AN EXPLICIT MATCHING THEOREM FOR LEVEL ZERO DISCRETE SERIES OF UNIT GROUPS OF $\mathfrak{p}$-ADIC SIMPLE ALGEBRAS

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Abstract. For $A|F$ a central simple algebra over a $\mathfrak{p}$-adic local field the group of units $A^\times \cong GL_m(D_d)$ is a general linear group over a central division algebra $D_d|F$ of index d. The product $n = dm$ being fixed, the Abstract Matching Theorem (AMT) implies the existence of bijective maps $\mathcal{JL}$ between the sets of discrete series representations of the groups $A^\times$ such that a character relation is preserved. In this paper we construct maximal level zero extended type components for every level zero discrete series representation of $A^\times$. Its maximal level zero extended type determines the discrete series representation uniquely (without any twist ambiguities as for the usual types) and, conversely, every level zero discrete series representation Π contains a maximal level zero extended type component $\tilde{\Sigma}(\Pi)$ which is unique up to conjugacy. In order to determine how $\mathcal{JL}$ matches the extended types we find certain regular elliptic elements where the characters of $\tilde{\Sigma}(\Pi)$ and Π are the same and we compute the character values at these elements by using a version of Shintani descent which we develop in Appendix B. Surprisingly, we find that AMT also implies explicit Shintani descent for irreducible characters of finite general linear groups which have cuspidal descent.

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Let F be a locally compact p-adic field, let D_d be a central division algebra of index d over F, and let $A = \mathrm{M}_m(D_d)$, the algebra of all $m \times m$ matrices with entries in D_d. Then A is a central simple F algebra of reduced degree $n := dm$ and the group of units $A^\times$ of A is the group of F points of a connected reductive algebraic group defined over F. The group $A^\times$ is separable, totally disconnected, and unimodular and the harmonic analysis of $A^\times$, as is well known, is linked to numerous problems in arithmetic.

It is interesting, and perhaps may be regarded as a version of the Abstract Matching Theorem (AMT) (see §0.2), that the representation theory of $A^\times$ depends upon the parameters d, m associated to A, but not more specifically on the isomorphism class of A. We follow the convention of writing D_d for any central F division algebra of index d; we also write D_n for any one of the $\varphi(n)$ isomorphism class representatives of central F division algebras of index n.

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This paper continues work in which, based on AMT, we have attempted to represent the Jacquet-Langlands correspondence ($\mathcal{JL}$) for the level zero discrete series of the unit groups of all central simple F algebras of reduced degree n "explicitly". Let A be such an algebra. In [GSZ] we associated to each Bernstein component of level zero representations of $A^\times$ a set of types in the sense of [BK2]. In [SZ2] we showed that at most one unramified twist class of discrete series representations occurs as a family of subquotients for a Bernstein component of $A^\times$ and that the set of components with discrete series subquotients associated to them is naturally parameterized by the set of Galois orbits of multiplicative characters of a finite field of degree n over the residual field of F. We showed that $\mathcal{JL}$ restricts to a bijection of level zero discrete series which preserves this parameter set ([SZ2]4.1; for a summary of the main results see §0.4 below).

In the present paper, to remove the "inertial ambiguity" of [SZ2]4.1, we construct for every unit group $A^\times$ a family of "maximal level zero extended type representations" such that every level zero discrete series representation contains, up to conjugacy, exactly one level zero extended type which in turn characterizes the representation. The level zero discrete series are canonically, via AMT, parameterized by level zero "Langlands parameters" and we find that, while the maximal level zero extended types are concretely and explicitly constructed, they too have a natural parameterization by the same set, and in this connection we will speak of the type parameters. The main result of this paper is to match the Langlands parameter of a level zero discrete series representation against its type parameter. "Explicit matching" (see Theorem 3) reduces the calculation of a level zero discrete series character on the regular elliptic set to a simple and well known calculation via Frobenius formula in the division algebra case. This would seem to be a first step toward a similar anticipated reduction for more general discrete series characters of $A^\times$. On the other hand the character of a level zero extended type equals the character of its host representation at certain regular elliptic elements and these values of the character distinguish the discrete series representation from its twists. Using the Abstract Matching Theorem and Shintani descent (see §B), which enables us to determine the characters of level zero extended types in the mixed case, we are able to determine the twist which converts the Langlands parameterization into the type parameterization.

Shintani descent (i.e. the inverse of base change for finite fields) enters in an essential way in the matching of the character of a maximal level zero extended type, via AMT, to a division algebra character. Given the existence theorem of Shintani [Sh], we discovered that AMT implies the explicit Shintani descent mapping for finite general linear groups for irreducible characters with cuspidal descent (see [Gy], where more general results are proved with certain restrictions on the characteristic of the finite field; Gyoja's proofs depend on the construction of characters in [DL]).

A more precise statement of our results is given in §0 as Theorems 1, 2, and 3. In §1 we use AMT to transport the formulas for the level zero discrete series characters on the regular elliptic set from $D_n^\times$ to $A^\times$. We see that the character formula of a discrete series representation depends explicitly on its Langlands parameter, which we introduce in §0.5 below. In §§2-4 we construct a set of maximal level zero extended types and prove their uniqueness up to conjugacy. In §4, while constructing a maximal level zero type representation for every level zero discrete series representation, we show that the set of Langlands parameters also serves as a set of type parameters. We then show that to any level zero discrete series representation of $A^\times$ there corresponds an open subset of $A^\times$ which consists of "good" regular elliptic elements, such that the character of the representation is constant in a large open neighborhood of a good element and is represented there by the character of its maximal level zero extended type (see 4.7). The proof of 4.7 involves a significant extension of results from [Zi] and proceeds largely in §A (see A.1 and A.2). In §5 we use AMT and Shintani descent to compute the character of the maximal level zero extended type at the good elements. It turns out that the character values of a discrete series representation at its "good" elements distinguish the representation from its unramified twists and we are able to use these character values to match the parameters. In 5.7, 5.8 we state our results concerning Shintani descent. We find that AMT, combined with Shintani's theorem asserting that characters descend to virtual characters ([Sh], Theorem 1), implies both explicit matching of discrete series in the level zero case and explicit descent for irreducible characters of finite general linear groups with cuspidal descent. Because we needed a Shintani descent theory which we could not easily adapt to the general set-up of Digne-Michel [DM2] we wrote §B to serve as the basis for our applications; for a more detailed discussion see the beginning of §B.

Fröhlich [Fr1] has constructed representations like our level zero extended type representations and recognized the role of the base-change lift (the inverse of the Shintani descent mapping) in parameterizing these representations. But Fröhlich did not match his types to discrete series representations.

Bushnell and Henniart, in several papers (e.g. [BH]), have also considered the problem of obtaining an explicit Jacquet-Langlands correspondence. And they have also found that this problem is intimately tied to the problem of explicit base change.[2]

§0 Notation, Preliminaries, and Statements of Results.

First we recall some notation and review some of the background of this paper as established in [GSZ, SZ2]. We introduce "level zero Langlands parameters" and "level zero extended types". We state our main results and set-up a strategy for proving them. In the remainder of the paper we give the proofs.

Our notation is mostly consistent with notation already introduced in [GSZ, SZ1]. We refer to results by number, equations by numbers enclosed in parentheses. Each section has its own numeration of equations. References to equations are prefixed by a section number if the equations referred to are not from the ambient section.

§0.1 Classes of Irreducible Representations.

For any unramified extension $F_\ell|F$ ($\ell \geq 1$) let $X_t(F_\ell^\times)$ $[X_u(F_\ell^\times)]$ denote the group of tame [unramified] multiplicative characters of F_ℓ, i.e. the group of characters which are trivial on the group $\mathfrak{U}^1_{F_\ell}$ of principal units of F_ℓ [trivial on the unit group $o_{F_\ell^\times}$]. Let $X(k_\ell^\times)$ denote the group of multiplicative characters of the finite field k_ℓ. For any $\eta \in X_t(F_\ell^\times)$ the *reduction* $\bar{\eta} \in X(k_\ell^\times)$ is defined by restricting η to $o_\ell^\times$ and factoring through the projection upon $k_\ell^\times$.

Let $\mathcal{R}(A^\times)$ $[\mathcal{R}_\mathbb{C}(A^\times)]$ denote the set of equivalence classes of irreducible unitary [respectively, not necessarily unitary] smooth representations of $A^\times$, let $\mathcal{R}^2(A^\times)$ $[\mathcal{R}^0(A^\times)]$ denote the subset of discrete series [unitary supercuspidal] representations of $A^\times$, and let $\mathcal{R}_0(A^\times) \subset \mathcal{R}(A^\times)$ $[\mathcal{R}_{0,\mathbb{C}}(A^\times) \subset \mathcal{R}_\mathbb{C}(A^\times)]$ denote the respective subsets consisting of level zero representations. (See the end of §0.3 for the definition

[2]The authors wish to thank Gopal Prasad and J.-K. Yu for a helpful conversation.

of "level zero representation".) Similarly, $\mathcal{R}_0^2(A^\times)$ $[\mathcal{R}_0^0(A^\times)]$ will denote the set of level zero discrete series [level zero supercuspidal representations]. Since $D_n^\times/F^\times$ is compact, $\mathcal{R}^0(D_n^\times) = \mathcal{R}^2(D_n^\times) = \mathcal{R}(D_n^\times)$.

Let $\Pi \in \mathcal{R}_{\mathbb{C}}(A^\times)$. We write ω_Π for the central quasi-character of Π. Then ω_Π is unitary if Π is unitary.

We sometimes write $Y^\wedge$ for the set of irreducible unitary representations of a group Y.

§0.2 The Abstract Matching Theorem (AMT).

The mapping $A^\times \ni x \mapsto \phi_x(T) \in F[T]$, which assigns to each element of $A^\times$ its minimal polynomial over F, induces a bijection:

$$(1)\qquad \left\{\begin{matrix}\text{regular elliptic conjugacy}\\ \text{classes of } A^\times\end{matrix}\right\} \quad\longleftrightarrow\quad \left\{\begin{matrix}\text{irreducible monic and separable}\\ \text{polynomials of degree } n \text{ over } F\end{matrix}\right\}.$$

We use this correspondence to identify the regular elliptic conjugacy classes of $A^\times$ for all A of reduced degree n over F.[3]

Let $C_c^\infty(A^\times)$ denote the convolution algebra consisting of all compactly supported locally constant complex-valued functions on $A^\times$. Let Π be an irreducible smooth representation of $A^\times$ in a complex vector space $V = V_\Pi$. Then $\Pi(f)$ has finite rank for every $f \in C_c^\infty(A^\times)$, so the mapping

$$(2)\qquad f \mapsto \Theta_\Pi(f) := \mathrm{trace}(\Pi(f)) \qquad (f \in C_c^\infty(A^\times)),$$

the *distributional character* of Π, is defined as a complex-valued linear functional on $C_c^\infty(A^\times)$.

Let $A^{\times\prime}$ denote the set of regular (semi-simple) elements of $A^\times$. A theorem of Harish-Chandra [HC1] implies that Θ_Π may be represented by a locally constant class function on $A^{\times\prime}$. We denote this function $\Theta_\Pi(x)$ and regard it as a measurable function defined a. e. on $A^\times$. Harish-Chandra [HC2] has proved, in the characteristic 0 case, that $\Theta_\Pi(x)$ is locally integrable on $A^\times$. More recently Lemaire [Lem] has proved that $\Theta_\Pi(x)$ is locally integrable on $A^\times$ with no restriction on the characteristic of F. Thus, $\Theta_\Pi(gxg^{-1}) = \Theta_\Pi(x)$ for any $x \in A^{\times\prime}$ and arbitrary $g \in A^\times$ and

$$(3)\qquad \Theta_\Pi(f) = \int_{A^\times} f(x)\Theta_\Pi(x)dx$$

for any $f \in C_c^\infty(A^\times)$.

Abstract Matching Theorem (AMT) [Ba][DKV][JL][Ro]**.** *Let $D_n|F$ be a central division algebra of index n and let $A|F$ ($A = M_m(D_d)$) be any central simple algebra of reduced degree n. Then there is a bijective correspondence, the "Jacquet-Langlands Correspondence",*

$$(4)\qquad \mathscr{JL}_{A,D_n} : \mathcal{R}(D_n^\times) \to \mathcal{R}^2(A^\times), \quad \Pi^A := \mathscr{JL}_{A,D_n}(\Pi^{D_n}),$$

such that

$$(5)\qquad \Theta_{\Pi^A}(x) = (-1)^{m-1}\Theta_{\Pi^{D_n}}(x)$$

for x regular elliptic.

[3] We call an element $x \in A^\times$ *regular elliptic* if $F(x)|F$ is a separable extension of degree n. Thus, the set of regular elliptic elements of $A^\times$ is the same as the set of all elements of A having minimal polynomials over F which are irreducible, separable, and of degree n.

§0.3 Hereditary Orders, Principal Orders, and Their Normalizers.

Let o_F denote the ring of integers of F, ϖ_F a prime element, $\mathfrak{p}_F := \varpi_F o_F$ the maximal ideal and let $k := k_F := o_F/\mathfrak{p}_F$ denote the residual field of F. Assume that k is a finite field, that $|k| = q = p^\mu$ with p the characteristic of k. For $F_\ell|F$ ($\ell \geq 1$) an unramified extension field of degree ℓ we write o_{F_ℓ}, $\mathfrak{p}_{F_\ell}$, and $k_\ell := k_{F_\ell} := o_{F_\ell}/\mathfrak{p}_{F_\ell}$ for the ring of integers, prime ideal, and residual field, respectively. For D_d any central F division algebra of index $d > 1$ we use the notations O_{D_d}, ϖ_{D_d}, $\mathfrak{p}_{D_d} := \varpi_{D_d} O_{D_d}$ and $k_{D_d} := O_{D_d}/\mathfrak{p}_{D_d}$ respectively. Usually F_d will denote an unramified extension of F which is contained in D_d and is stabilized under conjugation by ϖ_{D_d}. This implies that $\varpi_{D_d}^d$ is a prime element of F, and we assume that $\varpi_{D_d}^d = \varpi_F$. We also often identify $k_d := k_{D_d}$.

The mapping

$$\phi : x \mapsto {}^\phi x := \varpi_{D_d} x \varpi_{D_d}^{-1} \qquad (x \in F_d) \tag{6}$$

generates the Galois group $\mathrm{Gal}(F_d|F)$ and, by reduction, the group $\mathrm{Gal}(k_d|k)$. We identify these Galois groups.

Let $A = M_m(D_d)$, as above. We fix the maximal order $\mathfrak{A}_1 := M_m(O_{D_d})$; its Jacobson radical is $\mathfrak{P}_1 := M_m(\mathfrak{p}_{D_d})$. We also fix the minimal order $\mathfrak{A}_m \subset \mathfrak{A}_1$ which consists of those elements of $\mathfrak{A}_1$ with all entries below the main diagonal in $\mathfrak{p}_{D_d}$. The Jacobson radical $\mathfrak{P}_m$ of $\mathfrak{A}_m$ consists of those elements of $\mathfrak{A}_1$ with coefficients on or below the main diagonal in $\mathfrak{p}_{D_d}$. If $\mathfrak{A}$ is a hereditary order, we write $\mathfrak{P}_\mathfrak{A}$ for the Jacobson radical of $\mathfrak{A}$ and $\mathfrak{U}_\mathfrak{A}^1 := 1 + \mathfrak{P}_\mathfrak{A}$ for the group of principal units. When there is only one hereditary order, as in the case of D_ℓ or F_ℓ, we simplify the notation to write $\mathfrak{U}_{D_\ell}^1 := 1 + \mathfrak{p}_{D_\ell}$ or $\mathfrak{U}_{F_\ell}^1 := 1 + \mathfrak{p}_{F_\ell}$.

Every hereditary order $\mathfrak{A}$ is conjugate to a unique hereditary order $\mathfrak{A}'$ such that $\mathfrak{A}_m \subseteq \mathfrak{A}' \subseteq \mathfrak{A}_1$, and any such hereditary order $\mathfrak{A}'$ is called *standard.* The mapping $\mathfrak{A} \mapsto \mathfrak{A}/\mathfrak{P}$ sends the set of standard hereditary orders bijectively to the set of block diagonal matrix rings in $M_m(k_d)$. Thus, for any hereditary order $\mathfrak{A}$ we have a tuple of integers $s_1, \dots, s_r$ with sum $s_1 + \cdots + s_r = m$ such that the quotient ring $\mathfrak{A}/\mathfrak{P}_\mathfrak{A}$ is isomorphic to the semi-simple algebra consisting of ordered blocks

$$\bar{\mathfrak{A}} := \mathfrak{A}/\mathfrak{P}_\mathfrak{A} \cong M_{s_1}(k_d) \times \cdots \times M_{s_r}(k_d), \tag{7}$$

each factor being a complete matrix algebra over the residual field k_d of D_d. The multiplicative group of (7) is

$$\bar{\mathfrak{A}}^\times := (\mathfrak{A}/\mathfrak{P}_\mathfrak{A})^\times = \mathfrak{A}^\times/\mathfrak{U}_\mathfrak{A}^1 \cong \mathrm{GL}_{s_1}(k_d) \times \cdots \times \mathrm{GL}_{s_r}(k_d). \tag{8}$$

We call any subgroup of $\bar{\mathfrak{A}}_1^\times$ which contains $\mathfrak{A}_m^\times/\mathfrak{U}_{\mathfrak{A}_1}^1$ a *standard parabolic subgroup* of $\bar{\mathfrak{A}}_1^\times$. The mapping $\mathfrak{A}^\times \mapsto \mathfrak{A}^\times/\mathfrak{U}_{\mathfrak{A}_1}^1$ defines a bijection of the set of standard hereditary orders to the set of standard parabolic subgroups of $\bar{\mathfrak{A}}_1^\times$.

The unit group $\mathfrak{A}^\times$ always has a non-trivial normalizer $\mathfrak{K}_\mathfrak{A}$ which contains the group $D_d^\times I_m$; the group $\mathfrak{K}_\mathfrak{A}$ is an open compact mod center subgroup of $A^\times$ which also normalizes $\mathfrak{P}_\mathfrak{A}$ and $\mathfrak{U}_\mathfrak{A}^1$.

If $s_1 = \cdots = s_r = s$, the hereditary order $\mathfrak{A}$ is a *principal* order, which means that $\mathfrak{P}_\mathfrak{A}$ is a principal ideal. Such a hereditary order is conjugate to a unique standard principal order which we denote $\mathfrak{A}_r$ ($m = rs$) and we write $\mathfrak{P}_r := \mathfrak{P}_{\mathfrak{A}_r}$ for its Jacobson radical. The unit group of $\mathfrak{A}_r$ is

$$\mathfrak{A}_r^\times = \mathrm{GL}_s^r(O_{D_d}) \cdot \mathfrak{U}_{\mathfrak{A}_r}^1 \tag{9}$$

and its normalizer, denoted $\mathfrak{K}_r := \mathfrak{K}_{\mathfrak{A}_r}$, is a maximal compact modulo center (mcmc) subgroup of $A^\times$. Benz [Be] and Bushnell/Fröhlich [BF] have proved that every mcmc subgroup of $A^\times$ is conjugate to $\mathfrak{K}_r$ for some $r \mid m$. Concretely,

$$\mathfrak{P}_r = \mathfrak{A}_r t_r = t_r \mathfrak{A}_r \quad \text{and} \quad \mathfrak{K}_r \cong \mathfrak{A}_r^\times \rtimes \langle t_r \rangle, \tag{10}$$

semi-direct product, where

$$t_m = \begin{bmatrix} 0_{m-1,1} & I_{m-1} \\ \varpi_{D_d} & 0_{1,m-1} \end{bmatrix}, \quad t_r = t_m^s = \begin{bmatrix} 0_{r-1,1} & I_{r-1} \\ \varpi_{D_d} & 0_{1,r-1} \end{bmatrix} \otimes I_s \quad (rs = m), \tag{11}$$

with I_j the $j \times j$ identity matrix and $0_{j,k}$ the $j \times k$ zero matrix ($j, k \geq 1$). Clearly, $t_r^r = t_1 = \varpi_{D_d} I_m$ and $t_r^{dr} = \varpi_F I_m$.

Let $\Pi \in \mathcal{R}_{\mathbb{C}}(A^\times)$ act in the complex vector space V_Π. We call Π a *level zero* representation if V_Π contains a non-zero $\mathfrak{U}^1_{\mathfrak{A}_1}$-fixed vector, i.e. if $1_{\mathfrak{U}^1_{\mathfrak{A}_1}} \subset \Pi|_{\mathfrak{U}^1_{\mathfrak{A}_1}}$.

§0.4 Weak Explicit Matching of Level Zero Representations.

Let $\Omega(A^\times)$ denote the Bernstein spectrum of $A^\times$, i.e. the set of $A^\times$ conjugacy classes $[M, \pi]$ of pairs in which M is a standard Levi subgroup of $A^\times$ and π is an irreducible supercuspidal representation of M. Each $\Pi \in \mathcal{R}_{\mathbb{C}}(A^\times)$ has a *supercuspidal support* $CS(\Pi) \in \Omega(A^\times)$. We write $CS(\Pi) \in \Omega([M, \pi])$ to indicate that $CS(\Pi)$ is an unramified character twist of $[M, \pi]$, in other words that $CS(\Pi)$ belongs to the connected component $\Omega([M, \pi])$ of $\Omega(A^\times)$. Let $\mathcal{S}^A_\Omega$ denote the set of $\Pi \in \mathcal{R}^2(A^\times)$ such that $CS(\Pi) \in \Omega$, where Ω denotes a connected component of $\Omega(A^\times)$.

0.1 Fact. *The set $\mathcal{S}^A_\Omega$ is not empty and contains (consists entirely of) level zero representations if and only if $\Omega = \Omega([GL_s^r(D), \pi_s^{\otimes r}])$, where $rs = m$ and $\pi_s \in \mathcal{R}_0^0(GL_s(D))$. In this case $\mathcal{S}^A_\Omega$ comprises a single unramified twist class of discrete series representations. For $\Pi \in \mathcal{S}^A_\Omega$ set the inertial degree:*

$$f(\Pi) := |\{\lambda \in X_u(F^\times) \,:\, (\lambda \circ Nrd_{A|F}) \otimes \Pi \sim \Pi\}|.$$

Then $f(\Pi) = f(\pi_s)$.

Remark. The terms "inertial class" and "unramified twist class" of discrete series representations are used interchangeably. The "inertial degree" is a natural invariant of an inertial class of discrete series representations. Besides denoting the order of the subgroup of $X_u(F^\times)$ which fixes $\Pi \in S^A_\Omega$ [4] it also refers both to the inertial degree of the extension $F_f|F$ which is canonically associated to S^A_Ω and, in the level zero case, to the formal degree of Π with respect to a normalized Haar measure on $A^\times$ such that the Steinberg representation has formal degree one.

To recall the parameterization of [SZ2]2.9 for the set of components Ω such that $\mathcal{S}^A_\Omega \neq \emptyset$ let $k_n|k$ be the degree n extension of the residual field of F, let $X(k_n^\times)$ denote the group of multiplicative characters of k_n, let $\bar\chi \in X(k_n^\times)$, and let $[\bar\chi] := \mathrm{Gal}(k_n|k)\bar\chi$ be the $\mathrm{Gal}(k_n|k)$ orbit of $\bar\chi$. For $f = |[\bar\chi]|$ set $e := n/f$, $e' := (e, m)$, and $f' := f/(d, f) = m/(e, m)$ (see the proof of 3.2 below). Let $\sigma_{\bar\chi} \in \mathrm{GL}_{f'}(k_d)^\wedge$ be the cuspidal representation corresponding to the Green's parameter $\mathrm{Gal}(k_{df'}|k_d)\bar\chi_{df'}$, where $\bar\chi_{df'} \in X(k_{df'}^\times)$ is the unique character such that $\bar\chi = \bar\chi_{df'} \circ \mathrm{N}_{k_n|k_{df'}}$.

[4]This has prompted some authors (e.g. [AP]) to speak of the "torsion number" rather than the "inertial degree" of a discrete series representation.

0.2 Fact. *Let* $\Omega_{[\bar\chi]} = \Omega(GL_{f'}(D_d)^{e'}, \pi_{[\bar\chi]}^{\otimes e'}])$ *be the connected component of* $\Omega(A^\times)$ *such that* $\pi_{[\bar\chi]}$ *is a fixed representative chosen from the inertial class of supercuspidal representations of* $GL_{f'}(D_d)$ *such that the inflation* $Inf(\sigma_{\bar\chi}) \subset \pi_{[\bar\chi]}|_{\mathrm{GL}_{f'}(O_{D_d})}$. *Then* $Inf(\sigma_{\bar\eta}) \subset \pi_{[\bar\chi]}|_{\mathrm{GL}_{f'}(O_{D_d})}$ *for all* $\bar\eta \in [\bar\chi]$ *and the mapping*

$$Gal(k_n|k)\backslash X(k_n^\times) \ni [\bar\chi] \longmapsto \Omega_{[\bar\chi]} \subset \Omega(A^\times)$$

parameterizes the set of components $\Omega \subset \Omega(A^\times)$ *such that* $\emptyset \neq \mathcal{S}_\Omega^A \subset \mathcal{R}_0^2(A^\times)$. *The cuspidal components are obtained in the case* $e' = 1$ *which means* $f' = m$.

Abbreviating the notation of 0.2, we set $\mathcal{S}_{\bar\chi}^A := \mathcal{S}_{\Omega_{[\bar\chi]}}^A$. From Facts 1 and 2 we see that $\mathcal{R}_0^2(A^\times) = \coprod_{[\bar\chi]} \mathcal{S}_{\bar\chi}^A$, the disjoint union of inertial classes of discrete series representations, the set of inertial classes being parameterized by the set $\mathrm{Gal}(k_n|k)\backslash X(k_n^\times)$. For $\Pi \in \mathcal{S}_{\bar\chi}^A$ we have $f(\Pi) = f(\pi_{[\bar\chi]}) = |[\bar\chi]| = f$. The middle equality follows from the fact that $\pi_{[\bar\chi]} = \mathrm{cInd}_X^{\mathrm{GL}_{f'}(D_d)}(\tilde\sigma_{\bar\chi})$, where $\tilde\sigma_{\bar\chi}$ denotes an extension of $\mathrm{Inf}(\sigma_{\bar\chi})$ to $X = \langle \varpi_{D_d}^{(d,f)} \rangle \ltimes \mathrm{GL}_{f'}(O_{D_d})$ (see [GSZ] 5.1 and[SZ2] the proof of 2.8).

Let $\Omega \subset \Omega(A^\times)$. A *type* for Ω in the sense of [BK2] is a pair (K, ρ) consisting of a compact open subgroup $K \subset A^\times$ and a representation $\rho \in \mathcal{R}(K)$ such that: For all $\Pi \in \mathcal{R}_{\mathbb{C}}(A^\times)$ the supercuspidal support $CS(\Pi) \in \Omega$ if and only if $\rho \subset \Pi|_K$. A pair $(\mathfrak{A}^\times, \tau)$ is called a *cuspidal level zero pair* for $A^\times$ if $\mathfrak{A}$ is a standard hereditary order, $\bar\tau \in (\bar{\mathfrak{A}}^\times)^\wedge_{\mathrm{cusp}}$, and $\tau := \mathrm{Inf}(\bar\tau) \in (\mathfrak{A}^\times)^\wedge$. Define the following equivalence relation on the set of cuspidal level zero pairs: set $(\mathfrak{A}^\times, \tau) \sim ((\mathfrak{A}')^\times, \tau')$ if and only if there exists $g \in A^\times$ which acts via reduction such that $g^{-1}\bar{\mathfrak{A}}^\times g = (\bar{\mathfrak{A}}')^\times$ and $\bar\tau'(g^{-1}ag) = \bar\tau(a)$ for all $a \in \bar{\mathfrak{A}}^\times$. In [GSZ] the authors show that every Bernstein component which has level zero subquotients admits a cuspidal level zero pair as a type and that the assignment of cuspidal level zero pairs to Bernstein components defines a bijection between the set of equivalence classes of cuspidal level zero pairs and the set of all Bernstein components which have level zero subquotients.

0.3 Fact. *Let* $\Omega_{[\bar\chi]} \subset \Omega(A^\times)$ *and let* $\sigma := \sigma_{\bar\chi}$ *be as in Fact 2. For* $i = (i_1, \ldots, i_{e'}) \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$ *let* $\tau_i := Inf(\sigma^{\phi^{i_1}} \otimes \cdots \otimes \sigma^{\phi^{i_{e'}}}) \in (\mathfrak{A}_{e'}^\times)^\wedge$, *where we use the identification* $\bar{\mathfrak{A}}_{e'}^\times \cong GL_{f'}^{e'}(k_d)$ *as in (8).*
(i) The set of $(d,f)^{e'}$ *type representations* $(\mathfrak{A}_{e'}^\times, \tau_i)$ *(*$i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$*) comprise a single equivalence class of types which all correspond to the Bernstein component* $\Omega_{[\bar\chi]}$ *(see* [GSZ]*5.5,* [SZ2]*1.2).*
(ii) The discrete series representation $\Pi \in \mathcal{R}^2(A^\times)$ *belongs to* $\mathcal{S}_{\bar\chi}^A$ *if and only if* $\tau_i \subset \Pi|_{\mathfrak{A}_{e'}^\times}$ *for some* $i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$, *if and only if* $\tau_i \subset \Pi|_{\mathfrak{A}_{e'}^\times}$ *for all* $i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$. *Moreover, in this case each* τ_i *occurs in* $\Pi|_{\mathfrak{A}_{e'}^\times}$ *with multiplicity one (*[SZ2]*, 2.6).*

A central result of [SZ2] (see ibid. 3.2, 4.1) asserts that:

0.4 Fact (Weak Explicit Matching Theorem). *The Jacquet-Langlands correspondence (4) induces bijections*

$$\mathcal{R}_0(D_n^\times) \longleftrightarrow \mathcal{R}_0^2(A^\times), \qquad \mathcal{S}_{\bar\chi}^{D_n} \longleftrightarrow \mathcal{S}_{\bar\chi}^A$$

for all $[\bar\chi] \in Gal(k_n|k)\backslash X(k_n^\times)$.

§0.5 Langlands Parameters for $\mathcal{R}_0^2(A^\times)$.

The conjectures of Langlands[5], when combined with AMT, imply that the set of equivalence classes of n-dimensional indecomposable representations of the Weil-Deligne group W_F' which are trivial on the ramification subgroup and have unitary determinant character serve as a canonical parameter set for $\mathcal{R}_0^2(A^\times)$. In this section we want to set-up this formal parameterization.

Let $\bar{F}|F$ denote a fixed algebraic closure of F and let $F_f|F$ be the unramified extension of degree f contained in $\bar{F}$. A character $\chi_f \in X_t(F_f^\times)$ is called F *regular* if the Galois orbit $\mathrm{Gal}(\bar{F}|F)\chi_f = \mathrm{Gal}(F_f|F)\chi_f$ has length f. We shall regard the set $\mathcal{T}_0^n$ of "level zero Langlands parameters" concretely as follows:

0.5 Definition. Let $X_t'(F_f^\times)$ denote the set of F regular tame characters of $F_f^\times$ and let

$$\mathcal{T}_0^n := \mathrm{Gal}(\bar{F}|F)\backslash \coprod_{f|n} X_t'(F_f^\times)$$

denote the set of all Galois orbits

$$[\chi_f] := \mathrm{Gal}(\bar{F}|F)\chi_f = \mathrm{Gal}(F_f|F)\chi_f \qquad (\chi_f \in X_t'(F_f^\times);\ f \mid n).$$

0.6 Remark. We justify the "substitution" of character orbits as Langlands parameters in place of representations of the Weil/Deligne group as follows: The mapping $[\chi_f] \mapsto \mathrm{Ind}_{F_f\uparrow F}(\chi_f) \otimes \mathrm{sp}(e)$ sends the set of Galois orbits $\mathcal{T}_0^n$ bijectively to the set of n-dimensional indecomposable representations of $W_F' = \mathbb{C}^+ \rtimes W_F$ which are trivial on the ramification subgroup. The induction $\mathrm{Ind}_{F_f\uparrow F}$ signifies induction from W_{F_f} to W_F and $\mathrm{sp}(e)$ denotes the special representation of dimension $e := \frac{n}{f}$ of W_F' which has trivial determinant character. In the division algebra case we shall see that unitary representations correspond to orbits $[\chi_f]$ of unitary characters and, by AMT, this carries over to general A.

To proceed with defining the mapping $\mathcal{T}_0^n \to \mathcal{R}_0(A^\times)$ we begin with the division algebra case. We note that the assumption $f \mid n$ implies the existence of an embedding $F_f \to D_n$ which is unique up to conjugacy (Skolem/Noether). The centralizer of F_f is a division algebra $D_e|F_f$ which is of index $e = n/f$. For $\chi_f \in X_t'(F_f^\times)$ define $\tilde{\chi}_f := \chi_f \circ \mathrm{Nrd}_{D_e|F_f}$ of $D_e^\times$. Since χ_f is tame, $\tilde{\chi}_f$ extends to a character $\hat{\tilde{\chi}}_f$ of $D_e^\times \mathfrak{U}_{D_n}^1$. Moreover, $D_e^\times \mathfrak{U}_{D_n}^1 = D_e^\times O_{D_n}^\times$, since the residual fields k_{D_n} and k_{D_e} are naturally isomorphic. Set

$$\Pi_{\chi_f}^{D_n} := \mathrm{Ind}_{D_e^\times \mathfrak{U}_{D_n}^1}^{D_n^\times} (\hat{\tilde{\chi}}_f). \tag{12}$$

0.7 Definition. For $[\chi_f] \in \mathcal{T}_0^n$ set $\Pi_{\chi_f}^A := \mathcal{JL}_{A,D_n}(\Pi_{\chi_f}^{D_n})$.

0.7 is justified by:

Theorem 1. *(i) For every $\chi_f \in X_t(F_f^\times)$ such that $f \mid n$ the representation $\Pi_{\chi_f}^{D_n}$ is irreducible. Moreover, the mapping $\chi_f \mapsto \Pi_{\chi_f}^{D_n}$ factors through the projection $\chi_f \mapsto [\chi_f]$ and the induced mapping $\mathcal{L}_{D_n} : \mathcal{T}_0^n \to \mathcal{R}_0(D_n^\times)$ is bijective.*

[5]For an introduction to the local Langlands correspondence see [Kud]. M.Harris, R.Taylor, and G.Henniart have proved the local Langlands correspondence, but we will not use these results.

(ii) For any central simple algebra A of reduced degree n over F the mapping

$$\mathcal{JL}_{A,D_n} \circ \mathcal{L}_{D_n} : \mathcal{T}_0^n \to \mathcal{R}_0^2(A^\times) \qquad \mathcal{T}_0^n \ni [\chi_f] \mapsto \Pi_{\chi_f}^A \in \mathcal{R}_0^2(A^\times)$$

is a canonical bijection; in particular, for any pair of central division algebras D_n, D_n' of index n over F, $\mathcal{L}_{D_n'} = \mathcal{JL}_{D_n',D_n} \circ \mathcal{L}_{D_n}$.
(iii) Fix $[\chi_f] \in \mathcal{T}_0^n$ and a central simple algebra $A = M_m(D_d)$ $(dm = n)$ over F. For any separable extension $K|F$ of degree n let $\iota_K : K \hookrightarrow A$ be any embedding. If $K = F(y)$, then

$$\Theta_{\Pi_{\chi_f}^A}(\iota_K(y)) = (-1)^{m-1} \sum_{\eta \in [\chi_f]} \eta \circ N_{K|F_f}(y), \quad \text{if } F_f \subset K.$$

Moreover, if $\Pi \in \mathcal{R}^2(A^\times)$ satisfies this equation, then $\Pi \sim \Pi_{\chi_f}^A$.

Proof. See 1.1-3. $\square$

Remark. The bijective mapping

$$\operatorname{Ind}_{F_f \uparrow F}(\chi_f) \otimes \operatorname{sp}(e) = \varphi_{\chi_f} \longmapsto [\chi_f] \longmapsto \Pi_{\chi_f}^{D_n} \in \mathcal{R}_{0,\mathbb{C}}(D_n^\times)$$

from level zero indecomposable representations of the Weil-Deligne group W_f' to level zero irreducible representations of $D_n^\times$ is not always the Langlands correspondence because, in general, $\det(\varphi_{\chi_f})$ and the central character $\omega_{\Pi_{\chi_f}^{D_n}}$ are not the same and the ϵ-factors are not preserved. To correct the correspondence replace φ_{χ_f} by the twist $\varphi_{\omega_f^{f-1}\chi_f}$, where $\omega_f \in X_u(F_f^\times)$ has order two. For our purposes we do not need this because we want to compare AMT with the more explicit construction of dicrete series representations via maximal level zero extended types. So we match the Langlands parameterization $[\chi_f] \mapsto \Pi_{\chi_f}^A$ which commutes with $\mathcal{JL}$ by its very definition against the type parameterization $[\chi_f] \mapsto \tilde{\Sigma}_{\chi_f}^A \mapsto \Pi(\tilde{\Sigma}_{\chi_f}^A)$ which is to be defined below.

§0.6 Level Zero Extended Types for Discrete Series Representations.

A type in the sense of [BK2] characterizes an element of $\mathcal{R}_0^2(A^\times)$ up to unramified twist: For any $[\bar{\chi}] \in \operatorname{Gal}(k_n|k)\backslash X(k_n^\times)$ the set of discrete series characters $\mathcal{S}_{\bar{\chi}}^A$ comprises a single inertial class and corresponds to an equivalence class of types as in 0.3(ii). Now we want to introduce a finer set of invariants, the set of level zero extended types.

0.8 Definition. Let $\Pi \in \mathcal{R}^2(A^\times)$. A *level zero extended type* for Π is a pair (X, Σ) such that X is a compact mod center subgroup of $A^\times$, $\Sigma \in X^\wedge$, and:
(i) for some hereditary order $\mathfrak{A}$ the multiplicative group $\mathfrak{A}^\times \subset X$ and $1_{\mathfrak{U}_{\mathfrak{A}}^1} \subset \Sigma|_{\mathfrak{U}_{\mathfrak{A}}^1}$;
(ii) Σ occurs simply in $\Pi|_X$;
(iii) if $\Pi' \in \mathcal{R}^2(A^\times)$ and $\Sigma \subset \Pi'|_X$, then $\Pi' = \Pi$.

Let $\Pi \in \mathcal{R}^2(A^\times)$ and assume that (X, Σ) is a level zero extended type for Π. Let $X \subset X'$, where X' is also a compact mod center subgroup of $A^\times$, and assume that Σ' is an irreducible representation of X' such that $\Sigma \subset \Sigma'|_X$ and $\Sigma' \subset \Pi|_{X'}$. In this case, (X', Σ') is also a level zero extended type for Π. In particular, if $\operatorname{Ind}_X^{X'} \Sigma$ is irreducible, then $(X', \operatorname{Ind}_X^{X'} \Sigma)$ is a level zero extended type for Π. Since every compact mod center subgroup X is contained in some mcmc subgroup X', it follows that if (X, Σ) is a level zero extended type for Π, then for every mcmc $\mathfrak{K}$ such that $X \subset \mathfrak{K}$ there exists a unique level zero extended type $(\mathfrak{K}, \tilde{\Sigma}_{\mathfrak{K}})$ for Π such that $\Sigma \subset \tilde{\Sigma}_{\mathfrak{K}}|_X$. We call a level zero extended type $(\mathfrak{K}, \tilde{\Sigma}_{\mathfrak{K}})$ for Π such that $\mathfrak{K}$ is a mcmc subgroup of $A^\times$ a *maximal level zero extended type* for Π.

0.9 Lemma. *(i) Let* $\Pi \in \mathcal{S}^A_{\bar{\chi}}$ *and let* $\mathfrak{A}$ *be a standard hereditary order. Then* $1_{\mathfrak{U}^1_{\mathfrak{A}}} \subset \Pi|_{\mathfrak{U}^1_{\mathfrak{A}}}$ *if and only if the standard principal order* $\mathfrak{A}_{e'} \subset \mathfrak{A}$*, where* $e' = (e, m)$*,* $e = n/f$*, and* $f = \|[\bar{\chi}]\| = f(\Pi)$ *is the inertial degree of* Π.
(ii) Let $\Pi \in \mathcal{R}^2(A^\times)$ *and assume that* (X, Σ) *is a level zero extended type for* Π. *Then* $\Pi \in \mathcal{R}^2_0(A^\times)$ *and, up to conjugacy,* (X, Σ) *contains a level zero type* $(\mathfrak{A}^\times_{e'}, \tau_i)$*, where* e' *derives from* Π *as in (i).*

Proof. (i) Let $\mathfrak{P} \subset \mathfrak{P}' \subset \mathfrak{A}' \subset \mathfrak{A}$ be standard hereditary orders with their respective Jacobson radicals. Then $\bar{\mathfrak{A}'}^\times \subset \bar{\mathfrak{A}}^\times$ are Levi subgroups of standard parabolic subgroups of $\bar{\mathfrak{A}}^\times_1$. Let V denote a representation space for Π. Then the spaces of fixed vectors $V^{\mathfrak{U}^1_{\mathfrak{A}'}} \subset V^{\mathfrak{U}^1_{\mathfrak{A}}}$ are modules for the finite groups $\bar{\mathfrak{A}'}^\times$ and $\bar{\mathfrak{A}}^\times$, respectively, and Jacquet restriction maps $V^{\mathfrak{U}^1_{\mathfrak{A}}} \to V^{\mathfrak{U}^1_{\mathfrak{A}'}}$. If $\mathfrak{A}'$ is minimal such that $V^{\mathfrak{U}^1_{\mathfrak{A}'}} \neq (0)$, then $V^{\mathfrak{U}^1_{\mathfrak{A}'}}$ is a cuspidal representation of $\bar{\mathfrak{A}'}^\times$, so the inflations of its irreducible subrepresentations are types for $\Omega_{[\bar{\chi}]}$ (see 0.3 and preceding remarks). Since the only Levi subgroup of a standard parabolic subgroup of $\bar{\mathfrak{A}}^\times_1$ which is conjugate to $\bar{\mathfrak{A}}^\times_{e'}$ is $\bar{\mathfrak{A}}^\times_{e'}$, it follows that $V^{\mathfrak{U}^1_{\mathfrak{A}}} \neq (0)$ implies $\mathfrak{A}_{e'} \subset \mathfrak{A}$. Conversely, if $\mathfrak{A}_{e'} \subset \mathfrak{A}$ and if $V^{\mathfrak{U}^1_{\mathfrak{A}_{e'}}} \neq (0)$, then since $V^{\mathfrak{U}^1_{\mathfrak{A}}} \supseteq V^{\mathfrak{U}^1_{\mathfrak{A}_{e'}}}$, it is clear that $V^{\mathfrak{U}^1_{\mathfrak{A}}} \neq (0)$.
(ii) We may assume that the hereditary order associated to (X, Σ) by 0.8(i) is standard and minimal with the property of 0.8(i). Since (X, Σ) is a level zero extended type for Π, $\Sigma|_{\mathfrak{A}^\times}$ is a subrepresentation of $\Pi|_{\mathfrak{A}^\times}$ and therefore, by 0.8(i), Π is level zero. By the minimality of $\mathfrak{A}$, the reduction of $(\Sigma|_{\mathfrak{A}^\times}, V(\Sigma) \cap V^{\mathfrak{U}^1_{\mathfrak{A}}})$ is cuspidal, hence $\mathfrak{A}$ is also minimal such that $V^{\mathfrak{U}^1_{\mathfrak{A}}} \neq 0$. It follows that $\mathfrak{A} = \mathfrak{A}_{e'}$ with e' as in (i). $\square$

Theorem 2. *(i) If* $\Pi \in \mathcal{R}^2_0(A^\times)$*, then* Π *admits, up to conjugacy, exactly one maximal level zero extended type representation. Let* $\tilde{\Sigma}(\Pi)$ *denote a maximal level zero extended type representation of* Π *and assume that for* $\tilde{\Sigma}(\Pi)$ *the hereditary order of 0.8(i) is standard. Then* $\tilde{\Sigma}(\Pi) \in \mathfrak{K}^\wedge_r$*, where* $r := m/(f, m)$ *and* $f = f(\Pi)$ *is the inertial degree of* Π.
(ii) Let $t_r \in \mathfrak{K}_r$ *be the uniformizer as in (11) and let* $e = n/f$*,* $c = f/(f, m)$*, hence* $ce = dr$*. Then* $t_r^{ce} = \varpi_F I_m$*, and the irreducible polynomial*

$$F[T] \ni g(T) := \prod_{\zeta' \in \mathrm{Gal}(F_f|F)\zeta} (T^e - \zeta' \varpi_F), \tag{13}$$

where $\zeta \in k_f^\times \subset o_f^\times$ *is regular over* k*, has a root* $x = \alpha t_r^c \in \mathfrak{A}_r^\times t_r^c$*. The character* Θ_Π *is constant on* $x\mathfrak{U}^1_{\mathfrak{A}_r}$ *and*

$$\Theta_\Pi(y) = \Theta_{\tilde{\Sigma}(\Pi)}(x) \tag{14}$$

for $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$*; except possibly for a closed measure zero subset consisting of elements which are inseparable over* F *all elements of* $x\mathfrak{U}^1_{\mathfrak{A}_r}$ *are regular elliptic.*

Proof. The proof of (i) is completed in 4.1 and 4.7, for (ii) see A.2. $\square$

§0.7 The Explicit Matching Theorem.

Theorem 1(ii) implies the "Langlands parameterization"

$$\mathcal{T}^n_0 \ni [\chi_f] \longmapsto \Pi^A_{\chi_f} \in \mathcal{R}^2_0(A^\times)$$

and 4.6(iii) implies the parameterization

$$\mathcal{T}_0^n \ni [\chi_f] \longmapsto (\mathfrak{K}_r^A, \tilde{\Sigma}_{\chi_f}^A) \qquad (r = m/(f,m))$$

of the set of conjugacy classes of maximal level zero extended types of $A^\times$. Let $\Pi(\tilde{\Sigma}_{\chi_f}^A)$ denote the element of $\mathcal{R}_0^2(A^\times)$ which admits $\tilde{\Sigma}_{\chi_f}^A$. By 0.8(iii) and Theorem 2(i), $\tilde{\Sigma}_{\chi_f}^A \mapsto \Pi(\tilde{\Sigma}_{\chi_f}^A)$ induces a bijective mapping from the set of conjugacy classes of level zero extended types to $\mathcal{R}_0^2(A^\times)$.

Theorem 3. *Let $\omega \in X_u(F^\times)$ be any element such that $\omega^{2f} = 1$ and $\omega^f \neq 1$ and set $\tilde{\omega} := \omega \circ Nrd_{A|F}$. Then $\omega_f := \omega \circ N_{F_f|F} \in X_u(F_f^\times)$ is the unique element of order exactly two and for any Langlands parameter $[\chi_f] \in \mathcal{T}_0^n$ and for any $A|F$ of reduced degree $n = dm$*

$$\Pi_{\chi_f}^A = \tilde{\omega}^{m-(f,m)}\Pi(\tilde{\Sigma}_{\chi_f}^A) = \Pi(\tilde{\omega}^{m-(f,m)}\tilde{\Sigma}_{\chi_f}^A) = \Pi(\tilde{\Sigma}^A_{\omega_f^{m-(f,m)}\chi_f}). \tag{15}$$

Remark. Since the inertial degrees (see 0.1) satisfy $f(\Pi_{\chi_f}^A) = f(\Pi(\tilde{\Sigma}_{\chi_f}^A)) = f$, the character twist depends only upon the parity of $m - (f,m)$.

The proof of Theorem 3 is given in section 5. In the course of that proof we see that AMT implies the explicit Shintani descent mapping for irreducible characters of finite general linear groups which have cuspidal descent.

0.10 Examples: Two Very Special Cases. Let $[\chi_f] \in \mathcal{T}_0^n$ be such that $f = |[\chi_f]|$, let $\bar{\chi} = \bar{\chi}_f \circ \mathrm{N}_{k_n|k_f}$, and fix $A = \mathrm{M}_m(D_d)$.

1. It follows from 0.2 Fact that $\Pi_{\chi_f}^A \in \mathcal{R}_0^0(A^\times)$ if and only if $(e,m) = 1$, in which case $m \mid f$ and $e \mid d$, so $r = m/(f,m) = 1$ and $c = f/m = d/e = (d,f)$. The twist of (15) is trivial, since $m = (f,m)$. The maximal extended type $(\tilde{\Sigma}_{\chi_f}, \mathfrak{K}_1)$ of $\Pi_{\chi_f}^A$ may be constructed as follows. Let $\sigma(\bar{\chi}) \in \mathrm{GL}_m(k_d)^\wedge_{\mathrm{cusp}}$ be the representation with the Green's parameter $\mathrm{Gal}(k_n|k_d)\bar{\chi}$. Let $X = \langle \varpi_D^{(d,f)} \rangle \ltimes \mathfrak{A}_1^\times$ and define $\Sigma_{\chi_f} \in X^\wedge$ by setting $\Sigma_{\chi_f}|_{\mathfrak{A}_1^\times} := \mathrm{Inf}(\sigma(\bar{\chi}))$ and $\Sigma_{\chi_f}(\varpi_D^{(d,f)}) := \chi_f((-1)^{e-1}\varpi_F)J$, where J is the intertwining operator in the space of $\sigma(\bar{\chi})$ which fixes any $\mathrm{Gal}(k_d|k_{(d,f)})$ invariant Whittaker vector. Then Σ_{χ_f} is well defined, since $\varpi_D^{(d,f)}$ normalizes $\mathrm{Inf}(\sigma(\bar{\chi}))$, and $\Sigma_{\chi_f}(\varpi_F) = \chi_f((-1)^{e-1}\varpi_F)^e J^e = \chi(\varpi_F)I$. Finally, $\tilde{\Sigma}_{\chi_f} = \mathrm{Ind}_X^{\mathfrak{K}_1}\Sigma_{\chi_f}$. (For details in a more general context see §4.) If $A = D_n = \mathrm{M}_1(D_n)$, then for all $\chi_f \in X_t(F_f^\times)$ we have $\Pi_{\chi_f}^{D_n} \in \mathcal{R}_0^0(D_n^\times)$. In this case $X = O_D^\times\langle\varpi_D^f\rangle = \mathfrak{U}_D^1 D_e^\times$, $\Sigma_{\chi_f} = \hat{\tilde{\chi}}_f$, and $\tilde{\Sigma}_{\chi_f} = \Pi_{\chi_f}^{D_n}$ (cf. (12) and 4.8).
2. The Steinberg representation of $A^\times$ has the Langlands parameter $\chi_f \equiv 1$, so $f = 1$ and the twist ω_1^{m-1} is non-trivial whenever m is even. In this case, the maximal level zero extended type is, up to conjugacy, the inflation of $\omega_1^{m-1} \in \langle t_m \rangle^\wedge$ to $\mathfrak{K}_m = \langle t_m \rangle \ltimes \mathfrak{A}_m^\times$, where $\mathfrak{A}_m$ is the standard minimal principal order.

§1 The Proof of Theorem 1.

1.1 Proposition.

(i) Let $f \mid n$ and let $\chi_f \in X_t(F_f^\times)$ be F-regular. Then $\Pi_{\chi_f}^{D_n}$ is an irreducible level zero representation of dimension f and the class of $\Pi_{\chi_f}^{D_n}$ depends only upon the Galois orbit $[\chi_f]$. The mapping $[\chi_f] \mapsto \Pi_{\chi_f}^{D_n}$ defines a bijection of $\mathcal{T}_0^n$ to $\mathcal{R}_0(D_n^\times)$.

(ii) The character of $\Pi_{\chi_f}^{D_n}$ *is given by the formula*

$$\Theta_{\Pi_{\chi_f}^{D_n}}(x) = \begin{cases} 0, & \text{if } x \notin D_e^\times \mathfrak{U}_{D_n}^1 \\ \sum_{\eta\in[\chi_f]} \eta \circ N_{L|F_f}(x_0), & \text{if } x = x_0 u \in D_e^\times \mathfrak{U}_{D_n}^1, \end{cases}$$

where L *is a maximal subfield of* D_e *which contains* x_0.
(iii) Let $x \in D_n$ *be regular elliptic and assume that* $F_f \subset F(x)$. *Then*

$$\Theta_{\Pi_{\chi_f}^{D_n}}(x) = \sum_{\eta\in[\chi_f]} \eta \circ N_{F(x)|F_f}(x). \tag{2}$$

Proof. (i) Identifying the factor group

$$D_n^\times/\mathfrak{U}_{D_n}^1 \cong k_n^\times \rtimes \langle \varpi_{D_n} \rangle, \tag{3}$$

where the semi-direct product on the right is with respect to the Galois action on k_n which is induced by (0.6), we have a natural restriction map $\mathcal{R}_{\mathbb{C}}(D_n^\times/\mathfrak{U}_{D_n}^1) \to \mathrm{Gal}(k_n|k)\backslash X(k_n^\times)$. ¿From (1) and (3) it follows that $\Pi_{\chi_f}^{D_n}|_{k_n^\times}$ gives the character orbit $[\bar\chi_f \circ N_{k_n|k_f}] \in \mathrm{Gal}(k_n|k)\backslash X(k_n^\times)$, whence Wigner's little group method ([Se1], 8.2) applied to (3) implies that (1) is irreducible and that $[\chi_f] \mapsto \Pi_{\chi_f}^{D_n}$ gives the asserted bijection.

(ii) Since $\Pi_{\chi_f}^{D_n}$ is induced from a normal subgroup, it is easy to apply Frobenius's formula ([Se1], 7.2). We also use the fact that, for L maximal in D_e, $\mathrm{Nrd}_{D_e|F_f}|_L = \mathrm{N}_{L|F_f}$. Details are left to the reader.

(iii) follows from (ii) because $F(x) = L \subset D_e$ and $x = x_0$. □

1.2 Proposition. *(i) For any* $\eta \in X_t(F^\times)$ *and* $[\chi_f] \in \mathcal{T}_0^n$

$$(\eta \circ Nrd_{D_n|F}) \otimes \Pi_{\chi_f}^{D_n} = \Pi_{(\eta\circ N_{F_f|F})\chi_f}^{D_n}.$$

(ii) The central character of $\Pi_{\chi_f}^{D_n}$ *is* $\omega_{\Pi_{\chi_f}^{D_n}} = \chi_f^e|_{F^\times}$.
(iii) $\Pi_{\chi_f}^{D_n} \in \mathcal{S}_{\bar\chi}^{D_n}$, *where* $\bar\chi = \bar\chi_f \circ N_{k_n|k_f} \in X(k_n^\times)$ *and* $\mathcal{S}_{\bar\chi}^{D_n}$ *is as defined in §0.4. There are exactly* e *inequivalent unramified twists* $(\eta \circ Nrd_{D_n|F}) \otimes \Pi_{\chi_f}^{D_n}$ *such that* $\omega_{\Pi_{\chi_f}^{D_n}} = \omega_{(\eta\circ\mathrm{Nrd}_{D_n|F})\otimes\Pi_{\chi_f}^{D_n}}$.

Proof. We omit the proofs of (i) and (ii).
(iii) The map $X(k_n^\times) \ni \bar\chi \mapsto \sigma_{\bar\chi} \in \mathrm{GL}_{f'}(k_d)^\wedge_{\mathrm{cusp}}$ (§0.4 following 0.1) is the identity when $d = n$, so $f' = 1$. By (3), $\mathcal{S}_{\bar\chi}^{D_n}$ consists of all irreducible level zero representations Π such that $\bar\chi \subset \Pi|_{k_n^\times}$. The first assertion follows from the proof of 1.1(i); the second is left to the reader (or refer to [SZ2] 2.8). □

1.3 Proposition. *(i) For every central simple algebra* $A = M_m(D_d)$ *of reduced degree* $n = dm$ *the representation* $\Pi = \Pi_{\chi_f}^A := \mathcal{JL}_{A,D_n}(\Pi_{\chi_f}^{D_n})$ *is the unique discrete series representation of* $A^\times$ *such that*

$$\Theta_\Pi(x) = (-1)^{m-1} \sum_{\eta\in[\chi_f]} \eta(N_{F(x)|F_f}(x)) \tag{4}$$

for $x \in A^\times$ regular elliptic and such that F_f embeds in $F(x)$.

(ii) If $\lambda \in X_t(F^\times)$, then $(\lambda \circ Nrd_{A|F}) \otimes \Pi^A_{\chi_f} = \Pi^A_{(\lambda \circ N_{F_f|F})\chi_f}$. The central character of $\Pi^A_{\chi_f}$ is $\omega_{\Pi^A_{\chi_f}} = \chi_f^e|_{F^\times}$.

(iii) There are precisely f characters $\eta \in X_u(F^\times)$ such that $(\eta \circ Nrd_{A|F}) \otimes \Pi^A_{\chi_f} = \Pi^A_{\chi_f}$; there are exactly e inequivalent unramified twists $(\eta \circ Nrd_{A|F}) \otimes \Pi^A_{\chi_f}$ with equal central characters, such that $\omega_{\Pi^A_{\chi_f}} = \omega_{(\eta \circ \mathrm{Nrd}_{A|F}) \otimes \Pi^A_{\chi_f}}$.

(iv) The representation $\Pi^A_{\chi_f}$ is supercuspidal if and only if $(\frac{n}{f}, m) = 1$.

Proof. (i) By 1.1(iii), by the definition of $\Pi^A_{\chi_f}$, and by AMT we know that (4) must hold for $\Pi^A_{\chi_f}$. Conversely, if (4) is the character formula for $\Pi \in \mathcal{R}^2(A^\times)$, then $\mathcal{JL}_{D_n,A}(\Pi)$ is an irreducible representation of $D_n^\times$ with the corresponding character formula; thus AMT implies that it is enough to show that Π is uniquely determined by (4) in the case $A = D_n$. We therefore assume $A = D_n$, $m = 1$.

If $x \in D_n$ is very regular (i.e. if $x \in O_{D_n}^\times$ and $\bar{x}$ generates $k_n|k$) and if $u \in \mathfrak{U}^1_{D_n}$, then xu is very regular too. Therefore, since on regular elliptic elements Θ_Π has the same values as a character of a level zero representation, $\Theta_\Pi(x) = \Theta_\Pi(xu)$. From [SZ1] 1.1(ii) we know that the right side of (4) is not identically zero on very regular elements. Therefore Θ_Π is non-zero and constant on some $\mathfrak{U}^1_{D_n}$-coset, hence by Schur orthogonality Π is a level zero representation of $D_n^\times$; its dimension is f (look at $x = 1$). From 1.1(i) we see that $\Pi = \Pi^{D_n}_{\chi'_f}$ with $|[\chi'_f]| = f$; comparing 1.1(iii) and (4) yields

$$(5) \qquad \sum_{\eta' \in [\chi'_f]} \eta'(\mathrm{N}_{F(x)|F_f}(x)) = \sum_{\eta \in [\chi_f]} \eta(\mathrm{N}_{F(x)|F_f}(x))$$

for $x \in A^\times$ regular elliptic and such that $f \mid f_{F(x)|F}$. In particular, (5) holds for all x such that $F(x) = F_n$. Using [SZ1] 1.1(i) we see that $[\chi'_f \circ \mathrm{N}_{F_n|F}] = [\chi_f \circ \mathrm{N}_{F_n|F}] \in \mathrm{Gal}(F_n|F)\backslash X_t(F_n^\times)$, so $\chi'_f = \chi_f \cdot \lambda$, where λ is an unramified character of F_f of order dividing $e = \frac{n}{f}$. Since λ is $\mathrm{Gal}(F_f|F)$-invariant we obtain from (5) that

$$(6) \qquad (\lambda(\mathrm{N}_{F(x)|F_f}(x)) - 1)S(x) = 0,$$

where $S(x)$ denotes the right side of (5). Now consider $x = \alpha\varpi^f_{D_n}$ ($\alpha \in O^\times_{D_n}$) a root of the polynomial $g(T)$ of (0.13). In this case, $\mathrm{N}_{F(x)|F_f}(x) = (-1)^{e-1}\zeta\varpi_F$ up to $\mathrm{Gal}(F_f|F)$ conjugation. Using (6) we find that $(\lambda(\varpi_F)-1)\chi_f((-1)^{e-1}\varpi_F)\bar{S}(\bar{\zeta}) = 0$, where $\bar{S}(\bar{\zeta})$ denotes the finite field character sum defined by reducing $S(\zeta)$. By [SZ1] 1.1(ii) we know that $\bar{S}(\bar{\zeta}) \neq 0$. Therefore, $\lambda = 1$.
(ii),(iii) AMT implies that $\mathcal{JL}$ preserves central characters and commutes with character twists. Thus 1.2 implies these assertions. See [SZ2] 2.8 for a direct argument.
(iv) From 1.2(iii) and 0.4 Fact we see that $\Pi^A_{\chi_f} \in \mathcal{S}^A_{\bar{\chi}}$. By 0.2 Fact these representations are supercuspidal if and only if $e' = 1$. $\square$

§2 Level Zero $\mathfrak{A}_r^\times$ Submodules of Discrete Series Representations.

Let (Π, V) belong to the unramified twist class $\mathcal{S}^A_{\bar{\chi}}$ of level zero discrete series representations associated to $[\bar{\chi}] \in \mathrm{Gal}(k_n|k)\backslash X(k_n^\times)$ (§0.4 Facts 0.1 and 0.2). Recall that $f = f(\Pi) = |[\bar{\chi}]|$, $e = n/f$, and $e' = (e, m)$ and $\Pi|_{\mathfrak{A}^\times_{e'}}$ contains the type representations τ_i for all $i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$ (§0.4 Fact 0.3(ii)).

Now let $\mathfrak{A}_r$ be a standard principal order of A. Then the space $V_r := V^{\mathfrak{U}^1_{\mathfrak{A}_r}} \neq 0$ if and only if $r|e'$ (i.e. $\mathfrak{A}_{e'} \subset \mathfrak{A}_r$; see 0.9(i)). We write $\mathfrak{A}_r^{\times\wedge}(V_r)$ to denote the set of irreducible representations of $\bar{\mathfrak{A}}_r^\times$ occurring in V_r. In [SZ2], 2.6 & 3.1 we gave the decompositions of $V_{e'}$ and V_1 into irreducible representations of $\mathfrak{A}_{e'}^\times$ and $\mathfrak{A}_1^\times$, respectively. We want to reduce the classification of the components of the $\bar{\mathfrak{A}}_r^\times$ module V_r for arbitrary $r \mid e'$ to the known special cases.

We assign a value to s such that $rs = e'f' = m$ and we set $\delta := e'/r = s/f'$. Let $\mathfrak{S}_\delta^r$ denote the direct product of r copies of the symmetric group on δ letters $\mathfrak{S}_\delta$. We use the identifications of (0.8); in particular, $\bar{\mathfrak{A}}_{e'}^\times = GL_{f'}^{e'}(k_d)$ and $\bar{\mathfrak{A}}_r^\times = GL_s^r(k_d)$.

2.1 Proposition. *Let $\Pi \in \mathcal{S}_\chi^A$ and assume that $r \mid e'$. Divide each vector $i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$ into r subvectors, each a segment of length δ. Assume that any vector of permutations $(\pi_1, \dots, \pi_r) \in \mathfrak{S}_\delta^r$ acts on $i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$ by acting componentwise on the r segment subvectors of i, each segment being of length δ. Regard $\bar{\mathfrak{A}}_{e'}^\times$ as a Levi subgroup of $\bar{\mathfrak{A}}_r^\times$. Then the map*

$$(1) \qquad (\mathbb{Z}/(d,f)\mathbb{Z})^{e'} \ni i = (i_1, \dots, i_{e'}) \longmapsto \tau_i := \sigma^{\phi^{i_1}} \otimes \cdots \otimes \sigma^{\phi^{i_{e'}}} \in \mathfrak{A}_{e'}^{\times\wedge}(V_{e'})$$

induces bijections

$$(2) \qquad (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r \longleftrightarrow \mathfrak{A}_{e'}^{\times\wedge}(V_{e'})/(\mathcal{N}_{\bar{\mathfrak{A}}_r^\times}(\bar{\mathfrak{A}}_{e'}^\times)/\bar{\mathfrak{A}}_{e'}^\times) \longleftrightarrow \mathfrak{A}_r^{\times\wedge}(V_r),$$

the first bijection being induced by (1). In the middle term we consider the action of the factor group normalizer of $\bar{\mathfrak{A}}_{e'}^\times$ in $\bar{\mathfrak{A}}_r^\times$ divided by $\bar{\mathfrak{A}}_{e'}^\times$. In the second bijection, the mapping from right to left, is Jacquet restriction and, from left to right, parabolic induction with, in the reducible case, only the unique generic component occurring.[6] *Every irreducible constituent of V_r occurs with multiplicity one.*

Proof. For any factorization $rs = m$ we may consider the Levi subgroups $\bar{\mathfrak{A}}_r^\times \cong \mathrm{GL}_s^r(k_d)$ of $\bar{\mathfrak{A}}_1^\times$. If $r \mid e'$, then $\bar{\mathfrak{A}}_{e'}^\times$ is a Levi subgroup of $\bar{\mathfrak{A}}_r^\times$ and $V_{e'} \subset V_r \subset V_1$. The quotient group $\mathcal{N}_{\bar{\mathfrak{A}}_r^\times}(\bar{\mathfrak{A}}_{e'}^\times)/\bar{\mathfrak{A}}_{e'}^\times \cong \mathfrak{S}_\delta^r$ is the Weyl group which controls parabolic induction from $\bar{\mathfrak{A}}_{e'}^\times$ to $\bar{\mathfrak{A}}_r^\times$. Since the Jacquet (or Harish-Chandra) restriction mapping $r_{\bar{\mathfrak{A}}_{e'}^\times, \bar{\mathfrak{A}}_r^\times}$ is orthogonal projection upon the subspace $V_{e'} \subset V_r$, $r_{\bar{\mathfrak{A}}_{e'}^\times, \bar{\mathfrak{A}}_r^\times}$ is surjective. Since only $V_{e'}$ contains cuspidal components, we see from the transitivity of the restriction mappings that $r_{\bar{\mathfrak{A}}_{e'}^\times, \bar{\mathfrak{A}}_r^\times}$ does not annihilate any component of V_r. In particular, since $V_{e'}$ decomposes simply (see 0.3(ii)), V_r also decomposes simply and $r_{\bar{\mathfrak{A}}_{e'}^\times, \bar{\mathfrak{A}}_r^\times}$ partitions the set of irreducible components of $V_{e'}$ into a collection of non-empty subsets, each the $r_{\bar{\mathfrak{A}}_{e'}^\times, \bar{\mathfrak{A}}_r^\times}$ image of a component of V_r. We omit the proof that for any component of V_r this set is a full orbit under the action of $\mathfrak{S}_\delta^r$. We gave a proof for the case $r = 1$ in the proof of [SZ2]3.1(i); the proof of the general case is essentially the same. □

§3 The $\mathfrak{K}_r$ Decomposition of V_r.

In §2 we studied the $\mathfrak{A}_r^\times$ module structure on $V_r = V^{\mathfrak{U}^1_{\mathfrak{A}_r}}$ for $(\Pi, V) \in \mathcal{S}_\chi^A$. Since the normalizer $\mathfrak{K}_r$ of $\mathfrak{A}_r$ also normalizes $\mathfrak{U}^1_{\mathfrak{A}_r}$, it follows that V_r is also a $\mathfrak{K}_r$ submodule of V. We want to consider now the $\mathfrak{K}_r$ module structure on V_r for $r \mid e'$.

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[6] We do not prove this assertion here. In the case which concerns us (cf §4) the component we consider arises via an irreducible parabolic induction from a cuspidal representation of a Levi subgroup.

We consider $\mathfrak{K}_r = \mathfrak{A}_r^\times \rtimes \langle t_r\rangle$ with the uniformizer $t_r := (t_{e'})^{e'/r} = (t_{e'})^\delta$. We define an action of $\langle t_{e'}\rangle$ on $(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$ by setting

$$t_{e'}(i) = t_{e'}(i_1,\dots,i_{e'}) = (i_2,\dots,i_{e'},i_1+1), \tag{1}$$

for $i=(i_1,\dots,i_{e'})\in(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}$. Next we set

$$i=(j_1,\dots,j_r),\quad j_\ell=(i_{(\ell-1)\delta+1},\dots,i_{(\ell-1)\delta+\delta})\quad (1\le\ell\le r)$$

to decompose i into a vector of r subvector segments, each of length δ. We obtain from (1) that

$$t_r(i)=t_{e'}^\delta(i)=(j_2,\dots,j_r,j_1'),\quad j_1'=(i_1',\dots,i_\delta')=(i_1+1,\dots,i_\delta+1). \tag{2}$$

For any $s=(\pi_1,\dots,\pi_r)\in\mathfrak{S}_\delta^r$ we see that $t_rs(i)=(t_rst_r^{-1})t_r(i)$. Since

$$t_rst_r^{-1}=t_r(\pi_1,\dots,\pi_r)t_r^{-1}=(\pi_2,\dots,\pi_r,\pi_1)=s'\in\mathfrak{S}_\delta^r,$$

$\langle t_r\rangle$ acts on $(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$. We want to relate this action to the action of $\langle t_r\rangle/\langle\varpi_F\rangle=\mathfrak{K}_r/\mathfrak{A}_r^\times F^\times$ on $(\mathfrak{A}_r^\times)^\wedge(V_r)$ which is defined via conjugation. We set

$$(t_r\tau t_r^{-1})(x):=\tau(t_r^{-1}xt_r)\qquad (x\in\mathfrak{A}_r^\times,\tau\in(\mathfrak{A}_r^\times)^\wedge). \tag{3}$$

3.1 Lemma. *Let $r\mid e'$. Then the bijections of 2.1 are equivariant with respect to the action of $\langle t_r\rangle$ on both sides; in particular, $t_{e'}\tau_it_{e'}^{-1}=\tau_{t_{e'}(i)}$ for $i=(i_1,\dots,i_{e'})$ and τ_i as in (2.1).*

Proof. We first show that the mapping $\mu_{e'}:(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}\to\mathfrak{A}_{e'}^{\times\wedge}(V_{e'})$ of (2.1) is $\langle t_{e'}\rangle$ equivariant with respect to (1). Since $t_{e'}$ acts by conjugation on $\bar{\mathfrak{A}}_{e'}^\times=\mathrm{GL}_{f'}^{e'}(k_d)$ such that

$$t_{e'}^{-1}\mathrm{diag}(A_1,\dots,A_{e'})t_{e'}=\mathrm{diag}(\phi^{-1}(A_{e'}),A_1,\dots,A_{e'-1}), \tag{4}$$

where $\phi^{-1}(A_{e'})=\varpi_{D_d}^{-1}A_{e'}\varpi_{D_d}$, it follows from (2.1) that, for $i=(i_1,\dots,i_{e'})$,

$$(t_{e'}\tau_it_{e'}^{-1})(A_1,\dots,A_{e'})=\tau_i(\phi^{-1}(A_{e'}),A_1,\dots,A_{e'-1})\sim\tau_{i'}(A_1,\dots,A_{e'}) \tag{5}$$

with $i'=t_{e'}(i)=(i_2,\dots,i_{e'-1},i_1+1)$. We use here the convention $\sigma^\phi(A)=\sigma(\phi^{-1}(A))$ for the representation σ of $GL_{f'}(k_d)\ni A$. Next we show that this $\langle t_{e'}\rangle$ equivariant map induces via (2) an equivariant action of $\langle t_r\rangle$ on (2.2). Since t_r normalizes $\mathfrak{A}_r^\times$ and since Jacquet restriction is functorial, it follows that the second arrow of (2.2) is $\langle t_r\rangle$ equivariant. The equivariance of the first arrow follows from the identification $\mathfrak{S}_\delta^r\cong\mathcal{N}_{\bar{\mathfrak{A}}_r^\times}(\bar{\mathfrak{A}}_{e'}^\times)/\bar{\mathfrak{A}}_{e'}^\times$, which maps a permutation vector to a block diagonal permutation matrix. $\square$

By 3.1 we may study the $\langle t_r\rangle$ action on $(\mathfrak{A}_r^\times)^\wedge(V_r)$ by studying the combinatorial model $(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$. Clearly, $dr=[\mathfrak{A}_r^\times\langle t_r\rangle:\mathfrak{A}_r^\times F^\times]$. We have already seen that $V_r\ne(0)$ if and only if $r\mid e'$. Anticipating 4.2(ii), we assume both that $e\mid dr$ and that $r\mid e'$. Set

$$c:=dr/e\qquad\text{and}\qquad\delta:=e'/r. \tag{6}$$

We also fix the notations

$$\delta_0:=((d,f),e'),\qquad r_0:=e'/\delta_0,\qquad\text{and}\quad c_0:=dr_0/e. \tag{7}$$

3.2 Lemma. *(i) The properties* $e \mid dr$ *and* $r \mid e'$ *are equivalent to* $r_0 \mid r \mid e'$*. In particular,* c_0 *is an integer and* $c_0 \mid c \mid (d,f)$*.*
(ii) $c\delta = c_0\delta_0 = (d,f)$*.*
(iii) $\delta_0/\delta = c/c_0 = r/r_0$*; in particular, these quotients are integer.*
(iv) $c_0 = f/(f,m)$ *and* $r_0 = m/(f,m)$*; in particular,* $(c_0, r_0) = 1$*.*

Proof. (i) Since $(e/(e,m), m/(e,m)) = (d/(d,f), f/(d,f)) = 1$ and

$$\frac{e}{(e,m)}\frac{f}{(d,f)} = \frac{m}{(e,m)}\frac{d}{(d,f)},$$

(8)
$$\frac{e}{(e,m)} = \frac{d}{(d,f)} \qquad \text{and} \qquad \frac{f}{(d,f)} = \frac{m}{(e,m)}.$$

Using $(8)_1$ and $(7)_2$ we obtain:

$$e \mid dr \Leftrightarrow e' \cdot \frac{e}{e'} \mid \frac{d}{(d,f)} \cdot (d,f)r \Leftrightarrow e' \mid (d,f)r \Leftrightarrow r_0 \mid r.$$

Multiplying $r_0 \mid r \mid e'$ with d/e we obtain $c_0 \mid c \mid (d,f)$.

(ii) $c\delta = \frac{dr}{e} \cdot \frac{e'}{r} = d \cdot \frac{(d,f)}{d}$, and we may replace c, δ, r by c_0, δ_0, r_0.

(iii) follows from (i) and (ii).

(iv) Since $f' = f/(d,f)$, $(8)_2$ implies that $e'f' = m$ and that $e'f = (d,f)m$. Dividing both sides of the last equation by the product $((d,f),e')(f,m)$ and arguing as in the proof of (i), we find that

(9)
$$r_0 = e'/((d,f),e') = m/(f,m).$$

Using the identity $dm = ef$ and (9) we see that

(10)
$$c_0 = dr_0/e = fr_0/m = f/(f,m). \quad \square$$

Now let $C := c\mathbb{Z}/(d,f)\mathbb{Z}$. Then C is the subgroup of $\mathbb{Z}/(d,f)\mathbb{Z}$ of order δ. The bijective mapping $\mathbb{Z}/c\mathbb{Z} \ni \alpha \longmapsto \alpha + C$ from $\mathbb{Z}/c\mathbb{Z}$ to the quotient of $\mathbb{Z}/(d,f)\mathbb{Z}$ mod C identifies $\alpha \in \mathbb{Z}/c\mathbb{Z}$ with its whole coset, consisting of δ elements, of $\mathbb{Z}/(d,f)\mathbb{Z}$. We may fix an ordering of C and regard the coset $\alpha + C$ as a vector, e.g. $\alpha + C = (\alpha + 0, \alpha + c, \dots, \alpha + (\delta - 1)c)$.

More generally (recalling that $e' = \delta r$) we define the injective maps

(11)
$$\iota : (\mathbb{Z}/c\mathbb{Z})^r \to (\mathbb{Z}/(d,f)\mathbb{Z})^{e'} \qquad \bar{\iota} : (\mathbb{Z}/c\mathbb{Z})^r \to (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$$

by sending

(12)
$$(\mathbb{Z}/c\mathbb{Z})^r \ni (\alpha_1, \dots, \alpha_r) \overset{\iota}{\longmapsto} (\alpha_1 + C, \dots, \alpha_r + C) \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}.$$

In passing to $\bar{\iota}$ we "throw away" the ordering of C. Setting

$$t_r(\alpha_1, \dots, \alpha_r) = (\alpha_2, \dots, \alpha_r, \alpha_1 + 1),$$

we see that (12) becomes a t_r equivariant injection.

3.3 Proposition. *(i) The action of $\langle t_r\rangle$ partitions $(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$ into orbits which have lengths which are divisible by c and which divide $ce = dr$.*
(ii) In the case $\frac{c}{c_0} = \frac{r}{r_0} = \frac{\delta_0}{\delta} > 1$ there is no orbit of length c.
(iii) If $\frac{c}{c_0} = \frac{r}{r_0} = \frac{\delta_0}{\delta} = 1$, there exists precisely one orbit of length c and this is the orbit which contains $\bar\iota(\mathfrak{h}) = (C; h+C; \ldots ; (r-1)h+C)$, where $h = 0$ if $c = c_0 = 1$, $hr = 1 \bmod c\mathbb{Z}$ if $c = c_0 > 1$, and $\mathfrak{h} = (0, h, \ldots , (r-1)h) \in (\mathbb{Z}/c\mathbb{Z})^r$.

Proof. If $(d, f) = 1$, then $\delta_0 = \delta = 1$, $c_0 = c = 1$, and $r_0 = r = e'$. In this case, 3.3 is trivially true, so assume that $(d, f) > 1$.
(i) Consider the mapping $\varphi : (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r \to \mathbb{Z}/(d,f)\mathbb{Z}$ which is defined by setting $\varphi(\bar i) := \sum_{\nu=1}^{e'} i_\nu$ for any $i = (i_1, \ldots , i_{e'}) \in \bar i$. With respect to the action $t_r(x) = x + \delta$ for $x \in \mathbb{Z}/(d,f)\mathbb{Z}$, we see that φ is $\langle t_r\rangle$ equivariant. Therefore, $t_r^k(\bar i) = \bar i$ implies that

$$t_r^k(\varphi(\bar i)) = \varphi(t_r^k(\bar i)) = \varphi(\bar i) \quad \text{and} \quad \varphi(\bar i) + k\delta = \varphi(\bar i) \in \mathbb{Z}/(d,f)\mathbb{Z}.$$

This implies that $c \mid k$, since $c = (d, f)/\delta$. Moreover, $t_r^{ce} = t_r^{dr} = \varpi_F$ acts trivially, so the action of the cyclic group $\langle t_r\rangle$ factors through $\langle t_r\rangle/\langle t_r^{ce}\rangle$. This implies that each orbit length divides $ce = dr$ and completes the proof of (i).

For the proofs of (ii) and (iii) we need:

3.4 Lemma. *Any $\langle t_r\rangle$ orbit of length c in $(\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$ lies in the image of the embedding $\bar\iota$ defined in (11).*

Proof. Let S denote the set of non-negative integer valued functions ψ on $\mathbb{Z}/(d,f)\mathbb{Z}$ such that $\sum_{\nu\in\mathbb{Z}/(d,f)\mathbb{Z}} \psi(\nu) = \delta$ and let $\mathbb{Z}/(d,f)\mathbb{Z}$ act on S by translations:

$$(\mu + \psi)(\nu) := \psi(\nu - \mu) \qquad (\mu, \nu \in \mathbb{Z}/(d,f)\mathbb{Z}). \tag{13}$$

For $i = (i_1, \ldots , i_{e'}) = (j_1, \ldots , j_r) \in (\mathbb{Z}/(d,f)\mathbb{Z})^{\delta r}$ (see (2)) let $\psi_{\ell,i}(\nu)$ be the multiplicity of $\nu \in \mathbb{Z}/(d,f)\mathbb{Z}$ in the vector $j_\ell = (i_{(\ell-1)\delta+1}, \cdots , i_{\ell\delta})$ $(1 \le \ell \le r)$. Then the functions $\psi_{\ell,i}$ belong to S and depend only on $\bar i \in (\mathbb{Z}/(d,f)\mathbb{Z})^{\delta r}/\mathfrak{S}_\delta^r$. Now fix an orbit $\langle t_r\rangle\bar i \subset (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$ and consider the set J of functions $\psi = \psi_{\ell,t_r^k(i)}$ associated with that orbit. From (2) it follows that $1 + \psi \in J$; indeed $\psi_{\ell,t_r^r(i)} = 1 + \psi_{\ell,i}$ for all ℓ. If the orbit is of length c, then $\psi_{\ell,i} = \psi_{\ell,t_r^{cr}(i)} = c + \psi_{\ell,i}$, which implies that $\psi = c + \psi$ for all functions $\psi \in J$. Thus the support of ψ is a coset of $C = c\mathbb{Z}/(d,f)\mathbb{Z}$ in $\mathbb{Z}/(d,f)\mathbb{Z}$. Since $\psi = \psi_{\ell,i}$ for some $\ell = 1, \ldots , r$, we see that the subvectors j_ν of $i = (i_1, \ldots , i_{e'}) = (j_1, \ldots , j_r)$ have to be C cosets, and this means that $\bar i$ belongs to the image of $\bar\iota$. $\square$

We proceed with the proof of 3.3:
(ii) Let $\alpha = (\alpha_1, \ldots , \alpha_r) \in (\mathbb{Z}/c\mathbb{Z})^r$. If $t_r^c(\alpha) = \alpha$, then $t_r^{rc_0}(\alpha) = t_r^{r_0c}(\alpha) = \alpha$, hence $(\alpha_1 + c_0, \ldots , \alpha_r + c_0) = (\alpha_1, \ldots , \alpha_r) \in (\mathbb{Z}/c\mathbb{Z})^r$. But $c_0 < c$, by hypothesis, so this is impossible.
(iii) In this case, $c = c_0$, $r = r_0$, $\delta = \delta_0$, and $(c, r) = (f/(f, m), m/(f, m)) = 1$. Consider $\alpha = (\alpha_0, \ldots , \alpha_{r-1}) \in (\mathbb{Z}/c\mathbb{Z})^r$. Since $t_r^r(\alpha) = (\alpha_0 + 1, \ldots , \alpha_{r-1} + 1)$ and $(c, r) = 1$, for studying the $\langle t_r\rangle$ orbit of α we may assume that $\alpha_0 = 0 \in \mathbb{Z}/c\mathbb{Z}$. Similarly, since $(c, r) = 1$, we see that, if $t_r^c(\alpha) = \alpha$, then every component of α is determined by any single component. Thus there is at most one α which is fixed by t_r^c such that $\alpha_0 = 0$. Let $\alpha = \mathfrak{h} = (0, h, \ldots , (r-1)h)$. Then $t_r(\mathfrak{h}) = (h, \ldots , (r-1)h, 1) = (h, \ldots , (r-1)h, rh) \bmod c\mathbb{Z}$, since $rh = 1 \bmod c\mathbb{Z}$. It follows by induction that, for any $k \ge 1$, $t_r^k(\mathfrak{h}) = \mathfrak{h} + k(h, \ldots , h) \bmod c\mathbb{Z}$. Therefore, $t_r^k(\mathfrak{h}) = \mathfrak{h}$ if and only if $c \mid k$; in particular, $t_r^c(\mathfrak{h}) = \mathfrak{h}$. $\square$

§4 Existence and Uniqueness of Maximal Level Zero Extended Types.

Now we consider the unramified extension $F_n|F$ and a tame character $\chi \in X_t(F_n^\times)$ with reduction $\bar\chi \in X(k_n^\times)$. We note that the Galois orbits of χ and $\bar\chi$ with respect to F and k, respectively, have the same order f. The central character of $\Pi \in \mathcal{S}_{\bar\chi}^A$ is a tame character $\omega_\Pi \in X_t(F^\times)$ which has the reduction $\bar\omega_\Pi = \bar\chi|_{k^\times} \in X(k^\times)$. The pair $(\bar\chi, \omega_\Pi)$ determines a unique $\chi \in X_t(F_n^\times)$ with reduction $\bar\chi$ and restriction $\chi|_{F^\times} = \omega_\Pi$, and we write

$$\mathcal{S}_\chi^A := \{\Pi \in \mathcal{S}_{\bar\chi}^A \,:\, \omega_\Pi = \chi_F := \chi|_{F^\times}\} \tag{1}$$

for the subset consisting of all representations with the fixed central character χ_F. In [SZ2]2.8 we pointed out that $\mathcal{S}_\chi^A$ contains exactly $e = n/f$ inequivalent representations (see 1.3(iii)). Because $\mathcal{S}_{\bar\chi}^A = \mathcal{S}_{[\bar\chi]}^A$ only depends on the Galois orbit $[\bar\chi] \in \mathrm{Gal}(k_n|k)\backslash X(k_n^\times)$ we see that also $\mathcal{S}_\chi^A$ only depends on $[\chi] \in \mathrm{Gal}(F_n|F)\backslash X_t(F_n^\times)$, and the disjoint union $\mathcal{R}_0^2(A^\times) = \coprod_{[\bar\chi]} \mathcal{S}_{\bar\chi}^A$ has the refinement $\mathcal{R}_0^2(A^\times) = \coprod_{[\chi]} \mathcal{S}_\chi^A$.

For $\Pi \in \mathcal{S}_\chi^A$ the considerations of §§2,3 apply to the obvious $\mathfrak{A}_r^\times F^\times$ module structures on the spaces V_r. In particular, the types τ_i (see 0.3 and 2.1) extend by χ_F to representations of $\mathfrak{A}_{e'}^\times F^\times$. The bijections of (2.2) also apply to the extended structures; in this new context we write $(\mathfrak{A}_r^\times F^\times)^\wedge(V_r, \mathcal{S}_\chi^A)$ in place of $(\mathfrak{A}_r^\times)^\wedge(V_r)$.

For the definition and generalities concerning "level zero extended type representations" the reader should refer to §0.6.

4.1 Proposition.
(i) Every $\Pi \in \mathcal{R}_0^2(A^\times)$ admits, up to conjugacy, exactly one maximal level zero extended type representation $(\mathfrak{K}, \tilde\Sigma)$.
(ii) If $\Pi \in \mathcal{S}_{\bar\chi}^A$, then $\mathfrak{K} = \mathfrak{K}_{m/(f,m)}$ and $\tilde\Sigma$ is the only level zero $\mathfrak{K}_{m/(f,m)}$ component of Π whose restriction to $\mathfrak{A}_{m/(f,m)}^\times$ decomposes into the sum of exactly $f/(f,m)$ constituents.

Proof. Let $(\mathfrak{K}, \tilde\Sigma)$ be a pair such that $\mathfrak{K}$ is a mcmc subgroup of $A^\times$ and $\tilde\Sigma \in \mathfrak{K}^\wedge$ is level zero. By [BF](1.3.1), $\mathfrak{K}$ is the normalizer of a principal order $\mathfrak{A}$. It is enough to prove 4.1(i) for $\Pi \in \mathcal{S}_{\bar\chi}^A$, since $\mathcal{R}_0^2(A^\times) = \coprod_{[\bar\chi]\in \mathrm{Gal}(\bar k_n|k)\backslash X(k_n^\times)} \mathcal{S}_{\bar\chi}^A$ (0.2 Fact). Up to conjugacy we may assume that $\mathfrak{A} = \mathfrak{A}_r$, a standard principal order, and that $\mathfrak{K} = \mathfrak{K}_r$, the normalizer of $\mathfrak{A}_r$. First we prove a necessary condition:

4.2 Lemma. *If $(\mathfrak{K}_r, \tilde\Sigma)$ is a maximal level zero extended type for $\Pi \in \mathcal{S}_{\bar\chi}^A$, then:*
(i) $r \mid e'$.
(ii) $e \mid dr$, where $dr = [\mathfrak{K}_r : \mathfrak{A}_r^\times F^\times]$.
(iii) Let $\mathfrak{A}_r^\times F^\times \subset X \subset \mathfrak{K}_r$, where $\mathfrak{K}_r = \langle t_r\rangle \ltimes \mathfrak{A}_r^\times$ and X is the unique group such that $[X : \mathfrak{A}_r^\times F^\times] = e$. Assume that $(\mathfrak{A}_r^\times F^\times)^\wedge \ni \rho \subset \tilde\Sigma|_{\mathfrak{A}_r^\times F^\times} \subset \Pi|_{\mathfrak{A}_r^\times F^\times}$. Then there is a unique extension $\tilde\rho$ of ρ to X such that $\tilde\Sigma = \mathrm{Ind}_X^{\mathfrak{K}_r}\tilde\rho$. Equivalently, the $\langle t_r\rangle$ orbit of ρ is of length dr/e.

Proof. (i) Let $(\mathfrak{K}_r, \tilde\Sigma)$ be a maximal level zero extended type for $\Pi \in \mathcal{S}_{\bar\chi}^A$. Then $\tilde\Sigma|_{\mathfrak{A}_r^\times} \subset \Pi|_{\mathfrak{A}_r^\times}$ and $\tilde\Sigma|_{\mathfrak{A}_r^\times}$ has, by assumption, level zero components, so $r \mid e'$ (see 0.9(i)).
(ii) Let $(\mathfrak{A}_r^\times F^\times)^\wedge \ni \rho \subset \tilde\Sigma|_{\mathfrak{A}_r^\times F^\times} \subset \Pi|_{\mathfrak{A}_r^\times F^\times}$ and suppose that $X \subset \mathfrak{K}_r$ is the normalizer of ρ. Then there are precisely $[X : \mathfrak{A}_r^\times F^\times]$ inequivalent irreducible representations $\tilde\Sigma' \in \mathfrak{K}_r^\wedge$ such that $\rho \subset \tilde\Sigma'|_{\mathfrak{A}_r^\times F^\times}$; each of these representations may

be constructed by inducing an extension $\tilde{\rho} \in X^\wedge$ to $\mathfrak{K}_r$. The set of inequivalent representations $\tilde{\Sigma}'$ is also the set of inequivalent unramified twists of $\tilde{\Sigma}$ which satisfy the condition $\rho \subset \tilde{\Sigma}'|_{\mathfrak{A}_r^\times F^\times}$. For any $\lambda \in X_u(A^\times)$ which is trivial on $F^\times$ it follows from the compactness of $\mathfrak{A}_r^\times$ that $\rho \subset \lambda\tilde{\Sigma}|_{\mathfrak{A}_r^\times F^\times} \subset \lambda\Pi|_{\mathfrak{A}_r^\times F^\times}$. By the last statement of 2.1 ρ occurs simply in $\lambda\Pi|_{\mathfrak{A}_r^\times F^\times}$ for each λ. Thus we see that $\lambda\Pi \sim \Pi$ implies $\lambda\tilde{\Sigma} \sim \tilde{\Sigma}$. Moreover if $\tilde{\Sigma}$ is a level zero extended type for Π, then $\lambda\tilde{\Sigma}$ is a level zero extended type for $\lambda\Pi$, and therefore $\lambda\tilde{\Sigma} \sim \tilde{\Sigma}$ implies $\lambda\Pi \sim \Pi$. Therefore, $\lambda\tilde{\Sigma} \not\sim \tilde{\Sigma}$ if and only if $\lambda\Pi \not\sim \Pi$, so $[X : \mathfrak{A}_r^\times F^\times]$ equals both the number of inequivalent twists of $\tilde{\Sigma}$ by elements $\lambda \in X_u(A^\times)$ which are trivial on $F^\times$ and the cardinality $|\mathcal{S}_\chi^A|$, where χ has reduction $\bar{\chi}$. Since $|\mathcal{S}_\chi^A| = e$ (1.3(iii)), $[X : \mathfrak{A}_r^\times F^\times] = e$ and, since $dr = [\mathfrak{K}_r : \mathfrak{A}_r^\times F^\times] = [\mathfrak{K}_r : X][X : \mathfrak{A}_r^\times F^\times]$, we conclude that $e \mid dr$.
(iii) Since $t_r^{dr} = \varpi_F$ and $[X : \mathfrak{A}_r^\times F^\times] = e$, it follows that $X = \langle t_r^{dr/e}\rangle \ltimes \mathfrak{A}_r^\times$. Let $\Pi \in \mathcal{S}_{\bar{\chi}}^A$ have the level zero extended type $\tilde{\Sigma} \in \mathfrak{K}_r^\wedge$ and let $(\mathfrak{A}_r^\times F^\times)^\wedge \ni \rho \subset \tilde{\Sigma}|_{\mathfrak{A}_r^\times F^\times}$ have the unique extension $\tilde{\rho} \in X^\wedge$ such that $\tilde{\Sigma} = \mathrm{Ind}_X^{\mathfrak{K}_r}\tilde{\rho}$. Then, by (ii), $[\mathfrak{K}_r : X] = dr/e$ and, since X is the normalizer of ρ, ${}^{t_r^k}\rho \not\sim {}^{t_r^{k'}}\rho$ $(0 \le k < k' < dr/e)$. $\square$

We return to the proof of 4.1. From 4.2 we see that $\rho|_{\mathfrak{A}_r^\times}$ is an irreducible level zero component of $\Pi|_{\mathfrak{A}_r^\times}$ such that the $\langle t_r\rangle$ orbit of $\rho|_{\mathfrak{A}_r^\times}$ is of length $c = dr/e$, where $r \mid e'$. ¿From 3.3 we know that such an orbit exists if and only if $r = r_0 = m/(f,m)$ and in this case the orbit is unique. Therefore, if Π admits the maximal level zero extended type $(\mathfrak{K}_r, \tilde{\Sigma})$, then $r = m/(f,m)$ and $\tilde{\Sigma}$ is also uniquely determined.

Conversely, we must show that every $\Pi \in \mathcal{S}_\chi^A$ admits a level zero extended type. The equalities $[\mathfrak{K}_r : \mathfrak{A}_r^\times F^\times] = dr = ce$ together with 3.3(iii) imply that for $r = r_0$ and $c = c_0$ there exists, up to conjugacy by $\langle t_r\rangle$, a unique $\rho \in (\mathfrak{A}_r^\times F^\times)^\wedge(V_r, \mathcal{S}_\chi^A)$ such that $\mathrm{Ind}_{\mathfrak{A}_r^\times F^\times}^{\mathfrak{K}_r}(\rho) = \sum_{i=1}^e \tilde{\Sigma}_i$, where the $\tilde{\Sigma}_i$ are inequivalent irreducible representations of $\mathfrak{K}_r$. Exactly one $\tilde{\Sigma}_i$ $(i = i(\Pi))$ occurs as a $\mathfrak{K}_r$ component of Π because

$$\langle \mathrm{Ind}_{\mathfrak{A}_r^\times F^\times}^{\mathfrak{K}_r}(\rho), \Pi|_{\mathfrak{K}_r}\rangle_{\mathfrak{K}_r} = \langle \rho, \Pi|_{\mathfrak{A}_r^\times F^\times}\rangle_{\mathfrak{A}_r^\times F^\times} = 1.$$

Consider the map $\mathcal{S}_\chi^A \ni \Pi \mapsto \tilde{\Sigma}_{i(\Pi)}$. We obtain any $\tilde{\Sigma}_i$ as $\lambda\tilde{\Sigma}_{i(\Pi)}$ by some choice of $\lambda \in X_u(A^\times)$ such that λ is trivial on $F^\times$. Obviously ρ is an $\mathfrak{A}_r^\times F^\times$ component of $\lambda\Pi$, $\lambda\tilde{\Sigma}_{i(\Pi)} \subset \lambda\Pi|_{\mathfrak{K}_r}$, and $\lambda\Pi \in \mathcal{S}_\chi^A$. Thus the mapping $\Pi \mapsto \tilde{\Sigma}_{i(\Pi)}$ induces a surjective mapping of $\mathcal{S}_\chi^A$ to the set $\{\tilde{\Sigma}_i\}_{i=1}^e$. Thus the mapping is bijective and inequivalent elements of $\mathcal{S}_\chi^A$ correspond to distinct elements of $\{\tilde{\Sigma}_i\}$. Therefore every element of $\mathcal{S}_\chi^A$ admits a level zero extended type. $\square$

Now we give an explicit construction of the unique maximal level zero extended type of $\Pi \in \mathcal{S}_\chi^A$. *From now on we change the notation to write* $r := r_0 = m/(f,m)$, $s := (f,m)$, $c := c_0 = f/(f,m)$, $\delta := \delta_0 = e'/r = s/f' = (d,e,f,m)$(see (3.6-7)). As in 3.3(iii) let h be an inverse of r modulo c, with the exception $h = 0$ if $c = 1$.

4.3 Proposition.
(i) The set $(\mathfrak{A}_r^\times F^\times)^\wedge(V_r, \mathcal{S}_\chi^A)$ *contains a unique* $\langle t_r\rangle$ *orbit which consists of exactly* c *inequivalent representations* $\pi_1, \pi_2, \dots, \pi_c$ *and each representation of the orbit is normalized by* t_r^c. *These representations correspond to* $\bar{\iota}(\mathfrak{h})$ *and its* $\langle t_r\rangle$ *transforms (see (3.3), (3.12), and 3.3(iii)) under the bijections of (2.2). The numeration may be set such that* $\pi_1 = \mathit{Ind}_{\mathfrak{A}_{e'}^\times\langle\varpi_F\rangle}^{\mathfrak{A}_r^\times\langle\varpi_F\rangle}(\tau_{\iota(\mathfrak{h})}\chi_F)$, *where* $\iota(\mathfrak{h})$ *is defined by (3.12) and* $\mathfrak{h}$ *by 3.3(iii).*

(ii) Let $\pi := \pi_1$ and let $\bar{\pi} \in GL_s^r(k_d)^\wedge$ denote the restriction and reduction of π. Let $\sigma := \sigma_{\bar{\chi}} \in GL_{f'}(k_d)^\wedge_{\text{cusp}}$ (see the paragraph before 0.2), let $\Pi_{d/c}(\bar{\chi}) := I(\bigotimes_{\nu=0}^{\delta-1} \sigma^{\phi^{c\nu}})$ be the parabolic induction of $\bigotimes_{\nu=0}^{\delta-1} \sigma^{\phi^{c\nu}}$ from $GL_{f'}^\delta(k_d)$ to $GL_s(k_d)$, and let

$$\Pi_{d/c}(\bar{\chi})^\sharp := \Pi_{d/c}(\bar{\chi}) \otimes {}^{\phi^h}\Pi_{d/c}(\bar{\chi}) \otimes \cdots \otimes {}^{\phi^{h(r-1)}}\Pi_{d/c}(\bar{\chi}). \tag{2}$$

Then $\bar{\pi} = \Pi_{d/c}(\bar{\chi})^\sharp$.

Proof. (i) The restriction and reduction of $\mathrm{Ind}_{\mathfrak{A}_{e'}^\times\langle\varpi_F\rangle}^{\mathfrak{A}_r^\times\langle\varpi_F\rangle}(\tau_{\iota(\mathfrak{h})}\chi_F)$ is $\mathrm{Ind}_{\bar{\mathfrak{A}}_{e'}^\times}^{\bar{\mathfrak{A}}_r^\times}(\bar{\tau}_{\iota(\mathfrak{h})})$. Identifying $\bar{\mathfrak{A}}_r^\times = \mathrm{GL}_s^r(k_d)$ and $\bar{\mathfrak{A}}_{e'}^\times = \mathrm{GL}_{f'}^{e'}(k_d)$ as in (0.8) we see that the latter induction is a tensor product of r parabolic inductions from $\mathrm{GL}_{f'}^\delta(k_d)$ to $\mathrm{GL}_s(k_d)$ ($f'\delta = s$). Each of these parabolic inductions comes from the tensor product of δ inequivalent cuspidal representations of $\mathrm{GL}_{f'}(k_d)$ such that these cuspidal representations correspond to a coset of $C = c\mathbb{Z}/(d,f)\mathbb{Z}$ in $\mathbb{Z}/(d,f)\mathbb{Z}$. A C coset indexes δ inequivalent power of ϕ conjugates of $\sigma_{\bar{\chi}}$, the length of the $\langle\phi\rangle$ orbit of $\sigma_{\bar{\chi}}$ being (d,f). Applying the bijections of (2.2), we see that the length of the $\langle t_r\rangle$ orbit of the irreducible representation $\mathrm{Ind}_{\mathfrak{A}_{e'}^\times F^\times}^{\mathfrak{A}_r^\times F^\times}(\tau_{\iota(\mathfrak{h})}\chi_F)$ has the same length as the length of the $\langle t_r\rangle$ orbit of $\bar{\iota}(\mathfrak{h}) \in (\mathbb{Z}/(d,f)\mathbb{Z})^{e'}/\mathfrak{S}_\delta^r$, and this length is c. Clearly, all the representations in the orbit are constructed by similar irreducible parabolic inductions; the numeration may be set as indicated, so the set $\{\pi_1, \ldots, \pi_c\}$ is a $\langle t_r\rangle$ orbit with t_r^c normalizing each element of this orbit.
(ii) The explicit description of $\bar{\pi}$ follows immediately from the explicit description of $\bar{\iota}(\mathfrak{h})$ in 3.3(iii), using the bijections of 2.1. □

Next we construct and parameterize the e distinct extensions of π to X which induce the maximal level zero extended types for the representations of $\mathcal{S}_\chi^A$.

Since t_r^c normalizes π without centralizing $\mathfrak{A}_r^\times$, we need an intertwining operator to extend $\pi|_{\mathfrak{A}_r^\times}$ to $\mathfrak{A}_r^\times \rtimes \langle t_r^c\rangle$. Since $\pi|_{\mathfrak{A}_r^\times} = \mathrm{Ind}_{\mathfrak{A}_{e'}^\times}^{\mathfrak{A}_r^\times}(\tau_{\iota(\mathfrak{h})})$, it follows that, as a representation of $\bar{\mathfrak{A}}_r^\times = \mathrm{GL}_s^r(k_d)$, π is irreducibly parabolically induced from a cuspidal representation. In particular, we know that π is generic (see, for instance, [SZ1]§5). We use this property of π to fix our intertwining operator. For this consider the upper triangular unipotent subgroup $U_0^r(k_d) \subset \bar{\mathfrak{A}}_r^\times$ and the non-degenerate one-dimensional character $\varphi \in U_0^r(k_d)^\wedge$ defined such that

$$\varphi(u) = \varphi_0 \circ \mathrm{tr}_{k_d|k}(\sum_{i=1}^{m-1} u_{i,i+1}) \qquad (m = rs), \tag{3}$$

where φ_0 denotes a non-trivial additive character of k. Clearly, $\varphi(t_r^{-1}ut_r) = \varphi(u)$ for all $u \in U_0^r(k_d)$, as one sees using (0.11). Since π is generic, we can find a vector $v \in V_\pi$, unique up to scalar factors, such that $\pi(u)v = \varphi(u)v$ for all $u \in U_0^r(k_d)$. Therefore there is only one intertwining operator $J := J_\pi$ on the space V_π such that

$$\pi(u)v = \varphi(u)v, \qquad Jv = v, \qquad \text{and} \qquad \pi(t_r^c x t_r^{-c}) = J\pi(x)J^{-1}. \tag{4}$$

Since φ is $\langle t_r\rangle$ invariant, (4) remains true if we replace π in (4) by $t_r\pi t_r^{-1}$, where $(t_r\pi t_r^{-1})(x) := \pi(t_r^{-1}xt_r)$. Thus

$$J = J_\pi = J_{t_r^i \pi t_r^{-i}} \tag{5}$$

for all $i \in \mathbb{Z}$. Since $t_r^{ce} = \varpi_F I_m$ is central, the operator J^e is scalar; since $Jv = v$, it follows that $J^e = I_\pi$.

4.4 Remark. The element $t := t_r^c \bmod \varpi_F$ acts by conjugation as an automorphism of $G = \mathrm{GL}_s^r(k_d) = \bar{\mathfrak{A}}_r^\times$ which is of order e. We have a natural projection map $X \to \tilde{G}$, where $\tilde{G} := G \rtimes \langle t \rangle$, the semi-direct product. Consider π, by restriction and reduction, as an irreducible representation of G. Set $\tilde{\pi}(t) := J$ to define an extension $\tilde{\pi}$ of π to $\tilde{G}$ such that $\tilde{\varphi} \subset \tilde{\pi}|_{\tilde{U}}$, where $\tilde{\varphi}$ denotes the linear character of $\tilde{U} := U_0^r(k_d) \rtimes \langle t \rangle$ in the space $\mathbb{C}v$ which extends φ by one and satisfies $\tilde{\varphi}(t)v = \tilde{\pi}(t)v = Jv = v$. Therefore, $\tilde{\pi} \subset \tilde{\Gamma} := \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\tilde{\varphi})$, by Frobenius reciprocity. It is easy to generalize the proof of B3.3, which treats the case $G = \mathrm{GL}_s(k_d)$ with fixed point group $G^\phi = \mathrm{GL}_s(k)$, to the case $G = \mathrm{GL}_s^r(k_d)$ with fixed point group $G^t = \mathrm{GL}_s(k_c)^\sharp$ (see B2.1). One has to use here the methods of §B.4; we have omitted this generalization. Assuming this generalization, we see that B3.3(i), combined with the fact that $\tilde{\pi} \subset \tilde{\Gamma}$, implies that $\tilde{\pi}$ is the canonical extension of π, i.e. that $\mathrm{tr}(\tilde{\pi}(t)) = \mathrm{tr}(J) > 0$. Thus B3.3(i) implies that $\tilde{\pi}$ has proper Shintani descent and B3.3(ii) implies that $\mathrm{Sh}_t(\tilde{\pi})$ is a generic representation of G^t. These are crucial facts used in the proof of 5.1 (i.e. that (5.1) implies (5.2)). §B establishes the existence of a Shintani descent theory which is general enough for this application.

For a complex scalar ζ let π_ζ denote the extension of $\pi|_{\mathfrak{A}_r^\times}$ to X such that $t_r^c \mapsto \zeta J$. We construct π_ζ from the representation $\tilde{\pi}$ of 4.4 by inflating $\tilde{\pi}$ via the projection $X \to \tilde{G}$ and multiplying the inflation $\tilde{\pi}$ by the character of the cyclic group $\langle t_r^c \rangle$ such that $t_r^c \mapsto \zeta$. Since $\varpi_F I_m = (t_r^c)^e \mapsto \zeta^e I_\pi$, the representation π_ζ extends $\pi \in (\mathfrak{A}_r^\times F^\times)^\wedge(V_r, \mathcal{S}_\chi)$ if and only if $\chi(\varpi_F) = \zeta^e$.

Next we introduce the set of characters

$$X_t(F_f^\times, \chi) := \{\chi_f \in X_t(F_f^\times) \,:\, \chi_f \circ \mathrm{N}_{F_n|F_f} = \chi\}. \tag{6}$$

Inasmuch as $F_n|F_f$ is unramified of degree e, this set contains exactly e characters, and these characters are twists of each other by unramified characters of order dividing e. Obviously,

$$\chi(\varpi_F) = \chi_f(\varpi_F)^e = \chi_f((-1)^{e-1}\varpi_F)^e \tag{7}$$

for all $\chi_f \in X_t(F_f^\times, \chi)$. Thus we can choose $\zeta = \chi_f((-1)^{e-1}\varpi_F)$ to define an extension π_ζ of π.

4.5 Definition. Given $\pi \in (\mathfrak{A}_r^\times F^\times)^\wedge(V_r, \mathcal{S}_\chi)$ in the distinguished $\langle t_r \rangle$ orbit, $\chi_f \in X_t(F_f^\times, \chi)$, and J as in (4) and (5), let $\Sigma_{\chi_f,\pi}$ denote the extension of π to $X = \mathfrak{A}_r^\times \rtimes \langle t_r^c \rangle$ such that

$$\Sigma_{\chi_f,\pi}(\alpha t_r^c) := \pi(\alpha) J \chi_f((-1)^{e-1}\varpi_F) \qquad (\alpha \in \mathfrak{A}_r^\times). \tag{8}$$

Remark. Note that $\chi_f \in X_t(F_f^\times)$ is determined uniquely by $\bar{\chi}_f$ and $\chi_f|_{F^\times}$. Moreover, $\chi_f \in X_t(F_f^\times, \chi)$ implies that $\bar{\chi}_f$ is determined uniquely by $\bar{\chi}$. Therefore, the mapping

$$X_t(F_f^\times, \chi) \ni \chi_f \mapsto \chi_f|_{F^\times} \in X_t(F^\times)$$

defines a bijection to $\{\omega \in X_t(F^\times) \,:\, \bar{\omega} = \bar{\chi}_f|_{k^\times} \text{ and } \omega^e = \chi_F\}$. Thus $\Sigma_{\chi_f,\pi}$ is completely specified by (8), i.e. by specifying $\chi_f|_{F^\times}$. It would be equally meaningful to write $\Sigma_{\chi_f|_{F^\times},\pi}$ instead of $\Sigma_{\chi_f,\pi}$. This remark also justifies (9), since t_r centralizes $\chi_f|_{F^\times}$.

4.6 Proposition. *Fix π as in 4.3(ii). Then:*
(i) The mapping $X_t(F_f^\times,\chi) \ni \chi_f \mapsto \Sigma_{\chi_f,\pi} \in X^\wedge$ defines a bijection between $X_t(F_f^\times,\chi)$ and the set of distinct extensions of π to X. Moreover,

$$\Sigma_{\chi_f,t_r\pi t_r^{-1}} = t_r\Sigma_{\chi_f,\pi}t_r^{-1} \qquad (\textit{see}\,(3.3)). \tag{9}$$

(ii) For every $\lambda \in X_u(F^\times)$ such that $\lambda^n = 1$

$$\Sigma_{(\lambda\circ \mathrm{N}_{F_f|F})\chi_f,\pi} = (\lambda \circ \mathit{Nrd}_{A|F}) \otimes \Sigma_{\chi_f,\pi}. \tag{10}$$

(iii) Set $\tilde{\Sigma}_{\chi_f} := \mathit{Ind}_X^{\mathfrak{K}_r}(\Sigma_{\chi_f,\pi})$. The induction does not depend on the choice of $\pi \in \{\pi_1,\dots,\pi_c\}$ (see 4.3(i)) and the mapping $X_t(F_f^\times,\chi) \ni \chi_f \mapsto (\mathfrak{K}_r,\tilde{\Sigma}_{\chi_f})$ defines a parameterization of the set of maximal level zero extended types for $\mathcal{S}_\chi^A$, a bijection $\chi_f \mapsto \tilde{\Sigma}_{\chi_f} \mapsto \Pi = \Pi(\tilde{\Sigma}_{\chi_f})$ from $X_t(F_f^\times,\chi)$ to $\mathcal{S}_\chi^A$, which commutes with twisting by unramified characters $\lambda \in X_u(F^\times)$ such that $\lambda^n = 1$ in the sense that $(\lambda \circ N_{F_f|F})\chi_f \mapsto (\lambda\circ \mathit{Nrd}_{A|F})\otimes \Pi$. Finally, if $\eta_f \in [\chi_f]$, then $\tilde{\Sigma}_{\eta_f} = \tilde{\Sigma}_{\chi_f}$.

Proof. (i) We are mapping a set which contains e elements to another set of the same size, so to prove bijectivity it is enough to prove that the mapping is injective. Since the set $X_t(F_f^\times,\chi)$ consists of all characters $\lambda\chi_f$, where $\lambda \in X_u(F_f^\times)$ has order dividing e, the set of values $(\lambda\chi_f)((-1)^{e-1}\varpi_F)$ consists of all e-th roots of $\chi(\varpi_F) = \chi_f(\varpi_F)^e$. From each such choice we get a different extension, and this proves our first assertion. The second assertion follows from the fact that $t_r\Sigma_{\pi,\chi_f}t_r^{-1}$ is obviously an extension of $t_r\pi t_r^{-1}$ and, moreover, t_r commutes with t_r^c, so the extension is clearly the one associated to χ_f.
(ii) If $\lambda \in X_u(F^\times)$ and $\lambda^n = 1$, then $\lambda\circ \mathrm{N}_{F_f|F} \in X_u(F_f^\times)$ and $(\lambda\circ \mathrm{N}_{F_f|F})^e = 1$. Therefore, $(\lambda\circ \mathrm{N}_{F_f|F})\chi_f \in X_t(F_f^\times,\chi)$. Since both sides of (10) are extensions of π, it is enough to verify (10) for $x = \alpha t_r^c$ as in (8). Thus we apply (8) with $(\lambda\circ \mathrm{N}_{F_f|F})\chi_f$ in place of χ_f. This implies that

$$\Sigma_{(\lambda\circ \mathrm{N}_{F_f|F})\chi_f,\pi}(\alpha t_r^c) = \lambda\circ \mathrm{N}_{F_f|F}((-1)^{e-1}\varpi_F)\pi(\alpha)J\chi_f((-1)^{e-1}\varpi_F).$$

On the other hand, since λ is unramified,

$$((\lambda\circ \mathrm{Nrd}_{A|F})\otimes\Sigma_{\chi_f,\pi})(\alpha t_r^c) = \lambda\circ \mathrm{Nrd}_{A|F}(t_r^c)\pi(\alpha)J\chi_f((-1)^{e-1}\varpi_F),$$

so it suffices to check that

$$\lambda\circ \mathrm{N}_{F_f|F}((-1)^{e-1}\varpi_F) = (\lambda\circ \mathrm{Nrd}_{A|F})(t_r^c).$$

Since λ is trivial on units, $\lambda\circ \mathrm{N}_{F_f|F}((-1)^{e-1}\varpi_F) = \lambda(\varpi_F^f)$. ¿From $t_m^n = \varpi_F$ we see that $\mathrm{Nrd}_{A|F}(t_m) = (-1)^{n-1}\varpi_F$; therefore,

$$\mathrm{Nrd}_{A|F}(t_r^c) = \mathrm{Nrd}_{A|F}(t_m^{\frac{m}{r}c}) = (-1)^{(n-1)f}\varpi_F^f, \tag{11}$$

since $\frac{m}{r}c = f$. Since λ is unramified, $\lambda\circ \mathrm{Nrd}_{A|F}(t_r^c) = \lambda(\varpi_F^f)$.
(iii) In (i) we parameterized the set of extensions of π to X, and in 4.2(iii) we constructed a bijection between the set of such extensions and the set of maximal

level zero extended types corresponding to $\mathcal{S}_\chi^A$. Because of (9) the induction from X to $\mathfrak{K}_r$ does not depend on the choice of π. The twist property follows from (ii). In particular we see that $(\lambda \circ \mathrm{Nrd}) \otimes \Pi \sim \Pi$ if and only if $\lambda \circ \mathrm{N}_{F_f|F} = 1$, i.e. $\lambda^f = 1$, hence we recover the inertial degree $f(\Pi) = f$.

Now let $\eta_f \in [\chi_f]$ and set $\eta := \eta_f \circ \mathrm{N}_{F_n|F_f}$. Then we have $X_t(F_f^\times, \eta) \ni \eta_f \mapsto \pi' = \mathrm{Ind}\,(\tau_{\iota(h)}(\sigma_{\bar\eta})\chi_F)$. Clearly, $\tilde\Sigma_{\chi_f} \sim \tilde\Sigma_{\eta_f}$ if and only if $\Sigma_{\chi_f,\pi}$ and $\Sigma_{\eta_f,\pi'}$ are $\langle t_r\rangle$ conjugate. If π and π' are $\langle t_r\rangle$ conjugate, then, since $[\chi_f] = [\eta_f]$, the restrictions of χ_f and η_f to $\langle \varpi_F\rangle$ are identical, and this implies that the $\langle t_r\rangle$ conjugacy extends to $\Sigma_{\chi_f,\pi}$ and $\Sigma_{\eta_f,\pi'}$ (see (4), (5), and (8)). Thus to prove that $\tilde\Sigma_{\chi_f} \sim \tilde\Sigma_{\eta_f}$ it suffices to show that the reductions $\bar\pi$ and $\bar\pi'$ are $\langle t_r\rangle$ conjugate. The representation $\Pi_{d/c}(\bar\chi)^\sharp$ of 4.3(ii) is the same as $\bar\pi$ and it is enough to show that $\Pi_{d/c}(\bar\chi)^\sharp$ and $\Pi_{d/c}(\bar\eta)^\sharp$ are $\langle t_r\rangle$ conjugate. ¿From the way in which $\mathfrak{h}$ is defined (see 3.3(iii)) it follows that $\Pi_{d/c}(\bar\chi)$ is the irreducible parabolic induction from a tensor product of δ distinct Galois conjugates of $\sigma_{\bar\chi} \in \mathrm{GL}_{f'}(k_d)^\wedge_{\mathrm{cusp}}$. Moreover, the action of t_r on $\Pi_{d/c}(\bar\chi)^\sharp$ is equivalent to Galois conjugation by ϕ^h. Since $(c,h) = 1$, the length of the orbit is c. Using again the explicit definition of $\mathfrak{h}$ we see that the sets of cuspidal tensor factors in the parabolic inductions for the c $\langle\phi^h\rangle$ conjugates of $\Pi_{d/c}(\bar\chi)$ are distinct, since under (2.2) they correspond to distinct C cosets in $\mathbb{Z}/(d,f)\mathbb{Z}$. Therefore, the character orbits corresponding to the Green's parameters of the tensor factors consist of a subset of $[\chi_f]$ of cardinality $f'\delta c = \frac{f}{(d,f)}\frac{(d,f)}{c}c = f$. Thus we have the full orbit $[\chi_f]$. This means that $\sigma_{\bar\eta}$ occurs as a tensor factor for the induction of some $\langle\phi^h\rangle$ conjugate $\bar\pi'$ of $\bar\pi$. $\square$

Remark. The representations $\sigma_{\bar\eta} \in GL_{f'}(k_d)^\wedge_{\mathrm{cusp}}$ and $\mathrm{Ind}\,(\tau_{\iota(\mathfrak{h})}(\sigma_{\bar\eta})) \in (\bar{\mathfrak{A}}_r^\times)^\wedge$ are Galois conjugate to $\sigma_{\bar\chi}$ and $\mathrm{Ind}\,(\tau_{\iota(\mathfrak{h})}(\sigma_{\bar\chi}))$, respectively. Since the $\langle t_r\rangle$ orbit of $\mathrm{Ind}\,(\tau_{\iota(\mathfrak{h})}(\sigma_{\bar\chi}))$ is of length c and $(c,r) = 1$, the orbit is also generated by conjugating by powers of $t_r^r = \varpi_D$, i.e. by $\mathrm{Gal}(k_d|k)$ conjugation.

4.7 Proposition. *Let $\Pi \in \mathcal{S}_\chi^A$ and let $(\mathfrak{K}_r, \tilde\Sigma_{\chi_f})$ be the maximal level-zero extended type of Π defined by inducing (X, Σ_{χ_f}) (see (8)). Let $x = \alpha t_r^c \in \mathfrak{A}_r^\times t_r^c$ and assume that x is a root of the irreducible polynomial $g(T)$ defined by (0.13). Then the function $\Theta_\Pi(x)$ which represents the character of Π is defined and constant in the neighborhood $x\mathfrak{U}^1_{\mathfrak{A}_r}$ of x and the characters of Π and $\tilde\Sigma$ satisfy $\Theta_\Pi(x) = \Theta_{\tilde\Sigma}(x)$.*

Remark. We prove that the function Θ_Π is constant on an open set which consists of regular elliptic elements except for, possibly, a closed inseparable null subset.

Proof. Let $\Pi \in \mathcal{S}_\chi^A$ and let $x = \alpha t_r^c$ satisfy $g(x) = 0$. Then x is $(e, f, \mathfrak{A}_r)$-pure (see §A), since $g(T)$ is irreducible over F. Let ρ denote the characteristic function of the coset $x\mathfrak{U}^1_{\mathfrak{A}_r}$ and let "vol" denote the volume of this open compact set. Write $\tilde\Sigma := \tilde\Sigma_{\chi_f}$ and let $\Theta_\Pi(\rho)$ and $\Theta_{\tilde\Sigma}(\rho)$ denote the traces of the operators $\Pi(\rho)$ and $\tilde\Sigma(\rho)$ respectively (see (0.2) and (0.3)). We shall show that the characters satisfy

$$\Theta_\Pi(x) \overset{(I)}{=} \Theta_\Pi(\rho)/\mathrm{vol} \overset{(II)}{=} \Theta_{\tilde\Sigma}(\rho)/\mathrm{vol} \overset{(III)}{=} \Theta_{\tilde\Sigma}(x). \tag{12}$$

(I) is the most difficult part of the proof, so we have shifted the long argument we needed to §A, where the required result is stated as A.2. (III) follows from the fact that the normal, principal unit subgroup $\mathfrak{U}^1_{\mathfrak{A}_r}$ of $\mathfrak{K}_r$ lies in the kernel of the level zero representation $\tilde\Sigma$, so $\Theta_{\tilde\Sigma}$ is constant on $x\mathfrak{U}^1_{\mathfrak{A}_r}$.

We turn to the proof of (II). Since the support of the function ρ lies in $\mathfrak{K}_r$ and, since ρ is locally constant (which implies that $\Pi(\rho)$ is an operator of finite rank), we have a finite sum over characters of $\mathfrak{K}_r$:

$$\Theta_\Pi(\rho) = \Theta_{\Pi|_{\mathfrak{K}_r}}(\rho) = \sum_{\sigma \subset \Pi|_{\mathfrak{K}_r}} m_\sigma \theta_\sigma(\rho), \tag{13}$$

the sum of characters θ_σ over finitely many of the irreducible components σ of $\Pi|_{\mathfrak{K}_r}$ counting multiplicities m_σ, the multiplicities being finite because the representation Π is admissible. By Schur orthogonality $\theta_\sigma(\rho) = 0$ if $\mathfrak{U}^1_{\mathfrak{A}_r}$ does not lie in the kernel of σ, i.e. if σ is not level zero. Thus it suffices to sum over those σ which occur in the decomposition of $V_r = V^{\mathfrak{U}^1_{\mathfrak{A}_r}}$. In particular, 2.2 implies that $m_\sigma = 1$ for all such σ. By 3.3 and (2.2), we know that every component $\sigma \neq \tilde{\Sigma}$ is induced from a group $Y(\sigma)$ such that $\mathfrak{A}_r^\times F^\times \subseteq Y(\sigma) = \mathfrak{A}_r^\times \rtimes \langle t_r^{ac} \rangle \subsetneq X$ with $a > 1$ and $a \mid e$; this follows from the fact, proved in 3.3, that the length of an orbit of $\langle t_r \rangle$ acting on $[\mathbb{Z}/(d,f)\mathbb{Z}]/\mathfrak{S}^r_\delta$ is a proper multiple of c which divides e, except when $\bar{\iota}(\mathfrak{h})$ (see 3.3) is in the orbit, in which case the length of the orbit is exactly c. Therefore, $Y(\sigma)$ is, for $\sigma \neq \tilde{\Sigma}$, a proper subgroup of X and, by Frobenius's formula, the character θ_σ vanishes on the coset $x\mathfrak{U}^1_{\mathfrak{A}_r}$, since $x = \alpha t_r^c$. Therefore, $\tilde{\Sigma}$ is the only component of $\mathfrak{K}_r^\wedge(V_r, \mathcal{S}^A_\chi)$ with a character having support intersecting $x\mathfrak{U}^1_{\mathfrak{A}_r}$. $\square$

4.8 Remark. As [SZ2]2.10 points out, the types for a representation $\Pi^{D_n}_{\chi_f} \in \mathcal{S}^{D_n}_\chi$ are the f characters $\pi \in \mathrm{Gal}(F_n|F)\chi \subset X_t(F_n^\times)$, where $\chi = \chi_f \circ \mathrm{N}_{F_n|F_f}$. We have $\mathfrak{A}_r^\times F^\times = O_D^\times F^\times$, and via $O_D^\times F^\times/\mathfrak{U}^1_D \cong F_n^\times/\mathfrak{U}^1_{F_n}$ the conjugates π of χ are the distinguished orbit in $(O_D^\times F^\times)^\wedge$ which is of length $c = f$. The action is simply the Galois action because $t_r = t_1 = \varpi_D$. For each π there is a unique extension Σ_{π,χ_f} to $X = \langle \varpi^f_{D_n} \rangle \ltimes O_D^\times$ such that $\Sigma_{\pi,\chi_f}(\varpi^f_{D_n}) = \chi_f((-1)^{e-1}\varpi_F)$. Thus $\Sigma_{\pi,\chi_f}(\varpi_F) = \chi_f((-1)^{(e-1)e}\varpi_F^e) = \chi(\varpi_F)$. In fact $X = D_e^\times \mathfrak{U}^1_D$ and $\Sigma_{\chi_f,\pi} = \hat{\tilde{\chi}}_f$ (cf (0.12)). The representation $\Pi^{D_n}_{\chi_f} := \mathrm{Ind}^{D_n^\times}_X \Sigma_{\pi,\chi_f}$ is the maximal level zero extended type $\tilde{\Sigma}_{\chi_f}$ in the case of $D_n^\times$. Obviously, the restriction $\Pi^{D_n}_{\chi_f}|_X = \oplus_\pi \Sigma_{\pi,\chi_f}$.

§5 Explicit Jacquet-Langlands via Matching of Extended Types.

We consider a discrete series representation $\Pi \in \mathcal{R}^2_0(A^\times)$ with inertial degree $f = f(\Pi)$ (see 0.1).

(A) According to 4.1 the representation Π admits a unique maximal level zero extended type $(\mathfrak{K}_r, \tilde{\Sigma}(\Pi))$, where $\mathfrak{K}_r \subset A^\times$ is the standard mcmc subgroup of period $r = m/(f,m)$, and, according to 4.6, there is a well defined type parameter $[\chi_f] \in \mathcal{T}^n_0$ such that

$$\tilde{\Sigma}(\Pi) = \tilde{\Sigma}_{\chi_f} = \mathrm{Ind}^{\mathfrak{K}_r}_X(\Sigma_{\chi_f,\pi_i}),$$

for the distinguished $\langle t_r \rangle$ orbit $\{\bar{\pi}_1, \ldots, \bar{\pi}_c\} \subset (\bar{\mathfrak{A}}_r^\times)^\wedge$ which can be considered also as a $\mathrm{Gal}(k_d|k)$-orbit (see the Remark before 4.7). In particular this means that $\Pi \in \mathcal{S}^A_\chi \subset \mathcal{S}^A_{\bar{\chi}}$, where $\chi = \chi_f \circ \mathrm{N}_{F_n|F_f} \in X_t(F_n^\times)$ and $\bar{\chi} \in X(k_n^\times)$ is the reduction of χ.

(B) By 0.7 and Theorem 1 of §0.5 and 1.3(i), there is also a well defined Langlands parameter $[\chi'_f] \in \mathcal{T}^n_0$ such that $\Pi = \Pi^A_{\chi'_f}$.

As a first remark we observe:

5.1 Proposition. *The type parameter $[\chi_f]$ and the Langlands parameter $[\chi'_f]$ of $\Pi \in \mathcal{R}_0^2(A^\times)$ satisfy $[\chi'_f \circ N_{F_n|F_f}] = [\chi_f \circ N_{F_n|F_f}] \in Gal(F_n|F)\backslash X_t(F_n^\times)$. In particular, for $\chi := \chi_f \circ N_{F_n|F_f}$ and $\chi' := \chi'_f \circ N_{F_n|F_f}$, the restrictions satisfy $\chi|_{F^\times} = \chi'|_{F^\times}$.*

Proof. Clearly the last statement follows from the first. To prove that $[\chi'_f \circ \mathrm{N}_{F_n|F_f}] = [\chi_f \circ \mathrm{N}_{F_n|F_f}]$ let us begin by noting that 4.6(iii) implies that $\Pi \in \mathcal{S}_\chi^A$ if $[\chi_f]$ is the type parameter. By definition, $\Pi = \mathcal{JL}_{A,D_n}(\Pi_{\chi'_f}^{D_n})$ if the Langlands parameter is $[\chi'_f]$. From 1.2(ii)&(iii) it follows that $\Pi_{\chi'_f}^{D_n} \in \mathcal{S}_{\bar{\chi}'}^{D_n}$ with the central character $\chi'|_{F^\times}$, which implies that $\Pi_{\chi'_f}^{D_n} \in \mathcal{S}_{\chi'}^{D_n}$ (see (4.1)). From 0.4 and the fact that $\mathcal{JL}$ preserves central characters (a consequence of AMT) it follows that $\mathcal{JL}_{A,D_n}(\mathcal{S}_{\chi'}^{D_n}) = \mathcal{S}_{\chi'}^A$. Therefore, since $\Pi \in \mathcal{S}_{\chi'}^A \cap \mathcal{S}_\chi^A$, $[\chi] = [\chi']$ and the first statement is true. $\square$

Using 5.1, we may assume that $\chi_f \circ N_{F_n|F_f} = \chi'_f \circ N_{F_n|F_f} = \chi \in X_t(F_n^\times)$ which in terms of (4.6) means that $\chi_f,\ \chi'_f \in X_t(F_f^\times, \chi)$. In this section we want to combine (A) and (B) and, by applying arguments based on Shintani descent theory, to prove:

5.2 Proposition. *Let $\Pi \in \mathcal{S}_\chi^A$ and let $c, r, s = (f, m)$ be associated to $f = f(\Pi) = \|[\chi]\|$ as before. Let $\chi_f, \chi'_f \in X_t(F_f^\times, \chi)$ be the characters associated to $\Pi \in \mathcal{S}_\chi^A$ by (A) and (B).*
(i) Let $\{\bar{\pi}_i\}_{i=1}^c \subset (\bar{\mathfrak{A}}_r^\times)^\wedge = GL_s^r(k_d)^\wedge$ be the Galois orbit of representations which is associated to every element of $\mathcal{S}_\chi^A$ ($\bar{\pi}_i$ denotes the restriction and reduction of π_i (cf (A) above)). Shintani descent maps this set bijectively to the Galois orbit $Gal(k_c|k)\rho(\bar{\chi}_f)$, where $\rho(\bar{\chi}_f) \in GL_s(k_c)^\wedge_{\mathrm{cusp}}$ has the Green's parameter $Gal(k_f|k_c)\bar{\chi}_f$.
(ii) Let $\omega_f \in X_u(F_f^\times)$ be the unique order two unramified character of $F_f^\times$. Then $\chi'_f = \omega_f^{m-s}\chi_f$.

Proof. Let $\Pi \in \mathcal{S}_\chi^A$ and let $x := \alpha t_r^c$ be a root of the polynomial (0.13). For the proof of 5.1 we shall derive two formulas for $\Theta_\Pi(x)$, one from (A) and the other from (B).

First, (A) and 4.7 imply that

$$\Theta_\Pi(x) = \Theta_{\tilde{\Sigma}_{\chi_f}}(x) = \sum_{i=1}^{c} \theta_{\Sigma_{\chi_f,\pi_i}}(x), \tag{1}$$

and we obtain from (4.8) and from 4.4 ($\tilde{\pi}_i$ has proper Shintani descent)

$$\theta_{\Sigma_{\chi_f,\pi_i}}(x) = \mathrm{tr}(\pi_i(\alpha)J_{\pi_i})\chi_f((-1)^{e-1}\varpi_F) = \theta_{\mathrm{Sh}_t(\bar{\pi}_i)}(\mathcal{N}(\bar{\alpha}))\chi_f((-1)^{e-1}\varpi_F), \tag{2}$$

where t denotes the automorphism operator of order e induced on $\bar{\mathfrak{A}}_r^\times$ by t_r^c, $\bar{\pi}_i \in (\bar{\mathfrak{A}}_r^\times)^\wedge$ denotes the reduction of π_i, $\mathcal{N}(\bar{\alpha})$ denotes the matrix norm of the reduction $\bar{\alpha} \in \bar{\mathfrak{A}}_r^\times$ of $\alpha \in \mathfrak{A}_r^\times$, and $\mathrm{Sh}_t(\bar{\pi}_i)$ is the Shintani descent of $\bar{\pi}_i$ with respect to t as sketched in 4.4; for details see Appendix B. The second equation is the character relation B(1.6) for Shintani descent where we write $\theta_{\mathrm{Sh}_t(\bar{\pi}_i)}$ instead of $\mathrm{Sh}_t(f)$.

Set $\rho_i := \mathrm{Sh}_t(\bar{\pi}_i)$. In the proof of B2.1 we have seen that $t_r(x_0^\sharp) = \phi^h(x_0^\sharp)$ for $x_0 \in \mathrm{GL}_s(k_c)$, where $(c,h) = 1$. Therefore the equivariance property of Shintani descent (see remarks following B(1.6)) implies that $\mathrm{Sh}_t \circ t_r = \phi^h \circ \mathrm{Sh}_t$, hence that the set $\{\rho_i\}_{i=1}^c$ is a $\mathrm{Gal}(k_c|k)$ orbit. For $x = \alpha t_r^c$ a root of the polynomial $g(T)$ of (0.13) and thus an $(e, f, \mathfrak{A}_r)$-pure element of A (see §A1; $\alpha \in \mathfrak{A}_r$ may be embedded explicitly as in (A.9)) we see that the matrix norm $\mathcal{N}(\bar{\alpha})$ equals the reduction of $x^e \varpi_F^{-1}$. Since x is an $(e, f, \mathfrak{A}_r)$-pure element, we see that $\mathcal{N}(\bar{\alpha})$ embeds as a regular elliptic element in $\mathrm{GL}_s(k_c)$:

$$\mathcal{N}(\bar{\alpha}) = \bar{\zeta} \in (k_f^\times)^\sharp \subset \mathrm{GL}_s(k_c)^\sharp \cong \mathrm{GL}_s(k_c). \tag{3}$$

(See B2.1 for $\mathrm{GL}_s(k_c)^\sharp$.) Putting (1), (2), and (3) together we obtain

$$\Theta_\Pi(x) = \chi_f((-1)^{e-1}\varpi_F) \sum_{i=1}^c \theta_{\rho_i}(\bar{\zeta}). \tag{4}$$

¿From (B) we have $\Pi = \Pi^A_{\chi'_f}$, which is characterized by the character formula (1.4):

$$\Theta_\Pi(x) = \Theta_{\Pi^A_{\chi'_f}}(x) = (-1)^{m-1} \sum_{\eta \in \mathrm{Gal}(F_f|F)\chi'_f} \eta(\mathrm{N}_{F(x)|F_f}(x)). \tag{5}$$

Since the field extension $F(x)|F$ has inertial degree f and ramification e, it follows that

$$\mathrm{N}_{F(x)|F_f}(x) = (-1)^{e-1} x^e = (-1)^{e-1}\zeta\varpi_F. \tag{6}$$

Since every $\eta \in \mathrm{Gal}(F_f|F)\chi_f$ is tame and $\bar{\chi}'_f = \bar{\chi}_f$, it follows from (5) and (6) that

$$\Theta_\Pi(x) = \chi'_f((-1)^{e-1}\varpi_F)(-1)^{m-1} \sum_{\bar{\eta} \in \mathrm{Gal}(k_f|k)\bar{\chi}_f} \bar{\eta}(\bar{\zeta}), \tag{7}$$

where $\bar{\zeta}$ is the k regular element of k_f defined by equation (3). By varying x we can force $\bar{\zeta}$ to range over the set of all k regular elements of k_f.

Now by combining (4) and (7) we obtain

$$\chi_f((-1)^{e-1}\varpi_F) \sum_{i=1}^c \theta_{\rho_i}(\bar{\zeta}) = \chi'_f((-1)^{e-1}\varpi_F)(-1)^{m-1} \sum_{\bar{\eta} \in \mathrm{Gal}(k_f|k)\bar{\chi}_f} \bar{\eta}(\bar{\zeta}), \tag{8}$$

where $\bar{\zeta} \in k_f^\times$, which we identify with $(k_f^\times)^\sharp \subset \mathrm{GL}_s(k_c)^\sharp$; (8) holds for $\bar{\zeta} \in \mathcal{R}$, the set of k regular elements of k_f.

We write $S_2(\bar{\zeta}) := \sum_{\bar{\eta} \in \mathrm{Gal}(k_f|k)\bar{\chi}_f} \bar{\eta}(\bar{\zeta})$. It follows from [SZ1]1.1(ii) that $S_2(\bar{\zeta})$ cannot be identically zero on $\mathcal{R}$. Since $\mathcal{R} \subseteq \mathcal{R}'$, the set of k_c regular elements of k_f, we see, from (8), that the sum of the characters θ_{ρ_i} cannot be identically zero on $\mathcal{R}' \subset \mathrm{GL}_s(k_c)^\sharp$. In §4 we pointed out that, since they are irreducibly induced from cuspidal representations, the representations π_i are generic. Since Sh_t preserves genericity (§B3.3), the ρ_i are generic too. Since at least one of them has support on $\mathcal{R}'$, at least one of them is either generalized Steinberg or cuspidal. Since the ρ_i are Galois conjugate (see above), the whole set $\{\rho_i\}$ belongs to the same class of characters. In either case it follows from [SZ1]6.1, that each character ρ_i is represented on $\mathcal{R}'$ by a formula

$$\theta_{\rho_i}(\bar{\zeta}) = (-1)^{s-1} \sum_{\bar{\eta} \in \mathrm{Gal}(k_f|k_c)\psi_i} \bar{\eta}(\bar{\zeta}).$$

We substitute this on the left side of (8). Since the set $\{\rho_i\}_{i=1}^c$ is a $\mathrm{Gal}(k_c|k)$ orbit, we may rewrite (8) in the form

$$(9)\quad \chi_f((-1)^{e-1}\varpi_F)(-1)^{s-1}\sum_{\bar\eta\in\mathrm{Gal}(k_f|k)\bar\psi_f}\bar\eta(\bar\zeta)=\chi'_f((-1)^{e-1}\varpi_F)(-1)^{m-1}\sum_{\bar\eta\in\mathrm{Gal}(k_f|k)\bar\chi_f}\bar\eta(\bar\zeta),$$

where both $\mathrm{Gal}(k_f|k)\bar\psi_f$ and $\mathrm{Gal}(k_f|k)\bar\chi_f$ are Galois orbits in $X(k_f^\times)$. We write $S_1(\bar\zeta):=\sum_{\bar\eta\in\mathrm{Gal}(k_f|k)\bar\psi_f}\bar\eta(\bar\zeta)$ and we observe that, since $\chi'_f\chi_f^{-1}\in X_u(F_f^\times)$ is of order dividing e, hence trivial on $-1\in F_f^\times$, we have

$$(10)\qquad S_1(\bar\zeta)=\mu S_2(\bar\zeta)\qquad (\mu=(-1)^{m-s}(\chi'_f\chi_f^{-1})(\varpi_F))$$

for all $\bar\zeta\in\mathcal{R}$. We shall prove that $\mu=1$. Then $(10)_1$ and 5.3 imply 5.2(i) and $(10)_2$ implies 5.2(ii).

5.3 Lemma. *The following statements are equivalent:*
(i) $\mathrm{Gal}(k_f|k)\bar\psi_f=\mathrm{Gal}(k_f|k)\bar\chi_f$.
(ii) $S_1(\bar\zeta)=S_2(\bar\zeta)$ for $\bar\zeta\in k_f^\times$.
(iii) $S_1(\bar\zeta)=S_2(\bar\zeta)$ for $\bar\zeta\in\mathcal{R}$.
(iv) $\mu=1$.

Proof. The equivalence of (i), (ii), and (iii) follows from [SZ1], Theorem 1.1(i). It follows from (10) that (iv) implies (iii). Conversely, (iii) and (10) imply that $(1-\mu)S_2(\bar\zeta)=0$ for all $\bar\zeta\in\mathcal{R}$. In this case, if $\mu\neq1$, then $S_2(\bar\zeta)=0$ for all $\bar\zeta\in\mathcal{R}$, which contradicts [SZ1], Theorem 1.1(ii). □

We need another Lemma:

5.4 Lemma. $\{\rho_i\}_{i=1}^c\subset GL_s(k_c)^\wedge_{\mathrm{cusp}}$.

Proof. First let us recall that $\bar\pi_i\in\mathrm{GL}_s^r(k_d)^\wedge$ is irreducibly parabolically induced from a cuspidal character of a Levi subgroup and, hence, is generic. Since the support of the characters intersects $\mathcal{R}'$, we know that the set $\{\rho_i\}_{i=1}^c$ consists either of generalized Steinberg (GS) or cuspidal characters and, because the set is a $\mathrm{Gal}(k_c|k)$ orbit, the whole set belongs to only one of these two classes. We note that $\bar\pi_i=(\bar\pi)^\sharp$, where $\bar\pi\in\mathrm{GL}_s(k_d)^\wedge$ is also irreducibly induced from cuspidal (see 4.3(ii)). From B4.2, see B(4.5), we may assume without loss of generality that $r=1$ and $t=\phi^c$. Thus we fix $\bar\pi$ and $\rho:=\mathrm{Sh}_{\phi^c}(\bar\pi)$. It suffices to show that ρ is cuspidal. If ρ is not cuspidal, there is a proper parabolic subgroup P of GL_s with the Levi decomposition $P=L\ltimes U_P$ such that $\rho|_{P(k_c)}=\mathrm{Inf}(\Sigma)$, where $\Sigma\in L(k_c)^\wedge_{\mathrm{cusp}}$. If ρ is GS, then $\Sigma=\sigma^{\otimes\ell}$ with $\ell>1$ and $\sigma\in\mathrm{GL}_{s/\ell}(k_c)^\wedge_{\mathrm{cusp}}$. By B3.3(ii) there exists a unique $\gamma\in\mathrm{GL}_{s/\ell}(k_d)^\wedge_{\mathrm{gen}}$ which is ϕ^c invariant and such that $\mathrm{Sh}_{\phi^c}(\gamma)=\sigma$, hence such that $\mathrm{Sh}_{\phi^c}(\gamma^{\otimes\ell})=\sigma^{\otimes\ell}=\Sigma$. Now we use the fact that $\mathrm{Sh}_{\phi^c}(\bar\pi|_{P(k_d)})=\rho|_{P(k_c)}$ and that Sh_{ϕ^c} is an isometry. Therefore $\bar\pi|_{P(k_d)}$ contains the inflation of $\gamma^{\otimes\ell}$. By the transitivity of Jacquet restriction the cuspidal support of $\gamma^{\otimes\ell}$ is the same as the cuspidal support of $\bar\pi$. From this we deduce that the cuspidal support of $\bar\pi$ contains tensor factors which occur with multiplicities and this implies that $\bar\pi$ is a proper component of a parabolically induced from cuspidal representation. This contradicts that $\bar\pi$ is irreducibly parabolically induced from cuspidal and proves our claim. □

Now we are ready to prove that $\mu = 1$. From 5.4 and the fact that the length of the Galois orbit $\{\rho_i\}_{i=1}^c$ is exactly c, we conclude that all the ρ_i are different and we see that the orbit $\mathrm{Gal}(k_f|k)\bar{\psi}_f$ on the left side of (9) has order $c \cdot \frac{f}{c} = f$, the same as the order of $\mathrm{Gal}(k_f|k)\bar{\chi}_f$ on the right side. Suppose that $f = 1$. In this case, $k_f^\times = \mathcal{R} = k^\times$ and (10) becomes $\bar{\psi}_1(\bar{\zeta}) = \mu\bar{\chi}_1(\bar{\zeta})$ for all $\bar{\zeta} \in k^\times$. This implies that $\mu = 1$ because we can take $\bar{\zeta} = 1$.

Assume that $f > 1$ and $\mu \neq 1$. Then, by 5.3, the two orbits of characters are disjoint. Let $Y \subset X(k_f^\times)$ denote the subgroup consisting of all characters which are trivial on $k_f^\times - \mathcal{R}$. If $S_1 = \mu S_2$ on $\mathcal{R}$, then (the point-wise product) $\bar{\psi}(S_1 - \mu S_2) = S_1 - \mu S_2$ on all of $k_f^\times$ for all $\bar{\psi} \in Y$, since both sides vanish on $\mathcal{R}$ and $\bar{\psi} \equiv 1$ on $k_f^\times - \mathcal{R}$. Writing $S_1 = \bar{\psi}_1 + \cdots + \bar{\psi}_f$ and $S_2 = \bar{\chi}_1 + \cdots + \bar{\chi}_f$, we have for all $\bar{\psi} \in Y$

$$(11)\qquad \bar{\psi}(\bar{\psi}_1 + \cdots + \bar{\psi}_f - \mu\bar{\chi}_1 - \cdots - \mu\bar{\chi}_f) = \bar{\psi}_1 + \cdots + \bar{\psi}_f - \mu\bar{\chi}_1 - \cdots - \mu\bar{\chi}_f.$$

¿From the fact that the group $X(k_f^\times)$ comprises a linearly independent set of functions on $k_f^\times$ we deduce immediately that:

5.5 Lemma. *Assume that $f > 1$ and that $\mu \neq 1$. Then:*
(i) If $\mu \neq -1$, then the cyclic subgroup $Y \subset X(k_f^\times)$ acts by multiplication on each of the sets $\{\bar{\psi}_1, \dots, \bar{\psi}_f\}$ and $\{\bar{\chi}_1, \dots, \bar{\chi}_f\}$ by permutations with each orbit having cardinality $|Y|$.
(ii) If $\mu = -1$, then the same conclusion as 5.5(i) holds except that Y acts on the union of the two sets. $\square$

Since $X(k_f^\times)$ is a group under the pointwise multiplication of functions, it follows from 5.5 that $|Y| \mid f$, if $\mu \neq -1$, and $|Y| \mid 2f$, if $\mu = -1$. The result of Zsigmondy quoted in [SZ1], Lemma 1.2 implies that if the pair $(f, |k|)$ differs from (6,2) or (2,3), then there exists a prime $\ell > f$ such that $\ell \mid |Y|$. For such ℓ it is impossible that $\ell \mid |Y|$ and $|Y| \mid 2f$, so $\mu = 1$ provided that $(f, |k|) \neq (6, 2), (2, 3)$.

If $(f, |k|) = (6, 2)$, $X(k_f^\times) \cong (\mathbb{Z}/63\mathbb{Z})^+$ and $|Y| = 3$; the Frobenius corresponds to multiplication by 2 in $\mathbb{Z}/63\mathbb{Z}$ and Y is the subgroup generated by 21. As pointed out in [SZ1], mid-page 3343, there is exactly one Y stable Galois orbit consisting of regular characters. This excludes all possible values for $\mu \neq 1$ except $\mu = -1$. The Galois group $\mathrm{Gal}(k_6|k)$ normalizes Y and the semi-direct product H has order 18. Any Y orbit of a Galois orbit is also an H orbit, consequently a union of Galois orbits. It follows that $\mu = -1$ is impossible too, because $2f = 12$ does not divide 18.

If $(f, |k|) = (2, 3)$, then $X(k_f^\times) \cong (\mathbb{Z}/8\mathbb{Z})^+$, $|Y| = 4$, and the Frobenius acts as multiplication by 3 on $\mathbb{Z}/8\mathbb{Z}$. This implies that $\mu \neq \pm 1$ is impossible. If $\mu = -1$, then $S_1 - (-S_2) = \bar{\psi}_1 + \bar{\psi}_2 + \bar{\chi}_1 + \bar{\chi}_2$ can be stable under multiplication by Y, i.e. let $\{\bar{\psi}_1, \bar{\psi}_2, \bar{\chi}_1, \bar{\chi}_2\}$ correspond to the coset $1, 3, 5, 7$ in $\mathbb{Z}/8\mathbb{Z}$ of the cyclic subgroup Y of order 4. Thus the present argument definitely does not work in this case.

In the case that $cs = f = 2$ and $k = \mathbb{F}_3$ we have to give a direct proof of 5.2(i), that $\sum_{i=1}^c \mathrm{Sh}_t(\bar{\pi}_i) = \sum_{i=1}^c \rho_i$, where $\{\rho_i\}_{i=1}^c = \mathrm{Gal}(k_c|k)\rho(\bar{\chi}_f)$. The two sides are, respectively, $(-1)^{s-1}S_1(\bar{\zeta})$ and $(-1)^{s-1}S_2(\bar{\zeta})$ for $\bar{\zeta} \in \mathcal{R}$, so if we prove that they are equal, then 5.3 will imply that $\mu = 1$.

If $c = 2$ and $s = 1$, then $m = rs = r$ is odd, $f = 2$, and $2 \mid d$. In this case, $\delta f' = s = 1$ and, in (4.2), $\bar{\pi} = \bar{\pi}_1 = \bar{\chi}_{2,d}^\sharp$, where $\bar{\chi}_{2,d} = \bar{\chi}_2 \circ \mathrm{N}_{k_d|k_2} = \bar{\Pi}_{d/c}(\bar{\chi})$.

Applying B4.2 (see B(4.5)) we obtain $\mathrm{Sh}_t(\bar\pi) = \mathrm{Sh}_{\phi^2}(\bar\chi_{2,d})^\sharp$. It follows from the abelian case, trivially, that $\mathrm{Sh}_{\phi^2}(\bar\chi_{2,d}) = \bar\chi_2$, so $\mathrm{Sh}_t(\bar\pi) = \bar\chi_2^\sharp \in (\mathrm{GL}_1(k_2)^\sharp)^\wedge$.

Now let $c = 1$ and $s = 2$. Using B4.2 we immediately pass to the case $r = 1$ and consider $\mathrm{Sh}_\phi : \mathrm{GL}_2(k_d)^{\wedge,\phi} \to \mathrm{GL}_2(k)^\wedge$. Since $c = r = 1$, $\delta f' = s = m = 2$. We have to distinguish the cases $2 \mid d$, which implies that $\delta = 2$ and $f' = f/(d,f) = 1$, and $2 \nmid d$, which implies that $\delta = 1$ and $f' = 2$. From 4.3(ii) we read off the definition of $\bar\Pi_d(\bar\chi_2)$ and we have to prove only the following to complete the proof of 5.2(i):

5.6 Lemma. *Let $k := \mathbb{F}_3$ and let $\bar\chi_2 \in X(k_2^\times)$ be k regular. Set*

$$(12) \qquad \bar\Pi_d(\bar\chi_2) := \begin{cases} I(\bar\chi_{2,d}, {}^\phi\bar\chi_{2,d}), & \text{if } 2 \mid d \text{ and } \bar\chi_{2,d} = \bar\chi_2 \circ N_{k_d|k_2}; \\ \bar\pi(\bar\chi_{2,d}), & \text{if } 2 \nmid d \text{ and } \bar\chi_{2,d} = \bar\chi_2 \circ N_{k_{2d}|k_2}. \end{cases}$$

Then: $Sh_\phi(\bar\Pi_d(\bar\chi_2)) = \bar\pi(\bar\chi_2)$, where $\bar\pi(\bar\chi_2) \in GL_2(k)^\wedge_{\mathrm{cusp}}$ has the Green's parameter $Gal(k_2|k)\bar\chi_2$.

Remark. $|\mathrm{GL}_2(k)^\wedge_{\mathrm{cusp}}| = 3$ with one cuspidal character having the trivial character as its central character; we have to distinguish only the descents to the other two cuspidal characters. This follows from the fact, which we shall use again below, that if $\theta \in \mathrm{GL}_n(k_d)^\wedge$ is $\langle\phi\rangle$ invariant, then its central character is also $\langle\phi\rangle$ invariant and the Shintani descent of the central character of θ is the central character of $\mathrm{Sh}_\phi(\theta)$.

Proof. The case $2 \mid d$ is proved in [Sh]4-1; Shintani deals with a more general context and proves a more general statement than we need. The other case is asserted as [Sh]4-2 with the proof omitted. We can give a short proof for Shintani's Proposition 4-2, that $\mathrm{Sh}_\phi(\bar\pi(\bar\chi_{2,d})) = \bar\pi(\bar\chi_2)$ when d is odd, so we include it here. First, since $\bar\pi(\bar\chi_{2,d})$ is cuspidal and $\langle\phi\rangle$-invariant, an easy variant of the argument of 5.4 implies that $\mathrm{Sh}_\phi(\bar\pi(\bar\chi_{2,d}))$ is cuspidal, so we have only to identify the Green's parameter of the descent. Let Γ be a subtorus of GL_2 which is defined and anisotropic over k, so that $\Gamma(k) \cong k_2^\times$. Since d is odd, $\Gamma(k_d) \cong k_{2d}^\times$. Let $\Lambda_d(\bar\chi_{2,d}) := \bar\pi(\bar\chi_{2,d})|_{\Gamma(k_d)}$. It is well known that $\Lambda_d(\bar\chi_{2,d}) = \sum_{\bar\eta \in S_d(\bar\chi_{2,d})} \bar\eta$, where

$$S_d(\bar\chi_{2,d}) = \{\bar\eta \in \Gamma(k_d)^\wedge \,:\, \bar\eta^{-1}\bar\chi_{2,d}|_{k_d^\times} = 1, \bar\eta \neq \bar\chi_{2,d}, \bar\eta \neq {}^\phi\bar\chi_{2,d}\}$$

consists of $3^d - 1$ characters. Set $\tilde\Gamma := \mathrm{Gal}(k_d|k) \ltimes \Gamma(k_d)$, let $\tilde\pi$ denote the canonical extension of $\bar\pi(\chi_{2,d})$ to $\mathrm{Gal}(k_d|k) \ltimes \mathrm{GL}_2(k_d)$ (see B3.3), and set $\tilde\Lambda_d(\bar\chi_{2,d}) := \tilde\pi|_{\tilde\Gamma}$. The pair $(\Gamma(k_d), \phi)$ is a standard pair in the sense of B1.3 (see B2.2 Fact), so $\mathrm{Sh}_\phi : \mathcal{C}(\Gamma(k_d)\phi) \to \mathcal{C}(\Gamma(k))$ is an isometry of inner product spaces. Assume that $\mathrm{Sh}_\phi(\bar\pi(\bar\chi_{2,d})) = \bar\pi(\bar\chi_2')$. Then, since the central character of $\bar\pi(\bar\chi_{2,d})$ is $\bar\chi_{2,d}|_{k_d^\times}$, the central character of $\bar\pi(\bar\chi_2')$ is $\bar\chi_2|_{k^\times} = \bar\chi_2'|_{k^\times}$ and, by B1.4, $\tilde\Lambda_d(\bar\chi_{2,d})(x\phi) = \Lambda_1(\bar\chi_2')(\mathrm{N}_{\Gamma(k_d)|\Gamma(k)}(x))$ $(x \in \Gamma(k_d))$. For any $\bar\eta \in \Gamma(k_d)^{\wedge,\phi}$ such that $\bar\eta = \bar\eta_1 \circ \mathrm{N}_{\Gamma(k_d)|\Gamma(k)} = \bar\eta_1 \circ \mathrm{N}_{k_{2d}|k_2}$ with $\tilde\eta$ the canonical extension of $\bar\eta$ we have

$$\langle\tilde\eta, \tilde\Lambda_d(\bar\chi_{2,d})\rangle_{\Gamma(k_d)\phi} = \langle\bar\eta_1, \Lambda_1(\bar\chi_2')\rangle_{\Gamma(k)} = \begin{cases} 1, & \text{if } \bar\eta_1 \in S_1(\bar\chi_2'); \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for $\bar\eta_1 \in S_1(\bar\chi_2')$, $\langle\tilde\eta, \tilde\Lambda_d(\bar\chi_{2,d})\rangle_{\Gamma(k_d)\phi} \neq 0$ if and only if $\tilde\eta$ is the canonical extension of $\bar\eta = \bar\eta_1 \circ \mathrm{N}_{\Gamma(k_d)|\Gamma(k)} \in S_d(\bar\chi_{2,d}') \cap \Gamma(k_d)^{\wedge,\phi}$. On the other hand, for any $\bar\eta \in \Gamma(k_d)^{\wedge,\phi}$ Frobenius reciprocity implies that

$$\langle \mathrm{Ind}\,_{\Gamma(k_d)}^{\tilde\Gamma}\bar\eta, \tilde\Lambda\rangle_{\tilde\Gamma} = \langle\bar\eta, \Lambda_d(\bar\chi_{2,d}\rangle_{\Gamma(k_d)} = \begin{cases} 1, & \text{if } \bar\eta \in S_d(\bar\chi_{2,d}); \\ 0, & \text{otherwise.} \end{cases}$$

For $\zeta \in \mathbb{C}$ any d-th root of unity let $\varphi_\zeta \in \tilde{\Gamma}^{\wedge,\phi}$ denote the inflation of the character $\phi^j \mapsto \zeta^j$ in $\langle\phi\rangle^\wedge$. Then for any $\bar{\eta} \in S_d(\bar{\chi}_{2,d})$ with canonical extension $\tilde{\eta}$ we have $\mathrm{Ind}\,_{\Gamma(k_d)}^{\tilde{\Gamma}}\bar{\eta} = \sum_\zeta \varphi_\zeta\tilde{\eta}$ and, moreover, $\frac{1}{d}\sum_\zeta \zeta^{-1}\varphi_\zeta\tilde{\eta} = \mathrm{ext}_0(\tilde{\eta}|_{\Gamma(k_d)\phi})$, where ext_0 denotes the "extension by zero". According to B1.(9),(10) we have $\langle\tilde{\eta}, \tilde{\Lambda}_d(\bar{\chi}_{2,d})\rangle_{\Gamma(k_d)\phi} \neq 0$ if and only if $\langle\sum_\zeta \zeta^{-1}\varphi_\zeta\tilde{\eta}, \tilde{\Lambda}_d(\bar{\chi}_{2,d})\rangle_{\tilde{\Gamma}} \neq 0$, if and only if $\langle\mathrm{Ind}\,_{\Gamma(k_d)}^{\tilde{\Gamma}}\bar{\eta}, \tilde{\Lambda}_d(\bar{\chi}_{2,d})\rangle_{\tilde{\Gamma}} \neq 0$, if and only if $\bar{\eta} \in S_d(\bar{\chi}_{2,d}) \cap \Gamma(k_d)^{\wedge,\phi}$. Therefore, $S_d(\bar{\chi}_{2,d}) \cap \Gamma(k_d)^{\wedge,\phi} = S_d(\bar{\chi}'_{2,d}) \cap \Gamma(k_d)^{\wedge,\phi}$, which implies that $\bar{\chi}_2 = \bar{\chi}'_2$ or ${}^\phi\bar{\chi}'_2$. $\square$

We have proved that $\mu = 1$, and this completes the proof of 5.2. $\square$

Remarks. 1. We should verify that twisting by the unramified character ω_f^{m-s} stabilizes $\mathcal{S}_\chi^A$. As we have remarked in the proof of 4.6, twisting by $\lambda \in X_u(F_f^\times)$ preserves $X_t(F_f^\times, \chi)$ if and only if $\lambda^e = 1$. Thus, since ω_f has order 2, we must show that $(m-s)e$ is always even. Clearly, $m - s = m - (f, m)$ is even if m is odd or if f and m have the same parity. On the other hand, if m is even and f is odd, then e is even, since $ef = dm$.
2. From 5.2(ii) we see that, with $s = (f, m)$ and $rs = m$, $\Pi_{\chi_f}^A \in \mathcal{S}_\chi^A$ admits the level zero extended type $(\mathfrak{K}_r, \tilde{\Sigma}_{\chi_f\omega_f^{m-s}})$. Thus 5.2(ii) implies Theorem 3 of §0.7.

In order to complete the proof of 5.2 we had to deal with explicit Shintani descent in a very special case. Now we can use 5.2(i) to derive a much more general explicit Shintani descent assertion.

5.7 Corollary. *Let $\bar{\psi} \in X(k_s^\times)$ be k-regular, let $\delta = (d, s)$, let $\sigma_{\bar{\psi},d} \in GL_{s/\delta}(k_d)^\wedge_{\mathrm{cusp}}$ have the Green's parameter $Gal(k_dk_s|k_d)\bar{\psi}\circ N_{k_dk_s|k_s} \subset X((k_dk_s)^\times)$, and let $\Pi_d(\bar{\psi}) = I(\sigma_{\bar{\psi},d} \otimes \cdots \otimes \sigma_{\bar{\psi},d}^{\phi^{\delta-1}})$ be the parabolic induction from $GL_{s/\delta}^\delta(k_d)$ to $GL_s(k_d)$. Then $\Pi_d(\bar{\psi})$ has the Shintani descent $Sh_\phi(\Pi_d(\bar{\psi})) = \Pi_1(\bar{\psi}) = \sigma_{\bar{\psi},1} \in GL_s(k)^\wedge_{\mathrm{cusp}}$.*

Proof. We consider 5.2(i) in the case $c = r = 1$, i.e. $f = m = s$, $d = e$, and $\delta = e' = (d, m) = (d, s)$. We replace $\bar{\chi} \in X(k_n^\times)$ by $\bar{\psi} = \bar{\chi}_s = \bar{\chi}_f \in X(k_s^\times)$, where $\bar{\psi} \circ N_{k_n|k_s} = \bar{\chi}$. Then we have $\sigma_{\bar{\psi},d} = \sigma_{\bar{\chi}}$ as defined before 0.2 Fact in §0.4, and $\bar{\pi} = \bar{\pi}_1 = \Pi_d(\bar{\psi})$ in the sense of 4.3(ii). Now 5.2(i) says that $\mathrm{Sh}_t(\bar{\pi}) = \rho(\bar{\psi})$. But we have $t = t_r^c = t_1 = \varpi_D$, hence $\mathrm{Sh}_t = \mathrm{Sh}_\phi$, and $\rho(\bar{\psi}) = \sigma_{\bar{\psi},1}$. $\square$

Finally we use 5.7 and the methods of §B4, in particular B4.4, to make the bijective descent mapping of 5.2(i) explicit and unambiguous:

5.8 Corollary. *Let $\bar{\pi} = \bar{\pi}_1 = \Pi_{d/c}(\bar{\chi})^\sharp$ be the representation from 4.3(ii) which is part of the distinguished Galois orbit in $(\bar{\mathfrak{A}}_r^\times)^\wedge$ as in 5.2(i), and let $t = t_r^c$. The Shintani descent of $\bar{\pi}$ to the group $(\bar{\mathfrak{A}}_r^\times)^t = GL_s(k_c)^\sharp$ of t-fixed points is then $Sh_t(\bar{\pi}) = \rho(\bar{\chi}_f)^\sharp$ (in the sense of B4.1), where $\bar{\chi}_f \circ N_{k_n|k_f} = \bar{\chi}$.*

The proof follows from B4.4. $\square$

§A On the Constancy of Discrete Series Characters on Certain Cosets.

This Appendix, which depends upon and extends some of the results of [Zi], provides an essential step in the proof of 4.7. We begin by stating the main result, A.1, and deriving the important consequence, A.2. Afterwards we prove A.1 via a sequence of lemmas.

An element $y \in A$ is called an $(e, f, \mathfrak{A}_r)$-*pure* element if y generates a field $F(y)$, $F \subset F(y) \subset A$, of ramification index e and inertial degree f such that $ef = n$ and such that the multiplicative group $F(y)^\times$ normalizes $\mathfrak{A}_r$. By a theorem of H. Benz and A. Fröhlich this implies $r = m/(f,m)$. For y an $(e,f,\mathfrak{A}_r)$-pure element we write $F_{f,y}$ for the inertial subfield of $F(y)$. An $(e,f,\mathfrak{A}_r)$-pure element need not be separable; if it is separable, then it is also regular elliptic. The set of regular elliptic $(e,f,\mathfrak{A}_r)$-pure elements constitute an open, dense subset of the set of all $(e,f,\mathfrak{A}_r)$-pure elements (cf [DM1]14.9).

Remark. Although the fields $F_{f,y}$ are all isomorphic and are all isomorphic to F_f, in interesting cases these fields are embedded in A via rather unusual mappings. In A.8 we give some properties of the embedding in A of the inertial subfield $F_{f,x}$ of $F(x)$.

A.1 Theorem. *Let $x = \alpha t_r^c \in X \subset \mathfrak{K}_r$ (cf (0.13) and 4.7) be a root of the irreducible polynomial*

$$F[T] \ni g(T) := \prod_{\zeta' \in \mathrm{Gal}(F_f|F)\zeta} (T^e - \zeta' \varpi_F), \tag{1}$$

where $ef = n$ and ζ is a root of unity generating the unramified extension $F_f|F$. Then the coset $x\mathfrak{U}^1_{\mathfrak{A}_r} \subset X$ is comprised of $(e,f,\mathfrak{A}_r)$-pure elements, including an open dense subset consisting of regular elliptic elements. The group $\mathfrak{U}^1_{\mathfrak{A}_r}$ acts transitively by conjugation on the set of inertial subfields $\{F_{f,y} \mid y \in x\mathfrak{U}^1_{\mathfrak{A}_r}\}$, i.e. there exists $\kappa \in \mathfrak{U}^1_{\mathfrak{A}_r}$ such that $\kappa F_{f,y}\kappa^{-1} = F_{f,x}$.

It will not be necessary for us to prove that the set $x\mathfrak{U}^1_{\mathfrak{A}_r}$ contains a dense open subset comprised of regular elliptic elements, as this will be an automatic consequence of our proof that the set consists entirely of $(e,f,\mathfrak{A}_r)$-pure elements.

A.2 Corollary. *Let $\Pi \in \mathcal{R}_0^2(A^\times)$ be a representation which has the inertial degree $f(\Pi) = f$ and let $x = \alpha t_r^c$ be a root of the polynomial (1). Then the character Θ_Π of Π is constant on the coset $x\mathfrak{U}^1_{\mathfrak{A}_r}$.*

Proof. Since, by A.1, the coset $x\mathfrak{U}^1_{\mathfrak{A}_r}$ consists of $(e,f,\mathfrak{A}_r)$-pure elements, Harish-Chandra's theorem ([HC1]) implies that the function $\Theta_\Pi(g)$, which represents the character of Π, is defined and locally constant on the dense open subset of $x\mathfrak{U}^1_{\mathfrak{A}_r}$ consisting of regular elliptic elements. We want to show that this function is constant on this subset of $x\mathfrak{U}^1_{\mathfrak{A}_r}$. This will imply that we may regard the function Θ_Π as defined and constant on the whole of $x\mathfrak{U}^1_{\mathfrak{A}_r}$.

If $f(\Pi) = f$, then $\Pi = \Pi^A_{\chi_f}$ for some $[\chi_f] \in \mathcal{T}_0^n$. Therefore, by 1.3(i),

$$\Theta_\Pi(y) = (-1)^{m-1} \sum_{\eta \in \mathrm{Gal}(F_{f,y}|F)\chi_f} \eta(\mathrm{N}_{F(y)|F_{f,y}}(y)) \tag{3}$$

for all regular $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$. In fact the right side of (3) is defined for all $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$, and it is enough to prove that the right side is constant. The left side, where it is defined, is invariant under conjugation. Thus the right side of (3) is also invariant under conjugation, at least at regular elements, and, in particular, under conjugation by elements $\kappa \in \mathfrak{U}^1_{\mathfrak{A}_r}$; these conjugations preserve the coset $x\mathfrak{U}^1_{\mathfrak{A}_r}$ because x normalizes $\mathfrak{U}^1_{\mathfrak{A}_r}$. Therefore, applying A.1, which asserts that $\mathfrak{U}^1_{\mathfrak{A}_r}$ acts transitively by conjugation on the inertial subfields $F_{f,y}$, we may assume that $F(x)|F$ and $F(y)|F$ have the same inertial subfield $F_f := F_{f,x} = F_{f,y}$ and consider the right side of (3) only in this case. For x as in A.1 we have

$$(4) \qquad x^e = \varpi_F \zeta', \; F_f = F(\zeta'), \; \mathrm{N}_{F(x)|F_f}(x) = (-1)^{e-1}\varpi_F\zeta'.$$

Since y is an $(e, f, \mathfrak{A}_r)$-pure element, y is a prime element of $F(y)$ and y is a root of an Eisenstein polynomial $T^e + b_{e-1}T^{e-1} + \cdots + b_0 \in F_f[T]$; therefore, $-b_0/y^e = 1 + b_{e-1}/y + \cdots + b_1/y^{e-1} = \epsilon_2$ is a principal unit in $F(y)^\times \subset \mathfrak{K}_r$. Since $y = x\epsilon$ with $\epsilon \in \mathfrak{U}^1_{\mathfrak{A}_r}$, it follows that $y^e = x^e\epsilon_1$ with $\epsilon_1 \in \mathfrak{U}^1_{\mathfrak{A}_r}$, and

$$(5) \quad \mathrm{N}_{F(y)|F_f}(y) = (-1)^e b_0 = (-1)^{e-1}y^e\epsilon_2 = (-1)^{e-1}x^e\epsilon_1\epsilon_2 = \mathrm{N}_{F(x)|F_f}(x)\epsilon_1\epsilon_2,$$

so the norms of y and x differ by a factor which is a principal unit of F_f. But on the right side of (3) only tame characters of $F_f^\times$ occur and the group of principal units of $F_f^\times$ lies in the kernel of any tame character of $F_f^\times$. We conclude that the right side of (3) is constant on $x\mathfrak{U}^1_{\mathfrak{A}_r}$. $\square$

§A1 All Elements of $x\mathfrak{U}^1_{\mathfrak{A}_r}$ are $(e, f, \mathfrak{A}_r)$-pure Elements.

We begin the proof of A.1. Our first goal, achieved in A.3-A.5, will be to show that every $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$ is $(e, f, \mathfrak{A}_r)$-pure.

A.3 Lemma. *Let $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$. Then the minimal polynomial of y over F has degree $n = ef$.*

Proof. Since $y \in A$, the minimal polynomial of y is at most of degree n, so it is enough to show that the set of powers $1, y, y^2, \ldots, y^{n-1}$ is linearly independent over F. Set $z := y^e/\varpi_F$. With respect to $\mathfrak{P}_r = \mathfrak{A}_r t_r$ we have $\nu_{\mathfrak{P}_r}(z) = ce - dr = 0$, so $z \in \mathfrak{A}_r^\times \cap F[y]$. It suffices to show that the set of elements $\{y^i z^j\}$ for $i = 0, \ldots, e-1$ and $j = 0, \ldots, f-1$ is linearly independent over F. If this set is not linearly independent over F, then there exists an equation $\sum a_{i,j} y^i z^j = 0$ with coefficients $a_{i,j} \in F$. Assuming that such an equation exists, we rewrite it in the form

$$(6) \qquad f_0(z) + f_1(z)y + \cdots + f_{e-1}(z)y^{e-1} = 0,$$

where the polynomials $f_i(z) \in F[z]$ are of degree smaller than f. Assume that $f_i(z) \neq 0$ for at least one i; clearing denominators, we may also assume that all coefficients belong to the valuation ring o_F and that at least one coefficient is a unit. Hence there is at least one polynomial $f_i(z)$ such that not all its coefficients lie in the prime ideal $\mathfrak{p}_F$. Since x is a root of the polynomial $g(T)$ of (1) and since x normalizes $\mathfrak{U}^1_{\mathfrak{A}_r}$, it follows that for any $\epsilon \in \mathfrak{U}^1_{\mathfrak{A}_r}$ we have $y^e = (x\epsilon)^e = x^e\epsilon_1 = \varpi_F\zeta'\epsilon_1$, where ζ' is a root of unity which generates an unramified extension field of F of degree f and $\epsilon_1 \in \mathfrak{U}^1_{\mathfrak{A}_r}$. This means that $z = \zeta'\epsilon_1$ and $\bar{z} = \overline{\zeta'} \in \mathfrak{A}_r/\mathfrak{P}_r$. Thus, the set of reductions $1, \bar{z}, \cdots, \bar{z}^{f-1}$ is linearly independent over $k_F = \underline{o_F/\mathfrak{p}_F}$. Equation (6) implies that $f_0(z) \in \mathfrak{A}_r y \subset \mathfrak{P}_r$, and this implies that $\overline{f_0}(\bar{z}) := \overline{f_0(z)} = 0$. Thus $f_0(z) \in \mathfrak{p}_F[z]$ and therefore $\nu_{\mathfrak{P}_r}(f_0(z)) \geq dr$. This means that $\nu_{\mathfrak{P}_r}(f_0(z)/y) \geq dr - c$, which implies that $f_1(z) = -f_0(z)y^{-1} - f_2(z)y - \cdots - f_{e-1}(z)y^{e-2} \in \mathfrak{P}_r$. Therefore, $f_1(z) \in \mathfrak{p}_F[z]$ too. Converting the above argument for $i = 0, 1$ into an induction argument, we conclude that $f_i(z) \in \mathfrak{p}_F[z]$ for $0 \leq i < e$. Since this contradicts our hypothesis that at least one polynomial has a unit coefficient, we conclude that $f_i(z) = 0$ for all $i = 0, \ldots, e-1$. Since the set of reductions $1, \bar{z}, \ldots, \bar{z}^{f-1}$ is linearly independent over k, it follows that the set $1, z, \ldots, z^{f-1}$ is linearly independent over F, and therefore all coefficients $a_{i,j}$ must vanish. $\square$

Next we want to show that the minimal polynomial of y over F is irreducible. For this it is enough to show that the ring $R = \mathfrak{A}_r \cap F[y]$ is a discrete valuation ring, since a discrete valuation ring contains no zero divisors. By [Se2], I, Prop. 2 in order to show that R is a discrete valuation ring it is sufficient to show:

A.4 Lemma. *For $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$ the intersection $R := \mathfrak{A}_r \cap F[y]$ is a Noetherian local ring and its maximal ideal is generated by a non-nilpotent element.*

Proof. We shall show that R is a Noetherian local ring and that yR is its maximal ideal. Since $y \in A^\times$ is not nilpotent, this will prove both parts of A.4. Since $\mathfrak{A}_r$ is a finitely generated o_F module, the ring R is finitely generated as an o_F algebra; this implies that R is Noetherian.

Let us show by continuing the argument given in A.3 as in [Se2], III, Lemma 3 that the n elements $\{y^i z^j\}$ (see the proof of A.3) comprise an o_F module basis for R. From A.3 we know that R contains the vector space basis $\{y^i z^j\}$ for $F(y)|F$. Thus R is an o_F lattice in the n-dimensional vector space $F(y)|F$ and $R/\mathfrak{p}_F R$ is a k vector space of the same dimension. Applying the same argument as in A.3, we show that if $\sum a_{ij} y^i z^j \in \mathfrak{p}_F R$ and $a_{ij} \in o_F$, then $a_{ij} \in \mathfrak{p}_F$ too. This implies that $\{\overline{y}^i \overline{z}^j\}_{i,j}$ is a k basis for the k vector space $R/\mathfrak{p}_F R$. Using the k isomorphism $r \mapsto \varpi_F^\nu r$ to send $R/\mathfrak{p}_F R \to \mathfrak{p}_F^\nu R/\mathfrak{p}_F^{\nu+1} R$, we deduce that $\{\varpi_F^\nu y^i z^j\}_{i,j}$ is a k basis of $\mathfrak{p}_F^\nu R/\mathfrak{p}_F^{\nu+1} R$. Thus $\{y^i z^j\}$ generates R as an o_F module.

Now let us show that R is a local ring with the unique maximal ideal yR. It follows from A.3 that the minimal polynomial of y is of the form $u^n + a_{n-1}u^{n-1} + \cdots + a_0 \in F[u]$ with $a_0 \neq 0$, which implies that $y \in F[y]^\times$, since $(-a_0)^{-1}(y^{n-1} + a_{n-1}y^{n-2} + \cdots + a_1)y = 1$. From this we see that $y\mathfrak{A}_r \cap F[y] = yR$ and that $\mathfrak{P}_r \supseteq y\mathfrak{A}_r \supseteq \varpi_F \mathfrak{A}_r$ and $\mathfrak{P}_r \cap R \supseteq yR \supseteq \mathfrak{p}_F R$. From this it follows that there is a k vector space homomorphism $R/yR \longrightarrow \mathfrak{A}_r/\mathfrak{P}_r$. As pointed out in the proof of A.3, $\bar{z} = \overline{\zeta'} \in \mathfrak{A}_r/\mathfrak{P}_r$ generates a field of degree f over k. Therefore the k vector space isomorphism is in fact an isomorphism of R/yR to that field, hence $\mathfrak{P}_r \cap R = yR$ is a maximal ideal in R.

To see that yR is the unique maximal ideal of R consider $1 + yR = 1 + (\mathfrak{P}_r \cap R) = (1 + \mathfrak{P}_r) \cap R$. If $w \in \mathfrak{P}_r$, then $(1+w)^{-1} = 1 - w + w^2 - w^3 \pm \cdots$, the series being convergent in the topology of A, since the sequence $\{w^n\}$ converges to zero in A. If $w \in \mathfrak{P}_r \cap R$, then every partial sum of $(1+w)^{-1}$ lies in R. Since R is closed in $\mathfrak{A}_r$, $1 + yR \subset R^\times$. Since R/yR is a field, it follows that for all $w \in R - yR$ the congruence $ww' \equiv 1 \bmod yR$, has a solution $w' \in R$. This combined with the fact that every element of $1 + yR$ is a unit implies that every element of $R - yR$ is a unit. Thus, every non-trivial ideal of R is in yR. $\square$

A.5 Lemma. *Let $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$. Then $F(y)$ is a maximal subfield of A. The field $F(y)$ has ramification exponent e and inertial degree f and the multiplicative group $F(y)^\times$ normalizes $\mathfrak{A}_r$.*

Proof. It follows from A.3, A.4, and the proof of A.3 that $F(y) = F \cdot R$ is a field extension of degree n over F and R is a discrete valuation ring. By [Se2], II, §2, Cor. 2 there is a unique discrete valuation ring in $F(y)$ such that o_F is the intersection of that ring with F, hence $R = o_{F(y)}$. We have seen that yR is the prime ideal in R, i.e. y is a prime element of $F(y)$ and $z = y^e/\varpi_F$ is a unit. Therefore e is the ramification index of $F(y)|F$ and $f = n/e$ is the inertial degree. It follows that, inasmuch as $y \in \mathfrak{K}_r$ and $R \subset \mathfrak{A}_r$, we have $F(y)^\times = \langle y \rangle \cdot R^\times \subset \mathfrak{K}_r$. $\square$

§**A2 The Embedding of x Made Explicit.**

The following explicit information regarding the embedding of x into $\mathfrak{K}_r$ will be used in proving the part of A.1 which asserts that $\mathfrak{U}^1_{\mathfrak{A}_r}$ acts transitively on the set of inertial subfields associated to elements of $\mathfrak{U}^1_{\mathfrak{A}_r}$. As usual, ϕ denotes a generator of $\mathrm{Gal}(F_d|F)$ (see (0.6)).

A.6 Lemma. *(i) There exists an embedding $\iota : F_{df'} \to M_{f'}(F_d)$ such that the action of $\phi^{(d,f)} \in Gal(F_d|F_{(d,f)})$ extended to $M_{f'}(F_d)$ induces on $\iota(F_{df'})$ a generator of $Gal(F_{df'}|F_f)$.*
(ii) If $\phi' \in Gal(F_d|F_f)$ is arbitrary and if ι is an embedding as in (i), then the embedding $\phi' \circ \iota$ has the same properties.

Proof. (i) Choose any embedding ι such that $\iota(F_f) \subset \mathrm{M}_{f'}(F_{(d,f)})$. Then ϕ acts on $\mathrm{M}_{f'}(F_d)$ by acting on matrix coefficients, so $\phi^{(d,f)}$ fixes $\iota(F_f)$. Since $\phi^{(d,f)}$ stabilizes the scalar subfield F_d and fixes $\iota(F_f)$, it follows that $\phi^{(d,f)}$ stabilizes $\iota(F_{df'}) = \iota(F_f)F_d$. Since the order of $\phi^{(d,f)}$ on F_d is $d/(d,f)$ and since $F_{df'}|F_f$ is a field extension of degree $d/(d,f)$, it follows that $\phi^{(d,f)}$ acts on $\iota(F_{df'})$ as a generator of $\mathrm{Gal}(F_{df'}|F_f)$.

$\mathrm{F}_d \otimes_{F_{(d,f)}}$

(ii) This follows, as (i) does, from the fact that $\phi' \circ \iota$ is an embedding which injects F_f into $M_{f'}(F_{(d,f)})$. $\square$

A.7 Lemma. *The root $x = \alpha t_r^c$ of the polynomial $g(T)$ of (1) can be chosen such that $\alpha \in GL_s^r(o_d) \subset \mathfrak{A}_r^\times$ and such that $x^e \varpi_F^{-1} \in GL_{f'}^{e'}(o_d) \subset GL_s^r(o_d)$.*

Proof. Let

$$(9) \qquad \alpha := (\alpha_1, \ldots, \alpha_r) = (I_s, \ldots, I_s, \alpha_r), \qquad \alpha_r := \begin{bmatrix} 0_{s-f',f'} & I_{s-f'} \\ \xi & 0_{f',s-f'} \end{bmatrix},$$

where $\xi \in k_{df'}^\times \subset \iota(o_{df'})^\times \subset \mathrm{GL}_{f'}(o_d)$ with ι an embedding as in A.6 and where $\mathrm{N}_{k_{df'}|k_f}(\xi) = \zeta \in k_f^\times$. The element ζ is given in (1); we assume that the finite field multiplicative groups are embedded as roots of unity of order prime to p. Clearly this choice of α puts α into $\mathrm{GL}_s^r(o_d)$. We have to show that x is a root of the polynomial $g(T)$ and that

$$(10) \qquad x^e \varpi_F^{-1} = (\alpha t_r^c)^e \varpi_F^{-1} = \alpha(t_r^c \alpha t_r^{-c}) \cdots (t_r^{(e-1)c} \alpha t_r^{-(e-1)c}) \frac{t_r^{ce}}{\varpi_F} \in \mathrm{GL}_{f'}^{e'}(o_d).$$

Since $t_r^{ce} = \varpi_F$, the factor on the right in (10) is 1. As usual, we write $c := f/(f,m)$, $d' := d/(d,f)$, $\delta := (d,f)/c$, $e' := (e,m)$, $f' := f/(d,f)$, $r = m/(f,m)$, and $s := (f,m)$. Then $\delta = s/f'$ and $rs = e'f' = m$. We break up the "norm mapping" $x \mapsto x^e$ into three steps by factoring $e = d'\delta r$. For the first step we compute that

$$(11) \qquad (\beta_1, \ldots, \beta_r)\varpi_{D_d}^c := x^r = (\alpha t_r^c)^r = \alpha(t_r^c \alpha t_r^{-c}) \cdots (t_r^{(r-1)c} \alpha t_r^{-(r-1)c}) \varpi_{D_d}^c.$$

Since $(c,r) = 1$, we see from (9) and (11) that $\beta_r = \alpha_r$ and that for each i, $1 \le i \le r$, the component β_i is a Galois conjugate of α_r. Next, using the fact that $\delta c = (d,f)$, we obtain

$$(12) \qquad (\gamma_1, \ldots, \gamma_r)\varpi_{D_d}^{(d,f)} := x^{\delta r} = ((\beta_1, \ldots, \beta_r)\varpi_{D_d}^c)^\delta,$$

where $\gamma_i = \beta_i(\varpi_{D_d}^c \beta_i \varpi_{D_d}^{-c}) \cdots (\varpi_{D_d}^{(\delta-1)c} \beta_i \varpi_{D_d}^{-(\delta-1)c})$. Since the conjugation by ϖ_{D_d} gives $\varpi_{D_d} \beta_i \varpi_{D_d}^{-1} = \phi(\beta_i)$ (see (0.6)), we see, by using the explicit matrix embedding of (9), that

$$\text{(13)} \qquad \gamma_r = \alpha_r \phi^c(\alpha_r) \cdots \phi^{(\delta-1)c}(\alpha_r) = (\phi^{(d,f)-c}(\xi), \ldots, \phi^c(\xi), \xi) \in \mathrm{GL}_{f'}^{\delta}(o_d)$$

and, similarly, γ_i is for each i an element of $\mathrm{GL}_{f'}^{\delta}(o_d)$ which is Galois conjugate to γ_r. Finally, we see that

$$\text{(14)} \qquad x^e = ((\gamma_1, \ldots, \gamma_r)\varpi_{D_d}^{(d,f)})^{d'} = (\varepsilon_1, \ldots, \varepsilon_r)\varpi_F \in \mathrm{GL}_{f'}^{e'}(o_d)\varpi_F,$$

where $\varepsilon_i = \gamma_i \phi^{(d,f)}(\gamma_i) \cdots \phi^{(d-(d,f))}(\gamma_i)$. In particular, it follows from A.6 that

$$\text{(15)} \qquad \begin{aligned} \varepsilon_r &= \mathrm{diag}(\phi^{(d,f)-c}(\mathrm{N}_{F_{df'}|F_f}(\xi)), \ldots, \phi^c(\mathrm{N}_{F_{df'}|F_f}(\xi)), \mathrm{N}_{F_{df'}|F_f}(\xi)) \\ &= \mathrm{diag}(\phi^{(d,f)-c}(\zeta), \ldots, \phi^c(\zeta), \zeta) \in \mathrm{GL}_{f'}^{\delta}(o_d) \end{aligned}$$

and, similarly, for $1 \leq i < r$, ε_i is a block diagonal matrix in $\mathrm{GL}_{f'}^{\delta}(o_d)$ which is Galois conjugate to ε_r. Since each of the e' blocks of x^e/ϖ_F represents a Galois conjugate of ζ, it follows that x is a root of $g(T)$. $\square$

We write $F_c^\sharp$ for the fixed field of t_r^c in $(F_d I_s)^r$. This field, or more precisely its residual field $k_c^\sharp$, is determined and given an explicit description in §$B2$, see especially B2.1. The reader must substitute local unramified extension fields of F with the action of ϕ for finite extensions of k with the induced action in order to convert the discussion of §B to the context of A.8. The reader will also note that §B deals with GL_s^r instead of GL_1^r, but she can ignore this complication.

A.8 Corollary. *Assume that x is chosen as in (9). Then:*
(i) $F_{f,x} \cap (F_d I_s)^r = F_c^\sharp \subset (F_c I_s)^r$.
(ii) $F_{f,x} \otimes_{F_c^\sharp} (F_d I_s)^r$ is a vector space of dimension e' over $F_{df'}$ and is its own centralizer in $M_s^r(F_d)$.

Proof. (i) Equation (9) implies that $\alpha \in \mathrm{GL}_s^r(o_d) \subset \mathrm{M}_s^r(F_d)$. Clearly, $(F_d I_s)^r$ is the center of $\mathrm{M}_s^r(F_d)$. Therefore, α centralizes $(F_d I_s)^r$. Since $F_{f,x}$ is the inertial subfield of $F(x)$, $x = \alpha t_r^c$ centralizes $F_{f,x}$. Therefore, t_r^c centralizes $F_{f,x} \cap (F_d I_s)^r$. Since $F_c^\sharp$ is the centralizer of t_r^c in $(F_d I_s)^r$, by B2.1, we have $F_{f,x} \cap (F_d I_s)^r \subseteq F_c^\sharp$. Since $F_c^\sharp \subset (F_d I_s)^r$, we know that α centralizes $F_c^\sharp$, so x centralizes $F_c^\sharp$. But $F(x)$ is a maximal subfield of A, which implies that it is its own centralizer. Therefore, $F_c^\sharp \subset F(x)$. Since $F_c^\sharp|F$ is unramified and $F_{f,x}$ is the inertial subfield of $F(x)$, $F_c^\sharp \subseteq F_{f,x}$.
(ii) Let $F_{f,x} \supset E \supset F_c^\sharp$, where the field E satisfies the degree formula $[E : F_c^\sharp] = (d,f)/c = \delta$, and let $\iota : F_{f,x} \to \mathrm{M}_{f'}(F_d)$ be an embedding such that $\iota(F_{f,x}) \cap F_d I_{f'} = \iota(E)$. Let $\sigma_1, \ldots, \sigma_\delta \in \mathrm{Gal}(F_{f,x}|F_c^\sharp)$ be a set of representatives for $\mathrm{Gal}(E|F_c^\sharp)$. Then the mapping $\alpha \otimes \beta \mapsto \prod_{i=1}^{\delta} \iota(\sigma_i(\alpha))\beta$ of $F_{f,x} \otimes_{F_c^\sharp} F_d I_s \xrightarrow{\sim} (F_{df'})^\delta$ defines an isomorphism $F_{f,x} \otimes_{F_c^\sharp} (F_d I_s)^r \cong (F_{df'})^{\delta r} \subset \mathrm{M}_{f'}^{\delta r}(F_d) \subset \mathrm{M}_s^r(F_d)$. Therefore, $F_{f,x} \otimes_{F_c^\sharp} (F_d I_s)^r$, which is contained in $\mathrm{M}_{f'}^{e'}(F_d) \subset \mathrm{M}_s^r(F_d)$, is, in each case, the set of rational points of a maximal toral subalgebra, so it equals its centralizer. $\square$

§**A3 The Group $\mathfrak{U}^1_{\mathfrak{A}_r}$ Acts Transitively on the Set $\{F_{f,y} \subset F(y) \mid y \in x\mathfrak{U}^1_{\mathfrak{A}_r}\}$.**

Fix $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$. In the following we write $\mathfrak{A}_{F(y)|F} := \mathfrak{A}_r$ to signify that $\mathfrak{A}_r$ is the unique principal order in A which is normalized by $F(y)^\times$. We also write $\mathfrak{K}_{F(y)|F} := \mathfrak{K}_r$ for the normalizer of the order $\mathfrak{A}_{F(y)|F}$. We depend upon [Fr2], [Zi] for the properties of $\mathfrak{A}_{F(y)|F}$ and $\mathfrak{K}_{F(y)|F}$ which we use below.

A.9 Lemma. *There exists $\kappa \in \mathfrak{K}_r$ such that $\kappa F_{f,x}\kappa^{-1} = F_{f,y}$.*

Proof. Let $A_{F_{f,y}} \subset A$ denote the centralizer algebra of $F_{f,y}$. Then $\mathfrak{A}_{F(y)|F_{f,y}} := \mathfrak{A}_r \cap A_{F_{f,y}}$ and $\mathfrak{K}_{F(y)|F_{f,y}} := \mathfrak{K}_r \cap A_{F_{f,y}}$ are, respectively, the unique principal order in $A_{F_{f,y}}$ which is normalized by $F(y)^\times$ and the normalizer of that order. In $A_{F_{f,y}}$ there is a maximal field extension $K|F_{f,y}$ such that $K \cong_F F(x)$ and $K^\times \subset \mathfrak{K}_{F(y)|F_{f,y}}$. This follows from the fact that any field extension L of degree e of F_f can be embedded in $A_{F_{f,y}}$. Moreover, the principal order of $A_{F_{f,y}}$ which is normalized by L depends up to conjugation only on the ramification index and the inertial degree of $L|F_f$. In our case both $L \cong F(x)$ and $F(y)$ are fully ramified over F_f. Therefore up to conjugation $L^\times$ normalizes the order $\mathfrak{A}_{F(y)|F_{f,y}}$, i.e. $K^\times \subset \mathfrak{K}_{F(y)|F_{f,y}}$ for an appropriate conjugation $K = aLa^{-1}$. By Skolem/Noether the F isomorphism of K to $F(x)$ may be represented by a conjugation in $A^\times$. Thus we may assume that $x' = \kappa x \kappa^{-1}$ generates K. Since $K^\times \subset \mathfrak{K}_{F(y)|F_{f,y}} \subset \mathfrak{K}_r$, x' is an $(e, f, \mathfrak{A}_r)$–pure element in $\mathfrak{K}_r$. But x is also an $(e, f, \mathfrak{A}_r)$-pure element in $\mathfrak{K}_r$, since, according to A.3, $F(x)^\times$ normalizes $\mathfrak{A}_r$. Now we use the fact that two $(e, f, \mathfrak{A}_r)$–pure elements which are contained in $\mathfrak{K}_r$ and are conjugate in A must also be conjugate in $\mathfrak{K}_r$ ([Zi]6.). Therefore, $x' = \kappa x \kappa^{-1}$ with $\kappa \in \mathfrak{K}_r$ and, since $F_{f,y}$ and $F_{f,x}$ are the respective inertial subfields of $K|F$ and $F(x)|F$, $K = \kappa F(x)\kappa^{-1}$ and $F_{f,y} = \kappa F_{f,x}\kappa^{-1}$. □

A.10 Lemma. *For any $i \in \mathbb{Z}$ there exists $\beta_i \in \mathfrak{A}_r^\times$ such that $\beta_i t_r^i$ normalizes $F_{f,x}$.*

Proof. We let N and $Z := A^\times_{F_f}$ denote respectively the normalizer and the centralizer of F_f in $A^\times$. Then $N/Z \cong \mathrm{Gal}(F_f|F) \cong \mathbb{Z}/f\mathbb{Z}$. Arguing as in A.9, we find a complete set of representatives for N/Z in $\mathfrak{K}_r$. It follows that $(N \cap \mathfrak{K}_r)/(Z \cap \mathfrak{K}_r) \hookrightarrow N/Z$ is an isomorphism. Considering the exact sequence

$$(16) \quad 1 \to N \cap \mathfrak{A}_r^\times / Z \cap \mathfrak{A}_r^\times \to N \cap \mathfrak{K}_r / Z \cap \mathfrak{K}_r \to N \cap \mathfrak{K}_r/(N \cap \mathfrak{A}_r^\times)(Z \cap \mathfrak{K}_r) \to 1,$$

we see that the right side is a subquotient of $\mathfrak{K}_r/\mathfrak{A}_r^\times\langle\varpi_F\rangle$; since $x = \alpha t_r^c \in Z \cap \mathfrak{K}_r$, it is even a subquotient of $\mathfrak{K}_r/\mathfrak{A}_r^\times\langle t_r^c\rangle$. This implies that the order of the right side divides c. Since the order of the middle term is f, it suffices to show that the order of the left side is no larger than $s = f/c$. This will imply that the respective terms of the sequence have the orders s, f, and c; in other words, it will show that $N \cap \mathfrak{K}_r/(N \cap \mathfrak{A}_r^\times)(Z \cap \mathfrak{K}_r)$ is isomorphic to $\mathfrak{K}_r/\mathfrak{A}_r^\times\langle t_r^c\rangle$. Clearly, this implies that for any $i \in \mathbb{Z}$ we can choose $\beta_i \in \mathfrak{A}_r^\times\langle t_r^c\rangle$ such that $\beta_i t_r^i \in N \cap \mathfrak{K}_r$; multiplying by an appropriate power of $x = \alpha t_r^c$, which centralizes F_f, we obtain $\beta_i \in \mathfrak{A}_r^\times$. To complete the proof of A.10 we need:

A.11 Lemma. *The order of the quotient $(N \cap \mathfrak{A}_r^\times)/(Z \cap \mathfrak{A}_r^\times)$ does not exceed s.*

Proof. Let $k_f := ((F_{f,x} \cap \mathfrak{A}_r) + \mathfrak{P}_r)/\mathfrak{P}_r \subset \bar{\mathfrak{A}}_r$ and let $\bar{N}$ and $\bar{Z}$ in $\bar{\mathfrak{A}}_r^\times$ denote the normalizer and centralizer of k_f with respect to the action of $\bar{\mathfrak{A}}_r^\times$ on $\bar{\mathfrak{A}}_r$ by conjugation. Then we obtain the exact sequences

$$(17) \qquad 1 \to N \cap \mathfrak{U}^1_{\mathfrak{A}_r} \to N \cap \mathfrak{A}_r^\times \xrightarrow{p_N} \bar{N} \cap \bar{\mathfrak{A}}_r^\times,$$

$$1 \to Z \cap \mathfrak{U}^1_{\mathfrak{A}_r} \to Z \cap \mathfrak{A}_r^\times \overset{p_Z}{\to} \bar{Z} \cap \bar{\mathfrak{A}}_r^\times . \tag{18}$$

From the fact that $\mathrm{Gal}(F_f|F) \cong \mathrm{Gal}(k_f|k)$ it follows that $N \cap \mathfrak{U}^1_{\mathfrak{A}_r} = Z \cap \mathfrak{U}^1_{\mathfrak{A}_r}$. Let us show that the projection map p_Z is surjective. By [Zi], Prop. 1, we have $A_{F_{f,x}} \cong \mathrm{M}_{e'}(D')$, where $e' = (e, m)$ and where $D'|F_{f,x}$ is a division algebra of index $d' = d/(d, f)$. Since $F_{f,x}^\times \subset F[x]^\times \subset \mathfrak{K}_r$ and $F[x]|F_{f,x}$ is totally ramified, it follows from [Zi], Cor. 3 that $Z \cap \mathfrak{A}_r = \mathfrak{A}_{F[x]|F_{f,x}}$ has the period e'. ¿From the fact that the residual field of D' is $k_{df'}$ ($d'f = df'$) it follows that $(Z \cap \mathfrak{A}_r^\times)/(Z \cap \mathfrak{U}^1_{\mathfrak{A}_r}) \cong \mathrm{GL}_1(k_{df'})^{e'}$. To complete the proof that p_Z is surjective, we have to show that $\bar{Z} \cong \mathrm{GL}_1(k_{d'f})^{e'}$. If p_Z is surjective, then (17) and (18) imply that $N \cap \mathfrak{A}_r^\times / Z \cap \mathfrak{A}_r^\times \to \bar{N}/\bar{Z}$ is injective. Thus we finish the proof of A.11 by proving:

A.12 Lemma. *$\bar{Z} \cong GL_1(k_{df'})^{e'}$ and $\bar{N}/\bar{Z} \cong \mathbb{Z}/s\mathbb{Z}$. The projection p_N of (17) is surjective.*

Proof. That $\bar{Z} \cong \mathrm{GL}_1(k_{df'})^{e'}$ follows immediately from A.8(ii), noting that $\bar{\mathfrak{A}}_r \cong \mathrm{M}_s(k_d)^r$. From this we conclude that p_Z is surjective. Consider the injections $(N \cap \mathfrak{A}_r^\times)/(Z \cap \mathfrak{A}_r^\times) \hookrightarrow \bar{N}/\bar{Z} \hookrightarrow \mathrm{Gal}(k_f|k_c^\sharp) \cong \mathbb{Z}/s\mathbb{Z}$. Note that the second inclusion is defined because $k_c^\sharp$ lies in the center of $\bar{\mathfrak{A}}_r$. The existence of these mappings implies A.11 and completes the proof of A.10; it follows that $(N \cap \mathfrak{A}_r^\times)/(Z \cap \mathfrak{A}_r^\times)$ is precisely of order s and the inclusions are isomorphisms. By comparing (17) and (18) we see that p_N is also surjective. $\square$

For our final lemma we go back to the set-up of (4) and (5), <u>not</u> assuming A.1:

A.13 Lemma. *Let $y \in x\mathfrak{U}^1_{\mathfrak{A}_r}$ be arbitrary. Then the residual fields $o_{F_{f,x}} + \mathfrak{P}_r$ and $o_{F_{f,y}} + \mathfrak{P}_r$ are identical, i.e. the rings of integers $o_{F_{f,x}}$ and $o_{F_{f,y}}$ project on the same subfield of $\bar{\mathfrak{A}}_r$.*

Proof. We know that $o_{F_{f,x}}, o_{F_{f,y}} \subset \mathfrak{A}_r$; however, the two rings can be distinct. Nevertheless, (5) remains true provided we take $\mathrm{N}_{F(y)|F_{f,y}}$ as the norm on the left side. From (4) we obtain $\mathrm{N}_{F(y)|F_{f,y}}(y) = \mathrm{N}_{F(x)|F_{f,x}}(x)\epsilon_1\epsilon_2 = (-1)^{e-1}\varpi_F\zeta'\epsilon_1\epsilon_2$, where $\zeta'\epsilon_1\epsilon_2 \in F_{f,y}$ is a unit. This implies that $o_{F_{f,x}} + \mathfrak{P}_r = o_F[\zeta'] + \mathfrak{P}_r = o_F[\zeta'\epsilon_1\epsilon_2] + \mathfrak{P}_r \subseteq o_{F_{f,y}} + \mathfrak{P}_r$, and one inclusion suffices, since $F_{f,x}|F$ and $F_{f,y}|F$ are unramified extensions of the same degree. $\square$

Let us complete the proof of A.1. By A.9 there exists $\kappa \in \mathfrak{K}_r$ such that $\kappa F_{f,x}\kappa^{-1} = F_{f,y}$. By A.10 there is an element of $N \cap \mathfrak{K}_r$ with the same t_r exponent as κ, so we may assume that $\kappa \in \mathfrak{A}_r^\times$. Write $u \mapsto \bar{u}$ for the projection mapping $\mathfrak{A}_r^\times \to \bar{\mathfrak{A}}_r^\times$. By A.13 we have $\bar{\kappa} \in \bar{N}$ and by A.12 the projection mapping p_N is surjective, so we have $\kappa_1 \in N \cap \mathfrak{A}_r^\times$ such that $\bar{\kappa}_1 = \bar{\kappa}$. Therefore, $\kappa_2 := \kappa\kappa_1^{-1} \in \mathfrak{U}^1_{\mathfrak{A}_r}$ satisfies $\kappa_2 F_{f,x}\kappa_2^{-1} = F_{f,y}$.

§B An Introduction to Shintani Descent for $\mathbf{GL}_m$ and Its Levi Factors.

In the main part of this paper we show that the values of level-zero discrete series characters of $A^\times$ equal the values of level-zero "extended type" characters at elements which distinguish these characters. We then use Shintani descent theory to calculate the values of these level-zero extended type characters. Surprisingly, it turns out that AMT implies certain difficult and very important explicit results in Shintani descent theory for finite general linear groups. Stated succinctly, we are able to describe explicitly, in terms of the associated Green's parameters, the base change lifts (i.e. the preimages under Shintani descent) of all cuspidal characters of $\mathrm{GL}_m(k)$ (see 5.7 and B4.4), to prove a theorem which is hinted at already in the second "main theorem" of Shintani's paper [Sh]. Gyoja [Gy], working with Deligne-Lusztig's R_θ^T characters [DL, Car], has proved more general assertions.

The purpose of this appendix is to provide a basis for our applications of descent theory. The use of descent theory is crucial to our key results, which specify certain ± 1 twists of characters. Thus, among other things, we have to show that no sign ambiguities, due to "virtual" as opposed to "proper" or "true" descent of characters, enter into our use of descent theory. In addition, we need a more general descent theory, the finite field version of the descent theory studied in the local field case in work of Labesse [La]. We show that Labesse's descent, at least in our context, can always be brought back to the more familiar situation originally studied by Shintani [Sh]. On the other hand, Shintani only considered the Frobenius as a generator for $\mathrm{Gal}(k_d|k)$, whereas, for D_d an arbitrary F central division algebra, a prime element of D_d can act on k_d as any generator of $\mathrm{Gal}(k_d|k)$, so we need descent for an arbitrary generator of $\mathrm{Gal}(k_d|k)$. Gyoja, op. cit., had been the first to deal with arbitrary generators of $\mathrm{Gal}(k_d|k)$, but Gyoja deals with more general reductive groups than GL_m and he is mainly content to study virtual descent. We had to establish that for generic characters the descent is always proper, which we do in B3.3. And we had to relate Labesse's descent in the case of finite fields with Shintani's descent theory (see B4.2).

All these considerations led us to attempt to put together a coherent account, which can be followed from first principles to our applications, with precise references for the essential points we leave out. In B3.2 we generalize Shintani's fundamental theorem [Sh] Theorem 1, that characters descend to virtual characters, to the case in which an arbitrary generator ϕ of $\mathrm{Gal}(k_d|k)$ replaces the k Frobenius $\mathcal{F}$ of Shintani's formulation, leaving out only a crucial lemma [Sh]2-11 for which the proof in our set-up remains the same.

An earlier version of this appendix made use of an elegant and efficient approach to Shintani descent which we learned by reading [DM2]. We found in developing the underpinnings of the descent theory that showing the equivalence of Digne-Michel's norm to the matrix norm, which we had to return to for our applications, complicated our discussion more than dealing with the somewhat subtler questions which arose in establishing isometry and other fundamental properties of descent purely in the context of the matrix norm. For this reason we rewrote our development of the theory to avoid Digne-Michel's norm.

§B1 An Overview of Shintani Descent.

Let G be a finite group, t an automorphism of G of order e, $\tilde{G} := G \rtimes \langle t \rangle$, the semi-direct product, and G^t the subgroup of t fixed points. We write $t(g) = tgt^{-1}$. The coset Gt is stable under the inner action of G:

$$(1) \qquad x(gt)x^{-1} = xgt(x)^{-1} \cdot t \qquad (x, g \in G).$$

Since t is of order e, we may define

$$N_t(g) := (gt)^e = g \cdot t(g) \cdots t^{e-1}(g) \in Gt^e = G.$$

The "norm map"

$$Gt \ni gt \longmapsto N_t(g) \in G$$

is compatible with the inner action of G, so it induces a class map

$$\mathcal{N}_t : Gt/\sim \longrightarrow (G/\sim)^t \tag{2}$$

into the set of t-stable conjugacy classes of G, i.e. since

$$t(gt)^e t^{-1} = g^{-1}(gt)^e g \in [(gt)^e]$$

(square brackets denote the conjugacy class), the class $[(gt)^e]$ is t stable. The bijection

$$G \ni g \mapsto gt \in Gt$$

induces, via (1), the identification

$$G/\underset{t}{\sim} \longleftrightarrow (Gt)/\sim,$$

where $g \underset{t}{\sim} h$ means $h = xgt(x)^{-1}$ for some $x \in G$. Therefore we may replace (2) by the map

$$\mathcal{N}_t : G/\underset{t}{\sim} \longrightarrow (G/\sim)^t \tag{3}$$

and identify the class mappings (2) and (3).

Let $\mathcal{C}(Gt)$ denote the vector space of $\mathbb{C}$-valued central functions on the coset Gt, i.e. $f(x^{-1}(gt)x) = f(gt)$ for all $x, g \in G$ and define the inner product

$$\langle f_1, f_2 \rangle_{Gt} = \frac{1}{|G|} \sum_{x \in G} f_1(xt)\bar{f}_2(xt).$$

Let $G^\wedge$ denote the set of irreducible characters of G, $G^{\wedge,t}$ the subset consisting of all t invariant characters.

B1.1 Proposition. *For each $\theta \in G^{\wedge,t}$ let $\hat{\theta}$ denote a fixed extension to $\tilde{G}$. Consider the restriction $\hat{\theta}|_{Gt} \in \mathcal{C}(Gt)$. Then:*
(i): The set $\{\hat{\theta}|_{Gt} \,|\, \theta \in G^{\wedge,t}\}$ is an orthonormal basis of the inner product space $\mathcal{C}(Gt)$.
(ii): $|Gt/\sim| = |G^{\wedge,t}| = |(G/\sim)^t|$.

Proof. (i) From [Sh]1-1,1-2 we see that $\{\hat{\theta}|_{Gt} \,|\, \theta \in G^{\wedge,t}\}$ is an orthonormal set in $\mathcal{C}(Gt)$. We have to prove completeness. Let $\varphi(xt) \in \mathcal{C}(Gt)$. Since conjugation by t stabilizes Gt and since one sees that $\tilde{G}$ classes in Gt are the same as G classes, φ extends by zero to a class function $\hat{\varphi}$ on $\tilde{G}$. Therefore $\hat{\varphi} = \sum_{\pi \in \tilde{G}^\wedge} \langle \hat{\varphi}, \pi \rangle_{\tilde{G}} \pi$. If the restriction $\pi|_G$ is reducible, then π vanishes on the coset Gt (See for instance [Sh], loc. cit.). In this case $\langle \tilde{\varphi}, \pi \rangle_{\tilde{G}} = 0$. Finally if $\pi|_G$ is irreducible, then $\pi|_G \in G^{\wedge,t}$ and $\pi|_G$ has e different extensions to $\tilde{G}$, each differing on Gt by an e-th root of one factor. Therefore, one extension for each distinct $\theta \in G^{\wedge,t}$ suffices to produce a spanning set for $\mathcal{C}(Gt)$.
(ii) The first equality in (ii) follows from (i), the second from a theorem of Brauer [Is](6.32). $\square$

If G is abelian, then $G/\underset{t}{\sim}$ is the factor group of G modulo the t-augmentation subgroup I_tG and (3) is the homomorphism

$$(4) \qquad N_t : G/I_tG \longrightarrow G^t$$

of abelian groups. In this case, for any generator of the cyclic group $\langle t \rangle$ we obtain the same map since the factors commute.

We assume now that the class maps

$$(5) \qquad Gt/\sim \longrightarrow (G/\sim)^t \longleftarrow G^t/\sim,$$

where the first is induced by $gt \mapsto (gt)^e$ *and the second by the injection* $G^t \hookrightarrow G$, *are both bijections.* In this case, we interpret (5) as the map which takes the conjugacy class $[gt]$ to the conjugacy class

$$[(gt)^e] := [gt(g)\cdots t^{e-1}(g)] \cap G^t.$$

The class mappings (5) induce the Shintani descent map

$$(6) \qquad \begin{aligned} \mathrm{Sh}_t : \mathcal{C}(Gt) &\longrightarrow \mathcal{C}(G^t) \\ f &\longmapsto \mathrm{Sh}_t(f) \\ f([gt]) &= \mathrm{Sh}_t(f)([(gt)^e]) \end{aligned}$$

from the space of class functions on the coset Gt to the space of class functions on the group G^t. It follows from (5) that $\mathcal{N}_t : Gt/\sim \longrightarrow G^t/\sim$ is also bijective, and this implies that the inner product spaces $\mathcal{C}(Gt)$ and $\mathcal{C}(G^t)$ have the same dimension. Moreover, Sh_t is obviously also an algebra isomorphism with respect to the pointwise multiplication of functions:

$$\mathrm{Sh}_t(f_1 f_2) = \mathrm{Sh}_t(f_1)\mathrm{Sh}_t(f_2)$$

for all $f_1, f_2 \in \mathcal{C}(Gt)$.

Assume that ψ is an automorphism of $\tilde{G}$ which stabilizes G and fixes t. Then ψ also stabilizes Gt and G^t. Obviously, the maps (5) are ψ equivariant, and therefore the Shintani descent map (6) is ψ equivariant too. Thus, if $f \in \mathcal{C}(Gt)$ and $\psi(f)(x) := f(\psi^{-1}(x))$, then $\psi(f) \in \mathcal{C}(Gt)$ too; moreover,

$$\mathrm{Sh}_t(\psi(f)) = \psi(\mathrm{Sh}_t(f)),$$

and $\psi(\mathrm{Sh}_t(f)) \in \mathcal{C}(G^t)$.

B1.2 Lemma. *The Shintani descent mapping* $Sh_t : \mathcal{C}(Gt) \to \mathcal{C}(G^t)$ *is an isometry if and only if the centralizers of elements lying in the classes which correspond under (5) have the same order, i.e.* $|Z_G(yt)| = |Z_{G^t}(x)|$ *for* $y \in G$ *and* $x \in [(yt)^e]$.

Proof. Since, by assumption, the norm (3) induces a bijection $Gt/\sim \xrightarrow{\sim} G^t/\sim$, the set of characteristic functions f of classes in Gt comprise an orthogonal basis of $\mathcal{C}(Gt)$ and similarly $\mathrm{Sh}_t(f)$ ranges over an orthogonal basis of $\mathcal{C}(G^t)$. Therefore, to prove B1.2, it is enough to check that $\|f\|^2_{Gt} = \|\mathrm{Sh}_t(f)\|^2_{G^t}$ for f the characteristic function of a class $[yt] \subset Gt$ if and only if the orders of the centralizers are equal. We see that

$$\|f\|^2_{Gt} = \frac{|[yt]|}{G} \quad \text{and} \quad \frac{|[(yt)^e]|}{G^t} = \|\mathrm{Sh}_t(f)\|^2_{G^t}$$

and that

$$\frac{|[yt]|}{G} = |Z_G(yt)|^{-1} \quad \text{and} \quad \frac{|[(yt)^e]|}{G^t} = |Z_{G^t}([(yt)^e])|^{-1}.$$

The Proposition follows immediately from these observations. $\square$

B1.3 Definition. We call the pair (G,t) a *standard pair* if $\mathrm{Sh}_t : \mathcal{C}(Gt) \to \mathcal{C}(G^t)$ is an isometric bijection of inner product spaces, equivalently, if (5) is a bijection and the centralizer equalities of B1.2 hold.

B1.4 Proposition. *Assume that $H \subset G$ is a subgroup such that t induces an automorphism of H of the same order e and that the pair (H,t) is also standard. Then:*
(i) For any class function $f \in \mathcal{C}(Gt)$

$$Sh_t(Res_{Ht}(f)) = Res_{H^t}(Sh_t(f)).$$

(ii) Moreover, for $f \in \mathcal{C}(Ht)$ the induced class function

$$Ind_{Ht}^{Gt}(f)(yt) := \frac{1}{|H|} \sum_{x \in G : xytx^{-1} \in Ht} f(xytx^{-1})$$

satisfies the Frobenius formula on the cosets:

$$\langle Ind_{Ht}^{Gt}(f), g\rangle_{Gt} = \langle f, Res_{Ht}(g)\rangle_{Ht}$$

for all $g \in \mathcal{C}(Gt)$, and
(iii) Sh_t commutes with induction:

$$Sh_t(Ind_{Ht}^{Gt}(f)) = Ind_{H^t}^{G^t}(Sh_t(f))$$

for all $f \in \mathcal{C}(Ht)$.

Proof. We omit the proof of (i). For (ii) we consider the semi-direct product $\tilde{G} = G \rtimes \langle t\rangle$ and the embedding $\mathcal{C}(Gt) \to \mathcal{C}(\tilde{G})$, $f \mapsto \tilde{f} := f$ "extended by zero". Thus $\tilde{f} \in \mathcal{C}(\tilde{G})$. Clearly, $\langle f, g\rangle_{Gt} = e\langle \tilde{f}, \tilde{g}\rangle_{\tilde{G}}$, for all $f, g \in \mathcal{C}(Gt)$, and $[\mathrm{Ind}_{Ht}^{Gt}(f)]^{\sim} = \mathrm{Ind}_{\tilde{H}}^{\tilde{G}}(\tilde{f})$ for $f \in \mathcal{C}(Ht)$. This reduces (ii) to the usual Frobenius reciprocity for $\tilde{H} \subset \tilde{G}$.
(iii): For $f \in \mathcal{C}(Ht), g \in \mathcal{C}(Gt)$

$$\begin{aligned}
\langle \mathrm{Ind}_{H^t}^{G^t}(Sh_t(f)), Sh_t(g)\rangle_{G^t} &= \langle Sh_t(f), \mathrm{Res}_{H^t}(Sh_t(g))\rangle_{H^t} \\
&= \langle Sh_t(f), Sh_t(\mathrm{Res}_{Ht}(g))\rangle_{H^t} \\
&= \langle f, \mathrm{Res}_{Ht}(g)\rangle_{Ht} \\
&= \langle \mathrm{Ind}_{Ht}^{Gt}(f), g\rangle_{Gt} \\
&= \langle Sh_t(\mathrm{Ind}_{Ht}^{Gt}(f)), Sh_t(g)\rangle_{G^t}.
\end{aligned}$$

(iii) now follows from the orthogonality property, since g is arbitrary. $\square$

Let (G,t) be a standard pair and let $\theta \in G^{\wedge,t}$.

B1.5 Definition. We say that θ has *virtual descent* [has *descent* or *proper descent*] if there exists an extension $\hat{\theta} \in \tilde{G}^{\wedge}$ such that

$$\bar{\theta} := \mathrm{Sh}_t(\hat{\theta}) := \mathrm{Sh}_t(\hat{\theta}|_{Gt})$$

is a virtual [true] character of G^t.

Remarks. 1. Since $1 = \langle \theta, \theta \rangle_G = \langle \hat\theta, \hat\theta \rangle_{Gt}$ and since Sh_t is an isometry, $\langle \bar\theta, \bar\theta \rangle_{G^t} = 1$. Therefore up to sign $\bar\theta$ must be irreducible if θ has virtual descent.
2. If for every $\theta \in G^{\wedge,t}$, θ has virtual descent, then $\theta \mapsto \bar\theta$, $G^{\wedge,t} \to (G^t)^\wedge$ defines a bijection to a complete set of (signed) irreducible characters of G^t.

B1.6 Definition. An extension $\tilde\theta$ of θ is called a canonical extension if $\tilde\theta(t) > 0$.

Remarks. 3. If θ has virtual descent, then θ has proper descent if and only if θ has a canonical extension.
4. In this case, there is an extension $\tilde\theta \in \tilde G^\wedge$ and $\bar\theta \in (G^t)^\wedge$ such that

$$\tilde\theta(xt) = \bar\theta([(xt)^e]) \quad (x \in G); \qquad \tilde\theta(t) = \mathrm{Sh}_t(\tilde\theta)(1) = \dim(\bar\theta) > 0.$$

5. If $\hat\theta$ is any fixed extension of $\theta \in \tilde G^\wedge$ and $\lambda \in \langle t \rangle^\wedge$, i.e. $\lambda \in \tilde G^\wedge$ is constant on Gt, then

$$\mathrm{Sh}_t(\lambda\hat\theta) = \lambda(t)\mathrm{Sh}_t(\hat\theta), \tag{7}$$

since Sh_t is multiplicative. Thus θ can have at most one canonical extension.
6. If e is even, then θ has virtual descent if and only if θ has proper descent. If e is odd and θ has virtual descent, then either there exists an extension $\hat\theta$ such that $\mathrm{Sh}_t(\hat\theta|_{Gt})$ is a character or such that $-\mathrm{Sh}_t(\hat\theta|_{Gt})$ is a character, but not both.

Define the orthogonal projection mapping

$$p : \mathcal{C}(\tilde G) \to \mathcal{C}(\tilde G) \qquad \tilde f \mapsto \mathrm{ext}_0(\tilde f|_{Gt}) \quad (\tilde f \in \mathcal{C}(\tilde G)), \tag{8}$$

where $\mathrm{ext}_0(\tilde f|_{Gt})$ denotes the extension by zero of $\tilde f$ from the coset Gt. It is easy to see that

$$p(\tilde f) = \frac{1}{e} \sum_{\zeta:\zeta^e=1} \zeta^{-1}(\lambda_\zeta \tilde f), \tag{9}$$

where $\lambda_\zeta \in (\tilde G/G)^\wedge = \langle t \rangle^\wedge$ is defined by setting $\lambda_\zeta(t) = \zeta$. For $\tilde f, \tilde g \in \mathcal{C}(\tilde G)$

$$\begin{aligned} \langle \tilde f|_{Gt}, \tilde g|_{Gt} \rangle_{Gt} &= e\langle \mathrm{ext}_0(\tilde f|_{Gt}), \mathrm{ext}_0(\tilde g|_{Gt}) \rangle_{\tilde G} \\ &= e\langle p(\tilde f), p(\tilde g) \rangle_{\tilde G} \\ &= e\langle p(\tilde f), \tilde g \rangle_{\tilde G} \\ &\underset{(9)}{=} \sum_{\zeta:\zeta^e=1} \zeta^{-1}\langle \lambda_\zeta \tilde f, \tilde g \rangle_{\tilde G}. \end{aligned} \tag{10}$$

Now let $(H,t) \subset (G,t)$ be two standard pairs which are related as in B1.4. Let $\rho \in H^{\wedge,t}$ have descent and let $\tilde\rho$ denote the canonical extension of ρ. Set

$$\Lambda := \mathrm{Ind}\,_H^G(\rho), \quad \tilde\Lambda := \mathrm{Ind}\,_{\tilde H}^{\tilde G}(\tilde\rho), \quad \bar\Lambda := \mathrm{Ind}\,_{H^t}^{G^t}(\bar\rho) \qquad (\mathrm{Sh}_t(\tilde\rho|_{Ht}) = \bar\rho).$$

B1.7 Lemma. *With the above assumptions:*
(i) $\tilde\Lambda|_G = \Lambda,\ \ \tilde\Lambda|_{Gt} = Ind_{Ht}^{Gt}(\tilde\rho|_{Ht}),\ \ Sh_t(\tilde\Lambda) := Sh_t(\tilde\Lambda|Gt) = \bar\Lambda.$
(ii) Let $\theta \in G^{\wedge,t}$ *and* $\hat\theta \in \tilde G^{\wedge}$ *be an extension of* θ*. Assume that* $\langle\theta,\Lambda\rangle_G = \langle\hat\theta,\tilde\Lambda\rangle_{\tilde G}$*, i.e. if* $\langle\theta,\Lambda\rangle_G = \langle Ind_G^{\tilde G}(\theta),\tilde\Lambda\rangle_{\tilde G} \neq 0$*, then* $\hat\theta$ *is the only extension of* θ *occurring in* $\tilde\Lambda$*. Then:*

$$\langle\theta,\Lambda\rangle_G = \langle\hat\theta,\tilde\Lambda\rangle_{\tilde G} = \langle\hat\theta|_{Gt},\tilde\Lambda|_{Gt}\rangle_{Gt} = \langle Sh_t(\hat\theta|_{Gt}),\bar\Lambda\rangle_{G^t}. \tag{11}$$

(iii) Assume, in addition to the hypotheses of (ii), that θ *has virtual descent and* $\langle\theta,\Lambda\rangle_G \neq 0$*. Then* θ *has proper descent and* $\hat\theta = \tilde\theta$*, the canonical extension.*
(iv) Assume that for all $\theta \in G^{\wedge,t}$ *the hypotheses of (ii) and (iii) hold. Then* $\theta \mapsto Sh_t(\tilde\theta)$ *maps the set of* $\theta \in G^{\wedge,t}$ *such that* $\langle\theta,\Lambda\rangle_G \neq 0$ *bijectively to the set of* $\bar\theta \in (G^t)^{\wedge}$ *such that* $\langle\bar\theta,\bar\Lambda\rangle \neq 0$*, preserving multiplicities.*

Proof. (i) follows from Frobenius reciprocity, the last equality from 1.4(iii).
(ii) We assumed the first equality of (11), the last follows from the isometry property of Sh_t. Apply (10) for the straightforward proof of the middle equality
(iii) If θ has virtual descent, then $\mathrm{Sh}_t(\hat\theta|_{Gt}) = \zeta\bar\theta$, where $\bar\theta \in (G^t)^{\wedge}$ and ζ is a root of unity. But (ii) implies that $\langle\mathrm{Sh}_t(\hat\theta|_{Gt}),\bar\Lambda\rangle_{G^t}$ is a positive integer and this implies that $\zeta = 1$.
(iv) This is just a reformulation of (ii) and (iii). □

§B2 A Particular Standard Pair.

Let k be a finite field of characteristic p containing q elements, let $\bar k$ denote the algebraic closure of k and let $\mathcal F$ denote the k-Frobenius. For any $l \geq 1$ let $k_l|k$ be the l-th degree extension of k, $k \subseteq k_l \subset \bar k$. Let $k_d|k$ be a fixed extension and let $\phi := \mathcal F^i$ denote a positive power of $\mathcal F$, such that $(d,i) = 1$, so that ϕ, like $\mathcal F$, restricts to a generator of $\mathrm{Gal}(k_d|k)$. Let $G := \mathrm{GL}_s^r$ for some fixed $r,s \geq 1$ and let $\mathcal F$ and its powers act on $G(\bar k)$ by acting on the matrix coefficients. Define the automorphism $T_r : G(\bar k) \to G(\bar k)$ as follows:

$$T_r(x_0,\ldots,x_{r-1}) := (x_1,\ldots,x_{r-1},\phi(x_0)). \tag{1}$$

Obviously $T_r^r = \phi$ is the natural (diagonal) action of ϕ on $G(\bar k)$. Next let $c \geq 1$ and assume that $c|d$ and $(c,r) = 1$. Let $0 \leq h < r$ and set

$$\begin{aligned} h = h(c) &:= \begin{cases} 0, & \text{if } c = 1 \\ r^{-1} \bmod c, & \text{if } c > 1 \end{cases} \\ e &:= \frac{d}{c}\cdot r \\ T &:= T_r^c. \end{aligned} \tag{2}$$

Then $T^e = T_r^{dr} = \phi^d = 1$ on $G(k_d)$, and this implies that $t := T \bmod \phi^d$ is an automorphism of order e of $G(k_d)$.

We want to show that $(G(k_d),t)$ and a sufficiently rich class of its subgroups are standard pairs. We begin by describing the fixed point groups $G(\bar k)^T$ and $G(k_d)^t$. For this consider the embedding

$$x \longmapsto x^\sharp := (x,\phi^h(x),\ldots,\phi^{h(r-1)}(x)); \qquad \sharp : \mathrm{GL}_s(\bar k) \longrightarrow G(\bar k)$$

and let $\mathrm{GL}_s(\bar k)^\sharp$ denote its image.

B2.1 Proposition. $G(\bar{k})^T = GL_s(k_{ci})^\sharp$ *and* $G(k_d)^t = GL_s(k_c)^\sharp$.

Proof. The second equality follows from the first, since $k_{ci} \cap k_d = k_c$. If $c = 1$ and $h = 0$, then $G(\bar{k})^T = \mathrm{GL}_s(k_i)$ diagonally embedded in $G(\bar{k})$ and this is $\mathrm{GL}_s(k_{ci})^\sharp$ in the case $c = 1$. Assume $c > 1$. Let us first show that $\mathrm{GL}_s(k_{ci})^\sharp \subseteq G(\bar{k})^T$. If $x_0 \in \mathrm{GL}_s(k_{ci})$, then $\phi(x_0) = \phi^{hr}(x_0)$, since $1 = hr \bmod c$ and $\phi = \mathcal{F}^i$. Therefore, by (1), $T_r(x_0^\sharp) = \phi^h(x_0^\sharp)$ and $T(x_0^\sharp) = \phi^{ch}(x_0^\sharp) = x_0^\sharp$. Proving the opposite inclusion, we note that $G(\bar{k})^T \subseteq G(\bar{k})^{T^r} = G(k_{ci})$, since $T^r = (T_r^c)^r = \phi^c = \mathcal{F}^{ci}$. On the other hand, since $(c, r) = 1$, the relation $T_r^c(x_0, \ldots, x_{r-1}) = (x_0, \ldots, x_{r-1})$ creates a relation between every pair of components x_i, $x_{i'}$, so that any component of a fixed vector determines all other components. This implies that $G(\bar{k})^T \subseteq \mathrm{GL}_s(k_{ci})^\sharp$. $\square$

The class of subgroups we wish to consider is constructed as follows. Let $B_1 \subseteq M_s(k)$ a k-subalgebra containing $k = k \cdot I_s$ and let $B = B_1^r \subseteq M_s(k)^r$. For field extensions $k_l|k$ we put

$$(3) \qquad H_1(k_l) := (B_1 \underset{k}{\otimes} k_l)^\times, \quad H(k_l) := (B \underset{k}{\otimes} k_l)^\times.$$

Then $H = H_1^r$ is a connected algebraic subgroup of $G = \mathrm{GL}_s^r$ ([SpSt]III,3.23). The action (1) of T_r induces an action on $H(\bar{k})$ and B2.1 generalizes to produce the equality:

$$(4) \qquad H(k_d)^t = H_1(k_c)^\sharp.$$

We want to show that the pairs $(H(k_d), t)$ are standard pairs. We have to show that the mappings B1(5) are bijections. For this we define the *matrix norm* map

$$x \mapsto (xt)^e\,; \qquad N_t : H(k_d) \to H(k_d)$$

and prove that N_t induces a bijective class map

$$(5) \qquad [x]_t \mapsto [(xt)^e] \cap H^t(k_c)\,; \qquad \mathcal{N}_t : H(k_d)/\underset{t}{\sim}\longrightarrow (H(k_d)/\sim)^t \cap H^t(k_c)/\sim$$

between the set of t conjugacy classes of $H(k_d)$ and the set of ordinary conjugacy classes of $H^t(k_c)$.

For the abelian case we quote [Sh]2-1:

B2.2 Fact. *If A is a connected linear abelian algebraic group defined over k, then the class mapping induced by B1(4) with $G = A(k_d)$ and t any generator of $Gal(k_d|k)$ is bijective. In this case, N_t maps $A(k_d)$ surjectively to $A(k)$ and $(A(k_d), t)$ is a standard pair.*

By B1.2, the last assertion in B2.2 is equivalent to the equality of the orders $|Z_{A(k_d)}(xt)| = |A(k)|$ for any $x \in A(k_d)$; in fact $A(k_d)^t = A(k) = Z_{A(k_d)}(xt)$.

To prepare for the proof in B2.6 that (5) is a bijection we prove some technical lemmas:

B2.3 Lemma([Sh],[SpSt],[La]**).** *The class map $\iota_{c,d} : H_1(k_c)/\!\sim\,\to H_1(k_d)/\!\sim$, which is induced by the inclusion $i_{c,d} : H_1(k_c) \hookrightarrow H_1(k_d)$, is injective.*

Proof. It is enough to prove that the mapping $\iota_\ell : H_1(k_\ell)/\!\sim\,\to H_1(\bar{k})/\!\sim$ is injective for all $\ell \geq 1$, because in this case we can factor $\iota_c = \iota_d \circ \iota_{c,d}$ and conclude immediately that $\iota_{c,d}$ is injective too. To prove the injectivity of ι_ℓ we use the fact that $H_1(k_\ell) = (B_1 \underset{k}{\otimes} k_\ell)^\times$ is the multiplicative group of an associative algebra and therefore H_1 is a connected algebraic group ([SpSt]op. cit). Assume that

$$h' = xhx^{-1} \qquad (h, h' \in H_1(k_\ell), x \in H_1(\bar{k})). \tag{6}$$

Applying $\mathcal{F}^\ell$ we obtain $h' = \mathcal{F}^\ell(x)h\mathcal{F}^\ell(x)^{-1}$ and, comparing with (6), we see that $x^{-1}\mathcal{F}^\ell(x) \in Z_{H_1}(h)(\bar{k})$, the centralizer of h in $H_1(\bar{k})$. Since $Z_{H_1}(h)(\bar{k})$ is the group of invertible elements of an associative algebra, it is a connected algebraic group defined over k_ℓ and we can apply the Lang-Steinberg Theorem [DM1]3.10 to deduce that

$$x^{-1}\mathcal{F}^\ell(x) = u^{-1}\mathcal{F}^\ell(u)$$

has a solution $u \in Z_{H_1}(h)(\bar{k})$. Thus we have $y = ux^{-1} \in H_1(k_\ell)$ and

$$h' = xhx^{-1} = xu^{-1}hux^{-1} = y^{-1}hy,$$

which proves the injectivity of ι_ℓ. $\square$

B2.4 Lemma. *The mapping $H_1(k_d) \ni y \mapsto ext_1(y) := (y, 1, \dots, 1) \in H(k_d)$ induces a bijection $[ext_1] : H_1(k_d)/\underset{\phi^c}{\sim} \xrightarrow{\sim} H(k_d)/\underset{t}{\sim}$.*

Proof. For $\eta = (y_0, \dots, y_{r-1})$ and $\mu = (u_0, \dots, u_{r-1})$, elements of $H(k_d)$, consider

$$\begin{aligned} \mu^{-1}\eta t(\mu) &= (u_0^{-1}, \dots, u_{r-1}^{-1})\eta(t(\mu)_0, \dots, t(\mu)_{r-1}) \\ &= (u_0^{-1}y_0t(\mu)_0, u_1^{-1}y_1t(\mu)_1 \dots, u_{r-1}^{-1}y_{r-1}t(\mu)_{r-1}). \end{aligned}$$

To show that every t class in $H(k_d)$ has a representative $\mathrm{ext}_1(y)$ with $y \in H_1(k_d)$ it suffices to show that, for any η, the system of equations $u_jt(\mu)_j^{-1} = y_j$, $0 < j \le r-1$, has a solution μ. Since $(c, r) = 1$, the action of t cyclically permutes the components of μ with an additional Galois action, the order of the cycle being exactly r. Thus, $t(\mu)_j = \phi^{a(j)}(u_{b(j)})$, where $\{j \mapsto b(j)\}$ is a cyclic permutation of $\{0, 1, \dots, r-1\}$ of order r and $a(0) + \cdots + a(r-1) = c$. We must choose u_j such that $u_j = y_j\phi^{a(j)}(u_{b(j)})$ for $0 < j \le r-1$. Starting with $u = u_0 = u_{b^r(0)}$ arbitrary we find that the components $u_{b^j(0)} = y_{b^j(0)}\phi^{a(b^j(0))}(u_{b^{j+1}(0)})$ are uniquely determined and the solution μ satisfies $\mu\eta t(\mu)^{-1} \in \mathrm{ext}_1(H_1(k_d))$. Similarly, if $\eta \in \mathrm{ext}_1(H_1(k_d))$, then $\mu\eta t(\mu)^{-1} \in \mathrm{ext}_1(H_1(k_d))$ if and only if $u_j = t(\mu)_j = \phi^{a(j)}(u_{b(j)})$ $(j \ne 0)$ or $u_{b^j(0)} = \phi^{a(b^j(0))}(u_{b^{j+1}(0)})$ for $0 < j < r$ and $t(\mu)_0 = \phi^{a(0)}(u_{b(0)})$. Therefore, $\mu\eta t(\mu)^{-1}$ has the zero component $u^{-1}yt(\mu)_0 = u^{-1}y\phi^{a(0)+\cdots+a(r-1)}(u) = u^{-1}y\phi^c(u)$. $\square$

In the following, for any $x \in H_1(k_c)$ let $A_x(k_c) \subset B_1(k_c)$ be the k_c subalgebra generated by x. Then $A_x^\times$ is a connected commutative algebraic subgroup of H_1 which is defined over k_c and the group of k_d points of $A_x^\times$ is

$$A_x^\times(k_d) = (A_x(k_c) \underset{k_c}{\otimes} k_d)^\times.$$

B2.5 Lemma. *Let $x \in H_1(k_c)$, let $y \in A_x^\times(k_d)$ be chosen such that $N_{\phi^c}(y) = x$, and let $\zeta = ext_1(y) \in H(k_d)$. Then $Z_{H(k_d)}(\zeta t) = Z_{H_1(k_c)^\sharp}(x^\sharp)$.*

Proof. We begin with the case $r = 1$: $H = H_1$, $t = \phi^c$, and $\zeta = y$. Assume that $u \in Z_{H(k_d)}(y\phi^c)$. Then $x = (y\phi^c)^{d/c} = (u^{-1}y\phi^c u)^{d/c} = u^{-1}xu$, which implies that $u \in Z_{H(k_d)}(x)$. Therefore u centralizes every element of $A_x^\times(k_d)$, so $u \in Z_{H(k_d)}(y\phi^c) \cap Z_{H(k_d)}(y)$. Therefore, u centralizes ϕ^c, which implies that $u \in Z_{H(k_c)}(x)$. Conversely, if $u \in Z_{H(k_c)}(x)$, then since $y \in A_x^\times(k_d)$ and since ϕ^c centralizes u, we also have $u \in Z_{H(k_d)}(y\phi^c)$. Now assume that $r > 1$ and let $\mu = (u_0, \cdots, u_{r-1}) \in Z_{H(k_d)}(\zeta t)$. Then $\zeta = \mu\zeta t(\mu)^{-1}$, so $y = u_0 y t(\mu)_0^{-1}$ and $1 = u_i t(\mu_i)^{-1}$ for $i = 1, \ldots, r-1$. By B2.4, this implies that $t(\mu)_0 = \phi^c(u_0)$ and therefore $u_0 \in Z_{H_1(k_d)}(y\phi^c) = Z_{H_1(k_c)}(x)$, using the case $r = 1$ for the equality. Thus $u_0 \in H_1(k_c)$, $t(\mu)_0 = \phi^c(u_0) = u_0$, and, moreover, $t(\mu) = \mu$. By B2.1, $\mu = u_0^\sharp \in H_1(k_c)^\sharp$. But $u_0 \in Z_{H_1(k_c)}(x)$ and therefore $\mu \in Z_{H_1(k_c)^\sharp}(x^\sharp)$. Conversely, let $\mu \in Z_{H_1(k_c)^\sharp}(x^\sharp)$, so $\mu = u_0^\sharp$ and $u_0 \in H_1(k_c)$ centralizes x. Since $y \in A_x^\times(k_d)$, u_0 centralizes y too. Since $t(\mu) = \mu$, $\mu\zeta t(\mu)^{-1} = \mu\zeta\mu^{-1} = \zeta$, which implies that $\mu \in Z_{H(k_d)}(\zeta t)$. $\square$

B2.6 Proposition. *The following diagram is commutative and all the (class) maps are bijections:*

$$\begin{array}{ccccc}
H_1(k_d)/\underset{\phi^c}{\sim} & \xrightarrow{\mathcal{N}_{\phi^c}} & (H_1(k_d)/\sim)^{\phi^c} & \xleftarrow{\iota_{c,d}} & H_1(k_c)/\sim \\
\Big\downarrow{\scriptstyle [\mathrm{ext}_1]} & & \Big\downarrow{\scriptstyle [\sharp]_d} & & \Big\downarrow{\scriptstyle [\sharp]_c} \\
H(k_d)/\underset{t}{\sim} & \xrightarrow{\mathcal{N}_t} & (H(k_d)/\sim)^t & \xleftarrow{\iota_{c,d}^r} & H_1(k_c)^\sharp/\sim
\end{array}$$

The mapping $[ext_1]$ was defined in B2.4 and the class mappings $[\sharp]_c, [\sharp]_d$ are induced by sending: $y \mapsto y^\sharp := (y, {}^{\phi^h}y, \ldots, {}^{\phi^{(r-1)h}}y)$. For the norm maps see either B1(3) or B2(5).

Proof. We proved $\iota_{c,d}$ injective in B2.3 and $[\mathrm{ext}_1]$ bijective in B2.4. Let $\{x\}$ be a complete set of representatives for $H_1(k_c)/\sim$. B2.2 implies that for any $x \in \{x\}$ there exists $y(x) \in A_x^\times(k_d)$ such that $N_{\phi^c}(y(x)) = x$. Therefore we have $[y(x)] \in H_1(k_d)/\underset{\phi^c}{\sim}$ such that $\mathcal{N}_{\phi^c}([y(x)]) = \iota_{c,d}([x])$ ($[x] \in H_1(k_c)/\sim$). It follows from B2.3 that the restriction $\mathcal{N}_{\phi^c}|_{\cup_x[y(x)]}$ is an injective class mapping. By the case $r = 1$ of B2.5, we have $Z_{H_1(k_d)}(y(x)\phi^c) = Z_{H_1(k_c)}(x)$, from which we see that

$$1 = \sum_{[x]\in H_1(k_c)/\sim} \frac{1}{|Z_{H_1(k_c)}(x)|} = \sum_{[x]\in H_1(k_c)/\sim} \frac{1}{|Z_{H_1(k_d)}(y(x)\phi^c)|},$$

which implies that $\{y(x)\}$ is a complete set of representatives for $H_1(k_d)/\underset{\phi^c}{\sim}$. Thus $\mathcal{N}_{\phi^c}$ is injective and, by B1.1(ii), it is also surjective. We have proved that the horizontal arrows on the top line are bijections.

We now consider the other arrows of the diagram. B2.4 proves that the left vertical arrow is a bijection, obviously the right vertical arrow is a bijection, and, again by B1.1(ii), $(H(k_d)/\sim)^t$ has the same cardinality as the other sets. The right square of the diagram is obviously commutative; the bottom right arrow of the square maps into the space of t fixed classes of $H(k_d)$ because t fixes $y^\sharp$ for any $y \in H_1(k_c)$. The middle vertical arrow is injective because for any $y, y' \in H_1(k_d)$ obviously $y \not\sim y'$ implies $y^\sharp \not\sim (y')^\sharp$ in $H(k_d)$, since each of the components of two elements of $H(k_d)$ have to be conjugate in $H_1(k_d)$ for the elements to be conjugate in $H(k_d)$. Starting with $x \in H_1(k_c)$ and $y(x) \in A_x^\times(k_d)$ such that $N_{\phi^c}(y(x)) = x$, one sees that, since $A_x^\times(k_d)$ is an abelian group, $N_t(\mathrm{ext}_1(y(x))) = x^\sharp$. This implies the commutativity of the first square and the bijectivity of $\mathcal{N}_t$. $\square$

We have proved:

B2.7 Corollary. *Let $H = H_1^r \subseteq GL_s^r$ be a connected algebraic k-group which is defined as in B2(3) and let t be the restriction to $H(k_d)$ of the automorphism $T = T_r^c$. Then the pair $(H(k_d), t)$ is a standard pair and t is of order $e = \frac{d}{c} \cdot r$.*

§B3 Shintani's Main Theorem and Canonical Extensions of Characters.

For this section we assume that $r = 1$ and $G(k_d) = \mathrm{GL}_s(k_d)$, $\tilde{G}(k_d) = \mathrm{GL}_s(k_d) \rtimes \langle \mathcal{F} \rangle$. We also assume that $c = 1$, that $t = \phi = \mathcal{F}^i$ is a generator of $\mathrm{Gal}(k_d|k)$ and therefore that the order of $t = \phi$ is $d = e$.

First, we recall Shintani's main result:

B3.1 Theorem of Shintani ([Sh]Theorem 1). *Let $\phi \in Gal(k_d|k)$ be a generator of $Gal(k_d|k)$ and consider the standard pair $(G(k_d), \phi)$ $(G = GL_s)$. Then every $\theta \in G(k_d)^{\wedge,\phi}$ has virtual descent:*

$$\pm G(k)^\wedge \ni \bar{\theta} := Sh_\phi(\hat{\theta}|_{G(k_d)\phi})$$

for some extension $\hat{\theta} \in \tilde{G}(k_d)^\wedge$ of θ. The mapping $\theta \mapsto sign(\bar{\theta})\bar{\theta}$ defines a bijection $G(k_d)^{\wedge,\mathcal{F}} \to G(k)^\wedge$.

In fact, Shintani considered in his paper only the case $\phi = \mathcal{F}$, the k Frobenius, and proved his theorem only for this case. Using the results established in §2, the reader can generalize Shintani's proof to the case of arbitrary ϕ by following Shintani's proof verbatim.

Important results concerning a more general context, including more general generators of the Galois group, appeared in work of Gyoja [Gy] and Kawanaka [Ka].

In the following lemma we prove that, on a suitably chosen set of representatives for $G(k_d)/\underset{\phi}{\sim}$, the descent mappings Sh_ϕ do not depend upon the choice of generator of $\mathrm{Gal}(k_d|k)$. We have to assume only the original version of Shintani's theorem; the more general version alluded to above is a consequence of the following:

B3.2 Proposition. *Let $\mathcal{F}$ be the k Frobenius and let $\phi = \mathcal{F}^i$, where $(d, i) = 1$, be an arbitrary generator of $Gal(k_d|k)$. Let $\theta \in G(k_d)^{\wedge,\mathcal{F}}$ and let $\hat{\theta} \in \tilde{G}^\wedge$ be an extension of θ such that $\bar{\theta} := Sh_\mathcal{F}(\hat{\theta}|_{G(k_d)\mathcal{F}})$ is a virtual character of $G(k)$. Then:*

$$Sh_\phi(\hat{\theta}|_{G(k_d)\phi}) = Sh_\mathcal{F}(\hat{\theta}|_{G(k_d)\mathcal{F}}).$$

In other words, the induced mapping $G(k_d)^{\wedge,\phi} \to G(k)^\wedge$ is independent of the generator ϕ of $Gal(k_d|k)$.

Proof. Set $G = \mathrm{GL}_s(k_d)$, $\tilde{G} = \tilde{G}(k_d)$, and $\bar{\theta}_i = Sh_{\mathcal{F}^i}(\hat{\theta}|_{G\mathcal{F}^i})$. We must show that:

(1) $$\bar{\theta}_i(x) = \bar{\theta}(x)$$

for all $x \in G(k)$ and for $(d,i) = 1$. We fix x and consider the subgroup $A := A_x^\times(k_d) \subset G$. Since A is abelian, $\mathcal{N}_{A|A_x^\times(k)}$ is the same for $\mathcal{F}$ and $\mathcal{F}^i$ and it follows from [Sh]2-5 (see B1.3) that this norm is surjective. Therefore, we may choose $y \in A$ such that $\mathcal{N}_{A|A_x^\times(k)}(y) = x$. Now our assertion (1) is equivalent to:

$$\hat{\theta}(y\mathcal{F}^i) = \hat{\theta}(y\mathcal{F}). \tag{2}$$

Let V be an irreducible $\tilde{G}$ module for the character $\hat{\theta}$. Let χ be a linear character of A and let V_χ denote the χ-isotypy subspace of V. We compute the left side of (2) by restricting $\hat{\theta}$ to the subgroup $\tilde{A} \subset \tilde{G}$. If an irreducible representation ρ of $\tilde{A}$ occurs in V which has $\dim(\rho) > 1$, then $\mathrm{tr}(\rho(y\mathcal{F}^i) = 0$ for all i such that $(d,i) = 1$ because ρ must be induced. Therefore, $\hat{\theta}(y\mathcal{F}^i)$ is the sum over $\chi \in A^{\wedge,\mathcal{F}}$ of the traces $\mathrm{tr}(\mathcal{F}^i, V_\chi)$ in the isotypy spaces V_χ, i.e. for $(d,i) = 1$,

$$\hat{\theta}(y\mathcal{F}^i) = \sum_{\chi \in A^{\wedge,\mathcal{F}}} \chi(y)\mathrm{tr}(\mathcal{F}^i, V_\chi). \tag{3}$$

On the other hand, since $\bar{\theta}$ is a virtual character,

$$\hat{\theta}(y\mathcal{F}) = \bar{\theta}(x) = \sum_{\bar{\chi} \in A_x^\times(k)^\wedge} m_{\bar{\chi}}\bar{\chi}(x), \tag{4}$$

where the multiplicities $m_{\bar{\chi}}$ are integers. But $\bar{\chi}(x) = \bar{\chi}(\mathcal{N}(y)) = \chi(y)$, since the set of invariant characters χ of A is, via the norm map, in bijective correspondence with the characters $\bar{\chi}$ of $A_x^\times(k)$. From (3) and (4) we obtain

$$\sum_{\chi \in A^{\wedge,\mathcal{F}}} \mathrm{tr}(\mathcal{F}, V_\chi)\chi(y) = \sum_{\chi \in A^{\wedge,\mathcal{F}}} m_{\bar{\chi}}\chi(y)$$

for all $y \in A$. Since the characters of A are linearly independent, $\mathrm{tr}(\mathcal{F}, V_\chi) = m_{\bar{\chi}}$, where χ and $\bar{\chi}$ are related as above. In particular we see that, for every $\chi \in A^{\wedge,\mathcal{F}}$, $\mathrm{tr}(\mathcal{F}, V_\chi)$ is an integer.

Now we consider the representation of $\tilde{A}$ on the space V_χ. We may assume that the unitary operator $\mathcal{F}$ is diagonal and therefore $\mathrm{tr}(\mathcal{F}, V_\chi) = p(\zeta)$, where ζ is a d-th root of unity and $p(U) \in \mathbb{Z}[U]$ is a polynomial with integer coefficients. The map $\zeta \mapsto \zeta^i$ induces an automorphism of the cyclotomic field $\mathbb{Q}(\zeta)$, since $(d,i) = 1$. This automorphism takes $p(\zeta)$ to $p(\zeta^i) = \mathrm{tr}(\mathcal{F}^i, V_\chi)$. But $p(\zeta)$ is an integer. Therefore both numbers are the same, which means that $\mathrm{tr}(\mathcal{F}, V_\chi) = \mathrm{tr}(\mathcal{F}^i, V_\chi)$ for all invariant characters χ. Substituting into (3) we obtain (2). □

Next we want to show that generic (or "regular" (cf [DM1]14.39)) characters have proper descent. In other words, every generic character has a canonical extension (see B1.6). Let $H := U_0Z$, where U_0 is the upper triangular unipotent subgroup of GL_s and Z denotes the center. Then $(H(k_d), \phi)$ is a standard pair as in B2.7 (with $d = e$).

Let φ denote a generic linear character of $U_0(k_d)$. Let $\Gamma_d := \mathrm{Ind}_{U_0(k_d)}^{G(k_d)}(\varphi)$. It is known that Γ_d is independent of the choice of φ, so we may (and do) assume that φ is ϕ invariant. A character $\theta \in G(k_d)^\wedge$ is called *generic* if $\langle\theta, \Gamma_d\rangle_{G(k_d)} \neq 0$, in which case (see, for instance, [SZ1]Cor. 5.6) $\langle\theta, \Gamma_d\rangle_{G(k_d)} = 1$.

Inducing to $G(k_d)$ via $H(k_d)$ we see that $\Gamma_d = \sum_{\chi\in Z(k_d)^\wedge} \Gamma_{d,\chi}$, where $\Gamma_{d,\chi} := \mathrm{Ind}\,_{H(k_d)}^{G(k_d)}(\varphi\otimes\chi)$. If $\theta\in G(k_d)^{\wedge,\phi}$, then the central character of θ is also ϕ invariant and therefore $\langle\theta,\Gamma_{d,\chi}\rangle = 1$, where $\chi\in Z(k_d)^{\wedge,\phi}$. Since $Z(k_d)\cong k_d^\times$, the ϕ invariant characters of $Z(k_d)$ may be identified with characters of $k_d^\times$ such that there exists (uniquely) $\bar\chi\in(k^\times)^\wedge$ with $\chi=\bar\chi\circ \mathrm{N}_{k_d|k}$. Thus we can extend $\rho_\chi=\varphi\otimes\chi$ by 1 to $\tilde H := H(k_d)\rtimes\langle\phi\rangle$, to the canonical character $\tilde\rho_\chi = \tilde\varphi\otimes\tilde\chi\in\tilde H^\wedge$. Now we define $\tilde\Gamma_{d,\chi} := \mathrm{Ind}\,_{\tilde H}^{\tilde G}(\tilde\rho_\chi)$ and we note that

$$\begin{aligned}
\mathrm{Sh}_\phi(\tilde\Gamma_{d,\chi}) : &= \mathrm{Sh}_\phi(\tilde\Gamma_{d,\chi}|_{G(k_d)\phi})\\
&= \mathrm{Sh}_\phi(\mathrm{Ind}\,_{H(k_d)\phi}^{G(k_d)\phi}(\tilde\rho_\chi|_{H(k_d)\phi}))\\
&= \mathrm{Ind}\,_{H(k)}^{G(k)}(\mathrm{Sh}_\phi(\tilde\rho_\chi|_{H(k_d)\phi})) \qquad (\text{by B1.4(ii)})\\
&= \mathrm{Ind}\,_{H(k)}^{G(k)}(\mathrm{Sh}_\phi(\tilde\varphi)\otimes\bar\chi),
\end{aligned}$$

where $\mathrm{Sh}_\phi(\tilde\varphi)$ is a generic character of $U_0(k)$. Therefore, from $\chi=\bar\chi\circ\mathrm{N}_{k_d|k}$, we obtain:

$$\mathrm{Sh}_\phi(\tilde\Gamma_{d,\chi}|_{G(k_d)\phi}) = \Gamma_{1,\bar\chi},$$

which is the summand of the Gelfand-Graev character of $G(k)$ with the central character $\bar\chi$. Now we apply B1.7 with:

$$\begin{aligned}
(G,H,t) &= (G(k_d),H(k_d),\phi)\\
\rho &= \varphi\otimes\chi,\quad \chi=\bar\chi\circ\mathrm{N}_{k_d|k}\\
\Lambda &= \Gamma_{d,\chi},\quad \tilde\Lambda=\tilde\Gamma_{d,\chi},\quad \bar\Lambda=\Gamma_{1,\bar\chi}.
\end{aligned}$$

If $\theta\in G(k_d)^{\wedge,\phi}$, then θ has d different extensions $\hat\theta_i$ to $\tilde G(k_d)$ and $\sum_{i=1}^d\hat\theta_i = \mathrm{Ind}\,_{G(k_d)}^{\tilde G(k_d)}(\theta)$. If $\chi=\bar\chi\circ\mathrm{N}_{k_d|k}$ is the central character of θ, then:

$$\begin{aligned}
\sum_{i=1}^d\langle\hat\theta_i,\tilde\Gamma_{d,\chi}\rangle_{\tilde G(k_d)} &= \langle\mathrm{Ind}\,_{G(k_d)}^{\tilde G(k_d)}(\theta),\tilde\Gamma_{d,\chi}\rangle_{\tilde G(k_d)}\\
&= \langle\theta,\Gamma_{d,\chi}\rangle_{G(k_d)}\le 1,
\end{aligned}$$

since $\langle\theta,\Gamma_d\rangle_{G(k_d)}\le 1$. Applying B1.7 we conclude:

B3.3 Proposition.
(i) Let $\theta\in G(k_d)^{\wedge,\phi}$. If θ is generic, then there is precisely one extension $\tilde\theta$ of θ such that $\langle\tilde\theta,\tilde\Gamma_d\rangle_{\tilde G(k_d)}=1$ and this is the canonical extension of θ.
(ii) For any generator ϕ of Gal$(k_d|k)$ the mapping $\theta\mapsto Sh_{k_d|k}(\tilde\theta) := Sh_\phi(\tilde\theta|_{G(k_d)\phi})$ defines a bijection from the set of generic characters $\theta\in G(k_d)^{\wedge,\phi}$ to the set of generic characters $\bar\theta\in G(k)^\wedge$.

§B4 Some Reductions to the Case $r=1$.

We return to the standard pair $(G(k_d),t) = (\mathrm{GL}_s^r(k_d),T_r^c)$ of §B2. In the case $r=1$ we see that $G(k_d)=\mathrm{GL}_s(k_d)$ and the automorphism $t=\bar\phi^c=\phi^c$ on $\mathrm{GL}_s(k_d)$. The pairs $(G(k_d),t)$ and $(\mathrm{GL}_s(k_d),\bar\phi^c)$ are standard in the sense of §B1, and

$$G(k_d)^t = \mathrm{GL}_s(k_c)^\sharp\cong\mathrm{GL}_s(k_c) = \mathrm{GL}_s(k_d)^{\phi^c}.$$

We begin by proving a dual version of B2.1, which characterizes the set $G(k_d)^{\wedge,t}$ of t invariant characters. If $\Theta \in G(k_d)^\wedge$, then $\Theta(x) = \theta_0(x_0)\cdots\theta_{r-1}(x_{r-1})$, where $x = (x_0, \dots, x_{r-1}) \in G(k_d)$ and $\theta_i \in GL_s(k_d)^\wedge$ for $0 \le i \le r-1$.

Letting T_r act on $G(k_d)^\wedge$, we obtain

$$\begin{aligned}
{}^{T_r}\Theta(x_0, \dots, x_{r-1}) &= \Theta(T_r^{-1}(x_0, \dots, x_{r-1})) \\
&= \Theta({}^{\phi^{-1}}x_{r-1}, x_0, \dots, x_{r-2}) \qquad (\text{by B2(1)}) \\
&= {}^{\phi}\theta_0(x_{r-1})\theta_1(x_0)\cdots\theta_{r-1}(x_{r-2}) \\
&= \theta_1(x_0)\cdots\theta_{r-1}(x_{r-2})\,{}^{\phi}\theta_0(x_{r-1}).
\end{aligned} \tag{1}$$

We have derived the following analog of B2(1):

$$\begin{aligned}
{}^{T_r}\Theta(x) &= \prod_{j=1}^{r-1}\theta_j(x_{j-1})\,{}^{\phi}\theta_0(x_{r-1}), \\
{}^{\phi}\Theta(x) &= \prod_{j=0}^{r-1}{}^{\phi}\theta_j(x_j),
\end{aligned} \tag{2}$$

where the second equation in (2) is obvious. Using (2) we see that the proof of B2.1 proves:

B4.1 Lemma. *A character $\Theta \in G(k_d)^\wedge$ is t invariant if and only if*

$$\Theta(x) = \theta_0^\sharp(x) := \theta_0(x_0)\,{}^{\phi^h}\theta_0(x_1)\cdots{}^{\phi^{h(r-1)}}\theta_0(x_{r-1})$$

for some $\theta_0 \in GL_s(k_d)^\wedge$ such that ${}^{\phi^c}\theta_0 = \theta_0$ (See B2(2) for the number h.).

Concerning the proof of B4.1 we remark only that, just as in the proof of B2.1, one verifies that Θ is invariant if and only if ${}^{\phi^c}\Theta = \Theta$ and ${}^{T_r}\Theta = {}^{\phi^h}\Theta$.

We want to reduce the existence of the Shintani descent for $(G(k_d), t)$ to that for $(GL_s(k_d), \phi^c)$. We have the natural bijections

$$\begin{aligned}
\theta_0 \in \mathrm{GL}_s(k_d)^{\wedge,\phi^c} &\longmapsto \theta_0^\sharp \in G(k_d)^{\wedge,t}, \\
\rho \in GL_s(k_c)^\wedge &\longmapsto \rho^\sharp \in GL_s(k_c)^{\sharp\wedge},
\end{aligned} \tag{3}$$

where the first comes from B4.1 and the second is defined such that $\rho^\sharp(x_0^\sharp) = \rho(x_0)$.

B4.2 Lemma. *Consider the standard pair $(GL_s(k_d), \bar{\phi}^c)$ and let $\hat{\theta}_0$ be an extension of $\theta_0 \in GL_s(k_d)^{\wedge,\phi^c}$. Then $\Theta = \theta_0^\sharp \in G(k_d)^{\wedge,t}$ extends to a character $\hat{\Theta} \in \tilde{G}(k_d)^\wedge$ such that*

$$\hat{\Theta}(zt) = \hat{\theta}_0(y\bar{\phi}^c) \qquad (z = ext_1(y) \in G(k_d), \quad \textit{see B2.4}) \tag{4}$$

Restricting to the coset we consider $\hat{\Theta} \in \mathcal{C}(G(k_d)t)$. Then

$$Sh_t(\hat{\Theta}) = Sh_{\phi^c}(\hat{\theta}_0)^\sharp. \tag{5}$$

If θ_0 has virtual descent, then $\Theta = \theta_0^\sharp$ has virtual descent, and if θ_0 has a canonical extension $\tilde{\theta}_0$, then Θ has a canonical extension $\tilde{\Theta}$ such that $\tilde{\Theta}$ and $\tilde{\theta}_0$ satisfy (5).

Proof. First let us show that (4) implies (5). For $x^\sharp \in \mathrm{GL}_s(k_c)^\sharp$ the left side of (5) is:

$$\mathrm{Sh}_t(\hat{\Theta})(x^\sharp) = \hat{\Theta}(gt),$$

where $g \in G(k_d)$ satisfies

$$[x^\sharp] = [(gt)^e] \cap \mathrm{GL}_s(k_c)^\sharp. \tag{6}$$

On the right side of (5) we obtain:

$$\mathrm{Sh}_{\phi^c}(\hat{\theta}_0)^\sharp(x^\sharp) = \mathrm{Sh}_{\phi^c}(\hat{\theta}_0)(x) = \hat{\theta}_0(y\bar{\phi}^c) \qquad ([x] = [(y\bar{\phi}^c)^{d/c}] \cap \mathrm{GL}_s(k_c)). \tag{7}$$

Thus to prove (5) means that for any $x \in \mathrm{GL}_s(k_c)$ we have to find a pair (g, y) such that (6) and (7) are fulfilled and moreover

$$\hat{\Theta}(gt) = \hat{\theta}_0(y\bar{\phi}^c). \tag{8}$$

But in the last part of the proof of B2.5 we have seen that, given $x \in \mathrm{GL}_s(k_c)$ we can choose $y \in A_x^\times(k_d)$ such that $N_{\phi^c}(y) = x$ and can take $g = \mathrm{ext}_1(y) \in G(k_d)$. Then (6) and (7) hold, and (8) follows from (4).

Let us construct $\hat{\Theta} \in (G(k_d) \rtimes \langle t\rangle)^\wedge$ such that (4) holds. We do this by a variant of tensor induction. We set $H := \mathrm{GL}_s(k_d)$, $H_c := \mathrm{GL}_s(k_d) \rtimes \langle\bar{\phi}^c\rangle$, and $H_1 := \mathrm{GL}_s(k_d) \rtimes \langle\bar{\phi}\rangle$.

Let V_0 be an H_c module affording the character $\tilde{\theta}_0$. Then the induced character $\pi := \mathrm{Ind}_{H_c}^{H_1}\tilde{\theta}_0$ can be represented by the left action of H_1 on $V := \mathbb{C}[H_1] \underset{\mathbb{C}[H_c]}{\otimes} V_0$. If we consider V as an H_c space, we obtain the decomposition $V = V_0 \oplus \cdots \oplus V_{c-1}$ where the subspaces $V_l := \phi^l \otimes V_0$ depend only on l mod c.

We next consider the space $V^{\otimes r}$ and, as usual, we set

$$T_r(v^0 \otimes \cdots \otimes v^{r-1}) = v^1 \otimes \cdots \otimes v^{r-1} \otimes (\phi \otimes v^0),$$

where $v^i \in V$ for $0 \le i < r$. As a subspace we consider

$$V_0^\sharp := V_0 \otimes V_h \otimes \cdots \otimes V_{(r-1)h} \subset V^{\otimes r}, \tag{9}$$

and we are going to see that $V_0^\sharp$ may be taken as a representation space of $\Theta = \theta_0^\sharp$. To make a distinction between the tensor product of different copies of V and the tensor product which occurs in the definition of V we will use in V the notation $\phi^l * V_0 := \phi^l \otimes V_0 = V_l$, and we define an operator $\phi^\sharp : V_0^{\otimes r} \to V_0^\sharp$ by setting

$$\phi^\sharp(v_0 \otimes v_1 \otimes \cdots \otimes v_{r-1}) = v_0 \otimes (\phi^h * v_1) \otimes \cdots \otimes (\phi^{(r-1)h} * v_{r-1}). \tag{10}$$

We also write $v^\sharp := \phi^\sharp(v)$ for $v \in V_0^{\otimes r}$.

For $x = (x_0, \ldots, x_{r-1}) \in \mathrm{GL}_s^r(k_d)$ and $v = v_0 \otimes \cdots \otimes v_{r-1} \in V_0^{\otimes r}$ we have

$$x \cdot v^\sharp = x_0 v_0 \otimes x_1(\phi^h * v_1) \otimes \cdots \otimes x_{r-1}(\phi^{(r-1)h} * v_{r-1}) = (x \cdot v)^\sharp, \tag{11}$$

where $x \cdot v := x_0 v_0 \otimes {}^{\phi^{-h}}x_1 v_1 \otimes \cdots \otimes {}^{\phi^{-h(r-1)}}x_{r-1} v_{r-1}$ is a representation of $\mathrm{GL}_s^r(k_d)$ on the space $V_0^{\otimes r}$ which realizes the character $\Theta = \theta_0^\sharp$. Therefore the subspace

$V_0^\sharp \subset V^{\otimes r}$ can be identified as a representation space of Θ. On $V_0^\sharp$ we have the $\mathrm{GL}_s^r(k_d)$ invariant scalar product

$$(12)\quad \langle v^\sharp, {v'}^\sharp \rangle_{V_0^\sharp} = \langle v, v' \rangle_{V_0^{\otimes r}} = \prod_{j=0}^{r-1} \langle v_j, v'_j \rangle_{V_0} \quad (v = \otimes_{j=0}^{r-1} v_j, v' = \otimes_{j=0}^{r-1} v'_j \in V_0^{\otimes r}).$$

We choose an orthonormal basis $\mathcal{B}_0$ for V_0 which diagonalizes the unitary automorphism ϕ^c. We then define $\mathcal{B}$ to be the basis of $V_0^{\otimes r}$ such that $v \in \mathcal{B}$ if and only if $v = \otimes_{j=0}^{r-1} v_j$ with $v_j \in \mathcal{B}_0$ for all j. We may regard $\mathcal{B}^\sharp$ as an orthonormal basis for $V_0^\sharp$, i.e. $v^\sharp \in \mathcal{B}^\sharp$ if and only if $v \in \mathcal{B}$.

Next we note that T_r maps $V_0^\sharp$ to $\phi^h V_0^\sharp \subset V^{\otimes r}$:

$$(13)\quad \begin{aligned} T_r v^\sharp &= (\phi^h * v_1) \otimes (\phi^{2h} * v_2) \otimes \cdots \otimes (\phi^{(r-1)h} * v_{r-1}) \otimes (\phi * v_0) \\ &= \phi^h [\otimes_{j=1}^{r-1} (\phi^{(j-1)h} * v_j) \otimes (\phi^{(r-1)h} * \phi^{-cu} v_0)], \end{aligned}$$

where ϕ^h acts diagonally and we have used the relation $rh - cu = 1$ in manipulating $\phi * v_0$ on the right.

¿From (13) we see that $t = T_r^c$ stabilizes $V_0^\sharp$ because it maps $V_0^\sharp$ to $\phi^{ch} V_0^\sharp$, and the powers of ϕ^c preserve V_0 and therefore they also preserve $V_0^\sharp$. Thus the action of $\mathrm{GL}_s^r(k_d)$ on $V_0^\sharp$ extends to an action of $\tilde{G}$. This gives an explicit extension of Θ, and computing the trace of the operator zt on $V_0^\sharp$ will give us the equation (4). By the division algorithm, $c = \lfloor \frac{c}{r} \rfloor r + \nu$ $(0 < \nu < r)$ and therefore $\phi^{-c(h - u\lfloor \frac{c}{r} \rfloor)} t v^\sharp =$

$$(14)\quad v_\nu \otimes \cdots \otimes (\phi^{(r-1-\nu)h} * v_{r-1}) \otimes (\phi^{(r-\nu)h} * \phi^{-cu} v_0) \otimes \cdots \otimes (\phi^{(r-1)h} * \phi^{-cu} v_{\nu-1}),$$

where the left multiplier $\phi^{c(h - u\lfloor \frac{c}{r} \rfloor)}$ again acts diagonally. We want to compute the character $\tilde{\Theta}(zt) = \sum_{v \in \mathcal{B}} \langle zt \cdot v^\sharp, v^\sharp \rangle$ for $z = \mathrm{ext}_1(y)$. Using (14) we obtain $\langle zt \cdot v^\sharp, v^\sharp \rangle = \langle [z\phi^{c(h - u\lfloor \frac{c}{r} \rfloor)} (v_\nu \otimes \cdots \otimes v_{r-1} \otimes \phi^{-cu} v_0 \otimes \cdots \otimes \phi^{-cu} v_{\nu-1}]^\sharp, v^\sharp \rangle =$

$$(15)\quad \langle y\phi^{c(h - u\lfloor \frac{c}{r} \rfloor)} v_\nu, v_0 \rangle \prod_{j=\nu+1}^{r-1} \langle \phi^{c(h - u\lfloor \frac{c}{r} \rfloor)} v_j, v_{j-\nu} \rangle \prod_{j=0}^{\nu-1} \langle \phi^{c(h - u\lfloor \frac{c}{r} \rfloor - u)} v_j, v_{r+j-\nu} \rangle.$$

Since we have chosen an eigenbasis of V_0 for ϕ^c and since $(c, r) = 1$ (t permutes the v_j in a single cycle of length r), this product can differ from zero only if $v_j = v_0$ for all j. In this case $v^\sharp = \phi^\sharp(v_0^{\otimes r})$ is an eigenvector for t, as we see from (14), because v_0 is an eigenvector for ϕ^c. More precisely $t \cdot v^\sharp = \lambda v^\sharp$ where λ is the power of the eigenvalue $\lambda_0 = \langle \phi^c v_0, v_0 \rangle_{V_0}$ given by $r(h - u\lfloor \frac{c}{r} \rfloor) - \nu u = rh - cu = 1$. Thus for every $v_0 \in \mathcal{B}_0$ and $v = v_0^{\otimes r}$ we have $tv^\sharp = \lambda_0 v^\sharp = (\lambda_0 v)^\sharp = (\phi^c v_0 \otimes v_0 \otimes \cdots \otimes v_0)^\sharp$ and

$$zt \cdot v^\sharp = z \cdot (\phi^c v_0 \otimes \cdots \otimes v_0)^\sharp = [z(\phi^c v_0 \otimes \cdots \otimes v_0)]^\sharp = (y\phi^c v_0 \otimes \cdots \otimes v_0)^\sharp,$$

since $z = \mathrm{ext}_1(y)$. From (12) we obtain now that $\langle ztv^\sharp, v^\sharp \rangle_{V_0^\sharp} = \langle y\phi^c v_0, v_0 \rangle_{V_0}$, since $\mathcal{B}_0$ is an orthonormal basis of V_0. Summing over all $v = v_0^{\otimes r}$, where $v_0 \in \mathcal{B}_0$, we prove (4) and complete the proof of B4.2. $\square$

Remark. For the case $c = 1, h = 0$ we have $V_0 = V$ and $V_0^\sharp = V^{\otimes r}$. If, in addition, $d = 1$, then $\bar{\phi}$ acts trivially and $T_r = t$ acts by cyclic permutation. In this case $\tilde{G} = H^r \rtimes \langle t \rangle$ is a wreath product and $V_0^\sharp = V^{\otimes r}$ is the usual tensor induction.

In the following we consider the k Frobenius $\mathcal{F}$ and an arbitrary generator $\phi := \mathcal{F}^i$ of $\mathrm{Gal}(k_d|k)$, where $(d, i) = 1$. On $G(k_d)$ this gives the two automorphisms $t(\mathcal{F}) = T_r(\mathcal{F})^c$ and $t(\phi) = T_r(\phi)^c$, where each is of order $e = r \cdot \frac{d}{c}$ and each induces an operator as in B2(1). ¿From B2.1 we obtain two distinct isomorphic groups:

$$G(k_d)^{t(\mathcal{F})} = \mathrm{GL}_s(k_c)^{\sharp(\mathcal{F})} \cong \mathrm{GL}_s(k_c)$$
$$G(k_d)^{t(\phi)} = \mathrm{GL}_s(k_c)^{\sharp(\phi)} \cong \mathrm{GL}_s(k_c).$$

B4.3 Proposition. *Let $\theta_0 \in GL_s(k_d)^{\wedge,\phi^c}$ and let $\hat{\theta}_0 \in (GL_s(k_d) \rtimes \langle \bar{\phi}^c \rangle)^\wedge$ be an extension of θ_0 such that the descent $\bar{\theta}_0 = Sh_{k_d|k_c}(\hat{\theta}_0)$ is a virtual character (and therefore independent of the generator of Gal$(k_d|k_c)$). Then*

$$Sh_{t(\phi)}(\hat{\Theta}^{(\phi)}) = (\bar{\theta}_0)^{\sharp(\phi)},$$

where $\hat{\Theta}^{(\phi)}$ is the extension of the character $\theta_0^{\sharp(\phi)}$ such that $\hat{\Theta}^{(\phi)}(zt(\phi)) = \hat{\theta}_0(y\bar{\phi}^c)$ for $z = ext_1(y)$ (see B2.4).

Remark. In particular, if θ_0 has proper descent and if $\hat{\theta}_0 = \tilde{\theta}_0$, the canonical extension of θ_0, then the extension $\hat{\Theta}^{(\phi)}$ is the canonical extension of $\theta_0^{\sharp(\phi)}$. As we have already noted, $\hat{\theta}_0(y\bar{\phi}^c)$ does not depend upon the generator ϕ.

Proof. From (5) we see that $\mathrm{Sh}_{t(\phi)}(\hat{\Theta}^{(\phi)}) = [\mathrm{Sh}_{\phi^c}(\hat{\theta}_0)]^{\sharp(\phi)}$, and from B3.2 we know that $\bar{\theta}_0 = \mathrm{Sh}_{\phi^c}(\hat{\theta}_0)$ does not depend on ϕ^c. $\square$

Finally we want to use Lemma B4.2 in order to determine the Shintani descent $\mathrm{Sh}_t(\bar{\pi})$ from $\mathrm{GL}_s^r(k_d)$ to $\mathrm{GL}_s^r(k_d)^t = \mathrm{GL}_s(k_c)^\sharp$, where $\bar{\pi} = \Pi_{d/c}(\bar{\chi})^\sharp$ is the representation constructed in 4.3(ii). In 5.7 we saw that AMT implies the explicit descent of $\bar{\pi}$ in the special case $c = r = 1$.

B4.4 Corollary. *Let $f := cs$, let $f' := f/(d, f) = [k_d k_f : k_d]$, let $\bar{\chi}_f \in X(k_f^\times)$ be k-regular, let $\sigma_{\bar{\chi}_f} \in GL_{f'}(k_d)^\wedge_{\mathrm{cusp}}$ have the Green's parameter Gal$(k_d k_f|k_d)\bar{\chi}_f \circ N_{k_d k_f|k_f}$, and let $\Pi_{d/c}(\bar{\chi}_f)$ and $\Pi_{d/c}(\bar{\chi}_f)^\sharp$ be constructed from $\sigma = \sigma_{\bar{\chi}_f}$ as in 4.3(ii). Then:*

$$Sh_t(\Pi_{d/c}(\bar{\chi}_f)^\sharp) = Sh_{\phi^c}(\Pi_{d/c}(\bar{\chi}_f))^\sharp = \Pi_1(\bar{\chi}_f)^\sharp, \tag{16}$$

where $\Pi_1(\bar{\chi}_f) \in GL_s(k_c)^\wedge_{\mathrm{cusp}}$ has the Green's parameter Gal$(k_f|k_c)\bar{\chi}_f$.

Proof. We note that $\Pi_1(\bar{\chi}_f)$ is well defined because $cs = f$. Moreover we have the same representations as in 4.3(ii) because the character $\bar{\chi}$ which occurs there admits the factorization $\bar{\chi} = \bar{\chi}_f \circ \mathrm{N}_{k_n|k_f}$, where $\bar{\chi}_f$ is uniquely determined and k-regular. The first equation of (16) follows from B4.2 whereas the second is a consequence of §5.7 with the base field k replaced by k_c, i.e. we have to take $s = f/c$ and $\delta = (d/c, s) = (d, f)/c$. $\square$

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