THE Θ-MARUYAMA SCHEME FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED MEMORY TERM

Evelyn Buckwar *

Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6 10099 Berlin, Germany. email: buckwar@mathematik.hu-berlin.de

Abstract

We consider the problem of strong approximations of the solution of Itô stochastic functional differential equations involving a distributed delay term. The mean-square consistency of a class of schemes, the Θ-Maruyama methods, is analysed, using an appropriate Itô-formula. In particular, we investigate the consequences of the choice of a quadrature formula. Numerical examples illustrate the theoretical results.

1 Introduction

Consider Itô stochastic functional differential equations (SFDEs) of the form, for $t \in [0,T]$,

$$\begin{aligned} X(t) &= X(0)+\int_0^t F(s,X(s),Y(s))\,\mathrm{d}s+\sum_{j=1}^m \int_0^t G^j(s,X(s),Y(s))\,\mathrm{d}W^j(s), && (1)\\ X(s) &= \Psi(s), \quad s\in J, \quad \text{where } J:=[-\tau,0],\ \tau\geq 0. && (2)\end{aligned}$$

Here $Y(t)$ represents a memory term of the type

$$Y(t) = \int_{-\tau}^{0} K(t,s,X(t+s))\,\mathrm{d}s = \int_{t-\tau}^{t} K(t,s-t,X(s))\,\mathrm{d}s. \qquad (3)$$

Deterministic equations with distributed memory terms have been well studied. We refer, e.g., to [12] for models in population dynamics or to [11] for applications

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in physics, engineering or economy. However, more realistic mathematical models can be investigated by allowing for random perturbations incorporated into the equations. A system of equations like (1) modelling the spread of infectious diseases was investigated in [4]. A class of equations similar to (1) but with an additional discrete delay term is considered in [17], in particular with applications to finance. More models with a financial mathematics background based on related equations appear in Chang and Youree [7] and Hobson and Rogers [9].
In this article we investigate Θ-Maruyama schemes applied to simulate the solution of (1). For a statement of mean-square convergence of the method we rely on the corresponding theorem for general stochastic functional differential equations, stated and proved in [5], which extends the analysis in [2].
The standard technique to perform the error analysis for stochastic ordinary differential equations is to set up an Itô-Taylor expansion by application of the Itô-formula to the drift and diffusion functions, see [10, 14]. We note that although it is possible to prove convergence of the Euler-Maruyama method directly, the rigorous error analysis of the Θ-Maruyama necessitates the application of an Itô-formula. In [1], stochastic analysis for SFDEs is developed, in particular an Itô-formula for so-called quasi-tame functions is proved. We will apply a generalised version of this result.

2 Definitions and Preliminaries

Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and each $\{\mathcal{A}_t\},\ t \in [0,T]$ contains all $\mathbb{P}$-null sets in $\mathcal{A}$) and let $W(t) = (W^1(t), \ldots, W^m(t))^T$ be an m-dimensional standard Brownian motion on that probability space. Throughout the article $|.|$ denotes the Euclidean norm in $\mathbb{R}^d$. We denote the mean-square norm of a vector-valued square-integrable random variable $z \in L_2(\Omega; \mathbb{R}^d)$ by $\|z\|_{L_2} := (\mathbb{E}\,|z|^2)^{1/2}$, where $\mathbb{E}$ is expectation with respect to $\mathbb{P}$.
We assume that the drift and diffusion coefficients $F : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$, $G^j : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d,\ j = 1,\ldots,m$, are continuous and satisfy uniform Lipschitz and linear growth conditions with respect to their second and third argument. We require the same of the kernel function $K : [0,T]\times J\times\mathbb{R}^d \to \mathbb{R}^d$ with respect to its third argument. The initial path $\Psi(t) : J \to \mathbb{R}^d$ is assumed to be a continuous and $\mathcal{A}_0$-measurable random variable such that $(\mathbb{E}\ \sup_{s\in J}|\Psi(s)|^2)^{1/2} < \infty$. These conditions imply that there exists a path-wise unique strong solution to Eq. (1). Proofs of this can be found, e. g., in [13] and [16]. In addition, we assume that F, G and K have sufficient differentiability with respect to their arguments. We denote multiple Wiener integrals with some function $f(.)$ as an integrand by

$$I^{t,t+h}_{(j_1,j_2,\ldots,j_l)}(f) = \int_t^{t+h}\int_t^{s_l}\cdots\int_t^{s_2} f(s_1)\,\mathrm{d}W^{j_1}(s_1)\ldots\mathrm{d}W^{j_l}(s_l), \tag{4}$$

where $j_i \in \{0,1,\ldots,m\},\ i = 1,\ldots,l$ and $\mathrm{d}W^0(s) = \mathrm{d}s$. If $f \equiv 1$ we omit the argument (f). In what follows we denote by $D_i f(t_1,\ldots,t_m)$ and $D_i^2 f(t_1,\ldots,t_m)$ the first and second order derivative with respect to the i-th argument of a function f and define the abbreviation $f[t] := f(t, X(t), Y(t))$.

The following Itô-formula can be derived based on [8, Theorem 4.5.3]. For simplicity we state the scalar version, it can be extended to the multi-dimensional case. Let $f(t,x,y)$ be a continuous function in (t,x,y) where $t \geq 0,\ (x,y) \in \mathbb{R}^2$, together with its first t derivative and second x and y-derivatives. The Itô-formula for a function of the form $f(t,X(t),Y(t))$ with $t,t' \in [0,T], t > t'$ then reads:

$$f[t] - f[t'] = \int_{t'}^{t} \Big[\underbrace{D_1 f[s] + D_2 f[s]\ F[s] + \frac{1}{2}\ D_2^2 f[s]\ (G[s])^2}_{=:\ \Lambda_1 f[s]} + \underbrace{D_3 f[s]\ a_2(s)}_{=:\ \Lambda_3 f[s]} \Big]\ \mathrm{d}s$$

$$+ \int_{t'}^{t} \underbrace{D_2 f[s]\ G[s]}_{=:\ \Lambda_2 f[s]}\ \mathrm{d}W(s), \tag{5}$$

$$\text{where} \qquad a_2(t) := \frac{d}{dt} \int_{t-\tau}^{t} K(t, s-t, X(s))\ \mathrm{d}s$$

$$= \int_{t-\tau}^{t} D_1 K(t, s-t, X(s)) - D_2 K(t, s-t, X(s))\ \mathrm{d}s + K(t, 0, X(t)) - K(t, -\tau, X(t-\tau)).$$

We now introduce our numerical method. First we define a family of meshes on the interval $[0,T]$ as $\mathcal{T}_h^N := \{0 = t_0 < t_1 < t_2, \cdots < t_N\} \subseteq [0,T]$, with $t_n = nh,\ n = 0,\ldots,N,\ hN \leq T,\ N \in \mathbb{N}$. In addition, the choice of h is not arbitrary, and we require, with the lag τ given, that

$$h = \tau / N_\tau, \qquad N_\tau \in \mathbb{N}. \tag{6}$$

On $\mathcal{T}_h^N$ we consider strong approximations $\widetilde{X}_n$ of the solution to (1) using the Θ-Maruyama-method which is given by

$$\widetilde{X}_n = \widetilde{X}_{n-1} + h\ (\Theta\ F(t_n, \widetilde{X}_n, \widetilde{Y}_n) + (1-\Theta) F(t_{n-1}, \widetilde{X}_{n-1}, \widetilde{Y}_{n-1})) + \sum_{j=1}^{m} G^j(t_{n-1}, \widetilde{X}_{n-1}, \widetilde{Y}_{n-1}) I_j^{t_{n-1},t_n}. \tag{7}$$

The initial values are given by $\widetilde{X}_i := \Psi(t_i)$ for $i \leq 0$. The expression $\widetilde{Y}_m$ provides an approximation to the integral $Y(t_m)$. As in the deterministic case, the choice of a quadrature rule has consequences on the overall performance of the numerical method. We refer to the examples in [3, Section 3], where the second order convergence of the trapezium rule applied to a problem of the form (1) with $G \equiv 0$ could only be achieved by also using the (composite) trapezium rule as the quadrature rule for Y. The composite Euler quadrature did not suffice. Here, we also have the possibility to choose different quadrature rules in the drift and in the diffusion part, e. g. an implicit one in the drift and an explicit one in the diffusion term. This can be useful in the case of a small noise coefficient. Here we will discuss the composite Euler rule and the composite trapezium rule (at least in the drift part). These have the form, for $m \geq 0$,

$$\widetilde{Y}_m^E = h \sum_{\ell=m-N_\tau}^{m-1} K(t_m, t_\ell - t_m, \widetilde{X}_\ell) \quad \text{and} \quad \widetilde{Y}_m^T = h \sum_{\ell=m-N_\tau}^{m}{}'' K(t_m, t_\ell - t_m, \widetilde{X}_\ell), \tag{8}$$

respectively. As usual, the notation $\sum''$ means that the first and the last term in the sum are to be halved.
The method (7) is assumed to generate approximations $\widetilde{X}_n$ which are $\mathcal{A}_{t_n}$-measurable. The increment functions on the right-hand side of (7) satisfy uniform Lipschitz conditions, due to the corresponding assumptions on the functions F, G and K. These imply, for the step-size h sufficiently small, the solvability of the recurrence equation.
We will provide estimates of the local error, which is defined as the defect that is obtained when the exact solution values are inserted into the numerical scheme (see, e.g., [5, Def.1]). Here the local error takes the form

$$\begin{aligned}\delta_{n-1} = &\int_{t_{n-1}}^{t_n} F[s]\,\mathrm{d}s + \sum_{j=1}^{m}\int_{t_{n-1}}^{t_n} G^j[s]\,\mathrm{d}W^j(s)\\ &- h\,\Big(\Theta\; F(t_n, X(t_n), \widehat{Y}(t_n)) + (1-\Theta)\; F(t_{n-1}, X(t_{n-1}), \widehat{Y}(t_{n-1}))\Big)\\ &- \sum_{j=1}^{m} G^j(t_{n-1}, X(t_{n-1}), \widehat{Y}(t_{n-1}))I_j^{t_{n-1},t_n},\end{aligned} \tag{9}$$

where $\widehat{Y}(t_m)$ represents either of the quadrature formulas (8) with $\widetilde{X}_\ell$ replaced by the exact solution $X(t_\ell)$. We will consider consistency and convergence of our approximations in the following sense.

Definition 1 *The approximations $\{\widetilde{X}_n\}$ for the solution X of (1) are said to be* mean-square consistent with order p *($p>0$) if the following estimates hold:*

$$\max_{t_n\in\mathcal{T}_h^N} \|\mathbb{E}(\,\delta_n|\mathcal{A}_{t_n})\|_{L_2} \quad\leq\quad C\;h^{p+1}\quad as\ h\to 0, \tag{10}$$

and

$$\max_{t_n\in\mathcal{T}_h^N} \|\delta_n\|_{L_2} \quad\leq\quad C\;h^{p+\frac{1}{2}}\quad as\ h\to 0, \tag{11}$$

where the generic constant C does not depend on h, but may depend on T and the initial data.

Definition 2 *The approximations $\{\widetilde{X}_n\}$ for the solution X of equation (1), defined on $\mathcal{T}_h^N$ with the step-size h constraint by (6), are said to be* mean-square convergent with order p*, on the mesh-points, when*

$$\max_{t_n\in\mathcal{T}_h^N} \|X(t_n)\ -\ \widetilde{X}_n\|_{L_2} \quad\leq\quad C\;h^{p}\quad as\ h\to 0, \tag{12}$$

where $C<\infty$ is independent of h, but may depend on T and on the initial data.

By an application of Theorem 1 in [5] we obtain mean-square convergence of the Θ-Maruyama method, if the method is mean-square consistent and the increment functions on the right-hand side of (7) satisfy uniform Lipschitz conditions. The order of consistency implies the order of convergence.

3 Local error estimates

Our purpose in this section is to provide consistency estimates of the form (10) and (11) for the Θ-Maruyama method described by (7). For the sake of brevity and readability we will treat here only the scalar case $d = m = 1$. First we derive an appropriate expression for δ_n. Using (1) on $[t_{n-1}, t_n]$ instead of $[0, t]$, applying the Itô-formula (5) to the arguments $F[s]$ and $G[s]$, adding and subtracting $\Theta\ h\ F(t_n, X(t_n), Y(t_n))$ and applying again the Itô-formula (5) to $\Theta\ h\ F(t_n, X(t_n), Y(t_n))$, yields

$$\begin{aligned}
\delta_{n-1} &= (1-\Theta)\ h\ \Big(F(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - F(t_{n-1}, X(t_{n-1}), \widehat{Y}(t_{n-1}))\Big) \\
&+ \ \Theta\ h\ \Big(F(t_n, X(t_n), Y(t_n)) - F(t_n, X(t_n), \widehat{Y}(t_n))\Big) \\
&+ \ \Big(G(t_{n-1}, X(t_{n-1}), Y(t_{n-1})) - G(t_{n-1}, X(t_{n-1}), \widehat{Y}(t_{n-1}))\Big) I_1^{t_{n-1},t_n} \\
&+ \ R_1 + R_2,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= I_{0,0}^{t_{n-1},t_n}(\Lambda_1 F + \Lambda_3 F) - \Theta\ h\ I_0^{t_{n-1},t_n}(\Lambda_1 F + \Lambda_3 F) \\
R_2 &= I_{1,0}^{t_{n-1},t_n}(\Lambda_2 F) - \Theta\ h\ I_1^{t_{n-1},t_n}(\Lambda_2 F) \\
&+ \ I_{1,1}^{t_{n-1},t_n}(\Lambda_2 G) + I_{0,1}^{t_{n-1},t_n}(\Lambda_1 G + \Lambda_3 G).
\end{aligned}$$

Lemma 3.1 *Assume that the derivatives of F and G appearing in R_1 and R_2 exist for $t \in [0, T]$ and are uniformly bounded in modulus, and that the quadrature method satisfies*

a) $\max_{t_n \in \mathcal{T}_h^N} \|\mathbb{E}(\ Y(t_n) - \widehat{Y}(t_n)|\mathcal{A}_{t_n})\|_{L_2} \le C\ h^{\frac{1}{2}}$,

b) $\max_{t_n \in \mathcal{T}_h^N} \|Y(t_n) - \widehat{Y}(t_n)\|_{L_2} \le C\ h^{\frac{1}{2}}$,

where the generic constant C does not depend on h, but may depend on T and the initial data. Then the Θ-Maruyama method (7) is mean-square consistent with order $p = \frac{1}{2}$.

Proof: We will subsequently use estimates of multiple Itô-integrals to deal with R_1 and R_2, and we base them on Lemmas 2.1 and 2.2 in [14] and the discussion in that section.
To obtain the estimate (10) we apply the mean value theorem to F and, denoting $\xi_{t_m} := Y(t_m) + \zeta(\widehat{Y}(t_m) - Y(t_m)),\ 0 < \zeta < 1$, we have

$$\begin{aligned}
&|\mathbb{E}\,(\delta_{n-1}|\mathcal{A}_{t_{n-1}})| \\
&\quad \le (1-\Theta)\ h\ |\mathbb{E}\,\Big(D_3F(t_{n-1}, X(t_{n-1}), \xi_{t_{n-1}})(\widehat{Y}(t_{n-1}) - Y(t_{n-1}))|\mathcal{A}_{t_{n-1}}\Big)| \\
&\quad + \Theta\ h\ |\mathbb{E}\,\Big(D_3F(t_n, X(t_n), \xi_{t_n})(\widehat{Y}(t_n) - Y(t_n)|\mathcal{A}_{t_{n-1}})\Big)| + |\mathbb{E}\,(R_1|\mathcal{A}_{t_{n-1}})| \\
&\qquad\qquad \le (1-\Theta)\ h\ C\ |\,\mathbb{E}\,\big(Y(t_{n-1}) - \widehat{Y}(t_{n-1})|\mathcal{A}_{t_{n-1}}\big)| \\
&\qquad\qquad\qquad + \ \Theta\ h\ C\ |\,\mathbb{E}\big(Y(t_n) - \widehat{Y}(t_n)|\mathcal{A}_{t_{n-1}}\big)|\ +\ C\ h^2\ .
\end{aligned}$$

Thus,

$$\|\mathbb{E}\,(\delta_{n-1}|\mathcal{A}_{t_{n-1}})\|_{L_2} \le C\ h\ \|\ \mathbb{E}\,\big(Y(t_{n-1}) - \widehat{Y}(t_{n-1})|\mathcal{A}_{t_{n-1}}\big)\|_{L_2} \\ +\ C\ h\ \|\ \mathbb{E}\big(Y(t_n) - \widehat{Y}(t_n)|\mathcal{A}_{t_{n-1}}\big)\|_{L_2}\ +\ C\ h^2\ .$$

To obtain the estimate (11) we apply the inequality $(\sum_{i=1}^n a_i)^2 \le n\ \sum_{i=1}^n a_i^2$ and the Lipschitz conditions on F and G and arrive at

$$\mathbb{E}|\delta_{n-1}|^2 \le C\ (1-\Theta)^2\ h^2\ \mathbb{E}\ |Y(t_{n-1}) - \widehat{Y}(t_{n-1})|^2\ +\ C\ \Theta^2\ h^2\ \mathbb{E}\ |Y(t_n) - \widehat{Y}(t_n)|^2 \\ + h\ \ \mathbb{E}\ |Y(t_{n-1}) - \widehat{Y}(t_{n-1})|^2\ + C\ \mathbb{E}\,|R_1|^2 + C\ \mathbb{E}\,|R_2|^2.$$

Hence

$$\|\delta_{n-1}\|_{L_2} \le C\ h\ \|Y(t_{n-1}) - \widehat{Y}(t_{n-1})\|_{L_2} \\ + C\ h\ \|Y(t_n) - \widehat{Y}(t_n)\|_{L_2} + h^{\frac{1}{2}}\ \|Y(t_{n-1}) - \widehat{Y}(t_{n-1})\|_{L_2}\ + C\ h.$$

□

Remark 3.2 *Consider Eq. (1) with a small noise coefficient, i.e. when the diffusion terms G^j in (1) are given as $\epsilon\hat{G}^J$, where $\epsilon \ll 1$ is a small parameter, and $\hat{G}$ and its derivatives are assumed to have moderate values. Then one may want to derive expansions of the error in terms of the small parameter ϵ and the step-size, as has been done for stochastic ordinary differential equations and one-step methods in [15] and for multi-step methods in [6]. In this case the small diffusion term makes it unnecessary to use high order multiple stochastic integrals and Wiener increments will be sufficient if the step-size is not too small. If one uses an Itô-formula for mixed discrete and distributed delay (its development needs Malliavin calculus, which is beyond the scope of this article) to further expand the remainder term R_1, it is possible, e.g. for $\Theta = \frac{1}{2}$ to recover the second order deterministic order plus ϵh-terms. Of course, it is then not sufficient to use the Euler quadrature formula in the drift term. We will give numerical illustrations of this fact in the next section.*

Now we turn to the estimates concerning the quadrature formulas. First we derive an expansion of the error $Y(t_m) - \widehat{Y}(t_m)$ for the formulas (8) and here we have to distinguish two cases, *i)* $t_m - \tau < 0 < t_m$ and *ii)* $t_m \le 0$ (if $t_m - \tau \ge 0$ one would need to separate the integral).
Starting from the expressions

$$Y(t_m) - \widehat{Y}^E(t_m) = \sum_{\ell=m-N_\tau}^{m-1} \int_{t_\ell}^{t_{\ell+1}} K(t_m, s-t_m, X(s)) - K(t_m, t_\ell - t_m, X(t_\ell))\ \mathrm{d}s\ ,$$

$$Y(t_m) - \widehat{Y}^T(t_m) = \sum_{\ell=m-N_\tau}^{m-1} \int_{t_\ell}^{t_{\ell+1}} K(t_m, s-t_m, X(s)) \\ - \frac{1}{2}(K(t_m, t_{\ell+1} - t_m, X(t_{\ell+1})) + K(t_m, t_\ell - t_m, X(t_\ell)))\ \mathrm{d}s\ ,$$

we can for *i)*, by relying on the Lipschitz continuity of the kernel K, the conditions on the initial path Ψ, as well as the inequality $(\sum_{i=1}^{n} a_i)^2 \leq n \sum_{i=1}^{n} a_i^2$ and the Cauchy-Schwarz inequality, obtain for both quadrature formulas

$$\max_{t_m \leq 0} \|\mathbb{E}(\, Y(t_m) - \widehat{Y}(t_m))\|_{L_2} \leq C\, h, \quad \text{and} \quad \max_{t_m \leq 0} \|Y(t_m) - \widehat{Y}(t_m)\|_{L_2} \leq C\, h\,.$$

For *ii)* we will need another Itô-formula and we refer to [11, 5.1.2] for its formulation. Let $K_{t_m}[u]$ denote $K(t_m, u - t_m, X(u))$. Applying the Itô-formula to the terms $K(t_m, s - t_m, X(s))$ and $K(t_m, t_{\ell+1} - t_m, X(t_{\ell+1}))$ in the above expressions yields the remainder terms

$$Y(t_m) - \widehat{Y}^E(t_m) = \sum_{\ell=m-N_\tau}^{m-1} \Big\{ I_{0,0}^{t_\ell, t_{\ell+1}}(D_2 K_{t_m} + D_3 K_{t_m}\, F + \frac{1}{2} D_3^2 K_{t_m}\, G^2) + I_{1,0}^{t_\ell, t_{\ell+1}}(D_3 K_{t_m}\, G) \Big\}\,, \quad (13)$$

$$\text{and} \quad Y(t_m) - \widehat{Y}^T(t_m) = \sum_{\ell=m-N_\tau}^{m-1} \Big\{ I_{0,0}^{t_\ell, t_{\ell+1}}(D_2 K_{t_m} + D_3 K_{t_m}\, F + \frac{1}{2} D_3^2 K_{t_m}\, G^2)$$
$$- \frac{1}{2} \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{t_{\ell+1}} D_2 K_{t_m}[u] + D_3 K_{t_m}[u]\, F[u] + \frac{1}{2} D_3^2 K_{t_m}[u]\, (G[u])^2 \,\mathrm{d}u\, \mathrm{d}s$$
$$+ I_{1,0}^{t_\ell, t_{\ell+1}}(D_3 K_{t_m}\, G) - \frac{1}{2} \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{t_{\ell+1}} D_3 K_{t_m}[u]\, G[u]\, \mathrm{d}W(u)\, \mathrm{d}s \Big\}\,. \quad (14)$$

Lemma 3.3 *Assume that the kernel function K is uniformly Lipschitz continuous and sufficiently differentiable with its derivatives uniformly bounded in modulus. Then the composite Euler and the composite Trapezium quadrature satisfy for all $m \geq 0$ the estimates*

a) $\max_{t_m \in \mathcal{T}_h^N} \|\mathbb{E}(\, Y(t_m) - \widehat{Y}(t_m)|\mathcal{A}_{t_m})\|_{L_2} \leq C\, h$,

b) $\max_{t_m \in \mathcal{T}_h^N} \|Y(t_m) - \widehat{Y}(t_m)\|_{L_2} \leq C\, h^{\frac{1}{2}}$,

where C is a constant, not depending on h.

Proof: We obtain from (13) for $|\mathbb{E}(Y(t_m) - \widehat{Y}^E(t_m)|\mathcal{A}_{t_m})|$

$$\leq \sum_{\ell=m-N_\tau}^{m-1} |\mathbb{E}\Big(I_{0,0}^{t_\ell, t_{\ell+1}}(D_2 K[t_m] + D_3 K[t_m]\, F + \frac{1}{2} D_3^2 K[t_m]\, G^2)|\mathcal{A}_{t_m}\Big)|$$
$$\leq \sum_{\ell=m-N_\tau}^{m-1} C\, h^2 = C\, h.$$

The estimate for (14) follows analogously and then part a) follows.
For part b) we obtain again from (13) for $\mathbb{E}|Y(t_m) - \widehat{Y}^E(t_m)\ |^2$

$$\leq \ C \sum_{\ell=m-N_\tau}^{m-1} h \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{s} \mathbb{E}|D_2 K_{t_m}[u] + D_3 K_{t_m}[u]\ F[u] + \frac{1}{2} D_3^2 K_{t_m}[u]\ (G[u])^2|^2 \ \mathrm{d}u \ \mathrm{d}s$$

$$+ \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{s} \mathbb{E}|D_3 K_{t_m}[u]\ G[u]\ |^2 \ \mathrm{d}W(u) \ \mathrm{d}s \ \leq \ C \sum_{\ell=m-N_\tau}^{m-1} h^3 + h^2 \ \leq \ C\ h.$$

Again the estimate for (14) follows analogously and the theorem is proved. $\square$

Remark 3.4 *Consider again Eq. (1) with a small noise coefficient as in Remark 3.2. When deriving expansions of the error in terms of the small parameter ϵ and the step-size, it is clear that the first two terms in the sum in (14) cancel out after another application of the Itô-formula. However, the two stochastic integrals will not cancel out.*

4 Numerical experiments

We present some results of numerical experiments corresponding to an example of (1). Our objective is to illustrate the convergence of the Θ-Maruyama method with respect to decreasing step-size. In addition we present some numerical results concerning the behaviour of the Trapezium method ($\Theta = \frac{1}{2}$) and the choice of a quadrature method when Eq. (1) has a small parameter ϵ in the diffusion term.

Example 4.1 Consider the scalar equation

$$dX(t) = \Big[\int_{t-1}^{t} X(s)\ ds + \exp(-1) X(t) \Big]\ \mathrm{d}t + \epsilon X(t)\ \mathrm{d}W(t), \tag{15}$$

for $t \in [0, 2]$ and $X(s) = \exp(t)$ for $-1 \leq s \leq 0$.

We have chosen the Euler-Maruyama method (EM) with Euler quadrature for the integral, the Trapezium rule ($\Theta = \frac{1}{2}$) with Euler quadrature for the integral (TE), and with the Trapezium rule for the integral (TT). If we square both sides of (12) we obtain the mean-square error $\mathbb{E}|X(T) - \widetilde{X}_N|^2$ which should be bounded by $C\ h^{2p}$. An 'explicit solution' was computed on a very fine mesh (4096 steps). To illustrate the convergence of the method, 500 sample trajectories were simulated for the step-sizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$, and the values $\epsilon = 0,\ 0.005,\ 0.01,\ 0.1,\ 1$ and the error $\mathbf{e}(h)$ computed at the final time $T = 2$, where $\mathbf{e}(h) = \frac{1}{500} \sum_{j=1}^{500} |\widetilde{X}_N - X(T)|^2$. In addition, the ratio of terms $\mathbf{e}(\frac{h}{2})/\mathbf{e}(h)$ is computed, which approximates $\{\frac{1}{2}\}^{2p}$ for an appropriate p. In Table 1 the results of the experiments are presented.

Table 1: Example (15)

$\epsilon = 0$	EM $\mathbf{e}^h$	ratio	TE $\mathbf{e}^h$	ratio	TT $\mathbf{e}^h$	ratio
$h = 2^{-4}$	0.31847	*	0.051378	*	4.115E-05	*
2^{-5}	0.086627	0.27	0.013493	0.26	2.5664E-06	0.0625
2^{-6}	0.022621	0.26	0.0034587	0.256	1.6009E-07	0.0625
2^{-7}	0.0057817	0.26	0.00087568	0.256	9.9461E-09	0.0625
$\epsilon = 0.005$						
$h = 2^{-4}$	0.31929	*	0.051556	*	4.0527E-05	*
2^{-5}	0.086854	0.27	0.01354	0.26	2.6763E-06	0.07
2^{-6}	0.022679	0.26	0.0034706	0.256	2.265E-07	0.083
2^{-7}	0.0057966	0.256	0.00087869	0.256	2.9314E-08	0.13
$\epsilon = 0.1$						
$h = 2^{-4}$	0.33073	*	0.055101	*	0.0009071	*
2^{-5}	0.090033	0.27	0.014507	0.26	0.00028056	0.31
2^{-6}	0.023473	0.26	0.0037141	0.256	9.2505E-05	0.33
2^{-7}	0.0059809	0.256	0.0009407	0.256	3.6537E-05	0.4
$\epsilon = 1$						
$h = 2^{-4}$	11.633	*	10.568	*	10.579	*
2^{-5}	7.9148	0.67	6.9808	0.67	6.5775	0.625
2^{-6}	2.0541	0.256	1.8766	0.27	1.8248	0.278
2^{-7}	0.81361	0.4	0.79308	0.42	0.79424	0.43

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