ON HALANAY-TYPE ANALYSIS OF EXPONENTIAL STABILITY FOR THE θ–MARUYAMA METHOD FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS.

Dedicated to Donald Kershaw, Reader Emeritus, Lancaster University

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Abstract

Using an approach that has its origins in work of Halanay, we consider stability in mean square of numerical solutions obtained from the *θ–Maruyama discretization* of a test stochastic delay differential equation

$$dX(t) = \{f(t) - \alpha X(t) + \beta X(t-\tau)\}\mathrm{d}t + \{g(t) + \eta\; X(t) + \mu X(t-\tau)\}\; \mathrm{d}W(t),$$

interpreted in the Itô sense, where $W(t)$ denotes a Wiener process. We focus on demonstrating that we may use techniques advanced in a recent report by Baker and Buckwar to obtain criteria for asymptotic and exponential stability, in mean square, for the solutions of the recurrence

$$\widetilde{X}_{n+1} - \widetilde{X}_n = \theta h\{f_{n+1} - \alpha\widetilde{X}_{n+1} + \beta\widetilde{X}_{n+1-N}\} +$$
$$+(1-\theta)h\{f_n - \alpha\widetilde{X}_n + \beta\widetilde{X}_{n-N}\} + \sqrt{h}(g_n + \eta\widetilde{X}_n + \mu\widetilde{X}_{n-N})\xi_n \quad (\xi_n \in \mathcal{N}(0,1)).$$

θ-Maruyama scheme; asymptotic and exponential stability; stochastic delay differential & difference equations; Halanay-type inequalities.
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1 Introduction

This work is an extension of previous work of Baker & Buckwar [2, 3]. We indicate how results for stability of solutions obtained from a θ–Maruyama method applied to a linear stochastic delay differential equation (SDDE), that serves as a test equation, can be derived. The use of such a test equation is commonplace in numerical analysis; see *e.g.* [1, 5] for deterministic delay differential equations (DDDE s); for stochastic ordinary differential equations (SODE s) see *e.g.* [9]; for SDDE s see [2, 7]. Though it may seem that standard test equations are often chosen for their amenability to investigation, we here accept without discussion that such simple equations generate a 'test-bed' for obtaining insight into related non-trivial problems. At the same time, analysis of deterministic problems can yield understanding of the analysis of stochastic equations, and we exploit this.

We assume a little familiarity with the related literature, but seek to present a self-contained discussion. We employ a strategy presented for stability analysis in [3], where we illustrated the investigation of numerical stability by examining the *Euler-Maruyama method*. As we remarked in an aside in [3], the technique for analyzing stability that we illustrated by reference to the Euler-Maruyama method can also be applied to some other methods, including some that are semi-implicit (*i.e.* drift-implicit). We justify this remark here; a number of the other comments suggest further insight not found (or not easily found) in the existing literature.

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1.1 The test equation

With t_0, α, β, η, $\mu \in \mathbb{R}$ and $\tau \geq 0$, the Itô SDDE considered here is written

$$\begin{aligned} \mathrm{d}X(t) &= \{f(t) - \alpha X(t) + \beta\, X(t-\tau)\}\,\mathrm{d}t + \\ &\quad + \{g(t) + \eta\, X(t) + \mu X(t-\tau)\}\,\mathrm{d}W(t)\ (t \geq t_0), \qquad (1a) \end{aligned}$$

$$X(t) = \Phi(t), \quad t \in J, \quad J := [t_0 - \tau, t_0]. \qquad (1b)$$

(Note the sign attached to α in (1a).) If $\tau > 0$, we can, by a change of variable, normalize it to unity, and replace $\{\alpha, \beta, \eta, \mu, \tau\}$ by $\{\alpha\tau, \beta\tau, \eta\sqrt{\tau}, \mu\sqrt{\tau}, 1\}$.
For $t \in [t_0, \infty)$, $X(t) \equiv X(\Phi; t)$ denotes the solution of the SDDE (1a) for a given initial function Φ in (1b).
The discussion in [3] was presented in terms of a more general equation

$$\mathrm{d}X(t) = F(t, X(t), X(t-\tau))\,\mathrm{d}t + G(t, X(t), X(t-\tau))\,\mathrm{d}W(t). \qquad (2)$$

We use the linear inhomogeneous SDDE (1a), with $t_0 \leq t < \infty$, as a *test equation* for the discussion of stability and (on applying a numerical method) numerical stability. The functions $f(t)$ and $g(t)$ in (1a) satisfy conditions consistent with those normally assumed for F and G; their presence implies that the null function ($X(t) \equiv 0$ for $t \geq -\tau$) may not be a solution. We assume the standard infrastructure and notation [2, 3, 11]: (*i*) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathsf{P})$ is a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions, E denotes expectation with respect to P and $W(t)$ is a one-dimensional standard Wiener process on that probability space; (*ii*) the initial function $\Phi : J \times \Omega \to \mathbb{R}$ has continuous paths, is independent of the σ-algebra generated by $W(t)$ and satisfies $\mathsf{E}(\sup_{t \in J} |\Phi(t)|^2) < \infty$; (*iii*) there exists a unique strong solution of the SDDE (1) with $\mathsf{E}(\{X(t)\}^2) < \infty$ for bounded t. (A strong solution of (2) satisfies $X(t) = X(t_0) + \int_{t_0}^{t} F(s, X(s), X(s-\tau))\,\mathrm{d}s + \int_{t_0}^{t} G(s, X(s), X(s-\tau))\mathrm{d}W(s)$ almost surely, for $t \geq t_0$; for a full definition, and sufficient conditions for existence and uniqueness, see Mao [11] pp. 149–157. The convergence and stability results in [2, 3] require existence and uniqueness of solutions but do not require the global Lipschitz conditions in [11]. For a convergence proof for the θ-Maruyama method for general discrete delay SDDEs, including (2), see [6].)

1.2 The θ–Maruyama equations

Suppose that $\theta \in [0, 1]$ and choose a step $h = \tau/N$ where $N \in \mathbb{N}$. The Maruyama-type θ-method applied to (1) generates, where $\xi_n \in \mathcal{N}(0, 1)$ (ξ_n is normally distributed with zero mean and variance unity), the recurrence

$$\begin{aligned} \widetilde{X}_{n+1} - \widetilde{X}_n &= h f_n^\theta + h\Big\{\theta\{-\alpha\widetilde{X}_{n+1} + \beta\widetilde{X}_{n+1-N}\} + (1-\theta)\{-\alpha\widetilde{X}_n + \beta\widetilde{X}_{n-N}\}\Big\} + \\ &\quad + \sqrt{h}(g_n + \eta\,\widetilde{X}_n + \mu\,\widetilde{X}_{n-N})\ \xi_n, \qquad (3a) \end{aligned}$$

for approximations $\widetilde{X}_n \approx X(nh)$, where we use the shorthand notation

$$t_n = t_0 + nh,\ g_n = g(t_n),\ f_n = f(t_n) \text{ and } f_n^\theta := \theta f_{n+1} + (1-\theta) f_n\ (n \in \mathbb{Z}). \qquad (3b)$$

Eq. (3a) is a drift-implicit formula that (if h is not an "exceptional value" – *i.e.*, provided $1 + \theta\alpha h \neq 0$) generates the sequence $\{\widetilde{X}_n\}_{n \geq 1}$, when given

$$\widetilde{X}_{-\ell} = \Phi(t_{-\ell}) \text{ for } \ell \in \mathcal{J} \quad \text{where } \mathcal{J} := \{0, 1, \cdots, N\}. \qquad (3c)$$

To indicate the dependence on Φ we write $\widetilde{X}_n \equiv \widetilde{X}_n(\Phi)$, and our definition of stability relates to perturbations $\delta\widetilde{X}_n \equiv \delta\widetilde{X}_n(\Phi) := \widetilde{X}_n(\Phi+\delta\Phi) - \widetilde{X}_n(\Phi)$, that arise from perturbations $\delta\Phi(t_{-\ell})$ (for $\ell \in \mathcal{J}$) in the initial data. We will use the notation

$$\varrho_0 = \frac{1-(1-\theta)\alpha h}{1+\theta\alpha h},\ \varrho_1 = \frac{\theta\beta h}{1+\theta\alpha h},\ \varrho_2 = \frac{(1-\theta)\beta h}{1+\theta\alpha h},\ \varpi_0 = \frac{\eta\sqrt{h}}{1+\theta\alpha h},\ \varpi_1 = \frac{\mu\sqrt{h}}{1+\theta\alpha h}. \quad (4)$$

From (3a), if $1+\theta\alpha h \neq 0$ (as we assume),

$$\delta\widetilde{X}_{n+1} = (\varrho_0 + \varpi_0\xi_n)\delta\widetilde{X}_n + \varrho_1\delta\widetilde{X}_{n+1-N} + (\varrho_2 + \varpi_1\xi_n)\delta\widetilde{X}_{n-N}. \quad (5)$$

2 Exponential Stability of Solutions by a Halanay-type Technique

There is a variety of approaches to the investigation of stability; we cannot overemphasize that each approach has its merits or demerits and each has its adherents. Halanay [8] provided a technique for examining the exponential stability of solutions of DDDE s. This was modified for difference equations by Tang [12] (see also the related publications, *e.g.* [4]). Baker and Buckwar [3] progressed the Halanay-type theory by applying it to establish conditions for p-th moment exponential stability of solutions of SDDE s and certain discretized versions.

2.1 Stability definitions

Our definitions of stability, asymptotic stability, and exponential stability in mean-square of solutions of (1) are consistent with usual definitions[a] to be found in the literature; *cf.* [3], or [10, 11]; they are analogues of the definitions of (asymptotic, exponential) stability of solutions of stochastic *recurrence relations* or *difference equations.* However, the general stability definitions associated with (2) and its discretization can be simplified when considering (1), (3).

Definition 1 *A solution of (3) is said to be (a)* stable in mean-square *(*SMS*) if, for each $\varepsilon > 0$, there exists a corresponding value $\delta^+ > 0$ such that $\mathsf{E}(|\,\delta\widetilde{X}_n|^2) < \varepsilon$ for $n \in \mathbb{N}$, whenever $\mathsf{E}(\sup_{n\in\mathcal{J}}|\,\delta\Phi(t_n)|^2) < \delta^+$; (b)* asymptotically stable in mean-square *(*ASMS*) if it is stable in mean-square and $\mathsf{E}(|\,\delta\widetilde{X}_n|^2) \to 0$ as $n \to \infty$, whenever $\mathsf{E}(\sup_{n\in\mathcal{J}}|\,\delta\Phi(t_n)|^2)$ is bounded; (c)* exponentially stable in mean-square *(*ESMS*) if it is stable in the mean-square and if, given $\delta^+ > 0$, there exist a finite $C > 0$, and a value $\nu_h^+ > 0$ such that, whenever $\mathsf{E}(\sup_{n\in\mathcal{J}}|\,\delta\Phi(t_n)|^2) < \delta^+$, $\mathsf{E}(|\delta\widetilde{X}_n|^2) \leq C\exp\{-\nu_h^+(t_n - t_0)\}$ for all n sufficiently large. Given such a $\nu_h^+ > 0$, we then term the solution ν_h^+-*ESMS*, or* ESMS *with* exponent $-\nu_h^+$.

Emphasis in the numerical analysis literature is concentrated on (a) and (b) rather than (c); we consider results for ν-exponential stability (ν-ESMS). The definitions of stability for the *analytical solution* $X(\Phi;t)$ of (1) are natural analogues of those in Definition 1. Thus, exponential stability is defined as follows:

Definition 2 *The solution $X(\Phi;t)$ of the problem (1) is* exponentially mean-square stable*, with* exponent $-\nu^+$ *(ν^+-*ESMS*), if it is stable in the mean-square and if, given $\delta^+ > 0$, there exist a finite $C > 0$ and a value $\nu^+ > 0$ such that, whenever $\mathsf{E}(\sup_{t\in J}|\,\delta\Phi(t)|^2) < \delta^+$, $\mathsf{E}(|\delta X(t)|^2) \leq C\exp\{-\nu^+(t-t_0)\}$(where $\delta X(t) := X(\Phi+\delta\Phi;t) - X(\Phi;t)$) for all t sufficiently large.*

[a]Different notions of stability will not be considered here. (Other notions relate to almost sure behaviour of $\{\delta\widetilde{X}_n\}$, or stability in probability; another class of definitions correspond to *persistent perturbations* – perturbations in the inhomogeneous terms – *e.g.* in $\{f_n\}$ – rather than in Φ.)

The terms ν^+-ESMS and *exponent* $-\nu^+$ appear to be nonstandard. The restriction of the definitions to solutions of DDDE s (omitting the words "mean square") is clear.

2.2 A discrete inequality of Halanay type

We appeal to some results used in [3], to which we refer for discussion and proofs.

Lemma 1 *Denote by* $\mathcal{R}_N(\zeta;\ a,b)$ *the polynomial in* ζ:

$$\mathcal{R}_N(\zeta;\ a,b) := \zeta^{N+1} - (1-ah)\zeta^N - bh \quad (a,b\in\mathbb{R}; N\in\mathbb{N}), \tag{6a}$$

where $h = \tau/N > 0$. *If* $0 \le \beta_h < \alpha_h$ *and* $0 < \alpha_h h < 1$, *the polynomial* $\mathcal{R}_N(\zeta;\ \alpha_h,\beta_h)$ *has a single* positive *zero* ζ_h^+ *where*

$$\zeta_h^+ \in (1-(\alpha_h-\beta_h)h, 1), \quad \textit{if } \beta_h > 0, \quad \textit{and} \quad \zeta_h^+ = 1-\alpha_h h, \textit{ if } \beta_h = 0; \tag{6b}$$

further, $\zeta_h^+ = \exp(-\nu_h^+ h)$ *where* $\nu_h^+ = -\ell n(\zeta_h^+)/h$ *lies in* $(0,\alpha_h]$.

Theorem 1 *Suppose, for some fixed integer* $N \ge 0$, *that* $t_n = t_0 + nh$ *for some* $h > 0$ *and* $\{v_n\}_{-N}^{\infty}$ *is a sequence of positive numbers that satisfies, where*

$$0 \le \beta_h < \alpha_h \textit{ and } 0 < \alpha_h h < 1, \tag{7a}$$

the relation
$$\frac{v_{n+1}-v_n}{h} \le -\alpha_h v_n + \beta_h \max_{\ell\in\mathcal{J}} v_{n+\ell} \textit{ for } n\in\mathbb{N} \tag{7b}$$

with $N = 0$ *if* $\beta_h = 0$. *Then* $v_n \le \left\{\max_{\ell\in\mathcal{J}} v_\ell\right\}\exp\{-\nu_h^+(t_n-t_0)\}$ *where* $\nu_h^+ > 0$ *is the value occurring in Lemma 1.*

Theorem 1 is similar in spirit to a result obtained by Halanay [8] in the context of DDDE s. The form of the result $0 < \nu_h^+ \le \alpha_h$ explains the presence of the scaling factor $1/h$ in (7b) – so that $\{v_{n+1}-v_n\}/h$ then simulates a derivative.

3 Deterministic Insight

Results for deterministic problems yield insight. Consider the DDDE

$$x'(t) = f(t) - \alpha x(t) + \beta x(t-\tau) \quad (\alpha,\beta\in\mathbb{R}). \tag{8}$$

Theorem 2 *Given* $\nu^+ > 0$, *solutions of (8) are* ν^+*–exponentially stable* if and only if *the zeros of the function* $\mathcal{Q}(\zeta;\ \alpha,\beta,\tau) := \zeta + \alpha - \beta\exp(-\zeta\tau)$ *lie in the left half–plane* $\Re(\zeta) \le -\nu^+$; *a* sufficient condition *for* ν_+-ESMS *for some* $\nu_+ > 0$ *is* $|\beta| < -\alpha$.

> **Remark:** The special form of (3.1) allows use of a type of "method of D-partitions" (a boundary locus technique, *cf.* [10]) to determine, given $\nu^+ > 0$, exact regions in $(\alpha\tau,\beta\tau)$ parameter space for which solutions are ν^+-exponentially stable.

In the following proof of Theorem 3, we give an analysis for the deterministic case that can be modified for the stochastic case. Suppose $Nh = \tau$ ($N \in \mathbb{N}$) and $1+\alpha h\theta \ne 0$. The θ–method for (8) gives

$$x_{n+1} - x_n = hf_n^\theta - \alpha h\{\theta x_{n+1} + (1-\theta)x_n\} + \beta h\{\theta x_{n+1-N} + (1-\theta)x_{n-N}\}. \tag{9}$$

Perturbing $\{x_\ell\}_{\ell\in\mathcal{J}}$, we find the consequent perturbations $\{\delta x_\ell\}_{\ell\ge 1}$ satisfy

$$\delta x_{n+1} = \frac{1+(1-\theta)\alpha h}{1+\theta\alpha h}\,\delta x_n + \frac{\beta h\theta}{1+\alpha h\theta}\,\delta x_{n+1-N} + \frac{(1-\theta)\beta h}{1+\alpha h\theta}\,\delta x_{n-N}, \tag{10}$$

for $n \geq 0$. With $\rho_{0,1,2}$ as in (4), $\delta x_{n+1}^2 - \delta x_n^2$ can be expressed as $\Big(\{\varrho_0 + 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 x_{n-N}\Big) \times \Big(\{\varrho_0 - 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 \delta x_{n-N}\Big)$; we deduce that

$$\delta x_{n+1}^2 - \delta x_n^2 = (\varrho_0^2 - 1)\ \delta x_n^2 + 2\varrho_0\varrho_1\ \delta x_n \delta x_{n-N+1} + 2\varrho_0\varrho_2\ \delta x_n \delta x_{n-N} + 2\varrho_1\varrho_2\ \delta x_{n+1-N}\delta x_{n-N} + \varrho_1^2\ \delta x_{n+1-N}^2 + \varrho_2^2\ \delta x_{n-N}^2. \tag{11}$$

If $uv \neq 0$, $|suv| \leq \frac{1}{2}\{v^2 + s^2u^2\}$, with equality if, and only if, $s = v/u$. Thus $|uv| = \inf_{s\in(0,\infty)} \dfrac{1}{2s}\{v^2 + s^2u^2\} \leq \frac{1}{2}\{\dfrac{v^2}{r} + ru^2\}$ for all $r \in (0, \infty)$. Then,

$$|\varrho_r\varrho_s\delta x_j\delta x_k| \leq \frac{|\varrho_r\varrho_s|}{2}\{r_{jk}\delta x_j^2 + \frac{1}{r_{jk}}\delta x_k^2\} \text{ for arbitrary } r_{jk} \in (0,\infty), \tag{12}$$

with equality for some r_{jk}. From (11) and (12) we obtain the inequality

$$\delta x_{n+1}^2 - \delta x_n^2 \leq (\varrho_0^2 - 1)\ \delta x_n^2 + |\varrho_0\varrho_1|\{\frac{1}{r}\delta x_n^2 + r\delta x_{n-N+1}^2\} + |\varrho_0\varrho_2|\{\frac{1}{r'}\delta x_n^2 + r'\delta x_{n-N}^2\} + |\varrho_1\varrho_2|\{\frac{1}{r''}\delta x_{n+1-N}^2 + r''\delta x_{n-N}^2\} + \varrho_1^2\ \delta x_{n+1-N}^2 + \varrho_2^2\ \delta x_{n-N}^2, \tag{13}$$

for arbitrary $r, r', r'' \in (0, \infty)$. Hence,

$$\frac{\delta x_{n+1}^2 - \delta x_n^2}{h} \leq -A^\natural\ \delta x_n^2 + B^\natural\ \max_{\ell\in\mathcal{J}} \delta x_{n-\ell}^2 \quad (\mathcal{J} = \{0, 1, \cdots, N\}), \tag{14}$$

where, for arbitrary positive numbers $\{r, r'\}$ (and choosing $r'' = 1$) we may set

$$A_h^\natural \equiv A_h^\natural(r, r') = -\frac{1}{h}\{\varrho_0^2 - 1 + \frac{|\varrho_0\varrho_1|}{r} + \frac{|\varrho_0\varrho_2|}{r'}\}, \tag{15a}$$

$$B_h^\natural \equiv B_h^\natural(r, r') = \frac{1}{h}\{|\varrho_0\varrho_1|r + |\varrho_0\varrho_2|r' + (|\varrho_1| + |\varrho_2|)^2\}. \tag{15b}$$

($A_h^\natural$ and $B_h^\natural$ are functions of h and are $\mathcal{O}(1)$ as $h \to 0$.) We deduce:

Theorem 3 *For the deterministic case, the recurrence (9) is ν_+–exponentially stable for some $\nu_+ > 0$ if, for any choice of positive r,r', the values in (15) satisfy the conditions $hA^\natural \in (0, 1)$ and $0 \leq B^\natural < A^\natural$.*

> **Remark:** Theorem 3 provides a sufficient condition for ν_+–ESMS, for some $\nu_+ > 0$. However, the recurrence (10) is special; it yields $x_{n+1} = \varrho_0 x_n + \varrho_1 x_{n+1-N} + \varrho_2 x_{n-N} + hf_n^\theta/\{1 + \alpha h\theta\}$ (where $\tau = Nh$) and, given $\nu^+ > 0$, its solutions are ν^+–exponentially stable if and only if the zeros of $S_N(\zeta; \alpha, \beta, \tau) := \zeta^{N+1} - \varrho_0\zeta^N - \varrho_1\zeta - \varrho_2$ lie within or on the circle in the complex plane that is centered on 0 and has radius $\exp(-\nu^+h)$, any on the circle being simple. Thus, the parameters that correspond to ν^+–exponential stability can here be computed by a boundary-locus technique.

4 Simulation of Stability of $X(t)$ by that of $\{\widetilde{X}_n\}$.

We now advance to the stochastic problem. It is natural to ask to what extent the stability of $\{X_n(\Phi)\}$ corresponds to the stability of the true solution $X(\Phi; t)$ that it is assumed to approximate. For (1) we have the following result (see, *e.g.* [3]).

Theorem 4 *Every solution of (1) is ν_+-ESMS for some $\nu_+ > 0$ when (i) $|\beta| < \alpha - \{|\eta|^2 + |\mu|^2\}$, or (ii) $\mu = 0$ and $|\beta| < \alpha - \frac{1}{2}|\eta|^2$.*

We seek an analogue of Theorem 4 for stability of the numerical solutions, given $1+\theta\alpha h \neq 0$. To analyze mean-square stability we first derive a relationship between the expectations $\{\mathsf{E}(\delta\widetilde{X}_n^2)\}$, starting from (5). We seek a suitable relationship

$$\mathsf{E}(\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\delta\widetilde{X}^2_{n}) \leq -\alpha_h h \mathsf{E}(\delta\widetilde{X}^2_{n}) + \beta_h h \max_{\ell\in\mathcal{J}} \mathsf{E}(\delta\widetilde{X}^2_{n-\ell}). \tag{16}$$

Lemma 2 $\mathsf{E}(\,\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\,\delta\widetilde{X}^2_{n})$ *can be written*

$$\begin{aligned}&\Big\{(\varrho_0^2-1)\ \mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varrho_0\varrho_2\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n-N}) + 2\varrho_0\varrho_1\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n+1-N}) +\\ &+2\varrho_1\varrho_2\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varrho_2^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N}) + \varrho_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N})\Big\}\\ &+\Big(\varpi_0^2\mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varpi_0\varpi_1\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varpi_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\Big).\end{aligned} \tag{17}$$

Proof: $\delta\widetilde{X}_{n+1} \pm \delta\widetilde{X}_n = (\varrho_0 \pm 1 + \varpi_0\xi_n)\delta\widetilde{X}_n + \varrho_1\delta\widetilde{X}_{n+1-N} + (\varrho_2 + \varpi_1\xi_n)\,\delta\widetilde{X}_{n-N}$. Hence, for appropriate coefficients $\mathfrak{a}_i$ etc. that are functions of $\{\varrho_i\}$ and $\{\varpi_j\}$ (and hence of α, β, η, μ, and h), $\delta\widetilde{X}^2_{n+1} - \delta\widetilde{X}^2_{n} = \{\mathfrak{a}_0 + \mathfrak{a}_1\xi_n + \mathfrak{a}_2\xi_n^2\}\delta\widetilde{X}^2_{n} + \{\mathfrak{b}_0' + \mathfrak{b}_1'\xi_n + \mathfrak{b}_2'\xi_n^2\}\delta\widetilde{X}_n\delta\widetilde{X}_{n+1-N} + \{\mathfrak{b}_0''+\mathfrak{b}_1''\xi_n+\mathfrak{b}_2''\xi_n^2\}\delta\widetilde{X}_n\delta\widetilde{X}_{n-N} + \{\mathfrak{b}_0'''+\mathfrak{b}_1'''\xi_n+\mathfrak{b}_2'''\xi_n^2\}\delta\widetilde{X}_{n-N}\delta\widetilde{X}_{n-N+1}$ $+\{\mathfrak{c}_0+\mathfrak{c}_1\xi_n+\mathfrak{c}_2\xi_n^2\}\delta\widetilde{X}^2_{n+1-N} + \{\mathfrak{d}_0+\mathfrak{d}_1\xi_n+\mathfrak{d}_2\xi_n^2\}\delta\widetilde{X}^2_{n-N}$. (We note that $\mathfrak{b}_2' = \mathfrak{b}_2''' = \mathfrak{c}_2 = 0$, and the coefficients with index 0 arise in the deterministic case.) If $-N \leq r, s \leq n$ $(r, s \in \mathbb{N})$, we have $\mathsf{E}(\xi_n\delta\widetilde{X}_r\delta\widetilde{X}_s) = 0$ and $\mathsf{E}(\xi_n^2\delta\widetilde{X}_r\delta\widetilde{X}_s) = \mathsf{E}(\delta\widetilde{X}_r\delta\widetilde{X}_s)$, and so the coefficients with index 1 vanish when we take expectations in the expression for $\delta\widetilde{X}^2_{n+1} - \delta\widetilde{X}^2_{n}$, and obtain

$$\begin{aligned}&\mathsf{E}(\,\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\,\delta\widetilde{X}^2_{n}) =\\ &\{\mathfrak{a}_0 + \mathfrak{a}_2\}\ \mathsf{E}(\,\delta\widetilde{X}^2_{n}) + \mathfrak{b}_0'\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n-N}) + (\mathfrak{b}_0'' + \mathfrak{b}_2'')\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n+1-N}) +\\ &\mathfrak{b}_0'''\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \mathfrak{c}_0\mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N}) + \{\mathfrak{d}_0 + \mathfrak{d}_2\}\mathsf{E}(\,\delta\widetilde{X}^2_{n-N}).\end{aligned} \tag{18}$$

Expressing the coefficients in (18) in terms of (4) we establish the lemma.

4.1 Application of the general Halanay-type theory

Eq. (17) reduces to (11) in the deterministic case, and the first term – the term in braces $\Big\{\ \Big\}$ – in (17) can be treated in the manner used to prove (13) from (11), when obtaining (14). We now bound the terms in (18) that involve $\varpi_{0,1}$, using $|2\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N})| \leq \mathsf{E}(\,\delta\widetilde{X}^2_{n-N}) + \mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N})$, to obtain

$$\begin{aligned}&\left|\varpi_0^2\mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varpi_0\varpi_1\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varpi_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\right| \leq\\ &\quad(\varpi_0^2 + 2|\varpi_0\varpi_1| + \varpi_1^2)\sup\big\{\mathsf{E}(\,\delta\widetilde{X}^2_{n}), \mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N}), \mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\big\}.\end{aligned} \tag{19}$$

We thus obtain a delay-difference inequality of Halanay type, and hence, using Theorem 1, a condition for ν_+-ESMS:

Theorem 5 *Given arbitrary positive numbers* $\{r, r'\}$, *set* $A_h(r,r') = A_h^\natural(r,r')$, $B_h(r,r') = B_h^\natural(r,r') + (|\varpi_0| + |\varpi_1|)^2$, *where* $A_h^\natural(r,r')$ *and* $B_h^\natural(r,r')$ *are the values in (15), the deterministic case. Then*

$$\frac{\mathsf{E}(\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\delta\widetilde{X}^2_{n})}{h} \leq -A_h(r,r')\ \mathsf{E}(\delta\widetilde{X}^2_{n}) + B_h(r,r')\ \max_{\ell\in\mathcal{J}}\mathsf{E}(\delta\widetilde{X}^2_{n-\ell}), \tag{20}$$

where $\mathcal{J} = \{0, 1, \cdots, N\}$. *If, for any* $r, r' \in (0,\infty)$, $0 < hA_h(r,r') < 1$ *and* $0 \leq B_h(r,r') < A_h(r,r')$, $\{\widetilde{X}_n(\Phi)\}$ *is* $\nu^+(r,r')$-ESMS *for a value* $\nu^+(r,r') > 0$.

The condition $hA_h(r,r') \in (0,1)$ is a condition in Theorem 3.
Given (r,r') for which $hA_h(r,r') \in (0,1)$ and $B_h(r,r') \in (0, A_h(r,r'))$, an estimate of $\nu^+(r,r')$ can obtained (by Lemma 1) from the positive zero $\zeta^+(r,r')$ of $\mathcal{R}_N(\zeta; A_h(r,r'), B_h(r,r'))$; in principle, one can then seek the maximum value $\nu^+(r,r')$ over all such pairs (r,r').
It is clear that to emulate the result $|\beta| < -\alpha + |\eta|^2 + |\mu|^2$, with $\alpha \in (0,\infty)$, that holds in the case of the test equation (1) (*cf.* Theorem 4 (*i*)) it is advantageous if $\varrho_0 \to 0$ as $\alpha \to \infty$. Thus, when considering ν-ESMS properties, it appears that $\theta \in (\frac{1}{2}, 1]$ (corresponding to an underlying L-stable deterministic θ-formula) may be preferable to $\theta = \frac{1}{2}$ (where the deterministic formula is only A-stable and $|\varrho_0| \to 1$ as $\alpha \to \infty$) or to $\theta \in [0, \frac{1}{2})$. However, an L-stable formula can be stable when the DDDE is unstable.

5 Summary

Affine constant-coefficient test equations with constant lags (such as that in (1)) are special, and they allow a more complete stability analysis than is in general possible. A justification of the use of (1a) for insight for more general equations can be formulated if the theory of approximating linear equations is analyzed; theories involving approximation by deterministic problems can also be found in the literature [10]. Theorems 3 and 5 are new, and they accomplish our main objective of demonstrating the applicability of Halanay-type inequalities. However, restrictions on space have limited our discussion; it has not been possible to demonstrate the advantages of an approach based upon Halanay-type inequalities. These lie in the opportunity to consider solutions of test equations with time-dependent coefficients and lags and certain types of non-linearity. On the other hand, the perceived advantages come at a price, *e.g.* some loss of precision in special cases such as those where necessary and sufficient conditions can be found from other approaches.

References

[1] C. T. H. Baker, *Retarded differential equations*, *J. Comput. Appl. Math.* **125** (2000) 309–335.

[2] C. T. H. Baker and E. Buckwar, *Numerical analysis of explicit one-step methods for stochastic delay differential equations*, *LMS J. Comput. Math.* **3** (2000) 315–335. URL: `http://www.lms.ac.uk/jcm/`

[3] C. T. H. Baker and E. Buckwar, *Exponential Stability in p-th Mean of Solutions, and of Convergent Euler-type Solutions, of Stochastic Delay Differential Equations* (submitted for publication); see also MCCM Technical Report 390, 2001/2002. ISSN: 1360 1725. URL: `http://www.maths.man.ac.uk/~nareports/`

[4] C. T. H. Baker and A. Tang, *Stability analysis of continuous implicit Runge-Kutta methods for Volterra integro-differential systems with unbounded delays*, *Appl. Numer. Math.* **24** (1997) 153–173.

[5] A. Bellen and M. Zennaro, *Numerical methods for delay differential equations* (Oxford University Press, 2003).

[6] E. Buckwar and R. Winkler *Multi-step Maruyama methods for stochastic delay differential equations*, submitted for publication.

[7] Wanrong Cao, Mingzhu Liu and Zhencheng Fan, *MS-stability of the Euler-Maruyama method for stochastic differential delay equations*, *Applied Mathematics and Computation* (in press, available online 27 November 2003).

[8] A. Halanay, *Differential Equations — Stability, Oscillations, Time Lags.* (Academic Press, New York, 1966).

[9] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, *SIAM J. Numer. Anal.* **38** (2000) 753–769.

[10] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations* (Academic Press, New York, 1986).

[11] X. Mao, *Stochastic Differential Equations and their Applications* (Horwood Publishing Ltd., Chichester, 1997).

[12] A. Tang, *Analysis and numerics of delay Volterra integro-differential equations.* PhD thesis, Manchester University, 1995.