Index criteria for differential algebraic equations arising from linear-quadratic optimal control problems*

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Abstract

The index of DAE systems arising from linear quadratic optimal control problems is considered. Necessary and sufficient conditions ensuring regularity with tractability index one are proved. For problems with regular index zero or index one DAEs to be controlled, the DAE of the control problem is shown to be regular with tractability index one or three, depending on whether the control coefficient R is singular. Moreover, it is shown that, if the control problem DAE is regular with index one, and if the leading term of the DAE to be controlled is given by one full-column-rank- and one full-row-rank matrix, then it has a Hamiltonian inherent explicit ODE.
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1 Introduction

This paper deals with systems of differential algebraic equations (DAEs)

$$\begin{aligned} A(t)(B(t)x(t))' &= C(t)x(t) + D(t)u(t), && (1.1)\\ -B(t)^*(A(t)^*\psi(t))' &= W(t)x(t) + C(t)^*\psi(t) + S(t)u(t), && (1.2)\\ 0 &= S(t)^*x(t) + D(t)^*\psi(t) + R(t)u(t),\ t \in [0,T], && (1.3) \end{aligned}$$

which arise in optimal control problems given by a quadratic cost functional $J(u,x)$ subject to the constraint

$$\begin{aligned} A(t)(B(t)x(t))' &= C(t)x(t) + D(t)u(t), t \in [0,T], && (1.4)\\ A(0)B(0)x(0) &= z_0. && (1.5) \end{aligned}$$

If $A(t) \equiv I, B(t) \equiv I$, then equation (1.4) is an explicit ordinary differential equation (ODE) and the resulting system (1.1)-(1.3) is well known to be a semi-explicit DAE that has index one, supposed that the coefficient $R(t)$ remains nonsingular. However, if $R(t) \equiv 0, S(t) \equiv 0$, but $D(t)^*W(t)D(t)$ remains nonsingular, then the DAE (1.1)-(1.3)

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has index three (e.g. [BrCaPe]).

If the equation (1.4) to be controlled is a DAE itself, the situation will be more complicated. The investigation of control problems with DAEs or descriptor systems to be controlled began in parallel with the development of the DAE theory. First, systems (1.4) with constant coefficients (e.g. [BeLa], [Jo], [Me]) and systems (1.4) with constant A, B but time-varying C, D ([Ku]) were studied , later also systems with variable coefficients (e.g. [KuMe2], [KuMä]). While the investigations of DAEs (1.4) in [KuMe2] are based on special coefficients $B(t) \equiv I$, the paper [KuMä] deals with DAEs (1.4) for which $A(t) \equiv I$ holds true. In both cases, one does not obtain a system of the form (1.1)-(1.3) for the control problem, instead, equation (1.2) contains additional, distinct terms with derivatives of coefficients and projectors, respectively. Both papers provide sufficient conditions for the (modified) DAE (1.1)-(1.3) to be of index one.
According to the proposal of [BaMä] to formulate the leading coefficient of a DAE (1.4) by means of two well-matched coefficients A and B, in [Mä1] the system (1.1)-(1.3) for control problems with variable coefficient was derived, starting from (1.4) with a properly stated leading term.
In the present paper the concept of the tractability index from [Mä2] is applied to the DAE (1.1)-(1.3) in order to characterize its index.
In Section 3 we succeed in formulating conditions in terms of the problem data that are necessary and sufficient for the DAE (1.1)-(1.3) to be regular with tractability index one and, hence, to be accessible for numerical standard methods. Here, the DAE (1.4) to be controlled is not assumed to be regular or to have an index.
In Section 4 we introduce self-adjoint DAEs. We show them to have a Hamiltonian inherent regular ODE, provided that they are regular with index one and their leading terms are given by full-column-rank and full-row-rank matrices. As a consequence, also the regular DAEs (1.1)-(1.3) have a Hamiltonian inherent regular ODE under reasonable conditions.
In Section 5 the controlled DAE (1.4) itself is assumed to be regular with index zero or one and only the differentiable solution components, i.e., the state variables, are controlled. Generalizing the above statements for the case of $A = I, B = I$, we show that the resulting DAE (1.1)-(1.3) is regular with tractability index one if and only if R is nonsingular. In case of singular R, the DAE (1.1)-(1.3) cannot be regular with index two, not even in the general case. Reasonable conditions yield regularity with index three.
Finally, in Section 6 we show that an analogous generalization of the statements of Section 5 does not apply to regular DAEs (1.4) with index two.

2 Fundamentals

We consider the quadratic cost functional

$$J(u,x) := \frac{1}{2}\langle x(T), Vx(T)\rangle + \frac{1}{2}\int_0^T \left\langle \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\rangle dt, \tag{2.1}$$

with fixed time $T > 0$, to be minimized on solutions of the linear differential algebraic equation (DAE)

$$A(t)(B(t)x(t))' = C(t)x(t) + D(t)u(t),\ t \in \mathcal{I} := [0,T], \tag{2.2}$$

subject to the initial condition

$$A(0)B(0)x(0) = z_0. \tag{2.3}$$

The coefficients in (2.1), (2.2) are matrices $W(t) \in L(\mathbb{R}^m), R(t) \in L((\mathbb{R}^l), S(t) \in L(\mathbb{R}^l, \mathbb{R}^m)$, $A(t) \in L(\mathbb{R}^n, \mathbb{R}^k), B(t) \in L(\mathbb{R}^m, \mathbb{R}^n), C(t) \in L(\mathbb{R}^m, \mathbb{R}^k), D(t) \in L(\mathbb{R}^l, \mathbb{R}^k), t \in \mathcal{I}$, which depend continuously on t, and $V \in L(\mathbb{R}^m)$.
The matrix V is symmetric and positive semidefinite.
For all $t \in \mathcal{I}$, $W(t)$ and $R(t)$ are symmetric, and $\begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix}$ is positive semidefinite.
Here, $*$ denotes the transposition, $< .,. >$ stands for standard scalar products both in $L(\mathbb{R}^m)$ and $L(\mathbb{R}^{m+l})$.

The leading term of the DAE (2.2) is supposed to be properly stated in the sense that (cf. [BaMä]) the decomposition

$$ker A(t) \oplus im B(t) = \mathbb{R}^n, \; t \in \mathcal{I}, \tag{2.4}$$

holds true, and both subspaces forming this direct sum are spanned by basis functions that are continuously differentiable on $\mathcal{I}$.
A *solution* of the DAE (2.2) is a function $x : \mathcal{I} \to \mathbb{R}^m$ that belongs to the function space

$$C^1_B(\mathcal{I}, \mathbb{R}^m) := \{x \in C(\mathcal{I}, \mathbb{R}^m) : Bx \in C^1(\mathcal{I}, \mathbb{R}^n)\}$$

and satisfies (2.2) pointwise on $\mathcal{I}$.
An *admissible control* is a continuous function $u : \mathcal{I} \to \mathbb{R}^l$ such that the corresponding initial value problem (IVP) (2.2), (2.3) has a solution.
The number k of equations in (2.2) does not necessarily coincide with the number m of components of x. When $k = m$, we say for short that the DAE is quadratic.

Below, for an optimality criterion the following boundary value problem (BVP) will be used:

$$\begin{aligned} A(t)(B(t)x(t))' &= C(t)x(t) + D(t)u(t), && (2.5)\\ -B(t)^*(A(t)^*\psi(t))' &= W(t)x(t) + C(t)^*\psi(t) + S(t)u(t), && (2.6)\\ 0 &= S(t)^*x(t) + D(t)^*\psi(t) + R(t)u(t), t \in \mathcal{I}, && (2.7)\\ A(0)B(0)x(0) &= z_0, && (2.8)\\ B(T)^*A(T)^*\psi(T) &= Vx(T). && (2.9) \end{aligned}$$

In turn, system (2.5)-(2.7) forms a quadratic homogeneous DAE

$$\tilde{A}(t)(\tilde{B}(t)\tilde{x}(t))' = \tilde{C}(t)\tilde{x}(t), \; t \in \mathcal{I}, \tag{2.10}$$

with properly stated leading term. Namely, we have

$$\tilde{x}(t) := \begin{pmatrix} x(t) \\ \psi(t) \\ u(t) \end{pmatrix}, \quad \tilde{A}(t) := \begin{pmatrix} A(t) & 0 \\ 0 & B(t)^* \\ 0 & 0 \end{pmatrix}, \quad \tilde{B}(t) := \begin{pmatrix} B(t) & 0 & 0 \\ 0 & -A(t)^* & 0 \end{pmatrix},$$

$$\tilde{C}(t) := \begin{pmatrix} C(t) & 0 & D(t) \\ W(t) & C(t)^* & S(t) \\ S(t)^* & D(t)^* & R(t) \end{pmatrix}, \quad t \in \mathcal{I},$$

hence, $\tilde{m} = \tilde{k} = m + k + l, \tilde{n} = 2n, ker\tilde{A}(t) = kerA(t) \times (imB(t))^{\perp}, im\tilde{B}(t) = imB(t) \times (kerA(t))^{\perp}$.

Theorem 2.1 *If the triple $x_* \in C^1_B(\mathcal{I}, I\!R^m), \psi_* \in C^1_{A^*}(\mathcal{I}, I\!R^k), u_* \in C(\mathcal{I}, I\!R^l)$ solves the BVP (2.5)-(2.9), then u_* is an optimal control of the problem (2.1)-(2.3) and x_* is the corresponding optimal trajectory.*

Proof: Let $u \in C(\mathcal{I}, I\!R^l)$ be an arbitrarily admissible control and $x \in C^1_B(\mathcal{I}, I\!R^m)$ a corresponding solution of the IVP (2.2), (2.3). The bilinearity of the scalar products allows to compute the variation

$$\begin{aligned} J(u,x) - J(u_*, x_*) &= \frac{1}{2}\langle x(T) - x_*(T), V(x(T) - x_*(T))\rangle \\ &+\frac{1}{2}\int_0^T \left\langle \begin{pmatrix} x(t) - x_*(t) \\ u(t) - u_*(t) \end{pmatrix}, \begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x(t) - x_*(t) \\ u(t) - u_*(t) \end{pmatrix} \right\rangle dt, \end{aligned}$$

thus, due to positive semidefiniteness, $J(u,x) - J(u_*, x_*) \geq 0$. ◇

By means of the Hamiltonian function

$$\begin{aligned} H(x, \psi, u, t) &:= \langle \psi, C(t)x + D(t)u \rangle \\ &+\frac{1}{2}\{\langle x, W(t)x\rangle + 2\langle S(t)u, x\rangle + \langle u, R(t)u\rangle\}, \\ &x \in I\!R^m, \psi \in I\!R^k, u \in I\!R^l, t \in \mathcal{I}, \end{aligned}$$

system (2.5)-(2.7) may be rewritten as

$$\begin{aligned} A(t)(B(t)x(t))' &= H_\psi(x(t), \psi(t), u(t), t)^*, & (2.11) \\ -B(t)^*(A(t)^*\psi(t))' &= H_x(x(t), \psi(t), u(t), t)^*, & (2.12) \\ 0 &= H_u(x(t), \psi(t), u(t), t)^*, \ t \in \mathcal{I}. & (2.13) \end{aligned}$$

Remark 2.2 In [KuMe2] and [KuMä], DAEs of the forms

$$A(t)x'(t) = C(t)x(t) + D(t)u(t), t \in \mathcal{I} \quad (2.14)$$

resp.

$$(B(t)x(t))' = C(t)x(t) + D(t)u(t), t \in \mathcal{I} \quad (2.15)$$

are considered instead of (2.2). In both cases, the resulting optimality BVPs contain invisible terms with derivatives of coeffcients and projectors, respectively. The nicer form (2.5)-(2.7), which is well-known in case of constant coefficients, appears as a benefit from stating the leading term of a DAE by means of two well-matched matrix functions.
In [Mä1], BVP (2.5)-(2.9) as well as system (2.11)-(2.13) are formulated for the optimal control problem (2.1)-(2.3) with $m = k$.

3 Index one criteria for the DAE (2.10)

In this section we give necessary and sufficient conditions for the DAE (2.10) to be regular with tractability index one in terms of the coefficients in the given problem (2.1)-(2.3). Here we do not suppose that an index is defined for the DAE (2.2).

Put $G_0(t) := A(t)B(t), t \in \mathcal{I}$. Since the DAE (2.2) has a properly stated leading term, $G_0(t)$ has constant rank r_0, and $N_0(t) := kerG_0(t)$ has constant dimension $m - r_0$.
For $t \in \mathcal{I}$ let $Q_0(t) \in L(\mathbb{R}^m)$ and $Q_{*0}(t) \in L(\mathbb{R}^k)$ denote the orthoprojections onto $N_0(t)$ and $N_{*0}(t) := kerG_0(t)^* = (imG_0(t))^\perp$, respectively. Both, $Q_0(t)$ and $Q_{*0}(t)$ depend continuously on t. Put $P_0(t) := I - Q_0(t), P_{*0}(t) := I - Q_{*0}(t)$. Introduce further

$$G_1(t) := G_0(t) - C(t)Q_0(t),\ t \in \mathcal{I}.$$

By Definition A.1 and Lemma A.2, the DAE (2.10) is regular with tractability index one if the subspaces $\tilde{N}_0(t) := ker\tilde{G}_0(t)$ and $\tilde{S}_0(t) := \{\tilde{x} \in \mathbb{R}^{m+k+l} : \tilde{C}(t)\tilde{x} \in im\tilde{G}_0(t)\}$ with $\tilde{G}_0(t) := \tilde{A}(t)\tilde{B}(t)$ for all $t \in \mathcal{I}$ intersect transversally. This is why we take a closer look at the elements of $\tilde{N}_0(t) \cap \tilde{S}_0(t)$.
$\tilde{x} = \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_0(t) \cap \tilde{S}_0(t)$ means in detail

$$G_0(t)x = 0, \quad G_0(t)^*\psi = 0, \tag{3.1}$$
$$C(t)x + D(t)u = G_0(t)w_1, \tag{3.2}$$
$$W(t)x + C(t)^*\psi + S(t)u = G_0(t)^*w_2, \tag{3.3}$$
$$S(t)^*x + D(t)^*\psi + R(t)u = 0, \tag{3.4}$$

with some $w_1 \in \mathbb{R}^m, w_2 \in \mathbb{R}^l$.
Observe that, due to (3.1), the relations

$$\langle G_0(t)w_1, \psi\rangle = \langle w_1, G_0(t)^*\psi\rangle = 0,$$
$$\langle G_0(t)^*w_2, x\rangle = \langle w_2, G_0(t)x\rangle = 0$$

are valid. Therefore, taking the inner products of (3.2), (3.3), (3.4) with $-\psi, x$ and u, respectively, and adding the results, we obtain

$$\langle W(t)x, x\rangle + \langle S(t)u, x\rangle + \langle S(t)^*x, u\rangle + \langle R(t)u, u\rangle = 0,$$

that is,

$$\left\langle \begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \begin{pmatrix} x \\ u \end{pmatrix} \right\rangle = 0. \tag{3.5}$$

For a symmetric, positive semidefinite matrix G, the relation $\langle Gz, z\rangle = 0$ is equivalent to $Gz = 0$. We find that (3.5) is equivalent to system

$$W(t)x + S(t)u = 0, \quad S(t)^*x + R(t)u = 0. \tag{3.6}$$

Taking (3.6) into account, we derive condition

$$\begin{pmatrix} G_0(t)^* \\ Q_0(t)C(t)^* \\ D(t)^* \end{pmatrix} \psi = 0 \tag{3.7}$$

from (3.1), (3.3), (3.4). It turns out that, for all $\tilde{x} = \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_0(t) \cap \tilde{S}_0(t)$, condition (3.7) is valid. Derive

$$\begin{aligned} ker \begin{pmatrix} G_0(t)^* \\ Q_0(t)C(t)^* \\ D(t)^* \end{pmatrix} &= (im(G_0(t), C(t)Q_0(t), D(t)))^{\perp} \\ &= (im(G_0(t) - C(t)Q_0(t), D(t)))^{\perp} = (im(G_1(t), D(t)))^{\perp}. \end{aligned}$$

Lemma 3.1 *The condition*

$$im(G_1(t)\ D(t)) = I\!R^k,\ t \in \mathcal{I}, \tag{3.8}$$

is necessary for the DAE (2.10) to be regular with tractability index one.

Proof: If there is a $\bar{t} \in \mathcal{I}$ such that (3.8) is not true at this point, then there is a nontrivial $\psi \in I\!R^k$,

$$\psi \in ker \begin{pmatrix} G_0(\bar{t})^* \\ Q_0(\bar{t})C(\bar{t})^* \\ D(\bar{t})^* \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ \psi \\ 0 \end{pmatrix} \in I\!R^{m+k+l}$$

is a nontrivial element of $\tilde{N}_0(\bar{t}) \cap \tilde{S}_0(\bar{t})$. ◇

Next, if (3.8) is valid, $\begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_0(t) \cap \tilde{S}_0(t)$ implies, by (3.7), that $\psi = 0$, while x and u satisfy

$$\begin{pmatrix} G_0(t) & 0 \\ Q_{*0}(t)C(t) & Q_{*0}(t)D(t) \\ Q_0(t)W(t) & Q_0(t)S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0. \tag{3.9}$$

If this equation has a nontrivial solution $\begin{pmatrix} x \\ u \end{pmatrix} \in I\!R^{m+l}$, then $\begin{pmatrix} x \\ 0 \\ u \end{pmatrix} \in I\!R^{m+k+l}$ is a nontrivial element of the intersection $\tilde{N}_0(t) \cap \tilde{S}_0(t)$.

Lemma 3.2 *The condition*

$$im \begin{pmatrix} G_0(t)^* - C(t)^*Q_{*0}(t) & W(t)Q_0(t) & S(t) \\ -D(t)^*Q_{*0}(t) & S^*(t)Q_0(t) & R(t) \end{pmatrix} = I\!R^m \times I\!R^l, t \in \mathcal{I}, \tag{3.10}$$

is necessary for the DAE (2.10) to be regular with tractability index one.

Proof: With $\mathcal{N}(t) := ker \begin{pmatrix} G_0(t) & 0 \\ Q_{*0}(t)C(t) & Q_{*0}(t)D(t) \\ Q_0(t)W(t) & Q_0(t)S(t) \\ S(t)^* & R(t) \end{pmatrix}$ we have $\mathcal{N}^\perp = im \begin{pmatrix} G_0^* & C^*Q_{*0} & WQ_0 & S \\ 0 & D^*Q_{*0} & S^*Q_0 & R \end{pmatrix}$, and

$$\begin{aligned} G_0^* w_1 + C^*Q_{*0}w_2 &= (G_0^* - C^*Q_{*0})(P_{*0}w_1 - Q_{*0}w_2), \\ D^*Q_{*0}w_2 &= -D^*Q_{*0}(P_{*0}w_1 - Q_{*0}w_2), \end{aligned}$$

hence, $\mathcal{N}^\perp = im \begin{pmatrix} G_0^* - C^*Q_{*0} & WQ_0 & S \\ -D^*Q_{*0} & S^*Q_0 & R \end{pmatrix}$. Consequently, condition (3.10) means that $\mathcal{N} = 0$ is valid. $\diamond$

If both conditions (3.8) and (3.10) hold, then (3.1) - (3.4) yield $\psi = 0, u = 0, x = 0$, that is, $\tilde{N}_0(t) \cap \tilde{S}_0(t) = 0$. This proves the following theorem.

Theorem 3.3 *The DAE (2.10) is regular with tractability index one if and only if the conditions (3.8) and (3.10) are satisfied.*

If the DAE (2.2) has the semi-explicit form

$$\begin{aligned} A_1(t)x_1'(t) &= C_{11}(t)x_1(t) + C_{12}(t)x_2(t) + D_1(t)u(t), \\ 0 &= C_{21}(t)x_1(t) + C_{22}(t)x_2(t) + D_2(t)u(t), \end{aligned}$$

where $A_1(t) \in L(I\!R^n)$ is nonsingular, $A = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$, $B = (I \;\; 0)$, things become more transparent. Then, (3.8) resp. (3.10) mean nothing else but

$$\begin{aligned} & im(C_{22}(t) \;\; D_2(t)) = I\!R^{k-n} \quad \text{and} \\ & im \begin{pmatrix} C_{22}(t)^* & W_{22}(t) & S_2(t) \\ D_2(t)^* & S_2(t)^* & R(t) \end{pmatrix} = I\!R^{k-n} \times I\!R^l, \end{aligned}$$

or, equivalently,

$$ker \begin{pmatrix} C_{22}(t)^* \\ D_2(t)^* \end{pmatrix} = 0, \quad ker \begin{pmatrix} C_{22}(t) & D_2(t) \\ W_{22}(t) & S_2(t) \\ S_2(t)^* & R(t) \end{pmatrix} = 0.$$

Both conditions together imply the matrix

$$\hat{R}(t) := \begin{pmatrix} 0 & C_{22}(t) & D_2(t) \\ C_{22}(t)^* & W_{22}(t) & S_2(t) \\ D_2(t)^* & S_2(t)^* & R(t) \end{pmatrix}$$

to be invertible, and vice versa.
Note that this sort of conditions plays an important role in [BeLa], where constant coefficient problems are considered. The full-row-rank condition for $(C_{22}D_2)$ means in [BeLa] the controllability at ∞, and it is shown that the full-row-rank condition for $(C_{22}D_2)$ and

the invertibility of $\begin{pmatrix} W_{22} & S_2 \\ S_2^* & R \end{pmatrix}$ together are sufficient for $\hat{R}$ to be invertible. While in [BeLa] the matrix pencil $\lambda AB - C$ is assumed to be regular and R is positive definite, we realize now that we can do without those assumptions, our result is valid in the case of variable coefficients, too, and we have precise invertibility conditions for $\hat{R}(t)$.

Remark 3.4 In [KuMä], in a slightly different context, the conditions (3.8) and (3.10) are realized as sufficient index one conditions.
In [KuMe2], for problem (2.1), (2.14), (2.3), with $R(t)$ nonsingular, the corresponding necessary and sufficient index one condition is formulated somewhat implicitly via symplectic transformations into a semi-explicit form.

Corollary 3.5 *Let (3.8) be valid, and let* $\begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix}$ *be positive definite on* $N_0(t) \times U(t), U(t) := \{u \in I\!R^l : D(t)u \in imG_1(t)\},\ t \in \mathcal{I}$. *Then the DAE (2.10) is regular with tractability index one.*

Proof: We show that condition (3.10) is fulfilled. Let $x \in I\!R^m, u \in I\!R^l$ be a solution of the homogeneous system (3.9), i.e.,

$$G_0(t)x = 0, \quad Q_{*0}(t)C(t)x + Q_{*0}(t)D(t)u = 0, \tag{3.11}$$

$$\begin{pmatrix} Q_0(t)W(t) & Q_0(t)S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0. \tag{3.12}$$

From (3.11) we know that $x = Q_0(t)x, C(t)x + D(t)u = G_0(t)w$, with a certain $w \in I\!R^m$, hence $D(t)u = G_0(t)w - C(t)Q_0(t)x = (G_0(t) - C(t)Q_0(t))(P_0(t)w + x) = G_1(t)(P_0(t)w + x)$ are valid, i.e., $u \in U(t)$.
Moreover, due to (3.12) we obtain

$$\left\langle \begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \begin{pmatrix} x \\ u \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} Q_0(t)W(t) & Q_0(t)S(t) \\ S(t)^* & R(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \begin{pmatrix} x \\ u \end{pmatrix} \right\rangle = 0,$$

and consequently $x = 0, u = 0$. $\diamond$

Corollary 3.6 *If* $l = k$, *and if, for all* $t \in \mathcal{I}$, $\begin{pmatrix} W(t) & S(t) \\ S(t)^* & R(t) \end{pmatrix}$ *is positive definite and* $D(t)$ *is nonsingular, then both conditions (3.8) and (3.10) are satisfied.*

It should be stressed that Theorem 3.3 and the Corollaries 3.5 and 3.6 do not suppose regularity or special index properties of the DAE (2.2) that is to be controlled. They apply independently of those properties.
On the other hand, if the DAE (2.2) is regular with tractability index one or zero, then $k = m$ holds, and the matrix $G_1(t)$ remains nonsingular on $\mathcal{I}$. Therefore, condition (3.8) is valid a priori for those DAEs.
If (2.2) is a higher index regular DAE, then $rankG_1(t) < k$, and some extra assumptions on $D(t)$ are needed for meeting (3.8) (cf. Section 6 below).

If (2.2) is no longer a regular DAE in the sense of Definition A.1, but if it has tractability index one, then, to satisfy the conditions (3.8) and (3.10), additional properties have to be given for the coefficients.

Example 3.7 In [KuMe1], [KuMe2], quite general controlled linear DAEs are transformed into the form (cf. Example A.4)

$$\left.\begin{array}{rcll} x_1' & = & C_{13}x_3 & + \; D_1u \\ 0 & = & x_2 & + \; D_2u \\ 0 & = & & \; D_3u \end{array}\right\}. \tag{3.13}$$

The DAE (3.13) has strangeness index zero ([KuMe1]) and, provided that imC_{13} is constant, tractability index one in the sense of Definition A.3. Put $S = 0$, and let R be nonsingular. In this special case, we have

$$(G_1\ D) = \begin{pmatrix} I & 0 & -C_{13} & D_1 \\ 0 & -I & 0 & D_2 \\ 0 & 0 & 0 & D_3 \end{pmatrix},$$

$$\begin{pmatrix} G_0^* - C^*Q_{*0} & WQ_0 & 0 \\ -D^*Q_{*0} & 0 & R \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 & W_{12} & W_{13} & 0 \\ 0 & -I & 0 & 0 & W_{22} & W_{23} & 0 \\ 0 & 0 & 0 & 0 & W_{32} & W_{33} & 0 \\ 0 & -D_2^* & -D_3^* & 0 & 0 & 0 & R \end{pmatrix},$$

and (3.8) is equivalent to $imD_3^\perp = kerD_3^* = 0$, while (3.10) means $im(W_{32}W_{33})^\perp = ker\begin{pmatrix} W_{32}^* \\ W_{33}^* \end{pmatrix} = ker\begin{pmatrix} W_{23} \\ W_{33} \end{pmatrix} = 0$. ◇

We finish this section with a corollary that reflects a simple fact well-known for explicit ODEs (2.2) with $A(t) = I, B(t) = I$.

Corollary 3.8 *If the DAE (2.2) is regular with index zero, then the DAE (2.10) is regular with index one if and only if $R(t)$ is positive definite for all $t \in \mathcal{I}$.*

4 Adjoint and self-adjoint DAEs

In this section we introduce self-adjoint DAEs and show that the explicit inherent ODE of a selfadjoint index one DAE is a Hamiltonian one. Then the system (2.10) is transformed into a self-adjoint DAE by simply exchanging x and ψ.

For the DAE with properly stated leading term

$$A(t)(B(t)x(t))' = C(t)x(t), \quad t \in \mathcal{I}, \tag{4.1}$$

the adjoint equation is (cf. [BaMä])

$$-B(t)^*(A(t)^*y(t))' = C(t)^*y(t), \;\; t \in \mathcal{I}. \tag{4.2}$$

This is discussed in [BaMä] for $k = m$, but it keeps its value for $k \neq m$, too. If $x \in C_B^1(\mathcal{I}, I\!R^m)$ and $y \in C_{A^*}^1(\mathcal{I}, I\!R^k)$ are solutions of (4.1) resp. (4.2), then the derivative $(y(t)^*A(t)B(t)x(t))'$ vanishes identically, i.e., the generalized Lagrange identity

$$y^*(t)A(t)B(t)x(t) = \langle A(t)^*y(t), B(t)x(t)\rangle \equiv const \tag{4.3}$$

takes place.
The maps $\mathcal{L} : C^1_B(\mathcal{I}, I\!R^m) \to C(\mathcal{I}, I\!R^k)$, $\mathcal{L}^* : C^1_{A^*}(\mathcal{I}, I\!R^k) \to C(\mathcal{I}, I\!R^m)$ defined by

$$\begin{aligned} \mathcal{L}x &:= A(Bx)' - Cx, \; x \in C^1_B(\mathcal{I}, I\!R^m), \\ \mathcal{L}^*y &:= -B^*(A^*y)' - C^*y, \; y \in C^1_{A^*}(\mathcal{I}, I\!R^k) \end{aligned}$$

are linear and bounded if the involved function spaces are equipped with natural norms. With the product $(u, v) := \int_0^T \langle u(t), v(t) \rangle dt$ it holds that $(\mathcal{L}x, y) = (x, \mathcal{L}^*y)$ for all $x \in C^1_B(\mathcal{I}, I\!R^m), B(0)x(0) = 0, B(T)x(T) = 0$, and all $y \in C^1_{A^*}(\mathcal{I}, I\!R^k)$.

Definition 4.1 *The DAE (4.1) is said to be self-adjoint if* $m = k, C^1_B(\mathcal{I}, I\!R^m) = C^1_{A^*}(\mathcal{I}, I\!R^k)$, *and* $\mathcal{L} = \mathcal{L}^*$.

Remark 4.2 Self-adjoint DAEs of the form

$$iB(t)^*(B(t)x(t))' - C(t)x(t) = 0,$$

with $C(t) = C(t)^*$, $C(t)$ and $B(t)$ with complex entries, are first discussed in [Ab. et. all.]

Theorem 4.3 *If* $m = k, J \in L(I\!R^n)$ *is such that* $J^* = -J, J^2 = -I$ *and* $A(t) = B(t)^*J, C(t) = C(t)^*, \; t \in \mathcal{I}$, *then the DAE (4.1) is self-adjoint.*

Proof: Because of $m = k, A^* = -JB, B = JA^*$, the function spaces $C^1_B(\mathcal{I}, I\!R^m)$ and $C^1_{A^*}(\mathcal{I}, I\!R^k)$ coincide. For $x \in C^1_B(\mathcal{I}, I\!R^m)$, we derive

$$\begin{aligned} \mathcal{L}x &= A(Bx)' - Cx = B^*J(JA^*x)' - C^*x = -B^*(A^*x)' - C^*x \\ &= \mathcal{L}^*x. \end{aligned}$$

◇

For an index one DAE (4.1), the so-called inherent regular explicit ODE is uniquely determined as (cf. [BaMä])

$$u' = K'u + BG_1^{-1}CB^-u, \tag{4.4}$$

where $K(t) \in L(I\!R^n)$ denotes the continuously differentiable projector onto $imB(t)$ along $kerA(t)$ that realizes the decomposition (2.4), and the generalized inverse $B(t)^-$ is determined by $B(t)B(t)^- = K(t)$, $B(t)^-B(t) = P_0(t)$, $P_0(t) := I - Q_0(t)$, $t \in \mathcal{I}$.

Theorem 4.4 *Let* $m = k, J \in L(I\!R^n), J^* = -J, J^2 = -I, A(t) = B(t)^*J, C(t) = C(t)^*, kerA(t) = 0, t \in \mathcal{I}$, *and let the DAE (4.1) be regular with tractability index one. Then, the inherent ODE (4.4) is of the form*

$$u' = J^*E, \qquad \text{with} \qquad E = E^*. \tag{4.5}$$

Proof: Since $kerA(t) = 0$, due to the properties of the leading term, it holds that $K(t) = I, t \in \mathcal{I}$. Hence, (4.4) simplifies to

$$u' = BG_1^{-1}CB^-u.$$

The index one property yields $imQ_0CQ_0 = imQ_0 = N_0, kerQ_0CQ_0 = kerQ_0 = N_0^{\perp}$. Q_0 is the orthoprojector onto N_0, thus

$$(Q_0CQ_0)^+(Q_0CQ_0) = Q_0, \quad (Q_0CQ_0)(Q_0CQ_0)^+ = Q_0, \tag{4.6}$$

where "+" indicates the Moore-Penrose inverse.
Because of

$$\begin{aligned} G_0 &= AB = AJA^* = -(AJA^*)^* \\ &= -G_0^* \end{aligned}$$

it holds that $N_0 = N_{*0}$, hence $Q_0 = Q_{*0}$.
Additionally, since

$$\begin{array}{rclclclcl} imAJA^* &=& imA &=& imP_{*0} &=& N_{*0}^{\perp} &=& N_0^{\perp}, \\ kerAJA^* &=& kerA^* &=& imQ_{*0} &=& N_{*0} &=& N_0, \end{array}$$

it holds that

$$(AJA^*)^+(AJA^*) = P_0, \quad (AJA^*)(AJA^*)^+ = P_0.$$

Compute $G_1 = G_0 - CQ_0 = G_0 - Q_0CQ_0 - P_0CQ_0 = \mathcal{G}_1(I - G_0^+CQ_0)$ with

$$\begin{aligned} \mathcal{G}_1 &:= G_0 - Q_0CQ_0 = AJA^* - Q_0CQ_0, \\ \mathcal{G}_1^{-1} &:= (AJA^*)^+ - (Q_0CQ_0)^+. \end{aligned} \tag{4.7}$$

We also have $B^-B = P_0, BB^- = I$, and since P_0 is the orthoprojector, B^- is the Moore-Penrose inverse, i.e., $B^- = B^+$. With $B = JA^*, J^* = J^{-1}$, we find $B^+ = A^{*+}J^{-1} = A^{*+}J^* = [(A^*A)^{-1}A^*]^*J^* = A(A^*A)^{-1}J^*$.
Now we derive

$$\begin{aligned} M &:= BG_1^{-1}CB^+ = JA^*G_1^{-1}CA(A^*A)^{-1}J^* \\ &= JA^*(I + G_0^+CQ_0)\mathcal{G}_1^{-1}CA(A^*A)^{-1}J^* \\ &= (JA^* + JA^*G_0^+CQ_0)\mathcal{G}_1^{-1}CA(A^*A)^{-1}J^*, \end{aligned}$$

and, with $JA^*G_0^+ = JA^*(AJA^*)^+ = (A^*A)^{-1}A^*AJA^*(AJA^*)^+ = (A^*A)^{-1}A^*$, it follows that

$$\begin{aligned} M &= (JA^* + (A^*A)^{-1}A^*CQ_0)\mathcal{G}_1^{-1}CA(A^*A)^{-1}J^* \\ &= (A^*A)^{-1}A^*(AJA^* + CQ_0)\mathcal{G}_1^{-1}CA(A^*A)^{-1}J^* \\ &= -J\{J(A^*A)^{-1}A^*(AJA^* + CQ_0)\mathcal{G}_1^{-1}CA(A^*A)^{-1}J^*\} \\ &= -J\{J(A^*A)^{-1}A^*\mathcal{N}A(A^*A)^{-1}J^*\}, \end{aligned}$$

where (cf. (4.7))

$$\begin{aligned} \mathcal{N} &:= P_0(AJA^* + CQ_0)\mathcal{G}_1^{-1}CP_0 = P_0(AJA^* + CQ_0)((AJA^*)^+ - (Q_0CQ_0)^+)CP_0 \\ &= (P_0 - P_0CQ_0(Q_0CQ_0)^+)CP_0 \\ &= P_0CP_0 - P_0CQ_0(Q_0CQ_0)^+CP_0 \\ &= P_0CP_0 - P_0CQ_0(Q_0CQ_0)^+Q_0CP_0, \\ \mathcal{N}^* &= \mathcal{N}. \end{aligned}$$

Consequently, it holds that

$$M = J^*E, \quad E = J(A^*A)^{-1}A^*\mathcal{N}A(A^*A)^{-1}J^*$$

is symmetric. $\diamond$

Now we return to DAE (2.10). Transforming $\tilde{x} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \approx{x}$, we arrive at the DAE

$$\begin{pmatrix} A & 0 \\ 0 & B^* \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & B & 0 \\ -A^* & 0 & 0 \end{pmatrix} \approx{x} \right)' = \begin{pmatrix} 0 & C & D \\ C^* & W & S \\ D^* & S^* & R \end{pmatrix} \approx{x} \tag{4.8}$$

which obviously satisfies the assumptions of Theorem 4.3, since

$$\begin{pmatrix} A & 0 \\ 0 & B^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A \\ B^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Hence, the DAE (4.8) is self-adjoint.

Theorem 4.5 *Let the DAE (2.2) have a leading term with* $ker A(t) = 0, \; t \in \mathcal{I}$, *and let the conditions (3.8) and (3.10) be satisfied.*
Then, the composed DAE (2.10) is regular with tractability index one, and its inherent regular explicit ODE applies to the component $\tilde{u} := \begin{pmatrix} Bx \\ -A^*\psi \end{pmatrix}$, *and it is of the form*

$$\tilde{u}'(t) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tilde{E}(t)\tilde{u}(t), \quad t \in \mathcal{I}, \tag{4.9}$$

where $\tilde{E}(t) = \tilde{E}(t)^*$.

Proof: From (2.4) it follows that $ker B(t)^* = im B(t)^\perp = 0$, hence $ker \tilde{A}(t) = 0$. We just have to remark that the DAEs (4.8) and (2.10) have a common inherent ODE (cf. [Mä2]). $\diamond$

Remark 4.6 The condition $ker A(t) = 0$ can be realized by an appropriate refactorization of the leading term (cf. [Mä2]).

The use of a full-column-rank matrix $A(t)$ in (2.2) corresponds to a full-column-rank $\tilde{A}(t)$ in (2.10). With r denoting the rank of $A(t)$, we have then $n = r, \tilde{n} = 2n = 2r$, and under the conditions (3.8), (3.10), the inherent regular explicit ODE of the DAE (2.10) is given in minimal coordinates, i.e., in $\mathbb{R}^{2r}$. If one uses formulations of (2.2) with $n > r$, and, in particular, those with an $r-$dimensional time varying subspace $im B(t)$ of $\mathbb{R}^n$, one may lose the Hamiltonian property, as the following example shows. Then the inherent regular explicit ODE of (2.10) is given in $\mathbb{R}^{2n}$, but it is just relevant on its time-varying invariant subspace $im B(t) \times im A(t)^* \subset \mathbb{R}^{2n}$.

Example 4.7 Put $m = k = n = 2,\ r = 1,\ A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B(t) = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix},\ C(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ D(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ S(t) = 0,\ W(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ R(t) = 1$. We have $K(t) = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix},\ G_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ P_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B(t)^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ A(t)^{*-} = \begin{pmatrix} 0 & 0 \\ t & 1 \end{pmatrix},\ P_{*0}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The conditions (3.8),(3.10) are easily checked to be satisfied.
Compute

$$\tilde{A}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \tilde{B}(t) = \begin{pmatrix} 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},\ \tilde{B}(t)^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -t & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{G}_0(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \tilde{C}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},\ \tilde{Q}_0(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{G}_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix},\ \tilde{G}_1(t)^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{K}(t) = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t & 1 \end{pmatrix},\ \tilde{B}(t)\tilde{G}_1(t)^{-1}\tilde{C}(t)\tilde{B}(t)^- = \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -t & -1 \end{pmatrix}$$

The inherent regular ODE $\tilde{u}' = \tilde{K}'\tilde{u} + \tilde{B}\tilde{G}_1^{-1}\tilde{C}\tilde{B}^-\tilde{u}$ is given in $I\!R^4$, it has the special form

$$\tilde{u}'(t) = \begin{pmatrix} 0 & t+1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -t+1 & -1 \end{pmatrix} \tilde{u}(t),$$

which is not a Hamiltonian one.
If we turn back to this special DAE (2.2)

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} x(t) \right)' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

and refactorize the leading term (cf. [Mä2]) by means of $H(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ H(t)^- = (0, 1)$, we obtain the equivalent formulation of (2.2) $\bar{A}(\bar{B}x)' = \bar{C}x + \bar{D}u$ with the coefficients $\bar{A}(t) := A(t)H(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \bar{B}(t) := H(t)^- B(t) = (0, 1),\ \bar{C}(t) := C(t), \bar{D}(t) := D(t)$.

For the corresponding new DAE (2.10), it results that

$$\tilde{\tilde{A}}(t)=\begin{pmatrix}0&0\\1&0\\0&0\\0&1\\0&0\end{pmatrix},\quad \tilde{\tilde{B}}(t)=\begin{pmatrix}0&1&0&0&0\\0&0&0&-1&0\end{pmatrix},\quad \tilde{\tilde{B}}(t)^-=\begin{pmatrix}0&0\\1&0\\0&0\\0&-1\\0&0\end{pmatrix},\quad \tilde{\tilde{K}}(t)=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\tilde{\tilde{B}}(t)\,\tilde{\tilde{G}}_1(t)^{-1}\,\tilde{\tilde{C}}(t)\,\tilde{\tilde{B}}(t)^-=\tilde{\tilde{B}}(t)\,\tilde{\tilde{G}}_1(t)^{-1}\tilde{C}(t)\,\tilde{\tilde{B}}(t)^-=\begin{pmatrix}1&0\\2&-1\end{pmatrix},$$

and the inherent regular ODE that applies now to the components $\tilde{\tilde{u}}(t)=\begin{pmatrix}\bar{B}(t)x(t)\\-\bar{A}(t)^*\psi(t)\end{pmatrix}=\begin{pmatrix}x_2(t)\\-\psi_2(t)\end{pmatrix}$, is given in $I\!R^2$ as $\tilde{\tilde{u}}'(t)=\begin{pmatrix}1&0\\2&-1\end{pmatrix}\tilde{\tilde{u}}(t)$, i.e., it is Hamiltonian.
Consequently, in order to appropriately utilize the Hamiltonian properties it is advisable to arrange for $A(t)$ to have full-column-rank.

5 The DAE to be controlled is regular with index zero or one

If the DAE (2.2) is actually a regular explicit ODE, that is, for $n=m=k, A=I, B=I$, the resulting system (2.5)-(2.7) is well understood. In particular, for $S=0$, it reads

$$\left.\begin{array}{rcl}x' &=& Cx+Du\\ -\psi' &=& Wx+C^*\psi\\ 0 &=& D^*\psi+Ru\end{array}\right\}. \tag{5.1}$$

Obviously, (5.1) represents a semi-explicit DAE that has index one if and only if R is nonsingular. However, if $R=0$, then (5.1) is a Hessenberg size three DAE, supposed that D^*WD is a nonsingular block (e.g. [BrCaPe]).
In this section a similar situation will be shown to hold true if the controlled DAE (2.2) is a general regular index zero DAE, that is, for $n=m=k, A$ and B nonsingular, or a general regular DAE with tractability index one, supposed the control is directed only to the inherent regular explicit ODE.
In this section we put $m=k$. Recall that the coefficients of the DAE (2.10) under consideration are

$$\tilde{A}=\begin{pmatrix}A&0\\0&B^*\\0&0\end{pmatrix},\quad \tilde{B}=\begin{pmatrix}B&0&0\\0&-A^*&0\end{pmatrix},\quad \tilde{B}^-=\begin{pmatrix}B^-&0\\0&-A^{*-}\\0&0\end{pmatrix},\quad \tilde{C}=\begin{pmatrix}C&0&D\\W&C^*&S\\S^*&D^*&R\end{pmatrix}.$$

The generalized inverses B^-, A^{*-} are chosen so that $B^-B=:P_0, AA^-=P_{*0}$ are symmetric, $BB^-=K, A^-A=K$, where K realizes the decomposition (2.4) and $A^{*-}=A^{-*}$ (cf. [BaMä]).
We use the matrix function sequence provided in the appendix as a tool for investigating the index. Put

$$\tilde{G}_0:=\tilde{A}\tilde{B},\quad \tilde{C}_0:=\tilde{C}.$$

In the first case, if (2.2) is regular with tractability index zero, we have $n = m$, and A, B are nonsingular. Then, we continue the matrix function sequence with

$$\tilde{Q}_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \tilde{G}_1 := \tilde{G}_0 - \tilde{C}_0\tilde{Q}_0 = \begin{pmatrix} G_0 & 0 & -D \\ 0 & -G_0^* & -S \\ 0 & 0 & -R \end{pmatrix}, \tag{5.2}$$

where $G_0 := AB$. It is evident that $\tilde{G}_1$ is nonsingular if R is so, and vice versa. If R is actually invertible, equation (2.7) provides $u = -R^{-1}(D^*\psi + S^*x)$, and then (2.5), (2.6) determine the explicit ODE for the components Bx and $-A^*\psi$

$$\begin{pmatrix} Bx \\ -A^*\psi \end{pmatrix}' = \begin{pmatrix} A^{-1}(C - DR^{-1}S^*)B^{-1} & A^{-1}DR^{-1}D^*A^{*-1} \\ B^{*-1}(W - SR^{-1}S^*)B^{-1} & -B^{*-1}(C^* - SR^{-1}D^*)A^{*-1} \end{pmatrix} \begin{pmatrix} Bx \\ -A^*\psi \end{pmatrix}. \tag{5.3}$$

This is the so-called inherent regular explicit ODE of this special DAE (2.10), and it is obviously a Hamiltonian one (cf. Theorem 4.5).

In the second case we are dealing with in this section, in (2.2), n and m may be different, A and B are singular, $G_0 := AB$ is also singular but of constant rank, and the subspaces $N_0 := kerG_0$ and $S_0 := \{z \in I\!R^m : Cz \in imG_0\}$ intersect transversally (cf. Appendix). We use the orthoprojectors $Q_0 := I - P_0$, $Q_{*0} := I - P_{*0}$ onto $N_0 = kerG_0$ resp. $N_{*0} := kerG_0^* = imG_0^{\perp}$, and continue the matrix function sequence by

$$\tilde{Q}_0 = \begin{pmatrix} Q_0 & 0 & 0 \\ 0 & Q_{*0} & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \tilde{G}_1 = \tilde{G}_0 - \tilde{C}_0\tilde{Q}_0 = \begin{pmatrix} G_0 - CQ_0 & 0 & -D \\ -WQ_0 & -G_0^* - C^*Q_{*0} & -S \\ -S^*Q_0 & -D^*Q_{*0} & -R \end{pmatrix}. \tag{5.4}$$

Theorem 5.1 *Let the DAE (2.2) be regular with index zero or with index one, let*

$$imD \subseteq imAB, \quad imS \subseteq im(AB)^*. \tag{5.5}$$

Then, the control problem DAE (2.10) is regular with tractability index one if R is invertible and vice versa.

Proof: The conditions (5.5) imply $Q_{*0}D = 0$, $Q_0S = 0$, thus $D^*Q_{*0} = 0$, $S^*Q_0 = 0$. $G_1 := G_0 - CQ_0$ and $G_{*1} := -G_0^* - C^*Q_{*0}$ are nonsingular in both the index zero and the index one case. Consequently, $\tilde{G}_1$ is nonsingular exactly if R is so. ◇

Note that the conditions (5.5) are trivially satisfied for nonsingular AB. However, if $G_0 = AB$ is singular (i.e., in the index one case of (2.2)), the choice of D is restricted to the control of the inherent state component.

Example 5.2 Consider the special DAE (2.2)

$$\left.\begin{array}{rcl} x_1' &=& D_1u \\ 0 &=& x_2 + D_2u \end{array}\right\}.$$

Here we have $G_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, and condition (5.5) means that $D_2 = 0,\ S^* = (S_1^*\ 0)$. The resulting DAE (2.10) is

$$\begin{aligned}
x_1' &= D_1 u, \\
0 &= x_2, \\
-\psi_1' &= W_{11}x_1 + W_{12}x_2 + S_1 u, \\
0 &= W_{21}x_1 + W_{22}x_2 + \psi_2, \\
0 &= S_1^* x_1 + D_1^* \psi_1 + Ru,
\end{aligned}$$

which obviously has index one if R is invertible.

Remark 5.3 Theorem 5.1 confirms the corresponding special result provided by Theorem 3.3. Since G_1 is nonsingular, condition (3.8) is valid a priori. Condition (3.10) simplifies here to

$$im \begin{pmatrix} G_0^* - C^* Q_{*0} & WQ_0 & S \\ 0 & 0 & R \end{pmatrix} = I\!R^m \times I\!R^l,$$

and it is equivalent to $imR = I\!R^l$.

Now we turn to optimal control problems (2.1)-(2.3) with a singular coefficient R. Equation (2.2) is supposed to be regular with index zero or one.
We make use of the possibility to combine both cases by choosing $Q_0 := 0,\ Q_{*0} := 0$ for nonsingular G_0 in the index one formulation.
The general assumption made on $\begin{pmatrix} W & S \\ S^* & R \end{pmatrix}$ to be positive semidefinite leads to the inequality

$$\langle Su, x\rangle \le \frac{1}{2}\langle Wx, x\rangle, \text{ for all } \ x \in I\!R^m, u \in kerR.$$

Hence, the condition

$$kerR \subseteq kerS \tag{5.6}$$

has to be satisfied at the very beginning. Recall further that R is symmetric, and the orthogonal decomposition $I\!R^l = kerR \oplus imR$ is given. Let Q_R denote the orthoprojector that realizes the latter decomposition, i.e., $imQ_R = kerR,\ kerQ_R = imR$. Observe that $SQ_R = 0,\ Q_R S^* = 0$ is valid then due to condition (5.6).

By condition (5.5), we will show that the resulting DAE (2.10) with singular R fails to be regular with tractability index two. After that, we will show reasonable conditions to lead to a regular index three DAE as it is well known for the very special case (5.1) we discussed in the beginning of this section.

As mentioned in the proof of Theorem 5.1, condition (5.5) leads to the simpler matrix

$$\tilde{G}_1 = \begin{pmatrix} G_0 - CQ_0 & 0 & -D \\ -WQ_0 & -G_0^* - C^* Q_{*0} & -S \\ 0 & 0 & -R \end{pmatrix}. \tag{5.7}$$

The corresponding null-space $\tilde{N}_1 := ker\tilde{G}_1$ reads

$$\tilde{N}_1 = \left\{ \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in I\!R^{2m+l} : u = Q_R u, (G_0^* + C^* Q_{*0})\psi = -WQ_0 x, G_0 x - CQ_0 x = Du \right\}.$$

Because of (5.5), $G_0 x - CQ_0 x = Du$ implies $Q_0 x \in N_0 \cap S_0$, hence $Q_0 x = 0$. In the consequence, $\tilde{N}_1$ can be represented as follows

$$\tilde{N}_1 = \left\{ \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in I\!R^{2m+l} : u = Q_R u, Q_0 x = 0, \psi = 0, G_0 x = DQ_R u \right\}.$$

Next we compute $\tilde{N}_1 \cap \tilde{N}_0 = \left\{ \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in I\!R^{2m+l} : x = 0, \psi = 0, u = Q_R u, Du = 0 \right\}$, and we observe that the necessary regularity condition $\tilde{N}_1 \cap \tilde{N}_0 = 0$ (cf. Appendix) means nothing else but the relation

$$kerDQ_R = kerQ_R = imR. \tag{5.8}$$

On the other hand, condition (5.8) yields

$$(DQ_R)^+ DQ_R = Q_R, \quad (DQ_R)^+ = Q_R (DQ_R)^+,$$

where " + " indicates the Moore-Penrose inverse. With

$$H := (DQ_R)^+ G_0, \quad H^- := G_0^+ DQ_R \tag{5.9}$$

we have

$$H^- H H^- = H^-, \quad H H^- H = H, \quad H H^- = Q_R, \quad H^- H = G_0^+ DQ_R (DQ_R)^+ G_0,$$

and by

$$\tilde{Q}_1 := \begin{pmatrix} H^- H & 0 & 0 \\ 0 & 0 & 0 \\ H & 0 & 0 \end{pmatrix} \tag{5.10}$$

we define a projector onto $\tilde{N}_1$ that has the property $\tilde{Q}_1 \tilde{Q}_0 = 0$ such that we may continue the corresponding matrix function sequence. Moreover, since the blocks $G_0 - CQ_0$ and $G_0^* + C^* Q_{*0}$ in $\tilde{G}_1$ (cf. (5.7)) are invertible,

$$\tilde{\mathcal{W}}_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_R \end{pmatrix} \tag{5.11}$$

is the orthoprojector onto $im\tilde{G}_1^{\perp}$ along $im\tilde{G}_1$.

Theorem 5.4 *Let the DAE (2.2) be regular with tractability index zero or one. Let R be singular and (5.6) be valid. Then, under condition (5.5), the DAE (2.10) fails to be a regular index two DAE at all.*

Proof: If the regularity condition (5.8) is no longer valid, the DAE (2.10) fails to be a regular one at all.
Now let (5.8) be satisfied. Compute $\tilde{\mathcal{W}}_1\tilde{C}_1\tilde{Q}_1 = 0$, which leads to $\tilde{N}_1 \subseteq \tilde{S}_1 := ker\tilde{\mathcal{W}}_1\tilde{C}_0$, hence $\tilde{N}_1 \cap \tilde{S}_1 = \tilde{N}_1 \neq 0$. Because of this nontrivial intersection, by Lemma A.2, index two can never be reached. $\diamond$

The next question to deal with is whether (2.10) may be regular with tractability index three. To answer this question, we continue constructing the matrix function sequence by $\tilde{G}_2$ and $\tilde{G}_3$, and check the regularity conditions $\tilde{N}_2 \cap \tilde{N}_0 = 0$, $\tilde{N}_2 \cap \tilde{N}_1 = 0$ as well as the corresponding (cf. Appendix) smoothness. Finally, we check the index three condition $\tilde{N}_2 \cap \tilde{S}_2 = 0$.

Theorem 5.5 *Let the DAE (2.2) be regular with tractability index zero or one, and let condition (5.5) be satisfied.*
Let R be singular and condition (5.6) hold, further let R have constant rank.
Let the projector functions $\tilde{B}\tilde{P}_0\tilde{P}_1\tilde{B}^-$, $\tilde{B}\tilde{P}_0\tilde{P}_1\tilde{P}_2\tilde{B}^-$ defined below be continuously differentiable, and let the regularity conditions (5.8) and

$$ker\begin{pmatrix} W \\ S^* \end{pmatrix} \cap im(AB)^+DQ_R = 0 \tag{5.12}$$

be valid. Then, if the conditions

$$kerW \cap N_0 = 0 \text{ and } ker\begin{pmatrix} W & S \\ S^* & R \end{pmatrix} = ker\begin{pmatrix} W \\ S^* \end{pmatrix} \times kerR \tag{5.13}$$

hold true additionally, the DAE (2.10) is regular with tractability index three.

Before we verify Theorem 5.5, we formulate consequences showing the close relation to the well known results for (5.1) mentioned above.

Corollary 5.6 *Let (2.2) be regular with index zero or one, $imD \subseteq imAB$, $S = 0$, $R = 0$, $kerW \cap N_0 = 0$, $D^*G_0^{+*}WG_0^+D$ nonsingular. Then, if the relevant smoothness conditions are also satisfied, the DAE (2.10) is regular with tractability index three.*

Proof: Here, (5.5) and (5.6) are trivially satisfied. The relations $kerW \cap imG_0^+D = 0$, $kerD = 0$ are valid since $D^*G_0^{+*}WG_0^+D$ is nonsingular, hence (5.12) and (5.8) are also true. The second condition in (5.13) holds trivially. $\diamond$

Remark 5.7 If the DAE (2.2) is regular with index zero, $S = 0$, $R = 0$ and $D^*G_0^{*-1}WG_0^{-1}D$ is nonsingular, then, with the corresponding smoothness, the DAE (2.10) is regular with index three. If $G_0 = I$, which happens for $A = I$, $B = I$, but also for $A = B^{-1}$, it results that $D^*G_0^{*-1}WG_0^{-1}D = D^*WD$. In this way we have confirmed the well known conditions we started this section with.

Proof of Theorem 5.5: We have to form $\tilde{G}_2 := \tilde{G}_1 - \tilde{C}_1\tilde{Q}_1$, $\tilde{C}_1 := \tilde{C}_0\tilde{P}_0 + \tilde{G}_1\tilde{B}^-(\tilde{B}\tilde{P}_0\tilde{P}_1\tilde{B}^-)'\tilde{B}\tilde{P}_0$ (cf. Appendix). First we compute (cf. (5.9), (5.10))

$$\tilde{B}\tilde{P}_0\tilde{P}_1\tilde{B}^- = \begin{pmatrix} B(I-H^-H)B^- & 0 \\ 0 & K^* \end{pmatrix},$$

where K is the continuously differentiable basic projector realizing (2.4), $K = BB^-$.
The block $B(I-H^-H)B^-$ is again a projector. It decomposes $imB = imK$ into two further subspaces. Assuming $\tilde{B}\tilde{P}_0\tilde{P}_1\tilde{B}^-$ to be smooth means that these two further subspaces are spanned by C^1 base functions, too.
Now, with

$$G_1 := G_0 - CQ_0,\ G_{*1} := -G_0^* - C^*Q_{*0},\ \mathfrak{B} := (B(I-H^-H)B^-)'B$$

we obtain (cf. (5.7)) that

$$\tilde{G}_2 = \begin{pmatrix} G_1 & 0 & -D \\ -WQ_0 & G_{*1} & -S \\ 0 & 0 & -R \end{pmatrix} - \begin{pmatrix} CH^-H + A\mathfrak{B}H^-H & 0 & 0 \\ WH^-H & 0 & 0 \\ S^*H^-H & 0 & 0 \end{pmatrix}. \tag{5.14}$$

Note that $HQ_0 = 0$ follows immediately from (5.9).

Consider $\tilde{N}_2 \cap \tilde{N}_0$ and $\tilde{N}_2 \cap \tilde{N}_1$ to check the regularity conditions. First, $\begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_2 \cap \tilde{N}_0$ is characterized by $x = Q_0x,\ \psi = Q_{*0}\psi,\ G_1x - Du = 0,\ -WQ_0x + G_{*1}\psi - Su = 0,\ Ru = 0.$ This implies $-C_0x - Du = 0$, hence $x \in N_0 \cap S_0 = 0$, further $G_{*1}\psi = 0$, thus $\psi = 0$, and finally, $u = Q_Ru,\ DQ_Ru = 0$, thus $u = 0$ by (5.8). Therefore, we have $\tilde{N}_2 \cap \tilde{N}_0 = 0$.
Next, $\begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_2 \cap \tilde{N}_1$ is true by the equations $u = Q_Ru,\ \psi = 0,\ Q_0x = 0, G_0x - Du = 0,\ (C + A\mathfrak{B})H^-Hx = 0,\ WH^-Hx = 0,\ S^*H^-Hx = 0$. Now, condition (5.12) implies $H^-Hx = 0$. From $G_0x - DQ_Ru = 0$ we derive (cf. (5.9)) $u = Hx = HH^-Hx = 0$ and finally, $P_0x = 0$. It turns out that $\tilde{N}_2 \cap \tilde{N}_1 = 0$.
Now we are able to construct an appropriate projector $\tilde{Q}_2$ onto $\tilde{N}_2$ such that $\tilde{N}_0 \oplus \tilde{N}_1 \subseteq ker\tilde{Q}_2$. However, actually we do not compute $\tilde{Q}_2$ since Lemma A.2 allows us to check the index three property without explicitly using $\tilde{Q}_2$ and $\tilde{G}_3$.

Recall that $\tilde{G}_1$ has constant rank due to the constant rank property of R. Because of $\tilde{\mathcal{W}}_1\tilde{C}_0\tilde{Q}_1 = 0$, we find $im\tilde{G}_2 = im\tilde{G}_1 \oplus \tilde{\mathcal{W}}_1\tilde{C}_0\tilde{Q}_1 = im\tilde{G}_1$, hence $\tilde{G}_2$ has the same rank as $\tilde{G}_1$. Moreover, $\tilde{\mathcal{W}}_2 := \tilde{\mathcal{W}}_1$ is the orthoprojector onto $im\tilde{G}_2^{\perp}$ along $im\tilde{G}_2$. This yields

$$\begin{aligned} \tilde{S}_2 &:= ker\tilde{\mathcal{W}}_2\tilde{C}_0 = ker\tilde{\mathcal{W}}_1\tilde{C}_0 = \tilde{S}_1, \\ \tilde{S}_2 &= \left\{ \begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in I\!R^{2m+l} : Q_RD^*\psi = 0 \right\}. \end{aligned}$$

By Lemma A.2, instead of proving the invertibility of $\tilde{G}_3$ directly, it suffices to show that the subspaces $\tilde{S}_2$ and $\tilde{N}_2$ intersect transversally.
Consider an arbitrary element $\begin{pmatrix} x \\ \psi \\ u \end{pmatrix} \in \tilde{N}_2 \cap \tilde{S}_2$, i.e.,

$$Q_RD^*\psi = 0, \tag{5.15}$$

$$G_1x - Du - \{C + A\mathfrak{B}\}H^-Hx = 0, \tag{5.16}$$

$$-WQ_0x - WH^-Hx - Su + G_{*1}\psi = 0, \tag{5.17}$$

$$S^*H^-Hx + Ru = 0. \tag{5.18}$$

From (5.16) we derive by multiplying by Q_{*0} that

$$Q_{*0}C_0Q_0x + Q_{*0}C_0H^-Hx = 0. \tag{5.19}$$

Next we show that the relation

$$\varepsilon := \left\langle \begin{pmatrix} W & S \\ S^* & R \end{pmatrix} \begin{pmatrix} Q_0x + H^-Hx \\ u \end{pmatrix}, \begin{pmatrix} Q_0x + H^-Hx \\ u \end{pmatrix} \right\rangle = 0 \tag{5.20}$$

is true, and hence

$$\begin{pmatrix} W & S \\ S^* & R \end{pmatrix} \begin{pmatrix} Q_0x + H^-Hx \\ u \end{pmatrix} = 0. \tag{5.21}$$

Compute

$$\varepsilon = \langle WQ_0x + WH^-Hx + Su, Q_0x + H^-Hx\rangle + \langle S^*Q_0x + S^*H^-Hx + Ru, u\rangle.$$

Taking into account that $S^*Q_0 = 0$, by (5.17), (5.18) we obtain

$$\varepsilon = \langle G_{*1}\psi, Q_0x + H^-Hx\rangle = \langle G_{*1}\psi, Q_0x\rangle + \langle G_{*1}\psi, H^-Hx\rangle.$$

Using (5.19) we obtain that

$$\begin{aligned} \langle G_{*1}\psi, Q_0x\rangle &= \langle Q_0G_{*1}\psi, x\rangle = \langle -Q_0C_0^*Q_{*0}\psi, x\rangle \\ &= \langle -\psi, Q_{*0}C_0Q_0x\rangle = \langle \psi, Q_{*0}C_0H^-Hx\rangle \\ &= \langle C_0^*Q_{*0}\psi, H^-Hx\rangle, \end{aligned}$$

and, hence, by (5.15)

$$\begin{aligned} \varepsilon &= \langle C_0^*Q_{*0}\psi + G_{*1}\psi, H^-Hx\rangle = -\langle G_0^*\psi, H^-Hx\rangle \\ &= -\langle G_0^*\psi, G_0^+DQ_RHx\rangle = -\langle \psi, P_{*0}DQ_RHx\rangle \\ &= -\langle \psi, DQ_RHx\rangle = -\langle Q_RD^*\psi, Hx\rangle = 0. \end{aligned}$$

Since (5.20) is verified, we may use (5.21). Applying the conditions (5.12), (5.13) we derive from (5.21) that

$$u = Q_Ru, \quad Q_0x = 0, \quad H^-Hx = 0. \tag{5.22}$$

Now (5.17) reads $G_{*1}\psi = 0$, which implies $\psi = 0$. Finally, (5.16) simplifies to $G_0x - DQ_Ru = 0$, but this leads to $u = Hx = HH^-Hx = 0$, and $G_0x = 0$, hence $x = 0$. So the intersection $\tilde{N}_2 \cap \tilde{S}_2$ contains the trivial element only. $\diamond$

6 A remark on controlling index two DAEs

One could conjecture analogous results for regular higher index DAEs (2.2) as discussed in the previous section for the index zero and index one cases. Unfortunately, those results do not hold any longer. For instance, Theorem 5.1 fails just for index two DAEs (2.2). We make this clear for Hessenberg size two systems

$$\left.\begin{array}{rcll} x_1' & = & C_{11}x_1 + C_{12}x_2 & + \; D_1 u \\ 0 & = & C_{21}x_1 & + \; D_2 u \end{array}\right\}, \tag{6.1}$$

with nonsingular $C_{21}C_{12}$, which represent a very special case of regular index two DAEs. Let $S = 0,\ R$ be nonsingular. The resulting DAE (2.10) has the coefficients

$$\tilde{G}_0 = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{C}_0 = \begin{pmatrix} C_{11} & C_{12} & 0 & 0 & D_1 \\ C_{21} & 0 & 0 & 0 & D_2 \\ W_{11} & W_{12} & C_{11}^* & C_{21}^* & 0 \\ W_{21} & W_{22} & C_{12}^* & 0 & 0 \\ 0 & 0 & D_1^* & D_2^* & R \end{pmatrix},$$

and the elements of $\tilde{N}_0 \cap \tilde{S}_0$ are characterized by the equations

$$x_1 = 0,\ \psi_1 = 0,\ D_2 u = 0,\ W_{22}x_2 = 0,\ D_2^*\psi_2 + Ru = 0. \tag{6.2}$$

Obviously condition (5.5) means now that $D_2 = 0$.
However, if $D_2 = 0$, then $x_1 = 0,\ x_2 \in kerW_{22},\ \psi_1 = 0, \psi_2$ arbitrary, $u = 0$ form nontrivial elements of $\tilde{N}_0 \cap \tilde{S}_0$, that is, (2.10) fails to be regular with index one. Furthermore, additionally assuming W_{22} to be nonsingular or controlling just the inherent regular explicit ODE of (6.1) by supposing that $D_2 = 0,\ C_{12}(C_{21}C_{12})^{-1}C_{21}D_1 = 0$ does not change the situation essentially.

Considering (6.2) again, we observe that $\tilde{N}_0 \cap \tilde{S}_0 = 0$ can be obtained by letting $kerW_{22} = 0,\ kerD_2^* = 0$, i.e., $kerD_2R^{-1}D_2^* = 0$. A comparison with Theorem 3.3 above shows that these conditions coincide with the general index one condition (3.8), (3.10) specified for (6.1) and $S = 0,\ kerR = 0$. Namely, now it holds in detail that

$$(G_1\ D) = \begin{pmatrix} I & -C_{12} & D_1 \\ 0 & 0 & D_2 \end{pmatrix},$$

$$\begin{pmatrix} G_0^* - C^*Q_{*0} & WQ_0 & 0 \\ -D^*Q_{*0} & 0 & R \end{pmatrix} = \begin{pmatrix} I & -C_{21}^* & 0 & W_{12} & 0 \\ 0 & 0 & 0 & W_{22} & 0 \\ 0 & D_2^* & 0 & 0 & R \end{pmatrix},$$

and $kerD_2^* = 0,\ kerW_{22} = 0$ are nothing else but the full rank conditions (3.8) resp. (3.10) for these matrices.
For interesting special results concerning the case of controlled regular index two DAEs we refer to [Ba].

A Appendix: Tractability index

For the DAE

$$A(t)(B(t)x(t))' - C(t)x(t) = q(t),\ t \in \mathcal{I}, \tag{A.1}$$

with continuous matrix coefficients

$$A(t) \in L(\mathbb{R}^n, \mathbb{R}^k), B(t) \in L(\mathbb{R}^m, \mathbb{R}^n),\ C(t) \in L(\mathbb{R}^m, \mathbb{R}^k),$$

we form a sequence of matrix functions and subspaces as it was done in [Mä2] in order to define the tractability index. The leading term in (A.1) is supposed to be properly stated, i.e., the decomposition

$$ker A(t) \oplus im B(t) = \mathbb{R}^n,\ t \in \mathcal{I}, \tag{A.2}$$

is valid, and both subspaces forming this direct sum are spanned by continuously differentiable bases.
$K(t) \in L(\mathbb{R}^n)$ denotes the projector that realizes the decomposition (A.2), $ker A(t) = ker K(t),\ im K(t) = im B(t),\ t \in \mathcal{I}$. Since these subspaces have continuously differentiable bases, the projector $K(t)$ is continuously differentiable in t. Below, we drop the argument t, all relations are meant pointwise for $t \in \mathcal{I}$.

Put $G_0 := AB,\ N_0 := ker G_0,\ C_0 := C$ and introduce functions $Q_0, P_0 : \mathcal{I} \to L(\mathbb{R}^m)$ such that

$$Q_0^2 = Q_0,\ \ im Q_0 = N_0,\ \ P_0 := I - Q_0.$$

By this definition, Q_0 is a projector function onto N_0. Due to (A.2), it holds that $N_0 = ker B$, and G_0 has constant rank r_0 on $\mathcal{I}$. Therefore, we can choose Q_0 to be continuous on $\mathcal{I}$. Next, we determine the generalized inverse B^- of B by means of the conditions

$$B^-BB^- = B^-,\ \ BB^-B = B,\ \ BB^- = K,\ \ B^-B = P_0.$$

B^- is continuous on $\mathcal{I}$, too.
For $i \geq 1$, let

$$\begin{aligned} G_i &:= G_{i-1} - C_{i-1}Q_{i-1}, && \text{(A.3)} \\ N_i &:= ker G_i,\ \ Q_i^2 = Q_i,\ \ im Q_i = N_i, P_i := I - Q_i, \\ C_i &:= C_{i-1}P_{i-1} + G_i B^-(BP_0 \cdots P_i B^-)' BP_0 \cdots P_{i-1}. && \text{(A.4)} \end{aligned}$$

The matrix functions G_i, C_i have k rows and m columns. In [Mä2] this matrix function sequence is used for the case of $k = m$ only. When constructing the matrix function C_i one has to take care of the existence of the derivative involved in (A.4).
By construction, the rank of the G_i cannot decrease if the index i increases.
If one arrives at a certain index μ such that

$$N_\mu \subseteq ker C_\mu$$

is satisfied, then it follows immediately that $C_\mu Q_\mu = 0$, $G_{\mu+1} = G_\mu$, $N_{\mu+1} = N_\mu$, $N_{\mu+1} \subseteq kerC_{\mu+1}$, and, consequently,

$$N_j \subseteq kerC_j, \quad G_j = G_\mu \quad \text{for all} \quad j \geq \mu, \tag{A.5}$$

i.e., the sequence (A.3) becomes stationary for $j \geq \mu$.
If we have (A.5) with $m = k$, $N_\mu = 0$, then all matrix functions $G_j, j \geq \mu$, are nonsingular. Since the relations

$$N_i \cap kerC_i = N_i \cap N_{i+1} \subseteq N_{i+1} \cap kerC_{i+1}, \quad i \geq 0,$$

are given by construction, for obtaining a certain nonsingular G_μ it is necessary that all intersections $N_i \cap kerC_i$ are trivial, i.e.,

$$N_i \cap kerC_i = 0, \quad \text{i.e.,} \;\; N_i \cap N_{i+1} = 0, \; i \geq 0.$$

The concept of regular DAEs from [Mä2] is formally related to sequences (A.3) becoming stationary as well as nonsingular.

Definition A.1 *The DAE (A.1) is said to be a regular DAE with tractability index zero if $m = k$ and $r_0 = m$.*
The DAE (A.1) is said to be a regular DAE with tractability index $\mu \in \mathbb{N}$ if $m = k$ and there is a sequence (A.3), (A.4) such that for all $j \geq 1$

(i) G_j *has constant rank* r_j *on* $\mathcal{I}$,

(ii) $N_0 \oplus \cdots \oplus N_{j-1} \subseteq kerQ_j$,

(iii) $Q_j \in C(\mathcal{I}, L(\mathbb{R}^m)), \; BP_0 \cdots P_j B^- \in C^1(\mathcal{I}, L(\mathbb{R}^n))$

and $r_{\mu-1} < r_\mu = m$.
The DAE (A.1) is said to be regular if it is a regular DAE with tractability index $\mu \in \mathbb{N} \cup \{0\}$.

A regular index zero DAE may also be called a regular implicit ODE. For $A = I, B = I$ we have just an explicit ODE.
In addition to the subspaces N_i, the accompanying subspaces

$$S_i := \{z \in \mathbb{R}^m : Cz \in imG_i\} = ker\mathcal{W}_i C,$$

with $\mathcal{W}_i := I - G_i G_i^+$, G_i^+ the Moore-Penrose inverse of G_i, may be quite useful when checking properties of the DAE. Furthermore, the matrices

$$\begin{aligned} \mathcal{G}_{i+1} &:= G_i - \mathcal{W}_i C_i Q_i = G_i - \mathcal{W}_i C Q_i, \\ \mathring{G}_{i+1} &:= G_i - C_{i-1} P_{i-1} Q_i \end{aligned}$$

may be applied to simplify calculations.

Lemma A.2

(i) G_{i+1} *is injective* $\iff N_i \cap S_i = 0$,

(ii) G_{i+1} *is surjective* $\iff ker(\mathcal{W}_i C Q_i)^* = imG_i$,

(iii) G_{i+1} *is bijective* $\iff m = k$, *and* $\mathcal{W}_i C Q_i : N_i \to (imG_i)^\perp$ *is a bijection,*

(iv) $imG_{i+1} = im\mathring{G}_{i+1} = im\mathcal{G}_{i+1} = imG_i \oplus im\mathcal{W}_i C Q_i$.

Proof: By construction we have $\mathcal{W}_i C = \mathcal{W}_i C P_0 \cdots P_{i-1} = \mathcal{W}_i C_{i-1} P_{i-1} = \mathcal{W}_i C_i$ and $G_{i+1} = \mathcal{G}_{i+1} E_{i+1}$, $G_{i+1} = \mathring{G}_{i+1} F_{i+1}$ with nonsingular factors (cf. [Mä2])

$$\begin{aligned} E_{i+1} &:= I - G_i^- C_{i-1} P_{i-1} Q_i - P_i B^- (BP_0 \cdots P_i B^-)' BP_0 \cdots P_{i-1} Q_i, \\ F_{i+1} &:= I - P_i B^- (BP_0 \cdots P_i B^-)' BP_0 \cdots P_{i-1} Q_i. \end{aligned}$$

It holds that $ker\mathcal{G}_{i+1} = N_i \cap S_i$, $im\mathcal{G}_{i+1} = imG_i \oplus im\mathcal{W}_i C Q_i$, $N_{i+1} = E_{i+1}^{-1}(N_i \cap S_i)$, thus (iv) and (i) are verified.
If $im\mathcal{W}_i C Q_i = im\mathcal{W}_i$, then G_{i+1} is surjective, and vice versa. With $(imG_i)^\perp = im\mathcal{W}_i = im\mathcal{W}_i C Q_i = (ker(\mathcal{W}_i C Q_i)^*)^\perp$ we obtain assertion (ii). Assertion (iii) follows from the fact that G_{i+1} and $\mathcal{G}_{i+1}$ are bijective simultaneously.

As mentioned above, if certain $N_i \cap kerC_i$ are nontrivial, then the DAE (A.1) is no longer regular and, in particular, Condition (ii) in Definition A.1 cannot be satisfied. Considering non-regular DAEs we have to modify this condition.

Definition A.3 *The DAE (A.1) is said to be a non-regular one with tractability index* μ *if there is a sequence (A.3), (A.4) such that for* $j \geq 1$

(i) G_j *has constant rank on* $\mathcal{I}$,

(ii) $(N_0 + \cdots + N_{j-1}) \ominus ((N_0 + \cdots + N_{j-1}) \cap N_j) \subseteq kerQ_j$,

(iii) $Q_j \in C(\mathcal{I}, L(\mathbb{R}^m))$, $BP_0 \cdots P_j B^- \in C^1(\mathcal{I}, L(\mathbb{R}^n))$,

and $G_{\mu-1} \neq G_\mu$, $G_\mu = G_{\mu+1}, G_\mu$ *singular.*

Example A.4 The special DAE

$$\left.\begin{aligned} x_1' - C_{13} x_3 &= q_1 \\ -x_2 &= q_2 \\ 0 &= q_3 \end{aligned}\right\} \tag{A.6}$$

which is used in [KuMe1] as a canonical form, is obviously not a regular one. Supposed imC_{13} does not vary with time, it has tractability index one. Namely, we can compute $A = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}$, $B = (I\ 0\ 0)$, $B^- = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}$, $G_0 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $C_0 = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$, $G_1 = \begin{pmatrix} I & 0 & -C_{13} \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} C_{13}C_{13}^- & 0 & 0 \\ 0 & 0 & 0 \\ C_{13}^- & 0 & I - C_{13}^- C_{13} \end{pmatrix}$, $N_0 \cap N_1 = \{z : z_1 = 0, z_2 = 0, C_{13} z_3 = 0\}$, $N_0 \ominus (N_0 \cap N_1) = \{z : z_1 = 0, z_3 = C_{13}^- C_{13} z_3\}$, $BP_0 P_1 B^- = I - C_{13} C_{13}^-$, $C_1 = 0$ and, finally, $G_2 = G_1$.

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