Projectors for matrix pencils

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Abstract

In this paper, basic properties of projector sequences for matrix pairs which can be used for analyzing differential algebraic systems are collected.

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This paper collects, rearranges and completes results on matrix pairs and related projector sequences given in [Ga, GM1, GM2, M1, M2, M3, M4] in connection with the investigation of differential algebraic systems. It is organized as follows:

1. Basics (basic sequence of matrices, subspaces and projectors)
2. The case of regular matrix pencils
3. Admissible projector sequences (for regular matrix pencils)
4. Index-one criteria for regular pencils
5. s-admissible projector sequences (for general, possibly singular pencils)
6. Widely orthogonal s-admissible projectors
7. Admissible projectors using the subspaces S_i

1 Basics

For given ordered pairs $\{G, B\}$ of $k \times m$ matrices G, B we consider the following sequences of matrices, subspaces and projectors

$$G_0 := G, \; B_0 := B, \tag{1.1}$$
$$N_0 := kerG_0, \; Q_0^2 = Q_0, \; imQ_0 = N_0, \; P_0 := I - Q_0, \tag{1.2}$$

for $i \geq 0$:

$$G_{i+1} := G_i + B_i Q_i, \; B_{i+1} := B_i P_i, \tag{1.3}$$
$$N_{i+1} := kerG_{i+1}, \; Q_{i+1}^2 = Q_{i+1}, \; imQ_{i+1} = N_{i+1}, \; P_{i+1} := I - Q_{i+1}. \tag{1.4}$$

Thereby, Q_i, P_i are idempotent $m \times m$ matrices. We call such kind of projector matrices shortly projectors. The projectors Q_i, P_i in (1.2),(1.4) are not completely determined but just the range of Q_i (i.e., the nullspace of P_i) is fixed. Later on we will benefit from this flexibility. By construction, it holds that

$$G_{i+1} = G_0 + B_0Q_0 + \cdots + B_0P_0 \cdots P_{i-1}Q_i, \quad B_{i+1} = B_0P_0 \cdots P_i,$$

and $G_i = G_{i+1}P_i,\ B_iQ_i = G_{i+1}Q_i$, thus

$$imG_i \subseteq imG_{i+1}, \quad imB_iQ_i \subseteq imG_{i+1}. \tag{1.5}$$

Let $\mathcal{W}_i$ denote an additional $k \times k$ matrix such that $\mathcal{W}_i^2 = \mathcal{W}_i,\ ker\mathcal{W}_i = imG_i,\ i \geq 0$. Using these projectors we find the relation $\mathcal{W}_iB_i = \mathcal{W}_iB_0P_0 \cdots P_{i-1} = \mathcal{W}_iB_0P_0 \cdots P_{i-2}(I - Q_{i-1}) = \mathcal{W}_iB_0P_0 \cdots P_{i-2} = \mathcal{W}_iB_{i-1} = \cdots = \mathcal{W}_iB_1 = \mathcal{W}_iB_0 = \mathcal{W}_iB$.
We introduce the further subspaces

$$\begin{aligned} S_i &:= ker\mathcal{W}_iB_i = ker\mathcal{W}_iB \\ &= \{z : B_iz \in imG_i\} = \{z : Bz \in imG_i\},\ i \geq 0, \end{aligned} \tag{1.6}$$

which are independent of the special choice of the projectors $\mathcal{W}_i$.
Denoting by G_i^- the uniquely determined generalized inverse of G_i with

$$G_iG_i^-G_i = G_i,\ G_i^-G_iG_i^- = G_i,\ G_i^-G_i = P_i,\ G_iG_i^- = I - \mathcal{W}_i, \tag{1.7}$$

we may factorize G_{i+1} to

$$\begin{aligned} G_{i+1} &= \mathcal{G}_{i+1}F_{i+1}, \\ &\quad \mathcal{G}_{i+1} := G_i + \mathcal{W}_iB_iQ_i = G_i + \mathcal{W}_iBQ_i, \\ &\quad F_{i+1} := I + G_i^-B_iQ_i. \end{aligned} \tag{1.8}$$

Since F_{i+1} is nonsingular, $F_{i+1}^{-1} = I - G_i^-B_iQ_i$, the representation (1.8) leads to the relations

$$imG_{i+1} = im\mathcal{G}_{i+1} = imG_i \oplus im\mathcal{W}_iB_iQ_i,$$

$$rankG_{i+1} = rankG_i + rank\mathcal{W}_iB_iQ_i,$$

$$N_{i+1} = (I - G_i^-B_iQ_i)ker\mathcal{G}_{i+1}.$$

Proposition 1.1 *The following properties are satisfied for $i \geq 0$:*

(1) $S_i \subseteq S_{i+1}$,

(2) $N_0 + \cdots + N_i \subseteq S_{i+1}$,

(3) $N_i \cap kerB_i = N_i \cap N_{i+1}$,

(4) $N_i \cap N_{i+1} \subseteq N_{i+1} \cap N_{i+2}$,

(5) $ker\mathcal{G}_{i+1} = N_i \cap S_i,\ N_{i+1} = (I - G_i^-B_iQ_i)(N_i \cap S_i),\ dimN_{i+1} = dim(N_i \cap S_i)$,

(6) $S_{i+1} = N_i + S_i = N_0 + \cdots + N_i + S_0$.

Proof:

(1) For $z \in S_i$ it holds that $Bz = G_iw = G_{i+1}P_iw$, hence $z \in S_{i+1}$.

(2) Due to the definition $B_1 = B_0P_0,\ S_1 = \mathcal{W}_1B_1$ we have $N_0 \subseteq S_1$. For $i \geq 0,\ z \in N_0 + \cdots + N_i$, we use a decomposition $z = z_0 + \cdots + z_i,\ z_j \in N_j,\ j = 0, \ldots, i$, and derive

$$\mathcal{W}_{i+1}Bz = \mathcal{W}_{i+1}Bz_0 + \ldots + \mathcal{W}_{i+1}Bz_i = \mathcal{W}_{i+1}B_1z_0 + \cdots + \mathcal{W}_{i+1}B_{i+1}z_i = 0.$$

(3) $z \in N_i \cap N_{i+1} \Longleftrightarrow G_iz = 0, z = Q_iz,\ G_{i+1}z = 0 \Longleftrightarrow G_iz = 0,\ B_iz = 0.$

(4) $z \in N_i \cap N_{i+1}$ yields $z = Q_{i+1}z,\ z = Q_iz,$
$G_{i+2}z = (G_{i+1} + B_{i+1}Q_{i+1})z = B_{i+1}z = B_iP_iz = 0.$

(5) $z \in ker\mathcal{G}_{i+1}$ means $(G_i + \mathcal{W}_iB_iQ_i)z = 0$, or, equivalently, $G_iz = 0,\ \mathcal{W}_iB_iQ_iz = 0$, that is, $z = Q_iz,\ z \in ker\mathcal{W}_iB_i$, i.e., $z \in N_i \cap S_i$.
Due to the factorization (1.8), this leads to $N_{i+1} = F_{i+1}^{-1}(N_i \cap S_i)$, and both subspaces N_{i+1} and S_{i+1} must have the same dimension.

(6) Properties (1) and (2) imply the inclusion $N_i + S_i \subseteq S_{i+1}$. Conversely, for an arbitrary $z \in S_{i+1} = ker\mathcal{W}_{i+1}B_{i+1} = \mathcal{W}_{i+1}B_i$ we find a w such that $B_iz = G_{i+1}w$, i.e., $B_iz = G_iw + B_iQ_iw$, hence $B_i(z - Q_iw) = G_iw$. This shows $\tilde{z} := z - Q_iw$ to belong to S_i, and the representation $z = \tilde{z} + Q_iw \in S_i + N_i$ with $\tilde{z} \in S_i,\ Q_iw \in N_i$, becomes true. □

Corollary 1.2 *If, in the sequence (1.1)-(1.4), there arises a nontrivial intersection* $N_{i_*} \cap N_{i_*+1}$, *then none of the matrices* $G_i, i \geq 0$ *is injective.*

Proof:
This is a direct consequence of property (4). □

2 The case of regular matrix pencils

Definition: (e.g. [Ga]): The pair $\{G, B\}$ represents a *regular matrix pencil* if $m = k$ and the polynomial $p(\lambda) := det(\lambda G + B), \lambda \in \mathbb{C}$, does not vanish identically. Otherwise, $\{G, B\}$ is said to be a *singular matrix pencil.*
If $\{G, B\}$ represents a regular matrix pencil, then there are nonsingular matrices E, F (e.g. [Ga]) such that

$$EGF = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{matrix} \}s \\ \}m-s \end{matrix}, EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \}s \\ \}m-s \end{matrix}, \tag{2.1}$$

where J is a nilpotent $(m - s) \times (m - s)$ block. The nilpotency index $\mu \in \mathbb{N},\ J^\mu = 0,\ J^{\mu-1} \neq 0$ is called the Kronecker index of the pencil $\{G, B\}$, and (2.1) is the *Kronecker canonical form* of $\{G, B\}$.
If G is nonsingular, J is absent, $s = m$. Then $\{G, B\}$ has Kronecker index zero by definition.

Theorem 2.1 *(1) If $\{G,B\}$ is a regular matrix pencil with index μ, then the matrices $G_0,\ldots,G_{\mu-1}$ are singular, whereas G_μ is nonsingular. Conversely, if G_μ is nonsingular, then $\{G,B\}$ is a regular pencil.*
(2) If $\{G,B\}$ is a regular pencil with index μ, then, for $l=0,\ldots,\mu$, $\{G_l,B_l\}$ is a regular pencil with index $\mu-l$.

Proof: (1) = [GM1], Theorem 3, (2) = [GM1], Theorem 4. □

It should be stressed that Theorem 2.1 holds true independently of the special choice of the projectors $Q_0,\ldots,Q_{\mu-1}$.
Indicating by "$^-$" the sequence corresponding to $\bar{G}:=EGF,\ \bar{B}:=EBF,\ E,F$ nonsingular, with $\bar{Q}_i=F^{-1}Q_iF$, one realizes $\bar{G}_{i+1}=EG_{i+1}F,\ i\geq 0$.

Example 2.2 *Consider the pair $\{G,B\}$*

$$G_0=G=\left[\begin{array}{c|ccc|c} 1&0&0&0&0\\ \hline 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0 \end{array}\right],\quad B_0=B=\left[\begin{array}{c|ccc|c} \omega&0&0&0&0\\ \hline 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right]$$

which is regular with index 3 and has Kronecker normal form. Compute

$$Q_0=\left[\begin{array}{c|ccc|c} 0&0&0&0&0\\ \hline 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&1 \end{array}\right],\quad G_1=\left[\begin{array}{c|ccc|c} 1&0&0&0&0\\ \hline 0&1&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&1 \end{array}\right],\quad Q_1=\left[\begin{array}{c|ccc|c} 0&0&0&0&0\\ \hline 0&0&-1&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0 \end{array}\right],$$

$$B_1=\left[\begin{array}{c|ccc|c} \omega&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&0 \end{array}\right],\quad G_2=\left[\begin{array}{c|ccc|c} 1&0&0&0&0\\ \hline 0&1&1&0&0\\ 0&0&1&1&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&1 \end{array}\right],\quad Q_2=\left[\begin{array}{c|ccc|c} 0&0&0&0&0\\ \hline 0&0&0&1&0\\ 0&0&0&-1&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&0 \end{array}\right],$$

$$B_2=\left[\begin{array}{c|ccc|c} \omega&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&0 \end{array}\right],\quad G_3=\left[\begin{array}{c|ccc|c} 1&0&0&0&0\\ \hline 0&1&1&0&0\\ 0&0&1&1&0\\ 0&0&0&1&0\\ \hline 0&0&0&0&1 \end{array}\right].$$

Here the projectors fulfil the conditions $Q_1Q_0=0,\ Q_2Q_0=0,\ Q_2Q_1=0$, i.e., $N_0\subseteq kerQ_1,\ N_0+N_1=N_0\oplus N_1\subseteq kerQ_2$. Further, it holds that $rankG_0=3,\ rankG_1=rankG_2=4,\ rankG_3=5$.

Example 2.3 *(cf. [GM1], page 30)*
For the pair $G_0=G=\begin{bmatrix}1&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&1&0\end{bmatrix},\quad B_0=B=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$, *we obtain with*

$Q_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *the matrix* $G_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, *and with* $Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *we arrive at* $G_2 = G_0$, *and further* $G_{2j} = G_0, Q_{2j} = Q_0, G_{2j+1} = G_1,\ Q_{2j+1} = Q_1$. *All intersections* $N_{i+1} \cap N_i = N_1 \cap N_0 = 0,\ i \geq 0$, *are trivial ones. However, we have here* $Q_1Q_0 = Q_1 \neq 0,\ Q_2Q_0 = Q_0 \neq 0,\ Q_2Q_1 = Q_2 \neq 0,\ (N_0 \oplus N_1) \cap N_2 = (N_0 \oplus N_1) \cap N_0 = N_0 \neq 0$. *Note that* $p(\lambda) = det(\lambda G + B)$ *vanishes identically, i.e.,* $\{G, B\}$ *is a singular pencil.*

Example 2.4 *Consider the pair* $\{G, B\}$ *given in the previous example. Take* Q_0 *as above, leading to* G_1 *as before. However, choosing* $Q_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ *to meet the condition* $N_0 \subseteq kerQ_1$, *i.e.,* $Q_1Q_0 = 0$, *we find* $G_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $N_2 \subseteq kerB_2$.
Now, any choice of Q_2 *yields* $B_2Q_2 = 0,\ G_3 = G_2$.

As Example 2.3 shows, in case of a singular matrix pencil $\{G, B\}$, the intersections $(N_0 + \cdots + N_i) \cap N_{i+1}$ are not necessarily nontrivial.
Example 2.4 indicates that the intersections $(N_0 + \cdots + N_i) \cap N_{i+1}$ could serve as a necessary singularity criterion supposed that previous projectors $Q_0, \ldots, Q_i$ have the property $Q_lQ_k = 0,\ 0 \leq k < l \leq i$, so that $N_0 + \cdots + N_i = N_0 \oplus \cdots \oplus N_i$.

Lemma 2.5 *For two subspaces of* $\mathbb{R}^m(\mathbb{C}^M)$ $L = span\{l_1, \ldots, l_s\}$, $N = span\{n_1, \ldots, n_t\}, L \cap N = 0$, *there is a projector* $\mathcal{U}$ *such that* $im\mathcal{U} = N$, $ker\mathcal{U} \supseteq L$.

Proof: Denote by H the $m \times (s+t)$ matrix consisting of the columns $l_1, \ldots, l_s, n_1, \ldots, n_t$. Due to $L \cap N = 0$, $s + t$ is not greater than m, and $l_1, \ldots, l_s, n_1, \ldots, n_t$ are linearly independent. Then, H is injective, and we may construct the wanted projector $\mathcal{U}$ as

$$\mathcal{U} = H \left[\begin{array}{cc} 0 & \\ & I \end{array} \right] (H^*H)^{-1}.$$
$$\underbrace{\quad}_{s}\ \underbrace{\quad}_{t}$$

□

Proposition 2.6 *If* $\{G, B\}$ *is a regular matrix pencil, then the projectors in (1.1)-(1.4) can be chosen such that*

$$N_0 + \cdots + N_i = N_0 \oplus \cdots \oplus N_i \subseteq kerQ_{i+1},\ i \geq 0. \tag{2.2}$$

Proof: Since $\{G, B\}$ is a regular pencil, starting with any projector Q_0, we must have $N_0 \cap N_1 \neq 0$, and therefore $N_0 + N_1 = N_0 \oplus N_1$. Otherwise, by Corollary 1.2, there would be no nonsingular matrices among the G_i, but this contradicts the pencil regularity (cf. Theorem 2.1).

Applying Lemma 2.5 we determine Q_1 to meet the condition $Q_1Q_0 = 0$, i.e., $N_0 \subseteq kerQ_{1_*}$. Let, for some $i \geq 1$, the conditions $N_0+\cdots+N_i = N_0\oplus\cdots\oplus N_i,\ Q_lQ_k = 0,\ 0 \leq k < l \leq i$, be fulfilled. If $N_{i+1} \cap (N_0 \oplus \cdots \oplus N_i) = 0$, we may choose Q_{i+1} so that $N_0 \oplus \cdots \oplus N_i \subseteq kerQ_{i+1}$ and proceed. It remains to verify that N_{i+1} and $N_0 \oplus \cdots \oplus N_i$ intersect trivially. For $z \in N_{i+1} \cap (N_0 \oplus \cdots \oplus N_i)$ we use the decomposition $z = z_0 + \cdots + z_i,\ z_j \in N_j,\ j = 0, \ldots, i$, and compute $B_{i+1}z = B_0P_0 \cdots P_iz = B_0P_0 \cdots P_i(z_0 + \cdots + z_i) = B_0P_0 \cdots P_i(z_0+\cdots+z_{i-1}) = B_0P_0 \cdots P_{i-1}(z_0+\cdots+z_{i-1}) = B_0P_0 \cdots P_{i-1}(z_0+\cdots+z_{i-2}) = \ldots = B_0P_0z_0 = 0$.
This leads to the inclusion (cf. Proposition 1.1 (3))

$$N_{i+1} \cap (N_0 \oplus \cdots \oplus N_i) \subseteq N_{i+1} \cap kerB_{i+1} = N_{i+1} \cap N_{i+2}.$$

Because of $N_{i+1} \cap N_{i+2} = 0$ for regular pencils we are done. □
If the pencil $\{G, B\}$ is singular and $m = k$, all matrices $G_i,\ i \geq 0$, are singular, and their nullspaces N_i have dimensions $n_i > 0$. For reasons of dimension, there must be an $i_* \in \mathbb{N}$ and a nontrivial intersection $N_{i_*+1} \cap (N_0 \oplus \cdots \oplus N_{i_*})$.

3 Admissible projector sequences

We turn back to the general ordered pair $\{G, B\}$ of $k \times m$ matrices and the sequence (1.1)-(1.4).

Definition: The projectors $Q_i,\ i \geq 0$, in (1.1)-(1.4) are *admissible up to level* κ (the projectors $Q_0, \ldots, Q_\kappa$ are admissible) if $Q_jQ_l = 0$ for $0 \leq l < j \leq \kappa$, or, euqivalently,

$$N_0 + \cdots + N_j = N_0 \oplus \cdots \oplus N_j \subseteq kerQ_{j+1},\ j = 0, \ldots, \kappa - 1. \tag{3.1}$$

Proposition 3.1 *If the projectors $Q_0, \ldots, Q_\kappa$ are admissible, then the products*

$$P_0P_1 \cdots P_j,\ P_0P_1 \cdots P_{j-1}Q_j,\ j = 0, \ldots, \kappa, \tag{3.2}$$

are projectors, too, where

$$kerP_0P_1 \cdots P_j = N_0 \oplus \cdots \oplus N_j, \tag{3.3}$$

$$kerP_0P_1 \cdots P_{j-1}Q_j = kerQ_j. \tag{3.4}$$

Proof: From $Q_j = Q_jP_l,\ l = 0, \ldots, j - 1$, we derive $P_0P_1 \cdots P_{j-1}Q_j \cdot P_0P_1 \cdots P_{j-1}Q_j = P_0P_1 \cdots P_{j-1}Q_j \cdot Q_j = P_0P_1 \cdots P_{j-1}Q_j$, hence $P_0P_1 \cdots P_{j-1}Q_j$ is a projector.
Further, $P_0P_1 \cdots P_j \cdot P_0P_1 \cdots P_j = P_0P_1 \cdots P_{j-1}(I-Q_j)P_0P_1 \cdots P_j = P_0P_1 \cdots P_{j-1}P_0P_1 \cdots \cdots P_{j-1}P_j - P_0P_1 \cdots P_{j-1}Q_j \cdot P_j = (P_0P_1 \cdots P_{j-1})^2P_j = \ldots = P_0^2P_1 \cdots P_{j-1}P_j = P_0P_1 \cdots P_j$, i.e., $P_0 \cdots P_j$ is a projector.
To verify (3.3) we decompose $z \in N_0 \oplus \cdots \oplus N_j,\ z = z_0 + \cdots + z_j,\ z_l \in N_l, l = 0, \ldots, j$, and compute $P_0P_1 \cdots P_jz = P_0P_1 \cdots P_j(z_0 + \cdots + z_j) = P_0P_1 \cdots P_j(z_0 + \cdots + z_{j-1}) = P_0P_1 \cdots P_{j-1}(z_0 + \cdots + z_{j-1}) - P_0P_1 \cdots P_{j-1}Q_j(z_0 + \cdots + z_{j-1}) = P_0P_1 \cdots P_{j-1}(z_0 + \cdots + z_{j-1}) = \ldots = P_0z_0 = 0$. This provides the inclusion $N_0 \oplus \cdots \oplus N_j \subseteq kerP_0 \cdots P_j$.
Conversely, from $P_0P_1 \cdots P_jz = 0$, i.e., $(I - Q_0)(I - Q_1) \cdots (I - Q_j)z = 0$, we realize the relation

$$z = (Q_0 + Q_1 + \cdots + Q_j)z - (Q_0Q_1 + \cdots + Q_{j-1}Q_j)z + \cdots + (-1)^jQ_0Q_1 \cdots Q_jz$$

showing that z must belong to $N_0 \oplus \cdots \oplus N_j$, and we are done with (3.3).
As far as (3.4) is concerned, we have $Q_j = Q_j P_0 \cdots P_{j-1} Q_j$ and $Q_j = P_0 \cdots P_{j-1} Q_j + (I - P_0 \cdots P_{j-1}) Q_j = M_j P_0 \cdots P_{j-1} Q_j$, with $M_j := I + (I - P_0 \cdots P_{j-1}) Q_j$. Since the matrix M_j has the inverse $M_j^{-1} = I - (I - P_0 \cdots P_{j-1}) Q_j$, the product $M_j P_0 \cdots P_{j-1} Q_j$ has the same nullspace as $P_0 \cdots P_{j-1} Q_j$. $\square$

Next we compare two different sequences (1.1)-(1.4) constructed by the use of different projectors for a given pair $\{G, B\}$.
Denote these sequences by $G_0 := G,\ B_0 := B,\ Q_0, G_1 := G_0 + B_0 Q_0, \ldots$, and $\bar{G}_0 := G,\ \bar{B}_0 = B,\ \bar{Q}_0, \bar{G}_1 := \bar{G}_0 + \bar{B}_0 \bar{Q}_0, \ldots$, respectively.

Lemma 3.2 *If both projector sequences are admissible up to level κ, then, for $j = 0, \ldots, \kappa$, it holds that*

$$\bar{N}_0 \oplus \cdots \oplus \bar{N}_j = N_0 \oplus \cdots \oplus N_j \quad \textit{and}$$

$$\bar{G}_j = G_j Z_j, \quad \bar{B}_j = B_j + G_j \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl},$$

where

$$\begin{aligned} Z_j &:= \big(I + Q_{j-1} \bar{Q}_{j-1} P_{j-1} + \sum_{l=0}^{j-2} Q_l \mathfrak{A}_{j-1\,l}\, \bar{Q}_{j-1}\big) Z_{j-1}, \\ Z_1 &:= I + Q_0 \bar{Q}_0 P_0, \\ Z_0 &:= I, \end{aligned}$$

Z_j is nonsingular,

$$\mathfrak{A}_{j\,j-1} := \bar{P}_0 \cdots \bar{P}_{j-1},\ \ \mathfrak{A}_{jl} := \mathfrak{A}_{j-1\,l} \bar{P}_{j-1} = \mathfrak{A}_{j-1\,l} \bar{P}_0 \cdots \bar{P}_{j-1},\ \ l = 0, \ldots, j-2.$$

Proof:
We start with $\bar{G}_0 = G_0 = G,\ \bar{B}_0 = B_0 = B,\ im\bar{Q}_0 = \bar{N}_0 = N_0 = imQ_0$. Derive $\bar{G}_1 = \bar{G}_0 + \bar{B}_0 \bar{Q}_0 = G_0 + B_0 \bar{Q}_0 = G_0 + B_0 Q_0 \bar{Q}_0 = (G_0 + B_0 Q_0) Z_1$ with $Z_1 := P_0 + \bar{Q}_0 = I + \bar{Q}_0 P_0 = I + Q_0 \bar{Q}_0 P_0$, $Z_1^{-1} = I - Q_0 \bar{Q}_0 P_0$, and further, $\bar{N}_1 = Z_1^{-1} N_1$.
Since $N_0 = \bar{N}_0$, and each $\bar{z}_1 \in \bar{N}_1$ has the form $\bar{z}_1 = (I - Q_0 \bar{Q}_0 P_0) z_1 = z_1 - Q_0 \bar{Q}_0 P_0 z_1$, with $z_1 \in N_1$, we realize that $\bar{z}_1 \in N_1 \oplus N_0$, $\bar{N}_1 \subseteq N_0 \oplus N_1$, $\bar{N}_0 \oplus \bar{N}_1 \subseteq N_0 \oplus N_1$. For reasons of dimensions ($rank\bar{G}_1 = rankG_1$), $\bar{N}_0 \oplus \bar{N}_1 = N_0 \oplus N_1$ holds true. We have further $\bar{B}_1 = B_0 \bar{P}_0 = B_0 P_0 + B_0 Q_0 \bar{P}_0 = B_1 + G_1 Q_0 \bar{P}_0 = B_1 + G_1 Q_0 \mathfrak{A}_{10}$ with $\mathfrak{A}_{10} := \bar{P}_0$. Therefore, the assertion of Lemma 3.2 is valid for $j = 1$. Assume its validity up to the index $j = p$, and show it for $p + 1$. Compute

$$\bar{G}_{p+1} = G_p Z_p + \bar{B}_p \bar{Q}_p = G_p Z_p + B_p \bar{Q}_p + G_p \sum_{l=0}^{p-1} Q_l \mathfrak{A}_{pl} \bar{Q}_p,$$

and taking into consideration that $\bar{P}_0 \cdots \bar{P}_{p-1}\bar{Q}_p Z_p^{-1} = \bar{P}_0 \cdots \bar{P}_{p-1}\bar{Q}_p,\ kerB_pP_p \supseteq N_0 \oplus \cdots \oplus N_p$, we find

$$\begin{aligned}\bar{G}_{p+1}Z_p^{-1} &= G_p + B_pQ_p\bar{Q}_p + G_p\sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_p \\ &= G_{p+1} + B_pQ_p(\bar{Q}_p - Q_p) + G_p\sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_p \\ &= G_{p+1}\{I + Q_p(\bar{Q}_p - Q_p) + \sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_p\} = G_{p+1}Y_{p+1}\end{aligned}$$

with $Y_{p+1} := I + Q_p(\bar{Q}_p - Q_p) + \sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_p = I + Q_p\bar{Q}_pP_p + \sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_p,\ Y_{p+1}^{-1} := I - Q_p\bar{Q}_pP_p - \sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{Q}_pQ_p$, hence $\bar{G}_{p+1} = G_{p+1}Z_{p+1},\ \ Z_{p+1} := Y_{p+1}Z_p$. It follows that $\bar{N}_{p+1} = Z_{p+1}^{-1}N_{p+1} \subseteq N_0 \oplus \cdots \oplus N_{p+1}$, and for reasons of dimension $\bar{N}_0 \oplus \cdots \oplus \bar{N}_{p+1} = N_0 \oplus \cdots \oplus N_{p+1}$.
Finally we derive

$$\begin{aligned}\bar{B}_{p+1} &= \bar{B}_p\bar{P}_p = (B_p + G_p\sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl})\bar{P}_p \\ &= (B_pP_p + B_pQ_p + G_p\sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl})\bar{P}_p,\end{aligned}$$

and $B_pP_p\bar{P}_p = B_pP_0 \cdots P_p\bar{P}_p = B_pP_0 \cdots P_p\bar{P}_0 \cdots \bar{P}_p\bar{P}_p = B_pP_0 \cdots P_p\bar{P}_0 \cdots \bar{P}_p = B_pP_0 \cdots P_p = B_{p+1},\ \ B_pQ_p\bar{P}_p = G_{p+1}Q_p\bar{P}_p$, so that

$$\bar{B}_{p+1} = B_{p+1} + G_{p+1}\{Q_p\bar{P}_0 \cdots \bar{P}_{p-1}\bar{P}_p + \sum_{l=0}^{p-1} Q_l\mathfrak{A}_{pl}\bar{P}_p\} =: B_{p+1} + G_{p+1}\sum_{l=0}^{p} Q_l\mathfrak{A}_{p+1l},$$

with $\mathfrak{A}_{p+1p} := \bar{P}_0 \cdots \bar{P}_p,\ \mathfrak{A}_{p+1l} := \mathfrak{A}_{pl}\bar{P}_p = \mathfrak{A}_{pl}\bar{P}_0 \cdots \bar{P}_p,\ l = 0, \ldots, p-1$, holds true. □

Note that Lemma 3.2 and Theorem 3.3 below reflect the special case for constant matrices of [M4], Theorem 2.3. For time-varying matrices considered in [M4] the expressions $\mathfrak{A}_{pl}$ are much more complicated.

Theorem 3.3 *The subspaces $N_0 \oplus \cdots \oplus N_j, imG_j$, and S_j as well as the values $r_j := rankG_j$ are independent of the special choice of admissible projector sequences.*

Proof: By Lemma 3.2, $N_0 \oplus \cdots \oplus N_j$ is independent of the chosen admissible projectors, and for each two different admissible projector sequences, the relations $\bar{G}_j = G_jZ_j$ with a nonsingular Z_j are given, hence $im\bar{G}_j = imG_j,\ rank\bar{G}_j = rankG_j$, and further $\bar{S}_j = ker\bar{\mathcal{W}}_jB = ker\mathcal{W}_jB = S_j$. □

Theorem 3.4 *Assume that $m = k$.*

(1) *If the matrix pencil $\{G, B\}$ is regular with index μ, then there are admissible up to level $\mu - 1$ projector sequences, and $r_{\mu-1} < r_\mu = m$, and vice versa.*

(2) *If the matrix pencil is regular, there exist admissible up to level m projector sequences. If the matrix pencil is singular, then there is no admissible projector sequence up to level m.*

(3) *If, for some κ, $0 \leq \kappa \leq m$, $Q_0, \ldots, Q_\kappa$ are admissible, but $N_{\kappa+1} \cap (N_0 \oplus \cdots \oplus N_\kappa) \neq 0$, then the matrix pencil $\{G, B\}$ is singular.*

Proof: Assertion (1) is just a consequence of Theorem 2.1 and Proposition 2.6.

Turn to assertion (2). For regular pencils we may continue the admissible sequences $Q_0, \ldots, Q_{\mu-1}$ (cf. Assertion (1)) by trivial projectors $Q_\mu = 0, \ldots, Q_m = 0$. Now, let $\{G, B\}$ be a singular pencil. If there is an admissible projector sequence $Q_0, \ldots, Q_m$ with some trivial projectors, at least the last one, $Q_m = 0,\ dim N_m = 0,\ G_m$ must be nonsingular. However, this contradicts the pencil singularity. If $Q_0, \ldots, Q_m$ are admissible, and all of them are nontrivial, we have $dim N_j \geq 1,\ j = 0, \ldots, m,\ dim(N_0 \oplus \cdots \oplus N_m) \geq m + 1$, but this is impossible as the host space of these subspaces has just dimension m.

Assertion (3) follows from the inclusion

$$N_{\kappa+1} \cap (N_0 \oplus \cdots \oplus N_\kappa) \subseteq N_{\kappa+1} \cap ker B_{\kappa+1} \tag{3.5}$$

and Proposition 1.1.(3). To verify (3.5), we simply observe that $ker B_{\kappa+1} = ker B_0 P_0 \cdots P_\kappa \supseteq N_0 \oplus \cdots \oplus N_\kappa$. □

4 Index one criteria for regular pencils

Theorem 4.1 *Assume that $m = k$. The following assertions are equivalent:*

(1) *$\{G_0, B_0\}$ represents a regular matrix pencil with Kronecker index one.*

(2) *$S_0 \cap N_0 = 0$.*

(3) *$G_1 = G_0 + B_0 Q_0$ is nonsingular independently of the choice of the projector Q_0 onto N_0.*

(4) *$S_0 \oplus N_0 = \mathbb{R}^m$ ($\mathbb{C}^m$).*

Proof: ([GM1], Theorem A.13):
We realize (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2).
Supposed (2) is valid, we consider the homogeneous equation $(G_0 + B_0 Q_0)z = 0$, that is, $B_0 Q_0 z = -G_0 z$, hence $Q_0 z \in S_0 \cap N_0 = 0,\ G_0 z = 0,\ z = Q_0 z = 0$. Therefore, $G_0 + B_0 Q_0$ is nonsingular.
Let (3) be valid. With any Q_0 we form $Q_{0*} := Q_0 (G_0 + B_0 Q_0)^{-1} B_0$ and observe that $Q_{0*} Q_0 = Q_0 (G_0 + B_0 Q_0)^{-1} (G_0 + B_0 Q_0) Q_0 = Q_0,\ Q_{*0}^2 = Q_{*0},\ im Q_{*0} = N_0,\ \ Q_{*0}$ is a further projector onto N_0. Consider the nullspace of this projector.

$Q_{0*}z = 0$ means $(G_0 + B_0Q_0)^{-1}B_0z = P_0(G_0 + B_0Q_0)^{-1}B_0z$, i.e., $B_0z = G_0(G_0 + B_0Q_0)^{-1}B_0z \in imG_0$. Hence, $kerQ_{0*} \subseteq S_0$. Conversely, if $z \in S_0$, i.e., $B_0z = G_0w$, we compute

$$Q_{0*}z = Q_0(G_0 + B_0Q_0)^{-1}G_0w = Q_0(G_0 + B_0Q_0)^{-1}(G_0 + B_0Q_0)P_0w = Q_0P_0w = 0.$$

It comes out that S_0 and N_0 are the nullspace and the range of the projector Q_{0*}. Consequently, (4) must be true.

Next, let (4) be given. Taking Q_0 to be the projector onto N_0 along S_0, the relation

$$Q_0 = Q_0(G_0 + B_0Q_0)^{-1}B_0 \tag{4.1}$$

is satisfied. For $\lambda \in \mathbb{C}$ we consider the homogeneous system

$$(\lambda G_0 + B_0)z = 0. \tag{4.2}$$

Taking into account that $(G_0+B_0Q_0)^{-1}G_0 = P_0$, $(G_0+B_0Q_0)^{-1}B_0 = (G_0+B_0Q_0)^{-1}B_0(P_0+Q_0) = (G_0 + B_0Q_0)^{-1}B_0P_0 + Q_0$, we write (4.2) equivalently as

$$\lambda P_0z + Q_0z + (G_0 + B_0Q_0)^{-1}B_0P_0z = 0. \tag{4.3}$$

Due to (4.1), equation (4.3) splits into the system

$$Q_0z = 0, \quad (\lambda I + P_0(G_0 + B_0Q_0)^{-1}B_0)P_0z = 0. \tag{4.4}$$

Now we realize that, for all λ that do not belong to the spectrum of the $m \times m$ matrix $-P_0(G_0 + B_0Q_0)^{-1}B_0$, (4.2) implies $z = P_0z = 0$, hence, the matrix pencil $\{G_0, B_0\}$ is regular. To show the index to be one, we transform this regular pencil into Kronecker canonical form by nonsingular E, F,

$$\tilde{G_0} := EG_0F = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \; \tilde{B_0} := EB_0F = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}, \; J^\mu = 0, J^{\mu-1} \neq 0,$$

and derive
$\tilde{N_0} := ker\tilde{G_0} = kerG_0F = F^{-1}kerG_0 = F^{-1}N_0$, $\tilde{S_0} := ker\tilde{\mathcal{W}}_0\tilde{B_0} = ker\mathcal{W}_0B_0F = F^{-1}ker\mathcal{W}_0B_0 = F^{-1}S_0$, $\tilde{N_0} \cap \tilde{S_0} = F^{-1}(N_0 \cap S_0)$. From $N_0 \cap S_0 = 0$ we know that $\tilde{N_0} \cap \tilde{S_0} = 0$.
Looking at $\tilde{N_0} \cap \tilde{S_0} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u = 0, \; Jv = 0, \; v \in imJ \right\}$ we see that $\tilde{N_0} \cap \tilde{S_0} = 0$ implies $\mu = 1$, i.e., $J = 0$, since otherwise $\tilde{N_0} \cap \tilde{S_0}$ would be nontrivial.

Finally, assertion (1) implies assertion (2) via $0 = \tilde{N_0} \cap \tilde{S_0} = F^{-1}(N_0 \cap S_0)$. □

5 s-admissible projector sequences

For singular matrix pencils $\{G, B\}$ the admissible projector sequences do not fit in general. In order to reduce the multitude of possible projectors in (1.1)-(1.4) and to exclude

situations as in Example 2.3, respectively, we proceed as follows:
Decompose

$$N_0 + \cdots + N_{i-1} = [(N_0 + \cdots + N_{i-1}) \cap N_i] \oplus X_i, \quad X_i \subseteq N_0 + \cdots + N_{i-1},$$

and choose Q_i so that

$$X_i \subseteq kerQ_i, \quad i \geq 1. \tag{5.1}$$

Because of $X_i \cap N_i = 0$, projectors Q_i satisfying (5.1) do always exist (cf. Lemma 2.5). In particular, X_i can be chosen to be the orthogonal complement of $(N_0 + \cdots + N_{i-1}) \cap N_i$ in $N_0 + \cdots + N_{i-1}$, i.e., $X_i = \{z \in N_0 + \cdots + N_{i-1} :< z, w >= 0$ for all $w \in (N_0 + \cdots + N_{i-1}) \cap N_i\}$.

Definition:
The projectors $Q_i,\ i \geq 0$, in (1.1)-(1.4) are *s-admissible* (*singularly-admissible*) up to level κ if condition (5.1) is valid for $i = 1, \ldots, \kappa$.

For regular matrix pencils, the s-admissible projectors $Q_0, \ldots, Q_\kappa$ are admissible, and $N_0 \oplus \cdots \oplus N_j = N_0 + \cdots + N_j,\ j = 1, \ldots, \kappa$ (cf. Theorem 3.4).

Proposition 5.1 *The following properties are valid for s-admissible projector sequences:*

(1) $kerP_0 \cdots P_i = N_0 + \cdots + N_i$,

(2) $(N_0 + \cdots + N_{i-1}) \cap N_i \subseteq N_i \cap kerB_i = N_i \cap N_{i+1} \subseteq (N_0 + \cdots + N_i) \cap N_{i+1}$,

(3) $P_0 \cdots P_i$ *and* $P_0 \cdots P_{i-1}Q_i$ *are projectors,*

(4) $P_0 \cdots P_{i-1}Q_iQ_j = 0,\ j = 0, \ldots, i-1,\ kerP_0 \cdots P_{i-1}Q_i \supseteq N_0 + \cdots + N_{i-1}$.

Proof:
We verify assertion (1) by induction. $P_0P_1z = 0$ means $\tilde{z} := (I - Q_1)z \in N_0$, i.e., $z = \tilde{z} + Q_1z \in N_0 + N_1$. Each $z \in N_0 + N_1 = X_1 + (N_0 \cap N_1) + N_1$ may be written as $z = x_1 + z_{01} + z_1$ so that $P_0P_1z = P_0P_1x_1 = P_0x_1 = 0$. Let $kerP_0 \cdots P_{i-1} = N_0 + \cdots + N_{i-1}$ be given. From $z \in kerP_0 \cdots P_i$, i.e., $\tilde{z} := (I - Q_i)z \in N_0 + \cdots + N_{i-1}$ it follows that $z = \tilde{z} + Q_iz \in N_0 + \cdots + N_{i-1} + N_i$. Conversely, for $z \in N_0 + \cdots + N_i = ((N_0 + \cdots + N_{i-1}) \cap N_i) + X_i + N_i$ we use the decomposition $z = z_{0i} + x_i + z_i$ and derive $P_0 \cdots P_{i-1}P_iz = P_0 \cdots P_{i-1}P_ix_i = P_0 \cdots P_{i-1}x_i = 0$. Hence, we have $kerP_0 \cdots P_i = N_0 + \cdots + N_i$.
Turn to assertion (2). $z \in (N_0 + \cdots + N_{i-1}) \cap N_i$ means $P_0 \cdots P_{i-1}z = 0,\ z = Q_iz$, which leads to $G_{i+1}z = B_iz = B_iP_0 \cdots P_{i-1}z = 0$, i.e., $z \in N_{i+1},\ z \in kerB_i$, and we are done with assertion (2).
Next we show (3). Because of $kerP_0 \cdots P_i = N_0 + \cdots + N_i$ it holds that $P_0 \cdots P_iQ_j = 0,\ j = 0, \ldots, i$, and further $P_0 \cdots P_i = P_0 \cdots P_iP_j = 0,\ j = 0, \ldots, i,\ P_0 \cdots P_i \cdot P_0 \cdots P_i = P_0 \cdots P_i,\ P_0 \cdots P_{i-1}Q_iP_0 \cdots P_{i-1}Q_i = P_0 \cdots P_{i-1}P_0 \cdots P_{i-1}Q_i - P_0 \cdots P_{i-1}P_iP_0 \cdots P_{i-1}Q_i = P_0 \cdots P_{i-1}Q_i - P_0 \cdots P_{i-1}P_iQ_i = P_0 \cdots P_{i-1}Q_i$.
Finally, we consider assertion (4).
Since $N_0 + \cdots + N_{i-1} = [(N_0 + \cdots + N_{i-1}) \cap N_i] \oplus X_i$ we may write $z \in N_0 + \cdots + N_{i-1}$ as $z = z_{0i} + x_i$. This yields $P_0 \cdots P_{i-1}Q_iz = P_0 \cdots P_{i-1}z_{0i} = 0$. □

Corollary 5.2 *$G_{i+1}Q_j = B_jQ_j,\ j \le i$, is valid for s-admissible projectors, too.*

Proof:
$G_{i+1} = G_j + B_jQ_j + B_{j+1}Q_{j+1} + \cdots + B_iQ_i = G_j + B_jQ_j + B_0P_0\cdots P_jQ_{j+1} + \cdots + B_0P_0\cdots P_{i-1}Q_i$ yields $G_{i+1}Q_j = B_jQ_j$ for $j \le i$. □

Theorem 5.3 *The subspaces $N_0 + \cdots + N_j,\ imG_j$, and S_j as well as the values $r_j := rankG_j$ are independent of the special choice of s-admissible projector sequences.*

Proof:
Lemma 3.2 and Theorem 3.3 remain valid if we replace admissible projectors by s-admissible ones, use sums $N_0 + \cdots + N_j,\ \bar{N}_0 + \cdots + \bar{N}_j$ instead of the direct sums and if we use the somehow more complex invertible matrices

$$\begin{aligned} Z_j &:= \big(I + Q_{j-1}(\bar{P}_0\cdots\bar{P}_{j-2}\bar{Q}_{j-1} - P_0\cdots P_{j-2}Q_{j-1}) + \sum_{l=0}^{j-2} Q_l\mathfrak{A}_{j-1l}\bar{P}_0\cdots\bar{P}_{j-2}\bar{Q}_{j-1}\big)Z_{j-1}, \\ Z_1 &:= I + Q_0\bar{Q}_0P_0 = I + Q_0(\bar{Q}_0 - Q_0). \end{aligned}$$

Also the proof given for Lemma 3.3 can be applied if we take into account that, due to Corollary 5.2, it holds that $G_{p+1}Q_p\bar{P}_p = B_pQ_p\bar{P}_p = B_0P_0\cdots P_{p-1}Q_p\bar{P}_p = B_0P_0\cdots \cdots P_{p-1}Q_p\bar{P}_0\cdots\bar{P}_{p-1}\bar{P}_p = G_{p+1}Q_p\bar{P}_0\cdots\bar{P}_p$ and if we modify the matrices Y_p. □

Theorem 5.4 *If the matrix pencil $\{G, B\}$ is regular with index μ, then the s-admissible projector sequences are admissible ones, and $r_{\mu-1} < r_\mu = m$.*

6 Widely orthogonal s-admissible projectors

When realizing all free choices in s-admissible projector sequences in an orthogonal way, we provide uniquely determined projector sequences. Namely, we can start using the orthogonal projector Q_0 onto N_0, i.e.,

$$Q_0 = Q_0^T. \tag{6.1}$$

For $i \ge 1$, we determine

$$X_i := [(N_0 + \cdots + N_{i-1}) \cap N_i]^\perp \cap (N_0 + \cdots + N_{i-1}), \tag{6.2}$$

and choose Q_i to be the projector onto N_i along

$$[N_0 + \cdots + N_i]^\perp + X_i =: kerQ_i. \tag{6.3}$$

Due to $N_0 + \cdots + N_{i-1} + N_i = X_i \oplus N_i$, and since the decomposition $\mathbb{R}^m = [N_0 + \cdots + N_i] \oplus [N_0 + \cdots + N_i]^\perp = X_i \oplus N_i \oplus [N_0 + \cdots + N_i]^\perp$ holds true, Q_i is well defined by (6.2), (6.3).

Proposition 6.1 *The widely orthogonal choice (6.1) - (6.3) leads to*

$$\begin{gathered} im(P_0\cdots P_i) = [N_0 + \cdots + N_i]^\perp,\ \ ker(P_0\cdots P_i) = N_0 + \cdots + N_i, \\ P_0\cdots P_i = (P_0\cdots P_i)^\top,\ \ P_0\cdots P_{i-1}Q_i = (P_0\cdots P_{i-1}Q_i)^\top,\ \textit{for}\ i \ge 1. \end{gathered}$$

Proof: (6.1) yields $P_0 = P_0^\top,\ imP_0 = N_0^\perp,\ kerP_0 = N_0$. Due to Proposition 5.1 (1) we know the relation $kerP_0 \cdots P_i = N_0 + \cdots + N_i$ to be true, hence $P_0 \cdots P_{i-1}X_i = 0,\ imP_0 \cdots P_i = P_0 \cdots P_{i-1}imP_i = P_0 \cdots P_{i-1}([N_0 + \cdots + N_i]^\perp + X_i) = P_0 \cdots P_{i-1}([N_0 + \cdots + N_i]^\perp)$. From the inclusion $N_0 + \cdots + N_{i-1} \subseteq N_0 + \cdots + N_i$ we obtain $[N_0 + \cdots + N_i]^\perp \subseteq [N_0 + \cdots + N_{i-1}]^\perp$. Further, we have $imP_0P_1 = P_0imP_1 = P_0[N_0 + N_1]^\perp = P_0(N_0^\perp \cap N_1^\perp) = N_0^\perp \cap N_1^\perp = [N_0 + N_1]^\perp$, and supposing $imP_0 \cdots P_{i-1} = [N_0 + \cdots + N_{i-1}]^\perp$ we conclude $imP_0 \cdots P_i = P_0 \cdots P_{i-1}([N_0 + \cdots + N_i]^\perp) = [N_0 + \cdots + N_i]^\perp$.
The product $P_0 \cdots P_i$ represents a projector (cf. Proposition 5.1 (3)). Since it projects onto $[N_0 + \cdots + N_i]^\perp$ along $N_0 + \cdots + N_i$, it must be the orthoprojector, i.e., $P_0 \cdots P_i$ is symmetric. Finally, we derive $(P_0 \cdots P_{i-1}Q_i)^\top = (P_0 \cdots P_{i-1} - P_0 \cdots P_{i-1}P_i)^\top = P_0 \cdots P_{i-1} - P_0 \cdots P_{i-1}P_i = P_0 \cdots P_{i-1}Q_i$. □

7 Admissible projectors using the subspaces S_i

Since the susbpaces S_i are independent of the choice of admissible and s-admissible projectors, one can think of relying projectors also on these subspaces.
We have always (Proposition 1.1.(6))

$$S_{i+1} = N_0 + \cdots + N_i + S_0.$$

Theorem 4.1 says that $\{G, B\}$ with $m = k$ is a regular index one pencil if and only if $N_0 \cap S_0 = 0$. More generally, if $m = k,\ \{G, B\}$ is a regular pencil with index μ if and only if $N_{\mu-1} \cap S_{\mu-1} = 0$, but $N_j \cap S_j \neq 0,\ j = 0, \ldots, \mu - 2$. This is a consequence of Proposition 1.1.(5) and Theorem 2.1.
Decompose

$$S_0 = (N_0 \cap S_0) \oplus Z_0,\ \ Z_0 \subseteq S_0,$$

and choos Q_0 to satisfy the condition

$$Z_0 \subseteq kerQ_0. \tag{7.1}$$

It results that $Z_0 \subseteq imP_0$, i.e., $P_0Z_0 = Z_0$. Then we decompose $S_1 = S_0 + N_0$ into

$$S_1 = (N_1 \cap S_1) \oplus Z_1,\ \ N_0 \subseteq Z_1 \subseteq S_1.$$

This is possible if $N_0 \cap N_1 = 0$ is assumed. Now, Q_1 is chosen in such a way that

$$Z_1 \subseteq kerQ_1,$$

which implies $N_0 \subseteq kerQ_1,\ Q_1Q_0 = 0,\ P_1Z_1 = Z_1$. In general, for $i \geq 1$, assuming $(N_0 \oplus \cdots \oplus N_{i-1}) \cap N_i = 0$, we decompsoe

$$S_i = (S_i \cap N_i) \oplus Z_i,\ \ N_0 \oplus \cdots \oplus N_{i-1} \subseteq Z_i \subseteq S_i,$$

and choose Q_i so that the condition

$$Z_i \subseteq kerQ_i \tag{7.2}$$

is fulfilled.

For regular matrix pencils $\{G, B\}$ with index μ, this construction yields special admissible projector sequences with $S_{\mu-1} = Z_{\mu-1},\ S_{\mu-1} = kerQ_{\mu-1}$, so that $Q_{\mu-1}$ is the projector onto $N_{\mu-1}$ along $S_{\mu-1}$, and it holds that $Q_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B$.

References

[Ga] Gantmacher, F.R.: Teoriya matrits. Gos. Izdat. Techn.-Teor. Lit., Moskva 1953.

[GM1] Griepentrog, E., März, R.: Basic properties of some differential-algebraic equations. Zeitschrift für Analysis und ihre Anwendungen **8**, 25-40, 1989.

[GM2] Griepentrog, E., März, R.: Differential-algebraic equations and their numerical treatment. Teubner-Texte zur Mathematik **88**, Leipzig, 1986.

[M1] März, R.: A matrix chain for analyzing differential-algebraic equations. Humboldt-Universität zu Berlin, Inst. f. Mathematik, Preprint 162, 1987.

[M2] März, R.: Canonical projectors for linear differential algebraic equations. Computers Math. Applic. **31**, 121-135, 1996.

[M3] März, R.: The index of linear differential algebraic equations with properly stated leading terms. Results in Mathematics **42**, 308-338, 2002.

[M4] März, R.: Solvability of linear differential algebraic equations with properly stated leading terms. Results in Mathematics **45**, 88-105, 2004.