Two dimensional KP systems and their solvability

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Abstract

In this paper we introduce new various generalizations of the classical Kadomtsev-Petviashvili hierarchy in the case of operators in several variables. These generalizations are the candidates for systems that should play the role, analogous to the role of the KP hierarchy in the classical KP theory, in a generalized KP theory. In particular, they should describe flows of some generalized geometric datas, including those described in [14], for certain initial conditions. The unique solvability of the initial value problem for the generalized KP hierarchies is established. The connection of these systems with universal families of isospectral deformations of certain pairs of commuting differential operators is opened. To prove the solvability of the systems we generalize several results from the works of M.Mulase ([6]) and A.N.Parshin ([12]).

1 Introduction

In [12] A.N. Parshin offered a generalization of the classical KP-hierarchy and studied different properties of this system: the conversations laws, Zaharov-Shabat equations and so on. The generalized KP-hierarchy was there interpreted as a dynamical system on some infinite-dimensional variety. Recall that the classical KP-hierarchy has the following Lax form:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L)], \quad n \geq 1, \tag{1}$$

where $L = \partial + u_{-1}\partial^{-1} + \ldots \in P$ is a pseudodifferential operator, $u_i \in k((x))[\log(x)][[t_k]]$ and $P = P_+ \oplus P_-$, where $P = k((x))((\partial^{-1}))$ is the ring of pseudodifferential operators (in one variable), P_+ is the subring of differential operators.

The classical KP-hierarchy is only a starting point of a huge KP theory developed since 1970s or even earlier, which have, beyond other, a rich algebraic structure. Under the algebraic structure we mean here a so-called Krichever correspondence, which describes correspondences between certain solutions of the classical KP (KdV, etc.) equations and hierarchies, certain geometric datas (which consist of an algebraic complete curve, a point, a torsion free sheaf and a trivializations of it in the classical case), rings of commuting ordinary or matrix differential operators, points and moduli varieties in a universal grassmanian, θ-functions of jacobians of curves and τ-functions (for detailed explanation see, for example, the review [8] and other references cited there). A generalization of the KP-hierarchy should play a role of a system, which describe, for certain initial data, flows of some generalized geometric datas, which should include algebraic varieties of higher dimension. In works [14], [10] the so-called Krichever map was generalized. In classical case this is a map that sends a geometric Krichever data to a subspace of a one-dimensional local field, which can be interpreted as a point of an infinite-dimensional grassmanian.

One of the important steps in generalizing the classical KP-theory is studying the solutions of generalized KP-hierarchies and their connections with generalized geometric datas. In [6] M.Mulase solved the Cauchy problem for the classical and more complicated KP-hierarchies. Parshin's generalization deals with the ring of pseudodifferential operators in n variables. Except the properties proved in [12] there remain unsolved a lot of questions, in particular, it was not clear if the Cauchy problem has a solution in this case, is it true that the Parshin system is

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a master equation of all isospectral deformations of some differential operators in 2 variables, are there some connection with the problem of classification of all commutative subrings in the ring of differential operators in several variables, are there some geometrical solutions of these systems and so on.

In this paper we give the answers on these question for the Parshin generalization and define a series of modified Parshin's KP-hierarchies for the case of the ring of pseudodifferential operators over a (commutative) ring A, which satisfy certain properties listed in section 3.1. In particular, A can be equal to the ring $k[[x_1, x_2]]$ or $k((x_1))((x_2))[\log x_1, \log x_2]$. These hierarchies are parameterized by functions $\alpha : \mathbb{Z}_+ \to \mathbb{R}$ such that $\alpha(0) \le 0$. For given α such a system is called $(KP)_\alpha$. All these systems satisfy the same properties as the original Parshin system. For the constant function $\alpha = 0$ the system $(KP)_\alpha$ covers the classical KP-hierarchy, and for the constant function $\alpha = \infty$ the system $(KP)_\alpha$ is the original Parshin KP-hierarchy. We prove their unique solvability in a certain generalized ring of time-dependend pseudo-differential operators for all initial values. Also we prove that for all functions α except the constant function $\alpha = \infty$ the systems have nontrivial solutions. So, this paper is a strong generalization of the preprint [18].

We also show that the modified systems can be interpreted as master equations of all universal families of isospectral deformations of some pairs of commuting α-differential operators (the definition of this notion is given in section 6). Notably, the original Parshin system (which arise naturally in the framework of the theory of higher dimensional local (skew)-fields) appears as a part of a master equation of all universal families of isospectral deformations for any pair of normalized commuting differential operators that have some additional condition on their orders (conjecturally for pairs of completely integrable operators in the sense of [2], [1]). The whole master equation gives a necessary condition on a time-dependness of the ring of coefficients of operators. Recall that in one-dimensional situation this ring is obtained as a completion of the ring of polynomials in infinite many times with respect to a discrete valuation (see [6]). In two-dimensional case the ring is defined as a completion of the ring of polynomials in infinite many times with respect to a topology, whose model comes from the topology on a two-dimensional local field (see [4] or [3] for background on the theory of local fields). Such a topology is a weakest one, for which the unique solvability of the modified KP-systems is established.

To solve the Cauchy problem for the modified KP-systems we generalize the classical method. Namely, we first prove the equivalence of the modified KP-systems and obviously modified Sato-Wilson systems (in [6] the Cauchy problem was solved exactly for Sato-Wilson systems, in the case of commutative operator's coefficients they are equivalent to KP-systems, see the discussion after lemma 1.3 there). Then we find a solution of the last systems using a generalization of the *Birkhoff decomposition*. The original Birkhoff decomposition gives a factorization of a loop group into a product of subgroups of loops of special form ([15]); it was then generalized in [6], where the loop groups were replaced by groups of infinite order micro-differential operators. The last groups were defined as groups of certain invertible elements in a ring of extended time-dependent pseudo-differential operators. These operators can be represented as infinite series with certain valuation growth condition on their coefficients. We generalize the Birkhoff decomposition of [6] considering groups of certain invertible elements in a ring of extended time-dependent *iterated* pseudo-differential operators. These operators can be represented as iterated infinite series with certain more complicated valuation growth conditions on their coefficients. Now a solution of a system $(KP)_\alpha$ can be obtained from a solution U of the universal equation $dU = \omega_\alpha^{N_0} U$, where $\omega_\alpha^{N_0}$ is defined by formula (45) below, in the same way as in [6] (see section 4.2 below).

We would like to emphasize that the modified Sato-Wilson systems can not be solved just by reducing them to systems considered in [6], though each modified Sato-Wilson system can be represented, with help of some reordering of indeces, as a system from [6] with non-commutative coefficients. Actually, the main problem of these systems is that it is not clear a priory if there exists a reordering of indeces that brings a solution of a system from [6] to a solution of a modified Sato-Wilson system. So, we should again go through a generalization of the Birkhoff decomposition.

The modified KP-systems should play the role discussed above for a generalization of the classical KP theory. Nevertheless, we almost don't explain the connection between the solutions of systems and generalized geometric datas in this paper — this is a material of another paper.

Recently there have been made several attempts to generalize some other aspects of the KP theory to higher dimensions developing some ideas appeared in works of Nakayashiki, [9]. These are the works [16] and [5], where, in particular, some new systems that describe flows on Picard varieties (or their extensions) of higher dimensional varieties obtained. These are also the systems of KP type, but for operators with *matrix coefficients*. It would be interesting to compare various systems.

Here is a brief overview of this paper.

In section 2 we study solvability of the original Parshin KP-system and give some examples. In particular, in subsection 2.2 we show that a general system of such type, independently of time-dependentness of operator's coefficients, has only trivial solutions.

In section 3 we introduce a series of modified KP systems and develop some necessary technical tools. In particular, we define the ring of time-dependent operator's coefficients and the ring of extended iterated pseudo-differential operators discussed above.

In section 4.1 we prove that modified KP-systems are equivalent to a certainly defined modified Sato-Wilson systems. In section 4.2 we generalize some results of Mulase from [6] and show the unique solvability of these systems for arbitrary initial conditions. The solutions, nevertheless, may not belong to the ring of usual (not extended) iterated pseudo-differential operators.

In section 5 we give examples of modified systems and their initial conditions, whose solutions belong to the usual ring of iterated pseudo-differential operators, and show that the solutions are not trivial in general.

In section 6 we generalize the classical definition of a family of isospectral deformations of an ordinary monic differential operator to the case of a pair of monic commuting operators. We then derive, as in the classical case, that the problem of finding of a universal family of isospectral deformations for a pair of such operators (satisfying some additional condition on their order) is equivalent to the problem of finding a solution of an equation that have a Lax form, like a classical KP system. The original Parshin KP-hierarchy is a part of this equation. After that we introduce a notion of α-differential operators and show that the modified KP systems are the master equations of all universal families of isospectral deformations of certain pairs of monic commuting α-differential operators.

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2 Generalized KP-hierarchy

2.1 General setting

In this paper we use the notation from [12]. We will work with the following objects:
an associative algebra A over a field k of characteristic zero with the unity 1 and with two derivations (∂_1, ∂_2) such that $\partial_1\partial_2 = \partial_2\partial_1$ and $ker(\partial_1) \cap ker(\partial_2) = k$; everywhere, if another is not mentioned, the algebra A is assumed to be commutative;
the ring of formal pseudo-differential operators $E = A((\partial_1^{-1}))((\partial_2^{-1}))$.

Recall that such a ring can be defined iterately, and for any ring B and its derivation ∂ the ring $B((\partial^{-1}))$ is defined as a left B-module of all formal expressions

$$L = \sum_{i>-\infty}^{n} a_i\partial^i, \quad a_i \in B$$

with a multiplication defined according to the Leibnitz rule:

$$(\sum_i a_i\partial^i)(\sum_j b_j\partial^j) = \sum_{i,j,k\geq 0} C_i^k a_i \partial^k(b_j)\partial^{i+j-k},$$

where

$$C_i^k = \frac{i(i-1)\dots(i-k+1)}{k(k-1)\dots 1}, \quad \text{if } k>0\,,\, C_i^0 = 1.$$

If $L = \sum_{i\leq m} a_i\partial_2^i \in E$ and $a_m \neq 0$, then $m := \mathrm{ord}_{\partial_2}(L)$ will be called the order of the operator L. The function $\mathrm{ord}_{\partial_2}(.)$ defines a decreasing filtration $E_. : \dots \subset E_{-1} \subset E_0 \subset \dots$ of vector subspaces $E_i = \{L \in E : \mathrm{ord}_{\partial_2}(L) \leq i\} \subset E$. Analogously, one can define the function $\mathrm{ord}_{\partial_1}(.)$ on the ring $A((\partial_1^{-1}))$. Further we will sometimes use the notation ord instead of $\mathrm{ord}_{\partial_2}$.

We have the decomposition of E in a direct sum of subspaces

$$E = E_+ + E_-,$$

where $E_- = \{L \in E : \mathrm{ord}_{\partial_2}(L) < 0\}$ and E_+ consists of the operators containing only ≥ 0 powers of ∂_2.

Consider the space E^2 and consider the Lax system (6) on the page 14 in [12], namely, if $N \in E^2$, then it looks like

$$\frac{\partial N}{\partial t_k} = V_N^k, \tag{2}$$

where

$$V_N^k = ([(L^nM^m)_+, L], [(L^nM^m)_+, M])$$

if $N = (L, M)$ and $k = (n, m)$, $n, m \geq 0$. Note that we concider N as belonging to the extended phase space $\tilde{E}^2$, where $\tilde{E} = A[[\dots, t_k, \dots]]((\partial_1^{-1}))((\partial_2^{-1}))$ (to clarify t-dependence of operators, see section 3). Independently of t-dependence of operators, we can nevertheless prove some fundamental facts about the system. The first one is the following proposition.

Proposition 1. *Suppose* $\mathrm{ord}(LM) > 0$ *and* $N = (L, M)$ *satisfy the Lax system.*
Then $[L, M] = 0$.

Proof Since $\mathrm{ord}(LM) > 0$, either $\mathrm{ord}(L) > 0$ or $\mathrm{ord}(M) > 0$. Without loss of generality assume $\mathrm{ord}(M) > 0$. Then for $k = (1, n)$ with arbitrary large n we have:

$$\frac{\partial M}{\partial t_k} = ([L, M])M^n - [(LM^n)_-, M]$$

where from

$$([M, L])M^n = 0 \mod \wp^{-\mathrm{ord}(M)}$$

and $[M, L] = 0 \mod \wp^{\mathrm{ord}(M)(n-1)}$, where $\wp$ is the maximal ideal of the first (discrete) valuation in E. So, $[M, L] = 0$.
□

Further we will study some modifications of this system and we will look for solutions N with the property $\mathrm{ord}(M) = 1$, $\mathrm{ord}(L) = 0$. The propsition will remain true for all these systems because all these systems will depend on times indexed by infinite many different indices $k = (\dots, j)$ and j will appear as a power of M.

Nevertheless, for completeness of our reseach let's consider the following example.

Example. Let us try to find a solution of the system in the form:

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \dots$$

$$M = \partial + v_1\partial^{-1} + v_2\partial^{-2} + \dots$$

where $\partial = \partial_2$ and $u_i, v_i \in k((x_1))((x_2))[[\dots, t_{ij}, \dots]]((\partial_1^{-1}))$. First of all, note that the conditions

$$[(L^nM^m)_+, L]_i = 0, \quad [(L^nM^m)_+, M]_i = 0,$$

where $i \geq 0$ and $[.]_i$ denotes the i-th coefficient of an operator, are exactly the conditions for the operators L, M to commute, as we have seen above. It is not difficult to prove that for L, M these conditions are the following: $[L, M] = 0$ iff for all $n \geq 0$ holds

$$D(u_n - v_n) + \sum_{i,j\geq 1, i+j=n} [v_i, u_j] + \sum_{i,j,k\geq 1, i+j+k=n} (-1)^j C_{j+i-1}^{i-1}(v_i D^{(j)}(u_k) - u_i D^{(j)}(v_k)) = 0, \quad (3)$$

where $D() = \partial/\partial(x_2)$ and C_i^j are the binary coefficients.

The equations for $k = (1, n)$, $k = (n, 1)$ for all n give us the following property:

$$\frac{\partial u_1}{t_k} = \frac{\partial v_1}{t_k}$$

Indeed, this follows from (3) and the property: $[(L^nM^m)_+, L] = [(L^nM^m), L] - [(L^nM^m)_-, L]$. Therefore, $u_1 - v_1 =: c \in k((x_1))((\partial_1))$.

Now, the condition for the Lax system to have a solution appears if we will consider equations for u_l, v_l for $k = (i, j)$, $i + j + l \leq 4$. We have (below D will denote a derivation by x_2)

$$\frac{\partial u_1}{\partial t_{1,1}} = \left(D^{(2)}\right)(u_1) + v_1u_1 + 2\,\mathrm{D}(u_2) - u_1v_1$$

$$\frac{\partial v_1}{\partial t_{1,1}} = u_1v_1 + \left(D^{(2)}\right)(v_1) + 2\,\mathrm{D}(v_2) - v_1u_1$$

where from

$$2u_{22} + [v_1, u_1] = \frac{\partial u_1}{\partial t_{11}}$$

Then,

$$\frac{\partial u_2}{\partial t_{1,1}} = \left(D^{(2)}\right)(u_2) + [u_1, u_2] + [v_1, u_2] + 2\,\mathrm{D}(u_3) + 2u_1(\mathrm{D}(u_1))$$

$$\frac{\partial v_2}{\partial t_{1,1}} = 2v_1(\mathrm{D}(u_1)) + [v_1, v_2] + \left(D^{(2)}\right)(v_2) + 2\,\mathrm{D}(v_3) + [u_1, v_2]$$

where from

$$\frac{\partial(u_2 - v_2)}{\partial t_{1,1}} = [D(u_1), c] + [c, u_2] + [c, v_2]$$

(c were defined above). Differentiating by x_2 we get

$$[\frac{\partial u_1}{\partial t_{1,1}} - D^{(2)}, c] = 0$$

From other equations:

$$\frac{\partial u_1}{\partial t_{1,2}} = 3\,\mathrm{D}(u_3) + \left(D^{(3)}\right)(u_1) + 2\,[v_2,u_1] + 3\,\mathrm{D}(v_1)(u_1) + \\ 3\,\left(D^{(2)}\right)(u_2) + 2\,v_1(\mathrm{D}(u_1)) + 2\,[v_1,u_2] + u_1(\mathrm{D}(u_1))$$

$$\frac{\partial v_1}{\partial t_{1,2}} = 3\,\mathrm{D}(v_3) + \left(D^{(3)}\right)(v_1) + 3\,\left(D^{(2)}\right)(v_2) + \\ 3\,\mathrm{D}(v_1)(v_1) + 2v_1(\mathrm{D}(v_1)) + [u_1,v_2] + u_1(\mathrm{D}(v_1)) + [u_2,v_1]$$

$$\frac{\partial u_1}{\partial t_{2,1}} = \left(D^{(3)}\right)(u_1) + 3\,\mathrm{D}(u_3) + 2\,u_1(\mathrm{D}(u_1)) + v_1(\mathrm{D}(u_1)) + \\ [v_1,u_2] + [v_2,u_1] + 3\,\mathrm{D}(v_1)(u_1) + 3\,\left(D^{(2)}\right)(u_2)$$

$$\frac{\partial v_1}{\partial t_{2,1}} = v_1(\mathrm{D}(u_1)) + 3\,\mathrm{D}(v_3) + \left(D^{(3)}\right)(v_1) + 3\,\left(D^{(2)}\right)(v_2) + \\ 2\,u_1(\mathrm{D}(v_1)) + 2\,[u_1,v_2] + 3\mathrm{D}(u_1)(v_1) + 2\,[u_2,v_1]$$

we get

$$\frac{\partial u_1}{\partial t_{1,2}} - \frac{\partial u_1}{\partial t_{2,1}} = [v_2,u_1] + [v_1,u_2] - cD(u_1)$$

So, this expression includes u_2, v_2, which means that we get no equations on u_1.

This explains us that the system from this example contain not enough equations. This is, in fact, the corollary of taking the special form of solutions.

2.2 Parshin's KP-hierarchy

A more interesting question is: are there solutions of the form

$$L = u_0 + u_1\partial_2^{-1} + \dots \tag{4}$$

$$M = v_{-1}\partial_2 + v_0 + v_1\partial_2^{-1} + \dots \tag{5}$$

where u_0, v_{-1} are monic series with the orders $\mathrm{ord}_{\partial_1}(u_0) = 1$, $\mathrm{ord}_{\partial_1}(v_{-1}) = 0$.

This question was posed in one particular case in [12]. Also, as it will be shown in section 6, it arises by studying the existence of a universal family of isospectral deformations of a pair of differential operators in two variables. Also it is related to the question posed in Remark 1.7. in [6]. By proposition 1 we must have $[L,M]=0$, so,in particular, $[u_0,v_{-1}]=0$.

Note that, since $\frac{\partial L}{\partial t_{ij}} = [(L^iM^j)_+, L] = -[(L^iM^j)_-, L]$, we have $\partial/\partial t_{ij}(u_0) = 0$ and analogously $\partial/\partial t_{ij}(v_{-1}) = 0$. So, u_0, v_{-1} do not depend on times and therefore coincide with the first coefficients of the initial data. Since u_0, v_{-1} are invertible operators, the operators L, M of a solution will be also invertible.

Below we will prove that there are no nontrivial solutions of the form (4), (5) of our Lax system. By triviality of solutions we mean solutions representable either in the form (L_0, M_0) with $L_0 = (L_0)_+$, $M_0 = (M_0)_+$ or in the form (L, M), where L, M are series in variables L_0^{-1}, M_0^{-1} with constant coefficients. Obviously, all such series satisfy our system of equations.

Assume the converse. Consider the series of equations for $k=(n,0)$ and $k=(n,1)$. For the series $k=(n,0)$ we have

$$\frac{\partial L}{\partial t_k}=\sum_{i=0}^{\infty}([u_0^n,u_i]\partial_2^{-i}+u_i[u_0^n,\partial_2^{-i}]) \tag{6}$$

$$\frac{\partial M}{\partial t_k}=\sum_{i=-1}^{\infty}([u_0^n,v_i]\partial_2^{-i}+v_i[u_0^n,\partial_2^{-i}]) \tag{7}$$

Let's point out the following easy observation: since $\frac{\partial}{\partial t_k}$ and $[(L^iM^j)_+,.]$ are derivations, any series in variables L,M with constant coefficients satisfy all equations of our system if (L,M) is a solution of the system. Now, since our solutions are assumed to be nontrivial, there exists an index $i>0$ such that L and M belongs to $\wp^i+$ (some series in L,M with constant coefficients). Replacing L and M with $L'=L+$ (some series in L,M with constant coefficients) (correspondingly $M'=M+\dots$), we can assume that u_i',v_i' are the first nonzero coefficients of L',M' between all coefficients with indices $j>0$, and that u_i',v_i' are series in ∂_1 with *nonconstant* first coefficients, i.e. they don't belong to $\ker\partial_1\cap\ker\partial_2$ in any reasonable time-dependent ring $\tilde{E}$.

From equation (6) we derive that $\partial u_i'/\partial t_{n,0}=[u_0^n,u_i']$ for all n. Since the order of the left hand side is bounded from above, we obtain $[u_0^n,u_i']=0$, hence $[u_0,u_i']=0$. Analogously $[u_0,v_i']=0$. Using arguments from much more general lemma 6 below we obtain $[u_i',v_{-1}]=0$ and $[v_i',v_{-1}]=0$. Also we get that u_0,v_{-1},u_i',v_i' do not depend on $t_{n,0}$ for all n and that the first coefficients of u_i',v_i' belong to $\ker\partial_1$. Using the same arguments we obtain from (6), (7) that

$$[u_0,u_{i+1}']+iu_i'\partial_2(u_0)=0,\quad [u_0,v_{i+1}']+iv_i'\partial_2(u_0)=0. \tag{8}$$

Now consider the series of equations for $k=(n,1)$.

Lemma 1. *We have*

$$(L^nM)_+=u_0^nv_{-1}\partial_2+nu_1u_0^{n-1}v_{-1}+u_0^nv_0$$

Proof. By induction on n. For $n=0$ we have $M_+=v_{-1}\partial_2+v_0$. Since L contains only nonpositive order terms, and $L^{n-1}M$ has only one positive order term, namely $u_0^{n-1}v_{-1}\partial_2$, the plus-part of the operator L^nM will be

$$(L^nM)_+=u_1u_0^{n-1}v_{-1}+u_0((L^{n-1}M)_+)=u_0^nv_{-1}\partial_2+nu_1u_0^{n-1}v_{-1}+u_0^nv_0 \tag{9}$$

□

Now for $k=(n,1)$ we have by formula (9)

$$\frac{\partial u_i'}{\partial t_{n,1}}=[(L^nM)_+,L]_i=[u_0^nv_{-1}\partial_2+nu_1u_0^{n-1}v_{-1}+u_0^nv_0,L]_i=$$

$$[u_0^nv_{-1}\partial_2,u_i'\partial_2^{-i}]_i+[u_0^nv_{-1}\partial_2,u_{i+1}'\partial_2^{-i-1}]_i+[nu_1u_0^{n-1}v_{-1}+u_0^nv_0,u_i'\partial_2^{-i}]_i=$$

$$([u_0^nv_{-1},u_i'\partial_2^{-i}]\partial_2)_i+u_0^nv_{-1}\partial_2(u_i')+[u_0^nv_{-1},u_{i+1}']+u_0^n[v_0,u_i']=$$

$$iu_i'\partial_2(u_0^nv_{-1})+u_0^nv_{-1}\partial_2(u_i')+u_0^n[v_0,u_i']+u_0^n[v_{-1},u_{i+1}']-iu_i'\partial_2(u_0^n)v_{-1}=$$

$$u_0^n(iu_i'\partial_2(v_{-1})+v_{-1}\partial_2(u_i')+[v_0,u_i']+[v_{-1},u_{i+1}']), \tag{10}$$

where the transformations follows from the fact that $[u_1,u_i']=0$ (u_1 is either equal to u_i' or is representable as a series in u_0 with coefficients belonging to the ring $k[[\dots,t_{ij},\dots]]$ multiplied by v_{-1}^{-1} by the induction hypothesis). So, by usual arguments we obtain $\partial u_i'/\partial t_{n,1}=0$,

$$iu_i'\partial_2(v_{-1})+v_{-1}\partial_2(u_i')+[v_0,u_i']+[v_{-1},u_{i+1}']=0 \tag{11}$$

and analogously $\partial v'_i/\partial t_{n,1} = 0$,

$$iv'_i\partial_2(v_{-1}) + v_{-1}\partial_2(v'_i) + [v_0, v'_i] + [v_{-1}, v'_{i+1}] = 0. \tag{12}$$

Assume that $\mathrm{ord}_{\partial_1}(\partial_2(u'_i)) = \mathrm{ord}_{\partial_1}(u'_i)$, that is the first coefficient of u'_i does not belong to the ring $k[[\ldots, t_{ij}, \ldots]]$. Then $\mathrm{ord}_{\partial_1}(v_{-1}\partial_2(u'_i)) = \mathrm{ord}_{\partial_1}(u'_i)$.

Let's compare the orders of other summands in formula (11). Since u_0 is monic, we can write u'_i as a series in u_0^{-1}. This series will have coefficients belonging to $\ker \partial_1$, because $[u'_i, u_0] = 0$. The same is true also for the operator $v_{-1}u_0$. This observation imply that the orders of commutators with these operators will coinside with the orders of commutators with the first summands of these operators written as series in u_0^{-1}. From equation (7) we obtain (as in (8)) that $[u_0, v_0] - v_{-1}\partial_2(u_0) = 0$, where from

$$\mathrm{ord}_{\partial_1}([v_0, u'_i]) = \mathrm{ord}_{\partial_1}([v_0, u_0^{\mathrm{ord}_{\partial_1}(u'_i)}]) = \mathrm{ord}_{\partial_1}(v_{-1}\partial_2(u_0^{\mathrm{ord}_{\partial_1}(u'_i)})) < \mathrm{ord}_{\partial_1}(u'_i).$$

Now we have

$$\mathrm{ord}_{\partial_1}(iu'_i\partial_2(v_{-1})) < \mathrm{ord}_{\partial_1}(u'_i)$$

because of moniqueness of u_0,

$$\mathrm{ord}_{\partial_1}([v_{-1}, u'_{i+1}]) = \mathrm{ord}_{\partial_1}([v_{-1}u_0, u'_{i+1}]u_0^{-1} + v_{-1}u_0[u_0^{-1}, u'_{i+1}]) \leq$$

$$\max\{\mathrm{ord}_{\partial_1}([u_0, u'_{i+1}]u_0^{-1}), \mathrm{ord}_{\partial_1}(v_{-1}[u_0, u'_{i+1}]u_0^{-1})\} =$$

$$\max\{\mathrm{ord}_{\partial_1}(u'_i\partial_2(u_0)u_0^{-1}), \mathrm{ord}_{\partial_1}(v_{-1}u'_i\partial_2(u_0)u_0^{-1})\} < \mathrm{ord}_{\partial_1}(u'_i)$$

So, our assumption contradicts with equation (11). Denote the first coefficient of u'_i by c and consider the operator $L'' = L' - cL^{\mathrm{ord}_{\partial_1}(u'_i)}M^{-i}$. Since c does not depend on $t_{n,0}, t_{n,1}$, the operator L'' satisfy the equations (6) and (10) and $u''_i = u'_i - cu_0^{\mathrm{ord}_{\partial_1}(u'_i)}v_{-1}^{-i}$ has the order $\mathrm{ord}_{\partial_1}(u''_i) < \mathrm{ord}_{\partial_1}(u'_i)$. Repeating all the above arguments we conclude that u'_i can be written as a series in u_0 with coefficients in $k[[\ldots, t_{ij}, \ldots]]$ multiplied by v_{-1}^{-i}. Clearly, the same conclusion is true for v'_i.

Continuing this line of reasons we get that $L'_+ = L_+$ and $M'_+ = M_+$ can be written as series in L, M with coefficients in $k[[\ldots, t_{ij}, \ldots]]$ that don't depend on $t_{n,0}, t_{n,1}$, where from L, M can be written as series in L_+, M_+ with such coefficients. Since $[u_0, v_{-1}] = 0$, $[u_0, v_0] - v_{-1}\partial_2(u_0) = 0$, the operators L_+, M_+ commute. Therefore, $[(L^iM^j)_+, N] = 0$ for all i, j, where from we get $\partial N/\partial t_{ij} = 0$ for all i, j. So, (L, M) must be a trivial solution. Combining all together, we obtain

Proposition 2. *The system (2) has no nontrivial solutions of the form (4), (5) in any ring* $\tilde{E}$.

The proposition can be even more generalized, see remark after corollary 1.

3 Modified Parshin's KP-hierarchy

3.1 General setting

Now we introduce the following *modified* Lax systems:

$$\frac{\partial N}{\partial t_k} = V_N^k, \tag{$(KP)_\alpha$}$$

where

$$V_N^k = ([(L^nM^m)_+, L], [(L^nM^m)_+, M])$$

if $N=(L,M)$ and $k=(i,j)$, $j\ge 0$, $i\le \alpha j$, $i\in\mathbb{Z}$, and α is any function $\alpha:\mathbb{Z}_+\to\mathbb{R}$ such that $\alpha(0)\le 0$. Here and below we write αj instead of $\alpha(j)$. The main example of such function is a linear function, $j\mapsto \alpha j$ with $\alpha\in\mathbb{R}$.

We will look for a solution (L,M) of these systems with $\mathrm{ord}_{\partial_2}(L)=0$, $\mathrm{ord}_{\partial_2}(M)=1$ and with initial conditions

$$L_0=u_0+u_1\partial_2^{-1}+\ldots$$

$$M_0=v_{-1}\partial_2+v_0+v_1\partial_2^{-1}+\ldots,$$

where $u_i,v_i\in A((\partial_1^{-1}))$ are monic operators with $\mathrm{ord}_{\partial_1}(u_0)=1$, $\mathrm{ord}_{\partial_1}(v_{-1})=0$. As it will be explained in section 6, we can consider even more narrow set of initial conditions by assuming that $v_{-1}=\partial_1 u_0^{-1}$, $(v_0)_-\cap ker(\partial_1)=0$, where the last condition means that all monomials in all coefficients of $(v_0)_-$ do not belong to $ker(\partial_1)$.

From now on and until the end of the article we additionally assume that the following short sequences are exact:

$$A\xrightarrow{\partial_1}A\longrightarrow 0,\quad A\xrightarrow{\partial_2}A\longrightarrow 0,\quad \ker\partial_2\xrightarrow{\partial_1}\ker\partial_2\longrightarrow 0.$$

It is easy to see that these conditions imply also the exactness of the sequence $\ker\partial_1\xrightarrow{\partial_2}\ker\partial_1\longrightarrow 0$.

To clarify the notion of the ring to which the coefficients of solutions belong let's introduce the following notation.

First consider the ring $A_t:=A[\ldots,t_{ij},\ldots]$ of polynomials in infinite number of variables t_{ij}, $i\in\mathbb{Z}$, $j\in\mathbb{Z}_+$ with coefficients from the ring A. We assume that variables commute with each other and with elements of A. Let's define a valuation v on this ring,

$$v:A_t\backslash\{0\}\longrightarrow\mathbb{Z}\oplus\mathbb{Z}_+$$

by $v(t_{i,j})=(-i,j)$, $v(a)=0$ for $a\in A$. We define also a valuation $v_2:A_t\backslash\{0\}\longrightarrow\mathbb{Z}_+$ by $v_2(t_{ij})=j$. We assume here $(i_1,j_1)>(i,j)$ if $j_1>j$ or $j_1=j$ and $i_1>i$.

Now we introduce a group topology on A_t considering A_t as an abelian group. This topology is an appropriate model of the topology on a two-dimensional local field (see [4] for all details concerning the topologies on higher local fields, or also [17], [13]). Namely, we define the base of neigbourhoods of zero as the set of all sets of the following type

$$U:=\{\sum u_i\in A_t \text{ with } v_2(u_i)=i \text{ and } v(u_i)>(j_i,i)\},$$

where $\{j_i\}$ is a system of integer numbers with $j_i=-\infty$ for large i.

The completion $\bar{R}:=\hat{A}_t$ of the topological group A_t with respect to this topology has a structure of an associative k-algebra with the componentwise multiplication of fundamental sequences. Every element of this algebra can be thought of as a series, whose summands are monomials belonging to A_t, such that every neigbourhood of zero in A_t contains almost all summands of the series. The valuation v (and v_2) can be uniquely extended to the ring $\bar{R}$ by the rule

$$v(\sum a_i)=\min\{v(a_i)\},$$

where $\{a_i\}\in A_t$ are monomials. We extend the derivations ∂_1,∂_2 to the ring $\bar{R}$ in a usual way by assuming that all $t_{ij}\in\ker\partial_1\cap\ker\partial_2$. Now define

$$E_{\bar{R}}:=\bar{R}((\partial_1^{-1})),\quad E_R:=E_{\bar{R}}((\partial_2^{-1})),\quad E_k:=k((\partial_1^{-1}))((\partial_2^{-1})),$$

$$\mathcal{V}_{\bar{R}}:=\{1+E_{\bar{R}-}\},\quad \mathcal{V}_R:=\{1+E_{R-}\},$$

where the decomposition for the ring E_R in plus and minus parts is defined in the same way as in section 2.1, and the decomposition for the ring $E_{\bar{R}}$ is defined analogously with respect to ∂_1.

We extend the valuation v_2 from $\bar{R}$ to $E_{\bar{R}}$ by $v_2(\sum a_k\partial_1^k) = \min\{v_2(a_k)\}$. One can check immediately that this definition is correct. Now we can give an appropriate definition of the ring $\bar{R}\{\{\partial_1^{-1}\}\}$: we define

$$\widehat{E}_{\bar{R}} := \{L = \sum_{q\in\mathbb{Z}} b_q\partial_1^q | b_q \in \bar{R} \textit{ and for any integer } M \textit{ and positive integer } N$$
$$\textit{there exist only finite number of } b_q \textit{ with } q < M \textit{ such that } v_2(b_q) = N\}$$

Lemma 2. *The set* $\widehat{E}_{\bar{R}}$ *is a ring.*

Proof. Obviously, the set $\widehat{E}_{\bar{R}}$ is an abelian group. The multiplication of two series is defined by the same formula as for the ring $E_{\bar{R}}$. We must check only that it is well defined and the product of two series belong again to $\widehat{E}_{\bar{R}}$.

For two series $A = \sum_{q\in\mathbb{Z}} a_q\partial_1^q$, $B = \sum_{q\in\mathbb{Z}} b_q\partial_1^q$ we have

$$AB = \sum_{q\in\mathbb{Z}} g_q\partial_1^q,$$

where

$$g_q = \sum_{k\in\mathbb{Z}}\sum_{l\geq 0} C_k^l a_k\partial_1^l(b_{q+l-k}). \tag{13}$$

By definition of the set $\widehat{E}_{\bar{R}}$ for any integer M and positive integer N there exist integer M_1, M_2 such that $v_2(a_k) > N$ for any $k > M_1$ and $v_2(b_q) > N$ for any $q > M_2$. Since $v_2(\partial_1(b_q)) \geq v_2(b_q)$, we obtain that all summands in (13) for any $k > M_1$ and arbitrary l or for $k \leq M_1$ and $l > M_2 + k - q$ have valuation greater than N. So, the number of summands with valuation *less* than N is finite. Therefore, the series in (13) converges for any q. Moreover, if we take $q > M_2 + M_1$ then we obtain $v_2(g_q) > N$, where from we get that $AB \in \widehat{E}_{\bar{R}}$.

The associativity and distributivity can be easily deduced in the same way as for the ring $E_{\bar{R}}$.

□

We extend the valuation v_2 from $E_{\bar{R}}$ to $\widehat{E}_{\bar{R}}$ in the same way as for the ring $E_{\bar{R}}$. One can check immediately that this definition is also correct.

Now we give an appropriate definition of the ring $\bar{R}\{\{\partial_1^{-1}\}\}\{\{\partial_2^{-1}\}\}$: we define

$$\widehat{E}_R := \{L = \sum_{q\in\mathbb{Z}} a_q\partial_2^q | a_q \in \widehat{E}_{\bar{R}} \textit{ and there is a positive real number } C_L \textit{ and positive}$$
$$\textit{integer } M_L \textit{ such that } v_2(a_q) > C_L q \textit{ for all } q > M_L\},$$

Lemma 3. *The set* $\widehat{E}_R$ *is a ring.*

Proof. The proof is analogous to the proof of lemma 2. Obviously, the set $\widehat{E}_R$ is an abelian group. The multiplication of two series is defined by the same formula as for the ring E_R. We must check only that it is well defined and the product of two series belong again to $\widehat{E}_R$.

For two series $A = \sum_{q\in\mathbb{Z}} a_q\partial_2^q$, $B = \sum_{q\in\mathbb{Z}} b_q\partial_2^q$ we have

$$AB = \sum_{q\in\mathbb{Z}} g_q\partial_2^q,$$

where

$$g_q = \sum_{k\in\mathbb{Z}}\sum_{l\geq 0} C_k^l a_k \partial_2^l (b_{q+l-k}). \tag{14}$$

First we have to check that g_q belong to the ring $\widehat{E}_{\bar{R}}$. Since the linear growth condition on the valuation of coefficients is stronger than the growth condition from the definition of the ring $\widehat{E}_{\bar{R}}$, we obtain, as in lemma 2, that for any positive integer N there are only finite number of summands in (14) with valuation less than N. This means that this condition holds for each i th coefficient of each summand. So, $g_q = \sum_i g_{qi}\partial_1^i$ with $g_{qi}\in\bar{R}$. Since there are only finitely many summands in (14) with valuation less than N, for any integer M there exist only finite number of g_{qi} with $i<M$ and $v_2(g_{qi})\leq N$. Therefore, $g_q\in\widehat{E}_{\bar{R}}$ for any q.

Second we have to check that $v_2(g_q)$ satisfy some linear growth condition. But this is true because of general construction of the ring as it was made in [6]. Namely, if M_A, C_A, M_B, C_B are the numbers from the definition of the set $\widehat{E}_R$ for the elements A, B, then $M_{AB} = M_A + M_B$ and $C_{AB} = (\min\{C_A, C_B\})(\min\{M_A, M_B\})/(M_A + M_B)$ can be taken as the corresponding numbers for the element AB.

The rest of the proof is clear.

□

The valuation v_2 can be obviously extended also to the ring $\widehat{E}_R$. Now let's define the following rings and groups (analogs of corresponding objects from [6] defined in one dimensional case):

$$D_R := \{L = \sum_{n=0}^{\infty} a_n\partial_2^n \quad | \quad a_n\in\widehat{E}_{\bar{R}}, \quad a_n = 0 \quad \text{for } n\gg 0\},$$

$$\widehat{D}_R := \{L\in\widehat{E}_R | L_- = 0\}, \quad \widehat{\mathcal{V}}_R := \{1+\widehat{E}_{R-}\},$$

$$\widehat{\mathcal{V}}_k := \{L = 1 + \sum_{i=1}^{\infty} a_i\partial_2^{-i} | a_i = \sum_{q\in\mathbb{Z}} b_q\partial_1^q \in\widehat{E}_{\bar{R}} \quad \textit{and } b_q\in k[[\dots, t_{ij}, \dots]]\},$$

where $k[[\dots, t_{ij}, \dots]]$ is a subring of $\bar{R}$ defined in the same way;

$$\widehat{E}_R^{\times} := \{L\in\widehat{E}_R | L|_{t=0}\in\mathcal{V}_E \quad \text{and } (\pi(L))_+\in\mathcal{V}_{\bar{R}}\},$$

$$\widehat{D}_R^{\times} := \{L\in\widehat{D}_R | L|_{t=0} = 1 \quad \text{and } (\pi(L))_+\in\mathcal{V}_{\bar{R}}\},$$

where $\pi:\widehat{E}_R\to\widehat{E}_R/I$ is a projection, I is an ideal of valuation v_2 in $\widehat{E}_R$.

Lemma 4. *Every element $P\in\widehat{E}_R^{\times}$ is invertible in $\widehat{E}_R^{\times}$. More precisely, the Neumann series $\sum_{n=0}^{\infty}(1-P)^n$ gives a well-defined element in $\widehat{E}_R^{\times}$, which we denote P^{-1}.*

The proof of lemma will be given later. Now let's note that by this lemma $\widehat{E}_R^{\times}, \widehat{D}_R^{\times}$ are groups. Indeed, using formulas (14), (13) one can easily show that the product of two elements belong again to the same set. All other group laws follow from the laws of the ring $\widehat{E}_R$.

Using arguments of proposition 1, one gets that there can exist a nontrivial solution only if $[L_0, M_0] = 0$. So, *we will assume until the end of section that L_0, M_0 commute.* It is easy to see that the modified systems satisfy all the basic properties derived in [12] for the original system.

3.2 Technical tools

In this subsection we prove several important technical facts, which will be used later. These facts generalize well-known results from [12], [6].

The special choice of the coefficients u_0, v_0, v_{-1} mentioned in the beginning of section 3.1 is partially explained by the theorem below, which is a generalization of theorem 1 in [12]. Another part of the explanation will be given in section 6.

Theorem 1. *Let $L, M \in \widehat{E}_R$ be two monic operators satisfying the following condition:* $\mathrm{ord}_{\partial_2}(L) = 0$, $\mathrm{ord}_{\partial_1}(L_+) = 1$, $\mathrm{ord}_{\partial_2}(M) = 1$, $\mathrm{ord}_{\partial_1}((M\partial_2^{-1})_+) = 0$.
Then

(i) $[L, M] = 0$ if and only if there exists an operator $S \in \widehat{\mathcal{V}}_R$ such that

$$L = S^{-1}L_+S, \quad M = S^{-1}((M\partial_2^{-1})_+\partial_2 + v_0)S,$$

where v_0 is a uniquely defined by L, M element and $[L_+, (M\partial_2^{-1})_+\partial_2 + v_0] = 0$.

ii) If S, S' are two operators from i) then

$$S'S^{-1} \in 1 + k[[\ldots, t_{ij}, \ldots]]((L_{00}^{-1}))((M_{00}^{-1})) \cap \widehat{E}_{R-},$$

where $L_{00} = L_+$, $M_{00} = (M\partial_2^{-1})_+\partial_2 + v_0$.

iii) Suppose that the equation $\partial_1(x) = cx$ is solvable for any $c \in A$. Then there exists a zeroth order invertible operator $\bar{S} \in \widehat{E}_{\bar{R}}$ such that

$$L_+ = \bar{S}^{-1}u_0\bar{S}, \quad ((M\partial_2^{-1})_+\partial_2 + v_0) = \bar{S}^{-1}(v_{-1}\partial_2 + \tilde{v_0})\bar{S},$$

where u_0, v_{-1} are monic, $\mathrm{ord}_{\partial_1}(u_0) = 1$, $\mathrm{ord}_{\partial_1}(v_{-1}) = 0$, $v_{-1} = \partial_1 u_0^{-1}$, $(\tilde{v_0})_- \cap ker(\partial_1) = 0$.

If $\bar{S}'$ is another operator with these properties, then $\bar{S}\bar{S}'^{-1}$ is an operator with coefficients from $k[[\ldots, t_{ij}, \ldots]]$.

Proof. i) The "if" part is clear. To prove the "only if" part we will need the following lemmas:

Lemma 5. *Let $L, M \in \widehat{E}_R$ be two arbitrary operators of finite order and define $N = \max\{\mathrm{ord}_{\partial_2}(L), \mathrm{ord}_{\partial_2}(M)\}$. Then $[L, M] = 0$ if and only if*

$$\sum_{\substack{i+j+k=\alpha \\ j\ge 1, i,k\le N}} C_i^j(v_i\partial_2^{(j)}(u_k) - v_i\partial_2^{(j)}(u_k)) + \sum_{\substack{l+n=\alpha \\ l,n\le N}} [v_l, u_n] = 0$$

for all $\alpha \le \mathrm{ord}_{\partial_2}(L) + \mathrm{ord}_{\partial_2}(M)$, where $L = \sum u_i\partial_2^i$, $M = \sum v_j\partial_2^j$ and u_i, v_j are assumed to be zero if $i > \mathrm{ord}_{\partial_2}(L)$, correspondingly, $j > \mathrm{ord}_{\partial_2}(M)$.

Proof. The proof follows from easy calculations. From definition we have

$$ML = \sum_{i=\mathrm{ord}_{\partial_2}(M)}^{-\infty} \sum_{j=1}^{\infty} \sum_{k=\mathrm{ord}_{\partial_2}(L)}^{-\infty} v_i\partial_2^{(j)}(u_k)C_i^j\partial_2^{i-j+k} + \sum_{l=\mathrm{ord}_{\partial_2}(M)}^{-\infty} \sum_{n=\mathrm{ord}_{\partial_2}(L)}^{-\infty} v_lu_n\partial_2^{l+n},$$

$$LM = \sum_{i=\mathrm{ord}_{\partial_2}(L)}^{-\infty} \sum_{j=1}^{\infty} \sum_{k=\mathrm{ord}_{\partial_2}(M)}^{-\infty} u_i\partial_2^{(j)}(v_k)C_i^j\partial_2^{i-j+k} + \sum_{l=\mathrm{ord}_{\partial_2}(L)}^{-\infty} \sum_{n=\mathrm{ord}_{\partial_2}(M)}^{-\infty} u_lv_n\partial_2^{l+n},$$

where from

$$[M, L] = \sum_{i=N}^{-\infty}\sum_{j=1}^{\infty}\sum_{k=N}^{-\infty} C_i^j(v_i\partial_2^{(j)}(u_k) - u_i\partial_2^{(j)}(v_k))\partial_2^{i-j+k} + \sum_{l=N}^{-\infty}\sum_{n=N}^{-\infty}[v_l, u_n]\partial_2^{l+n}$$

and the rest of the proof is clear.
□

Lemma 6. *Let $u, v, r \in \widehat{E}_{\bar{R}}$ be three operators such that $\mathrm{ord}_{\partial_1}(u) = 1$ and u is monic. Then the conditions $[u, v] = 0$ and $[u, r] = 0$ imply $[v, r] = 0$. Moreover, v and r can be represented as series in u^{-1} with coefficients from $\ker\partial_1$.*

Proof. Clearly the first assertion follows from the second one. So, let's prove that, for example, r can be represented as a series in u^{-1} with coefficients from $\ker \partial_1$.

Since u is monic, each power of ∂_1 can be represented as a series in u^{-1}, where from r can be also represented as a series in u^{-1}, say $r = \sum b_k u^k$. Let $N = v_2(r)$ and let k be the maximal number such that $v_2(b_k) = N$. We have

$$0 = [u, r] = [\partial_1, r] + [u - \partial_1, r] \tag{15}$$

and if g_k is a corresponding coefficient of $[u-\partial_1, r]$, then $v_2(g_k) > N$, because $\mathrm{ord}_{\partial_1}(u-\partial_1) \ge 0$ and the ring A is commutative. From the other hand side, if we denote by g_k' the corresponding coefficient of $[\partial_1, r] = \partial_1(r)$, we obtain $g_k' = \partial_1(b_k)+$ "coefficients of valuation $> N$".

Therefore, the equation (15) implies that $v_2(\partial_1(b_k)) > N$, where from we obtain that $b_k = b_{k1} + b_{k2}$, where $v_2(b_{k1}) = N$, $v_2(b_{k2}) > N$ and $b_{k1} \in \ker \partial_1$. So, we can write $r = b_{k1}u^k + r'$, where the kth coefficient of r' has valuation greater that N.

Continuing this line of reasoning we obtain that $r = r_1 + r''$, where $r_1 = \sum_{i=k}^{-\infty} b_{i1}u^i$ and $v_2(r'') > N$. Repeating this procedure for r'' and so on, we get the representation $r = \sum_{i=1}^{\infty} r_i$, where $v_2(r_i) = N_i$, $N_{i+1} > N_i$ and $r_i = \sum_{q=k_i}^{-\infty} s_q u^q$ with $s_q \in \ker \partial_1$ for all q. Clearly, the series $\sum_{i=1}^{\infty} r_i$ with such properties converges in $\widehat{E}_{\bar{R}}$, so our representation is well defined. But this is exactly what we need.
□

Now let

$$L = u_0 + L_m = u_0 + u_m\partial_2^m + (< m),$$

$$M = v_{-1}\partial_2 + v_0' + M_m = v_{-1}\partial_2 + v_0' + v_{m+1}\partial_2^{m+1} + (< m+1), \tag{16}$$

where we assume that $v_{m+1} = 0$ if $m = -1$. If $S = 1 - P$, $P = b\partial_2^m$, $b \in \widehat{E}_{\bar{R}}$, then it is easy to check that

$$[u_0, P] = [u_0, b]\partial_2^m - mb\partial_2(u_0)\partial_2^{m-1} + (< m-1), \tag{17}$$

$$[v_{-1}\partial_2 + v_0', P] = [v_{-1}, b]\partial_2^{m+1} + (v_{-1}\partial_2(b) + [v_0', b] + mb\partial_2(v_{-1}))\partial_2^m + (< m) \tag{18}$$

Now we are ready to prove the theorem using subsequent approximations in powers of ∂_2. We will do this in several steps.

a) Let's introduce a function $F : \mathbb{Z} \times \mathbb{N} \times \widehat{E}_{\bar{R}} \longrightarrow \mathbb{N}$ by the rule

$F(M, N, a) = q - M$, where $q > M$ is the maximal number such that the coefficient

b_q of a, where $a = \sum b_k u_0^k$, satisfy the property $v_2(b_q) \le N$;

if there are no such coefficients, we put $F(M, N, a) = 0$.

The function F satisfy the following properties:

i) $F(M, N, a) \ge F(M, N, \partial_1(a))$;

ii) $F(M, N, \sum_{i=1}^{k} a_i) \le \max\{F(M, N, a_i)\}$;

iii) Let $[x, u_0] = 0$. Then for any M, N

$$F(M, N, \partial_1(x)) \le F(M, N, x), \tag{19}$$

where the equality holds only if $F(M, N, x) = 0$.

The first two properties are obvious. Let's prove the third one. For a given M, N let q be the number mentioned in the definition of F. By lemma 6 $b_q \in \ker \partial_1$, so we obtain $v_2(\partial_1(b_k)) > N$. Hence all the coefficients b_k' with $k \ge q$ of $\partial_1(x)$, where $\partial_1(x)$ is represented as a series in u_0, have valuations greater than N, where from we obtain (19).

b) Let the operators L, M have the form as in (16). We will look for an S of the form as above. We have

$$S^{-1}LS = (1 + P + P^2 + \ldots)L(1 - P) = L - [L, P] - P[L, P] - P^2[L, P] - \ldots =$$

$$L - [L, P] - P[L, P] - P^2[L, P] - \ldots =$$

$$u_0 + L_m - [u_0, P] - [L_m, P] - P[u_0, P] - \ldots \tag{20}$$

All the terms, except the first three, are the elements of the order less or equal to $2m$. Hence, we get

$$S^{-1}LS = u_0 + (u_m - [u_0, b])\partial_2^m + (< m).$$

To solve the equation $u_m - [u_0, b] = 0$ let's define a sequence of elements in $\widehat{E}_{\bar{R}}$ as follows. Let b_0 be a solution of the equation $u_m = \partial_1(b_0)$. It exists because of conditions on the ring A, and we can always choose the coefficients of b_0 in such a way that values of the valuation v_2 on corresponding coefficients of u_m, b_0 are equal, so the conditions from the definition of the ring $\widehat{E}_{\bar{R}}$ hold also for b_0.

Let b_1 be a solution of the equation $-[u_0 - \partial_1, b_0] = \partial_1(b_1)$, and, generally, let b_k be a solution of the equation $-[u_0 - \partial_1, b_{k-1}] = \partial_1(b_k)$. Since $\mathrm{ord}_{\partial_1}(u_0 - \partial_1) \le 0$ and the ring A is commutative, we have the following property for the elements $[u_0 - \partial_1, b_{k-1}]$ and therefore for b_k:

$$F(M, N, b_k) = F(M, N, [u_0 - \partial_1, b_{k-1}]) \le F(M, N, b_{k-1}),$$

where the equality holds only if $F(M, N, b_{k-1}) = 0$.

Therefore, the series $b := \sum_{k=0}^{\infty} b_k$ converges in $\widehat{E}_{\bar{R}}$ and gives a solution of the equation $u_m - [u_0, b] = 0$.

c) For the operator M we have

$$S^{-1}MS = v_{-1}\partial_2 + v_0' + M_m - [v_{-1}\partial_2 + v_0', P] - [M_m, P] - P[v_{-1}\partial_2 + v_0', P] - \ldots, \tag{21}$$

where from

$$S^{-1}MS = v_{-1}\partial_2 + v_0' + (v_{m+1} - [v_{-1}, b])\partial_2^{m+1} +$$
$$(v_m - (v_{-1}\partial_2(b) + [v_0', b] + mb\partial_2(v_{-1})))\partial_2^m + (< m).$$

From lemma 6 follows that $[v_{-1}, b]$ does not depend on the choice of b above. Therefore, we can define $v_0 := v_0' - [v_{-1}, b_{u_1}]$, where b_{u_1} is a solution of the equation $u_1 - [u_0, b] = 0$. The relation for $\alpha = 0$ from lemma 5 for operators L, M looks like

$$0 = v_{-1}\partial_2(u_0) + [v_0', u_0] + [v_{-1}, u_1] = v_{-1}\partial_2(u_0) + [v_0, u_0] \tag{22}$$

and the last equation is exactly the relation for $\alpha = 0$ for operators $L_+, ((M\partial_2^{-1})_+\partial_2 + v_0)$. All relations for $\alpha < 0$ are identically zero and the relation for $\alpha = 1$ coincide with the relation for operators L, M, so the operators L_+ and $((M\partial_2^{-1})_+\partial_2 + v_0)$ commute by lemma 5.

d) Now we can assume $m < -1$, so that $v_0' = v_0$, $u_{-1} = 0$ in (16). Since the operators M, L commute, let's consider the relation from lemma 5 for these operators for $\alpha = m+1$. We obtain

$$[v_{m+1}, u_0] = 0.$$

We claim that there exists an element $s \in \widehat{E}_{\bar{R}}$ such that $[u_0, s] = 0$ and

$$v_{m+1} - (v_{-1}\partial_2(s) + [v_0, s] + (m+1)s\partial_2(v_{-1})) = [v_{-1}, d],$$

where d is a solution of the equation $[u_0, d] = (m+1)s\partial_2(u_0)$.

First of all let us note that $[v_{-1}, d]$ does not depend on the choice of d because of lemma 6, so the above equation is uniquely determined. As it follows from formulas (17), (18), (20), (21), for the operators $S_1 := 1 + s\partial_2^{m+1}$, $S_2 := 1 + d\partial_2^m$ we obtain

$$S_2^{-1}S_1^{-1}LS_1S_2 = u_0 + (< m), \quad S_2^{-1}S_1^{-1}MS_1S_2 = v_{-1}\partial_2 + v_0 + (< m+1).$$

The operators S_i obtained in this way step by step can be multiplied inside the group $\widehat{\mathcal{V}}_R$ and the result will be a solution of our problem.

e) To find s we will construct a sequence of elements s_i such that $[s_i, u_0] = 0$ for each i and the sum $s := \sum s_i$ converges.

Let $v_{m+1} = \sum b_k u_0^k$. Let's first define the element $s_0 = \sum s_{0k}u_0^k$ to be a solution of the equation

$$\sum b_k u_0^k = \sum \partial_2(s_{0k})u_0^k.$$

We can always find a solution of such equation because of our assumptions on the ring A. But we will need a special kind of solutions, namely, we will look for a solution s_0 in such a way that

1) $v_2(b_k) = v_2(s_{0k})$ for all k;

2) $s_{0k} \in \ker \partial_1$ for all k.

Obviously, we can find a solution satisfying both conditions by the same reason. Note that such a special choice of a solution imply the property (19) for s_0 for any values M, N, because the condition 2) imply that $[u_0, s_0] = 0$.

Now put

$$v'_{m+1,0} := v_{m+1} - (\sum v_{-1}\partial_2(s_{0k}u_0^k) + \sum[v_0, s_{0k}u_0^k] + \sum(m+1)s_{0k}u_0^k\partial_2(v_{-1})) =$$

$$v_{m+1} - (\sum v_{-1}\partial_2(s_{0k})u_0^k + \sum(m+1)s_{0k}u_0^k\partial_2(v_{-1})),$$

where the last equality follows from relation (22) applied to u_0^k. For any M, N we have

$$F(M, N, v'_{m+1,0}) \le F(M, N, v_{m+1}), \tag{23}$$

where the equality holds only if $F(M, N, v_{m+1}) = 0$. Indeed, this is true for $v_{m+1} - \sum v_{-1}\partial_2(s_{0k})u_0^k$ because of the construction of s_0 and because v_{-1} is a monic zeroth order operator. This is also true for $\sum(m+1)s_{0k}u_0^k\partial_2(v_{-1})$, because $\mathrm{ord}_{\partial_1}((m+1)s_{0k}u_0^k\partial_2(v_{-1})) < k$ for all k. Now it is only remain to apply the property ii) of the function F.

Put $u_{m,0} := (m+1)s_0\partial_2(u_0)$. Again, since $\mathrm{ord}_{\partial_1}(\partial_2(u_0)) \ge 0$ we have

$$F(M, N, u_{m,0}) \le F(M, N, v_{m+1}). \tag{24}$$

Now define d_0 to be a solution of the equation $[u_0, d_0] = u_{m,0}$ satisfying the property 1) of the solution s_0. As we have seen above in the construction of such solutions, a solution with this property can be easily constructed. Since $\mathrm{ord}_{\partial_1}(v_{-1}) = 0$, we have for any M, N

$$F(M, N, [v_{-1}, d_0]) \le F(M, N, u_{m,0}) \le F(M, N, v_{m+1}), \tag{25}$$

where the equalities hold only if $F(M, N, v_{m+1}) = 0$.

Combining all these observations, we get for the element $v_{m+1,0} := v'_{m+1,0} - [v_{-1}, d_0]$ and any M, N

$$F(M, N, v_{m+1,0}) \le F(M, N, v_{m+1}), \tag{26}$$

where the equality holds only if $F(M, N, v_{m+1}) = 0$.

Moreover, as it follows from formulas (17), (18), (21), $v_{m+1,0}$ is the $(m+1)$th coefficient of the operator $S_2^{-1}S_1^{-1}MS_1S_2$, where $S_1 = 1 + s_0\partial_2^{m+1}$, $S_2 = 1 + d_0\partial_2^m$. Using the relation from lemma 5 for operators $S_2^{-1}S_1^{-1}MS_1S_2$, $S_2^{-1}S_1^{-1}LS_1S_2$ for $\alpha = m+1$ we get $[u_0, v_{m+1,0}] = 0$.

Now, repeating all the above arguments for $v_{m+1,0}$ instead of v_{m+1}, we obtain new elements $s_1, u_{m,1}, d_1, v_{m+1,1}$ with the same properties, where v_{m+1} is replaced by $v_{m+1,0}$. Continuing this line of reasoning we get a sequences of elements $v_{m+1,i}$, s_i, $u_{m,i}$, d_i. Note that the series $\sum s_i$, $\sum u_{m,i}$, $\sum d_i$ converge, because by (23), (24), (25) for any M, N there are only finite number of elements $v_{m+1,i}$, s_i, $u_{m,i}$, d_i such that $F(M,N,(.)) > 0$. Moreover, we have $\lim_i v_{m+1,i} = 0$. Indeed, by the formula (26) the function $F(M,N,v_{m+1,i})$ strictly decrease for any fixed M, N, so the sequences of k th coefficients of the operators $v_{m+1,i}$ converges to zero for any k.

Now we can define $s := \sum s_i$, $d := \sum d_i$. Then we have

$$S_1^{-1} L S_1 = u_0 + \sum u_{m,i} + (< m),$$

$$S_1^{-1} M S_1 = v_{-1}\partial_2 + v_0 + v_{m+1} - \\ (\sum_k v_{-1}\partial_2(\sum_i s_{ik}) u_0^k + \sum_k (m+1)(\sum_i s_{ik}) u_0^k \partial_2(v_{-1})) + (< m+1)$$

and

$$S_2^{-1} S_1^{-1} L S_1 S_2 = u_0 + (< m), \quad S_2^{-1} S_1^{-1} M S_1 S_2 = v_{-1}\partial_2 + v_0 + (\lim_i v_{m+1,i}) + (< m+1).$$

This proves i) of the theorem.

ii) In order to get the second statement of the theorem it is enough to note that the conditions imply (in the notation of the proof of i)

$$[S'S^{-1}, u_0] = 0, \quad [S'S^{-1}, v_{-1}\partial_2 + v_0] = 0.$$

Since u_0, v_{-1} are monic, we can represent the operator $S'S^{-1}$ as a series in $u_0, (v_{-1}\partial_2 + v_0)$ with some coefficients from R. By lemma 6, all these coefficients must belong to $\ker \partial_1$ because of the first condition above. The second condition together with the identity (22) then imply that all these coefficients belong also to $\ker \partial_2$, where from we obtain our statement.

iii) The third assertion is an easy consequence of the analogous theorem in one dimensional case. Namely, for the operator $(M\partial_2^{-1})_+ L_+$, because of the solvability of the equation $\partial_1(x) = cx$, there exists a zeroth order invertible operator $\bar{S}' \in \widehat{E}_{\bar{R}}$ such that $(M\partial_2^{-1})_+ L_+ = \bar{S}'^{-1} \partial_1 \bar{S}'$ (see, for example, Th.1 in [12] or lemma 7.5 in [7]). The coefficients of this operator are defined modulo elements from $\ker \partial_1$. So, $v_{-1} := \partial_1 \bar{S}' L_+^{-1} \bar{S}'^{-1} = \partial_1 u_0^{-1}$, where $u_0 := \bar{S}'^{-1} L_+ \bar{S}'$ does not depend on the choice of $\bar{S}'$.

Taking the decomposition of the element $(v_0)_- = (\tilde{v_0})_- + x$, where $x \in \ker \partial_1$ is a zeroth order operator and $(\tilde{v_0})_- \cap \ker \partial_1 = 0$, we can find a zeroth order invertible operator $\bar{S}'' \in \widehat{E}_{\bar{R}}$, $\bar{S}'' \in 1 + \widehat{E}_{\bar{R}-}$, satisfying the equation $\partial_2(\log \bar{S}'') = -x$ and belonging to $\ker \partial_1$. The coefficients of such operator will be defined modulo elements from $\ker \partial_1 \cap \ker \partial_2 = k[[\ldots, t_{ij}, \ldots]]$. So, the operator $\bar{S} := \bar{S}'\bar{S}''$ will satisfy the assertion and, as we have seen, it is defined modulo operators with coefficients from $k[[\ldots, t_{ij}, \ldots]]$.
□

Corollary 1. *If the operators L, M from the theorem belong to the ring E_R, then the operator S can be found in the group $\mathcal{V}_R$.*

Proof. To prove the corollary one can follow the proof of the theorem, (i) and note that the subsequent approximations of the operator S belong, in fact, to the group $\mathcal{V}_R$.
□

Remark. Combining the proof of proposition 2 with some arguments from the proof of item i) of theorem 1, one can easily prove that the system (2) has only trivial solutions of the form (4), (5) also in the ring $\widehat{E}_R$. Namely, one should understand $\mathrm{ord}_{\partial_1}(.)$ there as a function defined by the formula

$$\mathrm{ord}_{\partial_1}(a) = \min\{M | F(M, v_2(a), a) = 0\}.$$

With this definition and with lemma 6 all arguments remain true.

Now we can prove lemma 4.

Proof of lemma 4. Let $P = \sum a_k \partial_2^k$, $a_k \in \widehat{E}_{\bar{R}}$. Since $P \in \widehat{E}_R$, its coefficients satisfy a growth order condition

$$v_2(a_k) > C_P k \quad \text{for all } k > M_P.$$

Let $Q = 1 - P = \sum b_k \partial_2^k \in \widehat{E}_R$. Since $(\pi(Q\partial_2^{-1}))_+ = 0$, $v_2(b_k) \geq 1$ for all $k \geq 1$. Of course the coefficients of Q satisfy the same growth order condition

$$v_2(b_k) > C_P k \quad \text{for all } k > M_P.$$

Thus there exists a positive real number J such that

$$v_2(b_k) \geq Jk \quad \text{for all } k \geq 1.$$

Actually, $J = C_P/(1 + C_p M_P)$ will do. Let $Q^n = \sum b_{n,k} \partial_2^k \in \widehat{E}_R$. Then we have

Claim. i) For every $n, k \geq 1$ we have $v_2(b_{n,k}) \geq Jk$.

ii) The function $F(M, N, b_{n,k})$ satisfy the following property: for any given M, N, k there exists a natural number $T(M, N, k)$ such that $F(M, N, b_{n,k}) = 0$ for all $n > T(M, N, k)$.

Here we take the function F defined in step a) of the proof of theorem 1 with respect to $u_0 = \partial_1$.

Proof of the claim. 1) To prove the first assertion, we use an induction on n. Since $Q^{n+1} = Q \cdot Q^n$, by formula (14) of lemma 3 we have

$$b_{n+1,k} = \sum_{q \in \mathbb{Z}} \sum_{l \geq 0} C_q^l b_q \partial_2^l (b_{n,k+l-q}) = \sum_{q=k}^{\infty} \sum_{l \geq 0} C_q^l b_q \partial_2^l (b_{n,k+l-q}) + \tag{27}$$

$$\sum_{q=0}^{k-1} \sum_{l \geq 0} C_q^l b_q \partial_2^l (b_{n,k+l-q}) + \sum_{q=-\infty}^{-1} \sum_{l \geq 0} C_q^l b_q \partial_2^l (b_{n,k+l-q}).$$

In the first summation, we have $v_2(b_q \partial_2^l (b_{n,k+l-q})) \geq v_2(b_q) \geq Jq \geq Jk$ for all $q \geq k \geq 1$. In the second summation, since $v_2(\partial_2^l(b_{n,i})) \geq v_2(b_{n,i})$, by the induction hypothesis we have

$$v_2(b_q \partial_2^l (b_{n,k+l-q})) \geq v_2(b_q) + v_2(b_{n,k+l-q}) \geq Jq + J(k + l - q) \geq Jk$$

for all $0 \leq q \leq k-1$. In the third summation we have $v_2(b_q \partial_2^l (b_{n,k+l-q})) \geq v_2(b_{n,k+l-q}) \geq Jk$ for all $k \leq -1$. Hence $v_2(b_{n+1,k}) \geq Jk$.

2) To prove the second assertion we use an induction on N and $|k|$. First let's introduce a function $q : \mathbb{N} \times \widehat{E}_{\bar{R}} \to \mathbb{Z}$:

$q(N, a)$ is the maximal number such that the coefficient $b_{q(N,a)}$ of a

satisfy the property $v_2(b_{q(N,a)}) \leq N$; (28)

if there are no such coefficients, we put $q(N, a) = -\infty$

Using formula (13) of lemma 2 it is easy to obtain the following property:

$$q(N, ab) \leq q(N, a) + q(N, b). \tag{29}$$

If $v_2(a) > 0$ and $v_2(b) > 0$, we also have

$$q(N, ab) \leq q(N-1, a) + q(N-1, b). \tag{30}$$

Indeed, in this case by (13) we have: $v_2((ab)_k) > N$ for $k > q(N-1, a) + q(N-1, b)$ if $v_2(a_i) > N-1$ for $i > q(N-1, a)$ and $v_2(b_i) > N-1$ for $i > q(N-1, b)$.

Analogously, if $v_2(a) > 0$ and $v_2(b) = 0$, we have

$$q(N, ab) \leq q(N, a) + q(N-1, b), \tag{31}$$

and if $v_2(a) = 0$ and $v_2(b) > 0$

$$q(N, ab) \leq q(N-1, a) + q(N, b). \tag{32}$$

3) Note that it suffice to prove our assertion only for $M < 0$. For such M we have then the following corollaries of properties (29), (30):

$$F(M, N, ab) = 0 \text{ if } F(M, N, a) = F(M, N, b) = 0\,; \tag{33}$$

and if $v_2(a) > 0$ and $v_2(b) > 0$, we also have

$$F(M, N, ab) = 0 \text{ if } F(M, N-1, a) = F(M, N-1, b) = 0\,. \tag{34}$$

Indeed, the condition $F(M, N, a) = F(M, N, b) = 0$ means $M \geq q(N, a)$, $M \geq q(N, b)$. Therefore, by (29) $M > 2M \geq q(N, a) + q(N, b) \geq q(N, ab)$, where from $F(M, N, ab) = 0$. Analogously, the condition $F(M, N-1, a) = F(M, N-1, b) = 0$ together with (30) imply $F(M, N, ab) = 0$.

Obviously, by (31) we also have for $v_2(a) > 0$ and $v_2(b) = 0$

$$F(M, N, ab) = 0 \text{ if } F(M, N, a) = F(M, N-1, b) = 0\,, \tag{35}$$

and by (32) we have for $v_2(a) = 0$ and $v_2(b) > 0$

$$F(M, N, ab) = 0 \text{ if } F(M, N-1, a) = F(M, N, b) = 0\,. \tag{36}$$

4) Now let's prove the first step of our double induction. Let $N = 0$ and $|k| = 0$. Then we have $T(M, N, k) = |M|$ and $q(N, b_{n,k}) \leq -n$. Indeed, by (27)

$$F(M, 0, b_{n+1,0}) = F(M, 0, b_0 b_{n,0}) = \ldots = F(M, 0, b_0^{n+1}),$$

because $v_2(b_{j,l}) \geq Jl > 0$ for all $j, l \geq 1$ by the first assertion of the claim, so all the summands in (27) except $b_0 b_{n,0}$ have valuation greater than zero and therefore they don't change the value of the function $F(M, 0, .)$. Obviously, the same is true for the function $q(0, .)$. Since $\mathrm{ord}_{\partial_1} b_0 \leq -1$, we have $\mathrm{ord}_{\partial_1} b_0^n \leq -n$, where from $F(M, 0, b_0^{n+1}) = 0$ for all $n > |M|$ and $q(0, b_{n+1,k}) \leq -n-1$.

5) Now let $N = 0$ and $|k| > 0$. Since $v_2(b_{j,l}) > 0$ for all $j, l \geq 1$, we have $F(M, 0, b_{n,k}) = 0$, $q(0, b_{n,k}) = -\infty$ for all n and $k \geq 1$. So, we can assume $k < 0$. By the induction hypothesis $F(M, 0, b_{n,j}) = 0$ for all $n > T(M, 0, j)$ if $|j| < |k|$. Denote by N_0 the maximum of all such $T(M, 0, j)$. Now consider the elements $b_{n+N_0,k}$ with $n > N_0$. Since $Q^{n+N_0} = Q^n Q^{N_0}$, we have by the same reason as above

$F(M, 0, b_{n+N_0,k}) =$

$$F(M, 0, \sum_{i=k}^{0} \sum_{l=0}^{-k} C_i^l b_{n,i} \partial_2^l (b_{N_0, k+l-i})) \leq \max_{k \leq i \leq 0, 0 \leq l \leq -k} \{F(M, 0, b_{n,i} \partial_2^l (b_{N_0, k+l-i}))\}, \tag{37}$$

where the last inequality follows from the property ii) of the function F, see step a) of theorem 1.

6) Now it suffice to prove the assertion for each function $F(M,0,b_{n,i}\partial_2^l(b_{N_0,k+l-i}))$, where $k\le i\le 0, 0\le l\le -k$ and $k+l-i\le 0$, and then define the number $T(M,0,k)$ as a maximum of corresponding numbers for each pair (l,i).

The set of all pairs (l,i) can be divided in two subsets: $\{(0,0),(0,k)\}$ and all other pairs. For the pairs from the second subset we have $|i|<|k|$ and $|k+l-i|<|k|$. So, for all these pairs we have $F(M,0,b_{n,i})=0$ and $F(M,0,\partial_2^l(b_{N_0,k+l-i}))\le F(M,0,b_{N_0,k+l-i})=0$ (the last inequality holds by the analog of the property i) of the function F in step a) of the proof of theorem 1). Therefore, by (33) we obtain $F(M,0,b_{n,i}\partial_2^l(b_{N_0,k+l-i}))=0$ for all $n>N_0$ and all pairs from the second subset.

Now consider the pair $(0,0)$ from the first subset. By the property (29) we have

$$q(0,b_{n,0}\partial_2^l(b_{N_0,k}))\le q(0,b_{n,0})+q(0,\partial_2^l(b_{N_0,k}))\le q(0,b_{n,0})+q(0,b_{N_0,k})\le -n+q(0,b_{N_0,k}).$$

Therefore, for all $n>-M+q(0,b_{N_0,k})$ we have $F(M,0,b_{n,0}\partial_2^l(b_{N_0,k}))=0$.

7) Combining all together, we get for all $n>\max\{N_0,-M+q(0,b_{N_0,k})\}:=\tilde{T}$ (see (37)):

$$F(M,0,b_{n+N_0,k})\le F(M,0,b_{n,k}\partial_2^l(b_{N_0,0})). \tag{38}$$

Now put

$$O=(\max_{\tilde{T}\le j\le\tilde{T}+N_0}\{F(M,0,b_{j,k})\})$$

and put $T(M,0,k)=\tilde{T}+ON_0$. We claim, that for all $n>T(M,0,k)$ $F(M,0,b_{n,k})=0$.

Assume the converse. Let $n>T(M,0,k)$ be the number such that $F(M,0,b_{n,k})>0$. Then by (38) and (29) we have

$$0<F(M,0,b_{n,k})=q(0,b_{n,k})-M\le F(M,0,b_{n-N_0,k}\partial_2^l(b_{N_0,0}))=q(0,b_{n-N_0,k}\partial_2^l(b_{N_0,0}))-M$$

$$\le q(0,b_{n-N_0,k})+q(0,b_{N_0,0})-M\le q(0,b_{n-N_0,k})-M-N_0=F(M,0,b_{n-N_0,k})-N_0\le\dots$$

$$\le F(M,0,b_{n-\tilde{O}N_0,k})-\tilde{O}N_0,$$

where $\tilde{T}\le n-\tilde{O}N_0\le\tilde{T}+N_0$ and therefore $\tilde{O}>O$. So,

$$F(M,0,b_{n-\tilde{O}N_0,k})-\tilde{O}N_0\le O-\tilde{O}N_0\le 0,$$

a contradiction.

8) Now assume N is an arbitrary positive number. Let $k_0:=[N/J]$ be the integral part of the number N/J. For all $k>k_0$ we have $v_2(b_{n,k})\ge Jk>N$, so the assertion is trivial for all such k.

To prove the assertion we will use an inverse induction on $k\le k_0$. By the induction hypothesis on N there exists a natural number N_0 such that $F(M,N-1,b_{n,i})=0$ for all $n\ge N_0$ and $k_0\le i\le 0$ (with fixed M). We then have for all $n>N_0$ and $k=k_0$

$$F(M,N,b_{n+N_0,k_0})=F(M,N,\sum_{i=0}^{k_0}\sum_{l=0}^{k_0}C_i^l b_{n,i}\partial_2^l(b_{N_0,k_0+l-i}))\le$$

$$\max_{k_0\ge i\ge 0,0\le l\le k_0}\{F(M,N,b_{n,i}\partial_2^l(b_{N_0,k_0+l-i}))\}\le$$

$$\max\{F(M,N,b_{n,0}\partial_2^l(b_{N_0,k_0})),F(M,N,b_{n,k_0}\partial_2^l(b_{N_0,0}))\},$$

where the last inequality follows from (34). For all $n > -M + q(N, b_{N_0,k_0})$ we have, as above, $F(M, N, b_{n,0}\partial_2^l(b_{N_0,k_0})) = 0$. So, for all sufficiently large n

$$F(M, N, b_{n+N_0,k_0}) \leq F(M, N, b_{n,k_0}\partial_2^l(b_{N_0,0})),$$

and we can repeat the arguments of step 7) to get the proof of the assertion in the case $k = k_0$.

9) Let's prove the assertion for arbitrary $k < k_0$. By the induction hypothesis on N, k there exists a natural number N_0 such that $F(M, N-1, b_{n,j}) = 0$ for all $n \geq N_0$ and $k_0 \geq j \geq k-k_0$, and $F(M, N, b_{n,j}) = 0$ for all $n \geq N_0$ and $k_0 \geq j > k$ (again M is fixed). We then have for all $n > N_0$

$$F(M, N, b_{n+N_0,k}) = F(M, N, \sum_{i=k-k_0}^{k_0} \sum_{l=0}^{2k_0-k} C_i^l b_{n,i}\partial_2^l(b_{N_0,k+l-i})) \leq$$

$$\max_{k_0 \geq i \geq (k-k_0), 0 \leq l \leq 2k_0-k} \{F(M, N, b_{n,i}\partial_2^l(b_{N_0,k+l-i}))\} \leq$$

$$\max\{F(M, N, b_{n,0}\partial_2^l(b_{N_0,k})), F(M, N, b_{n,k}\partial_2^l(b_{N_0,0}))\},$$

where the last inequality follows from (33), (34), (35), (36). Repeating the arguments of the end of step 8), we obtain the assertion also in this case, hence in general.

The claim is proved.

By the claim we can conclude that $\sum_{n=0}^{\infty} b_{n,k} \in \widehat{E}_{\bar{R}}$ is a well-defined element for any k. Therefore

$$\sum_{n=0}^{\infty} Q^n = 1 + \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} b_{n,k}\partial_2^k = 1 + \sum_{k \in \mathbb{Z}} (\sum_{n=1}^{\infty} b_{n,k})\partial_2^k.$$

By the first assertion of the claim $v_2(\sum_{n=1}^{\infty} b_{n,k}) \geq Jk$ if $k > 0$. Obviously, we also have $b_{n,0}|_{t=0} = 0$ for all $n \geq 1$, where from $\sum_{n=1}^{\infty} b_{n,0})|_{t=0} = 0$. This implies that $\sum_{n=0}^{\infty} Q^n \in \widehat{E}_R^{\times}$. Since

$$1 = \sum_{n=0}^{\infty}(1-P)^n - \sum_{n=1}^{\infty}(1-P)^n = \sum_{n=0}^{\infty}(1-P)^n - (1-P)\sum_{n=0}^{\infty}(1-P)^n = P\sum_{n=0}^{\infty}(1-P)^n,$$

the Neumann series gives $P^{-1} \in \widehat{E}_R^{\times}$.

The lemma is proved.

□

4 Modified Sato-Wilson systems

4.1 Equivalence of modified KP and modified Sato-Wilson systems

In this subsection we will prove the equivalence of systems $(KP)_\alpha$ and appropriately generalized Sato-Wilson systems. This is a generalization of well known equivalence from the classical case. It is also the necessary step in solving the Cauchy problem for the systems $(KP)_\alpha$.

Let's fix a system $(KP)_\alpha$. For a given operator $N = (L, M)$ we define formal 1-forms by

$$Z_{\pm}^{\alpha} = \pm \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+, i \leq \alpha j} dt_{i,j}(L^i M^j)_{\pm}$$

Then it is easy to see that the Lax equation for the $(KP)_\alpha$ system is given by

$$d_\alpha N = [Z_+^\alpha, N] = [Z_-^\alpha, N], \tag{39}$$

where $d_\alpha N$ denotes the vector $(d_\alpha L, d_\alpha M)$, $[Z_\pm^\alpha, N]$ denotes the vector $([Z_\pm^\alpha, L], [Z_\pm^\alpha, M])$, and $d_\alpha = \sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+, i \leq \alpha j} dt_{i,j}\partial/\partial t_{i,j}$ denotes the exterior derivative in t.

Proposition 3. *The Lax equation for the* $(KP)_\alpha$ *system is equivalent to the integrability condition*

$$d_\alpha Z^\alpha_\pm = Z^\alpha_\pm \wedge Z^\alpha_\pm$$

Proof. In [12], prop. 4, i) one can find the proof that the solution of the $(KP)_\alpha$ system in the form (4), (5) also satisfies the system

$$\frac{\partial(L^iM^j)_+}{\partial t_{l,m}} - \frac{\partial(L^lM^m)_+}{\partial t_{i,j}} = [(L^lM^m)_+, (L^iM^j)_+], \tag{40}$$

where $i, l \in \mathbb{Z}$, $j, m \in \mathbb{Z}_+$, $i \le \alpha j$, $l \le \alpha m$, that is, to the equation $dZ^\alpha_+ = Z^\alpha_+ \wedge Z^\alpha_+$. Since for $Z^\alpha = Z^\alpha_+ - Z^\alpha_- = \sum dt_{i,j} L^iM^j$ we have $Z^\alpha \wedge Z^\alpha = 0 = Z^\alpha_+ \wedge Z^\alpha_+ + Z^\alpha_- \wedge Z^\alpha_-$, we have also $dZ^\alpha_- = Z^\alpha_- \wedge Z^\alpha_-$.

To prove the converse assertion let us consider first the equations (40) with arbitrary l, m, j and with $i = -|\alpha j|$. Clearly, $i \le \alpha j$ for any j. The equations (40) can be rewritten as

$$\frac{\partial(L^iM^j)}{\partial t_{l,m}} - [(L^lM^m)_+, (L^iM^j)] = -[(L^lM^m)_+, (L^iM^j)_-] + \frac{\partial(L^lM^m)_+}{\partial t_{i,j}} + \frac{\partial(L^iM^j)_-}{\partial t_{l,m}},$$

where from

$$\mathrm{ord}_{\partial_2}\left(\frac{\partial(L^iM^j)}{\partial t_{l,m}} - [(L^lM^m)_+, L^iM^j]\right) \le m$$

Since $\partial/\partial t_{l,m}$ and $[(L^lM^m)_+,]$ are derivations, we have

$$\left(\frac{\partial}{\partial t_{l,m}} - [(L^lM^m)_+,]\right) L^iM^j = \left(\sum_{k=i+1}^{0} L^k \left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^lM^m)_+, L^{-1}]\right) L^{i+1-k}\right) M^j +$$

$$L^i \sum_{k=0}^{j-1} M^k \left(\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]\right) M^{j-1-k},$$

where from

$$\mathrm{ord}_{\partial_2}\left(\sum_{k=i+1}^{0} L^k \left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^lM^m)_+, L^{-1}]\right) L^{i+1-k} M +\right.$$

$$\left. L^i \sum_{k=0}^{j-1} M^k \left(\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]\right) M^{-k}\right) \le m - j + 1 \tag{41}$$

Obviously, if (L, M) are solutions of (40) in the form (4), (5), then u_0, v_{-1} do not depend on times and therefore L, M are invertible. Hence, since the characteristic of the ground field k is zero, we have

$$\mathrm{ord}_{\partial_2}\left(L^i \sum_{k=0}^{j-1} M^k \left(\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]\right) M^{-k}\right) = \mathrm{ord}_{\partial_2}\left(\frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]\right) \tag{42}$$

and

$$\mathrm{ord}_{\partial_2}\left(\sum_{k=i+1}^{0} L^k \left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^lM^m)_+, L^{-1}]\right) L^{i+1-k}\right) = \mathrm{ord}_{\partial_2}\left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^lM^m)_+, L^{-1}]\right). \tag{43}$$

Let's denote by Q_L the coefficient of the series $W_1 = \frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^lM^m)_+, L^{-1}]$ by the power $\partial_1^{q(v_2(W_1), W_1)} \partial_2^{\mathrm{ord}_{\partial_2}(W_1)}$, and by Q_M the coefficient of the series $W_2 = \frac{\partial M}{\partial t_{l,m}} - [(L^lM^m)_+, M]$

by the power $\partial_1^{q(v_2(W_2),W_2)}\partial_2^{\mathrm{ord}_{\partial_2}(W_2)}$ (where the function q was defined in the proof of lemma 4, and we assume that $W_1, W_2 \neq 0$). Let Q_L', Q_M' be analogous coefficients of the series from the left hand sides of (43), (42). By the definition we have $v_2(Q_L') = v_2(Q_L) = v_2(W_1)$, $v_2(Q_M') = v_2(Q_M) = v_2(W_2)$. Then the equality of orders in (43), (42) exactly means that we have

$$v_2(Q_L' - iQ_L) > v_2(W_1), \quad v_2(Q_M' - jQ_M) > v_2(W_2).$$

Since j can be an arbitrary large number and the orders

$$\mathrm{ord}_{\partial_2}\left(\frac{\partial M}{\partial t_{l,m}} - [(L^l M^m)_+, M]\right), \quad \mathrm{ord}_{\partial_2}\left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^l M^m)_+, L^{-1}]\right)$$

don't depend on j, we obtain from (41)

$$\mathrm{ord}_{\partial_2}\left(\frac{\partial M}{\partial t_{l,m}} - [(L^l M^m)_+, M]\right) = \mathrm{ord}_{\partial_2}\left(\frac{\partial L^{-1}}{\partial t_{l,m}} - [(L^l M^m)_+, L^{-1}]\right) + 1$$

and since L, M are monic,

$$v_2(W_1) = v_2(W_2), \quad v_2(Q_M' - Q_L') > v_2(W_1),$$

where from

$$v_2(iQ_L - jQ_M) > v_2(W_1)$$

for all j (recall that i depend on j). The same arguments with $i' = -2|\alpha j|$ give us

$$v_2(i'Q_L - jQ_M) = v_2(2iQ_L - jQ_M) > v_2(W_1),$$

where from $v_2(W_1) < v_2(2iQ_L - iQ_L) = v_2(iQ_L) = v_2(W_1)$, a contradiction. Therefore, our assumption about W_1, W_2 is wrong, i.e. $W_1 = W_2 = 0$. But this means that the system (39) holds.
The proposition is proved.
□

Corollary 2. *The equations (39) are compatible, i.e.* $d_\alpha[Z_+^\alpha, N] = 0$.

The proof is clear.

Suppose that our $(KP)_\alpha$ system with initial condition $N_0 = (L_0, M_0)$ has a solution $N \in \widehat{E}_R$. Then by theorem 1, (i) there exists $S \in \widehat{\mathcal{V}}_R$ with $L = Su_0S^{-1}$, $M = S(v_{-1}\partial_2 + v_0)S^{-1}$. As we have already seen, u_0, v_{-1} do not depend on times and therefore coinside with the first coefficients of the initial condition N_0. Notably, v_0 constructed in the theorem, also don't depend on times. Indeed, if v_0' is the coefficient of M as in (16), we have

$$\frac{\partial v_0'}{\partial t_{ij}} = -[(L^i M^j)_-, M]_0 = -[x, v_{-1}],$$

where x is the first coefficient of $(L^i M^j)_-$ represented as series in ∂_2^{-1}. For x we have also another equation:

$$\frac{\partial u_1}{\partial t_{ij}} = -[(L^i M^j)_-, L]_1 = -[x, u_0]. \tag{44}$$

Therefore, from definition of v_0 follows

$$\frac{\partial v_0}{\partial t_{ij}} = \frac{\partial v_0'}{\partial t_{ij}} - [v_{-1}, \frac{\partial b_{u_1}}{\partial t_{ij}}] = 0,$$

because $\frac{\partial b_{u_1}}{\partial t_{ij}}$ can be an arbitrary solution of the equation (44).

So, denote $L_+ = u_0$ by L_{00} and $(v_{-1}\partial_2 + v_0)$ by M_{00}. As we have shown above, L_{00}, M_{00} are uniquely defined by N_0. They are also invertible because of our assumptions on the initial condition. Now let

$$\omega_\alpha^{N_0} = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+, i\le\alpha j} dt_{i,j} L_{00}^i M_{00}^j \tag{45}$$

It's clear that $S\omega_\alpha^{N_0}S^{-1} = \sum dt_{i,j}L^jM^j = Z^\alpha$. Let's introduce the additional notation:

$$E_{k,L_{00},M_{00}} := k((L_{00}^{-1}))((M_{00}^{-1})), \quad \mathcal{V}_{k,L_{00},M_{00}} := \{1 + E_{k,L_{00},M_{00},-}\},$$

$$\widehat{\mathcal{V}}_{k,L_{00},M_{00}} := \{L = 1 + \sum_{i=1}^{\infty} a_i M_{00}^{-i} | a_i = \sum_{q\in\mathbb{Z}} b_q L_{00}^q \in \widehat{E}_{\bar{R}} \quad \textit{and } b_q \in k[[\dots, t_{ij}, \dots]]\}.$$

From now on *we will make one more assumption.* Since we have fixed the system $(KP)_\alpha$ and this system contains no times t_{ij} with $i > \alpha j$ and since in general $(\partial/\partial t_{kl}N)(t_{ij} = 0) = \partial/\partial t_{kl}(N(t_{ij} = 0))$ for $i > \alpha j$ and $k \le \alpha l$ and since $[(L^kM^l)_+, N](t_{ij} = 0) = [(L(t_{ij} = 0)^k M(t_{ij} = 0)^l)_+, N(t_{ij} = 0)]$, we can *assume that solutions* (if they exist) *of the system do not depend on times t_{ij} with $i > \alpha j$*. Obviously, *we can also assume that solutions do not depend on t_{00}*.

Proposition 4. *The system $(KP)_\alpha$ with the initial condition N_0 is equivalent to the modified Sato-Wilson system:*

$$d_\alpha S = -(S\omega_\alpha^{N_0}S^{-1})_- S \tag{$(SW)_\alpha$}$$

with $S \in \widehat{\mathcal{V}}_R \cdot \widehat{\mathcal{V}}_{k,L_{00},M_{00}}$ and with initial condition $S(0) \in (1 + E_-)\cdot \mathcal{V}_{k,L_{00},M_{00}}$.

Proof. If S is a solution of the system $(SW)_\alpha$, then $N = (SL_{00}S^{-1}, SM_{00}S^{-1})$ gives a solution of the $(KP)_\alpha$, the proof is the same as in [12], prop.4, ii).

Let's prove the converse. Let N be a solution of the $(KP)_\alpha$, and let

$$N = (S_0 L_{00} S_0^{-1}, S_0 M_{00} S_0^{-1})$$

for some S_0. Put $Z_-^0 = S_0^{-1}Z_-^\alpha S_0 - S_0^{-1}d_\alpha S_0$. Then we have $d_\alpha Z_-^0 = Z_-^0 \wedge Z_-^0$. This follows from the same argument as in the classical case of the one dimensional (KP) system, namely, because Z_-^0 is a gauge transformation of the flat connection Z_-^α. We regard here $Z_\pm^\alpha$ as connections on the trivial bundle $E\times T$, where T is the space of deformation parameters $t = (t_{ij})$, on which the Lie algebra E acts by the commutator. One can check this also directly.

Now let us show that all operators in Z_-^0 (written as series in L_{00}^{-1}, M_{00}^{-1}) have only coefficients belonging to $k[[\dots, t_{ij}, \dots]]$:

$$[Z_-^0, L_{00}] = S_0^{-1}[S_0 Z_-^0 S_0^{-1}, S_0 L_{00} S_0^{-1}]S_0 = S_0^{-1}[Z_-^\alpha - d_\alpha S_0 S_0^{-1}, L]S_0 =$$

$$S_0^{-1}([Z_-^\alpha, L] - [dS_0S_0^{-1}, L])S_0 = S_0^{-1}(d_\alpha L - d_\alpha L)S_0 = 0$$

(and analogously for M_{00}), because

$$d_\alpha L = d_\alpha S_0 L_{00} S_0^{-1} - S_0 L_{00} S_0^{-1} d_\alpha S_0 S_0^{-1} = [d_\alpha S_0 S_0^{-1}, L].$$

Now consider the equation $d_\alpha C = Z_-^0 C$. Since $d_\alpha Z_-^0 = Z_-^0 \wedge Z_-^0$, this equation is compatible and therefore there exists a solution $C \in \widehat{\mathcal{V}}_{k,L_{00},M_{00}}$. Let's explain this fact, because here is important the choice of the topology in the ring $\bar{R}$ and our assumptions on the solutions of the system $(KP)_\alpha$.

To prove the existence of a solution, let's fix some bijection between the set of all pairs (i,j) with $i\in\mathbb{Z}$, $j\in\mathbb{Z}_+$, $i\le\alpha j$ and natural numbers, $f : \{(i,j)\} \to \mathbb{N}$, such that for any finite set of integers $(a_1,\dots a_n)$ there exists $J\in\mathbb{N}$ with the following property: for all $k>J$ $f^{-1}(k)=(i_k,j_k)$, where either $j_k>n$ or $i_k<a_{j_k}$. Such a bijection can be constructed, for example, by counting integral points (lying below the graphic of the function α) lying inside polygons formed by parallel lines that go through the points $(i,0)$, $(i-1,0)$, $i\le\alpha(0)$, and are parallel to the line going through the points $(\alpha(0)-1,0)$ and $(\alpha(1),1)$, the vertical lines $(.,k)$ and lines going through the points $(\alpha(k),k)$, $(\alpha(k+1),k+1)$.

Now $Z_-^0=\sum_{k=1}^{\infty}P_{f^{-1}(k)}dt_{f^{-1}(k)}$, where $\mathrm{ord}_{\partial_2}(P_{ij})\le -1$, and the solution $C=\exp(C')$, where C' is a solution of the equation $d_\alpha C'=Z_-^0$. To find a solution C' it suffice to find a solution C'' of the equation $d_\alpha C''=\sum_{k=2}^{\infty}P'_{f^{-1}(k)}dt_{f^{-1}(k)}$, where $P'_{f^{-1}(k)}=P_{f^{-1}(k)}-\partial/\partial t_{f^{-1}(k)}(\int P_{f^{-1}(1)}dt_{f^{-1}(1)})$, and then put $C'=C''+\int P_{f^{-1}(1)}dt_{f^{-1}(1)}$. Because of compatibility conditions on Z_-^0, $P'_{f^{-1}(k)}$ don't depend on $t_{f^{-1}(1)}$ for all $k\ge 2$. Continuing this line of reasoning, we obtain that C' must be equal to a sum of infinite number of elements from the ring $\widehat{E}_R$, where each summand is an integral on time and only finite number of summands depend on $t_{f^{-1}(k)}$ for any fixed k.

This sum converges if and only if the infinite sum of elements from the ring $\bar{R}$ with the same property converges in $\bar{R}$. By definition of the ring $\bar{R}$ a series converges if and only if for any neighborhood of zero there are only finite number of monomials of all summands not belonging to the neighborhood. By our assumption on the solutions of the system $(KP)_\alpha$ all summands don't depend on times t_{ij} with $i>\alpha j$. From the other hand side, since all summands are integrals on times, there are no free terms, i.e. all monomials depend on some other times t_{ij}. From the definition of the base of neighborhoods of zero in the ring A_t follows that each monomial that don't depend on t_{ij} with $i>\alpha j$ and don't belong to a neighborhood of zero, must depend on some t_{ij} from some *finite* set of times $\{t_{kl}\}$. It's easy to see that for each neighborhood such a finite set lies in the set $\{t_{f^{-1}(k)},k=1,\dots,m\}$ for some finite number m. Since in our sum there are only finite number of summands depending on finite number of such times and since all summands have no free terms, we obtain the convergence.

So, the solution C' exists. Since $\mathrm{ord}_{\partial_2}P_{f^{-1}(k)}\le -1$ for all k, we obtain $\mathrm{ord}_{\partial_2}(C')\le -1$. Therefore $C:=\exp(C')$ also exists.

Now put $S=S_0C$, then

$$d_\alpha S-Z_-^\alpha S=d_\alpha S_0C+S_0d_\alpha C-Z_-^\alpha S_0C=S_0(S_0^{-1}d_\alpha S_0+d_\alpha CC^{-1}-S_0^{-1}Z_-^\alpha S_0)C=0$$

and $SL_{00}S^{-1}=S_0L_{00}S_0^{-1}=L$, $SM_{00}S^{-1}=S_0M_{00}S_0^{-1}=M$.

At last, as it follows from corollary 1, the initial condition $S(0)\in(1+E_-)\cdot\mathcal{V}_{k,L_{00},M_{00}}$. $\square$

Remark. As it follows from the proof and corollary 1, if $N\in E_R$, then $S\in\mathcal{V}_R\cdot\widehat{\mathcal{V}}_{k,L_{00},M_{00}}$, and vice versa.

Proposition 5. *We have a bijection between the following two sets:*

$$Sol((KP)_\alpha)=\{N=(L,M),\quad L,M\in\widehat{E}_R(E_R)\quad|\quad N\quad \textit{satisfies } (KP)_\alpha \text{ and } L,M \textit{ don't depend on } t_{ij},\ i>\alpha j\}$$

and

$$Sol((SW)_\alpha)=\{S\in\widehat{\mathcal{V}}_R\cdot\widehat{\mathcal{V}}_{k,L_{00},M_{00}}(\mathcal{V}_R\cdot\widehat{\mathcal{V}}_{k,L_{00},M_{00}})\quad|\quad S\quad \textit{satisfies } (SW)_\alpha \text{ and does not depend on } t_{i,j},\ i>\alpha j\}/(\mathcal{V}_{k,L_{00},M_{00}})$$

Proof. The bijection is given by the formula $L=SL_{00}S^{-1}$, $M=SM_{00}S^{-1}$ and follows from the proposition 4. Indeed, suppose $S_1N_{00}S_1^{-1}=S_2N_{00}S_2^{-1}$ for $N_{00}=(L_{00},M_{00})$. Then $S_2=S_1C$, where $C\in\widehat{\mathcal{V}}_{k,L_{00},M_{00}}$. Therefore, from the equation $(SW)_\alpha$ we get

$$Z_-^\alpha(S_1C)=d_\alpha(S_1C)=d_\alpha S_1C+S_1d_\alpha C,$$

where from $S_1 d_\alpha C = 0$. Since S_1, S_2 don't depend on $t_{i,j}$, $i > \alpha j$, we conclude $C \in \mathcal{V}_{k,L_{00},M_{00}}$. The proposition is proved.
□

To complete the picture, we have to describe the set of all such initial conditions $S(0) \in (1+E_-)$ that give us trivial initial conditions (so, trivial solutions by theorem 4 below) of the system $(KP)_\alpha$. Following Mulase, [7], we give the following definition.

Definition 1. A pseudo-differential operator $T \in E$ is said to be (L_{00}, M_{00})-admissible if it is an invertible operator of order zero such that $TL_{00}T^{-1}, TM_{00}T^{-1} \in E_{k,L_{00},M_{00}}$. The set of all (L_{00}, M_{00})-admissible operators is denoted by $\Gamma_{a,L_{00},M_{00}}$.

Lemma 7. *Every (L_{00}, M_{00})-admissible operator T has the following form:*

$$T = S^{-1}T_0 a,$$

$$S = 1 + s_1 M_{00}^{-1} + s_2 M_{00}^{-2} + \dots,$$

where

$$a = e^{c_1 \tilde{x}_1 + c_2} c_0$$

(if such exponent exists, otherwise $a = c_0$), $0 \neq c_0 \in k$, $c_1 \in k$, $c_2 \in \ker \partial_1$ and $\tilde{x}_1 \in A[[L_{00}^{-1}]]$ is an element satisfying

$$[L_{00}, \tilde{x}_1] = 0, \quad [\partial_2 + \partial_2(\log V), \tilde{x}_1] = 0,$$

$V \in 1 + E_-$ is an operator such that $VL_{00}V^{-1} \in A[\partial_1]$,

$$s_n = \sum_{k=0}^{n} \tilde{x}_1^k \tilde{C}_k, \quad \tilde{C}_k = \sum_j a_j L_{00}^j \in \ker \partial_1((L_{00}^{-1})),$$

the coefficients a_j are defined recursively by formulas (56), (53) below.

The operator T_0 is the operator defined separately in each of the following two cases:

1) If $(M_{00}\partial_2^{-1})_+ \in k((L_{00}^{-1}))$, then $T_0^{-1} = \exp((\tilde{x}_1 k_1 + l_1)L_{00}^{-1} + \dots)$, where $k_i \in k$ satisfy the condition (61) and $l_i \in \ker \partial_1$ are defined by condition (59) below.

2) If $(M_{00}\partial_2^{-1})_+ \notin k((L_{00}^{-1}))$, then $T_0^{-1} = \exp(l_1 L_{00}^{-1} + l_2 L_{00}^{-2} + \dots)$, where $l_i \in \ker \partial_1$ are defined by condition (62) below.

Proof. Every (L_{00}, M_{00})-admissible operator $T \in E$ can be written as a product $T = S^{-1}T_0 a$, where $S \in 1 + E_-$, $T_0 \in 1 + A((L_{00}^{-1}))$, $a \in A$.

First of all, let's consider the action of the operators $S^{-1}, T_0 a$ on (L_{00}, M_{00}). For any operators L, M as in theorem 1 we have

$$(T_0 a L_+ a^{-1} T_0^{-1})_- = 0, \quad (S^{-1} L_+ S)_+ = L_+ \tag{46}$$

by formulas (20) and (17); and

$$(T_0 a M_+ a^{-1} T_0^{-1})_- = 0, \quad (S^{-1} M_+ S)_+ = M_+ - x, \quad x \in A((L_{00}^{-1})) \tag{47}$$

by formulas (21) and (18). Therefore we obtain two necessary conditions on the operator $T_0 a$:

$$T_0 a L_{00} a^{-1} T_0^{-1} \in k((L_{00}^{-1})),$$

$$T_0 a v_{-1} a^{-1} T_0^{-1} v_{-1}^{-1} \in k((L_{00}^{-1})).$$

Since v_{-1} commute with L_{00} and can be represented as a series in L_{00}^{-1} with coefficients from $\ker \partial_1$ by lemma 6, the necessary conditions above imply that either coefficients of v_{-1} belong to k or $T_0 a$ commute with L_{00}. So, we consider two cases:

Case 1. The coefficients of v_{-1} belong to k.

Case 2. There are coefficients of v_{-1} that do not belong to k.

Case 1. Assume that the equation $\partial_1(x) = cx$ with $c \in A$ has a solution in the ring $A' \supset A$. Formally we will not need this assumption, as we will see later, but it makes the proof more convenient and compact.

1) In this case there exists an invertible zeroth order operator $V \in E \otimes_A A'$ such that $VL_{00}V^{-1} = \partial_1$. Note that V can be found in the form $V = V_0 x$, where $V_0 \in 1 + E_-$ and $x \in A'$. Thus, V preserve the ring E, $VEV^{-1} \subset E$. Obviously, the operator T is (L_{00}, M_{00})-admissible if and only if VTV^{-1} is $(VL_{00}V^{-1}, VM_{00}V^{-1})$-admissible. Replacing now the operators L_{00}, M_{00} by $VL_{00}V^{-1}, VM_{00}V^{-1}$, we can assume $L_{00} = \partial_1$ and v_0 is a polynomial in ∂_1 with coefficients from $\ker \partial_1(A)$ (see theorem 1, iii)).

Because of (46) we must have

$$\partial_1 S = S(\partial_1 + k_1 M_{00}^{-1} + k_2 M_{00}^{-2} + \ldots),$$

where $k_i \in k((\partial_1^{-1}))$, where from

$$\partial_1(s_i) = \sum_{q=0}^{i-1} s_q k_{i-q}, \tag{48}$$

where $S = 1 + s_1 M_{00}^{-1} + s_2 M_{00}^{-2} + \ldots$. Let's fix some element $x_1 \in A$ with properties $\partial_1(x_1) = 1$, $\partial_2(x_1) = 0$. Such an element exists by the properties of the ring A, and it is defined up to a constant. So, from (48) we obtain that s_i is a polynomial in x_1 with coefficients from $\ker \partial_1((\partial_1^{-1}))$,

$$s_i = \sum_{q=0}^{i} x_1^q C_q, \tag{49}$$

where

$$C_i = \sum_j a_{ji} \partial_1^j \in \ker \partial_1((\partial_1^{-1})).$$

By (21) and (18) we have

$$S^{-1} M_{00} S = M_{00} + [v_{-1}, s_1] v_{-1}^{-1} + (< 0), \tag{50}$$

and $[v_{-1}, s_1]$ does not depend on the choice of s_1. Since v_{-1} is a series with constant coefficients and s_1 is a linear function in x_1, we obtain $[v_{-1}, s_1] \in k((\partial_1^{-1}))$.

Therefore, by (50) and (47), the operators $S^{-1}, T_0 a$ must be (L_{00}, M_{00})-admissible, and we can describe the conditions for these operators separately.

2) For the operator S we have the following condition:

$$M_{00} S = S M_{00} + \sum_{q=1}^{\infty} [M_{00}, s_q] M_{00}^{-q} =$$

$$S(M_{00} + l_0 + l_1 M_{00}^{-1} + \ldots) = S M_{00} + l_0 + \sum_{q=1}^{\infty} \sum_{m=0}^{q} s_m l_{q-m} M_{00}^{-q}, \tag{51}$$

where $l_i \in k((\partial_1^{-1}))$. We have by formula (18)

$$[M_{00}, s_q] = [v_{-1}, s_q] v_{-1}^{-1} M_{00} - [v_{-1}, s_q] v_{-1}^{-1} v_0 + v_{-1} \partial_2(s_q) + [v_0, s_q], \tag{52}$$

where from we get

$$\partial_2(s_q)=v_{-1}^{-1}(\sum_{m=0}^{q}s_ml_{q-m}-[v_0,s_q]+[v_{-1},s_q]v_{-1}^{-1}v_0-[v_{-1},s_{q+1}]v_{-1}^{-1}) \tag{53}$$

for all $q\geq 1$, and $l_0=[v_{-1},s_1]v_{-1}^{-1}=[v_{-1},x_1]v_{-1}^{-1}k_1$. Note that we have

$$[v_0,s_q]=\sum_{m=0}^{q-1}x_1^mB_m,\quad [v_{-1},s_q]v_{-1}^{-1}v_0=\sum_{m=0}^{q-1}x_1^mB_m', \tag{54}$$

where $B_m,B_m'\in\ker\partial_1((\partial_1^{-1}))$ depend on $C_{m+1},\dots C_q$ and do not depend on C_j , $j\leq m$. Moreover, we have

$$[v_{-1},s_{q+1}]v_{-1}^{-1}=(\sum_{m=0}^{q}s_mk_{q+1-m})[v_{-1},x_1]v_{-1}^{-1}+\sum_{m=0}^{q-1}x_1^mH_m, \tag{55}$$

where $H_m\in\ker\partial_1((\partial_1^{-1}))$ depend on $C_{m+1},\dots C_q$ and do not depend on C_j , $j\leq m$ (here C_j are the coefficients of the polynomial s_q). So, we obtain

$$s_ql_0-[v_{-1},s_{q+1}]v_{-1}^{-1}=(\sum_{m=0}^{q-1}s_mk_{q+1-m})[v_{-1},x_1]v_{-1}^{-1}+\sum_{m=0}^{q-1}x_1^mH_m,$$

where from, together with (54), the formula (53) can be rewritten as

$$\sum_{m=0}^{q-1}x_1^m(\sum_j\partial_2(a_{jm})\partial_1^j)=\sum_{m=0}^{q-1}x_1^mW_m, \tag{56}$$

where $W_m\in\ker\partial_1((\partial_1^{-1}))$ depend on $C_{m+1},\dots C_q$ and do not depend on C_j , $j\leq m$.

So, the equation (56) is solvable for arbitrary $q\geq 0$, and the set of all (L_{00},M_{00}) -admissible operators from the group $1+E_-$ can be described as a set of all operators S with coefficients s_q of the form (49) defined recursively by formulas (48), (56), (53) for arbitrary parameters $k_i,l_i\in k((L_{00}^{-1}))$, $i\geq 1$.

Note that formula (56) actually defines the *coefficients* of the series C_m , which belong to $\ker\partial_1(A)$. So, conjugating operators S with the operator V , we obtain the same description for admissible operators in general case just setting $\tilde{x}_1:=V^{-1}x_1V\in A((\partial_1))$ instead of x_1 . Actually $[L_{00},\tilde{x}_1]=0$ and $[\partial_2+\partial_2\log V,\tilde{x}_1]=0$.

3) Now for the operator $S_1^{-1}:=T_0a$ we have the following conditions:

$$S_1^{-1}\partial_1S_1=\partial_1+\partial_1(\log S_1)=\partial_1+k_0+\ldots\in k((\partial_1^{-1})), \tag{57}$$

where from $\log S_1=(x_1k_0+l_0)+(x_1k_1+l_1)\partial_1^{-1}+\ldots$, where $l_i\in\ker\partial_1$, and

$$S_1^{-1}M_{00}S_1=(S_1^{-1}v_{-1}S_1v_{-1}^{-1})M_{00}-(S_1^{-1}v_{-1}S_1v_{-1}^{-1})v_0+S_1^{-1}v_0S_1+(S_1^{-1}v_{-1}S_1)\partial_2(\log S_1), \tag{58}$$

where from

$$\partial_2(\log S_1)=$$

$$\sum_{i=0}^{\infty}\partial_2(l_i)\partial_1^{-i}\in k((\partial_1^{-1}))+(S_1^{-1}v_{-1}^{-1}S_1)((S_1^{-1}v_{-1}S_1v_{-1}^{-1})v_0-S_1^{-1}v_0S_1)\in\ker\partial_1((\partial_1^{-1})). \tag{59}$$

Clearly, this equation is solvable only if

$$v_{-1}^{-1}v_0-S_1v_{-1}^{-1}v_0S_1^{-1}\in\ker\partial_1[[\partial_1^{-1}]]\qquad\mod k((\partial_1^{-1})). \tag{60}$$

From two conditions (59), (57) we obtain that the coefficient a of all possible operators T_0a must be of the form $e^{c_1x_1+c_2}c_0$ (if such an exponent exists), where $c_0 \in k^{\times}$, $c_1 \in k$ and $c_2 \in \ker \partial_1$. Then the operator T_0^{-1} must be of the form $T_0^{-1} = \exp((x_1k_1 + l_1)\partial_1^{-1} + \ldots)$, where $k_i \in k$ satisfy the condition

$$v_{-1}^{-1}v_0 - T_0^{-1}v_{-1}^{-1}v_0T_0 \in (\ker \partial_1[[\partial_1^{-1}]])\partial \qquad \mod k((\partial_1^{-1})), \tag{61}$$

and $l_i \in \ker \partial_1$ are defined by condition (59). Obviously, the last condition is always compatible with any choice of parameters k_i. Note also that (61) don't depend on l_i, so our description is well defined.

Replacing x_1 with $\tilde{x}_1$ and ∂_1 with L_{00} in formulas (61), (59), we obtain, as in the case of operators S, the description of operators T_0 in general case.

Case 2. This case is very similar to the case 1. The difference is that the operators S, T_0a can be not (L_{00}, M_{00})-admissible. Repeating the proof of the case 1, we obtain the same conditions for the operator S, except the condition $l_0 \in k((\partial_1^{-1}))$. Now $l_0 \in \ker \partial_1((\partial_1^{-1}))$. This gives us a new condition for the operator T_0a. Namely, instead of condition (60), which holds automatically because T_0a commutes with ∂_1, we obtain

$$\partial_2(\log S_1) = \sum_{i=0}^{\infty} \partial_2(l_i)\partial_1^i \in v_{-1}^{-1}(k((\partial_1^{-1})) - [v_{-1}, s_1]v_{-1}^{-1}). \tag{62}$$

Replacing x_1 with $\tilde{x}_1$ and ∂_1 with L_{00} we obtain the description of operators T_0 in general case.

The lemma is proved.

□

4.2 Solvability of the modified SW-systems

In this subsection A need not be commutative.

To find a solution of the Cauchy problem for the modified Sato-Wilson systems we can follow the way described in [6]. Of course, for the modified systems the proofs will be more complicated.

The following theorem is a generalization of theorem 4.1. in [6].

Theorem 2. *Let $\Omega_{\alpha} = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}_+, i\leq \alpha j} P_{ij}dt_{ij}$ be a D_R-valued 1-form satisfying*

a) there is a positive real number $c > 0$ such that

$$\mathrm{ord}_{\partial_2} P_{ij} \leq \frac{j}{c} \quad \textit{for all } i, j.$$

b) $P_{ij} = P_{ij}|_{t_{m,n}=0, m>\alpha n}$.

c) Ω_{α} is integrable, i.e. $d_{\alpha}\Omega_{\alpha} = \Omega_{\alpha} \wedge \Omega_{\alpha}$.

Then for every given operator $Y(0) \in E_+$, there may exist only one solution $Y \in \widehat{D}_R$ of the linear total differential equation

$$d_{\alpha}Y = \Omega_{\alpha}Y$$

having $Y(0)$ as its initial value; $Y|_{t=0} = Y(0)$ and $Y|_{t_{m,n}=0, m>\alpha n} = Y$.

Remark. This version of theorem is weaker than in [6] or in [18], Th. 1. Nevertheless, for our aim this version is sufficient. The existence of a solution of a special system from this theorem will follow from a generalized Birkhoff decomposition, which we will prove below.

Proof. Let's fix some bijection between the set of all pairs (i, j) with $i \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $i \leq \alpha j$ and natural numbers, $\zeta : \{(i, j)\} \longrightarrow \mathbb{N}$, such that $\zeta(i, j) = k$, where $k \geq j$. Such a bijection can be constructed, for example, by counting integral points (lying below the graphic of the function α) lying inside polygons formed by parallel lines that go through the points $(i, 0)$,

$(i-1,0)$, $i \leq \alpha(0)$, and are parallel to the line going through the points $(\alpha(0)-1,0)$ and $(\alpha(1),1)$, the vertical lines $(.,k)$ and lines going through the points $(\alpha(k),k)$, $(\alpha(k+1),k+1)$.

Now, because of the condition b), the differential equation from the theorem can be rewritten as

$$dY = (\sum_{n\geq 1}^{\infty} P_{\zeta^{-1}(n)} dt_{\zeta^{-1}(n)})Y, \quad d = \sum_{n=1}^{\infty} dt_{\zeta^{-1}(n)} \frac{\partial}{\partial t_{\zeta^{-1}(n)}} \tag{63}$$

where $P_{\zeta^{-1}(n)}$ are considered as the elements with variables $t_{\zeta^{-1}(j)}$ instead of $t_{m,n}$.

Let's define the ring $\mathcal{R} = A((\partial_1^{-1}))[[t_{\zeta^{-1}(1)}, t_{\zeta^{-1}(2)}, \ldots]]$ as a projective limit ring with a valuation $v_2' : \mathcal{R}\backslash\{0\} \to \mathbb{N}$ defined by $v_2'(t_{\zeta^{-1}(n)} = n$. Let $\mathcal{D}$, $\mathcal{E}$, $\widehat{\mathcal{D}}$, $\widehat{\mathcal{E}}$ be rings defined with respect to $\mathcal{R}$ on pages 9,10 in [6].

Because of the valuation growth conditions on coefficients of elements from the ring $\widehat{E}_{\bar{R}}$ and because of special choice of the bijection, the elements $P_{\zeta^{-1}(n)}$ can be represented as series from the ring $\mathcal{D}$. Analogously, every element $Y \in \widehat{D}_R$ with the properties as in the theorem can be represented as an element from $\widehat{\mathcal{D}}$. Obviously, such a representation is uniquely defined, that is two different elements from $\widehat{D}_R$ give two different elements from the ring $\widehat{\mathcal{D}}$. Of course, the representation and the equation (63) depend on the choice of bijection.

By the property of the bijection

$$\mathrm{ord}_{\partial_2} P_{\zeta^{-1}(n)} = \mathrm{ord}_{\partial_2} P_{i,j} \leq \frac{j}{c} \leq \frac{n}{c}$$

for all n . So, the equation (63) is the differential equation from theorem 4.1 in [6], and therefore have a unique solution for any initial value $Y(0) \in E_+$. As we have seen, any solution of the equation from the theorem gives a uniquely defined

Now, if there were two different solutions of the differential equation from the theorem, the equation (63) would have two different solutions with the same initial value, a contradiction. □

Remark. Since an element from $\widehat{\mathcal{D}}$ can not be in general represented as an element of $\widehat{D}_R$, the existence of a solution of the equation (63) in $\widehat{\mathcal{D}}$ do not imply the existence of a solution of the differential equation from the theorem.

Now we can formulate a generalized version of the Birkhoff decomposition theorem.

Theorem 3. *For any element $U \in \widehat{E}_R^{\times}$ there exists a unique factorization*

$$U = S^{-1}Y,$$

where $S \in \widehat{\mathcal{V}}_R$ and $Y \in \widehat{D}_R^{\times}$.

Proof. The proof of this theorem is analogous to the proof of Theorem 3.2. in [6] with certain modifications.

The uniqueness is trivial. Indeed, if $S_1^{-1}Y_1 = S_2^{-1}Y_2$, then $S_1S_2^{-1} = Y_1Y_2^{-1} \in \widehat{\mathcal{V}}_R \cap \widehat{D}_R^{\times} = \{1\}$. Hence $S_1 = S_2$ and $Y_1 = Y_2$. To show the existence of the decomposition, we have to solve the equation $SU \in \widehat{D}_R$ for an unknown $S \in \widehat{\mathcal{V}}_R$.

So let $U = \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta$ and $S = 1 + \sum_{\gamma=1}^{\infty} s_\gamma \partial_2^{-\gamma}$, where $u_\beta, s_\gamma \in \widehat{E}_{\bar{R}}$. Then

$$\begin{aligned} SU &= (1+\sum_{\gamma=1}^{\infty} s_\gamma \partial_2^{-\gamma})(\sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta) \\ &= \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta + \sum_{\gamma=1}^{\infty}\sum_{\beta\in\mathbb{Z}}\sum_{i=0}^{\infty} C_{-\gamma}^i s_\gamma u_\beta^{(i)} \partial_2^{-\gamma+\beta-i} \\ &= \sum_{\beta\in\mathbb{Z}} u_\beta \partial_2^\beta + \sum_{\delta\in\mathbb{Z}} (\sum_{\gamma=1}^{\infty}\sum_{i=0}^{\infty} C_{-\gamma}^i s_\gamma u_{\delta+\gamma+i}^{(i)}) \partial_2^\delta, \end{aligned}$$

where $u^{(i)} = \partial_2^i(u)$.

Therefore the equation we have to solve is a system of algebraic equations

$$u_{-\beta} + \sum_{\gamma=1}^{\infty}\sum_{i=0}^{\infty} C^i_{-\gamma} s_\gamma u^{(i)}_{\gamma-\beta+i} = 0 \quad \text{for } \beta = 1, 2, 3, \dots \tag{64}$$

Define

$$\mathbf{u} = (u_{-1}, u_{-2}, u_{-3}, \dots), \qquad \mathbf{s} = (s_1, s_2, s_3, \dots)$$

and

$$M = \left[\sum_{i=0}^{\infty} C^i_{-\gamma} u^{(i)}_{\gamma-\beta+i}\right]_{\gamma,\beta=1,2,3,\dots}, \tag{65}$$

where M is a square matrix of infinite size with coefficients in $\widehat{E}_{\bar{R}}$. Equation (75) now reads

$$\mathbf{s}\, M = -\,\mathbf{u} \tag{66}$$

Therefore the solution $\mathbf{s}$ is given by $\mathbf{s} = -\,\mathbf{u}\, M^{-1}$. The idea is to define M^{-1} by the Neumann series $\sum_{n=0}^{\infty}(1-M)^n$, and use a similar technique developed in the proof of lemma 4 to establish well-definedness of $\sum_{n=0}^{\infty}(1-M)^n$ and $\mathbf{u}\, M^{-1}$. Since $\mathbf{s}$ determines all the coefficients of S, well-definedness of $\mathbf{u}\sum_{n=0}^{\infty}(1-M)^n$ implies the existence of S such that $SU \in \widehat{D}_R$.

Let $Q = 1 - M = [a_{\mu\nu}]_{\mu,\nu=1,2,3,\dots}$ and $Q^n = [a_{n,\mu\nu}]_{\mu,\nu=1,2,3,\dots}$. Since $a_{\mu\nu} = \delta_{\mu\nu} - \sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_{\mu-\nu+i}$, we have

$$(a_{\mu\mu})|_{t=0} = 1 - (\sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_i)|_{t=0} = 0,$$

because of definition of $\widehat{E}^{\times}_R$. Similarly, if $\mu > \nu$, then

$$\pi(a_{\mu\nu}) = \pi(-\sum_{i=0}^{\infty} C^i_{-\mu} u^{(i)}_{\mu-\nu+i}) = 0.$$

Because of the growth order condition for u_ν's, we can find a positive real number J such that

$$v_2(a_{\mu\nu}) \geq J(\mu-\nu) \quad \text{for all } \mu - \nu \geq 0\,. \tag{67}$$

as before.

Claim. i) For every $n \geq 1$ we have $v_2(a_{n,\mu\nu}) \geq J(\mu-\nu)$ if $\mu - \nu \geq 0$.

ii) The function $F(M, N, a_{n,\mu\nu})$ satisfy the following property: for any given M, N, μ, ν there exists a natural number $T(M, N, \mu, \nu)$ such that $F(M, N, a_{n,\mu\nu}) = 0$ for all $n > T(M, N, \mu, \nu)$.

Here we take the function F as in the proof of lemma 4.

Proof of the claim. 1) Let's prove the first assertion. If $n = 1$, then $a_{1,\mu\nu} = a_{\mu\nu}$ and i) follows from (67). Assume that i) holds for some $n \geq 1$. Since $Q^{n+1} = Q^n \cdot Q$,

$$a_{n+1,\mu\nu} = \sum_{l=1}^{\infty} a_{n,\mu l} a_{l\nu} = \sum_{l=1}^{\nu-1} a_{n,\mu l} a_{l\nu} + \sum_{l=\nu}^{\mu} a_{n,\mu l} a_{l\nu} + \sum_{l=\mu+1}^{\infty} a_{n,\mu l} a_{l\nu}. \tag{68}$$

We assume that $\mu - \nu \geq 0$. In the first term of (68), since $\nu > l$, we have $\mu - l > \mu - \nu \geq 0$, and hence

$$v_2(a_{n,\mu l} a_{l\nu}) \geq v_2(a_{n,\mu l}) \geq J(\mu - l) > J(\mu-\nu).$$

Therefore

$$v_2(\sum_{l=1}^{\nu-1} a_{n,\mu l} a_{l\nu}) \geq J(\mu-\nu).$$

In the second term of (68), since $\nu \le l \le \mu$, we have

$$v_2(a_{n,\mu l} a_{l\nu}) = v_2(a_{n,\mu l}) + v_2(a_{l\nu}) \ge J(\mu - l) + J(l - \nu) = J(\mu - \nu).$$

Therefore,

$$v_2(\sum_{l=\nu}^{\mu} a_{n,\mu l} a_{l\nu}) \ge J(\mu - \nu).$$

Finally, in the third item of (68), $l > \mu$ implies $l - \nu > \mu - \nu$. Therefore,

$$v_2(\sum_{l=\mu+1}^{\infty} a_{n,\mu l} a_{l\nu}) \ge v_2(a_{l\nu}) \ge J(\mu - \nu).$$

2) Now let's prove the second assertion. To prove it we use an induction on N and $|\mu - \nu|$, like in the proof of lemma 4. We will use here the function $q(N, a)$ and various properties used there.

It suffice to prove the assertion only for $M < 0$. Let's prove the first step of our double induction. Let $N = 0$ and $|\mu - \nu| = 0$. Then we have $T(M, N, \mu, \nu) = |M|$ and $q(N, a_{n,\mu\nu}) \le -n$. Indeed, by (68)

$$F(M, 0, a_{n+1,\mu\mu}) = F(M, 0, a_{n,\mu\mu} a_{\mu\mu}) = \ldots = F(M, 0, a_{\mu\mu}^{n+1}),$$

because $v_2(a_{j,\mu\nu}) \ge J(\mu - \nu) > 0$ for all $j, (\mu - \nu) \ge 1$ by the first assertion of the claim, so all the summands in (68) except $a_{n,\mu\mu} a_{\mu\mu}$ have valuation greater than zero and therefore they don't change the value of the function $F(M, 0, .)$. Obviously, the same is true for the function $q(0, .)$. Since $q(0, a_{\mu\mu}) \le -1$, we have by (29) $a_{\mu\mu}^n \le -n$, where from $F(M, 0, a_{\mu\mu}^{n+1}) = 0$ for all $n > |M|$ and $q(0, a_{n+1,\mu\mu}) \le -n - 1$.

3) Now let $N = 0$ and $|\mu - \nu| > 0$. Since $v_2(a_{j,\mu\nu}) > 0$ for all $j, (\mu - \nu) \ge 1$, we have $F(M, 0, a_{n,\mu\nu}) = 0$, $q(0, a_{n,\mu\nu}) = -\infty$ for all n and $(\mu - \nu) \ge 1$. So, we can assume $(\mu - \nu) < 0$.

By the induction hypothesis $F(M, 0, a_{n,\mu'\nu'}) = 0$ for all $n > T(M, 0, \mu', \nu')$ if $|\mu' - \nu'| < |k|$. Denote by N_0 the maximum of all $T(M, 0, \mu l), T(M, 0, l\nu)$ with $\mu < l < \nu$ (with some fixed μ, ν, $|\mu - \nu| = |k|$). Now consider the elements $a_{n+N_0,\mu\nu}$ with $n > N_0$. Since $Q^{n+N_0} = Q^n Q^{N_0}$, we have by the same reason as above

$$F(M, 0, a_{n+N_0,\mu\nu}) = F(M, 0, \sum_{l=\mu}^{\nu} a_{n,\mu l} a_{N_0,l\nu}) \le \max_{\mu \le l \le \nu} \{F(M, 0, a_{n,\mu l} a_{N_0,l\nu})\}, \tag{69}$$

where the last inequality follows from the property ii) of the function F, see step a) of theorem 1.

4) So, it suffice to prove the assertion for each function $F(M, 0, a_{n,\mu l} a_{N_0,l\nu}))$, where $\mu \le l \le \nu$, and then define the number $T(M, 0, \mu, \nu)$ as a maximum of corresponding numbers for each l.

For $\mu < l < \nu$ we have $F(M, 0, a_{n,\mu l}) = 0$ and $F(M, 0, a_{N_0,l\nu}) = 0$. Therefore, by (33) we obtain $F(M, 0, a_{n,\mu l} a_{N_0,l\nu}) = 0$ for all $n > N_0$.

Now consider the case $l = \mu$. By the property (29) we have

$$q(0, a_{n,\mu\mu} a_{N_0,\mu\nu}) \le q(0, a_{n,\mu\mu}) + q(0, a_{N_0,\mu\nu}) \le -n + q(0, a_{N_0,\mu\nu}).$$

Therefore, for all $n > -M + q(0, a_{N_0,\mu\nu})$ we have $F(M, 0, a_{n,\mu\mu} a_{N_0,\mu\nu}) = 0$.

5) Combining all together, we get for all $n > \max\{N_0, -M + q(0, a_{N_0,\mu\nu})\} := \tilde{T}$ (see (69)):

$$F(M, 0, a_{n+N_0,\mu\nu}) \le F(M, 0, a_{n,\mu\nu} a_{N_0,\nu\nu}). \tag{70}$$

Now put

$$O = (\max_{\tilde{T} \leq j \leq \tilde{T}+N_0} \{F(M,0,a_{j,\mu\nu})\})$$

and put $T(M,0,\mu,\nu) = \tilde{T} + ON_0$. We claim, that for all $n > T(M,0,\mu,\nu)$ $F(M,0,a_{n,\mu\nu}) = 0$.

Assume the converse. Let $n > T(M,0,\mu,\nu)$ be the number such that $F(M,0,a_{n,\mu\nu}) > 0$. Then by (70) and (29) we have

$$0 < F(M,0,a_{n,\mu\nu}) = q(0,a_{n,\mu\nu}) - M \leq F(M,0,a_{n-N_0,\mu\nu}a_{N_0,\nu\nu}) = q(0,a_{n-N_0,\mu\nu}a_{N_0,\nu\nu}) - M$$

$$\leq q(0,a_{n-N_0,\mu\nu}) + q(0,a_{N_0,\nu\nu}) - M \leq q(0,a_{n-N_0,\mu\nu}) - M - N_0 = F(M,0,a_{n-N_0,\mu\nu}) - N_0 \leq \dots$$

$$\leq F(M,0,a_{n-\tilde{O}N_0,\mu\nu}) - \tilde{O}N_0,$$

where $\tilde{T} \leq n - \tilde{O}N_0 \leq \tilde{T} + N_0$ and therefore $\tilde{O} > O$. So,

$$F(M,0,a_{n-\tilde{O}N_0,\mu\nu}) - \tilde{O}N_0 \leq O - \tilde{O}N_0 \leq 0,$$

a contradiction.

6) Now assume N is an arbitrary positive number. Let $k_0 := [N/J]$ be the integral part of the number N/J . For all $\mu - \nu > k_0$ we have $v_2(a_{n,\mu\nu}) \geq J(\mu - \nu) > N$, so the assertion is trivial for all such $(\mu - \nu)$.

To prove the assertion we will use an inverse induction on $(\mu - \nu) \leq k_0$. By the induction hypothesis on N there exists a natural number N_0 such that $F(M,N-1,a_{n,\mu l}) = 0, F(M,N-1,a_{n,l\nu}) = 0$ for all $n \geq N_0$ and $\nu \leq l \leq \mu$ (with fixed M,μ,ν). We then have for all $n > N_0$ and $\mu - \nu = k_0$

$$F(M,N,a_{n+N_0,\mu\nu}) = F(M,N,\sum_{l=\nu}^{\mu} a_{n,\mu l}a_{N_0,l\nu}) \leq$$

$$\max_{\nu \leq l \leq \mu} \{F(M,N,a_{n,\mu l}a_{N_0,l\nu})\} \leq \max\{F(M,N,a_{n,\mu\mu}a_{N_0,\mu\nu}), F(M,N,a_{n,\mu\nu}a_{N_0,\nu\nu})\},$$

where the last inequality follows from (34). For all $n > -M + q(N,a_{N_0,\mu\nu})$ we have, as above, $F(M,N,a_{n,\mu\mu}a_{N_0,\mu\nu}) = 0$. So, for all sufficiently large n

$$F(M,N,a_{n+N_0,\mu\nu}) \leq F(M,N,a_{n,\mu\nu}a_{N_0,\nu\nu}),$$

and we can repeat the arguments of step 7) to get the proof of the assertion in the case $\mu - \nu = k_0$.

7) Let's prove the assertion for arbitrary $\mu - \nu < k_0$. By the induction hypothesis on $N, k = (\mu - \nu)$ there exists a natural number N_0 such that $F(M,N-1,a_{n,\mu j}) = 0, F(M,N-1,a_{n,j\nu}) = 0$ for all $n \geq N_0$ and $\nu + k_0 \geq j \geq \mu - k_0$, $F(M,N,a_{n,\mu j}) = 0$ for all $n \geq N_0$ and $\nu > j \geq \mu - k_0$, and $F(M,N,a_{n,j\nu}) = 0$ for all $n \geq N_0$ and $\nu + k_0 \geq j > \mu$. We then have for all $n > N_0$

$$F(M,N,a_{n+N_0,\mu\nu}) = F(M,N,\sum_{l=\mu-k_0}^{\nu+k_0} a_{n,\mu l}a_{N_0,l\nu}) \leq$$

$$\max_{\mu-k_0 \leq l \leq \nu+k_0} \{F(M,N,a_{n,\mu l}a_{N_0,l\nu})\} \leq \max\{F(M,N,a_{n,\mu\mu}a_{N_0,\mu\nu})), F(M,N,a_{n,\mu\nu}a_{N_0,\nu\nu})\},$$

where the last inequality follows from (33), (34), (35), (36). Indeed, if $l \neq \mu,\nu$ we have either $\mu - l < k$ and therefore $l - \nu > 0$ or $l - \nu < k$ and therefore $\mu - l > 0$ or $\mu - l > k$, $l - \nu > k$. In the first case we apply (34) or (36), in the second case we apply (34) or (35), and in the third case we apply (33).

Repeating the arguments of the end of step 8), we obtain the assertion also in this case, hence in general.
The claim is proved.

By this claim we can conclude that $\sum_{n=0}^{\infty} a_{n,\mu\nu} \in \widehat{E}_{\bar{R}}$ is well defined for all $\mu, \nu = 1, 2, 3, \ldots$. Therefore,

$$M^{-1} = \sum_{n=0}^{\infty} (1-M)^n = \left[\sum_{n=0}^{\infty} a_{n,\mu\nu} \right]_{\mu,\nu}$$

is well defined. Let $M^{-1} = [b_{\mu\nu}]_{\mu,\nu=1,2,\ldots}$, namely $b_{\mu\nu} = \sum_{n=0}^{\infty} a_{n,\mu\nu}$. If $\mu > \nu$, then $v_2(b_{\mu\nu}) = v_2(\sum_{n=0}^{\infty} a_{n,\mu\nu}) \geq J(\mu - \nu)$. Therefore

$$s_\nu = -\sum_{\mu=1}^{\infty} u_{-\mu} b_{\mu\nu} = -\sum_{\mu=1}^{\nu} u_{-\mu} b_{\mu\nu} - \sum_{\mu=\nu+1}^{\infty} u_{-\mu} b_{\mu\nu}$$

is a well-defined element in $\widehat{E}_{\bar{R}}$. Thus we have established the existence of $S \in \widehat{\mathcal{V}}_R$ such that $SU \in \widehat{D}_R$.

Finally, let $Y = SU$. Then $Y \in \widehat{D}_R^{\times}$ and $U = S^{-1}Y$. This completes the proof of the theorem.

□

Let us illustrate how we use the Birkhoff factorization to solve the systems $(SW)_\alpha$. To solve the system $(SW)_\alpha$ with the initial value $S(0) \in 1 + E_-$ we take the explicit solution of the system

$$d_\alpha U = \omega_\alpha^{N_0} U, \tag{71}$$

given by

$$U = \exp\Big(\sum_{i \in \mathbb{Z}, j \in \mathbb{Z}_+, i \leq \alpha j} t_{i,j} L_{00}^i M_{00}^j \Big) S(0)^{-1} \in \widehat{E}_R^{\times} \tag{72}$$

and find, according to theorem 3, its unique decomposition $U = S^{-1}Y$. Define

$$Z_{\pm}^{\alpha} = \pm \sum_{i \leq \alpha j} dt_{i,j} (S L_{00}^i M_{00}^j S^{-1})_{\pm}.$$

We have

$$Z_+^\alpha - Z_-^\alpha = S\omega_\alpha S^{-1} = SdUU^{-1}S^{-1} = dYY^{-1} - dSS^{-1}$$

Since $\widehat{D}_R \cap \widehat{E}_{R-} = \{0\}$, we obtain $Z_+^\alpha = dYY^{-1}$ and $Z_-^\alpha = dSS^{-1}$. So, S gives a solution of $(SW)_\alpha$. Note that U (so, S) does not depend on $t_{i,j}$, $i > \alpha j$. Since $Y|_{t=0} = 1$,

$$S(0)^{-1} = U|_{t=0} = (S^{-1}|_{t=0})(Y|_{t=0}) = (S|_{t=0})^{-1},$$

namely, $S|_{t=0} = S(0)$.

Now let's prove the uniqueness of a solution S. Let $S' \in \widehat{\mathcal{V}}_R$ be another solution of $(SW)_\alpha$ with the initial value $S(0)$ such that S' does not depend on $t_{i,j}$, $i > \alpha j$. Then we also have $Z_{\pm}'^{\alpha} = \pm \sum_{i \leq \alpha j} dt_{i,j} (S' L_{00}^i M_{00}^j S'^{-1})_{\pm}$ satisfying, by propositions 4, 3 , $d_\alpha Z_{\pm}'^{\alpha} = Z_{\pm}'^{\alpha} \wedge Z_{\pm}'^{\alpha}$, and $Z_{\pm}'^{\alpha}$ do not depend on $t_{i,j}$, $i > \alpha j$. As we have seen in the proof of theorem 2, the system

$$d_\alpha Y' = Z_+'^{\alpha} Y'$$

can be represented, with help of some fixed bijection ζ, as a system (63), which has a unique solution in the ring $\widehat{\mathcal{D}}^{\times}$. The original solution S and the solution S' of the system $(SW)_\alpha$ can be also uniquely represented as elements of $\widehat{\mathcal{E}}^{\times}$ with help of ζ, as one can easily check. So, if we define $U' = S'^{-1} Y' \in \widehat{\mathcal{E}}^{\times}$, U' must satisfy the system of linear partial differential equations:

$$dU' = \omega_\alpha^{N_0} U', \quad d = \sum_{n=1}^{\infty} dt_{\zeta^{-1}(n)} \frac{\partial}{\partial t_{\zeta^{-1}(n)}}. \tag{73}$$

Indeed,

$$dU' = -S'^{-1}dS'S'^{-1}Y' + S'^{-1}dY' = S'^{-1}(dY'Y'^{-1} - dS'S'^{-1})S'S'^{-1}Y'$$
$$= S'^{-1}(Z'^{\alpha}_{+} - Z'^{\alpha}_{-})S'U' = S'^{-1}\sum_{i\le \alpha j} dt_{i,j}S'L_{00}^{i}M_{00}^{j}S'^{-1}S'U' = \omega_{\alpha}^{N_0}U'.$$

Lemma 8. *If the equation (73) has two solutions U and $V \in \widehat{\mathcal{E}}$ with the same initial value $U|_{t=0} = V|_{t=0}$, then $U = V$.*

Proof. Let $W = U - V$. Note that $W|_{t=0} = U|_{t=0} - V|_{t=0} = 0$. Since (73) is linear, we have $dW = \omega_{\alpha}^{N_0}W$, namely, $\partial W/\partial t_{\zeta^{-1}(n)} = L_{00}^{i}M_{00}^{j}W$. Therefore,

$$\frac{\partial}{\partial t_{\zeta^{-1}(n_1)}}\cdots\frac{\partial}{\partial t_{\zeta^{-1}(n_k)}}W = L_{00}^{i_1+\ldots+i_k}M_{00}^{j_1+\ldots+j_k}W$$

for any $n_1,\ldots n_k$, where from, since $[M_{00},W]|_{t=0} = [M_{00},W|_{t=0}]$, we have

$$(\frac{\partial}{\partial t_{\zeta^{-1}(n_1)}}\cdots\frac{\partial}{\partial t_{\zeta^{-1}(n_k)}}W)(0) = L_{00}^{i_1+\ldots+i_k}M_{00}^{j_1+\ldots+j_k}\cdot(W|_{t=0}) = 0.$$

This means that W does not depend on t. Since $W|_{t=0} = 0$, we can conclude that $W = 0$.
□

So, we obtain $S'^{-1}Y' = S^{-1}Y$ in the ring $\widehat{\mathcal{E}}^{\times}$. Since the Birkhoff decomposition is unique, we conclude $S' = S$ and $Y' = Y$. Therefore, $S' = S$ and $Y' = Y$ also in the ring $\widehat{E}_R^{\times}$. This completes the proof of the following theorem, which is an analog of the theorem 1.4 in [6]:

Theorem 4. *For every initial value $S(0) \in 1 + E_{-}$, there is a unique solution $S = S(t) \in \widehat{\mathcal{V}}_R$ of the system $(SW)_{\alpha}$ such that $S|_{t=0} = S(0)$.*

□

5 Existence of non-trivial solutions for the modified systems

In this section we will look for an answer on the following question: when the solution $S(t)$ of the system $(SW)_{\alpha}$ corresponds to a non-trivial solution of the system $(KP)_{\alpha}$ and belongs to the group $\mathcal{V}_R$. This question in general seems to be very difficult, and we don't know the answer. So, we will consider here only the case of a linear function $\alpha : j \mapsto \alpha\cdot j$, $\alpha \in \mathbb{R}$.

As we have seen in the section 2.1, in the case $\alpha = \infty$ the original system $(KP)_{\alpha}$ has only trivial solutions. The theorem below shows that $S(t) \in \mathcal{V}_R$ for some initial values $S(0)$. The theorem does not cover all possible initial values, but nevertheless, the examples described below will be important for applications. Other examples see in [18], section 4.

Theorem 5. *Suppose that $\alpha \le 0$ and $\mathrm{ord}_{\partial_1}(v_0) \le 0$ (recall that v_0 is the summand of M_{00}). Let $S(0) = 1 + \sum_{\gamma=1}^{\infty} w_{\gamma}M_{00}^{-\gamma}$, where $\mathrm{ord}_{\partial_1}(w_{\gamma}) \le -\alpha\gamma$.*

Then the system $(SW)_{\alpha}$ with the initial condition $S(0)$ has a solution $S(t) \in \mathcal{V}_R$.

The solution $S(t) \in \mathcal{V}_R$ corresponds to a non-trivial solution of the system $(KP)_{\alpha}$ for sufficiently general $S(0)$ (see lemma 7).

Proof. (i) The idea of the proof is to look at the proof of theorem 3 more carefully. Since the proof is very explicit, it is possible to check if the operator $S(t)$ constructed there belong to $\mathcal{V}_R$. The only difference with the proof of theorem 3 is that we have to work with operators U, S written as series in L_{00}^{-1}, M_{00}^{-1} from the beginning of the proof.

So let $\exp(\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+,i\le\alpha j} t_{ij}L_{00}^iM_{00}^j)S(0)^{-1} = U = \sum_{\beta\in\mathbb{Z}} u_\beta M_{00}^\beta$ and $S(=S(t)) = 1 + \sum_{\gamma=1}^\infty s_\gamma M_{00}^{-\gamma}$, where $u_\beta, s_\gamma \in \widehat{E}_{\bar{R}}$. Now, to proceed the proof of theorem 3, we need a formula for the commutator $[M_{00}^k, a]$, $a\in\widehat{E}_{\bar{R}}$ for arbitrary k, a. Such a formula is in general too complicated, but we will need only some general properties of it, and we can use the notation and some basic ideas from [19]. Let's recall here some definition and lemmas from [19].

Definition 2. Let us define linear maps ${}_m\delta_i : \widehat{E}_{\bar{R}} \to \widehat{E}_{\bar{R}}$, $m\in\mathbb{Z}$, $i\in\mathbb{N}$ as follows.

$$M_{00}^m a M_{00}^{-m} = {}_m\delta_0(a) + {}_m\delta_1(a)M_{00}^{-1} + {}_m\delta_2(a)M_{00}^{-2} + \ldots, \quad a\in\widehat{E}_{\bar{R}}.$$

If $m=0$, put ${}_m\delta_i = 0$.

Immediately from the definition follows

Lemma 9. *In the situation of definition 2 we have*
(i) for $|m|>1$

$$\begin{aligned}{}_m\delta_i(a) = {}_{sign(m)}\delta_0({}_{sign(m)(|m|-1)}\delta_i(a)) + {}_{sign(m)}\delta_i({}_{sign(m)(|m|-1)}\delta_0(a)) + \\ \sum_{j=1}^{i-1} {}_{sign(m)}\delta_j({}_{sign(m)(|m|-1)}\delta_{i-j}(a)),\end{aligned}$$

where $sign(m) = m/|m|$*;*
(ii) for any $m\neq 0$

$${}_{-m}\delta_0({}_m\delta_i) + {}_{-m}\delta_i({}_m\delta_0) + \sum_{j=1}^{i-1} {}_{-m}\delta_j({}_m\delta_{i-j}) = 0$$

Suppose that the element a from the definition belong to $E_{\bar{R}}$. Then, by formula (52) from the proof of lemma 7 (with s_q replaced by a), we can conclude that ${}_m\delta_i(a)\in E_{\bar{R}}$ for all i, m, $\mathrm{ord}_{\partial_1}({}_m\delta_0(a)) = \mathrm{ord}_{\partial_1}(a)$ for all m, and $\mathrm{ord}_{\partial_1}({}_1\delta_1(a)) \le \mathrm{ord}_{\partial_1}(a) + \mathrm{ord}_{\partial_1}(v_0)$. Therefore, by lemma 9, we obtain $\mathrm{ord}_{\partial_1}({}_m\delta_1(a)) \le \mathrm{ord}_{\partial_1}(a) + \mathrm{ord}_{\partial_1}(v_0)$ for all m, and, more generally,

$$\mathrm{ord}_{\partial_1}({}_m\delta_i(a)) \le \mathrm{ord}_{\partial_1}(a) + i\,\mathrm{ord}_{\partial_1}(v_0) \tag{74}$$

for all $i\ge 1$.

Now we have

$$\begin{aligned}SU &= (1+\sum_{\gamma=1}^\infty s_\gamma M_{00}^{-\gamma})(\sum_{\beta\in\mathbb{Z}} u_\beta M_{00}^\beta) \\ &= \sum_{\beta\in\mathbb{Z}} u_\beta M_{00}^\beta + \sum_{\gamma=1}^\infty \sum_{\beta\in\mathbb{Z}} \sum_{i=0}^\infty s_\gamma\, {}_{-\gamma}\delta_i(u_\beta) M_{00}^{-\gamma+\beta-i} \\ &= \sum_{\beta\in\mathbb{Z}} u_\beta M_{00}^\beta + \sum_{\delta\in\mathbb{Z}}(\sum_{\gamma=1}^\infty\sum_{i=0}^\infty s_\gamma\, {}_{-\gamma}\delta_i(u_{\delta+\gamma+i}))M_{00}^\delta.\end{aligned}$$

Now the equation we have to solve is a system of algebraic equations

$$u_{-\beta} + \sum_{\gamma=1}^\infty\sum_{i=0}^\infty s_\gamma\, {}_{-\gamma}\delta_i(u_{\gamma-\beta+i}) = 0 \quad \text{for } \beta = 1,2,3,\ldots \tag{75}$$

Define

$$\mathbf{u} = (u_{-1}, u_{-2}, u_{-3}, \ldots), \quad \mathbf{s} = (s_1, s_2, s_3, \ldots)$$

and

$$M=\left[\sum_{i=0}^{\infty}{}_{-\gamma}\delta_i(u_{\gamma-\beta+i})\right]_{\gamma,\beta=1,2,3,\dots}, \tag{76}$$

where M is a square matrix of infinite size with coefficients in $\widehat{E}_{\bar{R}}$. The solution $\mathbf{s}$ is given by $\mathbf{s}=-\mathbf{u}M^{-1}$.

Let $Q=1-M=[a_{\mu\nu}]_{\mu,\nu=1,2,3,\dots}$ and $Q^n=[a_{n,\mu\nu}]_{\mu,\nu=1,2,3,\dots}$. Since $a_{\mu\nu}=\delta_{\mu\nu}-\sum_{i=0}^{\infty}{}_{-\mu}\delta_i(u_{\mu-\nu+i})$, we have

$$(a_{\mu\mu})|_{t=0}=1-(\sum_{i=0}^{\infty}{}_{-\mu}\delta_i(u_{\mu-\nu+i}))|_{t=0}=0,$$

because of definition of $\widehat{E}_R^{\times}$, the property $\mathrm{ord}_{\partial_1}({}_m\delta_0(a))=\mathrm{ord}_{\partial_1}(a)$ and the obvious fact that the maps ${}_m\delta_i$ are $k[\dots,t_{ij},\dots]$-linear. Similarly, if $\mu>\nu$, then

$$\pi(a_{\mu\nu})=\pi(-\sum_{i=0}^{\infty}{}_{-\mu}\delta_i(u_{\mu-\nu+i}))=0.$$

Because of the growth order condition for u_ν, we can find a positive real number J such that

$$v_2(a_{\mu\nu})\geq J(\mu-\nu)\quad\text{for all }\ \mu-\nu\geq 0\,. \tag{77}$$

as before.

Claim. i) For every $n\geq 1$ we have $v_2(a_{n,\mu\nu})\geq J(\mu-\nu)$ if $\mu-\nu\geq 0$.

ii) The function $F(M,N,a_{n,\mu\nu})$ satisfy the following property: for any given M,N,μ,ν there exists a natural number $T(M,N,\mu,\nu)$ such that $F(M,N,a_{n,\mu\nu})=0$ for all $n>T(M,N,\mu,\nu)$.

iii) $\mathrm{ord}_{\partial_1}(a_{n,\mu\nu})\leq\alpha(\mu-\nu)$.

Proof of the claim. To prove i), ii) we can repeat the proof of the claim in theorem 3. Let us prove iii).

Let $U_0=\exp(\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_+,i\leq\alpha j}t_{ij}L_{00}^iM_{00}^j)=1+\sum_{\beta=1}^{\infty}u_\beta^0M_{00}^\beta$ and $S(0)^{-1}=1+\sum_{k=1}^{\infty}s_k^0M_{00}^{-k}$. Note that, because of (74) and the condition $\mathrm{ord}_{\partial_1}(v_0)\leq 0$, the coefficients s_k^0 satisfy the same condition as the coefficients w_k of the operator $S(0)$, i.e. $\mathrm{ord}_{\partial_1}(s_k^0)\leq-\alpha k$. Now

$$\begin{aligned}U=U_0S(0)^{-1}&=(1+\sum_{\beta=1}^{\infty}u_\beta^0M_{00}^\beta)(1+\sum_{k=1}^{\infty}s_k^0M_{00}^{-k})\\&=1+\sum_{\beta=1}^{\infty}u_\beta^0M_{00}^\beta+\sum_{k=1}^{\infty}s_k^0M_{00}^{-k}+\sum_{\beta=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}u_\beta^0{}_\beta\delta_i(s_k^0)M_{00}^{\beta-k-i}\\&=1+\sum_{\beta=1}^{\infty}u_\beta^0M_{00}^\beta+\sum_{k=1}^{\infty}s_k^0M_{00}^{-k}+\sum_{\gamma\in\mathbb{Z}}(\sum_{\beta=1}^{\infty}\sum_{i=0}^{\infty}u_\beta^0{}_\beta\delta_i(s_{\beta-i-\gamma}^0))M_{00}^\gamma,\end{aligned}$$

where ${}_\beta\delta_i(s_{\beta-i-\gamma}^0)=0$ if $\beta-i-\gamma\leq 0$. Since $\mathrm{ord}_{\partial_1}(v_0)\leq 0$, we have by (74)

$$\mathrm{ord}_{\partial_1}(u_\beta^0{}_\beta\delta_i(s_{\beta-i-\gamma}^0))=\mathrm{ord}_{\partial_1}(u_\beta^0)+\mathrm{ord}_{\partial_1}({}_\beta\delta_i(s_{\beta-i-\gamma}^0))\leq$$

$$\mathrm{ord}_{\partial_1}(u_\beta^0)+\mathrm{ord}_{\partial_1}(s_{\beta-i-\gamma}^0)\leq\alpha\beta+\alpha(-\beta+i+\gamma)\leq\alpha\gamma,$$

where from we get $\mathrm{ord}_{\partial_1}(u_\gamma)\leq\alpha\gamma$. Applying again (74), we get

$$\mathrm{ord}_{\partial_1}(a_{1,\mu\nu})\leq\mathrm{ord}_{\partial_1}(u_{\mu-\nu})\leq\alpha(\mu-\nu).$$

Assume now that iii) holds for some $n \geq 1$. Since $a_{n+1,\mu\nu} = \sum_{l=1}^{\infty} a_{n,\mu l} a_{l\nu}$, we get

$$\mathrm{ord}_{\partial_1}(a_{n+1,\mu\nu}) \leq \mathrm{ord}_{\partial_1}(a_{n,\mu l} a_{l\nu}) \leq \alpha(\mu - l) + \alpha(l - \nu) = \alpha(\mu - \nu).$$

This completes the proof of the claim.

By this claim we can conclude that $\sum_{n=0}^{\infty} a_{n,\mu\nu} \in E_{\bar{R}}$ is well defined for all $\mu, \nu = 1, 2, 3, \ldots$. Therefore,

$$M^{-1} = \sum_{n=0}^{\infty} (1 - M)^n = [\sum_{n=0}^{\infty} a_{n,\mu\nu}]_{\mu,\nu}$$

is well defined. Let $M^{-1} = [b_{\mu\nu}]_{\mu,\nu=1,2,3,\ldots}$, namely $b_{\mu\nu} = \sum_{n=0}^{\infty} a_{n,\mu\nu}$. Then

$$s_\nu = -\sum_{\mu=1}^{\infty} u_{-\mu} b_{\mu\nu} \in E_{\bar{R}}$$

is a well-defined element with $\mathrm{ord}_{\partial_1}(s_\nu) \leq \mathrm{ord}_{\partial_1}(u_{-\mu}) + \mathrm{ord}_{\partial_1}(b_{\mu\nu}) \leq -\alpha\nu$. Thus we have established the existence of $S = S(t) \in 1 + E_{R-}$ such that $SU \in \widehat{D}_R$.

At last, by lemma 7, the operator $S(0)$ of a sufficiently general type is not admissible, that is the solution $S(t)$ corresponds to a non-trivial solutions of the system $(KP)_\alpha$.
The theorem is proved.
□

6 Isospectral deformations and the modified KP systems

There is a natural question: can we obtain the Parshin system or the modified systems as defining equations of all isospectral deformations of some differential operators in two variables? We will show below that it is true for a pair of monic commuting differential operators.

The notion of isospectral deformations can be introduced in the same way as in [8], §4. Consider a family

$$\{P(t), t \in M\}$$

of operators, where the parameter space M is an open domain of $\mathbb{C}^N$ and $P(t) = (P_1(t), P_2(t))$, $P_i \in A[t]((\partial_1^{-1}))[\partial_2] = \tilde{D} \subset D_R$ is a pair of monic commuting "differential" operators depending on $t = (t_1, \ldots, t_N) \in M \subset \mathbb{C}^N$ analytically. As a specialization we can take, for example, $A = \mathbb{C}[[x_1, x_2]]$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$. For convenience, further we will work with this specialization.

Definition 3. We say $\{P(t), t \in M\}$ is a family of isospectral deformations if there exist "differential operators" $Q_1(t), Q_2(t), \ldots, Q_N(t) \in \tilde{D}$ depending on the parameter $t \in M$ analytically such that the following system of equations has a nontrivial solution $\psi(A, t; \lambda)$ for every eigenvalue $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ of $P(t)$:

$$\begin{cases} P(t)\psi(A, t; \lambda) = \lambda\psi(A, t; \lambda) \\ \frac{\partial}{\partial t_1}\psi(A, t; \lambda) = Q_1(t)\psi(A, t; \lambda) \\ \ldots \\ \frac{\partial}{\partial t_N}\psi(A, t; \lambda) = Q_N(t)\psi(A, t; \lambda) \end{cases} \tag{78}$$

The point here is that the eigenvalue λ in the first equation does not depend on the parameter t, i.e., it is preserved. Repeating the arguments from [8], §4, we obtain the compatibility conditions of the system (78):

$$0 = \frac{\partial}{\partial t_i}(P(t)\psi(x, t; \lambda) - \lambda\psi(x, t; \lambda)) = (\frac{\partial}{\partial t_i}P(t) - [Q_i(t), P(t)])\psi(x, t; \lambda),$$

where $x = (x_1, x_2)$. For every fixed $t \in M$, the eigenfunctions $\psi(x,t;\lambda)$ are linearly independent for distinct eigenvalues $\lambda \in \mathbb{C}$. Since $\frac{\partial}{\partial t_i} P_j(t) - [Q_i(t), P_j(t)]$, $j = 1, 2$ are pseudo-differential operators of finite order in ∂_2, they have mostly a countable (topological) basis of independent solutions. Therefore, by cardinality reason,

$$\frac{\partial}{\partial t_i} P(t) = [Q_i(t), P(t)] \tag{79}$$

Similarly, the condition $\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}\psi = \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_i}\psi$ gives

$$\frac{\partial}{\partial t_i} Q_j(t) - \frac{\partial}{\partial t_j} Q_i = [Q_i(t), Q_j(t)]. \tag{80}$$

The system of equations (79) and (80) is equivalent to the condition that equation (78) has a nontrivial solution for every $\lambda \in \mathbb{C}^2$. Therefore, finding a family $P(t)$ of isospectral deformations of a given pair $P(0)$ is equivalent to finding a solution of the Lax equation (79) for differential operators $Q_i(t)$ satisfying (80) together with the initial condition $P(t)|_{t=0} = P(0)$.

Without loss of generality assume that $\mathrm{ord}_{\partial_2}(P_1) \geq \mathrm{ord}_{\partial_2}(P_2)$. Let

$$(\mathrm{ord}_{\partial_1}(P_1 \partial_2^{-\mathrm{ord}_{\partial_2}(P_1)})_+, \mathrm{ord}_{\partial_2}(P_1)) = (p_1, q_1)$$

and

$$(\mathrm{ord}_{\partial_1}(P_2 \partial_2^{-\mathrm{ord}_{\partial_2}(P_2)})_+, \mathrm{ord}_{\partial_2}(P_2)) = (p_2, q_2).$$

For each pseudo-differential operator we will call such a pair of integers *the full order.*

Lemma 10. *Suppose $(p_1, q_1) \neq d(p_2/l, q_2/l)$ for any $d \in \mathbb{Z}$, where $l = gcd(p_2, q_2)$. Then equation (79) is equivalent to the equation*

$$\frac{\partial}{\partial t_i} L(t) = [Q_i(t), L(t)], \tag{81}$$

where $L = (L_1, L_2)$,

$$L_1 = u_0 + u_1 \partial_2^{-1} + \dots,$$

$u_i \in A[t]((\partial_1^{-1}))$, $\mathrm{ord}_{\partial_1}(u_0) = 1$, u_0 *is monic,*

$$L_2 = v_{-1}\partial_2 + v_0 + v_1 \partial_2^{-1} + \dots,$$

$u_i \in A[t]((\partial_1^{-1}))$, $\mathrm{ord}_{\partial_1}(v_{-1}) = 0$, v_{-1} *is monic.*

Proof. Consider the operator $P_1' := P_1^{q_2/(l \cdot gcd(q_1, q_2/l))} P_2^{-(1/l)q_1/gcd(q_1,q_2/l)}$. Let the full order of P_1' be equal to $(k, 0)$, $k \in \mathbb{Z}$.

If $k = 0$, this means that $p_1 q_2/(l \cdot gcd(q_1, q_2/l)) = p_2 q_1/(l \cdot gcd(q_1, q_2/l))$, where from p_2/l is divisible by $q_2/(l \cdot gcd(q_1, q_2/l))$ and p_1 is divisible by $q_1/gcd(q_1, q_2/l)$. Since $(p_2/l, q_2/l) = 1$, we have therefore $gcd(q_1, q_2/l) = q_2/l$ and $(p_1, q_1) = q_1 l/q_2 (p_2/l, q_2/l)$, a contradiction.

So, $k \neq 0$, and we put $L_1 := P_1'^{1/k}$. Since P_1, P_2 are monic operators, such a root exists. Then $L_2 := (P_2 L_1^{-p_2})^{1/q_2}$. Clearly, equation (81) implies equation (79). Let's prove the converse.

Since $\frac{\partial}{\partial t_i}$ and $[Q_i(t), .]$ are derivations, we have

$$0 = (\frac{\partial}{\partial t_i} - [Q_i(t), .])P_2(t) = \sum_{k=0}^{l-1} P_2(t)^{k/l} (\frac{\partial}{\partial t_i} P_2(t)^{1/l} - [Q_i(t), P_2(t)^{1/l}]) P_2(t)^{(l-1-k)/l}$$

Since $P_2(t)^{1/l}$ is monic, the last equality implies that $\frac{\partial}{\partial t_i}P_2(t)^{1/l} - [Q_i(t), P_2(t)^{1/l}] = 0$. Therefore, equation (79) is equivalent to the equation

$$\frac{\partial}{\partial t_i}P_2(t)^{1/l} = [Q_i(t), P_2(t)^{1/l}], \quad \frac{\partial}{\partial t_i}P_1(t) = [Q_i(t), P_1(t)]$$

Continuing this line of reasoning, we obtain the equivalence of the equation above with equation (81).
□

Remark. The condition on operators P_1, P_2 in lemma appears as an analog of condition of ellipticity for commuting differential operators in one variable (see [7], sect.5). In one-dimensional situation the existence of a monic operator in a ring of commuting operators implies the moniqueness of all commuting operators up to a constant. In our case the existence of two monic operators is not enough for having an analogous property in the ring of commuting operators. The extra condition from lemma imply this property as we will see from lemma below. Obviously, the operators from lemma are also algebraically independent. In another paper we are going to show that rings of commuting operators belonging to some subspaces of the ring of "differential" operators that have two such operators correspond to certain geometric data and vice versa. Some examples of such rings of commuting variables can be obtained as certain images of rings considered in [11], theorem 1, b) and remark 3. They appear as images of a generalized Krichever map constructed in [14], [10]. Other examples should (conjecturally) come from the examples considered in [2], [1], where the rings of commuting differential operators containing completely integrable operators of dimension two appeared. See also a discussion below.

The left hand side of equation (81) is a pair of operators of orders (in ∂_2) at most $(0,1)$. Therefore, the operator $Q_i(t)$ must satisfy

$$\mathrm{ord}_{\partial_2}([Q_i(t), L_1(t)]) \le 0, \quad \mathrm{ord}_{\partial_2}([Q_i(t), L_2(t)]) \le 1. \tag{82}$$

For simplicity, we can assume that operators L_1, L_2 (so, P_1, P_2) are normalized. By this we mean the operators obtained by conjugation by some invertible operator $\bar{S} \in \tilde{D}^{\times}$ as in theorem 1, iii) or in the proof of lemma 7. In this case canonically defined by L_1, L_2 elements L_{00}, M_{00} have some special form. Let's assume also that L_{1+}, L_{2+} do not depend on times and that $v_{-1} = 1$ (recall that $M_{00} = v_{-1}\partial_2 + v_0$). In this case (82) becomes

$$\mathrm{ord}_{\partial_2}([Q_i(t), L_1(t)]) < 0, \quad \mathrm{ord}_{\partial_2}([Q_i(t), L_2(t)]) < 0. \tag{83}$$

Lemma 11. *Let* $L = (L_1, L_2)$, $L_1, L_2 \in A((\partial_1^{-1}))((\partial_2^{-1}))$ *be arbitrary monic operators with* $\mathrm{ord}_{\partial_2}(L_1) = 0$, $\mathrm{ord}_{\partial_2}(L_2) = 1$, $\mathrm{ord}_{\partial_1}(L_{1+}) = 1$, $\mathrm{ord}_{\partial_1}((L_2\partial_2^{-1})_+) = 0$. *Then*

$$F_L = \{Q \in \tilde{D} | \, \mathrm{ord}_{\partial_2}([Q_i(t), L_1(t)]) < 0, \quad \mathrm{ord}_{\partial_2}([Q_i(t), L_2(t)]) < 0\}$$

coincides with the $\mathbb{C}$*-linear space (topologically) generated by the operators* $(L_1^i L_2^j)_+$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}_+$.

Proof. Obviously, the operators $(L_1^i L_2^j)_+$ belong to F_L. Conversely, let $Q \in F_L$ be an element of full order (m_1, m_2). We can represent Q as $S^{-1}Q'S$, where S is the operator from theorem 1 for L_1, L_2, and Q' is a series in M_{00}^{-1} with coefficients represented as series in L_{00}^{-1} ($L_{00} = \partial_1$ if L is normalized as in the proof of lemma 7). The condition $\mathrm{ord}_{\partial_2}([Q_i(t), L_1(t)]) < 0$ then implies that coefficients of all non-negative powers of M_{00} of this series (which are series in L_{00}^{-1}) commute with L_{00}. Since we assumed that L is normalized, v_0 also commutes with L_{00}, and therefore also with these coefficients. The second condition $\mathrm{ord}_{\partial_2}([Q_i(t), L_2(t)]) < 0$ then implies that coefficients of all non-negative powers of M_{00} commute with ∂_2, and therefore they are constant.

The last fact means that the leading coefficient of the leading coefficient of Q is constant, say $c \in \mathbb{C}$. Since $(L_1^{m_1}L_2^{m_2})_+$ is monic, the linear combination $Q - c(L_1^{m_1}L_2^{m_2})_+$ has order less than (m_1, m_2). Since $(Q - c(L_1^{m_1}L_2^{m_2})_+)$ satisfy (83), the lemma follows by induction on (m_1, m_2).

Note that we have got a representation of Q in an *infinite* sum, $Q = \sum_{j=0}^{m_2}\sum_{i=-\infty}^{i_j} c_{ij}(L_1^iL_2^j)_+$. This means that $(L_1^iL_2^j)_+$ are topological generators of the vector space F_L, where the topology comes from the topology of the two-dimensional local field $\mathbb{C}((L_1^{-1}))((L_2^{-2}))$. Indeed,

$$\sum_{j=0}^{m_2}\sum_{i=-\infty}^{i_j} c_{ij}(L_1^iL_2^j)_+ = (\sum_{j=0}^{m_2}\sum_{i=-\infty}^{i_j} c_{ij}L_1^iL_2^j)_+ = (S^{-1}(\sum_{j=0}^{m_2}\sum_{i=-\infty}^{i_j} c_{ij}L_{00}^iM_{00}^j)S)_+.$$

The lemma is proved.
□

By proposition 3 the equation

$$\frac{\partial}{\partial t_{ij}}L(t) = [(L_1(t)^iL_2(t)^j)_+, L(t)], \quad i \in \mathbb{Z}, j \in \mathbb{Z}_+$$

imply equation (80). Thus it is the master equation for the largest possible family of isospectral deformations of a pair of "differential" operators satisfying the condition of lemma 10 and assumptions above.

Note that the original Parshin's system (2) is a "half" of the system above. Therefore, as we have shown in section 2.2, it has only trivial solutions. This means that in general it is not possible to find a universal family. Nevertheless, it is possible to find some kind of restricted universal family of isospectral deformations, where we mean that the operators Q_i should belong not to the whole space of differential operators, but to some linear subspace generated by operators with some restrictions on the full order. Let's clarify the last assertion.

For any pair of operators $P_1, P_2 \in D_R$ satisfying the condition of lemma 10 the operators L_1, L_2 and the invertible operator $S \in \widehat{\mathcal{V}}_R$ are defined so that $L_1 = S^{-1}L_{00}S$, $L_2 = S^{-1}M_{00}S$. Then $SP_iS^{-1} = NP_i$, $i = 1, 2$, where NP_i are monomials in L_{00}, M_{00}. We will call this pair of operators *a normal form* of the pair of operators P_1, P_2.

For any function $\alpha : \mathbb{Z}_+ \to \mathbb{R}$, an operator $W \in \widehat{\mathcal{V}}_R$ and monic operators $L_{00} \in E_{\bar{R}}$, $M_{00} = v_{-1}\partial_2 + v_0$, $v_i \in E_{\bar{R}}$ let $D_R^{\alpha,W,L_{00},M_{00}}$ be a vector space defined as follows

$$D_R^{\alpha,W,L_{00},M_{00}} = \langle (W^{-1}QW)_+ | Q = \sum_{k=0}^{N} q_kM_{00}^k \in \bar{R}((L_{00}^{-1}))[M_{00}], \quad \mathrm{ord}_{\partial_1}(q_k) \le \alpha(k)\rangle.$$

Obviously, $D_R = \cup_\alpha D_R^{\alpha,W,L_{00},M_{00}}$ for any fixed W, L_{00}, M_{00}, where the union is taken over all possible functions α. Of course, in general the space $D_R^{\alpha,W,L_{00},M_{00}}$ is far from to be a ring. But in some partial cases, for example for linear functions and operators L_{00}, M_{00} considered in section 5, $W = 1$, this space has a natural ring structure, as one can easily check.

Now, let $P_1(0), P_2(0) \in A((\partial_1^{-1}))[\partial_2]$ be any pair of operators satisfying the condition of lemma 10. Let's fix the operators L_{00}, M_{00} defined by this pair and some function α such that $P_i(0) \in D_R^{\alpha,S(0),L_{00},M_{00}}$. Obviously, such a function exists. Moreover, since the normal form of $P(0)$ consists of monomials, we can assume that $\alpha(0) \le 0$. We will say that operators belonging to $D_R^{\alpha,\dots}$ are α*-differential operators.* All said above brings the following definition.

Definition 4. We say that a family $\{P(t), t \in M\}$ of pairs of monic commuting α-differential operators, whose full orders are constant and satisfy the condition of lemma 10, $P(t) = (P_1(t), P_2(t)), P_i(t) \in \tilde{D} \cap D_R^{\alpha,S(t),L_{00},M_{00}}$ is a family of isospectral deformations if there exist

α-differential operators $Q_1(t), Q_2(t), \dots, Q_N(t) \in \tilde{D} \cap D_R^{\alpha, S(t), L_{00}, M_{00}}$ depending on the parameter $t \in M$ analytically such that the following system of equations has a nontrivial solution $\psi(A,t;\lambda)$ for every eigenvalue $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ of $P(t)$:

$$\begin{cases} P(t)\psi(A,t;\lambda) = \lambda\psi(A,t;\lambda) \\ \frac{\partial}{\partial t_1}\psi(A,t;\lambda) = Q_1(t)\psi(A,t;\lambda) \\ \dots \\ \frac{\partial}{\partial t_N}\psi(A,t;\lambda) = Q_N(t)\psi(A,t;\lambda) \end{cases} \tag{84}$$

Repeating all arguments after definition 78 with $\tilde{D}$ replaced by $\tilde{D} \cap D_R^{\alpha, S(t), L_{00}, M_{00}}$ and indices $i \in \mathbb{Z}$, $j \in \mathbb{Z}_+$ replaced by $i \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, $i \le \alpha(j)$ in lemma 11, we come to a conclusion that the equation $(KP)_\alpha$ (see sect. 3.1) is the master equation for the largest possible family of isospectral deformations of a pair of α-differential operators satisfying the condition of lemma 10.

As we have shown in sections 3,4, this equation is uniquely solvable. By the same arguments as in [8], section 4 (equation (4.16)), we obtain

$$P(t) = S(t)NP(0)S(t)^{-1} = Y(t)P(0)Y(t)^{-1},$$

where $S(t)$ is the solution of the corresponding modified Sato-Wilson system, and $Y(t)$ is the operator coming from theorem 3. This shows that the pair of operators $P_i(t) \in D_R^{\alpha, S(t), L_{00}, M_{00}}$, because they have the same normal form as $P(0)$ and because $Y(t)$ has no negative order terms ($P_i(t) \in \tilde{D} \cap D_R^{\alpha, S(t), L_{00}, M_{00}}$ if $S(0)$ looks like in theorem 5).

So, the systems $(KP)_\alpha$ give for certain pairs P of α-differential operators a universal family of isospectral deformations.

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