STRUCTURAL STABILITY OF VECTOR OPTIMIZATION PROBLEMS

by

Paulo Mbunga
Institut für Mathematik
Humboldt-Universität zu Berlin

Abstract. — We study global stability properties for vector optimization problems of the type:

$\mathcal{VOP}(f,H,G)$: $$\min\{f(x)=(f_1(x),\dots,f_l(x)) \mid x\in M[H,G]\},$$

where

$$M[H,G]:=\{x\in\mathbb{R}^n \mid h_i(x)=0, \quad g_j(x)\le 0, \quad i\in I, j\in J\}$$

with

$$I:=\{1,\dots,m\}, \quad J:=\{1,\dots,s\}, \quad L:=\{1,\dots,l\}.$$

We extend Guddat/Jongen's **[5]** concept of structural stability of scalar nonlinear optimization problems to vector optimization problems. Under the assumption that $M[H,G]$ is compact we prove the necessary condition for the structural stability of a vector optimization problem, i.e. the scalar problem

$\mathcal{P}^{\max}(f,H,G)$: $$\min\left\{\max_{l\in L} f_l(x) \mid x\in M[H,G]\right\}$$

has to be structurally stable.

1. Introduction

We consider the vector optimization problem

$\mathcal{VOP}(f,H,G)$: $$\min\{f(x)=(f_1(x),\dots,f_l(x)) \mid x\in M[H,G]\},$$

where

$$M[H,G]:=\{x\in\mathbb{R}^n \mid h_i(x)=0, \quad g_j(x)\le 0, \quad i\in I, j\in J\}$$

Key words and phrases. — vector optimization, structural stability, strong stability, constraint qualification (Mangasarian-Fromovitz).

This work was supported by the Graduiertenkolleg Geometrie und Nichlineare Analysis of the Humboldt-Universität.

with

$$I := \{1, \ldots, m\}, \quad J := \{1, \ldots, s\}, \quad L := \{1, \ldots, l\}.$$

Unless otherwise statet, we will assume that the functions f_k, g_j, $k \in L, j \in J$ belong to $C^2(\mathbb{R}^n, \mathbb{R})$. The lower level set corresponding to $u \in \mathbb{R}^{|L|}$ will be denoted as follows:

$$\mathcal{L}^u(f, H, G) := \{x \in M[H, G] \mid f(x) \underset{D}{\leq} u\}, \quad u \in \mathbb{R}^{|L|}. \tag{1}$$

The partial ordering $\underset{D}{\leq}$ is induced by a convex cone D with

$$x \underset{D}{\leq} y \quad :\Longleftrightarrow \quad y - x \in D. \tag{2}$$

Definition 1. — *The vector optimization problems $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\tilde{f}, \tilde{H}, \tilde{G})$ are called equivalent if there exist continuous mappings $\phi : \mathbb{R}^{|L|} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\psi : \mathbb{R}^{|L|} \longrightarrow \mathbb{R}^{|L|}$ with the properties $P1 - P3$:*

P1 For every $u \in \mathbb{R}^{|L|}$ the mapping $\phi_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a homeomorphism from $\mathbb{R}^n$ onto itself, where $\phi_u := \phi(u, x)$.

P2 The mapping ψ is a homeomorphism from $\mathbb{R}^{|L|}$ onto itself and ψ is D-monotonically increasing; i.e.

$$q_1 \underset{D}{\leq} q_2 \text{ implies } \psi(q_1) \underset{D}{\leq} \psi(q_2) \text{ for } q_1, q_2 \in \mathbb{R}^{|L|}. \tag{3}$$

P3 $\phi_u[\mathcal{L}^u(f, H, G)] = \mathcal{L}^{\psi(u)}(\tilde{f}, \tilde{H}, \tilde{G})$ for all $u \in \mathbb{R}^{|L|}$.

Definition 2. — *The vector optimization problems $\mathcal{VOP}(f, H, G)$ is called* structurally stable if there exists a C^2_S-neighborhood $\mathcal{O}$ of (f, H, G) with the property that $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\tilde{f}, \tilde{H}, \tilde{G})$ are equivalent for all $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$.

The C^2_S-topology above for the product $\prod_{i=1}^r C^2(\mathbb{R}^n, \mathbb{R})$ will be the product topology generated by the strong (or Whitney-) C^2-topology C^2_S on each factor $C^2(\mathbb{R}^n, \mathbb{R})$ (cf. **[6]**,**[9]**). A typical base-neighbourhood of a function $\rho \in C^2(\mathbb{R}^n, \mathbb{R})$ is the set $\rho + \mathcal{W}_\epsilon$, where $\mathcal{W}_\epsilon$ is defined as follows with the aid of a continuous positive $\epsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\begin{aligned} \mathcal{W}_\epsilon = \{\eta \in C^2(\mathbb{R}^n, \mathbb{R}) \mid |\eta(x)| + \sum_i |\frac{\partial \eta}{\partial x_i}(x)| + \\ + \sum_{i,j} |\frac{\partial^2 \eta}{\partial x_i \partial x_j}(x)| < \epsilon(x), \forall x \in \mathbb{R}^n\}. \end{aligned} \tag{4}$$

We have the chance of a more local control for data corresponding to the scalar problem $\mathcal{P}(r, H, G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$ by putting for any subset U of $\mathbb{R}^n$:

$$\begin{aligned} &norm[(r, H, G), U] = \\ &\sup_{x \in U} \max_{\eta \in \{r, h_i, i \in I, g_j, j \in J\}} \left\{ |\eta(x)| + \sum_i |\frac{\partial \eta}{\partial x_i}(x)| + \sum_{i,j} |\frac{\partial^2 \eta}{\partial x_i \partial x_j}(x)| \right\}. \end{aligned} \tag{5}$$

Remark 1. — *If $|L| = 1$ and $D = \mathbb{R}_+^{|L|}$, then we have the structural stability for the scalar nonlinear problem introduced in* [**5**].

We consider the following scalar optimization problem

$$\mathcal{P}^{\max}(f, H, G): \qquad \min \left\{ \max_{l \in L} f_l(x) \mid x \in M[H, G] \right\}.$$

For the corresponding lower level set we shall use the notation

$$\mathcal{L}^t_{\max}(f, H, G) := \left\{ x \in M[H, G] \mid \max_{l \in L} f_l(x) \leq t \right\}, \quad t \in R.$$

We use the notations given in Weber [**14**] and denote the canonical projection of the space $\mathbb{R}^{n+1}$ on the first n components or on the last component by $pr_{(1,\ldots,n)}$ and pr_{n+1}, respectively; i.e.

$$\check{x} = \big(pr_{(1,\ldots,n)}(\check{x}), pr_{n+1}(\check{x})\big) = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}.$$

Let us now look at the following equivalent differentiable optimization problem which in particular reveals an objective function of class C^2.

$$\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G}): \qquad \min\{\check{f}(\check{x}) := pr_{n+1}(\check{x}) \mid x \in M[\check{H}, \check{G}]\},$$

where

$$M[\check{H}, \check{G}] := \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x \in M[H, G], f_l(x) - x_{n+1} \leq 0 \quad (l \in L) \right\},$$

namely with $\check{h}_i(\check{x}) := h_i(x)(i \in I)$ and

$$\check{g}_j(\check{x}) := \begin{cases} g_j(x), & \text{for } j \in J \\ f_l(x) - x_{n+1}, & \text{for } j = s + l, \quad l \in L. \end{cases}$$

$\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$ is strongly related to scalarization methods for vector optimization problems. As pointed out in Weidner [**16**], this scalarization method includes the known procedures of Brosowski and Conci [**3**], Helbig [**8**], Wierzbicki [**17**] and other papers cited in [**7**]. Roughly speaking, each minimal point of $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$ is weakly efficient and vice versa each weakly efficient and hence each efficient point solves the scalar problem $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$.
We note that the lower level sets $\mathcal{L}^t_{\max}(f, H, G)$ of $\mathcal{P}^{\max}(f, H, G)$ are just the projection in the sense of $pr_{(1,\ldots,n)}$ of the corresponding level sets $\mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$ due to $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$. Both sets are homotopy-equivalent (cf. Jongen et al. [**9**], Lemma 4.2.1) and $M[\check{H}, \check{G}]$ is just the part of the epigraph $E(\max_{l \in L} f_l)$ over the feasible set $M[H, G]$.

Definition 3. — *The Linear Independence Constraint Qualification (shortly LICQ) is said to hold at $x \in M[H, G]$ if the vectors $Dh_i(x), i \in I, Dg_j(x), j \in J_o(x)$, are linearly independent. The Mangasarian-Fromovitz Constarint Qualification (shortly MFCQ) is said to hold at $x \in M[H, G]$ if the following conditions MF1 and MF2 are satisfied:*

$MF1$ *The vectors $Dh_i(x), i \in I$ are linearly independent.*

$MF2$ There exists a vector $\xi \in \mathbb{R}^n$ satisfying:

$$Dh_i(x)\xi = 0, \quad i \in I \tag{6}$$

$$Dg_j(x))\xi \leq 0, \quad j \in J_o(x). \tag{7}$$

A vector ξ satisfying (6) and (7) will be called an MF-vector.

For $x \in \mathbb{R}^n$ and $\rho > 0$ let $B(x,\rho)$ denote the Euclidean ball in $\mathbb{R}^n$, centered at x with radius ρ. Following M. Kojima ([**11**]) we define:

Definition 4. — *Let $x^u \in M[H,G]$ be a Kuhn-Tucker point for $\mathcal{P}(r,H,G)$, $r \in C^2(\mathbb{R}^n,\mathbb{R})$. Then x^u is called* strongly stable if for some $\bar{\delta} > 0$ and each $\delta \in (0,\bar{\delta}]$ there exists an $\alpha > 0$ such that whenever $(\tilde{r},\tilde{H},\tilde{G})$ satisfies $norm[(r-\tilde{r}, H-\tilde{H}, G-\tilde{G}), B(x^u,\delta)] \leq \alpha$, the ball $B(x^u,\delta)$ contains a Kuhn-Tucker point for $\mathcal{P}(\tilde{r},\tilde{H},\tilde{G})$ which is unique in $B(x^u,\delta)$.

***Lemma 1* (Kojima [11]).** — *Let $\bar{x}^u \in M[H,G]$ be a Kuhn-Tucker point for $\mathcal{P}(r,H,G)$, $r \in C^2(\mathbb{R}^n,\mathbb{R})$.*

1. *If LICQ is satisfied at $\bar{x}^u$, then $\bar{x}^u$ is a strongly stable Kuhn-Tucker point if and only if the matrix $D^2L_{[\lambda,\mu]}(\bar{x}^u)$ has nonvanishing determinants with a common sign on the subspace $W(\bar{x}^u,\bar{J})$ for all $\bar{J}$ with $J_+(\bar{x}^u) \subset \bar{J} \subset J_o(\bar{x}^u)$, where*
$$\begin{aligned} J_+(\bar{x}^u) &= \{j \in J_o(\bar{x}^u) | \mu_j > 0\} \\ W(\bar{x}^u,\bar{J}) &= \{\xi \in \mathbb{R}^n | Dh_i(\bar{x}^u)\xi = 0, i \in I, Dg_j(\bar{x}^u)\xi = 0, j \in \bar{J}\} \\ L_{[\lambda,u]}(x) &:= r(x) + \sum_{i\in I} \lambda_i h_i(x) + \sum_{j \in J_o(\bar{x}^u)} u_j g_j(x). \end{aligned}$$
2. *Let MFCQ be satisfied at $\bar{x}^u$, but LICQ not. Then, $\bar{x}^u$ is a strongly stable Kuhn-Tucker point if and only if for every (λ,μ) the matrix $D^2L_{[\lambda,\mu]}(\bar{x}^u)$ is positive definite on the subspace $W(\bar{x}^u, J_+(\bar{x}^u))$.*

Firstly we note that the structural stability of the problem $\mathcal{P}^{\max}(f,H,G)$ was characterized in Weber [**14**].

***Theorem 1* (Main result).** — *Let $D = \mathbb{R}_+^{|L|}$ and the vector optimization problem $\mathcal{VOP}(f,H,G)$ with compact feasible set $M[H,G]$ be* structurally stable. Then the problem $\mathcal{P}^{\max}(f,H,G)$ is *structurally stable.*

2. Lemmas and preliminary results

Lemma 2. — *Let $t \in \mathbb{R}$, $D = \mathbb{R}_+^{|L|}$ and $u := (t,\ldots,t) = te$, $e = (1,\ldots,1) \in \mathbb{R}^{|L|}$. Then the following relations between lower level sets are fulfilled:*

$$\mathcal{L}^t_{\max}(f,H,G) = pr_{(1,\ldots,n)}\mathcal{L}^{\vee t}(\check{f},\check{H},\check{G}) = \mathcal{L}^u(f,H,G). \tag{8}$$

Proof. — The first equality is obvious. We show the second equality.
$\mathcal{L}^u(f,H,G) \subseteq pr_{(1,\ldots,n)}\mathcal{L}^{\vee t}(\check{f},\check{H},\check{G})$: Let $x \in \mathcal{L}^u(f,H,G)$. Then $x \in M[H,G]$, $f_l(x) \leq t$, $l \in \{1,\ldots,L\}$. This implies that $(x,t) \in M(\check{H},\check{G})$ and so $x \in pr_{(1,\ldots,n)}\mathcal{L}^{\vee t}(\check{f},\check{H},\check{G})$.

$\mathcal{L}^u(f,H,G) \supseteqq pr_{(1,\dots,n)}\mathcal{L}^{\vee t}(\check{f},\check{H},\check{G})$: Let $x \in pr_{(1,\dots,n)}\mathcal{L}^{\vee t}(\check{f},\check{H},\check{G})$. Then, there exists a $z \in \mathbb{R}$ with $(x,z) \in \mathcal{L}^{\vee t}(\check{f},\check{H},\check{G})$. This implies that $x \in M[H,G]$, $f_l(x) \leq z$, $l \in L$ and $z \leq t$ and so $f_l(x) \leq t$, $l \in L$. That means $x \in \mathcal{L}^u(f,H,G)$. □

Lemma 3. — *Let D be a pointed, closed and convex cone. Then there exists a $\bar{u} \in \mathbb{R}^{|L|}$ with:*

$$u_1 \underset{D}{\leq} \bar{u}, \quad u_2 \underset{D}{\leq} \bar{u} \tag{9}$$

for all $u_1, u_2 \in \mathbb{R}^{|L|}$.

Proof. — We follow the idea by Tanino in **[12]** (Proof of the Theorem 3.3.1). Pick

$$d \in intD$$

and define

$$y(\alpha) = \alpha d \qquad \alpha \in \mathbb{R}.$$

We claim that, for any $y \in \mathbb{R}^{|L|}$, there exists $\alpha > 0$ such that

$$y \in y(\alpha) - D.$$

If it is not true, then applying the standard separation theorem to the ray $\{y - \alpha d : \alpha > 0\}$ and the convex cone $-D$, we deduce the existence of $\nu \in \mathbb{R}^p \setminus \{0\}$ such that

$$\langle \nu, y - \alpha d\rangle \geq 0 \qquad \text{for all } \alpha > 0$$

and

$$\langle \nu, d'\rangle \geq 0 \qquad \text{for all } d' \in D.$$

Since $d \in intD$, $\langle \nu, d\rangle > 0$ from the latter inequality and so

$$\langle \nu, y - \alpha d\rangle < 0$$

for sufficiently large α, which is a contradiction.
We can therefore choose $\alpha_1, \alpha_2 > 0$ with

$$u_1 \in \alpha_1 d - D \qquad \text{and } u_2 \in \alpha_2 d - D.$$

Take $\alpha_{\max} := \max\{\alpha_1, \alpha_2\}$. Since $\alpha_i d \underset{D}{\leq} \alpha_{\max} d, i = 1, 2$, we can observe that the inequalities (9) are satisfied by $\bar{u} := \alpha_{\max} d$. □

3. Proof of the Necessity Part

Let us now assume that the problem $\mathcal{VOP}(f,H,G)$ is *structurally stable* according to the Definition 2. Then from the results given by Weber in **[10]** (see also **[14]** and **[5]**) we have to verify the following conditions:

($C1$) The Mangasarian-Fromovitz constraint qualification is satisfied at every point of $M[H,G]$.
($C2$) Every Kuhn-Tucker point of $\mathcal{P}^{\max}(f,H,G)$ is strongly stable.
($C3$) Different Kuhn-Tucker points have different ($f-$) values.

Proof of (C1): We show that the structural stability of $\mathcal{VOP}(f,H,G)$ implies (C1). Consider a C^2_s-neighborhood $\mathcal{O}$ of (f,H,G) with the property: for every $(\tilde{f},\tilde{H},\tilde{G}) \in \mathcal{O}$ the set $M[\tilde{H},\tilde{G}]$ is compact and moreover $\mathcal{VOP}(f,H,G)$ is equivalent with $\mathcal{VOP}(\tilde{f},\tilde{H},\tilde{G})$. Choose an arbitrary element $(\tilde{f},\tilde{H},\tilde{G}) \in \mathcal{O}$. Let ϕ and ψ be mappings which establish the equivalence according Definition 1. Take $\alpha, \tilde{\alpha} \in \mathbb{R}^{|L|}$ with

$$\alpha := \left(\alpha_1, \ldots, \alpha_{|L|}\right), \quad \alpha_i := \max_{x \in M[H,G]} f_i(x) \tag{10}$$

$$\tilde{\alpha} := \left(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{|L|}\right), \quad \tilde{\alpha}_i := \max_{x \in M[\tilde{H},\tilde{G}]} \tilde{f}_i(x) \tag{11}$$

and consider the vectors α and $\psi^{-1}(\tilde{\alpha})$. Lemma 9 implies the existence of a $\bar{u} \in \mathbb{R}^{|L|}$ such that $\alpha \underset{D}{\leq} \bar{u}$ and $\psi^{-1}(\tilde{\alpha}) \underset{D}{\leq} \bar{u}$. We note that $\mathcal{L}^{\bar{u}}(f,H,G) = M[H,G]$. Since ψ is monotonically increasing, we have $\tilde{\alpha} \underset{D}{\leq} \psi(\bar{u})$. Thus $\mathcal{L}^{\psi(\bar{u})}(\tilde{f},\tilde{H},\tilde{G}) = M[\tilde{H},\tilde{G}]$. From $\phi_{\bar{u}}(\mathcal{L}^{\bar{u}}(f,H,G)) = \mathcal{L}^{\psi(\bar{u})}(\tilde{f},\tilde{H},\tilde{G})$, we have $\phi_{\bar{u}}(M[H,G]) = M[\tilde{H},\tilde{G}]$. Hence, $\phi_{\bar{u}}$ establishes a homeomorphism between $M[H,G]$ and $M[\tilde{H},\tilde{G}]$. So, we have shown: for every $(\tilde{H},\tilde{G})$ in some C^2_S-neighborhood of (H,G) the sets $M[H,G]$ and $M[\tilde{H},\tilde{G}]$ are homeomorphic. Then, with the help of the Stability Theorem from [**5**] we may equivalently say that MFCQ is satisfied at every point $x \in M[H,G]$. Hence, the condition (C1) is valid.
Proof of (C2): Let us assume that there is a stationary point which is not strongly stable, i.e. condition (C2) is not valid. In order to arrive at a contradiction, the perturbation result from Weber [**14**] plays a central part.

***Lemma 4* (Perturbation Lemma, Weber [14]).** — *Let $M[H,G]$ be compact and suppose that (C1) holds. Suppose further, that $\mathcal{P}^{\max}(V,H,G)$ has a Kuhn-Tucker point $\bar{x}^u$ which is not strongly stable. Then, there exist a number $k \in \mathbb{N}$ and functions $(V^1,\tilde{H},\tilde{G})$, $(V^2,\tilde{H},\tilde{G})$, both arbitrarily C^2_S-near (V,H,G), with the following properties:*

(*i*) *Problem $\mathcal{P}^{\max}(V^1,\tilde{H},\tilde{G})$ has k Kuhn-Tucker points, exactly one of which is not strongly stable.*
(*ii*) *Problem $\mathcal{P}^{\max}(V^2,\tilde{H},\tilde{G})$ has at least $k+1$ Kuhn-Tucker points, all of them being strongly stable.*
(*iii*) *Conditions $C1$ and $C3$ hold for problems $\mathcal{P}^{\max}(V^i,\tilde{H},\tilde{G}), i = 1,2$.*

For $r \in C^2(\mathbb{R}^n,\mathbb{R})$ we denote the set $\{x \in M[H,G] \mid a \leq r(x) \leq b \in\}$ by $\mathcal{L}^a_b(r,H,G)$ for $a \leq b$, $a,b \in \mathbb{R}$.
The next Lemma is proved by Guddat and Jongen in [**5**].

***Lemma 5*.** — *Let the feasible set $M[H,G]$ corresponding to the problem $\mathcal{P}(r,H,G)$ be compact and suppose that Condition $C1$ is fulfilled. Let $a,b \in \mathbb{R}$ be given with $a < b$.*

(*i*) *If $\mathcal{L}^a_b(r,H,G)$ contains no Kuhn-Tucker point, then $\mathcal{L}^a(r,H,G)$ and $\mathcal{L}^b(r,H,G)$ are homeomorphic.*

(ii) *Suppose that* $\mathcal{L}_b^a(r, H, G)$ *contains exactly one Kuhn Tucker point* $\bar{x}^u$ *with* $a < r(\bar{x}^u) < b$, *and suppose that* $\bar{x}^u$ *is strongly stable. Then the sets* $\mathcal{L}^a(r, H, G)$ *and* $\mathcal{L}^b(r, H, G)$ *are not homeomorphic.*

Lemma 6. — *Let* $M[H, G]$ *be compact and suppose that Condition* $C1$ *is fulfilled. Let* $V^1, V^2 \in C^2(\mathbb{R}^n, \mathbb{R})^{|L|}$ *be functions satisfying Condition* $C3$ *w.r.t.* $\mathcal{P}^{\max}(V^i, H, G), i = 1, 2$, *with the following additional properties:*

(i) *Problem* $\mathcal{P}^{\max}(V^1, H, G)$ *has* k *Kuhn-Tucker points, exactly one of which is not strongly stable.*
(ii) *Problem* $\mathcal{P}^{\max}(V^2, H, G)$ *has at least* $k+1$ *Kuhn-Tucker points, all of them being strongly stable.*

Then, the vector optimization problems $\mathcal{VOP}(V^i, H, G), i = 1, 2$ *are not equivalent.*

Proof. — We refer to the n-dimensional problems $\mathcal{P}^{\max}(V^i, H, G), i = 1, 2$ due to the corresponding $(n+1)$-dimensional problems $\mathcal{P}^{\vee}(\check{f}^i, \check{H}, \check{G}), i = 1, 2$. For the two latter problems, we study the topological behaviour of the lower level sets, when the level parameter t increasingly traverses the line of real numbers. Taking account of Condition $C3$ formulated for both problems and the Lemma 5, it follows that the homeomorphy-type of the lower level set $\mathcal{L}^{\vee t}(\check{f}^2, \check{H}, \check{G})$ changes at least $k+1$, and at least $k-1$, but (only) at most k such changes for $\mathcal{L}^{\vee t}(\check{f}^1, \check{H}, \check{G})$. The pairwise homotopy equivalence between the lower level sets in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ gives rise to the analogous description in dimension n of the topological behavior of $\mathcal{L}^t_{\max}(V^i, H, G), i = 1, 2$. From Lemma 2 (8) it follows that $\mathcal{L}^{te}(V^i, H, G), i = 1, 2$ have two different numbers for the changes of the topological type in $\mathbb{R}^n$, as we move along the line $u(t) = te, e \in \mathbb{R}^{|L|}, t \in \mathbb{R}$. If the problems $\mathcal{VOP}(V^i, H, G), i = 1, 2$ would be equivalent, then these two numbers have to coincide. □

We now apply the perturbational result for our indirect proof of Condition $C2$. If this condition does not hold, we can apply Lemma 4 and Lemma 6 in order to obtain two non-equivalent problems $\mathcal{VOP}(f^i, \tilde{H}, \tilde{G}), i = 1, 2$ with each $(f^i, \tilde{H}, \tilde{G})$ arbitrarily C^2_S-near to (f, H, G). This is in contradiction with structural stability.

Proof of (C3): We know the validity of $C1$ and $C2$. If $\mathcal{P}^{\max}(f, H, G)$ does not satisfy Condition $C3$, whenever the equation $\max_{l \in L} f(\bar{x}^u_1) = \max_{l \in L} f(\bar{x}^u_2)$ holds for two different Kuhn-Tucker points of $\mathcal{P}^{\max}(f, H, G)$ then an addition of one small positive and one small negative constant to all functions $f_k \quad (k \in L)$ locally at $\bar{x}^u_1$ and $\bar{x}^u_2$, respectively, leads to a problem $\mathcal{P}^{\max}(\bar{f}, H, G)$ satisfying Condition C3. Let k the number of Kuhn-Tucker points for $\mathcal{P}^{\max}(\bar{f}, H, G)$. Then, reasoning as in the proof of Lemma 6, the homeomorphy-type of
$\mathcal{L}^t_{\max}(\bar{f}, H, G)$ for increasing t, changes exactly k-times. The corresponding number of changes w.r.t. $\mathcal{P}^{\max}(f, H, G)$ is less than k. So, as in Lemma 6, $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\bar{f}, H, G)$ are not equivalent. This leads again to a contradiction with the condition on structural stability of $\mathcal{VOP}(f, H, G)$.

References

[1] Bank, B.; Guddat, J.; Klatte, D.; Kummer, B. and Tammer, K.: Non-Linear Parametric Optimization. Akademie-Verlag. Berlin, 1982.

[2] Guddat, J., Jongen, H. Th., Rueckmann, J.: On stability and stationary points in nonlinear optimization, J. Aust. Math. Soc., Ser. B 28 (1986), 36 – 56.

[3] Brosowski, B. and Conci, A.: On vector optimization and Parametric Programming. Segundas Jornadas Latino Americanas de Matematica Aplicada, 2(1983), 483 – 495.

[4] Guddat, J.; Guerra Vasquez, F. and Jongen, H. Th.: Parametric Optimization, Singularities, Pathfollowing and Jumps. John Wiley, Chichester 1990.

[5] Guddat, Juergen and Jongen, H. Th.: Structural Stability in Nonlinear Optimization. In: Optimization 18 (1987)5, 617 – 631.

[6] Hirsch, M. W.: Differential Topology, Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[7] Helbig, S.: On Quatitative and Qualitative Stability of a Scalarization Method, I. In Parametric Optimization and Related Topics III, 255 – 270, Peter Lang Verlag, Franfurt am Main 1993.

[8] Helbig, S.: Approximation of the Efficient Point Set by Perturbation of the Ordering Cone, Zeitschrift für Operations Research 35(1991), 197 – 220.

[9] Jongen, H. Th.; Jonker, P., Twilt, F.: Nonlinear Optimization in $\mathbb{R}^n$, I. Morse Theory, Chebishev approximation, Peter Lang Verlag, Frankfurt, 1983.

[10] Jongen, H. Th and Weber, G.-W. : Nonlinear Optimization: Characterization of Structural Stability. In: J. Global Optimization 1 (1991), 47 – 64.

[11] Kojima, M.: Strongly stable stationary solutions in nonlinear programs. In: S.M. Robinson (ed.): Analysis and computation of fixed points. Academic Press, New York, 1980.

[12] Sawaragi Y., Nakayama H., Tanino T.: Theory of Multiobjective Optimization. Academic Press, Inc., Orlando 1985.

[13] The Luc, D.: Theory of Vector Optimization. Springer-Verlag, Berlin, 1989.

[14] Weber, G.: Minimization of a max-type function: Characterization of structural stability. In Parametric Optimization and Related Topics III, 519 – 538, Peter Lang Verlag, Franfurt am Main 1993.

[15] Weber, G.: Charakterisierung struktureller Stabilität in der nichtlinearen Optimierung. Aachener Beiträge zur Mathematik, Band 5, Aachen 1992.

[16] Weidner, P.: An approach to different scalarizations in vector optimization. Wissenschaftliche Zeitschrift der TH Ilmenau 3(1990), 103 – 110.

[17] Wierzbicki, A. P.: On the Completeness and Constructiveness of Parametric Characterization to Vector Optimization Problem, OR Spektrum 8(1986), 73 – 87.

[18] Wierzbicki, A. P.: Basic Properties of scalarizing functional for multiobjective optimization. Math. Oper.Stat. Optim. **8**, 55-60, 1977.

PAULO MBUNGA, INSTITUT FÜR MATHEMATIK, HUMBOLDT-UNIVERSITÄT ZU BERLIN, Campus Adlershof, Rudower Chaussee 25, Haus II, 4. Etage D-12489 Berlin
E-mail : `mbunga@mathematik.hu-berlin.de`