Solvability of the 2 dimensional KP systems

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1 Introduction

In [4] A.N. Parshin offered a generalization of the classical KP-hierarhy and studied different properties of this system: the conversations laws, Zhaharov-Shabat equations and so on. Recall that the classical KP-hierarhy has the following Lax form:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L)] \text{ n} \geq 1, \tag{1}$$

where $L = \partial + u_{-1}\partial^{-1} + \ldots \in P$ is a pseudodifferential operator, $u_i \in k((x))[\log(x)][[t_k]]$ and $P = P_+ \oplus P_-$, where $P = k((x))((\partial^{-1}))$ is the ring of pseudodifferential operators (in one variable), P_+ is the subring of differential operators.

In [2] M.Mulase solved the classical Cauchy problem for this and more complicated KP-hierarhies. Parshin's generalization deals with the ring of pseudodifferential operators in n variables. Except the properties proved in [4] there remain unsolved a lot of questions concerning another possible generalizations of the classical "KP theory" in the case of the ring of (pseudo)differential operators in several variables. In particular, it was not clear if the Cauchy problem has a solution in this case, is it true that the Parshin system is a master equation of all isospectral deformations of some differential operators in 2 variables, are there some connection with the problem of classification of all commutative subrings in the ring of differential operators in several variables, are there some geometrical solutions of these systems and so on.

In this paper we define a series of modified Parshin's KP-hierarchies for the case of the ring of pseudodifferential operators in 2 variables, which are parametrized by a real positive parameter α . For given α such a system is called $(KP)_\alpha$. All these systems satisfy the same properties as the original system. For the value $\alpha = 0$ the system $(KP)_\alpha$ is a classical KP-hierarhy, and for $\alpha = \infty$ $(KP)_\alpha$ is the original Parshin's generalized KP-hierarhy. We prove their unique solvability for some initial values. Also we prove that for all values α except $\alpha = \infty$ the systems have nontrivial solutions. Notably, the original Parshin system can be considered as a part of a master equation of all universal families of isospectral deformations for any pair of normalized commuting differential operators. So, it would be interesting to find some connections with the results described in the recent review [7]. We hope to do it in the future.

Here is a brief overview of this paper.

In section 2 we study solvability of the original Parshin KP-system and give some examples.

In section 3 we introduce a series of modified Parshin's systems. We generalize some results of Mulase from [1] and show the unique solvability of these systems for arbitrary

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initial conditions. The solutions, nevertheless, may not belong to the skew field introduced by Parshin.

In section 4 we show that the solutions of the modified systems belong to the original skew field of Parshin and that they are not trivial for certain initial conditions.

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2 Generalized Parshin's KP-hierarchy

2.1. In this paper we use the notation from [4]. Recall that the skew field $P=k((x_1))((x_2))((\partial_1^{-1}))((\partial_2^{-1}))$, where k is some field, $E=k[[x_1,x_2]]((\partial_1^{-1}))((\partial_2^{-1}))$.

If $L=\sum_{i\le m}a_i\partial_2^i$ and $a_m\ne 0$, then $m=\mathrm{ord}(L)$ will be called the order of the operator L. The function $\mathrm{ord}(.)$ defines a decreasing filtration $P_.:\ldots\subset P_{-1}\subset P_0\subset\ldots$ of vector subspaces $P_i=\{L\in P:\mathrm{ord}(L)\le i\}\subset P$. Analogously, one can define the function $\mathrm{ord}_{\partial_1}(.)$ on the skew field $k((x_1))((x_2))((\partial_1^{-1}))$.

We have the decomposition of P and, accordingly of E, in a direct sum of subspaces

$$P=P_++P_-$$

$$E=E_++E_-,\quad E_\pm=E\cap P_\pm,$$

where $P_-=\{L\in P:\mathrm{ord}(L)<0\}$ and P_+ consists of the operators containing only ≥ 0 powers of ∂_2.

Consider the space P^2 and consider the Lax system (6) on the page 14 in [4], namely, if $N\in P^2$, then it looks like

$$\frac{\partial N}{\partial t_k}=V_N^k,$$

where

$$V_N^k=([(L^nM^m)_+,L],[(L^nM^m)_+,M])$$

if $N=(L,M)$ and $k=(n,m)$, $n,m\ge 0$. Note that we concider N as belonging to the extended phase space $\tilde{P}^2$, where $\tilde{P}=k((x_1))((x_2))[[\ldots,t_k,\ldots]]((\partial_1^{-1}))((\partial_2^{-1}))$.

Proposition 1. *Suppose* $\mathrm{ord}(LM)>0$ *and* $N=(L,M)$ *satisfy the Lax system.*

Then $[L,M]=0$.

Proof Since $\mathrm{ord}(LM)>0$, either $\mathrm{ord}(L)>0$ or $\mathrm{ord}(M)>0$. Without loss of generality let $\mathrm{ord}(L)>0$. Then for $k=(1,1)$ we have:

$$\frac{\partial L}{\partial t_k}=L([M,L])-[(LM)_-,L]$$

where from

$$L([M,L])=0\mod \wp^{-\mathrm{ord}(L)}$$

and $[M,L]=0 \mod \wp^{-\operatorname{ord}(LM)}$, where $\wp$ is a maximal ideal of the first valuation in P. Now for $k=(n,1)$ we have similarly

$$L^n([M,L])=0 \mod \wp^{-\operatorname{ord}(L)}$$

where from $[M,L]=0 \mod \wp^{-\operatorname{ord}(LM)+n-1}$.
□

Example. Let us try to find a solution of the system in the form:

$$L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\dots$$

$$M=\partial+v_1\partial^{-1}+v_2\partial^{-2}+\dots$$

where $\partial=\partial_2$ and $u_i,v_i\in k((x_1))((x_2))((\partial_1^{-1}))$. First of all, note that the conditions

$$[(L^nM^m)_+,L]_i=0, \quad [(L^nM^m)_+,M]_i=0,$$

where $i\geq 0$ and $[.]_i$ denotes the i-th coefficient of an operator, are exactly the conditions for the operators L,M to be commute, as we have seen above. It is not difficult to prove that for L,M these conditions are the following: $[L,M]=0$ iff for all $n\geq 0$ holds

$$D(u_n-v_n)+\sum_{i,j\geq 1,i+j=n}[v_i,u_j]+\sum_{i,j,k\geq 1,i+j+k=n}(-1)^jC_{j+i-1}^{i-1}(v_iD^{(j)}(u_k)-u_iD^{(j)}(v_k))=0, \quad (2)$$

where $D()=\partial/\partial(x_2)$ and C_i^j are the binary coefficients.

The equations for $k=(1,n)$, $k=(n,1)$ for all n give us the following property:

$$\frac{\partial u_1}{t_k}=\frac{\partial v_1}{t_k}$$

Indeed, this follows from (2) and the property: $[(L^nM^m)_+,L]=[(L^nM^m),L]-[(L^nM^m)_-,L]$. Therefore, $u_1-v_1=:c\in k((x_1))((\partial_1))$.

Now, the condition for the Lax system to have a solution appears if we will consider equations for u_l,v_l for $k=(i,j)$, $i+j+l\leq 4$. We have (below D will denote a derivation by x_2)

$$\frac{\partial u_1}{\partial t_{1,1}}=\left(D^{(2)}\right)(u_1)+v_1u_1+2\,\mathrm{D}(u_2)-u_1v_1$$

$$\frac{\partial v_1}{\partial t_{1,1}}=u_1v_1+\left(D^{(2)}\right)(v_1)+2\,\mathrm{D}(v_2)-v_1u_1$$

where from

$$2u_{22}+[v_1,u_1]=\frac{\partial u_1}{\partial t_{11}}$$

Then,

$$\frac{\partial u_2}{\partial t_{1,1}}=\left(D^{(2)}\right)(u_2)+[u_1,u_2]+[v_1,u_2]+2\,\mathrm{D}(u_3)+2u_1(\mathrm{D}(u_1))$$

$$\frac{\partial v_2}{\partial t_{1,1}} = 2v_1(\mathrm{D}(u_1)) + [v_1, v_2] + \left(D^{(2)}\right)(v_2) + 2\,\mathrm{D}(v_3) + [u_1, v_2]$$

where from

$$\frac{\partial(u_2 - v_2)}{\partial t_{1,1}} = [D(u_1), c] + [c, u_2] + [c, v_2]$$

(c were defined above). Differentiating by x_2 we get

$$[\frac{\partial u_1}{\partial t_{1,1}} - D^{(2)}, c] = 0$$

From other equations:

$$\frac{\partial u_1}{\partial t_{1,2}} = 3\,\mathrm{D}(u_3) + \left(D^{(3)}\right)(u_1) + 2\,[v_2, u_1] + 3\,\mathrm{D}(v_1)(u_1) + \\ 3\,\left(D^{(2)}\right)(u_2) + 2\,v_1(\mathrm{D}(u_1)) + 2\,[v_1, u_2] + u_1(\mathrm{D}(u_1))$$

$$\frac{\partial v_1}{\partial t_{1,2}} = 3\,\mathrm{D}(v_3) + \left(D^{(3)}\right)(v_1) + 3\,\left(D^{(2)}\right)(v_2) + \\ 3\,\mathrm{D}(v_1)(v_1) + 2v_1(\mathrm{D}(v_1)) + [u_1, v_2] + u_1(\mathrm{D}(v_1)) + [u_2, v_1]$$

$$\frac{\partial u_1}{\partial t_{2,1}} = \left(D^{(3)}\right)(u_1) + 3\,\mathrm{D}(u_3) + 2\,u_1(\mathrm{D}(u_1)) + v_1(\mathrm{D}(u_1)) + \\ [v_1, u_2] + [v_2, u_1] + 3\,\mathrm{D}(v_1)(u_1) + 3\,\left(D^{(2)}\right)(u_2)$$

$$\frac{\partial v_1}{\partial t_{2,1}} = v_1(\mathrm{D}(u_1)) + 3\,\mathrm{D}(v_3) + \left(D^{(3)}\right)(v_1) + 3\,\left(D^{(2)}\right)(v_2) + \\ 2\,u_1(\mathrm{D}(v_1)) + 2\,[u_1, v_2] + 3\mathrm{D}(u_1)(v_1) + 2\,[u_2, v_1]$$

we get

$$\frac{\partial u_1}{\partial t_{1,2}} - \frac{\partial u_1}{\partial t_{2,1}} = [v_2, u_1] + [v_1, u_2] - cD(u_1)$$

So, this expression includes u_2, v_2 , which means that we get no equations on u_1 .

2.2. A more interesting question is are there solutions of the form

$$L = \partial_1 + u_1\partial_2^{-1} + \dots \tag{3}$$

$$M = \partial_2 + v_1\partial_2^{-1} + \dots \tag{4}$$

By proposition above $[L, M] = 0$, so by theorem 1 in [4] there exists $S \in 1 + E_-$ such that $L = S\partial_1 S^{-1}$, $M = S\partial_2 S^{-1}$.

Below we will prove that there are no nontrivial solutions of the form (3), (4) of the Lax system above.

Assume the converse. Consider the series of equations for $k = (n, 0)$ and $k = (n, 1)$. For the series $k = (n, 0)$ we have

$$\frac{\partial L}{\partial t_k} = \sum_{i=1}^{n} C_n^i \frac{\partial^i L}{\partial x_1^i} \partial_1^{n-i}$$

$$\frac{\partial M}{\partial t_k} = \sum_{i=1}^{n} C_n^i \frac{\partial^i M}{\partial x_1^i} \partial_1^{n-i}$$

Lemma 1. *We have*

$$\frac{\partial u_j}{\partial x_1} = 0, \frac{\partial v_j}{\partial x_1} = 0$$

for all j.

Proof. Indeed, suppose $\frac{\partial u_j}{\partial x_1} \neq 0$ for some j. Then there exists n such that $\mathrm{ord}_{\partial_1}(\frac{\partial u_j}{\partial x_1}\partial_1^{n-1}) > \mathrm{ord}_{\partial_1}(u_j)$. But from the equations above we have $\mathrm{ord}_{\partial_1}(\frac{\partial u_j}{\partial x_1}\partial_1^{n-1}) = \mathrm{ord}_{\partial_1}(\frac{\partial u_j}{\partial t_{n,0}}) \leq \mathrm{ord}_{\partial_1}(u_j)$, a contradiction.
□

Corollary 1. *We have* $[u_i, v_j] = 0$, $[u_i, u_j] = 0$, $[v_i, v_j] = 0$ *for all* i, j.

Now consider the series of equations for $k = (n, 1)$.

Lemma 2. *We have*

$$(L^n M)_+ = \partial_1^n \partial_2 + \sum_{i=1}^{n} C_n^i \frac{\partial^{i-1} u_1}{\partial x_1^{i-1}} \partial_1^{n-i}$$

Proof. By induction on n. Since L contains only nonpositive order terms, and $L^{n-1}M$ has only one positive order term, namely $\partial_1^{n-1}\partial_2$, the plus-part of the operator $L^n M$ will be

$$(L^n M)_+ = u_1\partial_1^{n-1} + \partial_1((L^{n-1}M)_+) = ((L^{n-1}M)_+)\partial_1 + u_1\partial_1^{n-1} + \frac{\partial((L^{n-1}M)_+ - \partial_1^{n-1}\partial_2)}{\partial x_1} =$$

$$\partial_1^n \partial_2 + n u_1 \partial_1^{n-1} + \sum_{i=2}^{n} (C_{n-1}^i + C_{n-1}^{i-1}) \frac{\partial^{i-1} u_1}{\partial x_1^{i-1}} \partial_1^{n-i} = \partial_1^n \partial_2 + \sum_{i=1}^{n} C_n^i \frac{\partial^{i-1} u_1}{\partial x_1^{i-1}} \partial_1^{n-i} \tag{5}$$

□

Now for $k = (1, 1)$ we have

$$\frac{\partial u_1}{\partial t_{1,1}} = \frac{\partial u_1}{\partial x_2}\partial_1 + \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\frac{\partial v_1}{\partial t_{1,1}} = [u_1, v_1] + \frac{\partial v_1}{\partial x_2}\partial_1 + \frac{\partial^2 v_1}{\partial x_1 \partial x_2} + \frac{\partial v_2}{\partial x_1}$$

Recall that $[L, M] = 0$ by prop. 1, and the nesessary and sufficient conditions for the operators L, M to be commute are

$$D(u_n) - D1(v_n) + \sum_{i,j\geq 1, i+j=n} [v_i, u_j] + \sum_{i,j,k\geq 1, i+j+k=n} (-1)^j C_{j+i-1}^{i-1} (v_i D^{(j)}(u_k) - u_i D^{(j)}(v_k)) = 0, \quad (6)$$

where $D() = \partial/\partial(x_2)$, $D1() = \partial/\partial(x_1)$.

So, by lemma 1, corollary 1 and formula (6) we get

$$\frac{\partial u_1}{\partial t_{1,1}} = 0$$

$$\frac{\partial v_1}{\partial t_{1,1}} = \frac{\partial v_1}{\partial x_2}\partial_1$$

Lemma 3. *We have*

$$\frac{\partial u_j}{\partial x_2} = 0, \frac{\partial v_j}{\partial x_2} = 0$$

for all j.

Proof. By formula (5), for all $n \geq 1$ we have

$$[(L^n M)_+, M]_i = [(L^{n-1}M)_+\partial_1 + u_1\partial_1^{n-1} + \frac{\partial(L^{n-1}M)_+}{\partial x_1}, M]_i =$$

$$[(L^{n-1}M)_+, M]_i\partial_1 + ((L^{n-1}M)_+\frac{\partial M}{\partial x_1})_i + [u_1\partial_1^{n-1}, M]_i + [\frac{\partial(L^{n-1}M)_+}{\partial x_1}, M]_i$$

By lemma 1, $\frac{\partial(L^{n-1}M)_+}{\partial x_1} = 0$, $\frac{\partial M}{\partial x_1} = 0$, $[u_1\partial_1^{n-1}, M]_i = u_1\partial_1^{n-1}v_i - v_i u_1\partial_1^{n-1} = 0$ because of corollary 1.

Therefore,

$$\frac{\partial v_i}{\partial t_{n,1}} = [(L^n M)_+, M]_i = [(L^{n-1}M)_+, M]_i\partial_1 = \frac{\partial v_i}{\partial t_{n-1,1}}\partial_1 = \frac{\partial v_i}{\partial x_2}\partial_1^n$$

for all n, where from

$$\frac{\partial v_i}{\partial x_2} = 0$$

for all i.

Now we apply formula (6) to get

$$\frac{\partial u_i}{\partial x_2} = 0$$

□

Corollary 2. *We have* $[(L^i M^j)_+, L] = 0$, $[(L^i M^j)_+, M] = 0$ *for all* $i, j \geq 0$. *So,*

$$\frac{\partial u_n}{\partial t_{i,j}} = 0, \frac{\partial v_m}{\partial t_{i,j}} = 0$$

for all n, m, i, j *and the Lax system has only trivial solutions of the form (3), (4).*

For the completeness we add the formula below. One can also get the same result as corollary 2 without considering the series of equations for $k = (n, 0)$.

Proposition 2. *For $n > 1$ we have*

$$\frac{\partial u_1}{\partial t_{n,1}} = (-1)^{n-1}\frac{\partial u_1}{\partial x_2}\partial_1^n + (-1)^{n-2}C_n^1\frac{\partial u_1}{\partial t_{1,1}}\partial_1^{n-1} + \dots$$
$$+ C_n^{n-1}\frac{\partial u_1}{\partial t_{n-1,1}}\partial_1 + \frac{\partial^n u_1}{\partial t_{1,1}\partial x_1^{n-1}} - \frac{\partial^n u_1}{\partial x_2 \partial x_1^{n-1}}\partial_1 + \frac{\partial r_n}{\partial x_1}$$

where

$$r_n = r_{n-1} + u_1\frac{\partial^{n-2}u_1}{\partial x_1}$$

and

$$r_2 = \frac{\partial u_1^2}{\partial x_1}$$

Remark. We want to note that recently there appear two works [5], [6] about Parshin's KP-hierarchy. Nevertheless, they don't contain a proof of the existence of a solution of the Parshin system. Moreover, as we have seen above, the baker function and tau function constructed there can not give a nontrivial solution!

3 Modified Parshin's KP-hierarchy

Now we introduce the following *modified* Lax systems:

$$\frac{\partial N}{\partial t_k} = V_N^k,$$

where $k = (i, j)$, $i, j \geq 0$, $i \leq \alpha j$, $\alpha > 0$. For a given α we will denote such a system by $(KP)_\alpha$.

We will look for a solution of these systems with initial conditions

$$L_0 = \partial_1 + u_1\partial_2^{-1} + \dots$$

$$M_0 = \partial_2 + v_1\partial_2^{-1} + \dots,$$

where $u_i, v_i \in k[[x_1, x_2]]((\partial_1^{-1}))$. Using the same arguments as in proposition 1, one can see that there is a nontrivial solution only if $[L_0, M_0] = 0$. So, we will assume that L_0, M_0 commute. In this case by [4], Th.1 there exists a unique up to an operator with constant coeffitients operator $S \in 1 + E_-$ such that

$$L = S\partial_1 S^{-1}, \quad M = S\partial_2 S^{-1}$$

It is easy to see that the modified systems satisfy all the main properties derived in [4] for the original system. Now, to solve the Cauchy problem we can just try to generalize the proof of Mulase in [1].

Let's fix a system $(KP)_\alpha$. For a given operator N we define formal 1-forms by

$$Z^N_\pm = \pm \sum_{i,j\geq 1, i\leq \alpha j} dt_{i,j}(L^iM^j)_\pm$$

Then it is easy to see that the Lax equation for the $(KP)_\alpha$ system is given by

$$dN = [Z^N_+, N] = [Z^N_-, N], \tag{7}$$

where dN denotes the vector (dL, dM), $[Z^N_\pm, N]$ denotes the vector $([Z^N_\pm, L], [Z^N_\pm, M])$, and $d = \sum_{i,j\geq 1, i\leq \alpha j} dt_{i,j}\partial/\partial t_{i,j}$ denotes the exterior derivative in t.

Proposition 3. *The Lax equation for the* $(KP)_\alpha$ *system is equivalent to the integrability condition*

$$dZ^N_\pm = Z^N_\pm \wedge Z^N_\pm$$

Proof. In [4], prop. 4, i) one can find the proof that the solution of the $(KP)_\alpha$ system also satisfies to the system

$$\frac{\partial (L^iM^j)_+}{\partial t_{l,m}} - \frac{\partial (L^lM^m)_+}{\partial t_{i,j}} = [(L^lM^m)_+, (L^iM^j)_+], \tag{8}$$

that is, to the equation $dZ^N_+ = Z^N_+ \wedge Z^N_+$. Since for $Z^N = Z^N_+ - Z^N_- = \sum dt_{i,j}L^iM^j$ we have $Z^N \wedge Z^N = 0 = Z^N_+ \wedge Z^N_+ + Z^N_- \wedge Z^N_-$, we have also $dZ^N_- = Z^N_- \wedge Z^N_-$.

To prove the converse assertion let us consider first the equations (8) with arbitrary l, m, j and with $i = 0$. Then we derive from these equations that

$$\frac{\partial (M^j)_+}{\partial t_{l,m}} - [(L^lM^m)_+, M^j] \in P^{\leq m-2}$$

Since $\partial/\partial t_{l,m}$ and $[(L^lM^m)_+,]$ are derivations, we have

$$\left(\frac{\partial}{\partial t_{l,m}} - [(L^lM^m)_+,]\right) M^j = j\left(\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]\right) M^{j-1} + \dots$$

where from

$$\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M] \in P^{\leq m-2-j+1} = P^{\leq m-j-1}.$$

Since j can be an arbitrary large number, we get the second equation in the Lax equation (7): $dM = [Z^N_+, M]$. To get the first one we need to concider now the equations (8) with arbitrary j, l, m and $i = 1$. Then we get by the same arguments

$$\frac{\partial L}{\partial t_{l,m}} - [(L^lM^m)_+, L] \in P^{\leq m-2-j}, \quad j \geq 1/\alpha$$

where from $dL = [Z^N_+, L]$.
□

Corollary 3. *The equations (7) are compatible, i.e.* $d[Z_+^N, N] = 0$.

The proof is clear.

Now let

$$\omega_\alpha = \sum_{i,j\geq 0, i\leq \alpha j} dt_{i,j}\partial_1^i\partial_2^j$$

It's clear that $S\omega_\alpha S^{-1} = \sum dt_{i,j}L^jM^j$.

Introduce the following notation. We define
$\bar{R} = k[[x_1,x_2]][[\dots,t_{i,j},\dots]]$ as a projective limit algebra with a valuation

$$v : \bar{R}\backslash\{0\} \longrightarrow \mathbb{N}\oplus\mathbb{N}$$

defined by $v(t_{i,j}) = (i,j)$, $i,j \geq 1$; we define also a valuation $v_2 : \bar{R}\backslash\{0\} \longrightarrow \mathbb{N}$: $v_2(t_{i,j}) := j$. We assume here $(i_1,j_1) > (i,j)$ if $j_1 > j$ or $j_1 = j$ and $i_1 > i$. In other words, each element of the algebra $\bar{R}$ can be represented as a formal series $\sum_{i,j\geq 1} a_{i,j}$, where $a_{i,j}$ is a finite sum of monomials from the ring $A[\dots,t_{i,j},\dots]$, $A = k[[x_1,x_2]]$, and each monomial has a valuation (i,j), i.e. $v(a_{i,j}) = (i,j)$ if $a_{i,j} \neq 0$, and $v(\sum_{i,j\geq 1} a_{i,j}) = \min\{v(a_{i,j})\}$.
Define $E_{\bar{R}} = R := \bar{R}((\partial_1^{-1}))$, $E_R := R((\partial_2^{-1}))$, $E_k := k((\partial_1^{-1}))((\partial_2^{-1}))$, $\mathcal{V}_R := \{1+E_{R-}\}$. We extend the valuation v from $\bar{R}$ to R by $v(\sum a_k\partial_1^k) = \min\{v(a_k)\}$. One can check immediately that this definition is correct. Now define

$$\widehat{E}_{\bar{R}} := \{L = \sum_{q\in\mathbb{Z}} b_q\partial_1^q | b_q \in \bar{R} \text{ and for any integer } M \text{ and positive integer } N$$
$$\text{there exist only finite number of } b_q \text{ with } q < M \text{ such that } v_2(b_q) = N\}$$

Note that $\widehat{E}_{\bar{R}}$ is a ring. We extend the valuation v (and v_2) to this ring by the obvious way. We define

$$\widehat{E}_R := \{L = \sum_{q\in\mathbb{Z}} a_q\partial_2^q | a_q \in \widehat{E}_{\bar{R}} \text{ and there is a positive real number } C_L \text{ and positive}$$
$$\text{integer } M_L \text{ such that } v_2(a_q) > C_Lq \text{ for all } q > M_L\},$$

$$D_R := \{L = \sum_{n=0}^{\infty} a_n\partial_2^n \quad | \quad a_n \in \widehat{E}_{\bar{R}}, \quad a_n = 0 \quad \text{for } n \gg 0\},$$

$\widehat{D}_R := \{L \in \widehat{E}_R | L_- = 0\}$, $\widehat{\mathcal{V}}_R := \{1 + \widehat{E}_{R-}\}$, $\widehat{\mathcal{V}}_{k[[t_{i,j}]]((\partial_1^{-1}))} := \{L = 1 + \sum_{i=1}^{\infty} a_i\partial_2^{-i} | a_i = \sum_{q\in\mathbb{Z}} b_q\partial_1^q \in \widehat{E}_{\bar{R}}$ and $b_q \in k[[\dots,t_{i,j},\dots]]$
$\widehat{E}_R^\times := \{L \in \widehat{E}_R | \pi(L) \in \mathcal{V}_{\widehat{E}_{\bar{R}}/I}\}$, $\widehat{D}_R^\times := \{L \in \widehat{D}_R | \pi(L) = 1\}$,
where I is a valuation ideal in $\widehat{E}_{\bar{R}}$ and $\pi : \widehat{E}_{\bar{R}} \to \widehat{E}_{\bar{R}}/I$ is a projection.

Lemma 4. *Every element* $L \in \widehat{E}_R^\times$ *is invertible in* $\widehat{E}_R^\times$. *More precisely, the Neumann series* $\sum_{n=0}^{\infty}(1-P)^n$ *gives a well-defined element in* $\widehat{E}_R^\times$, *which we denote* P^{-1}.

Proof. This is an analog of lemma 3.1. in [1]. The proof is the same, we only have to use the valuation v_2 instead of val_t there.

Proposition 4. *The system* $(KP)_\alpha$ *is equivalent to the modified Sato-Wilson system* $(SW)_\alpha$ *for* $S \in \mathcal{V}_R \cdot \widehat{\mathcal{V}}_{k[[t_{i,j}]]((\partial_1^{-1}))}$*:*

$$dS = Z_-^N S$$

Proof. If S is a solution of the system $(SW)_\alpha$, then $N = (S\partial_1 S^{-1}, S\partial_2 S^{-1})$ gives a solution of the $(KP)_\alpha$, see [4], prop.4, ii).

Let's prove the converse. Let N be a solution of the $(KP)_\alpha$, and $N = (S_0\partial_1 S_0^{-1}, S_0\partial_2 S_0^{-1})$ (such S_0 exists by th.1 in [4], namely, the proof of this theorem is valid also in this case). Put $Z_-^0 = S_0^{-1}Z_-^N S_0 - S_0^{-1}dS_0$. Then we have $dZ_-^0 = Z_-^0 \wedge Z_-^0$ (this follows from the same argument as in the classical case of the (KP) system, namely, because Z_-^0 is a gauge transformation of the flat connection Z_-^N).

Now let us show that all operators in Z_-^0 have only constant coeffitients:

$$[Z_-^0, \partial_1] = S_0^{-1}[S_0 Z_-^0 S_0^{-1}, S_0\partial_1 S_0^{-1}]S_0 = S_0^{-1}[Z_-^N - dS_0 S_0^{-1}, L]S_0 =$$

$$S_0^{-1}([Z_-^N, L] - [dS_0 S_0^{-1}, L])S_0 = S_0^{-1}(dL - dL)S_0 = 0$$

(and analogously for ∂_2), because

$$dL = dS_0\partial_1 S_0^{-1} - S_0\partial_1 S_0^{-1} dS_0 S_0^{-1} = [dS_0 S_0^{-1}, L].$$

Now consider the equation $dC = Z_-^0 C$. Since $dZ_-^0 = Z_-^0 \wedge Z_-^0$, this equation is compatible and therefore there exists a solution $C \in \widehat{\mathcal{V}}_{k[[t_{i,j}]]((\partial_1^{-1}))}$. Now put $S = S_0 C$, then

$$dS - Z_-^N S = dS_0 C + S_0 dC - Z_-^N S_0 C = S_0(S_0^{-1}dS_0 + dCC^{-1} - S_0^{-1}Z_-^N S_0)C = 0$$

and $S\partial_i S^{-1} = S_0\partial_i S_0^{-1} = L(M)$.
□

Proposition 5. *We have a bijection between the following two sets:*

$$Sol((KP)_\alpha) = \{N = (L, M), \quad L, M \in E_R \quad | \quad N \quad \textit{satisfies } (KP)_\alpha \\ \textit{and } L, M \textit{ don't depend on } t_{i,j},\ i > \alpha j\}$$

and

$$Sol((SW)_\alpha) = \{S \in \mathcal{V}_R \cdot \widehat{\mathcal{V}}_{k[[t_{i,j}]]((\partial_1^{-1}))} \quad | \quad S \quad \textit{satisfies } (SW)_\alpha \\ \textit{and does not depend on } t_{i,j},\ i > \alpha j\}/(1 + E_{k_-})$$

Proof. The bijection is given by the formula $L = S\partial_1 S^{-1}$, $M = S\partial_2 S^{-1}$ and follows from the proposition 4. Indeed, suppose $S_1\partial_i S_1^{-1} = S_2\partial_i S_2^{-1}$ for $i = 1, 2$. Then $S_2 = S_1 C$, where $C \in \widehat{\mathcal{V}}_{k[[t_{i,j}]]((\partial_1^{-1}))}$. Therefore, from the equation $(SW)_\alpha$ we get

$$Z_-^\alpha(S_1 C) = d(S_1 C) = dS_1 C + S_1 dC,$$

where from $S_1 dC = 0$. Since S_1, S_2 don't depend on $t_{i,j}$, $i > \alpha j$, we conclude $C \in (1 + E_{k-})$.
The proposition is proved.
□

It would be interesting to describe the set of all such operators $S \in (1 + E_-)$ that give us trivial initial conditions (so, trivial solutions by theorem 3 below) of the system $(KP)_\alpha$. Following Mulase, [2], we give the following definition.

Definition 1. A pseudo-differential operator $T \in E$ is said to be admissible if it is an invertible operator of order zero such that $T\partial_i T^{-1} \in E_k$ for $i = 1, 2$. The set of all admissible operators is denoted by Γ_a.

Lemma 5. *Every admissible operator T has the following form:*

$$T = c_0 T_0 T_{00} e^{c_1 x_1 + c_2 x_2},$$

$$T_0 = 1 + t_1 \partial_2^{-1} + t_2 \partial_2^{-2} + \ldots,$$

where $0 \neq c_0 \in k$, $c_1, c_2 \in k$, $T_{00} \in k[[x_1]]((\partial_1^{-1}))$ is an admissible operator from [2], Def. 4.3 and t_n is a polynomial in x_1, x_2 with coefficients in $k((\partial_1^{-1}))$ of (homogenious) degree less than or equal to n.

Let $S_0 = T_0^{-1} = 1 + s_1 \partial_2^{-1} + s_2 \partial_2^{-2} + \ldots$. Then, if there is at least one s_i depending on x_i, $i = 1$ or $i = 2$, there exist infinitely many elements s_j depending on x_i.

Proof. Every admissible operator $T \in E$ can be written as a product $T = SS_0 a$, where $S \in 1 + E_-$, $S_0 \in 1 + k[[x_1, x_2]]((\partial_1^{-1}))$, $a \in k[[x_1, x_2]]$. It is clear that a must be of the form $e^{c_1 x_1 + c_2 x_2} c_0$. It is easy to see that S_0 does not depend on x_2, so it must be an admissible operator from the one dimensional case, as in [2], section 4.

So, to prove the lemma it suffice to solve the following system for arbitrary $u_i, v_i \in k((\partial_1^{-1}))$:

$$\partial_1 S_0 = S_0(\partial_1 + v_1 \partial_2^{-1} + v_2 \partial_2^{-2} + \ldots)$$
$$\partial_2 S_0 = S_0(\partial_2 + u_1 \partial_2^{-1} + u_2 \partial_2^{-2} + \ldots),$$

where $S_0 = T_0^{-1} = 1 + s_1 \partial_2^{-1} + s_2 \partial_2^{-2} + \ldots$, $s_i \in k[[x_1, x_2]]((\partial_1^{-1}))$. Put $s_0 = 1$, $u_0 = v_0 = 0$. If we compare the coefficients of the series from the left and right hand sides, we get the following system of equations:

$$\frac{\partial s_i}{\partial x_1} = \sum_{k=0}^{i} s_k v_{i-k} \quad \frac{\partial s_i}{\partial x_2} = \sum_{k=0}^{i} s_k u_{i-k} \tag{9}$$

for $i \geq 1$. This system is solvable if the compatibility conditions hold:

$$\frac{\partial}{\partial x_1}(\sum_{k=0}^{i} s_k u_{i-k}) = \frac{\partial}{\partial x_2}(\sum_{k=0}^{i} s_k v_{i-k})$$

For $i=1$ the compatibility condition is trivial. For arbitrary i we have

$$\frac{\partial}{\partial x_1}(\sum_{k=0}^{i} s_k u_{i-k}) = \sum_{k=0}^{i-1} s_k(\sum_{q=1}^{i-1-k} v_q u_{i-k-q}) = \sum_{k=0}^{i-1} s_k(\sum_{q=1}^{i-1-k} u_q v_{i-k-q}) = \frac{\partial}{\partial x_2}(\sum_{k=0}^{i} s_k v_{i-k})$$

So, our system is solvable for arbitrary u_i, v_i. Then it is easy to see that for each n s_n is a polynomial in x_1, x_2 with coefficients in $k((\partial_1^{-1}))$ of (homogenious) degree less than or equal to n:

$$s_n = \sum_{k=0}^{n}\sum_{q=0}^{k} x_1^q x_2^{k-q} C_{k,q}, \quad \text{where } C_{k,q} \in k((\partial_1^{-1})).$$

Obviously, the operator $T_0 = S_0^{-1}$ satisfy the same property for its coefficients.

Now assume there is at least one element s_j depending on x_i. Without lose of generality we can assume $i=2$. Let s_j be the first such element. Let u_k be the first nonzero element (such an element exists, otherwise s_j would not depend on x_2). Then from the system (9) follows that s_{j+k} must depend on x_2, moreover, the degree of s_{j+k} is greater that the degree of s_j. Continuing this line of reasons, we obtain that the elements s_{j+nk}, $n \ge 1$ depend on x_2 and their degree monotonically increase.

The lemma is proved.

□

The following theorem is a generalization of theorem 4.1. in [1].

Theorem 1. *Let $\omega_\alpha = \sum_{i,j\ge 0}^{\infty} P_{i,j} dt_{i,j}$ be a D_R-valued 1-form satisfying*

a) there is a positive real number $c>0$ such that

$$\mathrm{ord}_{\partial_2} P_{i,j} \le \frac{j}{c} \quad \textit{for all } i,j \ge 0.$$

b) $P_{i,j} = P_{i,j}|_{t_{m,n}=0, m>\alpha n}$.

c) ω_α is integrable, i.e. $d\omega_\alpha = \omega_\alpha \wedge \omega_\alpha$.

Then for every given operator $Y(0) \in E_+$, there is a unique solution $Y \in \widehat{D}_R$ of the linear total differential equation

$$dY = \omega_\alpha Y$$

having $Y(0)$ as its initial value; $Y|_{t=0} = Y(0)$ and $Y|_{t_{m,n}=0, m>\alpha n} = Y$.

Proof. Let's define a map $\zeta : (\mathbb{N}\cup\{0\}) \times (\mathbb{N}\cup\{0\}) \longrightarrow \mathbb{N}\cup\{0\}$ by the following recurrent formula

$$\zeta(0,0)=0, \quad \zeta(0,j) = \zeta(0,j-1) + |\alpha(j-1)| + 1 \quad \text{for } j>1, \quad \zeta(i,j) = \zeta(0,j) + i \text{ for } i>0.$$

It is clear that $\zeta|_{(i,j), i\le \alpha j}$ is a bijection and that $\zeta(i,j) \ge j$ for all $i,j \ge 0$.

Consider now the equation

$$dY = (\sum_{n\ge 0}^{\infty} P_{\zeta^{-1}(n)} dt_n) Y, \quad d = \sum_{n=1}^{\infty} dt_n \frac{\partial}{\partial t_n} \tag{10}$$

where $P_{\zeta^{-1}(n)}$ are considered as the elements with variables $t_{\zeta^{-1}(m,n)}$ instead of $t_{m,n}$. By condition b), if we apply ζ to all indices in $P_{\zeta^{-1}(n)}$, we get again the elements $P_{i,j}$. By theorem 4.1 in [1], there exists a unique solution $Y \in \widehat{\mathcal{D}}$ of (10), where $\widehat{\mathcal{D}}$ is the algebra defined in [1] with coefficients in the noncommutative ring $k[[x_1,x_2]]((\partial_1^{-1}))[[\dots,t_n,\dots]]$. So,

$$Y = \sum_{n=0}^{\infty} y_n \partial_2^n,$$

where $y_n \in k[[x_1,x_2]]((\partial_1^{-1}))[[\dots,t_n,\dots]]$ and there is some linear growth condition on the values $val_t(y_n)$, where val_t is a valuation defined in [1]. Note that every y_n can be written as:

$$y_n = \sum_{k=-\infty}^{\infty} a_k \partial_1^k,$$

where $a_k \in k[[x_1,x_2]][[\dots,t_n,\dots]]$ and for every integer M and positive integer N there exist only finite number of a_k with $k < M$ such that $val_t(a_k) = N$. So, if we apply the map ζ^{-1} to each index of t_m in y_n, we get an element $y_n' \in \widehat{E}_{\bar{R}}$, because the map ζ is monotone, i.e. for every positive integer N there exists a positive integer N' such that $v_2(\zeta^{-1}(a)) \geq N$ if $val_t(a) \geq N'$. Therefore, if we apply ζ^{-1} to each index in Y, we get an element of $\widehat{D}_R$, which is a unique solution of the original system with the conditions $Y|_{t=0} = Y(0)$ and $Y|_{t_{m,n}=0, m>\alpha n} = Y$.
□

Now we can formulate a generalized version of the Birkhoff decomposition theorem.

Theorem 2. *For any element $U \in \widehat{E}_R^{\times}$ there exists a unique factorization*

$$U = S^{-1}Y,$$

where $S \in \widehat{\mathcal{V}}_R$ and $Y \in \widehat{D}_R^{\times}$.

Proof. Unfortunately, this theorem is not the same as in [1], Th.3.2, because Mulase proved this theorem for the coefficient ring $R = A[[\dots,t_n,\dots]]$, where A is some algebra (may be not commutative). Since $\widehat{E}_{\bar{R}} \neq k[[x_1,x_2]]((\partial_1^{-1}))[[\dots,t_{i,j},\dots]]$ (for example, the element $a = \sum_{i=1}^{\infty} t_{i,1}\partial_1^{-i} \in k[[x_1,x_2]]((\partial_1^{-1}))[[\dots,t_{i,j},\dots]]$ does not belong to $\widehat{E}_{\bar{R}}$), we have to prove the theorem again. Nevertheless, the algebra $\widehat{E}_{\bar{R}}$ is almost equal to the algebra $k[[x_1,x_2]]((\partial_1^{-1}))[[\dots,t_j,\dots]]$ because of the special growth order condition for elements in $\widehat{E}_{\bar{R}}$. So, in particular, one can just repeat the proof of Th.3.2., [1] of Mulase taking the valuation v_2 instead of the valuation val_t there (see also the proof of the theorem 4 below).
□

Let us illustrate how we use this factorization to solve the systems $(SW)_\alpha$. Suppose we have a solution $S \in \widehat{\mathcal{V}}_R$ such that S does not depend on $t_{i,j}$, $i > \alpha j$. Then we also have $Z_{\pm}^{\alpha} = \pm \sum_{i \leq \alpha j} dt_{i,j}(S\partial_1^i\partial_2^j S^{-1})_{\pm}$ satisfying $dZ_{\pm} = Z_{\pm} \wedge Z_{\pm}$ and $Z_{\pm}^{\alpha}$ do not depend on $t_{i,j}$, $i > \alpha j$. Now by theorem 1 we have a unique element $Y \in \widehat{D}_R^{\times}$ such that

$$dY = Z_+^{\alpha} Y \tag{11}$$

If we define $U = S^{-1}Y \in \widehat{E}_R^\times$, U must satisfy the system of linear partial differential equations of constant coefficients:

$$dU = \omega_\alpha U. \tag{12}$$

Indeed,

$$dU = -S^{-1}dSS^{-1}Y + S^{-1}dY = S^{-1}(dYY^{-1} - dSS^{-1})SS^{-1}Y$$
$$= S^{-1}(Z_+^\alpha - Z_-^\alpha)SU = S^{-1}\sum_{i\leq\alpha j} dt_{i,j}S\partial_1^i\partial_2^j S^{-1}SU = \omega_\alpha U$$

Lemma 6. *If the equation (12) has two solutions U and $V \in \widehat{E}_R$ with the same initial value $U|_{t=0} = V|_{t=0}$, then $U = V$.*

Proof. Let $W = U - V$. Note that $W|_{t=0} = U|_{t=0} - V|_{t=0} = 0$. Since (12) is linear, we have $dW = \omega_\alpha W$, namely, $\partial W/\partial t_{i,j} = \partial_1^i\partial_2^j W$. Therefore,

$$\frac{\partial}{\partial t_{i_1,j_1}}\cdots\frac{\partial}{\partial t_{i_k,j_k}}W = \partial_1^{i_1+\ldots+i_k}\partial_2^{j_1+\ldots+j_k}W,$$

where from

$$(\frac{\partial}{\partial t_{i_1,j_1}}\cdots\frac{\partial}{\partial t_{i_k,j_k}}W)(0) = \partial_1^{i_1+\ldots+i_k}\partial_2^{j_1+\ldots+j_k}(W(0)).$$

This means that W does not depend on t. Since $W|_{t=0} = 0$, we can conclude that $W = 0$.

□

Thus a solution of the system $(SW)_\alpha$ gives rise to a solution U of the universal equation (12). The factorization theorem 2 gives a way to go back. So let us start with a solution U of (12). By theorem 2 we have a unique decomposition $U = S^{-1}Y$. Then we have

$$Z_+^\alpha - Z_-^\alpha = S\omega_\alpha S^{-1} = SdUU^{-1}S^{-1} = dYY^{-1} - dSS^{-1}$$

Since $\widehat{D}_R \cap \widehat{E}_{R-} = \{0\}$, we obtain $Z_+ = dYY^{-1}$ and $Z_- = dSS^{-1}$.

Now to solve the system $(SW)_\alpha$ with the initial value $S(0) \in 1 + E_-$ we take the explicit solution of the system (12) given by

$$U = \exp(\sum_{i,j\geq 1, i\leq\alpha j} t_{i,j}\partial_1^i\partial_2^j)S(0)^{-1} \in \widehat{E}_R^\times \tag{13}$$

and find its unique decomposition $U = S^{-1}Y$. Then, as we have seen, S gives a solution of $(SW)_\alpha$. Note that U (so, S) does not depend on $t_{i,j}$, $i > \alpha j$. Since $Y|_{t=0} = 1$,

$$S(0)^{-1} = U|_{t=0} = (S^{-1}|_{t=0})(Y|_{t=0}) = (S|_{t=0})^{-1},$$

namely, $S|_{t=0} = S(0)$. So, we get the following analog of the theorem 1.4 in [1]:

Theorem 3. *For every initial value $S(0) \in 1 + E_-$, there is a unique solution $S = S(t) \in \widehat{\mathcal{V}}_R$ of the system $(SW)_\alpha$ such that $S|_{t=0} = S(0)$.*

To prove the uniqueness in this theorem one can repeat the proof of the uniqueness in the Th.1.4 in [1].

4 Existence of non-trivial solutions for the modified systems

Now we want to know, when the solution $S(t)$ of the system $(SW)_\alpha$ is not trivial and belong to the group $\mathcal{V}_R$. As we have seen above, in the case $\alpha = \infty$ the original system $(KP)_\alpha$ has only trivial solutions. The theorem below shows that $S(t) \in \mathcal{V}_R$ for some initial values $S(0)$. It would be interesting to find all such values. Unfortunately, we don't know the answer on this question now.

Theorem 4. *(i) Suppose* $S(0)^{-1} \in 1 + k[[x_1]][x_2][[\partial_1^{-1}]][\partial_2^{-1}]\partial_1^{-2}$. *Then there exists a value* $\alpha_{S(0)}$ *such that for all* $\alpha < \alpha_{S(0)}$ *the system* $(SW)_\alpha$ *with the initial condition* $S(0)$ *has a solution* $S(t) \in \mathcal{V}_R$.

(ii) For given α *there exist initial conditions* $S(0)$, $S(0)^{-1} \in k[[x_1]][x_2][[\partial_1^{-1}]][\partial_2^{-1}]$ *such that the system* $(SW)_\alpha$ *with these initial conditions has a solution* $S(t) \in \mathcal{V}_R$.

In both cases the solutions $S(t) \in \mathcal{V}_R$ *correspond to non-trivial solutions of the system* $(KP)_\alpha$ *if the initial conditions* $S(0)^{-1}$ *are not trivial, i.e. their coefficients depend on* x_i, $i = 1$ *or* $i = 2$.

Proof. (i) As we have seen above, a solution of the system $(SW)_\alpha$ one obtains as a part of the decomposition of the operator U from the formula (13). To prove our theorem we take the original proof of Mulase in [1] and show that the solution $S(t)$ constructed there belongs to the ring $\mathcal{V}_R$.

To construct the solution $S = S(t)$, we have to solve the equation $SU \in \widehat{D}_R^{\times}$ for an unknown $S \in \widehat{\mathcal{V}}_R$.

So let $U = \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta$ and $S = 1 + \sum_{\gamma=1}^{\infty} s_\gamma \partial_2^{-\gamma}$, where $u_\beta, s_\gamma \in \widehat{E}_{\bar{R}}$. Note that if $S(0)^{-1} \in k[x_1, x_2][\partial_1^{-1}, \partial_2^{-1}]$, then $u_\beta = 0$ for almost all $\beta < 0$ and $u_\beta \in k[x_1, x_2][\partial_1^{-1}, \partial_1]$ for all $\beta \in \mathbb{Z}$. Then

$$
\begin{aligned}
SU = (1 + \sum_{\gamma=1}^{\infty} s_\gamma \partial_2^{-\gamma})(\sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta) \\
= \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta + \sum_{\gamma=1}^{\infty}\sum_{\beta\in\mathbb{Z}}\sum_{i=0}^{\infty} C^i_{-\gamma} s_\gamma u_\beta^{(i)} \partial_2^{-\gamma+\beta-i} \\
= \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta + \sum_{\delta\in\mathbb{Z}}(\sum_{\gamma=1}^{\infty}\sum_{i=0}^{\infty} C^i_{-\gamma} s_\gamma u^{(i)}_{\delta+\gamma+i})\partial_2^\delta,
\end{aligned}
$$

where $u^{(i)} = \partial^i u / \partial x_2^i$ and C_i^j are the binary coefficients.

Therefore the equation we have to solve is a system of algebraic equations

$$
u_{-\beta} + \sum_{\gamma=1}^{\infty}\sum_{i=0}^{\infty} C^i_{-\gamma} s_\gamma u^{(i)}_{\gamma-\beta+i} = 0 \quad \text{for } \beta = 1, 2, 3, \ldots \tag{14}
$$

Define

$$
\mathbf{u} = (u_{-1}, u_{-2}, u_{-3}, \ldots), \quad \mathbf{s} = (s_1, s_2, s_3, \ldots)
$$

and

$$M = \left[\sum_{i=0}^{\infty} C^i_{-\gamma} u^{(i)}_{\gamma-\beta+i} \right]_{\gamma,\beta=1,2,3,\ldots}, \tag{15}$$

where M is a square matrix of infinite size with coeffitients in $\widehat{E}_{\bar{R}}$.

Let us denote by $d_{x_2}(S(0)^{-1})$ *the order* of the polynomial $S(0)^{-1}$ with respect to the variable x_2, i.e. $\partial^{d_{x_2}+1}(S(0)^{-1})/\partial x_2^{d_{x_2}+1} = 0$. Then it is easy to see that $d_{x_2}(u_\beta) \le d_{x_2}(S(0)^{-1})$ for all β. Therefore, the coefficients of M belong to the ring $k[x_1, x_2][\partial_1, \partial_1^{-1}]$. Equation (14) now reads

$$\mathbf{s}\, M = -\,\mathbf{u} \tag{16}$$

Therefore the solution $\mathbf{s}$ is given by $\mathbf{s} = -\,\mathbf{u}\, M^{-1}$. The idea is to define M^{-1} by the Neumann series $\sum_{n=0}^{\infty}(1-M)^n$, and use a similar technique developed in [1] to establish well-definedness of $\sum_{n=0}^{\infty}(1-M)^n$ and $\mathbf{u}\, M^{-1}$. Since $\mathbf{s}$ determines all the coefficients of S, well-definedness of $\mathbf{u}\sum_{n=0}^{\infty}(1-M)^n$ imlies the existence of S such that $SU \in \widehat{D}_R$.

Let $N = 1 - M = [a_{\mu\nu}]_{\mu,\nu=1,2,3,\ldots}$ and $N^n = [a_{n,\mu\nu}]_{\mu,\nu=1,2,3,\ldots}$. Since $a_{\mu\nu} = \delta_{\mu\nu} - \sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_{\mu-\nu+i}$, we have

$$\pi(a_{\mu\mu}) = \pi(1 - \sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_i) = \pi(-\sum_{i=1}^{\infty} C^i_{-\mu} u^{(i)}_i) = 0,$$

because $\pi(u_i) = 0$ for all $i > 0$. Similarly, if $\mu > \nu$, then

$$\pi(a_{\mu\nu}) = \pi(-\sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_{\mu-\nu+i}) = 0.$$

Because of the growth order condition for u_ν's, we can find a positive real number J such that

$$v_2(a_{\mu\nu}) \ge J(\mu - \nu + 1) \quad \text{for all } \mu - \nu \ge -1\,. \tag{17}$$

Actually, $J = C_U/(1 + C_U M_U)$.

Now put $\alpha_{S(0)} = 1/(d_{x_2}(S(0)^{-1}) + deg_{\partial_2^{-1}}(S(0)^1))$, where $deg_{\partial_2^{-1}}(S(0)^1))$ is the degree of the polynomial $S(0)^{-1}$ with respect to the variable ∂_2^{-1}, and assume $\alpha \le \alpha_{S(0)}$.

Claim. For every $n \ge 1$ we have

$$v_2(a_{n,\mu\nu}) \ge J(\mu - \nu + 1) \quad \text{for all } \mu - \nu \ge -1 \tag{18}$$

$$\mathrm{ord}_{\partial_1}(a_{n,\mu\nu}) \le \alpha(\mu - \nu) \tag{19}$$

Proof of the claim. To prove (18) we can repeat the proof of the claim on p.27 in [1] using v_2 instead of val_t there. Let us prove (19).

Let $U_0 = \exp(\sum_{i,j\geq 1, i\leq \alpha j} t_{i,j}\partial_1^i\partial_2^j) = 1+\sum_{\beta=1}^{\infty} u_\beta^0 \partial_2^\beta$ and $S(0)^{-1} = 1+\sum_{k=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})} s_k^0\partial_2^{-k}$. Then

$$U = U_0 S(0)^{-1} = (1+\sum_{\beta=1}^{\infty} u_\beta^0\partial_2^\beta)(1+\sum_{k=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})} s_k^0\partial_2^{-k})$$

$$= 1+\sum_{\beta=1}^{\infty} u_\beta^0\partial_2^\beta + \sum_{k=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})} s_k^0\partial_2^{-k} + \sum_{\beta=1}^{\infty}\sum_{k=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})}\sum_{i=0}^{\beta} C_\beta^i u_\beta^0 s_k^{0(i)}\partial_2^{\beta-k-i}$$

$$= 1+\sum_{\beta=1}^{\infty} u_\beta^0\partial_2^\beta + \sum_{k=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})} s_k^0\partial_2^{-k} + \sum_{\gamma=-deg_{\partial_2^{-1}}(S(0)^{-1})}^{\infty} (\sum_{\beta=1}^{\infty}\sum_{i=0}^{\beta} C_\beta^i u_\beta^0 s_{\beta-i-\gamma}^{0(i)})\partial_2^\gamma,$$

where $s_{\beta-i-\gamma}^{0(i)} = 0$ if $\beta-i-\gamma > deg_{\partial_2^{-1}}(S(0)^{-1})$ or $i > deg_{x_2}(S(0)^{-1})$ or $\beta-i-\gamma\leq 0$. Therefore,

$$u'_\gamma = \sum_{\beta=1}^{\infty}\sum_{i=0}^{\beta} C_\beta^i u_\beta^0 s_{\beta-i-\gamma}^{0(i)} = \sum_{\beta=1}^{\gamma+deg_{x_2}(S(0)^{-1})+deg_{\partial_2^{-1}}(S(0)^{-1})}\sum_{i=0}^{\beta} C_\beta^i u_\beta^0 s_{\beta-i-\gamma}^{0(i)}, \tag{20}$$

where from we get $\mathrm{ord}_{\partial_1}(u'_\gamma)\leq \mathrm{ord}_{\partial_1}(u_\gamma^0)-1 < \mathrm{ord}_{\partial_1}(u_\gamma^0)\leq \alpha\gamma$ if $\gamma\geq 0$ and $\mathrm{ord}_{\partial_1}(u'_\gamma)\leq -1\leq \alpha\gamma$ otherwise. So, we get $\mathrm{ord}_{\partial_1}(u_\gamma)\leq\alpha\gamma$.

Note that for $\gamma\geq 0$ we have

$$\mathrm{ord}_{\partial_1}(\frac{\partial u_\gamma}{\partial x_2}) \leq \alpha\gamma-1\leq \alpha(\gamma-1). \tag{21}$$

Indeed, $\partial u_\gamma/\partial x_2 = \partial u'_\gamma/\partial x_2$ and $\mathrm{ord}_{\partial_1}(\partial u'_\gamma/\partial x_2)\leq \mathrm{ord}_{\partial_1}(u'_\gamma)$. Therefore, we get

$$\mathrm{ord}_{\partial_1}(a_{1,\mu\nu})\leq \mathrm{ord}_{\partial_1}(u_{\mu-\nu})\leq\alpha(\mu-\nu).$$

Assume now that (19) holds for some $n\geq 1$. Since $N^{n+1} = N^n\cdot N$,

$$a_{n+1,\mu\nu} = \sum_{l=1}^{\infty} a_{n,\mu l}a_{l\nu}.$$

Therefore,

$$\mathrm{ord}_{\partial_1}(a_{n+1,\mu\nu})\leq \mathrm{ord}_{\partial_1}(a_{n,\mu l}a_{l\nu})\leq \alpha(\mu-l)+\alpha(l-\nu) = \alpha(\mu-\nu).$$

This completes the proof of the claim.

By this claim we can conclude that $\sum_{n=0}^{\infty} a_{n,\mu\nu}\in R$ is well defined for all $\mu,\nu = 1,2,3,\dots$. Therefore,

$$M^{-1} = \sum_{n=0}^{\infty}(1-M)^n = [\sum_{n=0}^{\infty} a_{n,\mu\nu}]_{\mu,\nu}$$

is well defined. Let $M^{-1}=[b_{\mu\nu}]_{\mu,\nu=1,2,3,\dots}$, namely $b_{\mu\nu}=\sum_{n=0}^{\infty}a_{n,\mu\nu}$. Then

$$s_\nu = -\sum_{\mu=1}^{\infty}u_{-\mu}b_{\mu\nu} = -\sum_{\mu=1}^{deg_{\partial_2^{-1}}(S(0)^{-1})}u_{-\mu}b_{\mu\nu}\in R$$

is a well defined element. Thus we have established the existence of $S=S(t)\in 1+E_{R-}$ such that $SU\in\widehat{D}_R$. Note that $\mathrm{ord}_{\partial_1}(s_\nu)\le \mathrm{ord}_{\partial_1}(u_{-\mu})+\mathrm{ord}_{\partial_1}(b_{\mu\nu})\le -\alpha\nu\le 0$.

(ii) Take $S(0)^{-1}\in k[x_1,x_2][\partial_1^{-1},\partial_2^{-1}]\partial_1^{-[3\alpha]-1}$ with $deg_{\partial_2^{-1}}(S(0)^{-1})=deg_{x_2}(S(0)^{-1})=1$, where $[.]$ denotes the integral part. In this case it suffice to prove (see Claim above) that $\mathrm{ord}_{\partial_1}(u_\gamma)\le\alpha\gamma$ for all γ and for $\gamma\ge 0$ holds

$$\mathrm{ord}_{\partial_1}(\frac{\partial u_\gamma}{\partial x_2})\le\alpha(\gamma-1),$$

as in formula (21). From formula (20) we get $\mathrm{ord}_{\partial_1}(u'_\gamma)\le\mathrm{ord}_{\partial_1}(u^0_\gamma)-[\alpha]-1<\mathrm{ord}_{\partial_1}(u^0_\gamma)\le\alpha\gamma$ if $\gamma\ge 0$ and $\mathrm{ord}_{\partial_1}(u'_\gamma)\le -[2\alpha]-1\le\alpha\gamma$ otherwise. So, both properties are true.

At last, by lemma 5, for every element $S(0)^{-1}\in k[x_1,x_2][\partial_1^{-1},\partial_2^{-1}]$ with coefficients depending on x_i the element $S(0)$ is not admissible. Therefore, the solutions $S(t)$ correspond to non-trivial solutions of the system $(KP)_\alpha$.

The theorem is proved.

□

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