On Émery's inequality and a variation-of-constants formula

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Abstract

A generalization of Émery's inequality for stochastic integrals is shown for convolution integrals of the form $(\int_0^t g(t-s)Y(s-)\,\mathrm{d}Z(s))_{t\geqslant 0}$, where Z is a semimartingale, Y an adapted càdlàg process, and g a deterministic function. The function g is assumed to be absolutely continuous with bounded derivative. The function g may also have jumps, provided that the jump sizes are absolutely summable. The inequality is used to prove existence and uniqueness of solutions of equations of variation-of-constants type. As a consequence, it is shown that the solution of a semilinear delay differential equation with functional Lipschitz diffusion coefficient and driven by a general semimartingale satisfies a variation-of-constants formula.

Key words: Émery's inequality, functional Lipschitz coefficient, linear drift, semimartingale, stochastic delay differential equation, variation-of-constants formula

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1 Introduction

The variation-of-constants formula is a key tool in the study of long term behavior of semilinear stochastic differential equations. It describes the diffusion and nonlinearity in the drift as perturbations of a deterministic linear equation and thus enables to reveal information on the long term behavior (see for instance [1, 11]). In the case of stochastic *delay* differential equations driven by semimartingales, however, such a variation-of-constants formula seemed to be unknown. We will prove in this paper a variation-of-constants formula for stochastic delay differential equations with linear drift and a functional Lipschitz diffusion coefficient driven by a general semimartingale. Our proof includes the extension of several other important results. In particular, we present an extension of Émery's inequality for stochastic integrals.

Consider the stochastic delay differential equation

$$\left\{\begin{array}{ll} \mathrm{d}X(t) = \displaystyle\int_{(-\infty,0]} X(t+a)\mu(\mathrm{d}a)\,\mathrm{d}t + F(X)(t-)\,\mathrm{d}Z(t), & t\geqslant 0, \\ X(t)=\Phi(t), & t\leqslant 0, \end{array}\right. \tag{1.1}$$

where μ is a finite signed Borel measure on $(-\infty,0]$, Z is a semimartingale, F is a functional Lipschitz coefficient, and $(\Phi(t))_{t\leqslant 0}$ is a given suitable initial process. We want to show that the solution of (1.1) satisfies the variation-of-constants formula

$$X(t) = \int_0^t g(t-s)\,\mathrm{d}J(s) + \int_0^t g(t-s)F(X)(s-)\,\mathrm{d}Z(s), \quad t \geqslant 0, \tag{1.2}$$

where the semimartingale J given by

$$J(t) = \Phi(0) + \int_0^t \int_{(-\infty,-s]} \Phi(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \tag{1.3}$$

contains the initial condition. The function g is the fundamental solution of the underlying deterministic delay equation, that is,

$$\begin{cases} g|'_{[0,\infty)} = \int_{(-\infty,0]} g(\bullet + a)\mu(\mathrm{d}a) \text{ Lebesgue a.e. on } [0,\infty), \\ g(0) = 1,\ g(t) = 0,\ t < 0. \end{cases} \tag{1.4}$$

It is well known (see [5]) that equation (1.4) has indeed a unique solution g with the property that $g|_{[0,\infty)}$ is absolutely continuous.

If a solution of (1.2) exists, it can be shown by a Fubini argument that it also satisfies the original stochastic delay differential equation (1.1). Since (1.1) is known to admit a unique solution, we infer that this solution then satisfies (1.2). Thus, it remains to prove existence of solutions of (1.2). Our proof of existence (and uniqueness) of solutions of (1.2) is an extension of the proof for stochastic differential equations presented in [10]. The idea there is to use localization arguments in order to reduce to a Banach fixed point argument in a suitable space of stochastic processes. The key estimate to obtain a contraction is an inequality due to Émery, see [4] or [10, Theorem V.3]. It says that for an adapted càdlàg process Y and a semimartingale Z the size of the stochastic integral can be estimated by

$$\left\| \int_0^\bullet Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} \leqslant \|Y\|_{S^p} \|Z\|_{H^q}, \tag{1.5}$$

for certain suitable norms on spaces of processes and semimartingales.

It turns out that for the more general equations of variation-of-constants type an extension of Émery's inequality is needed, namely for integral processes of the form $\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s)$, where g is a deterministic function. We will show that for a large class of functions g the inequality

$$\left\| \int_0^\bullet g(t-s)Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} \leqslant R\|Y\|_{S^p} \|Z\|_{H^q} \tag{1.6}$$

holds, where R is a constant independent of Y and Z. We establish an even more general inequality for integrals of the form $\int_0^\bullet Y(\bullet, s-)\,\mathrm{d}Z(s)$, where Y belongs to a class of processes with two parameters.

With the inequality (1.6) we can prove that the variation-of-constants type equation

$$X(t) = J(t) + \int_0^t g(t-s)F(X)(s-)\,\mathrm{d}Z(s), \quad t \geqslant 0, \tag{1.7}$$

has a unique (up to indistinguishability) adapted càdlàg solution X, for any semimartingales J and Z. The nonlinear coefficient F is here assumed to be functional Lipschitz. With the aid of the solution of (1.7) we are then able to prove the next variation-of-constants formula for stochastic delay differential equations. For abbreviation, we denote by $\mathbb{D}$ the space of all adapted càdlàg processes on a filtered probability space that satisfies the usual conditions.

Theorem 1.1. *Let μ be a finite signed Borel measure on $(-\infty, 0]$ and let $g : \mathbb{R} \to \mathbb{R}$ be the unique solution of* (1.4) *with $g|_{[0,\infty)}$ absolutely continuous. Let $F : \mathbb{D} \to \mathbb{D}$ be functional Lipschitz. Let J and Z be semimartingales. Then a process $X \in \mathbb{D}$ satisfies*

$$\begin{aligned} X(t) \quad &= \quad J(t) + \int_0^t \int_{(-s,0]} X(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\qquad + \int_0^t F(X)(s-)\,\mathrm{d}Z(s), \quad t \geqslant 0, \end{aligned} \tag{1.8}$$

if and only if it satisfies the variation-of-constants formula (1.2)*. Moreover, there exists one and only one $X \in \mathbb{D}$ satisfying* (1.8) *and* (1.2)*.*

The outline of the paper is as follows. In order to make the paper self-contained, Section 2 settles notation and briefly reviews the basic constructions and tools that we need in the sequel. In Section 3 we prove an inequality of Emery type for stochastic integrals of two parameter processes. Section 4 then derives the inequality (1.6). Existence and uniqueness of solutions of equation (1.7) is discussed in Section 5. Finally in Section 6 we prove the variation-of-constants formula, Theorem 1.1.

2 Preliminaries

2.1 Processes

All random variables and stochastic processes are assumed to be defined on a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_t$ satisfies the usual conditions (see [8, Definition 1.2.25]). Let $I \subset [0, \infty)$ be an interval and let $\mathbb{D}(I)$ denote the set of all adapted processes $(X(t))_{t\in I}$ with paths that are almost surely *càdlàg* (that is, right-continuous and the left limit exists at every $t \in I$ distinct from the left endpoint of I). If $X, Y \in \mathbb{D}(I)$ satisfy $X(t) = Y(t)$ a.s. for every $t \in I$, then they are *indistinguishable*, that is, $X(t) = Y(t)$ for all $t \in I$ a.s. (see [8, Problem 1.1.5]). We will identify processes that are indistinguishable. Every process $X \in \mathbb{D}(I)$ is jointly measurable from $\Omega \times I \to \mathbb{R}$ (see [8, Remark 1.1.14]). For a process $X \in \mathbb{D}(I)$, where $0 \in I$, and a stopping time T we define the *stopped process* X^T by

$$(X^T)(t)(\omega) = X(t \wedge T(\omega))(\omega), \quad \omega \in \Omega,\ t \in I,$$

and X^{T-} by

$$(X^{T-})(t)(\omega) = \begin{cases} X(t)(\omega)1_{\{0\leqslant t<T(\omega)\}} + X(t \wedge T(\omega)-)(\omega)1_{\{t\geqslant T(\omega)\}} & \text{if } T(\omega) > 0, \\ 0 & \text{if } T(\omega) = 0, \end{cases}$$

$\omega \in \Omega$, $t \in I$. Here $(X(t \wedge T(\omega)-))(\omega) = \lim_{s\uparrow T(\omega)} X(t \wedge s)(\omega)$ for $\omega \in \Omega$ with $T(\omega) > 0$. Stopping times are allowed to attain the value ∞. The jumps of a process $X \in \mathbb{D}(I)$, where $I \subset [0, \infty)$ is an interval with left endpoint a, are defined by $(\Delta X)(t) = X(t) - X(t-)$ for $t \in I$, $t \neq a$, and $(\Delta X)(a) = X(a)$. Further, by convention, $X(a-) = 0$.

For an interval I and a function $f : I \to \mathbb{R}$ we define the *total variation of f over I* as

$$\mathrm{Var}_I(f) = \sup \sum_{k=0}^{m-1} |f(t_{k+1}) - f(t_k)|$$

where the supremum is taken over all $t_0, \ldots, t_m \in I$ with $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_m$ and all $m \in \mathbb{N}$. A process $X \in \mathbb{D}(I)$ with paths that have almost surely finite total variation over each bounded subinterval of I will be called an *FV-process* and $\mathrm{Var}_I(X)$ is defined pathwise.

Let L^p denote the Lebesgue space $L^p(\Omega, \mathcal{F}, \mathbb{P})$, where $1 \leqslant p \leqslant \infty$. For a process $(X(t))_{t\in I}$ define

$$\|X\|_{S^p(I)} = \left\| \sup_{t\in I} |X(t)| \right\|_{L^p}$$

(possibly ∞) and

$$S^p(I) = \{X \in \mathbb{D}(I) : \|X\|_{S^p(I)} < \infty\}.$$

If the interval of definition is clear from the context, we will simply write S^p and $\|\bullet\|_{S^p}$.

2.2 Semimartingales

We adopt the definitions and notation of [2, 6, 7]. Recall that a process $X \in \mathbb{D}[0,\infty)$ is called a *semimartingale* if there exist a local martingale M and an FV-process A such that $X(t) = X(0) + M(t) + A(t)$ a.s. for all $t \geqslant 0$. For two semimartingales X and Y we denote by $[X, Y]$ their covariation (see [2, VII.42] or [7, p. 519]). For any semimartingale X the process $[X, X]$ is positive and increasing (see [7, Theorem 26.6(ii)]). We denote $[X, X]_\infty = \sup_{t\geqslant 0}[X, X](t)$.

We will use the above terminology also for processes $X \in \mathbb{D}[a, b]$, where $0 \leqslant a \leqslant b$. We say that $X \in \mathbb{D}[a, b]$ is a local martingale (or semimartingale) if there exists a local martingale (or semimartingale) $Y \in \mathbb{D}[0,\infty)$ such that $X(t) = Y(t)$ for all $t \in [a, b]$. If $X_1, X_2 \in \mathbb{D}[a, b]$ are semimartingales and $Y_1, Y_2 \in \mathbb{D}[0,\infty)$ are semimartingales such that $X_i(t) = Y_i(t)$ for all $t \in [a, b]$, $i = 1, 2$, then we define $[X_1, X_2](t) := [Y_1^b - Y_1^{a-}, Y_2^b - Y_2^{a-}](t)$, $t \in [a, b]$.

For a semimartingale $Z \in \mathbb{D}[a, b]$ with $Z(a) = 0$ we define

$$\begin{aligned}\|Z\|_{H^p[a,b]} = \inf\{\|[M, M](b)^{1/2} + \mathrm{Var}_{[a,b]}(A)\|_{L^p} : Z = M + A \text{ with}\\ M \text{ a local martingale, } A \text{ an FV-process,}\\ \text{and } M(a) = A(a) = 0\}\end{aligned} \tag{2.1}$$

(possibly ∞) and let

$$H^p[a, b] := \{Z \text{ semimartingale} : Z(a) = 0,\ \|Z\|_{H^p[a,b]} < \infty\}.$$

The space $H^p[0,\infty)$ is defined similarly by replacing the norm in (2.1) by $\|[M, M]_\infty^{1/2} + \mathrm{Var}_{[0,\infty)}(A)\|_{L^p}$. Observe that for any stopping time T and any $Z \in H^p[a, b]$ we have $Z^{T-} \in H^p[a, b]$ and $\|Z^{T-}\|_{H^p} \leqslant \|Z\|_{H^p}$.

Theorem 2.1. *Let $1 \leqslant p < \infty$ and $0 \leqslant a \leqslant b$. The spaces $(S^p[a, b], \|\bullet\|_{S^p[a,b]})$ and $(H^p[a, b], \|\bullet\|_{H^p[a,b]})$ are Banach spaces. Moreover, if $Z \in H^p[a, b]$ then $Z \in S^p[a, b]$ and there exists a constant $c_p > 0$ (independent of a and b) such that*

$$\|Z\|_{S^p[a,b]} \leqslant c_p \|Z\|_{H^p[a,b]} \text{ for all } Z \in H^p[a, b].$$

Proof. It is said in [10, p.188–189] that $\|\bullet\|_{S^p[0,\infty)}$ and $\|\bullet\|_{H^p[0,\infty)}$ are norms. It is straightforward that $(S^p[0,\infty), \|\bullet\|_{S^p[0,\infty)})$ is complete. Completeness of $H^p[0,\infty)$ endowed with $\|\bullet\|_{H^p[0,\infty)}$ is mentioned in [2, VII.98(e)]. The sets $\{X \in S^p[0,\infty) : X^b = X,\ X^{a-} = 0\}$ and $\{X \in H^p[0,\infty) : X^b = X,\ X^a = 0\}$ are closed subspaces of $S^p[0,\infty)$ and $H^p[0,\infty)$, respectively, and they are isometrically isomorphic to $S^p[a, b]$ and $H^p[a, b]$. The existence of c_p is the content of [10, Theorem V.2]. ☐

The next statement easily follows from [10, Theorem V.1, Corollary, p. 189–190].

Corollary 2.2. *Let $1 \leqslant p \leqslant \infty$ and $0 \leqslant a \leqslant b$. If $Z \in H^p[a,b]$, then $[Z,Z](b)^{1/2} \in L^p$ and*

$$\|[Z,Z](b)^{1/2}\|_{L^p} \leqslant \|Z\|_{H^p[a,b]}.$$

Further, if $M \in \mathbb{D}[a,b]$ is a local martingale with $M(a)=0$ and $[M,M](b)^{1/2} \in L^p$, then $M \in H^p[a,b]$ and

$$\|M\|_{H^p[a,b]} = \|[M,M](b)^{1/2}\|_{L^p}.$$

2.3 Stochastic integrals

We use the stochastic integral as presented in [2, 6, 7]. Let us summarize the properties which we need. The *predictable σ-algebra* $\mathcal{P}$ is the σ-algebra in $[0,\infty)\times\Omega$ generated by the processes $(X(t))_{t\geqslant 0}$ that are adapted to $(\mathcal{F}_{t-})_{t\geqslant 0}$ and which have paths that are left-continuous on $(0,\infty)$. Here $\mathcal{F}_{t-}$ is the σ-algebra generated by $\mathcal{F}_s$ with $s<t$ if $t>0$, and $F_{0-} := \mathcal{F}_0$. A process X is *predictable* if $(t,\omega)\mapsto X(t,\omega)$ is measurable with respect to the predictable σ-algebra. For an interval I containing 0, a process $(X(t))_{t\in I}$ is *locally bounded* if there exist stopping times $T_k \uparrow \infty$, that is, $0 = T_0 \leqslant T_1 \leqslant \dots$ with $\sup_k T_k = \infty$ a.s., such that for each k there is a constant c_k with, a.s., $|X^{T_k}(t) - X(0)| \leqslant c_k$ for all $t \in I$. For any process $X \in \mathbb{D}[0,\infty)$ the process $t \mapsto X(t-)$ is both predictable and locally bounded. We consider the class $\mathcal{E}$ of processes of the form

$$H(t) = H_{-1}1_{\{0\}}(t) + H_0 1_{(0,t_1]}(t) + \dots + H_{n-1}1_{(t_{n-1},\infty)}(t), \quad t \geqslant 0, \tag{2.2}$$

where H_{-1}, H_0 are $\mathcal{F}_0$-measurable and H_i are $\mathcal{F}_{t_i}$-measurable random variables for $i \geqslant 1$ such that $\operatorname{essup}|H_i| < \infty$ and where $0 = t_0 \leqslant t_1 \leqslant \dots \leqslant t_n = \infty$, $n \in \mathbb{N}$. For a semimartingale X and a process H given by (2.2) the stochastic integral is defined by

$$(H\bullet X)(t) = \int_0^t H(s)\,\mathrm{d}X(s) := \sum_{i=1}^n H_{i-1}\Big(X(t_i\wedge t) - X(t_{i-1}\wedge t)\Big), \quad t \geqslant 0.$$

The next theorem (see [2, VIII.3 and 9], [6, Theorem I.4.31 and I.4.33-37], or [7, Theorem 26.4], and [7, Theorem 26.6(ii) and (v)]) extends the stochastic integral to all locally bounded predictable processes.

Theorem 2.3. *Let X be a semimartingale. The map $H \mapsto H\bullet X$ on $\mathcal{E}$ has a unique linear extension (also denoted by $H \mapsto H\bullet X$) on the space of all predictable locally bounded processes into the space of adapted càdlàg processes such that if $(H^n)_n$ is a sequence of predictable processes with $|H^n(t)| \leqslant K(t)$ for all $t \geqslant 0$, $n \in \mathbb{N}$ and for some locally bounded predictable process K and $H^n(t)(\omega) \to H(t)(\omega)$ for all $t \geqslant 0$, $\omega \in \Omega$ and for some process H, then*

$$(H^n\bullet X)(t) \to (H\bullet X)(t) \text{ in probability for all } t \geqslant 0.$$

Moreover, for every locally bounded predictable processes H and K the following statements hold:

(a) *$H\bullet X$ is a semimartingale;*

(b) *$K(H\bullet X)$ and $(KH)\bullet X$ are indistinguishable;*

(c) *$\Delta(H\bullet X)$ and $H\Delta X$ are indistinguishable and $(H\bullet X)(0) = \Delta(H\bullet X)(0) = H(0)X(0)$;*

(d) *if X is a local martingale then $H \bullet X$ is a local martingale and*

$$[H \bullet X, H \bullet X](t) = \int_0^t H(s)^2 \, \mathrm{d}[X,X](s) \text{ for all } t \geqslant 0;$$

(e) *if X is of bounded variation then $H \bullet X$ is of bounded variation and*

$$\operatorname{Var}_{[0,\infty)}(H \bullet X) \leqslant \sup_{t \geqslant 0} |H(t)| \operatorname{Var}_{[0,\infty)}(X);$$

(f) *if T is a stopping time, then $1_{[0,T]} \bullet X = X^T$ and $(H \bullet X)^T = (H 1_{[0,T]}) \bullet X = H \bullet X^T$ up to indistinguishability.*

It follows that the stochastic integral $\int_0^t H(s-) \, \mathrm{d}X(s)$, $t \geqslant 0$, is well defined for any $H \in \mathbb{D}[0,\infty)$ and any semimartingale X. If X is an FV-process and $H \in \mathbb{D}[0,\infty)$, then the stochastic integral $\int_0^t H(s-) \, \mathrm{d}X(s)$ equals the pathwise defined Stieltjes integral, where the convergence of the Riemann-Stieltjes sums holds only in the sense of convergence of nets that are indexed by the set of partitions directed by the refinement relation.

The precise formulation of Émery's inequality reads as follows (see [4] or [10, Theorem V.3]).

Theorem 2.4 (Émery's inequality). *Let $p,q,r \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (with the convention that $\frac{1}{\infty} = 0$). Let $T > 0$. For every process $(Y(t))_{t \in [0,T]}$ in $S^p[0,T]$ and every semimartingale $(Z(t))_{t \in [0,T]}$ in $H^q[0,T]$ the process $(\int_0^\bullet Y(s-) \, \mathrm{d}Z(s))_{t \in [0,T]}$ is in $H^r[0,T]$ and*

$$\left\| \int_0^\bullet Y(s-) \, \mathrm{d}Z(s) \right\|_{H^r[0,T]} \leqslant \|Y\|_{S^p[0,T]} \|Z\|_{H^q[0,T]}.$$

In Section 3 and Section 6 we need the following stochastic Fubini theorem, which we collect from [10].

Theorem 2.5. *Let $(A,\mathcal{A})$ be a measurable space and let μ be a finite signed measure on $\mathcal{A}$. Let $\Phi\colon A \times [0,\infty) \times \Omega \to \mathbb{R}$ be an $\mathcal{A} \otimes \mathcal{P}$-measurable map, where $\mathcal{P}$ denotes the predictable σ-algebra in $[0,\infty) \times \Omega$. Let Z be a semimartingale with $Z(0) = 0$. If for each $a \in A$ the process $\Phi(a,\bullet)$ is locally bounded, then*

(i) *for every $a \in A$ there exists an adapted càdlàg version $(I_a(t))_{t \geqslant 0}$ of the stochastic integral*

$$\left(\int_0^t \Phi(a,s) \, \mathrm{d}Z(s) \right)_{t \geqslant 0}$$

such that the map $(a,t,\omega) \mapsto I_a(t,\omega)$ is $\mathcal{A} \times \mathcal{B}([0,\infty)) \otimes \mathcal{F}$-measurable;

(ii) *if moreover the process $\int_A \Phi(a,\bullet)^2 |\mu|(\mathrm{d}a)$ is locally bounded, then a.s.*

$$\int_A \left(\int_0^t \Phi(a,s) \, \mathrm{d}Z(s) \right) \mu(\mathrm{d}a) = \int_0^t \left(\int_A \Phi(a,s) \mu(\mathrm{d}a) \right) \mathrm{d}Z(s), \ t \geqslant 0,$$

where for the inner integral at the left hand side the versions of (i) *are chosen.*

Proof. Due to [10, Theorem IV.15, p.134], [10, Corollary IV.44, p.159] yields (i). Observe that the measurability conditions yield that the process $\int_A \Phi(a,\bullet)^2 |\mu|(\mathrm{d}a)$ is predictable. Assertion (ii) follows therefore by linearity from [10, Theorem IV.46, p.169], again due to [10, Theorem IV.15, p.134]. □

3 A class of processes with two parameters

In this section we prove the next theorem.

Theorem 3.1. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $1 \leqslant r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $T > 0$ and let $(Y(t,s))_{t,s\in[0,T]}$ be a process such that $Y(t,\bullet)$ is an adapted càdlàg process for every $t \in [0,T]$ and such that there exists a process $(Y_1(t,s))_{t,s\in[0,T]}$ with*

$$Y(t,s-) = Y(0,s-) + \int_0^t Y_1(u,s)\,\mathrm{d}u \quad \textit{for all } t \in [0,T] \textit{ a.s.}$$

for each $s \in [0,T]$, where $(t,s,\omega) \mapsto Y_1(t,s,\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable, $\sup_{s\in[0,T]} \int_0^T Y_1(u,s)^2\,\mathrm{d}u < \infty$ a.s., $\int_0^T Y_1(u,\bullet)^2\,\mathrm{d}u$ is a locally bounded process, and $\sup_{u\in[0,T]} \|Y_1(u,\bullet)\|_{S^p[0,T]} < \infty$. Then for every $Z \in H^q[0,T]$ the process $\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}Z(s)$ is a semimartingale and

$$\left\| \int_0^\bullet Y(\bullet,s-)\,\mathrm{d}Z(s) \right\|_{H^r[0,T]} \leqslant \Gamma_p(Y)\|Z\|_{H^q[0,T]},$$

where

$$\Gamma_p(Y) = \|Y(\bullet,\bullet-)\|_{S^p[0,T]} + (1+c_r)T \sup_{t\in[0,T]} \|Y_1(t,\bullet)\|_{S^p[0,T]}. \tag{3.1}$$

If $\Gamma_p(Y) < \infty$, then $\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}Z(s) \in H^r[0,T]$.

We use the convention that $\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}Z(s)$ denotes the process $(\int_0^t Y(t,s-)\,\mathrm{d}Z(s))_{t\in[0,T]}$ and, similarly, $Y(\bullet,\bullet-)$ denotes $(Y(t,t-))_{t\in[0,T]}$.

Throughout the section let p, q, r, T, and $(Y(t,s))_{t,s\in[0,T]}$ and $(Y_1(t,s))_{t,s\in[0,T]}$ be as in Theorem 3.1. Let further $Z \in \mathbb{D}[0,T]$ be a semimartingale with $Z = M + A$, $M(0) = A(0) = 0$, where M is a local martingale with $[M,M](T)^{1/2} \in L^q$ and A an FV-process with $\mathrm{Var}_{[0,T]}(A) \in L^q$. We divide the proof of Theorem 3.1 into several lemmas. We will consider the integral process $\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}Z(s)$, substitute Y_1, and apply stochastic Fubini. We begin by estimating the quadratic variation and total variation of the ensuing terms.

Lemma 3.2. *The map $s \mapsto Y(s,s-)$ is a predictable locally bounded process and*

$$\left[\int_0^\bullet Y(s,s-)\,\mathrm{d}M(s), \int_0^\bullet Y(s,s-)\,\mathrm{d}M(s)\right](T) \leqslant \sup_{s\in[0,T]} |Y(s,s-)|^2[M,M](T).$$

Proof. As $(s,\omega) \mapsto Y(0,s-,\omega)$ is $\mathcal{P}$-measurable and $(t,s,\omega) \mapsto Y_1(t,s,\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable, we have that $(t,s,\omega) \mapsto Y(t,s-,\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable. Hence $(s,\omega) \mapsto Y(s,s-,\omega)$ is $\mathcal{P}$-measurable. Further, $(Y(0,s))_{s\in[0,T]}$ is adapted and càdlàg, so that $Y(0,\bullet-)$ is locally bounded, and $(\int_0^s Y_1(u,s)\,\mathrm{d}u)_{s\in[0,T]}$ is locally bounded since $(\int_0^T Y_1(u,s)^2\,\mathrm{d}u)_{s\in[0,T]}$ is locally bounded. Hence $(Y(s,s-))_{s\in[0,T]}$ is locally bounded. The inequality follows from Theorem 2.3(d). □

Lemma 3.3. *The process $\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}A(s)$ is an FV-process and*

$$\mathrm{Var}_{[0,T]}\left(\int_0^\bullet Y(\bullet,s-)\,\mathrm{d}A(s)\right) \leqslant \sup_{s\in[0,T]} \Big(|Y(s,s-)| + \mathrm{Var}_{[s,T]}(Y(\bullet,s-))\Big)\,\mathrm{Var}_{[0,T]}(A).$$

Proof. If we consider the stochastic integrals as pathwise Stieltjes integrals we find for a partition $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$ a.s.

$$\sum_{k=0}^{m-1} \left| \int_0^T \left(1_{[0,t_{k+1}]}(s) Y(t_{k+1}, s-) - 1_{[0,t_k]}(s) Y(t_k, s-) \right) \mathrm{d}A(s) \right| \\ \leqslant \mathrm{Var}_{[0,T]}(A) \sup_{s \in [0,T]} \mathrm{Var}_{[0,T]} \left(1_{[0,\bullet]}(s) Y(\bullet, s-) \right).$$

By continuity of $Y(\bullet, s-)$,

$$\mathrm{Var}_{[0,T]} \Big(1_{[0,\bullet]}(s) Y(\bullet, s-) \Big) \leqslant |Y(s,s-)| + \mathrm{Var}_{[s,T]} \Big(Y(\bullet, s-) \Big),$$

and we obtain the desired inequality. Since $(\int_0^T Y_1(u,s)^2 \, \mathrm{d}u)_{s \in [0,T]}$ has a.s. bounded paths and $Y(0, \bullet)$ is càdlàg, it follows that $\sup_{s \in [0,T]} |Y(s,s-)| < \infty$ a.s. Thus $\int_0^\bullet Y(\bullet, s-) \, \mathrm{d}A(s)$ is an FV-process. □

In the next lemma we need Émery's inequality for integrands that are not càdlàg. It is easy to verify that Émery's proof in [4] establishes the inequality

$$\left\| \int_0^\bullet V(s) \, \mathrm{d}Z(s) \right\|_{H^r[0,T]} \leqslant \|V\|_{S^p[0,T]} \|Z\|_{H^q[0,T]}. \tag{3.2}$$

for any predictable process $(V(t))_{t \in [0,T]}$ with $\|V\|_{S^p} < \infty$ and any $Z \in H^q$.

Lemma 3.4. *The process* $\int_0^\bullet \left(\int_0^u Y_1(u,s) \, \mathrm{d}M(s) \right) \mathrm{d}u$ *is an FV-process with*

$$\left\| \mathrm{Var}_{[0,T]} \left(\int_0^\bullet \left(\int_0^u Y_1(u,s) \, \mathrm{d}M(s) \right) \mathrm{d}u \right) \right\|_{L^r} \leqslant c_r T \sup_{u \in [0,T]} \|Y_1(u, \bullet)\|_{S^p} \|M\|_{H^q}.$$

Proof. With aid of a well known inequality from the theory of Bochner integration (see [3, Lemma III.11.16(b) and Theorem III.2.20]) and Émery's inequality (3.2) we obtain

$$\begin{aligned} \left\| \int_0^T \left| \int_0^u Y_1(u,s) \, \mathrm{d}M(s) \right| \mathrm{d}u \right\|_{L^r} &\leqslant \int_0^T \left\| \int_0^u Y_1(u,s) \, \mathrm{d}M(s) \right\|_{L^r} \mathrm{d}u \\ &\leqslant \int_0^T \left\| \int_0^\bullet Y_1(u,s) \, \mathrm{d}M(s) \right\|_{S^r} \mathrm{d}u \leqslant c_r \int_0^T \left\| \int_0^\bullet Y_1(u,s) \, \mathrm{d}M(s) \right\|_{H^r} \mathrm{d}u \\ &\leqslant c_r T \sup_{u \in [0,T]} \|Y_1(u, \bullet)\|_{S^p} \|M\|_{H^q}. \end{aligned}$$

In particular, $\int_0^\bullet | \int_0^u Y_1(u,s-) \, \mathrm{d}M(s)| \, \mathrm{d}u$ is a.s. bounded. For absolutely continuous functions the total variation is given by the L^1-norm of the weak derivative and thus the assertions follow. □

Lemma 3.5. *The process* $\int_0^\bullet Y(\bullet, s-) \, \mathrm{d}Z(s)$ *is a semimartingale and*

$$\int_0^\bullet Y(\bullet, s-) \, \mathrm{d}Z(s) = \left(\int_0^\bullet Y(s,s-) \, \mathrm{d}M(s) \right) \\ + \left(\int_0^\bullet Y(\bullet, s-) \, \mathrm{d}A(s) + \int_0^\bullet \int_0^u Y_1(u,s) \, \mathrm{d}M(s) \, \mathrm{d}u \right),$$

where the process inside the first pair of parentheses is a local martingale and the process inside the second pair of parentheses an FV-process.

Proof. We will use the stochastic Fubini Theorem 2.5. Let $t \in [0,T]$. The process $(a,s,\omega) \mapsto 1_{[s,T]}(a) Y_1(a,s)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable, so $\int_0^t 1_{[\bullet,T]}(a) Y_1(a,\bullet)^2 \, \mathrm{d}a$ is predictable. The latter process is also locally bounded, as $\int_0^t Y_1(a,\bullet)^2 \, \mathrm{d}a$ is locally bounded. Hence we obtain a.s.

$$\begin{aligned} &\int_0^t \left(\int_0^\vartheta 1_{[s,T]}(a) Y_1(a,s) \, \mathrm{d}M(s) \right) \mathrm{d}a \\ &\qquad = \int_0^\vartheta \left(\int_0^t 1_{[s,T]}(a) Y_1(a,s) \, \mathrm{d}a \right) \mathrm{d}M(s) \quad \text{for all } \vartheta \in [0,T]. \end{aligned}$$

For $\vartheta = t$ the right hand side equals

$$\int_0^t Y(t,s-) \, \mathrm{d}M(s) - \int_0^t Y(s,s-) \, \mathrm{d}M(s).$$

As both sides are adapted càdlàg processes, the desired identity is established. Lemma 3.2, Theorem 2.3(d), and Lemmas 3.3 and 3.4 complete the proof. □

Proposition 3.6. *We have*

$$\left\| \int_0^\bullet Y(\bullet, s-) \, \mathrm{d}Z(s) \right\|_{H^r[0,T]} \leqslant \Gamma_p(Y) \| [M,M](T)^{1/2} + \mathrm{Var}_{[0,T]}(A) \|_{L^q},$$

where $\Gamma_p(Y) \in [0,\infty]$ *is given by* (3.1).

Proof. First observe that

$$\left\| \int_0^T \sup_{s \in [0,T]} |Y_1(t,s)| \, \mathrm{d}t \right\|_{L^p} \leqslant \int_0^T \left\| \sup_{s\in[0,T]} |Y_1(t,s)| \right\|_{L^p} \mathrm{d}t = \int_0^T \|Y_1(t,\bullet)\|_{S^p} \, \mathrm{d}t,$$

which is clear for $p = \infty$ and similar to the first step in the proof of Lemma 3.4 for $p < \infty$. Hence

$$\begin{aligned} \left\| \sup_{s\in[0,T]} \mathrm{Var}_{[0,T]}(Y(\bullet,s)) \right\|_{L^p} &= \left\| \sup_{s\in[0,T]} \int_0^T |Y_1(t,s)| \, \mathrm{d}t \right\|_{L^p} \\ &\leqslant \int_0^T \|Y_1(t,\bullet)\|_{S^p} \, \mathrm{d}t \leqslant T \sup_{t\in[0,T]} \|Y_1(t,\bullet)\|_{S^p}. \end{aligned}$$

Next, Lemmas 3.2–3.5 together with Hölder's inequality and Corollary 2.2 yield

$$\begin{aligned} &\left\| \int_0^\bullet Y(\bullet,s-) \, \mathrm{d}Z(s) \right\|_{H^r[0,T]} \\ &\quad \leqslant \Bigg\| \left[\int_0^\bullet Y(s,s-) \, \mathrm{d}M(s), \int_0^\bullet Y(s,s-) \, \mathrm{d}M(s) \right]^{1/2} (T) \\ &\qquad\qquad + \mathrm{Var}_{[0,T]} \left(\int_0^\bullet Y(\bullet,s-) \, \mathrm{d}A(s) + \int_{(0,\bullet]} \left(\int_0^u Y_1(u,s) \, \mathrm{d}M(s) \right) \mathrm{d}u \right) \Bigg\|_{L^r} \\ &\quad \leqslant \Gamma_p(Y) \| [M,M](T)^{1/2} + \mathrm{Var}_{[0,T]}(A) \|_{L^q}. \end{aligned}$$

□

Finally, Theorem 3.1 follows from Proposition 3.6 by taking the infimum over the semimartingale representations $Z = M + A$.

4 Application to convolutions

This section concerns an Émery inequality for convolutions of the form

$$\int_0^t g(t-s)Y(s-)\,\mathrm{d}Z(s), \quad t \in [0,T],$$

where g is a deterministic function, Y an adapted càdlàg process, and Z a semimartingale. If the function g is right-continuous, then the integral is well defined.

Let $W^{1,\infty}[a,b]$ denote the space of absolutely continuous functions h from the interval $[a,b]$ into $\mathbb{R}$ whose derivative h' is in $L^\infty[a,b]$. Let further the vector space of pure jump functions of bounded variation $PJBV[a,b]$ consist of all $j : [a,b] \to \mathbb{R}$ such that

$$j(t) = \sum_{i=1}^\infty \alpha_i 1_{[t_i,b]}(t), \quad t \in [a,b], \tag{4.1}$$

for some $t_i \in [a,b]$ and $\alpha_i \in \mathbb{R}$, $i \in \mathbb{N}$, with $\sum_{i=1}^\infty |\alpha_i| < \infty$.

Theorem 4.1. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $1 \leqslant r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $T > 0$. If $g : [0,T] \to \mathbb{R}$ is such that $g = h + j$ with $h \in W^{1,\infty}[0,T]$ and $j \in PJBV[0,T]$, then there exists a constant $R \geqslant 0$ such that for every $Y \in S^p[0,T]$ and every $Z \in H^q[0,T]$ we have $\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \in H^r[0,T]$ and*

$$\left\| \int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r[0,T]} \leqslant R\|Y\|_{S^p[0,T]}\|Z\|_{H^q[0,T]}.$$

If j is given by (4.1), then we have

$$R = |h(0)| + (1+c_r)T\|h'\|_\infty + \sum_{i=1}^\infty |\alpha_i|,$$

where c_r is the constant of Theorem 2.1.

The proof is divided into the next two lemmas. We will first study absolutely continuous functions g by means of Theorem 3.1 and then consider pure jump functions. We write H^r as shorthand for $H^r[0,T]$, S^p for $S^p[0,T]$, etc.

Lemma 4.2. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $1 \leqslant r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $T > 0$. If $g : [0,T] \to \mathbb{R}$ is absolutely continuous with derivative $g' \in L^\infty[0,T]$, then for every $Y \in S^p$ and every $Z \in H^q$ one has $\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \in H^r$ and*

$$\begin{aligned}\left\| \int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} \\ \leqslant \Big(|g(0)| + (1+c_r)T\|g'\|_\infty\Big)\|Y\|_{S^p}\|Z\|_{H^q}.\end{aligned}$$

Proof. We begin by extending g by setting $g(t) := g(0)$ for $t \in (-\infty, 0)$. Then g is absolutely continuous on $(-\infty, T]$, $g'(t) = 0$ for $t < 0$, and the supremum norms of g and g' are not changed by the extension. Define

$$Y(t,s) := g(t-s)Y(s) \quad \text{and} \quad Y_1(t,s) := g'(t-s)Y(s-), \quad t,s \in [0,T].$$

Since g is continuous, $Y(t,\bullet)$ is an adapted càdlàg process for every $t \in [0,T]$. Further, $Y_1(\bullet, s) \in L^\infty[0,T]$ a.s. and for $t \in [0,T]$,

$$\int_0^t Y_1(u,s)\,\mathrm{d}u = \int_0^t g'(u-s)Y(s-)\,\mathrm{d}u = Y(t,s-) - Y(0,s-).$$

Also, $(t,s,\omega) \mapsto Y_1(t,s,\omega)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable, $\int_0^T Y_1(u,\bullet)^2 \, \mathrm{d}u = \int_0^T g'(u-\bullet)^2 \, \mathrm{d}u Y(\bullet-)^2$ has a.s. bounded paths and is a locally bounded process, and

$$\sup_{u \in [0,T]} \|Y_1(u,\bullet)\|_{S^p} \leqslant \|g'\|_\infty \|Y\|_{S^p} < \infty.$$

Moreover, $\Gamma_p(Y) \leqslant \Big(|g(0)| + (1+c_r)T\|g'\|_\infty\Big)\|Y\|_{S^p} < \infty$. Hence, an application of Theorem 3.1 completes the proof. $\square$

The next lemma concerns pure jump functions.

Lemma 4.3. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $1 \leqslant r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $T > 0$ and let $g : [0,T] \to \mathbb{R}$ be given by*

$$g(t) = \sum_{i=1}^\infty \alpha_i 1_{[t_i,T]}(t), \quad t \in [0,T],$$

where $t_i \in [0,T]$ and $\alpha_i \in \mathbb{R}$, $i \in \mathbb{N}$, are such that $\sum_{i=1}^\infty |\alpha_i| < \infty$. Then for every $Y \in S^p$ and every $Z \in H^q$ one has $\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \in H^r$ and

$$\left\| \int_0^\bullet g(\bullet - s) Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} \leqslant \left(\sum_{i=1}^\infty |\alpha_i| \right) \|Y\|_{S^p} \|Z\|_{H^q}.$$

Proof. Let $Y \in S^p$ and $Z \in H^q$. Observe that g is a càdlàg function. For $m \in \mathbb{N}$ let

$$g_m(t) := \sum_{i=1}^m \alpha_i 1_{[t_i,T]}(t), \quad t \in [0,T].$$

Émery's inequality (Theorem 2.4) yields

$$\begin{aligned} \left\| \int_0^\bullet g_m(\bullet - s) Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} &\leqslant \sum_{i=1}^m |\alpha_i| \left\| \int_0^{(\bullet - t_i)^+} Y(s-)\,\mathrm{d}Z(s) \right\|_{H^r} \\ &\leqslant \sum_{i=1}^m |\alpha_i| \|Y\|_{S^p} \|Z\|_{H^q}. \end{aligned}$$

If we apply for $n > m$ the previous inequality to $g_n - g_m$ instead of g_m, we obtain that $\big(g_n(\bullet - s)Y(s-)\,\mathrm{d}Z(s)\big)_n$ is a Cauchy sequence in H^r. As H^r is complete (Theorem 2.1), it follows that $\int_0^\bullet g_n(\bullet - s)Y(s-)\,\mathrm{d}Z(s)$ converges in H^r to some $H \in H^r$. For fixed $t \in [0,T]$ we have

$$|(g_m(t-s) - g(t-s))Y(s-)| \leqslant \|g_m - g\|_\infty |Y(s-)|$$

for $0 \leqslant s \leqslant t$ and $\|g_m - g\|_\infty \to 0$ as $m \to \infty$. Hence by Theorem 2.3,

$$\int_0^s g_m(t-u)Y(u-)\,\mathrm{d}Z(u) \to \int_0^s g(t-u)Y(u-)\,\mathrm{d}Z(u)$$

in probability for all $s \in [0,t]$, and in particular for $s = t$. It follows that $H(t) = \int_0^t g(t-u)Y(u-)\,\mathrm{d}Z(u)$ and that the desired inequality holds. $\square$

5 Existence for equations of variation-of-constants type

In this section we will exploit the extended Émery inequality to show existence and uniqueness for stochastic equations of variation-of-constants type. We follow the proof of existence and uniqueness for stochastic differential equations as given in [10].

Definition 5.1. Let $I \subset [0,\infty)$ be an interval. A map $F : \mathbb{D}(I) \to \mathbb{D}(I)$ is called *functional Lipschitz* if there exists an increasing (finite) process $(K(t))_{t\in I}$ such that for all $X, Y \in \mathbb{D}(I)$,

(i) $X^{T-} = Y^{T-} \implies F(X)^{T-} = F(Y)^{T-}$ for every stopping time T,

(ii) $|F(X)(t) - F(Y)(t)| \leqslant K(t) \sup_{s\in I\cap[0,t]} |X(s) - Y(s)|$ a.s. for all $t \in I$.

Recall that equalities of processes such as in (i) are meant up to indistinguishability. It is contained in (ii) that a functionally Lipschitz map F is well-defined this way. Indeed, if X and Y in $\mathbb{D}(I)$ are indistinguishable, then (ii) yields that $F(X)(t) = F(Y)(t)$ a.s. for all $t \in I$ and hence $F(X)$ and $F(Y)$ are indistinguishable.

We will establish the next result by a sequence of lemmas.

Theorem 5.2. *Let $(Z(t))_{t\geqslant 0}$ be a semimartingale, let $J \in \mathbb{D}$, let $F : \mathbb{D} \to \mathbb{D}$ be a functional Lipschitz map, and let $g : [0,\infty) \to \mathbb{R}$ be such that $g|_{[0,T]} = h + j$ with $h \in W^{1,\infty}[0,T]$ and $j \in PJBV[0,T]$ for every $T > 0$. Then the equation*

$$X(t) = J(t) + \int_0^t g(t-s)F(X)(s-)\,\mathrm{d}Z(s), \quad t \geqslant 0,$$

has a unique (up to indistinguishability) solution $X \in \mathbb{D}$. If J is a semimartingale, then so is X.

Given constants $1 \leqslant p < \infty$, $t_0 > 0$, and $R > 0$, we will use the following property of a function $g : [0,t_0] \to \mathbb{R}$:

$$\begin{aligned}&g \text{ is càdlàg and for every } Y \in S^p[0,t_0] \text{ and } Z \in H^\infty[0,t_0] \text{ we have}\\ &\textstyle\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s) \in H^p[0,t_0] \text{ and}\\ &\textstyle\|\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s)\|_{H^p[0,t_0]} \leqslant R\|Y\|_{S^p[0,t_0]}\|Z\|_{H^\infty[0,t_0]}.\end{aligned} \tag{5.1}$$

Lemma 5.3. *Let $1 \leqslant p < \infty$, let $t_0 > 0$, and let $J \in S^p[0,t_0]$. Let $F : \mathbb{D}[0,t_0] \to \mathbb{D}[0,t_0]$ be functional Lipschitz as in Definition 5.1 with $F(0) = 0$ and $\sup_{t\in[0,t_0]} |K(t)| \leqslant k$ a.s. for some constant k. Let $g : [0,t_0] \to \mathbb{R}$ be a function and $R > 0$ be a constant such that (5.1) is satisfied. Let $Z \in H^\infty[0,t_0]$ be such that $\|Z\|_{H^\infty[0,t_0]} \leqslant 1/2\gamma$, where $\gamma = c_p kR$. Let T be a stopping time. Then the equation*

$$X(t) = J^{T-}(t) + \left(\int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)^{T-}(t), \quad 0 \leqslant t \leqslant t_0,$$

has a unique solution X in $S^p[0,t_0]$ and $\|X\|_{S^p[0,t_0]} \leqslant 2\|J\|_{S^p[0,t_0]}$.

Proof. Define

$$\Lambda(X)(t) := J(t) + \int_0^t g(t-s)F(X)(s-)\,\mathrm{d}Z(s), \quad t \in [0,t_0],\ X \in S^p[0,t_0].$$

By the assumption (5.1), the assumption $F(0)=0$, and the fact that $H^p\subset S^p$ we have that $\Lambda(X)\in S^p$ for every $X\in S^p$. Further, for $X,Y\in S^p$ we have

$$\Lambda(X)-\Lambda(Y)=\int_0^\bullet g(\bullet-s)\Big(F(X)(s-)-F(Y)(s-)\Big)\,\mathrm{d}Z(s)\in H^p.$$

Moreover, due to assumption (5.1),

$$\begin{aligned}\|\Lambda(X)^{T-}-\Lambda(Y)^{T-}\|_{H^p}&\leqslant\|\Lambda(X)-\Lambda(Y)\|_{H^p}\\&\leqslant R\|F(X)-F(Y)\|_{S^p}\|Z\|_{H^\infty}\leqslant\frac{1}{2c_pk}k\|X-Y\|_{S^p},\end{aligned}$$

so $\|\Lambda(X)^{T-}-\Lambda(Y)^{T-}\|_{S^p}\leqslant c_p\|\Lambda(X)^{T-}-\Lambda(Y)^{T-}\|_{H^p}\leqslant\frac12\|X-Y\|_{S^p}$. Since S^p is complete, we find a unique fixed point $X\in S^p$ of $\Lambda(\bullet)^{T-}$. This X is the solution as asserted, and

$$\|X-J\|_{S^p}=\|\Lambda(X)^{T-}-\Lambda(0)^{T-}\|_{S^p}\leqslant\frac12\|X\|_{S^p},$$

so that $\|X\|_{S^p}\leqslant\|X-J\|_{S^p}+\|J\|_{S^p}\leqslant\frac12\|X\|_{S^p}+\|J\|_{S^p}$ and hence $\|X\|_{S^p}\leqslant 2\|J\|_{S^p}$. ☐

Definition 5.4. (see [10, p.192]) Let $I=[0,\infty)$ or $I=[a,b]$ for some $0\leqslant a\leqslant b$. Let $Z\in H^\infty(I)$, and $\alpha>0$. Then Z is called *α-sliceable*, denoted by $Z\in\mathcal{S}(\alpha)$, if there exist stopping times $0=T_0\leqslant T_1\leqslant\cdots\leqslant T_k$ such that $Z=Z^{T_k-}$ and $\|(Z-Z^{T_i})^{T_{i+1}-}\|_{H^\infty(I)}\leqslant\alpha$ for $i=0,\dots,k-1$.

Theorem 5.5. (see [10, Theorem V.5, p.192]) *Let $Z\in\mathbb{D}[0,\infty)$ be a semimartingale with $Z(0)=0$ a.s.*

(i) *If $\alpha>0$, $Z\in\mathcal{S}(\alpha)$, and T is a stopping time, then $Z^T\in\mathcal{S}(\alpha)$ and $Z^{T-}\in\mathcal{S}(\alpha)$.*

(ii) *For every $\alpha>0$ there exist stopping times $T_k\uparrow\infty$, that is, $0=T_0\leqslant T_1\leqslant\cdots$ and $\sup_k T_k=\infty$ a.s., such that $Z^{T_k-}\in\mathcal{S}(\alpha)$ for all k.*

It follows that for every $\alpha>0$, $t_0>0$, and $Z\in H^\infty[0,t_0]$ there exist stopping times $T_k\uparrow\infty$ such that $Z^{T_k-}\in\mathcal{S}(\alpha)$ for all k.

The next lemma extends Lemma 5.3 to more general semimartingales.

Lemma 5.6. *The existence and uniqueness assertions of Lemma 5.3 remain true if the condition $\|Z\|_{H^\infty}\leqslant 1/(2\gamma)$ is relaxed to $Z\in\mathcal{S}(1/(2\gamma))$.*

Proof. Let $S_0,S_1,\dots,S_\ell$ be stopping times such that $0=S_0\leqslant S_1\leqslant\cdots\leqslant S_\ell$, $Z=Z^{S_\ell-}$, and $\|(Z-Z^{S_i})^{S_{i+1}-}\|_{H^\infty}\leqslant 1/2\gamma$ for $0\leqslant i\leqslant\ell-1$ (these exist because $Z\in\mathcal{S}(1/2\gamma)$). Let $T_i:=S_i\wedge T$, $i=0,\dots,\ell$. Then $0=T_0\leqslant T_1\leqslant\cdots\leqslant T_\ell$ and $\|(Z-Z^{T_i})^{T_{i+1}-}\|_{H^\infty}=\|((Z-Z^{S_i})^{S_{i+1}-})^{T-}\|_{H^\infty}\leqslant 1/2\gamma$. We argue by induction on i. If the equation stopped at T_i- has a unique solution, we first show that the equation stopped at T_i has a unique solution and then we show existence and uniqueness for the equation stopped at $T_{i+1}-$.

Suppose that $i\in\{0,\dots,\ell-1\}$ is such that the equation

$$X(t)=J^{T_i-}(t)+\left(\int_0^\bullet g(\bullet-s)F(X)(s-)\,\mathrm{d}Z^{T_i-}(s)\right)^{T_i-}(t),\quad 0\leqslant t\leqslant t_0,\tag{5.2}$$

has a unique solution $X\in S^p$. In order to simplify notation, we extend $F(U)(t):=F(U)(t_0)$, $J(t):=J(t_0)$, and $Z(t):=Z(t_0)$ for $t\geqslant t_0$ and $U\in\mathbb{D}[0,t_0]$. Further we interpret $[c,b]=\emptyset$ if $c>b$. Let

$$Y:=X+\left(\Delta J(T_i)+\Delta\Big(\int_0^\bullet g(\bullet-s)F(X)(s-)\,\mathrm{d}Z(s)\Big)(T_i)\right)1_{[T_i,t_0]}.$$

Observe that $Y^{T_i-}=X^{T_i-}$ and $Y^{T_i}=Y$.

▷ *Claim:* Y is the unique solution in S^p of

$$V(t) = J^{T_i}(t) + \left(\int_0^\bullet g(\bullet - s)F(V)(s-)\,\mathrm{d}Z^{T_i}(s)\right)^{T_i}(t), \quad 0 \leqslant t \leqslant t_0. \tag{5.3}$$

Proof. We have $Y^{T_i-} = X^{T_i-}$, so $F(Y)^{T_i-} = F(X)^{T_i-}$ and

$$\begin{aligned} J^{T_i}(t) &+ \left(\int_0^\bullet g(\bullet - s)F(Y)(s-)\,\mathrm{d}Z(s)\right)^{T_i}(t) \\ &= X(t) + \Delta J(T_i)1_{[T_i,t_0]}(t) \\ &\quad + \Delta\left(\int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)(T_i)1_{[T_i,t_0]}(t) \\ &= Y(t), \quad 0 \leqslant t \leqslant t_0, \end{aligned}$$

so Y satisfies (5.3). Further,

$$\left\| \sup_{s\in[0,t_0]} |Y(s)| \right\|_{L^p} \leqslant \|X\|_{S^p} + \|J\|_{S^p} + 2R\|F(X)\|_{S^p}\|Z\|_{H^\infty},$$

so $Y \in S^p$. To see uniqueness, suppose that $V \in S^p$ is another solution of (5.3). Then V^{T_i-} satisfies the equation for X, so $V^{T_i-} = X^{T_i-} = Y^{T_i-}$. From (5.3) it is clear that $V = V^{T_i}$ and that

$$\begin{aligned} V^{T_i} - V^{T_i-} &= \left(\Delta J(T_i) + \Delta\left(\int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)(T_i)\right)1_{[T_i,t_0]} \\ &= Y^{T_i} - Y^{T_i-}, \end{aligned}$$

so that $V = Y$. □

Let us introduce $D_iU := (U - U^{T_i})^{T_{i+1}-}$ and $G(U) = F(Y+U) - F(Y)$, $U \in \mathbb{D}[0,t_0]$. Consider the equation

$$\begin{aligned} U(t) = \Bigg(D_iJ(t) &+ \left(\int_0^\bullet g(\bullet - s)F(Y)(s-)\,\mathrm{d}Z^{T_{i+1}-}(s)\right)^{T_{i+1}-}(t) \\ &- \left(\int_0^\bullet g(\bullet - s)F(Y)(s-)\,\mathrm{d}Z(s)\right)^{T_i\wedge T_{i+1}-}(t)\Bigg) \\ &+ \left(\int_0^\bullet g(\bullet - s)G(U)(s-)\,\mathrm{d}D_iZ(s)\right)^{T_{i+1}-}(t), \quad 0 \leqslant t \leqslant t_0, \end{aligned} \tag{5.4}$$

▷ *Claim:* (a) equation (5.4) has a unique solution U in S^p, and (b) the process $V := U + Y^{T_{i+1}-}$ is the unique solution of

$$V(t) = J^{T_{i+1}-}(t) + \left(\int_0^\bullet g(\bullet - s)F(V)(s-)\,\mathrm{d}Z^{T_{i+1}-}(s)\right)^{T_{i+1}-}(t), \quad 0 \leqslant t \leqslant t_0.$$

Proof. (a) Observe that the sum of the first three terms of (5.4) is a member of S^p, G is functional Lipschitz with $G(0) = 0$ and satisfying the same estimates as F, and

$$\|D_iZ(s)\|_{H^\infty} = \|(Z - Z^{T_i})^{T_{i+1}-}\|_{H^\infty} < 1/2\gamma.$$

Now apply Lemma 5.3.

(b) From (5.4) it is clear that $U = U^{T_{i+1}-}$ and $U^{T_i-} = 0$. Consequently, $F(Y+U)^{T_i-} = F(Y)^{T_i-}$. Due to (5.3) we can easily check that

$$U + Y^{T_{i+1}-} = J^{T_{i+1}-} + \left(\int_0^\bullet g(\bullet - s)F(Y+U)(s-)\,\mathrm{d}Z^{T_{i+1}-}(s)\right)^{T_{i+1}-}.$$

For any solution V, the process $V - Y^{T_{i+1}-}$ satisfies (5.4) and therefore equals U. □

We conclude that if equation (5.2) has a unique solution X in S^p, then the equation (5.2) with T_i replaced by T_{i+1} has a unique solution in S^p as well. As for $i = 0$, $X = 0$ is the unique solution X in S^p, we find that there exists a unique solution $V \in S^p$ of (5.2) with $i = \ell$.

Finally, let

$$X(t) := J^{T-}(t) + \left(\int_0^\bullet g(\bullet - s)F(V)(s-)\,\mathrm{d}Z(s)\right)^{T-}(t), \quad t \in [0, t_0].$$

Because $T_\ell = T \wedge S_\ell$ and $Z^{S_\ell-} = Z$ we have $X^{T_\ell-} = V$ and hence

$$\left(\int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)^{T-}(t) = \left(\int_0^\bullet g(\bullet - s)F(X^{T_\ell-})(s-)\,\mathrm{d}Z^{T_\ell-}(s)\right)^{T-}$$
$$= \left(\int_0^\bullet g(\bullet - s)F(V)(s-)\,\mathrm{d}Z(s)\right)^{T-} = X(t) - J^{T-}(t),\ t \in [0, t_0].$$

□

We will increase the generality of the assumptions building on Lemma 5.6 in Proposition 5.8 below. The next lemma is needed in the proof of Proposition 5.8.

Lemma 5.7. *Let $Y \in \mathbb{D}[0,\infty)$ and let $Z \in \mathbb{D}[0,\infty)$ be a semimartingale. Let $1 \leqslant p < \infty$ and let $g : [0,\infty) \to \mathbb{R}$ be a function such that for every $t_0 > 0$ there exists a constant $R > 0$ such that* (5.1) *is satisfied. Then*

$$\left(\int_0^t g(t-s)Y(s-)\,\mathrm{d}Z(s)\right)_{t \geqslant 0}$$

is a semimartingale.

Proof. By convention, $Y(0-) = 0$, so we may assume $Z(0) = 0$. Observe that there exist stopping times $T_k \uparrow \infty$ such that $Y^{T_k-} \in S^p[0,\infty)$ for all k. Use Theorem 5.5 to choose the stopping times T_k such that also $Z^{T_k-} \in H^\infty[0,\infty)$ for each k. Then for each $t_0 > 0$, $(\int_0^\bullet g(\bullet - s)Y^{T_k-}(s-)\,\mathrm{d}Z^{T_k-}(s))_{t\in[0,t_0]} \in H^p[0,t_0]$. Hence

$$\left(\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s)\right)^{T_k \wedge t_0 -} = \left(\int_0^\bullet g(\bullet - s)Y^{T_k-}(s-)\,\mathrm{d}Z^{T_k-}(s)\right)^{T_k \wedge t_0 -}$$

equals a stopped semimartingale. It follows that $\int_0^\bullet g(\bullet - s)Y(s-)\,\mathrm{d}Z(s)$ is a local semimartingale and hence a semimartingale by [6, Proposition I.4.25(a) and (b)]. □

Proposition 5.8. *Let Z be a semimartingale, $J \in \mathbb{D}[0,\infty)$, and let $F : \mathbb{D}[0,\infty) \to \mathbb{D}[0,\infty)$ be functional Lipschitz. Let $g : [0,\infty) \to \mathbb{R}$ be a function such that for every $t_0 > 0$ there exists a constant $R > 0$ such that* (5.1) *is satisfied. Then*

$$X(t) = J(t) + \int_0^t g(t-s)F(X)(s-)\,\mathrm{d}Z(s), \tag{5.5}$$

$t \geqslant 0$, has a unique solution X in $\mathbb{D}[0,\infty)$. If J is a semimartingale, then so is X.

Proof. We use the notation of Definition 5.1. As $F(X)(0-) = 0$ for all X, we may assume that $Z(0) = 0$. We begin by replacing J by $J + \int_0^\bullet g(\bullet - s)F(0)(s-)\,\mathrm{d}Z(s)$ and F by $F(\bullet) - F(0)$. Thus we may assume that $F(0) = 0$.

▷ *Claim:* Let $t_0 > 0$. Suppose that $|K(t,\omega)| \leqslant k$ for a.e. ω and all $0 \leqslant t \leqslant t_0$. Let S be a stopping time. Then there is a unique process $X \in \mathbb{D}$ such that

$$X(t) = J^{S-}(t) + \left(\int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)^{S-}(t), \quad 0 \leqslant t \leqslant t_0. \tag{5.6}$$

Proof. Let $R > 0$ be a constant corresponding to t_0 such that (5.1) is satisfied. Let $\gamma := c_p k R$. For every stopping time T such that $J^{T-} \in S^2$ and $Z^{T-} \in \mathcal{S}(1/2\gamma)$, Lemma 5.6 says that there is a unique $X_T \in S^2$ such that

$$X_T = (J^{T-})^{S-} + \left(\int_0^\bullet g(\bullet - s)F(X_T)(s-)\,\mathrm{d}Z^{T-}(s)\right)^{S-}.$$

By uniqueness we have for any two such stopping times T_1 and T_2 that $X_{T_1}^{T_3-} = X_{T_2}^{T_3-}$, where $T_3 = T_1 \wedge T_2$. Due to Theorem 5.5, there exist stopping times $T_\ell \uparrow \infty$ such that $J^{T_\ell-} \in S^2$ and $Z^{T_\ell-} \in \mathcal{S}(1/2\gamma)$ for all ℓ. Define

$$X(t) := \sum_{\ell=1}^{\infty} X_{T_\ell}(t) 1_{[T_{\ell-1},T_\ell)}(t), \quad 0 \leqslant t \leqslant t_0.$$

Then $(X(t))_{t\geqslant 0}$ is an adapted càdlàg process and $X^{S-} = X$. Further, for $\ell \geqslant 1$, we have $X^{T_\ell-} = X_{T_\ell}$ and by (i) of Definition 5.1,

$$\begin{aligned}
&\left(\left(J + \int_0^\bullet g(\bullet - s)F(X)(s-)\,\mathrm{d}Z(s)\right)^{S-}\right)^{T_\ell-} \\
&\quad = (J^{T_\ell-})^{S-} + \left(\left(\int_0^\bullet g(\bullet - s)F(X_{T_\ell})(s-)\,\mathrm{d}Z(s)\right)^{T_\ell-}\right)^{S-} \\
&\quad = X_{T_\ell} = X^{T_\ell-}.
\end{aligned}$$

It follows that X satisfies (5.6). □

To show uniqueness, let Y be another adapted càdlàg solution of (5.6). There exist stopping times $S_\ell \uparrow \infty$ with $Y^{S_\ell-} \in S^2$ for all ℓ. Then $Y^{(S_\ell\wedge T_\ell)-}$ satisfies the same equation as $X_{T_\ell}^{S_\ell-}$ and by uniqueness we obtain $Y^{(S_\ell\wedge T_\ell)-} = X_{T_\ell}^{S_\ell-} = X^{(S_\ell\wedge T_\ell)-}$. Since $\sup_\ell S_\ell \wedge T_\ell = \infty$ a.s., it follows that $X = Y$. □

Next, fix $t_0 > 0$. For $n = 1, 2, 3, \ldots$ define the stopping time

$$T_n(\omega) := \inf\{t \in [0, t_0] : K(t,\omega) > n\}, \quad \omega \in \Omega,$$

where $\inf \emptyset := \infty$. Then $T_n \uparrow \infty$. Define

$$F_n(X)(t) := F(X)^{T_n-}(t), \quad 0 \leqslant t \leqslant t_0,\ X \in \mathbb{D}[0,t_0],\ n = 1, 2, \ldots.$$

Then $F_n : \mathbb{D}[0,t_0] \to \mathbb{D}[0,t_0]$ is functional Lipschitz, $|F_n(X)(t) - F_n(Y)(t)| \leqslant n \sup_{0\leqslant s\leqslant t} |X(s) - Y(s)|$, for all $X, Y \in \mathbb{D}[0,t_0]$, and $F_n(0) = 0$. By the first claim, there exists therefore for each n a unique $X_n \in \mathbb{D}[0,t_0]$ such that

$$X_n(t) = J^{T_n-}(t) + \left(\int_0^\bullet g(\bullet - s)F_n(X_n)(s-)\,\mathrm{d}Z(s)\right)^{T_n-}(t), \quad 0 \leqslant t \leqslant t_n.$$

Then by uniqueness, $X_m^{T_n-} = X_n$ for every $m \geqslant n$. Define

$$X(t) := \sum_{n=1}^{\infty} X_n(t) 1_{[T_{n-1}, T_n)}(t), \quad 0 \leqslant t \leqslant t_0.$$

Then $X \in \mathbb{D}[0, t_0]$ and $X^{T_n-} = X_n$ for $n = 1, 2, \ldots$. Further, $F_n(X_n)^{T_n-} = F(X)^{T_n-}$, so

$$X^{T_n-} = J^{T_n-} + \left(\int_0^{\bullet} g(\bullet - s) F(X)(s-) \, \mathrm{d}Z(s) \right)^{T_n-},$$

for each n. Hence X satisfies (5.5) for $0 \leqslant t \leqslant t_0$. Moreover, X is the unique solution of the latter equation. Indeed, if V is a solution as well, then V^{T_n-} satisfies the defining equation for X_n and therefore $V^{T_n-} = X_n = X^{T_n-}$ for all n. This implies $V = X$.

Finally, we can vary t_0 and glue solutions together to obtain a unique $X \in \mathbb{D}$ such that (5.5) holds for all $t \geqslant 0$. It follows from Lemma 5.7 that X is a semimartingale whenever J is a semimartingale. □

Theorem 5.2 follows from Theorem 4.1 and Proposition 5.8. Notice that the function g in Theorem 5.2 need not be continuous. In this way Theorem 5.2 generalizes [9].

6 Variation-of-constants formula for SDDE with linear drift

It is the aim of this section to prove Theorem 1.1. It is well known that (1.4) has a unique solution $g : \mathbb{R} \to \mathbb{R}$ with $g|_{[0,\infty)}$ absolutely continuous (see [5]). Then $\int_{(-\infty,0]} g(\bullet + a)\mu(\mathrm{d}a)$ is bounded on $[0, T]$ and hence $g|_{[0,T]} \in W^{1,\infty}[0, T]$ for every $T > 0$.

The proof of Theorem 1.1 proceeds as follows. Due to Theorem 5.2, there exists a solution of (1.2). By means of a stochastic Fubini argument, we will show that this solution also satisfies (1.8). As equation (1.8) has only one solution, we then know that the solutions of (1.8) and (1.2) coincide and the proof is complete. The Fubini argument is given next.

Lemma 6.1. *Let μ be a finite signed Borel measure on $(-\infty, 0]$ and let $g : \mathbb{R} \to \mathbb{R}$ be the solution of (1.4) with $g|_{[0,\infty)}$ absolutely continuous. Let $F : \mathbb{D}[0, \infty) \to \mathbb{D}[0, \infty)$ be functional Lipschitz. Let $(Z(t))_{t \geqslant 0}$ and $(J(t))_{t \geqslant 0}$ be semimartingales. If $X \in \mathbb{D}[0, \infty)$ satisfies*

$$\begin{aligned} X(t) &= g(t)X(0) + \int_0^t g(t-s) \, \mathrm{d}J(s) \\ &\quad + \int_0^t g(t-s) F(X)(s-) \, \mathrm{d}Z(s), \quad t \geqslant 0, \end{aligned} \tag{6.1}$$

then

$$\begin{aligned} X(t) &= X(0) + J(t) + \int_0^t \int_{(-s,0]} X(s+a)\mu(\mathrm{d}a) \, \mathrm{d}s \\ &\quad + \int_0^t F(X)(s-) \, \mathrm{d}Z(s), \quad t \geqslant 0, \end{aligned}$$

Proof. Observe that we may assume that $Z(0) = 0$ and $J(0) = 0$. We will first apply stochastic Fubini twice to prove the identity

$$\int_0^T \int_{(-\infty,0]} \int_0^{(s+a)^+} g(s+a-m)F(X)(m-)\,\mathrm{d}Z(m)\mu(\mathrm{d}a)\,\mathrm{d}s \tag{6.2}$$
$$= \int_0^T g(T-m)F(X)(m-)\,\mathrm{d}Z(m) - \int_0^T F(X)(m-)\,\mathrm{d}Z(m),$$

for any $T \geqslant 0$ and any $X \in \mathbb{D}$. Since we will only evaluate g on $(-\infty, T]$, we may assume that g is bounded. Fix an $s \in [0,T]$. The map $(a,t,\omega) \mapsto g(s+a-t)F(X)(t-)(\omega)$ is bounded and $\mathcal{B}((-\infty,0]) \otimes \mathcal{P}$-measurable, where $\mathcal{P}$ denotes the predictable σ-algebra. Further, the process $F(X)(\bullet-)$ is predictable and locally bounded, the function g is Borel measurable and bounded, and

$$\int_{(-\infty,0]} g(s+a-\bullet)^2 F(X)(\bullet-)^2|\mu|(\mathrm{d}a) \leqslant \|g\|_\infty^2 |\mu|((-\infty,0])F(X)(\bullet-)^2.$$

The stochastic Fubini theorem (Theorem 2.5) therefore yields that

$$\int_{(-\infty,0]} \left(\int_0^t g(s+a-m)F(X)(m-)\,\mathrm{d}Z(m)\right)\mu(\mathrm{d}a)$$
$$= \int_0^t \left(\int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)F(X)(m-)\right)\mathrm{d}Z(m) \text{ a.s.}$$

for every $t \geqslant 0$. Since $g(\vartheta) = 0$ for $\vartheta < 0$, the inner integral at the left hand side of the previous equality runs only up to $(s+a)^+$ if $t \geqslant s$. The map $(s,m,\omega) \mapsto \int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)F(X)(m-)$ is measurable with respect to $\mathcal{B}([0,T]) \otimes \mathcal{P}$, because $(s,m) \mapsto \int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable. Moreover, for each $s \geqslant 0$ the processes $\int_{(-\infty,0]} g(s+a-\bullet)\mu(\mathrm{d}a)F(X)(\bullet-)$ and $\int_0^T (\int_{(-\infty,0]} g(s+a-\bullet)\mu(\mathrm{d}a)F(X)(\bullet-))^2\,\mathrm{d}s$ are locally bounded, since g is bounded, μ is a finite measure, and $F(X)(\bullet-)$ is locally bounded. Hence, again by the stochastic Fubini theorem (Theorem 2.5), we have

$$\int_0^T \left(\int_0^t \int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)F(X)(m-)\,\mathrm{d}Z(m)\right)\mathrm{d}s$$
$$= \int_0^t \left(\int_0^T \int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)F(X)(m-)\,\mathrm{d}s\right)\mathrm{d}Z(m) \text{ a.s.}$$

for every $t \geqslant 0$. Next, we substitute $t = T$ in the previous equality, use that $g(\vartheta) = 0$ for $\vartheta < 0$, and rewrite the right hand side by observing that for $m \geqslant 0$,

$$\int_m^T \int_{(-\infty,0]} g(s+a-m)\mu(\mathrm{d}a)\,\mathrm{d}s = \int_0^{T-m} g'(s)\,\mathrm{d}s = g(T-m) - 1.$$

Thus we arrive at the identity (6.2).

Similarly, we have for any $T \geqslant 0$ that

$$\int_0^T \int_{(-\infty,0]} \int_0^{(s+a)^+} g(s+a-m)\,\mathrm{d}J(m)\mu(\mathrm{d}a)\,\mathrm{d}s \tag{6.3}$$
$$= \int_0^T g(T-m)\,\mathrm{d}J(m) - \int_0^T \mathrm{d}J(m).$$

Next assume that $X \in \mathbb{D}$ satisfies (6.1). Set $X(t) := 0$ for $t < 0$. Then

$$\begin{aligned} X(u) &= g(u)X(0) + \int_0^{u^+} g(u-s)\,\mathrm{d}J(s) \\ &\qquad + \int_0^{u^+} g(u-s)F(X)(s-)\,\mathrm{d}Z(s), \quad \text{for all } u \in \mathbb{R}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^t \int_{(-s,0]} X(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s = \int_0^t \int_{(-\infty,0]} X(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\quad = \int_0^t \int_{(-\infty,0]} g(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s X(0) \\ &\qquad + \int_0^t \int_{(-\infty,0]} \int_0^{(s+a)^+} g(s+a-m)\,\mathrm{d}J(m)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\qquad + \int_0^t \int_{(-\infty,0]} \int_0^{(s+a)^+} g(s+a-m)F(X)(m-)\,\mathrm{d}Z(m)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\quad = \int_0^t g'(s)\,\mathrm{d}s X(0) + \int_0^t g(t-m)\,\mathrm{d}J(m) - J(t) \\ &\qquad + \int_0^t g(t-m)F(X)(m-)\,\mathrm{d}Z(m) - \int_0^t F(X)(m-)\,\mathrm{d}Z(m) \\ &\quad = \Big(g(t)-1\Big)X(0) + \int_0^t g(t-m)\,\mathrm{d}J(m) - J(t) \\ &\qquad + \int_0^t g(t-m)F(X)(m-)\,\mathrm{d}Z(m) - \int_0^t F(X)(m-)\,\mathrm{d}Z(m) \\ &\quad = X(t) - X(0) - J(t) - \int_0^t F(X)(m-)\,\mathrm{d}Z(m), \end{aligned}$$

for all $t \geqslant 0$. Here the third equality is justified by the identities (6.2) and (6.3), which yields the assertion. $\square$

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