, \ldots, g_m)$, which we call the drift-implicit Milstein scheme for (1.6).
We point out the explicit use of the inner derivative $x_u = I + \hat{v}_u$ and the scaling A^- in the last term. The inner derivative can be expressed as

$$x_u = I + \hat{v}_u = I - (A + \lambda R f_x)^{-1}\lambda R f_x = I - I + (A + \lambda R f_x)^{-1}A = (A + \lambda R f_x)^{-1}A$$

with a free parameter $\lambda \neq 0$. Choosing $\lambda = h$, it may be approximated via

$$\begin{aligned} x_u &= (A + hRf_x)^{-1}A \qquad = (A + hf_x)^{-1}A \quad + \mathcal{O}(h) \\ x_u A^- &= (A + hRf_x)^{-1}(I - R) = (A + hf_x)^{-1}(I - R) + \mathcal{O}(h) \end{aligned}$$

by means of the Jacobian of Newton's method. Hence, the term $x_u A^-$ can be substituted by $(A + hRf_x)^{-1}(I - R)$ without changing the order of the scheme. Penski's approach [13] results in a similar approximation to the Milstein scheme in a more specialized setting.
The higher order 1 of strong convergence of these schemes has to be paid for with the use of a large number of double stochastic integrals and the use of the derivatives of the diffusion coefficients. In an application with a large number of small noise sources one has to pay much for a mostly theoretical gain in accuracy.

4.3 Trapezoidal Rule

The trapezoidal rule is widely used to integrate oscillatory solutions of ODEs. It is A-stable and convergent of order 2. It is also applied to index 1 DAEs of the form

$$Ax'(t) + f(x(t),t) = 0 \tag{4.3}$$

via the following reformulation: Formally transforming (4.3) to the augmented semi-explicit system

$$\begin{aligned} x'(t) - y(t) &= 0 \\ Ay(t) + f(x(t),t) &= 0, \end{aligned}$$

discretizing the differential equations by the trapezoidal rule

$$\begin{aligned} \frac{x_\ell - x_{\ell-1}}{h} - \frac{1}{2}\{y_\ell + y_{\ell-1}\} &= 0 \\ Ay_\ell + f(x_\ell, t_\ell) &= 0, \end{aligned}$$

and reformulating this system to

$$y_\ell := -y_{\ell-1} + 2\frac{x_\ell - x_{\ell-1}}{h}, \quad A(-y_{\ell-1} + 2\frac{x_\ell - x_{\ell-1}}{h}) + f(x_\ell, t_\ell) = 0$$

implicitly realizes the trapezoidal rule for the inherent regular ODE. The augmented system is of index 2 since its constraints $Ay(t) + f(x(t),t) = 0$ are not solvable for the variables y. However, this does not matter since the index 2 components do not enter the implicit formula in x_ℓ. They are of no interest here. Implementing this scheme requires only residuals.

A stochastic counterpart of the trapezoidal rule for the integration of SDEs (3.1) is given by

$$\frac{x_\ell - x_{\ell-1}}{h_\ell} = \frac{1}{2}(f(x_\ell, t_\ell) + f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell. \tag{4.4}$$

It is strongly convergent of order $\gamma = 1/2$ like the other Euler schemes. For small noise the error is bounded by $O(h^2 + \epsilon h + \epsilon^2 h^{1/2})$ (see [15], or [12] for related results).

An adaptation of this scheme to SDAEs, analogously to the deterministic case, would lead to an implicit discretization of the diffusion term. A way out is to create explicit constraints. This can be done by a suitable scaling of (1.6): We scale the system by a non-singular matrix $\tilde{D}$ such that

$$\tilde{D}A = \begin{pmatrix} \tilde{D}_1 A \\ \tilde{D}_2 A \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ 0 \end{pmatrix} \quad \text{or} \quad \tilde{D}R = \begin{pmatrix} 0 \\ \tilde{R}_2 \end{pmatrix}, \ \operatorname{rank} \tilde{A}_1 = \operatorname{rank} A,$$

and denote analogously

$$\tilde{D}f = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \quad \tilde{D}G = \begin{pmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{pmatrix}.$$

Then the trapezoidal rule is realized by

$$\begin{aligned} \tilde{A}_1 \frac{x_\ell - x_{\ell-1}}{h} + \frac{1}{2}\{\tilde{f}_1(x_\ell, t_\ell) + \tilde{f}_1(x_{\ell-1}, t_{\ell-1})\} + \tilde{G}_1(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell &= 0 \\ \tilde{f}_2(x_\ell, t_\ell) &= 0. \end{aligned}$$

The iterates satisfy the constraints at the current time-point, and the trapezoidal rule for the inherent SDE is realized. Since the differential equations and the constraints are now decoupled, it is possible to use a different scaling for both parts, which leads to a better conditioned system:

$$\begin{aligned} \tilde{A}_1(x_\ell - x_{\ell-1}) + \frac{h_\ell}{2}\{\tilde{f}_1(x_\ell, t_\ell) + \tilde{f}_1(x_{\ell-1}, t_{\ell-1})\} + \tilde{G}_1(x_{\ell-1}, t_{\ell-1})\Delta w_\ell &= 0 \\ \tilde{f}_2(x_\ell, t_\ell) &= 0. \end{aligned}$$

Here, the Jacobian with respect to the new iterate is

$$\begin{pmatrix} \tilde{A}_1 + h_\ell \tilde{f}_{1x}'(x,t)/2 \\ \tilde{f}_{2x}'(x,t) \end{pmatrix}.$$

It is non-singular for sufficiently small stepsizes and its condition number is bounded independently of the stepsizes.

5 Numerical Results

The drift-implicit Euler scheme has been used to simulate a ring-oscillator built of three coupled inverter steps with simple mosfet-models. Such an oscillator was also used for test runs in [13]. Thermal noise in the mosfets and in the resistors are modelled by multiplicative and additive white noise sources. The circuit diagram is given in Fig. 3. The corresponding noise-free circuit is a free running oscillator.

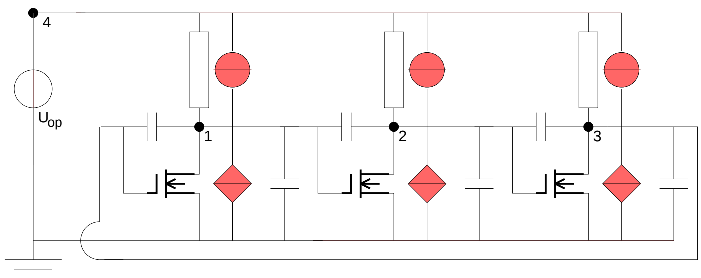


Figure 3: Thermal noise sources in a mosfet ring-oscillator model

The unknowns in the MNA system are the charges for the six capacities, the four nodal potentials and the current through the voltage source. The system is of index 1, but, formally, has direct noise. The three thermal resistance noise sources directly affect the current through the voltage source. However, the direct noise occurring in this current does not influence other variables. Omitting the corresponding variable together with the nodal equation for node 4 would lead to a system without direct noise. The diffusion

coefficients have been scaled (by a factor 10^3) to make the noise effects more visible.
In Fig. 4 we present numerical results obtained with the drift-implicit Euler scheme. A mean-square estimate of the dominating local error term was used to control the stepsize according to the relative tolerance 10^{-4}. Realizations of the Wiener increments Δw_ℓ were simulated by a normal random number generator of the RANLIB library (of Fortran routines for random number generation). Fig. 4 shows the nodal potential at node 1: the dark solid lines correspond to two different paths of the stochastic potential and the dashed line to the noise-free potential. The solid grey lines give the mean function μ of 100 sample paths and the boundaries of the interval $[\mu - 3\sigma, \mu + 3\sigma]$, where σ denotes the standard deviation. The paths exhibit a highly visible phase noise and, hence, can hardly be considered as small perturbations of the deterministic potential. The mean function appears damped and differs considerably from the noise-free potential.

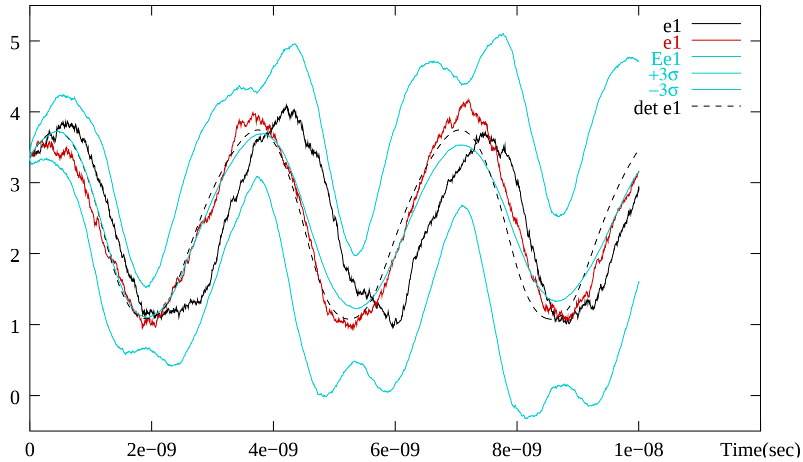


Figure 4: 2 sample paths of the voltage in node 1 (e1), the mean over 100 sample paths (E e1), the 3σ range ($\pm 3\sigma$), and the noiseless voltage (det e1)

Appendix

For the proof of Theorem 3.1 we need a discrete analogue of Gronwall's inequality.
Lemma: Let a_ℓ, $\ell = 1, \ldots, N$, and C_1, C_2 be nonnegative real numbers and assume that the inequalities

$$a_\ell \leq C_1 + C_2 \frac{1}{N} \sum_{i=1}^{\ell-1} a_i, \quad \ell = 1, \ldots, N,$$

are valid. Then we have $\max\limits_{\ell=1,\ldots,N} a_\ell \leq C_1 \exp(C_2)$.

Proof: (of Theorem 3.1)
Let $\tilde{d}_\ell$ be $\mathcal{F}_{t_\ell}$-measurable having a p-th order moment for each $\ell = 1, \ldots, N$. If the function φ does not depend on the variable x, the discretization scheme is explicit and the new iterate $\tilde{x}_\ell$ is given by

$$\tilde{x}_\ell = \tilde{x}_{\ell-1} + \varphi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell) + \psi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell, I_{t_{\ell-1}, h_\ell}) + \tilde{d}_\ell$$

for $\ell = 1, \ldots, N$. Otherwise, the scheme is implicit and the new iterate $\tilde{x}_\ell$ is given by the implicit equation (3.3). We assume that $h^0 > 0$ is chosen such that $h^0 L_2 < 1$. Due to the global Lipschitz condition (A1), the equation

$$x = \tilde{x}_{\ell-1} + \varphi(\tilde{x}_{\ell-1}, x; t_{\ell-1}, h_\ell) + b_\ell$$

is uniquely solvable by the contraction principle since $h_\ell L_2 \leq h^0 L_2 < 1$. Moreover, the solution $\tilde{x}_\ell$ depends on $\tilde{x}_{\ell-1}$ and on b_ℓ in a Lipschitz continuous way (with a constant $L_4 > 0$). Since $b_\ell := \psi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell, I_{t_{\ell-1}, h_\ell}) + \tilde{d}_\ell$ is a $\mathcal{F}_{t_\ell}$-measurable random variable, $\tilde{x}_\ell$ is also $\mathcal{F}_{t_\ell}$-measurable. Furthermore, $\tilde{x}_\ell$ has a p-th order moment. The latter fact is a consequence of the estimates

$$\begin{aligned}
(\mathbb{E}|\tilde{x}_\ell|^p)^{\frac{1}{p}} &\leq (\mathbb{E}|\tilde{x}_\ell - x_\ell^0|^p)^{\frac{1}{p}} + |x_\ell^0| \\
&\leq L_4\{(\mathbb{E}|\tilde{x}_{\ell-1}|^p)^{\frac{1}{p}} + (\mathbb{E}|b_\ell|^p)^{\frac{1}{p}}\} + |x_\ell^0|,
\end{aligned}$$

where x_ℓ^0 is the unique solution of the equation $x = \varphi(0, x; t_{\ell-1}, h_\ell)$, and

$$\begin{aligned}
(\mathbb{E}|b_\ell|^p)^{\frac{1}{p}} &\leq (\mathbb{E}|\psi(\tilde{x}_{\ell-1}; t_{\ell-1}, h_\ell, I_{t_{\ell-1}, h_\ell})|^p)^{\frac{1}{p}} + (\mathbb{E}|\tilde{d}_\ell|^p)^{\frac{1}{p}} \\
&\leq h^{\frac{1}{2}} L_3\{(\mathbb{E}|\tilde{x}_{\ell-1}|^p)^{\frac{1}{p}} + (\mathbb{E}|\psi(0; t_{\ell-1}, h_\ell, I_{t_{\ell-1}, h_\ell})|^p)^{\frac{1}{p}}\} + (\mathbb{E}|\tilde{d}_\ell|^p)^{\frac{1}{p}}
\end{aligned}$$

and of condition (A4).
Next we derive the stability estimate (3.4). The estimate (3.5) was shown in [22] for $p = 2$, but its proof carries over to the more general situation $p \geq 1$.
Let d_ℓ^* and $\tilde{d}_\ell$ for $\ell = 1, \ldots, N$ be perturbations of the discrete system and let x_ℓ^* and $\tilde{x}_\ell$, $\ell = 1, \ldots, N$, be their unique solutions. We introduce the following notations for $i = 1, \ldots, N$

$$\begin{aligned}
e_i &:= x_i^* - \tilde{x}_i, \quad \triangle\varphi_i := \varphi(x_{i-1}^*, x_i^*; t_{i-1}, h_i) - \varphi(\tilde{x}_{i-1}, \tilde{x}_i; t_{i-1}, h_i), \\
d_i &:= d_i^* - \tilde{d}_i, \quad \triangle\psi_i := \psi(x_{i-1}^*; t_{i-1}, h_i, I_{t_{i-1}, h_i}) - \psi(\tilde{x}_{i-1}; t_{i-1}, h_i, I_{t_{i-1}, h_i}),
\end{aligned}$$

and obtain from (3.3) and Hölder's inequality that

$$\begin{aligned}
e_i &= e_{i-1} + \triangle\varphi_i + \triangle\psi_i + d_i = e_0 + \sum_{k=1}^{i} \triangle\varphi_k + \sum_{k=1}^{i} \triangle\psi_k + \sum_{k=1}^{i} d_k, \\
a_\ell &:= \mathbb{E}(\max_{i=1,\ldots,\ell} |e_i|^p) \leq 4^{p-1}\Big\{\mathbb{E}(|e_0|^p) + \mathbb{E}\Big(\max_{i=1,\ldots,\ell} |\sum_{k=1}^{i} \triangle\varphi_k|^p\Big) + \\
&\qquad \mathbb{E}\Big(\max_{i=1,\ldots,\ell} |\sum_{k=1}^{i} \triangle\psi_k|^p\Big) + \mathbb{E}\Big(\max_{i=1,\ldots,\ell} |\sum_{k=1}^{i} d_k|^p\Big)\Big\}
\end{aligned}$$

holds for each $i, \ell = 1, \ldots, N$. For the second summand in the right-hand side of the latter estimate we continue by using (A1) and $ih \leq a(T - t_0)$

$$|\sum_{k=1}^{i} \triangle\varphi_k|^p \leq i^{p-1} \sum_{k=1}^{i} |\triangle\varphi_k|^p \leq (2i)^{p-1} h^p \sum_{k=1}^{i} \{L_1^p |e_{k-1}|^p + L_2^p |e_k|^p\} \leq \hat{L}_2 \frac{1}{N} \sum_{k=0}^{i} |e_k|^p.$$

where $\hat{L}_2 := 2^p(a(T-t_0))^p \max\{L_1^p, L_2^p\}$. Hence, we obtain the estimate

$$\mathbb{E}\Big(\max_{i=1,\dots,\ell}|\sum_{k=1}^{i}\triangle\varphi_k|^p\Big) \le \hat{L}_2\frac{1}{N}\mathbb{E}\Big(\sum_{k=0}^{\ell}|e_k|^p\Big) \le \hat{L}_2\frac{1}{N}\Big(\sum_{k=0}^{\ell-1}a_k + a_\ell\Big).$$

For estimating the third summand we observe that the discrete parameter process $\{M_i := \sum_{k=1}^{i}\triangle\psi_k, \mathcal{F}_{t_{i-1}}\}_{i=1}^N$ is a martingale due to (A2) and has finite p-th order moments. Then Burkholder's and Davis's inequalities [18, Chapter VII.3] yield the estimate

$$\mathbb{E}\Big(\max_{i=1,\dots,\ell}|\sum_{k=1}^{i}\triangle\psi_k|^p\Big) \le B_p\mathbb{E}\Big(\sum_{k=1}^{\ell}|\triangle\psi_k|^2\Big)^{\frac{p}{2}}$$

for each $\ell = 1,\dots,N$ with some universal constant $B_p > 0$. For instance, $B_p := \frac{18p^{\frac{5}{2}}}{(p-1)^{\frac{3}{2}}}$ is such a constant if $p > 1$. Hence, we obtain

$$\begin{aligned}
\mathbb{E}\Big(\max_{i=1,\dots,\ell}|\sum_{k=1}^{i}\triangle\psi_k|^p\Big) &\le B_p\ell^{\frac{p}{2}-1}\sum_{k=1}^{\ell}\mathbb{E}|\triangle\psi_k|^p \le B_p\ell^{\frac{p}{2}-1}\sum_{k=0}^{\ell-1}h_k^{\frac{p}{2}}L_3^p\mathbb{E}|e_k|^p \\
&\le B_pL_3^p\ell^{\frac{p}{2}-1}h^{\frac{p}{2}}\sum_{k=1}^{\ell}\mathbb{E}|e_k|^p \\
&\le B_pL_3^p(a(T-t_0))^{\frac{p}{2}}\frac{1}{N}\sum_{k=0}^{\ell-1}a_k.
\end{aligned}$$

for $\ell = 1,\dots,N$ by using Hölder's inequality and condition (A3). Setting $\hat{L}_3 := B_pL_3^p(a(T-t_0))^{\frac{p}{2}}$ we arrive, altogether, at the estimate

$$a_\ell \le 4^{p-1}\Big\{\mathbb{E}|e_0|^p + \hat{L}_2\frac{1}{N}\Big(\sum_{k=0}^{\ell-1}a_k + a_\ell\Big) + \hat{L}_3\frac{1}{N}\sum_{k=0}^{\ell-1}a_k + \mathbb{E}\Big(\max_{i=1,\dots,\ell}|\sum_{k=1}^{i}d_k|^p\Big)\Big\}$$

for $\ell = 1,\dots,N$. If necessary, we choose h^0 smaller such that $4^{p-1}\hat{L}_2\frac{1}{N} \le \frac{3}{4}$ holds if $h < h^0$. We conclude that

$$a_\ell \le 4^p\Big\{\mathbb{E}(|e_0|^p) + (\hat{L}_2+\hat{L}_3)\frac{1}{N}\sum_{k=0}^{\ell-1}a_k + \mathbb{E}\Big(\max_{i=1,\dots,\ell}|\sum_{k=1}^{i}d_k|^p\Big)\Big\}$$

holds for $\ell = 1,\dots,N$. By applying the lemma this leads to the semifinal estimate

$$\max_{\ell=1,\dots,N}a_\ell = \mathbb{E}(\max_{i=1,\dots,N}|e_i|^p) \le 4^p\exp(4^p(\hat{L}_2+\hat{L}_3))\Big\{\mathbb{E}|e_0|^p + \mathbb{E}\max_{i=1,\dots,N}|\sum_{k=1}^{i}d_k|^p\Big\}.$$

It remains to decompose the perturbation difference d_k into $d_k = r_k + s_k$ with $\mathbb{E}(s_k|\mathcal{F}_{t_{k-1}}) = 0$ for $k = 1,\dots,N$. Then $\{\sum_{k=1}^{i}s_k, \mathcal{F}_{t_{i-1}}\}_{i=1}^N$ is a martingale having finite p-th order moments. Appealing again to Burkholder's and Davis's inequalities provides

$$\begin{aligned}
\mathbb{E}\Big(\max_{i=1,\ldots,N}|\sum_{k=1}^{i}s_k|^p\Big) &\leq B_p\mathbb{E}\Big(\sum_{k=1}^{N}|s_k|^2\Big)^{\frac{p}{2}} \leq B_pN^{\frac{p}{2}}\max_{k=1,\ldots,N}\mathbb{E}|s_k|^p \\
&\leq B_p(a(T-t_0))^{\frac{p}{2}}\frac{\max_{k=1,\ldots,N}\mathbb{E}|s_k|^p}{h^{\frac{p}{2}}}.
\end{aligned}$$

Summarizing we obtain the final estimate

$$\begin{aligned}
\mathbb{E}(\max_{i=1,\ldots,N}|e_i|^p) &\leq \hat{S}^p\Big\{\mathbb{E}|e_0|^p+\mathbb{E}\max_{i=1,\ldots,N}|\sum_{k=1}^{i}s_k|^p+\mathbb{E}\max_{i=1,\ldots,N}|\sum_{k=1}^{i}r_k|^p\Big\} \\
&\leq \hat{S}^p\Big\{\mathbb{E}|e_0|^p+B_p(a(T-t_0))^{\frac{p}{2}}\frac{\max_{k=1,\ldots,N}\mathbb{E}|s_k|^p}{h^{\frac{p}{2}}} \\
&\qquad +(a(T-t_0))^p\frac{\mathbb{E}\max_{k=1,\ldots,N}|r_k|^p}{h^p}\Big\}
\end{aligned}$$

where $\hat{S}^p := 4^p2^{p-1}\exp(4^p(\hat{L}_2+\hat{L}_3))$. This completes the proof. #

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