To the multidimensional tame symbol

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1 Introduction

Let $K = k((t))$ be a 1-dimensional local field. Then the tame symbol

$$(f,g)_K = (-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{\nu(f)}} \mod t \cdot k[[t]], \tag{1}$$

where f, g are from K^*, ν is the discrete valuation of K. By the Weyl reciprocity law the product of tame symbols of rational functions over all the points of a projective curve is equal to 1.

In [2] the tame symbol is obtained as commutator of central extension of some group of k-linear operators in K. By this way the reciprocity law was proved too. This method is the multiplicative analog of the Tate method for the presentation of residues of differentials on curves via the traces of infinite-dimensional operators ([18]).

Let $K = k((t_1))((t_2))$ be the two-dimensional local field. In [5] was given the generalisation of Tate's method to the multidimensional local fields.

In this article we give the construction of the 2-dimensional tame symbol as the commutator of group-like monoidal groupoid which is obtained from some group of k-linear operators in K. We give also the hypothetical method for the proof of 2-dimensional Parshin reciprocity laws.

In section 2 we give the construction of the group $G_{K/k}$ of k-linear operators acting in K. This construction is from [5].

In section 3 we introduce some identifications in the category of 1-dimensional k-vector spaces and give the definition of k^*-gerbe and the definition of morphism between k^*-gerbes.

In section 4 we describe the commensurability for the pairs of k-subspaces which generalises the commensurability from [18], [2]. From the pair of such subspaces we construct k^*-groupoid and $\mathbb{Z}$-torsor by means of Kapranov's determinantal and dimensional theories, [14].

In section 5 we recall the notion of group-like monoidal groupoid and the connection of such categories with cohomology of groups [9]. There is the generalisation of commutator map for such categories ([7]). We give explicit formulae from [7].

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In section 6 we construct the group-like monoidal groupoid from the action of the group $G_{K/k}$ in two-dimensional local field K. The obtained category corresponds to some element from $H^3(G_{K/k}, k^*)$. The commutator of this category gives us the two-dimensional tame symbol up to sign. We obtain the sign by expression from commutators of lifting elements in the central extension of the group G by $\mathbb{Z}$.

In section 7 we give the hypotetical formula connecting the commutators obtained from group-like monoidal categories connected with $K_1 \oplus K_2$, K_1 and K_2, where K_1, K_2 are various two-dimensional local fields. By this formula we reduce the reciprocity law to the adelic ring on the surface around the point and along the curve. We prove that the sign expression and group-like monoidal groupoid connected with this adelic rings are trivial.

We hope that one can remove all the constructions of this article to the case of local artinian rings instead of the ground field k to obtain as derivation the Beilinson construction of residues and the simple proofs of explicit reciprocity laws with the value in Witt vectors, see [1] for the case of curves.

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2 Construction of the group

The constructions of this section are from [5]. Let K/k be a n-dimensional local field of equal characteristic, i.e. $K \simeq k((t_1))\dots((t_n)) = \bar{K}((t_n))$ after the choice of local parameters, $\bar{K}$ is the first residue field of K. Denote by $\mathcal{O}_K \subset K$ the discrete valuation ring with respect to the discrete valuation on K. We have $\mathcal{O}_K = \bar{K}[[t_n]]$. For a finite dimensional over K vector space V we will call by $\mathcal{O}_K$-lattice a $\mathcal{O}_K$-submodule $L \subset V$ such that $L \otimes_{\mathcal{O}_K} K = V$, $L \ne 0$, $L \ne V$.

Let V, $\tilde{V}$ be finite dimensional vector spaces over K. We will define subspace

$$E_{K/k}(V, \tilde{V}) \subset \mathrm{Hom}_k(V, \tilde{V}).$$

Let $A : V \to \tilde{V}$ be a k-linear operator. Let $L' \subseteq L \subset V$, $\tilde{L} \subseteq \tilde{L}' \subset \tilde{V}$ be $\mathcal{O}_K$-lattices such that

$$A(L) \subseteq \tilde{L}', A(L') \subseteq \tilde{L}.$$

Then we get an induced morphism

$$\bar{A} \in \mathrm{Hom}_k(L/L', \tilde{L}'/\tilde{L})$$

and $L/\tilde{L}'$ and $\tilde{L}'/\tilde{L}$ are vector spaces of finite dimension over $\bar{K}$.

Definition 1 *Let V, $\tilde{V}$ be as above. We define the set $E_{K/k}(V, \tilde{V})$ by the following two properties:*

1. *If* $K = k$ *, then* $E_{K/k}(V, \tilde{V}) = \mathrm{Hom}_k(V, \tilde{V})$ *.*

2. *Let* K/k *be a local field of dimension at least one. Then*

$$A \in E_{K/k}(V, \tilde{V}) \quad \overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{l} \text{for all lattices } L \subset V,\ \tilde{L} \subset \tilde{V} \text{ there exist} \\ \text{lattices } L' \subseteq L\ ,\ \tilde{L}' \supseteq \tilde{L} \text{ as above} \\ \text{such that } \bar{A} \in E_{\bar{K}/k}(L/L', \tilde{L}'/\tilde{L}). \end{array}$$

Proposition 1 *The following properties are satisfied:*

1. $\mathrm{Hom}_K(V, \tilde{V}) \subseteq E_{K/k}(V, \tilde{V})$ *.*

2. $E_{K/k}(V, \tilde{V}) \subseteq \mathrm{Hom}_k(V, \tilde{V})$ *is a* k *-subspace.*

3. $E_{K/k}(\tilde{V}, \hat{V}) \circ E_{K/k}(V, \tilde{V}) \subseteq E_{K/k}(V, \hat{V})$ *. In particular,* $E_{K/k}(V, V)$ *is an (in general noncommutative) algebra with unit.*

4. *The definition of* $E_{K/k}$ *does not depend on the choice of* L' *and* $\tilde{L}'$ *.*

The proof of this proposition is in [17].

Let K be a local field of dimension 1. Then it is not difficult to see that $E_{K/k}(V, \tilde{V})$ coincides with the space of continuous k-linear operators, if the topology on V and $\tilde{V}$ is induced by the discrete valuation topology on K.

Remark, that after the choice of local parameters we can describe $E_{K/k}(K, K)$ in the following way. Let $K = \bar{K}((t_n))$, and let $A \in \mathrm{End}_k(K)$. Then consider the matrix $(A_{ij})_{i,j \in Z}$ given by

$$A(\bar{x} t_n^i) = \sum_j A_{ij}(\bar{x}) t_n^j \quad \text{with} \quad A_{ij} \in \mathrm{End}_k(\bar{K}),\ \bar{x} \in \bar{K}.$$

Then $E_{K/k}(K, K) = \{A \in \mathrm{End}_k(K)$ and the following conditions hold:

1. The linear maps A_{ij} lye in $E_{\bar{K}/k}(\bar{K})$ for all i, j;

2. The set of indices i, j s.t. $A_{i,j} \neq 0$ is contained in a domain with a boundary a monotonely increasing curve $j = j(i)$ such that $j(i) \to \infty$ if $i \to \infty$. $\}$

Definition 2 *When* K *is a* 2*-dimensional local field, denote by* $G_{K/k}$ *the group of invertible elements of* $E_{K/k}(K, K)$ *.*

3 1-dimensional vector spaces and k^*-gerbes

Let A, B, C be 1-dimensional k-vector spaces. Then we have the canonical isomorphism:

$$(A \otimes_k B) \otimes_k C \to A \otimes_k (B \otimes_k C),$$

such that for any four 1-dimensional k-vector spaces the following diagram is commutative

$$\begin{array}{ccc}
A \otimes_k (B \otimes_k (C \otimes_k D)) & \longrightarrow & A \otimes_k ((B \otimes_k C) \otimes_k D) \\
\downarrow & & \downarrow \\
(A \otimes_k B) \otimes_k (C \otimes_k D) & & (A \otimes_k (B \otimes_k C)) \otimes_k D \\
\searrow & & \swarrow \\
& ((A \otimes_k B) \otimes_k C) \otimes_k D &
\end{array}$$

We have also the following canonical morphisms:

$$A \otimes_k A^* \to k \qquad A \otimes_k k \to A \qquad k \otimes_k A \to A \qquad (A \otimes_k B)^* = B^* \otimes_k A^*$$

We will identify the 1-dimensional k-vector spaces with respect to the above canonical morphisms. All these identifications don't lead to the contradiction when we consider these identifications in the chain of morphisms of the tensor product of 1-dimensional k-vector spaces. It follows from the diagram above and some other easy diagrams. (In fact, we have from these diagrams that the category of 1-dimensional k-vector spaces with the tensor product and the operation of dual space is the group-like monoidal groupoid.)

Definition 3 *A category C is a k^*-gerbe, if*

1. *For any $c_1, c_2 \in \mathrm{Ob}(C)$ $Hom_C(c_1, c_2)$ is a k^*-torsor and for any $c_3 \in \mathrm{Ob}(C)$ the composition $\mathrm{Hom}_C(c_1, c_2) \otimes \mathrm{Hom}_C(c_2, c_3) \to \mathrm{Hom}_C(c_1, c_3)$ is bilinear. $\mathrm{Hom}_C(c_1, c_1)$ is the trivial k^*-torsor.*

2. *For any k^*-torsor E, for any $c \in \mathrm{Ob}(C)$ there exists a unique $c' \in \mathrm{Ob}(C)$ such that $E = \mathrm{Hom}_C(c, c')$ as k^*-torsors. c' is denoted $E \otimes c$.*

Definition 4 *Let C_1 and C_2 be a k^*-gerbes. Then $F \in \mathrm{Hom}(C_1, C_2)$ iff*

1. *F is a functor, which is an equivalence of categories;*

2. *$F(\mathrm{Hom}_{C_1}(c_1, c_2)) = \mathrm{Hom}_{C_2}(F(c_1), F(c_2))$ as k^*-torsors for any $c_1, c_2 \in \mathrm{Ob}(C_1)$.*

Remark 1
1. Any k^*-gerbe C after the choice of an object c is isomorphic to the category of k^*-torsors: $\tilde{c} \mapsto \mathrm{Hom}(c, \tilde{c})$.

2. For any k^*-gerbes C_1, C_2, any $F \in \mathrm{Hom}(C_1, C_2)$ is defined uniquely by the value on one $c \in \mathrm{Ob}(C_1)$: $F(\tilde{c}) = \mathrm{Hom}_{C_1}(c, \tilde{c}) \otimes F(c)$

3. For any k^*-gerbes C_1 and C_2 $\mathrm{Hom}(C_1, C_2)$ is a k^*-gerbe as well, where for any $F_1, F_2 \in \mathrm{Hom}(C_1, C_2)$ $\mathrm{Hom}_{\mathrm{Hom}(C_1, C_2)}(F_1, F_2)$ are natural transformations between functors F_1 and F_2.

The more information about gerbes is in [6], [8], [9].

4 Commensurability

Recall the following definitions from [18], [2] and their modification from [1].

Let V be a k-vector space. Let A and B be k-subspaces. Then

$$A \sim B \qquad \text{iff} \qquad \dim_k \frac{A}{A \cap B} < \infty \quad \text{and} \quad \dim_k \frac{B}{A \cap B} < \infty.$$

If $A \sim B$, then

$$[A \mid B] \quad \stackrel{\text{def}}{=} \quad \dim_k \frac{B}{A \cap B} - \dim_k \frac{A}{A \cap B}.$$

If W is a finite-dimensional vector space over k, then let $\det W$ be the top exterior power of W. Then

$$(A \mid B) \quad \stackrel{\text{def}}{=} \quad \varinjlim_{\substack{C \sim A \sim B \\ C \subset A, C \subset B}} \quad \mathrm{Hom}_k \left(\det(A/C),\ \det(B/C)\right)$$

is a 1-dimensional k-space, where for the passing to the direct limit we need the identities: for $C' \subset C$

$$\det(A/C') = \det(A/C) \otimes_k \det(C/C')$$
$$\det(B/C') = \det(B/C) \otimes_k \det(C/C').$$

And $f \in \mathrm{Hom}_k(\det(A/C), \det(B/C)) \longmapsto f' \in \mathrm{Hom}_k(\det(A/C'), \det(B/C'))$, where $f'(a \otimes c) \stackrel{\text{def}}{=} f(a) \otimes c$, a is any from $\det(A/C)$, c is any from $\det(C/C')$.

Proposition 2 *1. If $A \sim B$, $B \sim C$, then $A \sim C$.*

2. Let A, B, C be as above, then

$$[A \mid B] + [B \mid C] = [A \mid C].$$

3. There is a canonical isomorphism

$$\alpha \ : \ (A \mid B) \otimes_k (B \mid C) \to (A \mid C)$$

such that the following diagram of associativity is commutative:

$$\begin{array}{ccc}
(A \mid B) \otimes_k (B \mid C) \otimes_k (C \mid B) & \to & (A \mid C) \otimes_k (C \mid D) \\
\downarrow & & \downarrow \\
(A \mid B) \otimes_k (B \mid D) & \to & (A \mid D)
\end{array}$$

Proof The proofs of items 1, 2 of the lemma are not difficult and can be found in [2, §1]. For item 3 remark that we have a canonical map:

$$\begin{aligned}\mathrm{Hom}_k(\det(A/C'), \det(B/C')) \quad &\otimes_k \\ &\mathrm{Hom}_k(\det(B/C'), \det(C/C')) \longrightarrow \mathrm{Hom}_k(\det(A/C'), \det(C/C')),\end{aligned} \tag{2}$$

which satisfies the associativity diagram. And this map commutes with the direct limit from the definition of $(\mid)$. We obtain the map α after the passing to the direct limit in (2).

Now we give the following definitions from [14].

Let V be a k-space with the filtration with finite-dimensional over k factors. Let $Gr(V)$ be the set of all k-subspaces of V which are commensurable (like $\sim$) with subspaces of filtration.

Definition 5 *Let V be a k-space with the filtration with the finite-dimensional factors. A dimension theory on V is a map $d : Gr(V) \to \mathbb{Z}$ such that, whenever $U_1, U_2 \in G(V)$, we have*

$$d(U_2) = d(U_1) + [U_1 \mid U_2].$$

The set of dimension theories will be denoted $\mathrm{Dim}(V)$. The group $\mathbb{Z}$ acts on $\mathrm{Dim}(V)$ by adding constant functions and makes $\mathrm{Dim}(V)$ into a $\mathbb{Z}$-torsor.

Definition 6 *Let V be a k-space with the filtration with the finite-dimensional factors. A determinantal theory on V is a rule Δ which associates to each $U \in Gr(V)$ a 1-dimensional k-vector space $\Delta(U)$, to each pair $U_1, U_2 \in G(V)$, an isomorphism*

$$\Delta_{U_1 U_2} \;:\; \Delta(U_1) \otimes_k (U_1 \mid U_2) \to \Delta(U_2)$$

so that for any $U_1, U_2, U_3 \in G(V)$ the obvious diagram

$$\begin{array}{ccc} \Delta(U_1) \otimes_k (U_1 \mid U_2) \otimes_k (U_2 \mid U_3) & \to & \Delta(U_1) \otimes_k (U_1 \mid U_3) \\ \downarrow & & \downarrow \\ \Delta(U_2) \otimes_k (U_2 \mid U_3) & \to & \Delta(U_3) \end{array}$$

is commutative.

We denote by $\mathrm{Det}(V)$ the category (groupoid) formed by all determinantal theories on V. If we fix $U \in Gr(V)$, then

$$\mathrm{Hom}_{\mathrm{Det}(V)}(\Delta, \Delta') = \Delta'(U) \otimes_k \Delta(U)^* \setminus 0.$$

Define the twisting for any k^*-torsor E, for any determinantal theory $\Delta \in \mathrm{Det}(V)$

$$(E \otimes \Delta)(U) \stackrel{\mathrm{def}}{=} E \otimes_k \Delta(U).$$

One easily sees that

Proposition 3 $\mathrm{Det}(V)$ *is a k^*-gerbe.*

Remark 2 Any element $\tilde{U} \in Gr(V)$ gives $d_{\tilde{U}} \in \mathrm{Dim}(V)$ and $\Delta_{\tilde{U}} \in \mathrm{Det}(V)$ by the rule

$$d_{\tilde{U}}(U) = [\tilde{U} \mid U] \qquad , \qquad \Delta_{\tilde{U}}(U) = (\tilde{U} \mid U).$$

Denote by $\otimes$ the tensor product of k^*-torsors, and by $\odot$ for $\mathbb{Z}$-torsors.

For any k^*-gerbes $\mathcal{C}', \mathcal{C}''$ we denote $\mathcal{C}' \boxtimes \mathcal{C}''$ the category (groupoid, i.e., every morphism is invertible) whose class of objects is $\mathrm{Ob}(\mathcal{C}') \times \mathrm{Ob}(\mathcal{C}'')$ and

$$\mathrm{Hom}_{\mathcal{C}' \boxtimes \mathcal{C}''}((x', x''), (y', y'')) = \mathrm{Hom}_{\mathcal{C}'}(x', y') \otimes \mathrm{Hom}_{\mathcal{C}''}(x'', y'').$$

Remark that under this definition $\mathcal{C}' \boxtimes \mathcal{C}''$ is not a k^*-gerbe, since we have not property 2 from the definition 3.

One calls a sequence of k-spaces with filtrations with finite-dimensional factors

$$0 \longrightarrow V' \xrightarrow{\alpha} V \xrightarrow{\beta} V'' \longrightarrow 0 \qquad (3)$$

admissible, if filtration on V' is induced from the filtration on V, and filtration on V'' is the factor filtration of filtration on V. We will also speak about admissible filtrations $V_1 \subset V_2 \subset \ldots \subset V_n$.

Proposition 4 *1. For each admissible short exact sequence (3) we have a natural identification of $\mathbb{Z}$-torsors*

$$\mathrm{Dim}(V') \odot \mathrm{Dim}(V'') \longrightarrow \mathrm{Dim}(V)$$

and these identifications are associative in any admissible filtration of length 2

2. for an admissible short exact sequence(3) there is a functor between groupoids

$$\delta_{V'VV''} \ : \ \mathrm{Det}(V') \boxtimes \mathrm{Det}(V'') \longrightarrow \mathrm{Det}(V)$$

and the following diagram is commutative for any admissible filtration of $V_1 \subset V_2 \subset V_3$ of length 2:

$$\begin{array}{ccc} \mathrm{Det}(V_1) \boxtimes \mathrm{Det}(V_2/V_1) \boxtimes \mathrm{Det}(V_3/V_2) & \longrightarrow & \mathrm{Det}(V_1) \boxtimes \mathrm{Det}(V_3/V_1) \\ \downarrow & & \downarrow \\ \mathrm{Det}(V_2) \boxtimes \mathrm{Det}(V_3/V_2) & \longrightarrow & \mathrm{Det}(V_3). \end{array}$$

Proof (see [14, §2]). Given dimension theories d' on V' and d'' on V'', we have a dimension theory d on V given by

$$d(U) = d'(\alpha^{-1}(U)) + d''(\beta(U)).$$

Given determinantal theories Δ' on V' and Δ'' on V'', we have a determinantal theory $\Delta = \delta_{V'VV''}(\Delta', \Delta'')$ on V defined by

$$\Delta(U) = \Delta'(\alpha^{-1}(U)) \otimes_k \Delta''(\beta(U)).$$

The diagram is commutative after the our agreements on identifications of 1-dimensional k-vector spaces (see the beginning of section 3). (Without this agreement on identifications of 1-dimensional k-vector space this diagram is commutative up to some natural transformation, and these transformations fit into a commutative cube for any admissible length 3 filtration, as in [14, §2].)

Let K/k be a 2-dimensional local field.

Definition 7 *Let* V *be a finite-dimensional vector space over* K. *For* k*-subspaces* $A, B \in V$ *one calls* $A \approx B$ *iff there are* $\mathcal{O}_K$ *lattices* $L \subseteq M \subset V$ *such that*

$$L \subseteq A \subseteq M \qquad \text{and} \qquad L \subseteq B \subseteq M.$$

Definition 8 *Let a* k*-vector space* $V = \prod_{l \in I} K_l$, *where every* K_l/k *is a* 2*-dimensional local field. For two* k*-subspaces* $A, B \in V$ *one calls* $A \approx B$ *iff there is a finite set* $J \subset I$ *such that for every* $j \in J$ *there are* $\mathcal{O}_{K_j}$*-lattices* $L_j \subseteq M_j \subset K_j$, *there is a* k*-subspace* $W \subset \prod_{l \in I \setminus J} K_l$ *such that*

$$\prod_{j \in J} L_j \times W \ \subseteq \ A \ \subseteq \ \prod_{j \in J} M_j \times W$$

and

$$\prod_{j \in J} L_j \times W \ \subseteq \ B \ \subseteq \ \prod_{j \in J} M_j \times W.$$

Remark that if k-spaces $A \approx B \subset V$, $A \supset B$, then A/B is a space with filtration with finite-dimensional factors. For example, in the case V is finite-dimensional over K, the filtration is induced from the filtration of the $\bar{K}$-space M/L by $\mathcal{O}_{\bar{K}}$-lattices.

Definition 9 *Let* k*-subspaces* $A \approx B \subset V$. *Then define*

$$[[A \mid B]] \quad \stackrel{\mathrm{def}}{=} \quad \varinjlim_{\substack{C \approx A \approx B \\ C \subset A, C \subset B}} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Dim}(A/C),\ \mathrm{Dim}(B/C)\right)$$

The possibility of the passing to the direct limit follows from the identities: for $C' \subset C$

$$\mathrm{Dim}(A/C') = \mathrm{Dim}(A/C) \odot \mathrm{Dim}(C/C')$$

$$\mathrm{Dim}(B/C') = \mathrm{Dim}(B/C) \odot \mathrm{Dim}(C/C').$$

And $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(A/C), \mathrm{Dim}(B/C)) \longmapsto f' \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(A/C'), \mathrm{Dim}(B/C'))$, where $f'(a \odot c) \stackrel{\mathrm{def}}{=} f(a) \odot c$, a is any from $\mathrm{Dim}(A/C)$, c is any from $\mathrm{Dim}(C/C')$.

Proposition 5 1. $[[A \mid B]]$ *is a* $\mathbb{Z}$*-torsor for any* k*-subspaces* $A \approx B \subset V$.

2. *For any* k*-subspaces* $A \approx B \approx C \subset V$ *there is a canonical isomorphism of* $\mathbb{Z}$*-torsors*

$$[[A \mid B]] \odot [[B \mid C]] \longrightarrow [[A \mid C]]$$

and these isomorphisms are associative for any 4 *subspaces* $A \approx B \approx C \approx D \subset V$.

Proof is the same as for 1-dimensional k-spaces in item 3 of proposition 2.

Let k-subspaces $A \approx B \approx P \subset V$, $P \subset A$, $P \subset B$. Define

$$((A, B, P)) = \mathrm{Hom}(\mathrm{Det}(A/P), \mathrm{Det}(B/P)).$$

Then $((A, B, P))$ is a k^*-gerbe. We have a natural functor for a k-subspace $C \supset P$, $C \approx P$

$$((A, B, P)) \boxtimes ((B, C, P)) \to ((A, C, P)) \tag{4}$$

Let $A \approx B \approx P \approx Q$, $A \supset P \supset Q$, $B \supset P \supset Q$. We construct the functor $\mathcal{F}_{P,Q}$ between k^*-gerbes: $((A, B, P)) \to ((A, B, Q))$. We have the exact sequences:

$$0 \to P/Q \to A/Q \to A/P \to 0 \qquad \text{and} \qquad 0 \to P/Q \to B/Q \to B/P \to 0.$$

Therefore we have the functors $\delta_{P/Q,A/Q,A/P}$ and $\delta_{P/Q,B/Q,B/P}$

$$\mathrm{Det}(P/Q) \boxtimes \mathrm{Det}(A/P) \to \mathrm{Det}(A/Q) \qquad \text{and} \qquad \mathrm{Det}(P/Q) \boxtimes \mathrm{Det}(B/P) \to \mathrm{Det}(B/Q)$$

Choose any $e \in \mathrm{Ob}(\mathrm{Det}(P/Q))$. Then for $\phi \in ((A, B, P))$ define

$$\mathcal{F}_{P,Q}(\phi) = \phi',$$

where $\phi' \in ((A, B, Q))$ is defined by the following rule:

$$\phi'(\delta_{P/Q,A/Q,A/P}(e \boxtimes a)) = \delta_{P/Q,B/Q,B/P} e \boxtimes \phi(a) \qquad \text{for any} \qquad a \in \mathrm{Ob}(\mathrm{Det}(A/P)),$$

Since $\mathrm{Det}(A/Q)$ and $\mathrm{Det}(B/Q)$ are k^*-gerbes, the functor ϕ' is determined by the value on one object.

The functor ϕ' does not depend on the choice of e, since for any other $e' \in \mathrm{Ob}(\mathrm{Det}(P/Q))$ we have

$$\mathrm{Hom}_{\mathrm{Det}(A/Q)}(e \boxtimes a, e' \boxtimes a) = \mathrm{Hom}_{\mathrm{Det}(P/Q)}(e, e').$$

From item 2 of property 4 we have the exact equality of functors

$$\mathcal{F}_{Q,T}\mathcal{F}_{P,Q} = \mathcal{F}_{P,T}$$

for any $P \supset Q \supset T$, $P \approx Q \approx T$. Therefore the following definition is correct.

Definition 10 *For k-subspaces $A \approx B \subset V$ define the category $((A \mid B))$ as*

$$\mathrm{Ob}(((A \mid B))) \stackrel{\mathrm{def}}{=} \varinjlim_{P} \mathrm{Ob}(((A, B, P))), \textit{ and}$$

$$\mathrm{Hom}_{((A|B))}(\{c_P\}, \{c'_P\}) \stackrel{\mathrm{def}}{=} \varinjlim_{P} \mathrm{Hom}_{((A,B,P))}(c_P, c'_P),$$

where the direct limit is given with respect to the functors $\mathcal{F}_{P,Q}$.

Proposition 6 *1.* $((A \mid B))$ *is a* k^* *-gerb for any* k *-subspaces* $A \approx B \subset V$ *.*

2. There is a functor between groupoids

$$\delta_{A,B,C} \; : \; ((A \mid B)) \boxtimes ((B \mid C)) \longrightarrow ((A \mid C))$$

and the following diagram is commutative for any k *-subspaces* $A \approx B \approx C \approx D \subset V$

$$\begin{array}{ccc} ((A \mid B)) \boxtimes ((B \mid C)) \boxtimes ((C \mid D)) & \longrightarrow & ((A \mid B)) \boxtimes ((B \mid D)) \\ \downarrow & & \downarrow \\ ((A \mid C)) \boxtimes ((C \mid D)) & \longrightarrow & ((A \mid D)). \end{array}$$

Proof . Item 1 follows from the definition of $((A \mid B))$. Item 2 follows from the analogous statements for $((A, B, P))$, which are obvious (see expression (4)). The diagram is commutative, since the composition of functors between categories is strictly associative. The passing to the direct limit conserves these statements.

Remark that for any $s \in [[A \mid B]]$ there is a well-defined $s^{-1} \in [[B \mid A]]$ such that $s^{-1} \odot s$ and $s \odot s^{-1}$ are the identity maps in $[[A \mid A]] = \lim\limits_{\rightarrow} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(A/P), \mathrm{Dim}(A/P))$ and $[[B \mid B]] = \lim\limits_{\rightarrow} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(B/P), \mathrm{Dim}(B/P))$.

For $S \in \mathrm{Ob}(((A \mid B)))$ there is always a well-defined $S^{-1} \in \mathrm{Ob}(((B \mid A)))$ such that $\delta_{A,B,A}(S^{-1} \boxtimes S)$ and $\delta_{B,A,B}(S \boxtimes S^{-1})$ are the identity functors from $((A \mid A)) = \lim\limits_{\rightarrow} \mathrm{Hom}(\mathrm{Det}(A/P), \mathrm{Det}(A/P))$ and $((B \mid B)) = \lim\limits_{\rightarrow} \mathrm{Hom}(\mathrm{Det}(B/P), \mathrm{Det}(B/P))$ correspondingly.

Let H be a subgroup of $E^*_{K/k(V,V)}$, if V is a finite-dimensional vector space over K . Let H be a subgroup of all $G_{K_l/k}$ for $l \in I$, if $V = \prod_{l \in I} K_l$. Then we have an action of the group H on V (diagonal action of H in the second case), such that for any $h \in H$, for any $A \approx B \subset V$ $hA \approx hB$ and if $A \supset B$, then h induces a well-defined map $Gr(A/B) \to Gr(hA/hB)$.

Then for any $h \in H$ we have a map (and a functor)

$$h \quad : \quad \mathrm{Dim}(A/P) \to \mathrm{Dim}(hA/hP) \qquad , \qquad \mathrm{Det}(A/P) \to \mathrm{Det}(hA/hP),$$

where for any $U \in Gr(hA/hP)$, $d \in \mathrm{Dim}(A/P)$, $\Delta \in \mathrm{Det}(A/P)$

$$(h \circ d)(U) \stackrel{\mathrm{def}}{=} d(h^{-1}U) \qquad \text{and} \qquad (h \circ \Delta)(U) \stackrel{\mathrm{def}}{=} \Delta(h^{-1}U).$$

Therefore we have a map

$$h \quad : \quad \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(A/P), \mathrm{Dim}(B/P)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(hA/hP), \mathrm{Dim}(hB/hP))$$

and a functor

$$h \quad : \quad ((A, B, P)) \to ((hA, hB, hP)),$$

where for any F from $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(A/P), \mathrm{Dim}(B/P))$ or from $((A, B, P))$ we put $g \circ F \stackrel{\mathrm{def}}{=} gFg^{-1}$. We pass to the direct limit and obtain a map (and a funcor)

$$h \quad : \quad [[A \mid B]] \to [[hA \mid hB]] \qquad \text{and} \qquad ((A \mid B)) \to ((hA \mid hB))$$

such that $h[[A \mid B]] \odot h[[B \mid C]] = h[[A \mid C]]$ and the following diagram is commutative

$$\begin{array}{ccc} ((A \mid B)) \boxtimes ((B \mid C)) & \longrightarrow & ((A \mid C)) \\ \downarrow & & \downarrow \\ h((A \mid B)) \boxtimes h((B \mid C)) & \longrightarrow & h((A \mid C)). \end{array}$$

Remark that for any $h_1, h_2 \in H$ we have

$$h_2 \circ (h_1 \circ [[A \mid B]]) = (h_2 h_1) \circ [[A \mid B]] \quad , \quad h_2 \circ (h_1 \circ ((A \mid B))) = (h_2 h_1) \circ ((A \mid B)).$$

5 Group-like monoidal groupoids and cohomology of groups

In this section we recall the notion of group-like monoidal groupoid, connection with the group cohomology and analog of commutator map [9], [7].

The groupoid is a category, in which every morphism is invertible. The group-like monoidal groupoid (or groupoid with tensor product, or gr-category) is a groupoid C with tensor product, i.e., a category equipped with a composition law, which is a functor $\otimes : C \times C \to C$, denoted by $(X, Y) \mapsto X \otimes Y$, together with an associativity constraint, which is a functorial isomorphism

$$c_{X,Y,Z} \ : \ X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$$

and a unit object I for which there are given functorial isomorphisms

$$g_X \ : \ I \otimes X \to X, \quad d_X \ : \ X \otimes I \longrightarrow X.$$

The following diagrams are required to be commutative:

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \longrightarrow & X \otimes (I \otimes Y) \\ & \searrow \quad \swarrow & \\ & X \otimes Y & \end{array}$$

and

$$\begin{array}{ccc} X \otimes (Y \otimes (Z \otimes W)) & \longrightarrow & X \otimes ((Y \otimes Z) \otimes W) \\ \downarrow & & \downarrow \\ (X \otimes Y) \otimes (Z \otimes W) & & (X \otimes (Y \otimes Z)) \otimes W \\ \searrow & & \swarrow \\ & ((X \otimes Y) \otimes Z) \otimes W & \end{array}$$

It is required that every object X admits an "inverse" X^* for which there is an isomorphism $\epsilon_X \ : \ X \otimes X^* \to I$. There is also, therefore, a well-defined isomorphism: $\nu_X \ : \ I \to X^* \otimes X$.

The set $\pi_0(C)$ of isomorphism classes of objects is a group under tensor product. Let $\pi_1(C)$ denote the group $Aut_C(I)$, where the group law is induced by the tensor product. It follows that $\pi_1(C)$ is abelian. The group $\pi_0(C)$ operates on $\pi_1(C)$ as follows: for X

an object of C and γ an automorphism of I, let $[X]\cdot\gamma$ denote the automorphism of $I \simeq X\otimes(I\otimes X^*)$ given by $Id_X\otimes(\gamma\otimes Id_{X^*})$. By theorem of Sinh we have a canonical class in the cohomology group $H^3(\pi_0(C),\pi_1(C))$ which represents the obstruction to finding an assignment $g\in\pi_0(C)\mapsto P_g\in \mathrm{Ob}(C)$, together with isomorphisms c_{g_1,g_2} : $P_{g_1g_2}\to P_{g_1}\otimes P_{g_2}$ for all g_1, g_2 in $\pi_0(C)$ such that for any three elements of this group a natural associativity diagram holds. Choose objects P_g and isomorphisms c_{g_1,g_2}. Then with $c=c_{P_{g_1},P_{g_2},P_{g_3}}$ we have the equality

$$c\circ(Id\otimes c_{g_2,g_3})\circ c_{g_1,g_2g_3}=f(g_1,g_2,g_3)\circ(c_{g_1,g_2}\otimes Id)\circ\psi g_1g_2,g_3 \tag{5}$$

for a unique $f(g_1,g_2,g_3)\in\pi_1(C)$. The cohomology class of f in $H^3(\pi_0(C),\pi_1(C))$ is independent of all choices.

We say that group-like monoidal groupoids C_1 and C_2 with given π_0 and π_1 are equivalent, if there is a functor $F:C_1\to C_2$ together with functorial isomorphisms $\lambda: F(X\otimes Y)\to F(X)\otimes F(Y)$ and $\mu: F(I)\to I$, which are compatible with the associativity isomorphisms and the identity isomorphisms in C_1 and C_2; it is also required that F induces the identity maps on π_0 and π_1.

There is the following proposition (see [9]).

Proposition 7 *By attaching to a group-like monoidal groupoid its invariant $f(g_1,g_2,g_3)$ from (5), we obtain an isomorphism between the group of equivalence classes of group-like monoidal groupoids C, for which $\pi_0(C)=H$ and $\pi_1(C)=M$, with given action of H on M, and the cohomology group $H^3(H,M)$*

Consider any abelian subgroup $D\subset\pi_0(C)$ such that D-module structure on $\pi_1(C)$ is trivial. For each $g\in\pi_0(C)$ let P_g be a representative object of C in the isomorphism class of g. To C one can associate the $\pi_1(C)$-torsor E above $D\times D$, whose fibre above $(g,h)\in D^2$ is the set

$$E_{g,h}=\mathrm{Hom}_C(P_h\otimes P_g,P_g\otimes P_h).$$

Composing the elements of $E_{g,h}$ on the right with automorphisms of $P_h\otimes P_g$, viewed as elements of $\pi_1(C)$, makes E into a right $\pi_1(C)$-torsor on $D\times D$. (Alternate choices for the reperesentative objects P'_g and P'_h of g and h yield an $\pi_1(C)$-torsor E' on $D\times D$ isomorphic to E.) For each elements $g,h\in\pi_0(C)$ choose an element $c_{g,h}\in Hom(P_{gh},P_g\otimes P_h)$.

The $\pi_1(C)$-torsor E is endowed with a pair of partial multiplication laws:

$$+_1\ :\ E_{g,h}\otimes E_{g',h}\longrightarrow E_{gg',h}\ ;\qquad\qquad +_2\ :\ E_{g,h}\otimes E_{g,h'}\longrightarrow E_{g,hh'},$$

where for $u\in E_{g,h}$, $v\in E_{g',h}$, $w\in E_{g,h'}$ the partial sum $u+_1v$ is defined as the following composition of isomorphisms

$$P_h\otimes P_{gg'}\overset{c_{g,g'}}{\to}P_h\otimes(P_g\otimes P_{g'})\to(P_h\otimes P_g)\otimes P_{g'}\overset{u}{\to}(P_g\otimes P_h)\otimes P_{g'}\to$$

$$\to P_g\otimes(P_h\otimes P_{g'})\overset{v}{\to}P_g\otimes(P_{g'}\otimes P_h)\to(P_g\otimes P_{g'})\otimes P_h\overset{c_{g,g'}{}^{-1}}{\to}P_{gg'}\otimes P_h$$

Here the unlabelled arrows are the associativity isomorphisms. The partial sum $u +_2 w$ is defined in an analogous way:

$$P_{hh'} \otimes P_g \xrightarrow{c_{h,h'}} (P_h \otimes P_{h'}) \otimes P_g \to P_h \otimes (P_{h'} \otimes P_g) \xrightarrow{\omega} P_h \otimes (P_g \otimes P_{h'}) \to$$

$$\to (P_h \otimes P_g) \otimes P_{h'} \xrightarrow{u} (P_g \otimes P_h) \otimes P_{h'} \to P_g \otimes (P_h \otimes P_{h'}) \xrightarrow{c^{-1}_{h,h'}} P_g \otimes P_{hh'}$$

These definitions don't depend on the choice of $c_{g,g'}$ and $c_{h,h'}$.

Proposition 8 ([7]) *These partial multiplication laws on E give the structure of a weak biextension, i.e. these laws are associative and compatible with each other.*

Now consider elements $g_1, g_2, g_3 \in \pi_0(C)$ such that these elements commute with each other. Fix any corresponding objects $P_{g_1}, P_{g_2}, P_{g_3} \in C$ and any morphisms:

$$e_{g_1,g_2} \in E_{g_1,g_2} \qquad e_{g_1,g_3} \in E_{g_1,g_3} \qquad e_{g_2,g_3} \in E_{g_2,g_3}.$$

We consider an automorphism of the object $P_{g_3} \otimes P_{g_2} \otimes P_{g_1}$, which follows from the composition of morphisms in the following diagram:

$$\begin{tikzcd}[column sep=small]
 & P_{g_3} \otimes P_{g_2} \otimes P_{g_1} \arrow[dr] & \\
P_{g_3} \otimes P_{g_1} \otimes P_{g_2} \arrow[ur] & & P_{g_2} \otimes P_{g_3} \otimes P_{g_1} \arrow[d] \\
P_{g_1} \otimes P_{g_3} \otimes P_{g_2} \arrow[u] & & P_{g_2} \otimes P_{g_1} \otimes P_{g_3} \arrow[dl] \\
 & P_{g_1} \otimes P_{g_2} \otimes P_{g_3} \arrow[ul] &
\end{tikzcd} \tag{6}$$

Where morphisms in this diagram by modulo the obvious associativity isomorphisms are given consequently as: $e_{g_2,g_3} \otimes P_{g_1}$, $P_{g_2} \otimes e_{g_1,g_3}$, $e_{g_1,g_2} \otimes P_{g_3}$, $P_{g_1} \otimes e^{-1}_{g_3,g_2}$, $e^{-1}_{g_1,g_3} \otimes P_{g_2}$, $P_{g_3} \otimes e^{-1}_{g_1,g_2}$. .

Remark 3 In [15] the diagram (6) is named Yang-Baxter hexagon. This diagram correspond to the 2-dimensional permutohedron, i.e. convex polytope, whose vertices correspond to all permutations of 3 letters.

We have the following proposition (see [7]).

Proposition 9 *1. The automorphism obtained from diagram (6) belongs to k^* and depends only on the elements g_1, g_2, g_3 and the class of the category C in $H^3(\pi_0(C), \pi_1(C))$. It is denoted $\phi_C(g_1, g_2, g_3)$.*

2. ϕ_C is a trilinear alternating map.

3.

$$\phi_C(g_1, g_2, g_3) = \frac{f(g_1, g_2, g_3) f(g_3, g_1, g_2) f(g_2, g_3, g_1)}{f(g_1, g_3, g_2) f(g_3, g_2, g_1) f(g_2, g_1, g_3)}. \tag{7}$$

If $\pi_0(C)$ is abelian, then ϕ_C is evaluation of the 3-cocycle f on the triple Pontrjagin product cycle $g_1.g_2.g_3 \in H_3(\pi_0(C))$ of classes $g_1, g_2, g_3 \in H_1(\pi_0(C)) = \pi_0(C)$.

4. ϕ_C is trivial iff the both partial group laws $+_1$ and $+_2$ of E are commutative.

Remark 4 The map ϕ_C is an analog of the commutator map for H^2-cohomology of groups. Let we have the central extension of groups:

$$1 \longrightarrow A \longrightarrow G \xrightarrow{p} B \longrightarrow 1. \tag{8}$$

Consider $b_1, b_2 \in B$ such that they commute. Fix any $b'_1, b'_2 \in G$ such that $p(b'_1) = b_1$, $p(b'_2) = b_2$. Define $\psi_G(b_1, b_2) = [b'_1, b'_2]$. We have the following property of ψ_G:

1. $\psi_G(b_1, b_2)$ belongs to A and depends only on the elements b_1, b_2 and the class of extension (12) in $H^2(B, A)$.

2. ψ_G is a bilinear alternating map.

3. $$\psi_G(b_1, b_2) = \frac{f(b_1, b_2)}{f(b_2, b_1)}, \tag{9}$$

   where f is a 2-cocycle of extension (12). If B is abelian, then ψ_G is simply evaluation of the 2-cocycle f on the double Pontrjagin product cycle $b_1.b_2 \in H_2(B)$ of classes $b_1, b_2 \in H_1(B) = B$.

4. If B is abelian, the ψ is trivial iff G is abelian.

For an abelian subgroup $D \subset \pi_0(C)$ we fix any $h, g \in D$. Then we have two central extensions:

$$1 \longrightarrow \pi_1(C) \longrightarrow E_{1,h} \longrightarrow D \longrightarrow 1 \tag{10}$$

and

$$1 \longrightarrow \pi_1(C) \longrightarrow E_{2,g} \longrightarrow D \longrightarrow 1, \tag{11}$$

where the group law in $E_{1,h}$ is given from the partial group law $+_1 : E_{b_1,h} \otimes E_{b_2,h} \to E_{b_1 b_2,h}$ for any b_1, b_2 from D; the group law in $E_{2,g}$ is given from the partial group law $+_2 : E_{g,b_1} \otimes E_{g,b_2} \to E_{g,b_1 b_2}$ for any b_1, b_2 from D.

Proposition 10 *Let $g_1, g_2, g_3 \in \pi_0(C)$ commute with each other. Then*

$$\phi_C(g_1, g_2, g_3) = \psi_{E_{1,g_3}}(g_1, g_2) = \psi_{E_{2,g_1}}(g_2, g_3)^{-1}.$$

Proof From [7] we have the following explicit expressions for a cocycle $f_{h,1}$ of extension $E_{1,h}$ and for a cocycle $f_{g,2}$ of extension $E_{2,g}$

$$f_{h,1}(b_1, b_2) = \frac{f(b_1, b_2, h) f(h, b_1, b_2)}{f(b_1, h, b_2)} \quad , \quad f_{g,2}(b_1, b_2) = \frac{f(b_1, g, b_2)}{f(g, b_1, b_2) f(b_1, b_2, g)}.$$

We compare now the last expressions and explicit expression for ϕ_C and ψ of formulae (7) and (9). It gives the proof of proposition.

6 Two-dimensional symbol as commutator of group-like monoidal categorie

Let $K = \bar{K}((t))$ be a 2-dimensional local field. Let k be a residue field of $\bar{K}$. We have a discrete valuation of rank 2.

$$(\nu_1, \nu_2) : K^* \to \mathbb{Z} \oplus \mathbb{Z},$$

where ν_2 is the discrete valuation with respect to the local parameter t, and $\nu_1(b) \stackrel{\mathrm{def}}{=} \nu_{\bar{K}}(b\bar{t}^{-\nu_2(b)})$. ν_1 depends on the choice of local parameter t. Let $\wp_K$ be the discrete valuation ideal of K with respect to ν_2, $\wp_{\bar{K}}$ the discrete valuation ideal of $\bar{K}$.

Define a map:

$$\nu_K \quad : \quad K^* \times K^* \longrightarrow Z$$

as the composition of maps:

$$K^* \times K^* \longrightarrow K_2(K) \xrightarrow{\partial_2} \bar{K}^* \xrightarrow{\partial_1} \mathbb{Z},$$

where ∂_i is the boundary map in algebraic K-theory. ∂_2 coincides with tame symbol (1) with respect to discrete valuation ν_2. ∂_1 coincides with the discrete valuation $\nu_{\bar{K}}$.

Define a map:

$$(\ ,\ ,\ )_K \quad : \quad K^* \times K^* \times K^* \longrightarrow k^*$$

as the composition of maps

$$K^* \times K^* \times K^* \longrightarrow K_3^M(K) \xrightarrow{\partial_3} K_2(\bar{K}) \xrightarrow{\partial_2} k^*,$$

where K_3^M is the Milnor K-group.

There are the following explicit expressions for these maps (see [11]):

$$\nu_K(f,g) = \nu_1(f)\nu_2(g) - \nu_2(f)\nu_1(g)$$

$$(f,g,h)_K = \mathrm{sign}_K(f,g,h) f^{\nu_K(g,h)} g^{\nu_K(h,f)} h^{\nu_K(f,g)} \bmod_{\wp_K} \bmod_{\wp_{\bar{K}}}$$

$$\mathrm{sign}_K(f,g,h) = (-1)^B,$$

where $B = \nu_1(f)\nu_2(g)\nu_2(h) + \nu_1(g)\nu_2(f)\nu_2(h) + \nu_1(h)\nu_2(g)\nu_2(f) + \nu_2(f)\nu_1(g)\nu_1(h) + \nu_2(g)\nu_1(f)\nu_1(h) + \nu_2(h)\nu_1(f)\nu_1(g)$.

Proposition 11 *For any* $f, g, h \in K^*$

$$\mathrm{sign}_K(f,g,h) = (-1)^A \qquad , \textit{ where}$$

$$A = \nu_K(f,g)\nu_K(f,h) + \nu_K(f,g)\nu_K(f,h) + \nu_K(f,g)\nu_K(f,h) + \nu_K(f,g)\nu_K(f,h)\nu_K(g,h).$$

Proof follows from direct calculations modulo 2 with A and B using the explicit expressions above.

Let V be either a finite-dimensional over K vector space, or $\prod_{l\in I} K_l$, where every K_l is a 2-dimensional local field. Let a k-subspace $L \subset V$ be an $\mathcal{O}_K$-lattice in the first case, and $\prod_{l\in I} L_l \subset V$ in the second case, where L_l is $\mathcal{O}_{K_l}$-lattice for every $l \in I$. Let a group H be a subgroup of $E_{K/k}(V,V)^*$ in the first case, and a subgroup of $G_{K_l/l}$ for every $l \in I$ in the second case such that for any $h \in H$ for almost all $l \in I$ we have $hL_l = L_l$.

Now we can well define the following central extension:

$$0 \longrightarrow \mathbb{Z} \longrightarrow G_{V,L} \longrightarrow H \longrightarrow 1, \qquad (12)$$

where elements of $G_{V,L}$ are the pairs (h,d) with $h \in H$, $d \in [[L \mid hL]]$. The multiplication law is $(h,d)(g,d') = (hg, d \odot (h \circ d'))$. The unit is (e, id), where e is unit of H, and id is the identity map from $((L \mid L))$.

Proposition 12 *1. For an other $L' \in V$ such that $L' \approx L$ there is a canonical isomorphism between central extensions $G_{V,L}$ and $G_{V,L'}$*

2. If for any $h \in H$ we have $hl = L$, then the extension $G_{V,L}$ is splittable.

Proof We prove item 1. We choose any $s \in [[L' \mid L]]$. An isomorphism between central extensions $G_{V,L}$ and $G_{V,L'}$ is given as $(h,d) \mapsto (h, s \odot d \odot (h \circ s^{-1}))$. It is clear that this isomorphism does not depend on the choice of s.

Now we prove item 2. The splitting is constructed as following: $h \mapsto (h, id)$, where id is the identity map from $((L \mid L))$. It finisches the proof.

From the last proposition we have $\psi_{G_{V,L}}(f,g) = \psi_{G_{V,L'}}(f,g)$ for any commuting elements $f, g \in H$. We denote

$$\psi_V(f,g) = \psi_{G_{V,L}}(f,g).$$

Theorem 1 *Let $V = K$ and $H = K^*$. Then in central extension (12) we have for any $f, g \in K^*$ the commutator of lifting of these elements in $G_{V,L}$*

$$\psi_V(f,g) = \nu_K(f,g).$$

Proof We take $L = \mathcal{O}_K$. Both ψ_V and $\nu_K(f,g)$ are bimultiplicative and skew-symmetric. There is a multiplicative decomposition

$$K^* = \bar{K}^* \times t^{\mathbb{Z}} \times \mathcal{U}_K^1,$$

where $\mathcal{U}_K^1 = 1 + \wp_K t$. Therefore to proof the theorem it is enough to consider the following cases.

1. Let $f, g \in \bar{K}^* \times \mathcal{U}_K^1$. Then $\nu_K(f,g) = 0$. We have $f\mathcal{O}_K = \mathcal{O}_K$, $g\mathcal{O}_K = \mathcal{O}_K$. Therefore by item 2 of proposition 12 we have $\psi_V(f,g) = 1$.

2. Let $f \in \bar{K}^*$, $g = t^{-1}$. Then $\nu_K(f, t^{-1}) = -\nu_{\bar{K}}(f)$. We fix any $d \in \mathrm{Dim}(\bar{K}) =$ $= \mathrm{Dim}(\mathcal{O}_K / t^{-1}\mathcal{O}_K) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(0), \mathrm{Dim}(\mathcal{O}_K / t^{-1}\mathcal{O}_K)) = [[\mathcal{O}_K \mid t^{-1}\mathcal{O}_K]]$.
   Let $\hat{f} = (f, id)$, $\hat{g} = (g, d)$ be from $G_{V,L}$. We fix any $U \in Gr(\bar{K})$.
   Then $\hat{f}\hat{g} = (ft^{-1}, f \circ d)$ and $\bar{g}\bar{f} = (ft^{-1}, d)$. Therefore
   $\psi_{G_{V,L}}(f, g) = [\hat{f}, \hat{g}] = (f \circ d)(U) - d(U) = d(f^{-1}(U)) - d(U) = [U \mid f^{-1}U] = -\nu_{\bar{K}}(f)$.

Let V, L, H be the same as in the definition of central extension (12). Now we construct the group-like monoidal groupoid $C_{V,L}$ with $\pi_0(C_{V,L}) = H$ and $\pi_1(C_{V,L}) = k^*$ and the trivial action of H on k^*.

$$\mathrm{Ob}(C_{V,L}) = \{(h, F) \mid h \in H, F \in ((L \mid hL))\}$$

$$\mathrm{Hom}_{C_{V,L}}((h_1, F_1), (h_2, F_2)) = \begin{cases} \emptyset, & \text{if} \quad h_1 \neq h_2; \\ \mathrm{Hom}_{((L|h_1L))}(F_1, F_2), & \text{if} \quad h_1 = h_2 \end{cases}$$

$$(h_1, F_1) \otimes (h_2, F_2) = (h_1h_2, \delta_{L, h_1L, h_1h_2L}(F_1 \boxtimes (g_1 \cdot F_2)))$$

$$I = (e, Id),$$

where e is the unit element of H and Id is the identy equivalence from $((L \mid L))$.

$$\text{If} \quad X = (h, F), \quad \text{then} \quad X^* = (h^{-1}, F^{-1}).$$

Then $C_{V,L}$ is the group-like monoidal category with the strict associtiavity and unit, i.e., $c_{X,Y,Z}$, d_X and g_X from the axioms of group-like monoidal category are the identity morphisms.

Remark 5 The definition of this group-like monoidal groupoid is very similar to the defintion of central extension of groups above (12) and from [2]. Also it is similar to the definition of group-like monoidal category, constructed from the action of the group $SU(2)$ on $S^3 = SU(2)$ (see [9, §7.3]). On S^3 there exists the gerbe C_p connected with a point $p \in S^3$. The cohomology class of this gerbe gives the generater of the group $H^3(S^3, \mathbb{Z}) = \mathbb{Z}$ (after the choice of orientation of S^3). Then the obstruction to the lifting of the action of the group $SU(2)$ on C_p is a cohomology class from $H^3(SU(2), \mathbb{C}^*)$. And this cohomology class is presented by the group-like monoidal category C as following. Let the point p be the unit e of the group $SU(2)$. $\mathrm{Ob}(C)$ are the pairs (g, γ), where $g \in G$, and γ is a path from e to $g \cdot e$. A morphism from (g, γ_1) to (g, γ_2) is a homotopy class of maps $\sigma : [0,1] \times [0,1] \to S^3$ such that $\sigma(0, y) = 1$, $\sigma(1, y) = g \cdot e$, $\sigma(x, 0) = \gamma_1(x)$ and $\sigma(x, 1) = \gamma_2(x)$. The composition of morphisms is given by vertical juxtaposition of squares, and the tensor product by horizontal juxtaposition of squares.

Proposition 13 *1. For any other $L' \in V$ such that $L' \approx L$ the group-like monoidal groupoid $C_{V,L'}$ is canonically equivalent (in the sence of group-like monoidal groupoids) to the group-like monoidal groupoid $C_{V,L}$.*

2. If for any $h \in H$ we have $hl = L$, then the category $G_{V,L}$ is splited over H (and the class of this category in $H^3(H, k^*)$ is trivial).

Proof We proof item 1. Fix any $S \in \mathrm{Ob}(((L' \mid L)))$. Then a functor of equivalence of group-like monoidal groupoids from $C_{V,L}$ to $C_{V,L'}$ is given as

$$(h, F) \mapsto (h, \delta_{L',hL,hL'}(\delta_{L',L,hL}(S \boxtimes F) \boxtimes (h \circ S^{-1}))).$$

We have $\mathrm{Hom}_{((L'|L))}(S, S') \otimes \mathrm{Hom}_{((L'|L))}(S^{-1}, S'^{-1}) = k^*$ and
$\mathrm{Hom}_{((L'|L))}(h \circ S^{-1}, h \circ S'^{-1}) = \mathrm{Hom}_{((L'|L))}(S^{-1}, S'^{-1})$. Therefore

$$\mathrm{Hom}_{((L',hL'))}(\delta_{L',hL,hL'}(\delta_{L',L,hL}(S \boxtimes F) \boxtimes (h \circ S^{-1})), \delta_{L',hL,hL'}(\delta_{L',L,hL}(S' \boxtimes F) \boxtimes (h \circ S'^{-1})) = k^*.$$

Therefore from the definition of k^*-gerb we have

$$\delta_{L',hL,hL'}(\delta_{L',L,hL}(S \boxtimes F) \boxtimes (h \circ S^{-1})) = \delta_{L',hL,hL'}(\delta_{L',L,hL}(S' \boxtimes F) \boxtimes (h \circ S'^{-1}))$$

Thus the functor above doesn't depend on the choice of $S \in \mathrm{Ob}(((L' \mid L))$.

Now we prove item 2. The splitting is constructed as following: $h \mapsto (h, Id)$, where Id is the identity equivalence from $((L \mid L))$. It finisches the proof of the proposition.

For any commuting elements $f, g, h \in H$ we have from proposition (13) $\phi_{C_{V,L}}(f, g, h) = \phi_{C_{V,L'}}(f, g, h)$. We denote

$$\phi_V(f, g, h) = \phi_{C_{V,L}}(f, g, h).$$

Lemma 1 *Let $f(L) = L$, $g(L) = L$. Then $\phi_V(f, g, h)$ corresponds to the computing of the automorphism of $F \in \mathrm{Ob}(((L \mid hL)))$ from the following diagram*

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & g \circ F \\ \uparrow {\scriptstyle \beta^{-1}} & & \downarrow {\scriptstyle g \circ \beta} \\ f \circ F & \xleftarrow{f \circ \alpha^{-1}} & gf \circ F, \end{array} \tag{13}$$

where $F \in ((L \mid hL))$, $\alpha \in \mathrm{Hom}(F, g \circ F)$ and $\beta \in \mathrm{Hom}(F, f \circ F)$ are any. This diagram corresponds to the computing of the commutator $\phi_{E_{1,h}}$ in central extension (10).

Proof We take $P_f = (f, Id)$, $P_g = (g, Id)$ and $P_h = (h, F)$, where F is any from $((L \mid hL))$. Then we take morphisms $Id \in E_{f,g}$ and any $\alpha \in E_{g,h}$, $\beta \in E_{f,h}$. Then diagram (6) is reduced to diagram (13). It proves the lemma.

Theorem 2 *Let $V = K$ and $H = K^*$. Then for any $f, g, h \in K^*$ we have "the commutator" of lifting of these elements in $C_{V,L}$*

$$\phi_V(f, g, h) = f^{\nu_K(g,h)} g^{\nu_K(h,f)} h^{\nu_K(f,g)} \bmod_{\wp_K} \bmod_{\wp_{\bar{K}}}. \tag{14}$$

Proof We take $L = \mathcal{O}_K$. Both hand sides of (14) are trilinear and skew-symmetric. Let $K = k((t_1))((t_2))$ There is a multiplicative decomposition

$$K^* = t_1^{\mathbb{Z}} \times t_2^{\mathbb{Z}} \times O_K^*,$$

where $O_K^* = k^* + \wp_{\bar{K}} t_1 + \wp_K t_2$. Therefore to prove the theorem it is enough to consider the following cases.

1. Let $f \in O_K^*$, $g \in O_K^*$ and $h \in O_K^*$ or $h = t_1$. Then the right hand side of (14) is equal to 1. We have $fL = L$, $gL = L$, $gL = L$. Therefore from item 2 of proposition 13 we have $\phi_V(f,g,h) = 1$.

2. Let $f \in O_K^*$, $g \in O_K^*$, $h = t_2^{-1}$. Then the right hand side of (14) is equal to 1. Let us check $\phi_V(f,g,t_2^{-1})$ in this case. We have

$$((\mathcal{O}_K \mid t_2^{-1}\mathcal{O}_K)) = \mathrm{Hom}(\mathrm{Det}(0), \mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)) = \mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K).$$

   We take $U = t_2^{-1}O_K/\mathcal{O}_K \in Gr(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)$. Then $fU = U$, $gU = U$. Therefore for any $\Delta \in \mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)$ we have

$$g \circ \Delta(U) = \Delta(g^{-1}U) = \Delta(U),$$

$$f \circ \Delta(U) = \Delta(U)$$

$$fg \circ \Delta(U) = \Delta(U).$$

   Thus we have

$$\mathrm{Hom}_{\mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)}(\Delta, g \circ \mathrm{Tr}) = k^*,$$

$$\mathrm{Hom}_{\mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)}(\Delta, f \circ \Delta) = k^*.$$

   And in diagram (13) we can take $\alpha = id$, $\beta = id$ for $F = \Delta$. Therefore by lemma 1 we obtain $\psi_V(f,g,t_2^{-1}) = 1$.

3. Let $f^{-1} \in O_K^*$, $g = t_1$, $h = t_2^{-1}$. Then the right hand side of (14) is equal to $f \bmod \wp_K \bmod \wp_{\bar{K}}$. Let us check $\phi_V(f^{-1}, t_1, t_2^{-1})$ in this case. We have $((\mathcal{O}_K \mid t_2^{-1}\mathcal{O}_K)) = \mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)$. Let $U = t_2^{-1}O_K/\mathcal{O}_K \in Gr(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)$. Let $\Delta_U \in \mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)$ be induced by U. Then

$$f^{-1} \circ \Delta_U(U) = \Delta_U(fU) = \Delta_U(U) = k^*,$$

$$t_1 \circ \Delta_U(U) = \Delta_U(t_1^{-1}U) = \Delta_U(U) \otimes (U \mid t_1^{-1}U) = (U \mid t_1^{-1}U).$$

   Therefore

$$\mathrm{Hom}_{\mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)}(\Delta, f^{-1} \circ \Delta) = k^*,$$

$$\mathrm{Hom}_{\mathrm{Det}(t_2^{-1}\mathcal{O}_K/\mathcal{O}_K)}(\Delta, t_1 \circ \Delta) = (U \mid t_1^{-1}U).$$

In giagram (13) for $F = \Delta$ we take $\beta = id$ and α induced by an element $t_1^{-1}t_2^{-1} \in (U \mid t_1^{-1}U)$. Then $f(\alpha^{-1})\alpha = f \bmod \wp_K \bmod \wp_{\bar{K}}$. By lemma 1 we obtain that $\phi_V(f^{-1}, t_1, t_2^{-1}) = f \bmod \wp_K \bmod \wp_{\bar{K}}$. The theorem is proved.

For any commuting elements $f, g, h \in H$ we define

$$(f,g,h)_V = \mathrm{sign}_V(f,g,h)\phi_V(f,g,h) \in k^*,$$

where

$$\mathrm{sign}_V(f,g,h) = (-1)^{\psi_V(f,g)\psi_V(f,h) + \psi_V(g,f)\psi_V(g,h) + \psi_V(h,f)\psi_V(h,g) + \psi_V(f,g)\psi_V(f,h)\psi_V(g,h)}.$$

Theorem 3 *Let* $V = K$ *and* $H = K^*$. *Then for any* $f, g, h \in K^*$ *we have*

$$(f,g,h)_V = (f,g,h)_K.$$

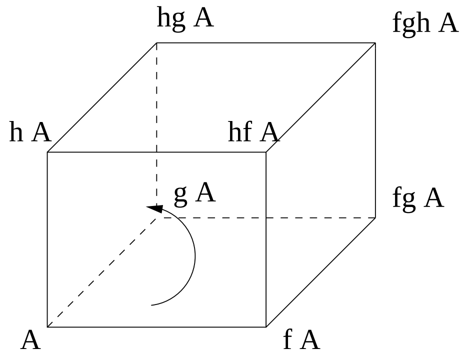
Proof follows from theorems 1 and 2.

Remark 6 We can reformulate the expressions $\psi_V(f,g)$ and $\phi_V(f,g,h)$ in the following geometrical terms. Define a simplicial set where Δ^0-simplices are k-subspaces $A \subset V$ such that $A \approx L$. Δ^1-simplices are elements of $[[A \mid B]]$ with the boundary Δ^0-simplices A and B. We define a combinatorial $\mathbb{Z}$-sheaf $\mathcal{F}_L$ (see [13], [3], [4]) on this simplicial set such that the stalk $\mathcal{F}_{LA}$ is the $\mathbb{Z}$-torsor $[[L \mid A]]$ and every 1-simplex $d \in [[A \mid B]]$ gives an isomorphism $\mathcal{F}_{LA} \to \mathcal{F}_{LB} : [[L \mid A]] \xrightarrow{\odot d} [[L \mid B]]$. The group H acts on this simplicial set. Then $\psi_V(f,g)$ is the monodromy of $\mathcal{F}_L$ on the following square:

$$\begin{array}{ccc} gA & \xleftarrow{g\circ\alpha} & gfA \\ \downarrow \beta & & \uparrow f\circ\beta^{-1} \\ A & \xrightarrow{\alpha} & fA \end{array}$$

This intrepretation is similar to the "template" diagram from [1].

We construct now a bisimplicial set, where Δ^0-simplices are the same as above, Δ^1-simplices are objects of $((A \mid B))$, $\Delta^1 \times \Delta^1$-simplices are elements of $\mathrm{Hom}_{((A|C))}(\delta_{A,B,C}(F_1 \boxtimes F_2), \delta_{A,D,C}(F_3 \boxtimes F_4))$, where the boundary Δ^1-simplices are $F_1 \in ((A \mid B))$, $F_2 \in ((B \mid C))$, $F_3 \in ((A \mid D))$, $F_4 \in ((D \mid C))$. We define a combinatorial gerbe (see [13], [3], [4]) $\mathcal{G}_L$ on this bisimplicial set such that a stalk $\mathcal{G}_{LA}$ is the k^*-gerbe $((L \mid A))$, for every Δ^1-simplex $F \in ((A \mid B))$ we have an equivalence between stalks : $\mathcal{G}_{LA} \longrightarrow \mathcal{G}_{LA} \boxtimes F \xrightarrow{\delta_{L,A,B}} \mathcal{G}_{LB}$, and 2-cells give the natural transformations of this equivalences. We have the action of the group H on this bisimplicial set. Then $\phi_V(f,g,h)$ is the monodromy of combinatorial gerbe $\mathcal{G}_L$ on the following cube:



where we have to choose and fix 3 edges and 3 faces of cube which have the boundary vertice A. Then other edges and faces of the cube are obtained by action of elements f, g, h and their combinations. The arrow gives the orientation of the front face and the cube.

7 Reciprocity laws

Let X be a 2-dimensional algebraic surface over a field k. We assume that k is an algebraically closed field and X is a normal surface. For any point $x \in X$ and any formal irreducible germ C at x of some curve one associates a canonical 2-dimensional local field $K_{x,C}$ (see [16], [11]).

Fix an irreducible projective curve $C \subset X$. Let $t_C \in k(X)$ be the local parametr of the curve C on some open $U \subset X$. Then for any point $x \in C$ we have $K_{x,C} = \hat{k(C)}_x((t_C))$. We introduce an adelic ring:

$$\mathbb{A}_C = \{f_x\} \in \prod_{x \in C} K_{x,C} \quad \text{such that}$$

for $f_{x,C} = \sum\limits_i a^i_{x,C} t^i_C$ we have for each i the collection $\{a^i_{x,C} \in \hat{k(C)}_x\}$ is the usual adele on the curve C. (It means that for the every fixed i for almost all points $x \in C$ we have $a^i_{x,C} \in \mathcal{O}_{\hat{k(C)}_x}$.)

Let $\hat{k(X)}_C$ be the completion of the field of rational functions $k(X)$ with respect to the discrete valuation given by the curve C. For a divisor $D \subset X$ we consider a complex $\mathcal{A}_C(D)$:

$$\hat{k(X)}_C \ \times \ (\prod_{x \in C} (\mathcal{O}_{K_{x,C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))) \cap \mathbb{A}_C \longrightarrow \mathbb{A}_C.$$

Lemma 2 *Let $C \subset X$ be an irreducible projective curve. Then the cohomology groups of the complex $\mathcal{A}_C(D)$ are k-vector spaces with the filtration with finite-dimensional over k factors.*

Proof We denote a sheaf $\mathcal{F} = \mathcal{O}_X(D)$. Then the complex $\mathcal{A}_C(D)$ is the passing with respect to m to projective limit and then with respect to n to injective limit of the adelic complexes of the sheafs $J^n_C \mathcal{F} / J^{n+m}_C$ on the 1-dimensional scheme $(C, \mathcal{O}_X / J^m_C)$. Here $J_C \subset \mathcal{O}_X$ is the ideal sheaf of the curve C, $(C, \mathcal{O}_X / J^m_C)$ is the scheme with the topological space C and the structure sheaf $\mathcal{O}_X / J^m_C$.(About the adelic complexes see [5], [12]). These adelic complexes calculate the cohomology groups of the sheafs $J^n_C \mathcal{F} / J^{n+m}_C$ on the schemes $(C, \mathcal{O}_X / J^m_C)$. And from C is projective curve it follows that these cohomology groups are finite dimensional over the field k spaces. Thus the powers n of the sheaf J_C give the filtration of the cohomology groups of the complex $\mathcal{A}_C(D)$.

After the lemma we can define a $\mathbb{Z}$-torsor

$$\mathrm{Dim}(\mathcal{A}_C(D)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(H^1(\mathcal{A}_C(D)))\ ,\ \mathrm{Dim}(H^0(\mathcal{A}_C(D))))$$

and a k^*-gerbe

$$\mathrm{Det}(\mathcal{A}_C(D)) = \mathrm{Hom}(\mathrm{Det}(H^1(\mathcal{A}_C(D)))\ ,\ \mathrm{Det}(H^0(\mathcal{A}_C(D)))).$$

The divisor $D \subset X$ defines a k-subspace $D = (\prod\limits_{x \in C} \mathcal{O}_{K_{x,C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)) \cap \mathbb{A}_C \subset \mathbb{A}_C$.

Proposition 14 *1. For any two divisors $D, E \subset X$ we have a canonical isomorpism of $\mathbb{Z}$-torsors*

$$[[D \mid E]] \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(D), \mathrm{Dim}(\mathcal{A}_C(E))), \tag{15}$$

and this isomorphism transfers $\odot$-product of $\mathbb{Z}$-torsors to the composition of Hom *for any 3 divisors $D, E, F \subset X$.*

2. For any two divisors $D, E \subset X$ we have a canonical equivalence of k^*-gerbs

$$((D \mid E)) \longrightarrow \mathrm{Hom}(\mathrm{Det}(\mathcal{A}_C(D), \mathrm{Det}(\mathcal{A}_C(E)))$$

and this equivalence transfers $\boxtimes$-product of k^*-gerbes to the composition of Hom *for any 3 divisors $D, E, F \subset X$.*

Proof We proof item 1. We recall that

$$[[D \mid E]] = \varinjlim_{\substack{F \approx D \approx E \\ F \subset D, C \subset E}} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Dim}(D/F),\ \mathrm{Dim}(E/F)\right).$$

Therefore we fix some $F \subset E$, $F \subset D$. We have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Dim}(\mathcal{A}_C(F) \longrightarrow \mathrm{Dim}(\mathcal{A}_C(D) \longrightarrow D/F \longrightarrow 0.$$

The last complex is the space D/F in 0-position. From the long exact cohomological sequence and item 1 of proposition 4 we obtain an isomorphism:

$$\mathrm{Dim}(D/F) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(F), \mathrm{Dim}(\mathcal{A}_C(D))).$$

In the similar way we obtain an isomorphism:

$$\mathrm{Dim}(E/F) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(F), \mathrm{Dim}(\mathcal{A}_C(E))).$$

From these isomorphisms and passing to the direct limit on F we obtain the isomorphism (15). Item 2 of the proposition is proved in the same way.

Fix a point $x \in X$. We introduce an adelic ring:

$$\mathbb{A}_x = \left\{ \{f_C\} \in \prod_{C \ni x} K_{x,C} \quad \text{such that} \quad f_C \in \mathcal{O}_{K_{x,C}} \quad \text{for almost all} \quad C \ni x. \right\}$$

Let $\mathrm{Frac}(\hat{\mathcal{O}}_x)$ be the fraction field of the completion of the local ring $\mathcal{O}_x$ at the point x. For a divisor $D \subset X$ we consider a complex $\mathcal{A}_x(D)$:

$$\mathrm{Frac}(\hat{\mathcal{O}}_x) \ \times \ \prod_{C \ni x} (\mathcal{O}_{K_{x,C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)) \longrightarrow \mathbb{A}_x.$$

Lemma 3 *The cohomology groups of the complex $\mathcal{A}_x(D)$ are k-vector spaces with the filtration with finite-dimensional over k factors.*

Proof We denote a sheaf $\mathcal{F} = \mathcal{O}_X(D)$. Then the complex $\mathcal{A}_x(D)$ is the adelic complex of the sheaf $\mathcal{F}$ on the 1-dimensional scheme $\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x$, see [5], [12]. Therefore the cohomology groups of the complex $\mathcal{A}_x(D)$ coinsides with the cohomology groups $H^*(\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x, \mathcal{F})$. But for any i

$$H^i(\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x,\, \mathcal{F}) = \varinjlim_n \mathrm{Ext}^i(m_x^n\,,\, \hat{\mathcal{F}}_x).$$

(It follows from $H^0(\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x, \mathcal{G}) = \varinjlim_n \mathrm{Hom}(m_x^n, j_*\mathcal{G})$, where $\mathcal{G}$ is any quasicoherent sheaf on $\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x$ and $j : \mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x \hookrightarrow \mathrm{Spec}\, \hat{\mathcal{O}}_x$.)

Now from an exact sequence

$$0 \longrightarrow m_x^n \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/m_x^n\mathcal{F} \longrightarrow 0$$

we obtain

$$H^0(\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x,\, \mathcal{F}) = \hat{\mathcal{F}}_x \quad \text{and} \quad H^1(\mathrm{Spec}\, \hat{\mathcal{O}}_x \setminus x, \mathcal{F}) = \varinjlim_n \mathrm{Ext}^2(\hat{\mathcal{F}}_x/m_x^n\hat{\mathcal{F}}_x\,,\, \hat{\mathcal{F}}_x).$$

The powers of the maximal ideal m_x at the point x give the filtartion of the cohomology groups of the complex $\mathcal{A}_x(D)$.

After the lemma we can define a $\mathbb{Z}$-torsor

$$\mathrm{Dim}(\mathcal{A}_x(D)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(H^1(\mathcal{A}_x(D)))\ ,\ \mathrm{Dim}(H^0(\mathcal{A}_x(D))))$$

and a k^*-gerbe

$$\mathrm{Det}(\mathcal{A}_x(D)) = \mathrm{Hom}(\mathrm{Det}(H^1(\mathcal{A}_x(D)))\ ,\ \mathrm{Det}(H^0(\mathcal{A}_x(D)))).$$

The divisor $D \subset X$ defines a k-subspace $D = \prod\limits_{C \ni x} \mathcal{O}_{K_{x,C}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \subset \mathbb{A}_x$.

Proposition 15 *1. For any two divisors $D, E \subset X$ we have a canonical isomorpism of $\mathbb{Z}$-torsors*

$$[[D \mid E]] \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_x(D), \mathrm{Dim}(\mathcal{A}_x(E))),$$

and this isomorphism transfers $\odot$-product of $\mathbb{Z}$-torsors to the composition of Hom *for any 3 divisors $D, E, F \subset X$.*

2. For any two divisors $D, E \subset X$ we have a canonical equivalence of k^*-gerbs

$$((D \mid E)) \longrightarrow \mathrm{Hom}(\mathrm{Det}(\mathcal{A}_x(D), \mathrm{Det}(\mathcal{A}_x(E)))$$

and this equivalence transfers $\boxtimes$-product of k^*-gerbes to the composition of Hom *for any 3 divisors $D, E, F \subset X$.*

Proof is the same as the proof of proposition 14.

Theorem 4 *1. Fix any irreducible projective curve $C \subset X$ and any divisor $D \subset X$. Let $H = k(\hat{X})^*_C$. Then the central extension $G_{A_C,D}$ and the group-like monoidal groupoid $C_{A_C,D}$ are splited over H.*

2. Fix any point $x \in X$ and any divisor $D \subset X$. Let $H = \mathrm{Frac}(\hat{\mathcal{O}}_x)^*$. Then the central extension $G_{A_x,D}$ and the group-like monoidal groupoid $C_{A_x,D}$ are splitted over H.

Proof We proof item 1. We have an action of the group H on complexes:

$$h \in H \ : \ \mathcal{A}_C(D) \longrightarrow \mathcal{A}_C(hD).$$

This action induces an action on $\mathbb{Z}$-torsors: $\mathrm{Dim}(\mathcal{A}_C(D)) \longrightarrow \mathrm{Dim}(\mathcal{A}_C(hD))$ such that the following diagram is commutativ

$$\begin{array}{ccc} [[D \mid E]] & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(D)), \mathrm{Dim}(\mathcal{A}_C(E))) \\ \downarrow & & \downarrow \\ [[hD \mid hE]] & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(hD)), \mathrm{Dim}(\mathcal{A}_C(hE))). \end{array}$$

We define the central extension $G'_{A_C,D}$ over H. Elements of this group are pairs (h, f) where $h \in H$ and f is from $\mathbb{Z}$-torsor $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(D)), \mathrm{Dim}(\mathcal{A}_C(hD)))$. And the multiplication in this group is given $(h, f_1)(g, f_2) = (hg, f_1 \odot h(f_2))$. The isomorphism of $\mathbb{Z}$-torsors

$$[[D \mid gD]] \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(D)), \mathrm{Dim}(\mathcal{A}_C(hD))).$$

gives the isomorphism of central extensions:

$$G_{A_C,D} \longrightarrow G'_{A_C,D}.$$

But the central extension $G'_{A_C,D}$ has a canonical slitting given by multiplication on $h \in H$ the complex $\mathrm{Dim}(\mathcal{A}_C(D))$, which gives the element from $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Dim}(\mathcal{A}_C(D)), \mathrm{Dim}(\mathcal{A}_C(hD)))$. The splitting of the group-like monoidal groupoid $C_{A_C,D}$ and item 2 of theorem can be proved in the same way.

Corollary

1. *For any $f, g, h \in k(\hat{X})^*_C$ we have*

$$\psi_{A_C}(f, g) = 1 \qquad \textit{and} \qquad (f, g, h)_{A_C} = 1.$$

2. *For any $f, g, h \in \mathrm{Frac}(\hat{\mathcal{O}}_x)^*$ we have*

$$\psi_{A_x}(f, g) = 1 \qquad \textit{and} \qquad (f, g, h)_{A_x} = 1.$$

We consider k-spaces $V_1 = \prod_{l \in I_1} K_l$ and $V_2 = \prod_{l \in I_2} K_j$. We fix any $\mathcal{O}_{K_i}$-lattices $L_i \in K_i$ for $i \in I_1 \cup I_2$. We consider an group H such that H is a subgroup of $G_{K_i/k}$ for every $i \in I_1 \cup I_2$ and for any $h \in H$ for almost all $i \in I_1 \cup I_2$ $hL_i = L_i$.

Proposition 16 (Abstract reciprocity law for $\psi_V(\, , \,)$.) *For any commuting elements f, g from H we have*

$$\psi_{V_1 \oplus V_2}(f,g) = \psi_{V_1}(f,g) + \psi_{V_2}(f,g)$$

Proof It is clear that the central extension $G_{V_1 \oplus V_2, L_1 \oplus L_2}$ is $G_{V_1,L_1} \times_H G_{V_2,L_2}/\mathbb{Z}$. Therefore we obtain the formula in the proposition.

We can recover the following reciprocity law, see [11].

Corollary (Reciprocity laws for $\nu_K(\, , \,)$)

1. *Fix a projective curve $C \subset X$ and $f,g \in k(\hat{X})_C^*$. Then a number of points $x \in C$ with non-zero $\nu_{K_{x,C}}(f,g)$ is finite and*

$$\sum_{x \in C} \nu_{K_{x,C}}(f,g) = 0.$$

2. *Fix a point $x \in X$ and $f,g \in \mathrm{Frac}(\hat{\mathcal{O}}_x)^*$. Then a number of germs C with non-zero $\nu_{K_{x,C}}(f,g)$ is finite and*

$$\sum_{C \ni x} \nu_{K_{x,C}}(f,g) = 0.$$

Proof We prove item 2. We fix some divisor $D \in X$. It defines a k-space $D \subset \mathbb{A}_C$. For almost all $x \in C$ we have both elements $f,g,h \in \mathcal{O}^*_{K_{x,C}}$. Let $V_1 \subset \mathbb{A}_C$ be the sum of $K_{x,C}$ over such x. Let V_2 be the rest part of $\mathbb{A}_C$. The k-space V_2 consists of the finite sum of 2-dimensional local fields. We have $fD \cap V_1 = D \cap V_1$, $gD \cap V_2 = D \cap V_2$. Therefore from item 2 of proposition 12 we have the splitting $G_{V_1, V_1 \cap D}$ over the group generated by f and g. Therefore $\psi_{V_1}(f,g) = 1$. Also for any $K_{x,C} \subset V_1$ we have $\psi_{K_{x,C}}(f,g) = 1$. Now from proposition 16 we have $\psi_V(f,g) = \psi_{V_2}(f,g)$. Now we apply proposition 16 some times to V_2 and from theorem 1 we obtain the reciprocity law. Item 1 can be proved in the same way with the ring $\mathbb{A}_C$.

We consider k-spaces V_1 and V_2 and a group H the same as before proposition 16.

Hypothesis 1 (Abstract reciprocity law for $(\, , \, , \,)_V$) *For any commuting elements f, g, h from H we have*

$$(f,g,h)_{V_1 \oplus V_2} = (f,g,h)_{V_1} (f,g,h)_{V_2}.$$

At least, it is clear that the equality of this hypothesis is true up to some sign. Maybe, is it possible to prove the statement by some induction, i.e., to reduce by means of biextensions and proposition 10 the formula of this hypothesis to the case of 1-dimensional situation, where the analogous formula is true and follows from the long computations in the exterior algebra, see [2] ?

We can recover the following Parshin reciprocity laws, see [11].

Corollary (Reciprocity laws for $(\, , \, , \,)_K$)

1. *Fix a projective curve* $C \subset X$ *and* $f, g, h \in k(\hat{X})^*_C$. *Then a number of points* $x \in C$ *with non-unit* $(f,g,h)_{K_{x,C}}$ *is finite and*

$$\prod_{x \in C} (f,g,h)_{K_{x,C}} = 0.$$

2. *Fix a point* $x \in X$ *and* $f, g, h \in \mathrm{Frac}(\hat{\mathcal{O}}_x)^*$. *Then a number of germs* C *with non-unit* $(f,g,h)_{K_{x,C}}$ *is finite and*

$$\prod_{C \ni x} (f,g,h)_{K_{x,C}} = 0.$$

Proof By corollary of theorem 4 and hypothesis 1 the proof is on the same way as the above proof of the reciprocity laws for $\nu_{K_{x,C}}$.

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