On Artin's L-functions. II: Dirichlet Coefficients

By *Florin Nicolae* at Berlin and Bucharest

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Let a_n, $n \geq 1$, be complex numbers. If for a real number θ the summatory function $\sum_{n \leq x} a_n$ is $\mathbf{O}(x^\theta)$ as $x \to \infty$, then the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges in the half-plane $\mathrm{Re}\,(s) > \theta$. For a non-principal Dirichlet character modulo m it holds for every $x \geq 1$

$$|\sum_{n \leq x} \chi(n)| \leq \varphi(m),$$

so $\sum_{n \leq x} \chi(n)$ is $\mathbf{O}(1)$ and the series $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges for $\mathrm{Re}\,(s) > 0$. The main result of this paper is

Theorem. *Let $K/\mathbb{Q}$ be a finite Galois extension, let χ be a character of the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ which does not contain the principal character, let $L_{ur}(s, \chi, K/\mathbb{Q})$ be the unramified part of the corresponding Artin L-function, and let*

$$L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{\chi(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for $\mathrm{Re}\,(s) > 1$. *Then:*
(i) The coefficients a_n are algebraic numbers of the field $\mathbb{Q}(e^{\frac{2\pi i}{|G|}})$ and $|a_n| \leq 1$ for every $n \geq 1$;
(ii) The summatory function $\sum_{n \leq x} a_n$ is $\mathbf{o}(x)$ as $x \to \infty$.

The theorem says in the case $K = \mathbb{Q}(e^{\frac{2\pi i}{m}})$ that $\sum_{n \leq x} \chi(n)$ is $\mathbf{o}(x)$ as $x \to \infty$, which is weaker than: $\sum_{n \leq x} \chi(n)$ is $\mathbf{O}(1)$. It should be seen as an assertion about a general Artin L-function in the case of a non-abelian extension $K/\mathbb{Q}$.

An evaluation $\mathbf{O}(x^\theta)$ with $\theta < 1$ of the summatory function of the Dirichlet coefficients of an Artin L-function had as a consequence that its Dirichlet series converges in $\mathrm{Re}\,(s) > \theta$, so the L-function is holomorphic in that half plane, which would be an important step towards Artin's holomorphy conjecture. This is not reached here. It is the motivation for future research.

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P r o o f of the theorem: Let d_K be the discriminant of the algebraic number field K, and let ([2], P. 297, (7))

$$L_{ur}(s,\chi,K/\mathbb{Q})) := \prod_{(p,d_K)=1} \frac{1}{\det(I - \frac{A_p}{p^s})} \tag{1}$$

be the product of the Euler factors of $L(s,\chi,K/\mathbb{Q})$ corresponding to the unramified primes p. This is absolutely convergent in $\mathrm{Re}\,(s) > 1$. The dimension of the matrix A_p is $r = \chi(1)$, and it holds

$$\det(I - \frac{A_p}{p^s}) = \prod_{j=1}^{r}(1 - \frac{\zeta_{j,p}}{p^s}),$$

where $\zeta_{1,p},\ldots,\zeta_{r,p}$ are the eigenvalues of A_p, which are roots of unity of order dividing the order $|G|$ of the Galois group G. By Newton's binomial formula it holds for $\mathrm{Re}\,(s) > 0$

$$\frac{1}{\det(I - \frac{A_p}{p^s})^{\frac{1}{r}}} = \prod_{j=1}^{r} \frac{1}{(1 - \frac{\zeta_{j,p}}{p^s})^{\frac{1}{r}}} = \prod_{j=1}^{r} (1 - \frac{\zeta_{j,p}}{p^s})^{-\frac{1}{r}} = \tag{2}$$

$$= \prod_{j=1}^{r}[1 + \sum_{n=1}^{\infty} \frac{-\frac{1}{r}\cdot(-\frac{1}{r}-1)\cdot\ldots\cdot(-\frac{1}{r}-n+1)}{1\cdot 2\cdot\ldots\cdot n}(-\frac{\zeta_{j,p}}{p^s})^n] =$$

$$= \prod_{j=1}^{r}[1 + \sum_{n=1}^{\infty} \frac{1\cdot(1+r)\cdot\ldots\cdot(1+(n-1)r)}{r^n\cdot n!}\frac{\zeta_{j,p}^n}{p^{ns}}] = 1 + \sum_{k=1}^{\infty}\frac{a_{p^k}}{p^{ks}},$$

where for $k \geq 1$

$$a_{p^k} = \sum_{k_1\geq 0,\ldots,k_r\geq 0, k_1+\ldots+k_r=k} \prod_{j=1}^{r} c_{k_j}\zeta_{j,p}^{k_j} = \sum_{k_1\geq 0,\ldots,k_r\geq 0, k_1+\ldots+k_r=k} \prod_{j=1}^{r} c_{k_j} \cdot \prod_{j=1}^{r}\zeta_{j,p}^{k_j},$$

with

$$c_0 = 1,\ c_l = \frac{\prod_{u=1}^{l}(1+(u-1)r)}{r^l\cdot l!},\ l \geq 1, \tag{3}$$

hence
(4)
$$|a_{p^k}| \leq \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} |\prod_{j=1}^{r} c_{k_j} \cdot \prod_{j=1}^{r} \zeta_{j,p}^{k_j}| = \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \prod_{j=1}^{r} c_{k_j}.$$

Lemma. *For $r \geq 1$ and $k \geq 1$ it holds*

$$\sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \prod_{j=1}^{r} c_{k_j} = 1. \tag{5}$$

P r o o f of the lemma: For $l \geq 0$ let $d_l = r^l \cdot l! \cdot c_l$. By (3) it holds

$$d_0 = 1,\ d_l = \prod_{u=1}^{l}(1+(u-1)r),\ l \geq 1. \tag{6}$$

For $k_1 \geq 0, \ldots, k_r \geq 0$ with $k_1 + \ldots + k_r = k$ it holds

$$\prod_{j=1}^{r} c_{k_j} = \frac{\prod_{j=1}^{r} d_{k_j}}{r^k \cdot k_1! \cdot \ldots \cdot k_r!} = \frac{1}{r^k \cdot k!}\binom{k}{k_1 \ldots k_r}\prod_{j=1}^{r} d_{k_j},$$

where

$$\binom{k}{k_1 \ldots k_r} = \frac{k!}{k_1! \cdot \ldots \cdot k_r!}$$

is the multinomial coefficient, so (5) is equivalent to

$$\sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \binom{k}{k_1 \ldots k_r}\prod_{j=1}^{r} d_{k_j} = r^k \cdot k!. \tag{7}$$

For $k = 1$ it holds by (6)

$$d_1 + \ldots + d_1 = r \cdot d_1 = r \cdot 1,$$

so (7) is true. Suppose that (7) is true for a number $k \geq 1$. Then

$$r^{k+1} \cdot (k+1)! = r(k+1)r^k \cdot k! = \tag{8}$$

$$= r(k+1) \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \binom{k}{k_1 \ldots k_r}\prod_{j=1}^{r} d_{k_j} =$$

$$= \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} (rk+r)\binom{k}{k_1 \ldots k_r}\prod_{j=1}^{r} d_{k_j} =$$

$$= \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \binom{k}{k_1 \ldots k_r}[(rk_1+1)+\ldots+(rk_r+1)]\prod_{j=1}^{r} d_{k_j}.$$

By (6) it holds for $l \geq 0$

$$d_{l+1} = (1+lr)d_l,$$

so (8) is equivalent to

$$(9)\quad r^{k+1}\cdot(k+1)! = \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \binom{k}{k_1 \ldots k_r}\sum_{j=1}^{r} d_{k_1}\cdot\ldots\cdot d_{k_j+1}\cdot\ldots\cdot d_{k_r}.$$

It holds

$$(10)\qquad \sum_{k_1\geq 0,\ldots,k_r\geq 0,k_1+\ldots+k_r=k} \binom{k}{k_1 \ldots k_r}\sum_{j=1}^{r} d_{k_1}\cdot\ldots\cdot d_{k_j+1}\cdot\ldots\cdot d_{k_r} =$$

$$= \sum_{l_1\geq 0,\ldots,l_r\geq 0,l_1+\ldots+l_r=k+1} [\sum_{j=1,\, l_j\geq 1}^{r} \binom{k}{l_1 \ldots l_j-1 \ldots l_r}]d_{l_1}\cdot\ldots\cdot d_{l_r} =$$

$$= \sum_{l_1\geq 0,\ldots,l_r\geq 0,l_1+\ldots+l_r=k+1} \binom{k+1}{l_1 \ldots l_r}d_{l_1}\cdot\ldots\cdot d_{l_r}.$$

From (8), (9) and (10) it follows that (7) is true for $k+1$. □

By (4) and the preceding lemma it holds for every $k \geq 1$

$$(11)\qquad |a_{p^k}| \leq 1.$$

Let $a_1 := 1$, $a_n := 0$ for $(n, d_K) \neq 1$, $a_n := a_{p_1}^{k_1}\cdot\ldots\cdot a_{p_m}^{k_m}$ for $n = p_1^{k_1}\cdot\ldots\cdot p_m^{k_m} > 1$, $(n, d_K) = 1$. Then it holds by (1) and (2)

$$L_{ur}(s,\chi,K/\mathbb{Q})^{\frac{1}{\chi(1)}} = \sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

with

$$(12)\qquad a_n \in \mathbb{Q}(e^{\frac{2\pi i}{|G|}}),\ |a_n| \leq 1,$$

for every $n \geq 1$. The first part of the theorem is proved. For the second part apply

The Tauberian Theorem of Wiener-Ikehara. *Let $G(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ be a Dirichlet series. Suppose there exists a Dirichlet series $H(s) = \sum_{n=1}^{\infty}\frac{b_n}{n^s}$ with positive real coefficients such that*
a) $|a_n| \leq b_n$ for all $n \geq 1$;

b) the series $H(s)$ converges for $\mathrm{Re}\,(s) > 1$*;*
c) the function $H(s)$ (respectively $G(s)$) can be extended to a meromorphic function in the region $\mathrm{Re}\,(s) \geq 1$ *having no poles except (respectively except possibly) for a simple pole at $s = 1$ with residue $C \geq 0$ (respectively c).*
Then

$$\sum_{n \leq x} a_n = cx + \mathbf{o}(x)$$

as $x \to \infty$. In particular, if $G(s)$ is holomorphic at $s = 1$, then $c = 0$ and $\sum_{n \leq x} a_n = \mathbf{o}(x)$ as $x \to \infty$. ([3], Theorem 1.1, P. 7)

Take

$$G(s) = L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{\chi(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$H(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The assumptions of the tauberian theorem of Wiener-Ikehara are satisfied:
a) It was proved above that $|a_n| \leq 1$ for every $n \geq 1$;
b)the series $H(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $\mathrm{Re}\,(s) > 1$;
c)the Riemann zeta function $H(s) = \zeta(s)$ is meromorphic in the whole complex plane with exactly one pole at $s = 1$, which is simple with residue $C = 1 \geq 0$. The Artin L-function $L_{ur}(s, \chi, K/\mathbb{Q})$ is holomorphic and has no zeroes for $\mathrm{Re}\,(s) \geq 1$ ([1], Satz 3, P. 105), so the function $L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{\chi(1)}}$ is defined and holomorphic for $\mathrm{Re}\,(s) \geq 1$. It has no pole at $s = 1$: $c = 0$.
By the theorem of Wiener-Ikehara it holds

$$\sum_{n \leq x} a_n = \mathbf{o}(x).$$

□

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Humboldt-Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25, D-10099 Berlin
e-mail:nicolae@mathematik.hu-berlin.de

and

Institute of Mathematics of the Romanian Academy, P.O.BOX 1-764 RO 70700 Bucharest 1