On linear differential-algebraic equations with properly stated leading term.[1]

I: Regular points

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Abstract

We consider in this work linear, time-varying differential-algebraic equations (DAEs) of the form $A(t)(D(t)x(t))' + B(t)x(t) = q(t)$ through a projector approach. Our analysis applies in particular to linear DAEs in standard form $E(t)x'(t) + F(t)x(t) = q(t)$. Under mild smoothness assumptions, we introduce local regularity and index notions, showing that they hold uniformly in intervals and are independent of projectors. Several algebraic and geometric properties supporting these notions are addressed. This framework is aimed at supporting a complementary analysis of so-called critical points, where the assumptions for regularity fail. Our results are applied here to the analysis of a linear time-varying analogue of Chua's circuit with current-controlled resistors, displaying a rich variety of indices depending on the characteristics of resistive and reactive devices.

Keywords: differential-algebraic equation, index, projector, Chua's circuit.

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1 Introduction

We deal in this paper with linear, time-varying differential-algebraic equations of the form

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{J}, \tag{1}$$

where $\mathcal{J} \subseteq \mathbb{R}$ is an interval, and the matrix coefficients $A(t) \in L(\mathbb{R}^n, \mathbb{R}^m)$, $D(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$, $B(t) \in L(\mathbb{R}^m)$ depend continuously on t. The special form of the leading term $A(t)(D(t)x(t))'$ has been recently introduced coming from symmetry demands in adjoint problems, and also from applications in control and circuit theory. This formulation displays nice numerical properties, and opens a way for the study of abstract DAEs and linear PDAEs. See [1, 6, 10, 11, 12, 17].

Note that the form (1) encompasses (and therefore all results will be applicable to) "classical" or standard form linear DAEs

$$E(t)x'(t) + F(t)x(t) = q(t) \tag{2}$$

with $E(t),\ F(t) \in L(\mathbb{R}^m)$, as discussed in Section 2. The special form of the leading term in (1) helps to figure out exactly which components need to be differentiated in $x(t)$; a brief digression in subsection 3.3 will illustrate the importance of this when we look at (1) or (2) as input-output systems, specially if we are also interested in formulating inverse models.

Recent works [10, 11, 12] introduce a notion of regularity, a tractability index, and solvability results for problems (1) defined on a given interval. Generally speaking, these are based on the construction of a matrix chain $\{G_i\}$ supported on a sequence of projectors $\{Q_i\}$, and the decoupling of the DAE in terms of certain components of x once an invertible G_μ is met. This emanates from previous projector techniques [5, 9], but the form (1) allows for improved results and much better clarity, as well as for the treatment of problems with arbitrary index.

In this context, the main goal of this paper stems from the remark that the regularity and index notions in [10, 11, 12] apply to linear DAEs defined *on a given time interval.* We introduce here the local version of this, defining *regular points* in a way which will be proved independent of projectors. We will show that this local notion is *uniform*, in the sense that if every point in a given interval is regular, then the DAE is regular on the whole interval with the same characteristic values, and can be handled via operators which are well-defined on the whole interval. This will be carried out in a constructive manner and under mild smoothness requirements. An important consequence is that a uniform treatment is possible in maximal regularity intervals, and that this yields a unique and independent of projectors decomposition of the interval in a (possibly infinite) number of well-defined regularity subintervals. The ultimate aim of this approach is to define consistently and analyze *critical points*; this is addressed in the companion work [13].

Reduction techniques [14, 15] and the strangeness index framework [7, 8] rely on certain algebraic (typically constant rank) conditions. We prove certain results which clarify the algebraic conditions supporting the above-mentioned matrix chain construction. These conditions have in turn a geometrical counterpart, related with certain linear subspaces and, specifically, how these intersect and merge among each other. The technicalities involved in this can be roughly seen as the result of a somehow "minimalistic" spirit, which is apparent in the fact that only one (partly arbitrary) operator $Q_i(t)$ needs to be introduced per step, and also in the moderate smoothness requirements.

At every level of the chain construction, a transversality hypothesis will be needed in combination with a constant rank one. This will lead to the concepts of *preadmissible* projectors and *algebraically nice* DAEs at a given level. The notion of an algebraically nice DAE will be proved independent of the specific choice of preadmissible projectors, and will be shown to be characterizable in pure "constant rank" terms, somehow bridging the gap with strangeness and reduction approaches. Note in this regard that derivative-array techniques [2, 3] are different in spirit and require stronger smoothness assumptions. When smoothness requirements are met, the above-mentioned concepts will be transformed into *admissible* projectors and *nice* DAEs.

The rest of the paper is structured as follows. Section 2 introduces the concept of a properly stated leading term and some additional basic notions. In Section 3 we discuss extensively the notions of preadmissible and admissible projectors, as well as algebraically nice, nice and regular DAEs, and show how to unveil the behavior of the DAE in terms of a decoupling, for regular problems with arbitrary index. Section 4 addresses local niceness and regularity notions, and proves the above-mentioned uniformity property. This framework is applied to a linear time-varying analogue of Chua's circuit in Section 5, displaying several different indices depending on device characteristics. Finally, we stress in Section 6 that the dynamics in all regular subintervals can be described in terms of a uniquely defined inherent explicit ODE, in virtue of the fact that this description is feasible in the original setting (loosely speaking, the x-coordinates, without transformations) of the DAE. This holds regardless of the (possibly different) characteristic values and index in different subintervals; the discontinuities, singularities, etc. of this universal explicit ODE at the boundaries of regularity intervals provide additional motivation for the study of critical points carried out in [13].

2 Properly stated leading terms

When considering linear DAEs of the form (1), a natural question concerns their relation with the more common formulation (2). Focusing on the latter, if there exists a C^1 projector $P(t)$ along $\ker E(t)$, we may write

$$E(t)(P(t)x(t))' + [F(t) - E(t)P'(t)]x(t) = q(t), \tag{3}$$

and the equation takes the form (1). Such a projector P exists in particular if E is C^1 with constant rank. We emphasize that (3) is a particular instance of (1) and that $D(t)$ need not be a projector.

Conversely, if $D(t)$ is C^1, and we restrict the attention to C^1 solutions, (1) can be rewritten in the form (2) simply as

$$A(t)D(t)x'(t) + [A(t)D'(t) + B(t)]x(t) = q(t). \tag{4}$$

We avoid doing this even if $D(t)$ is C^1 for the reasons discussed in 3.3.

Definition 1 *The leading term of the DAE (1) is properly stated on the interval $\mathcal{I} \subseteq \mathcal{J}$ if the coefficients A and D are well matched there in the sense that*

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \quad t \in \mathcal{I}, \tag{5}$$

holds true, and both subspaces are spanned by continuously differentiable basis functions, i.e., $\operatorname{im} D(t) = span\{\eta_1(t), \ldots, \eta_r(t)\}$, $\ker A(t) = span\{\eta_{r+1}(t), \ldots, \eta_n(t)\}$, with $\eta_i \in C^1(\mathcal{I}, \mathbb{R}^n)$, $i = 1, \ldots, n$.

If $\mathcal{I} \subseteq \mathcal{J}$ is an interval where the leading term is properly stated, then $A(t), D(t)$ and $A(t)D(t)$ have constant rank r on $\mathcal{I}$. Denoting by $\mathcal{H}(t)$ the $n \times n$ matrix consisting of the columns $\eta_1(t), \ldots, \eta_n(t)$ we find

$$R(t) := \mathcal{H}(t) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \mathcal{H}(t)^{-1} \tag{6}$$

to be continuously differentiable in t, and $R(t)^2 = R(t)$, $\operatorname{im} R(t) = imD(t)$, $\ker R(t) = \ker A(t)$, $t \in \mathcal{I}$, i.e., R is the projector function that realizes the decomposition (5).

The matrices $A(t)$ and $D(t)$ are singular in general. They may be rectangular. Invertible $A(t)$ and $D(t)$ are just very special cases. Clearly, if $A(t)$ and $D(t)$ are invertible on $\mathcal{I}$, we have simply $n = m$, $R = I_n$.

Often, semi-explicit DAEs

$$M(t)x_1'(t) + B_{11}(t)x_1(t) + B_{12}(t)x_2(t) = q_1(t) \tag{7a}$$
$$B_{21}(t)x_1(t) + B_{22}(t)x_2(t) = q_2(t), \tag{7b}$$

are considered, where $t \in \mathcal{J}$ and $M(t)$ is an $r \times r$ matrix that is invertible for all t. Taking either

$$n = r, \qquad A = \begin{bmatrix} M \\ 0 \end{bmatrix}, \quad D = [I_r \;\; 0], \quad R = I_r, \quad \text{or}$$

$$n = m, \qquad A = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

we realize (2.4) to be properly stated on $\mathcal{J}$.

Definition 2 *A continuous function $x : \mathcal{I} \to \mathbb{R}^m$ is said to be a solution of equation (1) on $\mathcal{I}$ if $Dx \in C^1(\mathcal{I}, \mathbb{R}^n)$ and equation (1) is satisfied for all $t \in \mathcal{I}$. Denote by*

$$C^1_D(\mathcal{I}, \mathbb{R}^m) := \{x \in C(\mathcal{I}, \mathbb{R}^m) : Dx \in C^1(\mathcal{I}, \mathbb{R}^n)\}$$

the corresponding function space.

Definition 3 *For $i \in \mathbb{N} \cup \{0\}$, a time-varying subspace $L(t) \subset \mathbb{R}^l$, $t \in \mathcal{I}$, is said to be a C^i-subspace on $\mathcal{I}$ if $L(t)$ has constant dimension and is spanned by basis functions that belong to $C^i(\mathcal{I}, \mathbb{R}^l)$.*

Example 1 Consider the DAE

$$\alpha(t)(\delta(t)x_1(t))' + x_1(t) = q_1(t), \tag{8a}$$

$$x_2(t) = q_2(t), \tag{8b}$$

with $t \in [-1, 1]$, α, $\delta \in C([-1, 1], \mathbb{R})$. This DAE has the form (1) with $A = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, $D = [\delta \;\; 0]$, $B = I$, $m = 2$, $n = 1$.

On all subintervals where the product $\alpha(t)\delta(t)$ remains either positive or negative, and on subintervals where both functions vanish identically, the leading term is properly stated. Border points between those intervals will be treated as critical ones. In particular, if $\alpha(t) = t^2$, $\delta(t) = 1$, this DAE has a properly stated leading term on $[-1, 0)$ and on $(0, 1]$, and $t_* = 0$ is a critical point that corresponds to a singular point of the ODE (8a) with respect to x_1. Note that the case $\alpha(t) = t^2$, $\delta(t) = t$ shows that the direct sum in (5) can hold for all t in $\mathcal{I}$ even without constant rank on A, D. If $\alpha(t) = \delta(t) = 0$ for $t \in [-1, 0]$, $\alpha(t) = t^2, \delta(t) = t$ for $t \in (0, 1]$, the DAE has a properly stated leading term on $[-1, 0]$ and $(0, 1]$. Again, $t_* = 0$ is a critical point. Now it corresponds not only to a singularity of the ODE with respect to x_1, but also to a change of degree of freedom.

3 Nice and regular DAEs on intervals

Following [10] we form a sequence of subspaces and matrix functions for the DAE (1). Assuming the leading term of (1) to be stated properly on the interval $\mathcal{I} \subseteq \mathcal{J}$, we continue to use the projector function R given by (6). Additionally, we choose a continuous in t generalized inverse $D(t)^-$ of $D(t)$ such that

$$D(t)D(t)^-D(t) = D(t), \;\; D(t)^-D(t)D(t)^- = D(t)^-, \;\; D(t)D(t)^- = R(t),$$

for all $t \in \mathcal{I}$.

For shortness and more transparency we mostly drop the argument t; the given relations are meant pointwise then. Introduce

$$G_0 := AD, \;\; N_0 := \ker G_0, \;\; P_0 := D^- D, \;\; Q_0 := I - P_0, \;\; B_0 := B, \tag{9}$$

and, for $i \geq 0$,

$$\begin{aligned} G_{i+1} &:= G_i + B_i Q_i, \\ & \quad N_{i+1} := \ker G_{i+1}, \\ & \quad Q_{i+1}^2 = Q_{i+1}, \ \operatorname{im} Q_{i+1} = N_{i+1}, \ P_{i+1} := I - Q_{i+1}, \\ B_{i+1} &:= B_i P_i - G_{i+1} D^- (D P_0 \cdots P_{i+1} D^-)' D P_0 \cdots P_i. \end{aligned} \tag{10, 11}$$

By construction, the sequence (10)-(11) depends on how the projectors Q_0 (resp. D^-) and Q_i, $i \geq 1$, are chosen. By now, the projectors Q_i, $i \geq 1$, are determined just by their range $\operatorname{im} Q_i = N_i$. In order for them to be useful in decoupling the DAE, we restrict the remaining multiple possibilities to choose these projectors by the demand that they verify $Q_i Q_j = 0$, for all $0 \leq j < i$ and all $t \in \mathcal{I}$. This condition is required to ensure that the products $P_0 P_1, \ldots, P_0 \cdots P_i, P_0 Q_1, \ldots, P_0 \cdots P_{i-1} Q_i$ are projectors, too. Such a sequence $Q_0, \ldots, Q_k$ will generate a decomposition of $\mathbb{R}^m$ given by

$$I = P_0 \cdots P_k + P_0 \cdots P_{k-1} Q_k + \cdots + P_0 Q_1 + Q_0. \tag{12}$$

In the sequel, all subspaces related to (12) will be continuous. The decomposition (12) in its turn induces the decomposition

$$R = D P_0 D^- = D P_0 \cdots P_k D^- + D P_0 \cdots P_{k-1} Q_k D^- + \cdots + D P_0 Q_1 D^- \tag{13}$$

of the C^1-subspace $\operatorname{im} D$.

We elaborate on this key condition $Q_i Q_j = 0$ in the next subsection. To build the matrix chain (10)-(11), we will additionally require a constant rank condition in G_i, and will also have to take care of the existence of the derivative involved in (11), what is in turn closely related to the decomposition depicted in (13).

3.1 Admissible projector sequences

To get some insight into the above-mentioned conditions $Q_i Q_j = 0$, assume we have chosen a projector Q_0 onto $N_0 = \ker G_0$. In the next step, we wonder if there exists a Q_1 onto $N_1 = \ker G_1$ satisfying $Q_1 Q_0 = 0$. If it does, the next step raises the double condition $Q_2 Q_0 = 0$, $Q_2 Q_1 = 0$. If no such a Q_2 exists, we wonder if a different choice of Q_0, Q_1 could make these conditions hold; note that, changing Q_0, Q_1, we would have to check again the previous condition $Q_1 Q_0 = 0$, etc.

Fortunately, this problem is simpler than it seems. First, we remark that, given Q_0, the existence of a Q_1 onto N_1 satisfying $Q_1 Q_0 = 0$ means $N_0 \subseteq \ker Q_1$, and only requires $N_0 \cap N_1 = \{0\}$. In turn, this supports writing $N_0 + N_1$ as $N_0 \oplus N_1$, and makes it possible to state the double condition $Q_2 Q_0 = 0$, $Q_2 Q_1 = 0$ as $N_0 \oplus N_1 \subseteq \ker Q_2$, relying on $(N_0 \oplus N_1) \cap N_2 = \{0\}$, etc. This is acknowledged in the next definition; it will be then shown in Proposition 2 that these transversality conditions $(N_0 \oplus \cdots \oplus N_{i-1}) \cap N_i = \{0\}$ are independent of the choice of the projectors up to Q_{i-1}.

Definition 4 *Let the DAE (1) have a properly stated leading term on $\mathcal{I} \subseteq \mathcal{J}$. Any continuous projector Q_0 onto $N_0 = \ker G_0$ will be said to be* admissible *or* admissible up to level 0.

The sequence $Q_0, \ldots, Q_i$ with $i \in I\!N$, built for the DAE (1) on $\mathcal{I}$, is said to be preadmissible *(on $\mathcal{I}$) up to level k, $1 \le k \le i$, if*

(i) it is admissible up to level $k-1$;

(ii) G_k has constant rank r_k, and Q_k is continuous;

(iii) $N_0 \oplus \cdots \oplus N_{k-1} \subseteq \ker Q_k$.

Additionally, it is called admissible *(on $\mathcal{I}$) up to level k if it is preadmissible up to level k and*

(iv) $DP_0 \cdots P_k D^-$ is continuously differentiable.

The admissibility definition for Q_0 is supported on the fact that, if the leading term is properly stated, then G_0 has constant rank $r_0 = r$, the continuity of D^- implies that of Q_0, and $DP_0D^- = DD^- = R$ is continuously differentiable. In this context, the demand for a preadmissible sequence $Q_0, \ldots, Q_k$ to be admissible accounts for the smoothness requirement in the matrix chain construction, and means that the related further partition of the C^1-subspace $\operatorname{im} D$ is given by C^1-subspaces related to $DP_0 \cdots P_k D^-$ and $DP_0 \cdots P_{k-1} Q_k D^- = DP_0 \cdots P_{k-1} D^- - DP_0 \cdots P_k D^-$.

The following description of the (invariant) subspace $N_0 \oplus \cdots \oplus N_{k-1}$ will be useful later.

Proposition 1 *Let $Q_0, \ldots, Q_{k-1}$ be an admissible up to level $k-1$ projector sequence on $\mathcal{I}$. Then*

$$\ker P_0 \cdots P_{k-1} = N_0 \oplus \cdots \oplus N_{k-1}. \tag{14}$$

Proof: We verify relation (14) by induction. $P_0P_1z = 0$ means $z_0 := (I - Q_1)z \in N_0$, hence $z \in \ker P_0P_1$ implies $z = Q_1z + z_0 \in N_0 \oplus N_1$. Conversely, decomposing $z \in N_0 \oplus N_1$ as $z = Q_0w_0 + Q_1w_1$, we find $P_1Q_1w_1 = 0$, and $P_0P_1Q_0w_0 = P_0(I - Q_1)Q_0w_0 = P_0Q_0w_0 = 0$, so that $P_0P_1z = 0$ and therefore $N_0 \oplus N_1 \subseteq \ker P_0P_1$.

Let $i \le k-1$, and suppose now $\ker P_0 \cdots P_{i-1} = N_0 \oplus \cdots \oplus N_{i-1}$. We conclude from $P_0 \cdots P_i z = 0$, i.e., $\tilde{z} := (I - Q_i)z \in N_0 \oplus \cdots \oplus N_{i-1}$, that $z = \tilde{z} + Q_i z$ belongs to $N_0 \oplus \cdots \oplus N_i$. On the other hand, for $z \in N_0 \oplus \cdots \oplus N_i = (N_0 \oplus \cdots \oplus N_{i-1}) \oplus N_i$ we use a decomposition of the form $z = z_* + z_i$, with $z_* \in N_0 \oplus \cdots \oplus N_{i-1}$ and $z_i \in N_i$. Now $Q_i z_* = 0$ because $N_0 \oplus \cdots \oplus N_{i-1} \subseteq \ker Q_i$ by the admissibility assumption, so that $P_i z_* = z_*$; also $P_i z_i = 0$ since $z_i \in N_i$. Hence $P_0 \cdots P_i z = P_0 \cdots P_{i-1} P_i z_* + P_0 \cdots P_i z_i = P_0 \cdots P_{i-1} z_* = 0$, i.e., $z \in \ker P_0 \cdots P_{i-1}$. Identity (14) is then proved. $\square$

The key objects in the preadmissibility Definition 4 are independent of projectors: this follows from the particular case $\operatorname{rk} G_k =$ constant, $\dim (N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = 0$ in the result below.

Proposition 2 *Let $Q_0, \ldots, Q_{k-1}$, and $\overline{Q}_0, \ldots, \overline{Q}_{k-1}$, $k \ge 1$, be admissible (up to level $k-1$) projector sequences on a given subinterval $\mathcal{I}$. Then for all t in $\mathcal{I}$ it holds that* *(i)* $\operatorname{rk} G_k = \operatorname{rk} \overline{G}_k$,

(ii) $\dim (N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \dim (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{k-1}) \cap \overline{N}_k$.

Proof: along the lines defined in [11, Th. 2.3], we may check that $\overline{G}_i = G_i Z_i$ for $1 \leq i \leq k-1$, with a nonsingular factor which is defined as follows:

$$Z_i = (I + Q_{i-1}\overline{Q}_{i-1}P_{i-1} + \sum_{j=0}^{i-2} Q_j Z_{ij} P_0 \ldots P_{i-2}) Z_{i-1},$$

for appropriately chosen Z_{ij}.

With regard to [11, Th. 2.3], it is worth emphasizing that Z_k defined as above with $i = k$ does only depend on the sequences $Q_0, \ldots, Q_{k-1}$, and $\overline{Q}_0, \ldots, \overline{Q}_{k-1}$, defined up to level $k-1$, not involving any projector at level k, so that the reasoning there holds also for $i = k$. From this particular case, the relation $\overline{G}_k = G_k Z_k$ shows that $\operatorname{im} G_k = \operatorname{im} \overline{G}_k$, making (i) obvious.

Concerning (ii), we can check that $N_0 \oplus \cdots \oplus N_i = \overline{N}_0 \oplus \cdots \oplus \overline{N}_i$ for all $i \leq k-1$ by using $Z_0, \ldots, Z_i$ and noting that $N_i = Z_i \overline{N}_i$. Due to the form of Z_i, it follows that

$$N_i \subseteq (N_0 \oplus \cdots \oplus N_{i-1}) + \overline{N}_i = (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{i-1}) + \overline{N}_i \tag{15}$$

and, conversely,

$$\overline{N}_i \subseteq (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{i-1}) + N_i = (N_0 \oplus \cdots \oplus N_{i-1}) + N_i. \tag{16}$$

It then follows that $(N_0 \oplus \cdots \oplus N_{i-1}) + N_i = (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{i-1}) + \overline{N}_i$, and hence

$$N_0 \oplus \cdots \oplus N_i = \overline{N}_0 \oplus \cdots \oplus \overline{N}_i,$$

holding in particular for $i = k-1$. All direct sums rely on the admissibility hypotheses on $Q_0, \ldots, Q_{k-1}$ and $\overline{Q}_0, \ldots, \overline{Q}_{k-1}$.

With a small remark, we can do something closely related for $i = k$: note again that $N_k = Z_k \overline{N}_k$ does not involve any projector at level k. Due to the form of Z_k, it follows that (15) and (16) hold also for $i = k$, so that

$$(N_0 \oplus \cdots \oplus N_{k-1}) + N_k = (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{k-1}) + \overline{N}_k,$$

although now we cannot restate the sum as a direct one.

Additionally, in the light of (i) we have $\dim N_k = \dim \overline{N}_k$. Therefore,

$$\begin{aligned}
&\dim (N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \\
&\qquad = \dim N_0 \oplus \cdots \oplus N_{k-1} + \dim N_k - \dim ((N_0 \oplus \cdots \oplus N_{k-1}) + N_k) = \\
&\qquad = \dim (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{k-1}) + \dim \overline{N}_k - \dim ((\overline{N}_0 \oplus \cdots \oplus \overline{N}_{k-1}) + \overline{N}_k) = \\
&\qquad = \dim (\overline{N}_0 \oplus \cdots \oplus \overline{N}_{k-1}) \cap \overline{N}_k
\end{aligned}$$

and (ii) is proved. □

3.2 Nice and algebraically nice DAEs

Definition 5 *The DAE (1) is called* algebraically nice *(on $\mathcal{I}$) at level 0 if both $A(t)$ and $D(t)$ have constant rank on $\mathcal{I}$, and* $\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n$ *for all $t \in \mathcal{I}$. It is called* nice *(on $\mathcal{I}$) at level 0 if, additionally, both spaces are C^1.*

It is called algebraically nice *(on $\mathcal{I}$) at level k, $k \geq 1$, if*

(i) it is nice (on $\mathcal{I}$) at level $k-1$;

(ii) G_k has constant rank r_k on $\mathcal{I}$, for some (hence any) admissible up to level $k-1$ sequence $Q_0, \dots, Q_{k-1}$;

(iii) $(N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \{0\}$ on $\mathcal{I}$, for some (hence any) admissible up to level $k-1$ sequence $Q_0, \dots, Q_{k-1}$.

The invariant numbers $r_0, \dots, r_k$ will be called characteristic values *of the DAE up to level k.*

The DAE is called nice *(on $\mathcal{I}$) at level k if it is algebraically nice on $\mathcal{I}$ at level k and there exists a C^0 projector Q_k onto N_k, satisfying the preadmissibility condition $N_0 \oplus \cdots \oplus N_{k-1} \subseteq \ker Q_k$, for which*

(iv) $DP_0 \cdots P_k D^- \in C^1(\mathcal{I}, L(\mathbb{R}^n))$.

We emphasize below the invariance property which follows from Proposition 2.

Corollary 1 *Neither the definition of an algebraically nice (at level k) DAE nor the characteristic values $r_0, \dots, r_k$ are dependent on the specific choice of the admissible (up to $k-1$) projector sequence $Q_0, \dots, Q_{k-1}$.*

In the light of Definition 5, the DAE is nice at level 0 if and only if it has a properly stated leading term on $\mathcal{I}$; in particular, $G_0 := AD$ will have constant rank, and $DP_0D^- = DD^-$ will be C^1. A DAE will be algebraically nice at level k iff there exists a sequence $Q_0, \dots, Q_k$ which is preadmissible up to level k: it suffices to "lift" the admissible (up to $k-1$) sequence $Q_0, \dots, Q_{k-1}$ by taking a continuous Q_k such that $N_0 \oplus \cdots \oplus N_{k-1} \subseteq \ker Q_k$, what is allowed by condition (iii). Obviously, the DAE will be nice at level k if and only if it admits an admissible up to k sequence.

For problems with sufficiently smooth coefficients A, D, B, the DAE is nice if and only if it is algebraically nice, as shown below.

Proposition 3 *Assume that the coefficients $A(t)$, $D(t)$, $B(t)$ in the DAE (1) are C^r, $r \geq 1$. If the DAE is algebraically nice at level $k \leq r$, then it is nice at level k.*

Proof: We have to check that, if the DAE is algebraically nice at a given level k, then the smoothness requirement (iv) can be met. This follows from the fact that we can take Q_0 in the class C^r, so that $G_1 = G_0 + BQ_0$ is also C^r. If $k \geq 1$ the DAE is algebraically nice at level 1, and then we may choose a preadmissible Q_1 in C^r, so that Q_1 will be actually admissible and the DAE is nice at level 1. Then B_1 and so G_2 will be in the class C^{r-1}.

For $k \geq 2$, we can continue up the admissible sequence with Q_2 in C^{r-1}, and later on with Q_k in C^{r-k+1} also in admissible manner. Note that the last Q_r is C^1 and hence admissible. □

The transversality condition $(N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \{0\}$ can be rephrased as a maximal rank one. Recall that x, N_i, etc. lie on $\mathbb{R}^m$.

Proposition 4 *Denote $L_k = P_0 P_1 \cdots P_{k-1}$, $k \geq 1$. For a nice up to level $k-1$ DAE, it holds that*

$$(N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \{0\} \iff \operatorname{rk}[L_k^T L_k + G_k^T G_k] = m \tag{17a}$$

$$\Longleftarrow \operatorname{rk}[B_k^T B_k + G_k^T G_k] = m, \tag{17b}$$

for any admissible up to level $k-1$ sequence $Q_0, \ldots, Q_{k-1}$.

Proof: We will use the property $\ker P_0 \cdots P_{k-1} = N_0 \oplus \cdots \oplus N_{k-1}$ proved in Proposition 1. The equivalence (17a) becomes clearer if we denote $L_k = P_0 P_1 \cdots P_{k-1}$ and check that

$$\ker L_k \cap \ker G_k = \ker (L_k^T L_k + G_k^T G_k), \tag{18}$$

because then (17a) amounts to the case in which both sides of the identity (18) are trivial. It is obvious that $\ker L_k \cap \ker G_k \subseteq \ker (L_k^T L_k + G_k^T G_k)$; to see the converse we only need to remark that $(L_k^T L_k + G_k^T G_k)v = 0 \Rightarrow v^T (L_k^T L_k + G_k^T G_k)v = (L_k v)^T L_k v + (G_k v)^T G_k v = 0$, i.e., $L_k v = G_k v = 0$ and therefore $v \in \ker L_k \cap \ker G_k$.

In turn, (17b) follows from the identity $\ker (B_k^T B_k + G_k^T G_k) = \ker B_k \cap \ker G_k$ and the remark that $N_0 \oplus \cdots \oplus N_{k-1} = \ker P_0 \cdots P_{k-1} \subseteq \ker B_k$. This is due to the expression for B_k depicted in (11), which makes

$$\begin{aligned}
B_k &= B_{k-1} P_{k-1} - G_k D^- (D P_0 \cdots P_k D^-)' D P_0 \cdots P_{k-1} \\
&= B_{k-2} P_{k-2} P_{k-1} - [G_{k-1} D^- (D P_0 \cdots P_{k-1} D^-)' D + G_k D^- (D P_0 \cdots P_k D^-)' D] P_0 \cdots P_{k-1} \\
&\vdots \\
&= [B - G_1 D^- (D P_0 P_1 D^-)' D - G_2 D^- (D P_0 P_1 P_2 D^-)' D - \ldots] P_0 P_1 \cdots P_{k-1},
\end{aligned}$$

making it apparent that $\ker P_0 \cdots P_{k-1} \subseteq \ker B_k$. If $\ker B_k \cap \ker G_k = \{0\}$ then $(N_0 \oplus \cdots \oplus N_{k-1}) \cap N_k = \{0\}$ and (17b) is proved. □

Corollary 2 *The DAE (1) is algebraically nice on $\mathcal{I}$ up to level k if and only if it is nice up to level $k-1$ and the following two constant rank conditions hold on $\mathcal{I}$:*

(i) $\operatorname{rk} G_k$ is constant, *and*

(ii) $\operatorname{rk}[L_k^T L_k + G_k^T G_k] = m$.

Note that the maximal rank condition on both $L_k^T L_k + G_k^T G_k$ and $B_k^T B_k + G_k^T G_k$ can be stated also as a maximal rank condition on the matrices $(G_k^T \; L_k^T)$ and $(G_k^T \; B_k^T)$, respectively. Maximal rank in these enlarged matrices together with constant rank in G_k^T defines these transposed pairs as *regular* for Rabier and Rheinboldt [15, Def. 12.1]. Hence:

Corollary 3 *Let the DAE (1) be nice up to level $k-1$. Then it is algebraically nice up to level k*

(i) if and only if the pair (G_k^T, L_k^T) is regular in the sense of Rabier and Rheinboldt; or

(ii) if the pair (G_k^T, B_k^T) is regular, in the same sense.

3.3 Regular DAEs and decoupling

Definition 6 *Let the DAE (1) have a properly stated leading term on $\mathcal{I} \subseteq \mathcal{J}$. If both A and D are invertible on $\mathcal{I}$, then (1) is said to be a* regular DAE with tractability index zero *(on $\mathcal{I}$).*

The DAE (1) is said to be regular with tractability index $\mu \in I\!N$ *on $\mathcal{I}$ if there exists an admissible projector sequence $Q_0, \ldots, Q_{\mu-1}$, and $r_{\mu-1} < r_\mu = m$.*

Equation (1) is said to be a regular DAE *(on $\mathcal{I}$) if it is regular with tractability index $\mu \in I\!N \cup \{0\}$. The ranks $0 \leq r_0 \leq r_1 \leq \cdots \leq r_{\mu-1} < r_\mu = m$ as well as μ and $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$ are said to be* characteristic values *of a regular DAE.*

The direct sum appearing in item (iii) of Definition 5 shows that, if the maximum rank m is met, then this is done in no more than m steps, i.e., $\mu \leq m$. A DAE will be regular with index μ if and only if it nice or algebraically nice at level μ, $N_\mu = \{0\}$, and μ is minimal with respect to the latter. Note that the transversality and smoothness conditions (iii) and (iv) in Definition 5 become trivial at level μ. Remark also that a constant-coefficient DAE is regular with tractability index μ if and only if the matrix pencil $\{G_0, B\}$ is regular with Kronecker index μ [10].

Regularity is defined in an invariant manner owing to Proposition 2, as acknowledged below.

Corollary 4 *Neither the definition of a regular DAE nor the characteristic values $r_0, r_1, \ldots, r_{\mu-1}$, $r_\mu = m$, μ, d are dependent on the specific choice of the admissible projector sequence $Q_0, \ldots, Q_{\mu-1}$.*

Each regular index zero DAE (1) can be rewritten as an explicit ODE

$$u' + DG_0^{-1}BD^{-1}u = DG_0^{-1}q \tag{19}$$

with respect to $u = Dx$, and solutions of (1) are obtained by $x = D^{-1}u$. We discuss below the decoupling of DAEs with index ≥ 1.

Decoupling index-1 DAEs. Regular index one DAEs (1) are known [11] to be equivalent to

$$u' - (DP_0D^-)'u + DP_0G_1^{-1}BD^-u = DP_0G_1^{-1}q, \tag{20a}$$

$$v_0 = -\mathcal{K}_0D^-u + \mathcal{L}_0q, \tag{20b}$$

with $u = Dx,\ v_0 = Q_0x$. More precisely, x is a solution of (1) in C_D^1 if and only if it can be written as

$$x = D^-u + v_0, \tag{21}$$

where u is a C^1 solution of (20a) in the invariant space $\operatorname{im} D$, and v_0 is given by (20b).

The coefficients $\mathcal{K}_0$ and $\mathcal{L}_0$ are given by $\mathcal{K}_0 := Q_0G_1^{-1}BP_0,\ \mathcal{L}_0 = Q_0G_1^{-1}$. A special, smart choice of the projector Q_0 (and the generalized inverse D^-, respectively) leads to $\mathcal{K}_0 = 0$, that is, to a complete decoupling of the DAE (1) into two parts (20a) and (20b) independent of each other.

A functional digression. At this point, we have the necessary tools to illustrate that the properly stated leading term in (1) is of great help in providing precise input-output functional descriptions of DAEs. Let us consider the standard form (2) and assume for the moment that E is the identity and F is continuous. We are faced with a linear explicit ODE, and C^0 excitations q are well-known to be mapped *bijectively* onto C^1 solutions x: more precisely, the fixed initial condition $x(t_0) = x_0$ yields a bijection between the C^0 space of excitations q and the space $\{x \in C^1 :\ x(t_0) = x_0\}$. This bijection makes it possible to work with an inverse model of the ODE, defined by $x \to q = x' + Fx$. This can be obviously extended to $C^i \leftrightarrow C^{i+1}$ bijections, if F is C^i.

In contrast, it is well-known that solutions x of an index-1 DAE in the classical form (2) cannot be guaranteed to be C^1 if the excitation q is just C^0, even in the constant-coefficient case (see e.g. [16]). One way to circumvent this is to assume that q is C^1; this is reasonable in many applications, and we can then guarantee (in index-1 problems with sufficiently smooth E, F) that solutions are C^1. But suppose we are interested in working with an inverse model of the linear DAE, mapping x to q. Such an inverse, for C^1 mappings x, would read $x \to q = Ex' + Fx$. However, $Ex' + Fx$ cannot be guaranteed in general to be C^1, even if E and F are C^1. It seems that, in order to overcome this, we should require x to be C^2. But then we would be faced with the above-indicated problem, one degree of smoothness further: not every C^1 excitation q guarantees a C^2 solution x.

This means that, in a functional framework, C^i spaces are not the right ones to frame DAEs from an input-output perspective. Instead, notice that, if x is C^1, then $Ex' + Fx$ will belong to some space C_*^1 verifying $C^1 \subset C_*^1 \subset C^0$, provided that E, F are at least continuous. In the same way, we can actually reconsider just continuous excitations, by expanding the space in which solutions are sought. In this direction we go back to a reformulation such as (3), in fact to general problems of the form (1). We define, for $i \geq 1$, the space $C_D^i(\mathcal{I}, I\!R^m) := \{x \in C^{i-1}(\mathcal{I}, I\!R^m) : Dx \in C^i(\mathcal{I}, I\!R^n)\}$.

Now, an excitation $q \in C^i$ ($i = 0$, $i = 1$ corresponding to the cases considered above) yields, if the problem is regular with index 1, a solution $x \in C_D^{i+1}$ defined by (21), given an initial condition $D(t_0)(x(t_0) - x_0) = 0$. A bijection and an inverse model can be then precisely defined. Also, (20b) makes it clear that the above-mentioned inverse image C_*^1 of C^1 is actually $C_{\mathcal{L}_0}^1$. See Figure 1.

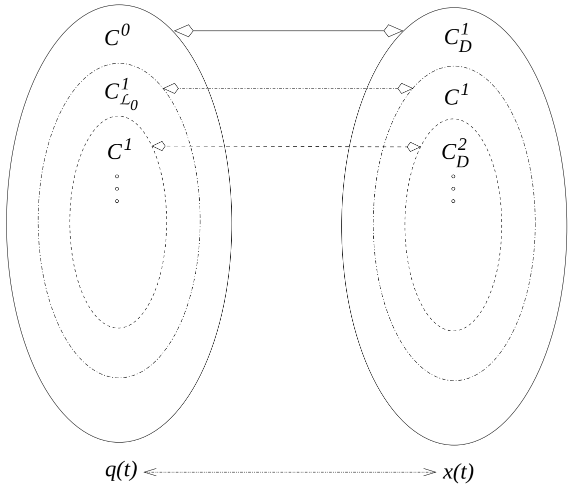

Figure 1: Bijective functional transformations for linear index-1 DAEs.

This has been achieved by setting the problem in the form (1), which allows one to figure out exactly which components of the solution need to be differentiated. In turn, the decoupling (20) explicitly shows the necessary smoothness requirements on q.

Decoupling DAEs with arbitrary index. As shown in [11], a regular index μ DAE decouples into the system

$$u' - (DP_0 \cdots P_{\mu-1}D^-)'u + DP_0 \cdots P_{\mu-1}G_\mu^{-1}BD^-u = DP_0 \cdots P_{\mu-1}G_\mu^{-1}q, \tag{22a}$$

$$v_{\mu-1} = -\mathcal{K}_{\mu-1}D^-u + \mathcal{L}_{\mu-1}q, \tag{22b}$$

$$v_k = -\mathcal{K}_k D^- u + \mathcal{L}_k q + \sum_{j=k+1}^{\mu-1} \mathcal{N}_{kj}(Dv_j)' + \sum_{j=k+2}^{\mu-1} \mathcal{M}_{kj} v_j, \quad k = \mu-2, \dots, 1, 0, \tag{22c}$$

where $u = DP_0 \cdots P_{\mu-1}x,\ v_0 = Q_0 x,\ v_i = P_0 \cdots P_{i-1}Q_i x,\ i = 1, \dots, \mu-1$, and solutions have the form

$$x = D^-u + v_0 + \cdots + v_{\mu-1}. \tag{23}$$

The coefficients $\mathcal{K}_i,\ \mathcal{L}_i,\ \mathcal{N}_{kj},\ \mathcal{M}_{kj}$ are given in the appendix below. All of them are a priori continuous. A special, smart choice of the admissible sequence $Q_0, \dots, Q_{\mu-1}$ yields a so-called fine decoupling with vanishing coefficients $\mathcal{K}_1, \dots, \mathcal{K}_{\mu-1}$. Thereby, Q_0 can be chosen arbitrarily. It is also possible to construct an admissible sequence $Q_0, \dots, Q_{\mu-1}$ in such a way that a complete decoupling ($\mathcal{K}_i = 0, i = 0, 1, \dots, \mu-1$) results (cf. [12]). The fine decoupling projectors may require some additional smoothness, which can be shown to hold if the problem is sufficiently smooth [12].

For a homogeneous regular DAE (1) the geometric solution space

$$S_{can\ \mu}(t) := \{x(t) \in \mathbb{R}^m : x \in C^1(\mathcal{I}, \mathbb{R}^m),\ A(Dx)' + Bx = 0\},\ t \in \mathcal{I},$$

is continuous and, for every t, has dimension $d := m - \sum_{i=0}^{\mu-1}(m - r_i)$. This is a consequence of the existence of fine decouplings (cf. [12]). With fine decoupling projectors $Q_0, \dots, Q_{\mu-1}$ and

$$\Pi_{can\ \mu} := (I - \mathcal{K}_0)P_0 \cdots P_{\mu-1} \tag{24}$$

we find that $\Pi_{can\ \mu}$ is also a projector, and

$$im\Pi_{can\ \mu} = S_{can\ \mu}, \ \ker \Pi_{can\ \mu} = N_{can\ \mu} := N_0 \oplus \cdots \oplus N_{\mu-1}. \tag{25}$$

While $S_{can\ \mu}$ is just a C^0-subspace in $\mathbb{R}^m$, the subspace $\operatorname{im} DP_0 \cdots P_{\mu-1}D^-$ is a C^1-subspace in $\mathbb{R}^n$. Recall that $\operatorname{im} DP_0 \cdots P_{\mu-1}D^-$ is the basic invariant subspace of the so-called inherent explicit ODE (22a). This invariant subspace as well as the coefficients in (22a) do not depend on the special choice of a fine decoupling sequence $Q_0, \ldots, Q_{\mu-1}$ (cf. [12]).

The function space

$$\begin{aligned} C_{adm\ \mu}(\mathcal{I}, \mathbb{R}^m) \ := \ & \{q \in C(\mathcal{I}, \mathbb{R}^m) : D\mathcal{L}_{\mu-1}q =: s_{\mu-1} \in C^1(\mathcal{I}, \mathbb{R}^n), \\ & s_{\mu-2} := D\mathcal{L}_{\mu-2}q + D\mathcal{N}_{\mu-2\mu-1}s'_{\mu-1} \in C^1(\mathcal{I}, \mathbb{R}^n), \\ & s_k := D\mathcal{L}_k q + \sum_{j=k+1}^{\mu-1} D\mathcal{N}_{kj}s'_j + \sum_{j=k+2}^{\mu-1} D\mathcal{M}_{kj}D^- s_j \in C^1(\mathcal{I}, \mathbb{R}^n), \ k = \mu-3, \ldots, 1\} \end{aligned}$$

contains the admissible right-hand sides q leading to solvable (in C^1_D) equations. Conversely, if we put $q := A(Dx)' + Bx$ for given $x \in C^1_D(\mathcal{I}, \mathbb{R}^m)$, then q belongs to $C_{adm\ \mu}(\mathcal{I}, \mathbb{R}^m)$. The functional input-output discussion carried out above for index-1 cases can be extended to DAEs with arbitrary index along these lines. Note finally that, in this non-homogeneous setting, for a given $q \in C_{adm\ \mu}(\mathcal{I}, \mathbb{R}^m)$ the solution manifold

$$\mathcal{M}(t) := \{x(t) : x \in C^1_D(\mathcal{I}, \mathbb{R}^m), \ \ A(Dx)' + Bx = q\}, \ \ t \in \mathcal{I},$$

is an affine subspace of dimension d, for every fixed t.

4 Local analysis

So far, all concepts and results apply to the DAE (1) "globally" on some interval $\mathcal{I}$. In the sequel we consider local information; we define a *regular point* in a way such that, if all points are regular in a given interval, then the DAE itself is regular *on the interval*, say, in a uniform manner with common characteristic values. The same will hold for *algebraically nice* and *nice* points. Additionally, this will pave the way for a systematic treatment of critical points in [13].

In the case defined by $n = m$ and $D(t) \equiv I$, equation (1) reads

$$A(t)x'(t) + B(t)x(t) = q(t), \ \ t \in \mathcal{J}, \tag{26}$$

and $t_* \in \mathcal{J}$ is a regular point if $A(t_*)$ is nonsingular. Recall that, if $A(t_*)$ is nonsingular, the leading term is properly stated around t_*.

Definition 7 *A point $t_* \in \mathcal{J}$ is called*

- algebraically nice *at level $k \geq 0$, or*
- nice *at level $k \geq 0$, or*
- regular*,*

for (1) if the DAE itself is so in some open interval $\mathcal{I} \subseteq \mathcal{J}$ with $t_* \in \mathcal{I}$.

Clearly, if the DAE (1) is regular with tractability index μ (resp. algebraically nice or nice at level k) on a given interval, then all inner points of this interval are regular (resp. algebraically nice or nice at level k) with common characteristic values $r_0, \ldots, r_\mu$, μ and d (resp. $r_0, \ldots, r_k$). The opposite is also true as the next propositions will show.

Proposition 5 *Let $t_1, t_2 \in \mathcal{J}$ be two different regular (resp. algebraically nice or nice at level k) points of the DAE (1) with regularity intervals $\mathcal{I}_1$, $\mathcal{I}_2$, and characteristic values μ_1, μ_2, $r_{1,0}, \ldots, r_{1,\mu_1}$, $r_{2,0}, \ldots, r_{2,\mu_2}$ (resp. $r_{1,0}, \ldots, r_{1,k}$, $r_{2,0}, \ldots, r_{2,k}$). If the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ is nontrivial, then*

(i) $\mu_1 = \mu_2$, $r_{1,i} = r_{2,i}$, $i = 0, \ldots, \mu_1$ *(resp* $r_{1,i} = r_{2,i}$, $i = 0, \ldots, k$*), and*

(ii) the DAE is regular (resp. algebraically nice or nice at level k) on $\mathcal{I}_1 \cup \mathcal{I}_2$ with these characteristics.

Proof:

1. The assertion is trivial for algebraically nice or nice points at level 0. Assume w.l.o.g. $t_1 < t_2$. Then, there is a closed interval $[a, b] \subset \mathcal{I}_1 \cap \mathcal{I}_2$, $t_1 \leq a < b \leq t_2$. Let λ denote a scalar C^1-function such that $\lambda(t) = 0$ for $t \leq a$, $\lambda(t) = 1$ for $t \geq b$, $\lambda'(a) = \lambda'(b) = 0$. If the DAE is regular on $\mathcal{I}_1$ and on $\mathcal{I}_2$, there are admissible sequences $\overset{1}{Q}_0, \ldots, \overset{1}{Q}_{\mu_1 - 1}$ and $\overset{2}{Q}_0, \ldots, \overset{2}{Q}_{\mu_2 - 1}$. Both sequences are admissible on $[a, b]$. If the DAE is (algebraically) nice at level $k \geq 1$ on $\mathcal{I}_1$ and on $\mathcal{I}_2$, there are (pre)admissible sequences $\overset{1}{Q}_0, \ldots, \overset{1}{Q}_k$ and $\overset{2}{Q}_0, \ldots, \overset{2}{Q}_k$. Again, both sequences are (pre)admissible on $[a, b]$. Since the characteristic values are independent of the special choice of admissible functions, it follows that $\mu_1 = \mu_2 =: \mu$ and $r_{1,i} = r_{2,i}$, $i = 0, \ldots, \mu$ for the regular case, and $r_{1,i} = r_{2,i}$, $i = 0, \ldots, k$ for algebraically nice and nice cases.

2. Next, for the regular case we build an admissible sequence $Q_0, \ldots, Q_{\mu-1}$ for the DAE on $\mathcal{I}_1 \cup \mathcal{I}_2$ in such a way that it coincides with $\overset{1}{Q}_0, \ldots, \overset{1}{Q}_{\mu-1}$ for $t \leq a$, and with $\overset{2}{Q}_0, \ldots, \overset{2}{Q}_{\mu-1}$ for $t \geq b$. For (algebraically) nice at level k cases we stop when the (pre)admissible projector Q_k is constructed.

Notice that, given two projectors Q, $\tilde{Q}$ onto a given subspace, a linear combination of the form $\alpha Q + (1 - \alpha)\tilde{Q}$ is also a projector onto this subspace for any real α. Write

$$\begin{aligned} Q_0 &:= \begin{cases} \overset{2}{Q}_0, & \text{for } b \leq t, \\ \lambda \overset{2}{Q}_0 + (1 - \lambda) \overset{1}{Q}_0, & \text{for } a \leq t \leq b, \\ \overset{1}{Q}_0, & \text{for } t \leq a, \end{cases} \\ P_0 &:= I - Q_0, \end{aligned}$$

so that Q_0 is continuous on $\mathcal{I}_1 \cup \mathcal{I}_2$. Compute $\overset{1}{G}_1 = G_0 + B_0 \overset{1}{Q}_0$, $\overset{2}{G}_1 = G_0 + B_0 \overset{2}{Q}_0$ on $\mathcal{I}_1$ resp. $\mathcal{I}_2$. For $t \leq a$ and $t \geq b$ we have $G_1 = \overset{1}{G}_1$ resp. $G_1 = \overset{2}{G}_1$.

The matrix function $G_1 = G_0 + B_0Q_0$ is then continuous and has constant rank $r_{1,0} = r_{2,0}$ on $\mathcal{I}_1 \cup \mathcal{I}_2$. For regular DAEs with index 1 the proof would be completed at this point.

3. Write $Q_1 := \overset{1}{Q}_1$ for $t \le a$ and $Q_1 := \overset{2}{Q}_1$ for $t \ge b$. Below we construct a projector Q_1 on $[a,b]$ such that $Q_1(a) = \overset{1}{Q}_1(a)$, $Q_1(b) = \overset{2}{Q}_1(b)$. So far, we have on $[a,b]$:

$$\begin{aligned} G_1 &= \overset{1}{G}_1 Z_1 \quad \text{with} \quad Z_1 := I + Q_0 \overset{1}{P}_0, \quad N_1 = Z_1^{-1} \overset{1}{N}_1, \\ \overset{2}{G}_1 &= \overset{1}{G}_1 \tilde{Z}_1 \quad \text{with} \quad \tilde{Z}_1 := I + \overset{2}{Q}_0 \overset{1}{P}_0, \quad \overset{1}{N}_1 = \tilde{Z}_1 \overset{2}{N}_1 . \end{aligned}$$

Now, $\lambda \tilde{Z}_1 \overset{2}{Q}_1 \tilde{Z}_1^{-1} + (1-\lambda) \overset{1}{Q}_1$ is a projector onto $\overset{1}{N}_1$, hence

$$Q_1 := Z_1^{-1}(\lambda \tilde{Z}_1 \overset{2}{Q}_1 \tilde{Z}_1^{-1} + (1-\lambda) \overset{1}{Q}_1) Z_1 \tag{27}$$

projects onto N_1. Taking a closer look at (27), due to $\overset{1}{Q}_1 \overset{i}{Q}_0 = 0$, $\overset{2}{Q}_1 \overset{i}{Q}_0 = 0$, $i = 1, 2$, we find $\overset{2}{Q}_1 \tilde{Z}_1^{-1} = \overset{2}{Q}_1$, $\overset{2}{Q}_1 Z_1 = \overset{2}{Q}_1$, $\overset{1}{Q}_1 Z_1 = \overset{1}{Q}_1$, thus

$$Q_1 = Z_1^{-1}(\lambda \tilde{Z}_1 \overset{2}{Q}_1 + (1-\lambda) \overset{1}{Q}_1),$$

and then $Q_1Q_0 = 0$. This means $N_0 \subseteq \ker Q_1$, and yields $N_0 \cap N_1 = \{0\}$, so that $N_0 + N_1 = N_0 \oplus N_1$.

The sequence Q_0, Q_1 is preadmissible and the proof for algebraically nice points at level 1 would be completed at this stage.

4. Moreover, because of $Z_1(a) = I$, $Z_1(b) = I + \overset{2}{Q}_0(b) \overset{1}{P}_0(b)$, $\tilde{Z}_1(b) = I + \overset{2}{Q}_0(b) \overset{1}{P}_0(b)$, we find

$$Q_1(a) = \overset{1}{Q}_1(a), \quad Q_1(b) = Z_1(b)^{-1} \tilde{Z}_1(b) \overset{2}{Q}_1(b) = \overset{2}{Q}_1(b).$$

Next we show that $DP_0P_1D^-$ is continuously differentiable on $[a,b]$ and $(DP_0P_1D^-)'(a) = (D \overset{1}{P}_0 \overset{1}{P}_1 \overset{1}{D}^-)'(a)$, $(DP_0P_1D^-)'(b) = (D \overset{2}{P}_0 \overset{2}{P}_1 \overset{2}{D}^-)'(b)$, so that $DP_0P_1D^-$ is C^1 on $\mathcal{I}_1 \cup \mathcal{I}_2$. Compute (on $[a,b]$) $P_0P_1 = \lambda P_0 \overset{2}{P}_1 + (1-\lambda) P_0 \overset{1}{P}_1$,

$$DP_0P_1D^- = \lambda DP_0 \overset{2}{P}_1 D^- + (1-\lambda) DP_0 \overset{1}{P}_1 D^-.$$

From $R = DD^- = D \overset{1}{D}^- = D \overset{2}{D}^-$ it follows that $D^- = P_0 \overset{1}{D}^- = P_0 \overset{2}{D}^-$. Deriving $DP_0 \overset{2}{P}_1 D^- = DP_0D^-D \overset{2}{P}_0 \overset{2}{P}_1 D^- = DP_0D^-D \overset{2}{P}_0 \overset{2}{P}_1 P_0 \overset{2}{D}^- = DP_0D^-D \overset{2}{P}_0 \overset{2}{P}_1 \overset{2}{D}^-$, and analogously, $DP_0 \overset{1}{P}_1 D^- = DP_0D^-D \overset{1}{P}_0 \overset{1}{P}_1 \overset{1}{D}^-$, we obtain the representation

$$DP_0P_1D^- = DP_0D^-[\lambda D \overset{2}{P}_0 \overset{2}{P}_1 \overset{2}{D}^- + (1-\lambda) D \overset{1}{P}_0 \overset{1}{P}_1 \overset{1}{D}^-], \tag{28}$$

which shows $DP_0P_1D^-$ to be continuously differentiable.

From (28) it follows that $(DP_0P_1D^-)'(a) = (D \overset{1}{P}_0 \overset{1}{P}_1 \overset{1}{D}^-)'(a)$, $(DP_0P_1D^-)'(b) = (D \overset{2}{P}_0 \overset{2}{P}_1 \overset{2}{D}^-)'(b)$. Consequently, Q_0, Q_1 are admissible on $\mathcal{I}_1 \cup \mathcal{I}_2$ projectors. At this point we would be done for nice points at level 1.

5. In the next steps we proceed analogously. If $Q_0, \dots, Q_{i-1}$ is an admissible up to level $i-1$ sequence, we have [11]

$$\begin{aligned} G_i &= \overset{1}{G}_i\, Z_i, \;\; Z_i := (I + \overset{1}{Q}_{i-1}\, Q_{i-1}\, \overset{1}{P}_{i-1} + \sum_{j=0}^{i-2} \overset{1}{Q}_j\, Z_{i-1j} Q_{i-1}) Z_{i-1}, \\ \overset{2}{G}_i &= G_i \tilde{Z}_i, \;\; \tilde{Z}_i := (I + \overset{1}{Q}_{i-1} \overset{2}{Q}_{i-1} \overset{1}{P}_{i-1} + \sum_{j=0}^{i-2} \overset{1}{Q}_j\, \tilde{Z}_{i-1j}\, \overset{2}{Q}_{i-1}) \tilde{Z}_{i-1}, \end{aligned}$$

so that G_i is continuous with constant rank. This would complete the proof for problems with index $i \geq 2$.

6. If G_i has not maximal rank, define

$$Q_i := Z_i^{-1} (\lambda \tilde{Z}_i\, \overset{2}{Q}_i\, \tilde{Z}_i^{-1} + (1-\lambda)\, \overset{1}{Q}_i) Z_i$$

which is a projector function onto N_i, and

$$\begin{aligned} Q_i &= Z_i^{-1} (\lambda \tilde{Z}_i\, \overset{2}{Q}_i + (1-\lambda)\, \overset{1}{Q}_i), \\ Q_i Q_j &= 0, \; j = 0, \dots, i-1. \end{aligned}$$

If $i = k$, this completes the proof for algebraically nice points at level $k \geq 2$.

7. Notice that $\overset{1}{N}_0 \oplus \cdots \oplus \overset{1}{N}_{i-1} = \overset{2}{N}_0 \oplus \cdots \oplus \overset{2}{N}_{i-1} = N_0 \oplus \cdots \oplus N_{i-1}$ [11], thus $\overset{1}{P}_0 \cdots \overset{1}{P}_{i-1}, \overset{2}{P}_0 \cdots \overset{2}{P}_{i-1}$ and $P_0 \cdots P_{i-1}$ are all projectors along the same subspace.

Compute

$$\begin{aligned} P_0 \cdots P_i &= \lambda P_0 \cdots P_{i-1}\, \overset{2}{P}_i + (1-\lambda) P_0 \cdots P_{i-1}\, \overset{1}{P}_i, \\ D P_0 \cdots P_i D^- &= \lambda D P_0 \cdots P_{i-1}\, \overset{2}{P}_i\, D^- + (1-\lambda) D P_0 \cdots P_{i-1}\, \overset{1}{P}_i\, D^-, \\ D P_0 \cdots P_{i-1}\, \overset{2}{P}_i\, D^- &= D P_0 \cdots P_{i-1}\, \overset{2}{P}_0 \cdots \overset{2}{P}_{i-1} \overset{2}{P}_i\, D^- = D P_0 \cdots P_{i-1} D^- D\, \overset{2}{P}_0 \cdots \overset{2}{P}_i \overset{2}{D}^-, \end{aligned}$$

and, analogously,

$$D P_0 \cdots P_{i-1}\, \overset{1}{P}_i\, D^- = D P_0 \cdots P_{i-1} D^- D\, \overset{1}{P}_0 \cdots \overset{1}{P}_i \overset{1}{D}^-.$$

This leads to

$$D P_0 \cdots P_i D^- = D P_0 \cdots P_{i-1} D^- [\lambda D\, \overset{2}{P}_0 \cdots \overset{2}{P}_i \overset{2}{D}^- + (1-\lambda) D\, \overset{1}{P}_0 \cdots \overset{1}{P}_i \overset{2}{D}^-],$$

hence $D P_0 \cdots P_i D^-$ is continuously differentiable. It holds that $(D P_0 \cdots P_i D^-)'(a) = (D\, \overset{1}{P}_0 \cdots \overset{1}{P}_i \overset{1}{D}^-)'(a)$, $(D P_0 \cdots P_i D^-)'(b) = (D\, \overset{2}{P}_0 \cdots \overset{2}{P}_i \overset{2}{D}^-)'(b)$, hence, $Q_0, \dots, Q_i$ is an admissible sequence on $\mathcal{I}_1 \cup \mathcal{I}_2$.

This way we construct on $\mathcal{I}_1 \cup \mathcal{I}_2$ an admissible sequence $Q_0, \dots, Q_k$ to prove the nice at level $k \geq 2$ case, thus completing the proof. □

Proposition 6 *If all points of an open interval $\mathcal{I} \subseteq \mathcal{J}$ are (algebraically) nice at level k or regular, then they have uniform characteristic values $r_0, \ldots, r_k$ or $\mu, r_0, \ldots, r_{\mu-1}$ and $r_\mu = m$, and the DAE is (algebraically) nice at level k or regular on $\mathcal{I}$ with these characteristic values.*

Proof: Let us first consider a (for the moment arbitrary) compact subinterval $\hat{\mathcal{I}}$ within $\mathcal{I}$. The subinterval $\hat{\mathcal{I}}$ can be covered by a finite number of (algebraically) nice or regularity intervals; it is straightforward to adapt the proof of Proposition 5 to show that uniform characteristic values are displayed and uniform projectors can be constructed in $\hat{\mathcal{I}}$.

Now, we describe $\mathcal{I}$ as a denumerable union of compact subintervals $\overset{j}{\mathcal{I}}$ such that $\overset{j}{\mathcal{I}} \cap \overset{l}{\mathcal{I}} \neq \emptyset$ if and only if $l-1 \leq j \leq l+1$. To achieve this it suffices e.g. to take an exhaustive sequence of compact intervals $[a_k, d_k]$ in $\mathcal{I}$, meaning that $\mathcal{I} = \bigcup_{k \in \mathbb{N} \cup \{0\}} [a_k, d_k]$ with $a_{k+1} < a_k < d_k < d_{k+1}$; choose b_1, c_1 such that $a_0 < b_1 < c_1 < d_0$, and then for $k \geq 2$ take b_k, c_k such that $a_{k-1} < b_k < a_{k-2}$ and $d_{k-2} < c_k < d_{k-1}$. The partition $\mathcal{I} = [a_0, d_0] \cup (\bigcup_{k \in \mathbb{N}} [a_k, b_k] \cup [c_k, d_k])$ satisfies the above indicated intersection requirement.

These compact subintervals will have associated projectors $\overset{j}{Q}_i$ and, since only pairwise non-trivial intersections have to be considered, they can be modified as in Proposition 5 to yield well-defined and (pre)admissible projectors Q_i on the whole $\mathcal{I}$. □

In the light of Definition 7, the *regular set* $\mathcal{I}_{\text{reg}}$ or set of regular points is obviously open in $\mathcal{J}$, and therefore can be described as a (possibly infinite) number of disjoint open subintervals. This partition is independent of any actual choice of projectors. Propositions 5 and 6 prove that in each of these subintervals a uniform structure is given, and hence a uniform treatment is possible.

5 A linear time-varying analogue of Chua's circuit

Figure 2 displays a linear time-varying analogue of Chua's circuit [4]. We assume that resistors are current-controlled (in contrast to the usual voltage-control assumption in this context) since we will be interested in cases in which $R_1(t) = 0$ and/or $R_2(t) = 0$.

Modified Nodal Analysis (MNA) equations read for this circuit

$$(C_1(t)e_1)' - i_{r_1} + i_{r_2} = 0 \tag{29a}$$

$$(C_2(t)e_2)' + i_l + i_{r_1} = 0 \tag{29b}$$

$$(L(t)i_l)' - e_2 = 0 \tag{29c}$$

$$e_2 - e_1 - R_1(t)i_{r_1} = 0 \tag{29d}$$

$$e_1 - R_2(t)i_{r_2} = 0. \tag{29e}$$

Resistors' currents i_{r_1} and i_{r_2} explicitly appear in the model due to the current-control assumption. Note that the eventual vanishing of $R_1(t)$ and $R_2(t)$ (at isolated points or even on intervals) precludes using a voltage-controlled representation and hence rules out the standard state-space

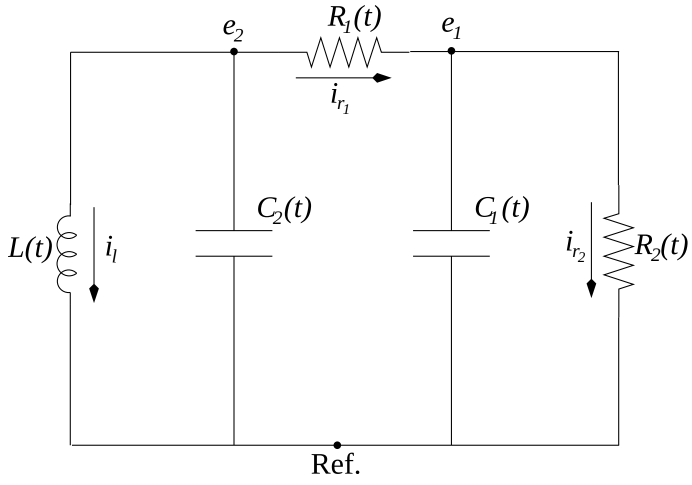

Figure 2: Linear time-varying Chua's circuit with current-controlled resistors.

form of Chua's equation [4]. The DAE framework seems to be of interest for this case: actually, as detailed below, this circuit will exhibit a rich variety of indices depending on the actual shapes of devices' characteristics. These characteristics are only assumed to be continuous, and we emphasize that this assumption will need not be strengthened in the analysis.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 \end{bmatrix}, \; G_0 = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

and

$$B_0 = B = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -R_1 & 0 \\ 1 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{31}$$

At points where C_1, C_2 and L do not vanish, we are led to a properly stated leading term and the DAE is nice at level 0.

Index 1. Let $R_1 \neq 0 \neq R_2,\ C_1 \neq 0 \neq C_2,\ L \neq 0$. For later use, we remark that the kernel N_0 is defined by $x_1 = x_2 = x_3 = 0$. Take

$$D^- = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 1/C_2 & 0 \\ 0 & 0 & 1/L \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so that } P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{32}$$

and then

$$G_1 = G_0 + B_0 Q_0 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & C_2 & 0 & 1 & 0 \\ 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & -R_1 & 0 \\ 0 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{33}$$

From the assumed non-vanishing in the reactances, we conclude that the problem is regular with index 1 on intervals where $R_1 \neq 0 \neq R_2,\ C_1 \neq 0 \neq C_2,\ L \neq 0$. Note that, under these assumptions, a voltage-controlled description of resistors and a state reduction of (29) in terms of e_1, e_2, i_l are straightforward; this defines the linear time-varying analogue of the standard setting for Chua's circuit. Nevertheless, additional interesting cases can be found beyond these conditions, as discussed below.

Index 2. Consider intervals where $R_1 = 0,\ R_2 \neq 0,\ 0 \neq C_1 \neq -C_2 \neq 0,\ L \neq 0$; the first condition describes a persistent short-circuit in R_1. In this situation G_1 has constant rank and the kernel N_1 is easily checked to be defined by the conditions $C_1x_1 - x_4 = 0$, $C_2x_2 + x_4 = 0$, $x_3 = 0$, $x_5 = 0$, so that the intersection $N_0 \cap N_1$ is trivial. This means that, as long as R_2 does not vanish, the DAE is algebraically nice at level 1. We may take the preadmissible projector

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -C_1/C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

This preadmissible projector can be easily be checked to be admissible without additional smoothness requirements. After some computations, we get

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},\ G_2 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & C_2 & 0 & 1 & 0 \\ C_1/C_2 & 0 & L & 0 & 0 \\ -1 - C_1/C_2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{35}$$

Since C_2, L, and R_2 do not vanish, we conclude that this matrix is non-singular if and only if $1 + C_1/C_2$ is not zero, i.e., if $C_1 \neq -C_2$. The DAE is then regular with index 2 on intervals where $R_1 = 0,\ R_2 \neq 0,\ 0 \neq C_1 \neq -C_2 \neq 0,\ L \neq 0$.

Remark that $R_1 = 0$ yields a capacitor loop, which in conventional MNA still is index ≤ 1 [17]; the use of resistors' currents changes this. We emphasize that the model (29) is intended to be used for all t, regardless of the actual values of resistances on subintervals.

Index 3. Let $R_1 = 0,\ R_2 \neq 0,\ L \neq 0$, and assume additionally that $C_1 = -C_2 \neq 0$ in some interval. In this situation, G_2 reads

$$G_2 = \begin{bmatrix} C_1 & 0 & 0 & -1 & 1 \\ 0 & -C_1 & 0 & 1 & 0 \\ -1 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{36}$$

Looking e.g. at the first four columns we realize that $\operatorname{rk} G_2 = 4$. It is not difficult to check that $(N_0 \oplus N_1) \cap N_2 = \ker P_0 P_1 \cap \ker G_2 = \{0\}$. This means that the DAE is algebraically nice at level 2. A preadmissible projector Q_2 is

$$Q_2 = \begin{bmatrix} 0 & 0 & L & 0 & 0 \\ 0 & 0 & L[1 + 1/(R_2 C_1)] & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & L[C_1 + 1/R_2] & 0 & 0 \\ 0 & 0 & L/R_2 & 0 & 0 \end{bmatrix}. \tag{37}$$

To simplify computations we make use of the property that G_{k+1} is non-singular if N_k and $S_k = \{x \in \mathbb{R}^m : B_k x \in \operatorname{im} G_k\}$ intersect trivially [5, Th. A.13]. Computing $S_2 = \{x \in \mathbb{R}^5 : x_1 = x_2\}$ and $N_2 \cap S_2 = \{0\}$ we realize that G_3 must be non-singular. The DAE is hence regular with index 3 on intervals where $R_1 = 0,\ R_2 \neq 0,\ C_1 = -C_2 \neq 0,\ L \neq 0$.

In this regard, note that an isolated two-capacitors loop with $C_1 = -C_2 \neq 0$ would yield an underdetermined problem, since any voltage $e(t)$ defines a solution with loop current $i = (C_1(t)e)'$. Roughly speaking, this is not the case here because the voltage $e_1 = e_2$ is determined by the external L-R_2 loop.

6 Concluding remarks

We finish this analysis with some remarks aimed at motivating the study of critical or singular points, where the above-discussed regularity conditions fail. In Section 4 it has been shown that the interval $\mathcal{J}$ can be partitioned in a unique manner in *regularity intervals* where the linear DAE (1) has well-defined (and possibly distinct from one subinterval to another) characteristic values and index. The union of these regularity intervals defines the regular set $\mathcal{I}_{\text{reg}}$.

On the other hand, as stated in Section 3 and detailed in [11, 12], on every regularity interval the dynamics of the DAE can be unraveled through the inherent explicit ODE (22a),

$$u' - (DP_0 \cdots P_{\mu-1} D^-)' u + DP_0 \cdots P_{\mu-1} G_\mu^{-1} B D^- u = DP_0 \cdots P_{\mu-1} G_\mu^{-1} q,$$

together with the additional conditions (22b)-(22c). For a particular choice of (so-called fine decoupling) admissible projectors, the coupling coefficients $\mathcal{K}_i$ do vanish for $i \geq 1$, and this can be extended to $\mathcal{K}_0$ through the concept of a complete decoupling. Furthermore, the operators $(DP_0 \cdots P_{\mu-1} D^-)'$, $DP_0 \cdots P_{\mu-1} G_\mu^{-1} B D^-$ and $DP_0 \cdots P_{\mu-1} G_\mu^{-1}$ appearing in the fine-decoupling inherent ODE, as well as the basic invariant space $\operatorname{im} DP_0 \cdots P_{\mu-1}D^-$, are actually independent of the specific choice of fine projectors.

Regardless of the actual value μ of the index on any of these regularity intervals, we note that, for $k \geq \mu$, $Q_k = 0$ and therefore $G_k = G_\mu$, $P_k = I$. Hence, the inherent explicit ODE can be rewritten as

$$u' - (DP_0 \cdots P_{m-1}D^-)'u + DP_0 \cdots P_{m-1}G_m^{-1}BD^-u = DP_0 \cdots P_{m-1}G_m^{-1}q. \tag{38}$$

Remark that the index μ cannot exceed m, as acknowledged after Definition 6. This means that (38) is a uniquely defined inherent explicit ODE holding for the whole regular set $\mathcal{I}_{\text{reg}}$, independently of the characteristic values on regularity subintervals. Of course, at critical points delimiting regularity intervals the operators in (38) may undergo discontinuities (typically, in index transitions), isolated singularities (such as poles in analytic problems), etc. Note also that not necessarily critical points define a discrete set, nor a smooth behavior through them is precluded. A full description of the behavior of the linear DAE (1) should take into account these critical points, considering the phenomenon from which the singularity stems. This is addressed in the work [13].

Appendix: The coefficients of formula (22)

For $k = 1, \ldots, \mu - 1,\ j = k+2, \ldots, \mu - 1$:

$$\begin{aligned}
\mathcal{L}_k &= P_0 \cdots P_{k-1}Q_kP_{k+1} \cdots P_{\mu-1}G_\mu^{-1}, \\
\mathcal{K}_k &= -P_0 \cdots P_{k-1}Q_kP_{k+1} \cdots P_{\mu-1}G_\mu^{-1}BP_0 \cdots P_{\mu-1} \\
&\quad -P_0 \cdots P_{k-1}Q_kP_{k+1} \cdots P_{\mu-1}P_kD^-(DP_0 \cdots P_{\mu-1}D^-)'DP_0 \cdots P_{\mu-1},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{k\,k+1} &= P_0 \cdots P_{k-1}Q_kQ_{k+1}D^-, \\
\mathcal{N}_{kj} &= P_0 \cdots P_{k-1}Q_kP_{k+1} \cdots P_{j-1}Q_jD^-,
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{kj} &= -P_0 \cdots P_{k-1}Q_k\{Q_{k+1}D^-(DP_0 \cdots P_kQ_{k+1}D^-)' + P_{k+1}Q_{k+2}D^-(DP_0 \cdots P_{k+1}Q_{k+2}D^-)' \\
&\quad + \cdots + P_{k+1} \cdots P_{\mu-2}Q_{\mu-1}D^-(DP_0 \cdots P_{\mu-2}Q_{\mu-1}D^-)'\}DP_0 \cdots P_{j-1}Q_j \\
&\quad - \sum_{l=1}^{i} P_0 \cdots P_{k-1}Q_kP_{k+1} \cdots P_{\mu-1} \cdots P_lD^-(DP_0 \cdots P_lD^-)'DP_0 \cdots P_{j-1}Q_j.
\end{aligned}$$

For $k = 0$, in the top of these expressions, $P_0 \cdots P_{k-1}Q_k$ has to be replaced by Q_0.

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