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Transformation of LEBESGUE Measure and Integral by LIPSCHITZ Mappings

Transformation of Lebesgue Measure and Integral by Lipschitz Mappings

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Abstract We first show that LIPSCHITZ mappings transform measurable sets into measurable sets. Then we prove the following theorem:

Let $E \subseteq \mathbb{R}^n$ be open, and let $\phi : E \to \mathbb{R}^n$ be continuous. If ϕ is differentiable at $x_0 \in E$, then

$$\lim_{r \to 0} \frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} = \Big| \det \phi'(x_0) \Big|.$$

From this result the change of variables formula for injective and locally LIPSCHITZ mappings is easily derived by using the RADON-NIKODYM theorem. We finally discuss the transformation of L^p functions by LIPSCHITZ mappings.

Contents

Notations

$$
\begin{aligned}
x &= (x_1, \ldots, x_n) \in \mathbb{R}^n \\
|x| &= \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}, \ |x|_\infty = \max_{i=1,\ldots,n} |x_i| \\
B_r(x_0) &= \{x \in \mathbb{R}^n \mid |x - x_0| < r\} \\
Q_r(x_0) &= \{x \in \mathbb{R}^n \mid |x - x_0|_\infty < r\} \\
\lambda_n &= \text{Lebesgue measure in } \mathbb{R}^n \\
\nabla u &= \Big(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\Big)
\end{aligned}
$$

1. Transformation of Measurable Sets

It is well-known that continuous mappings can fail to transform sets of measure zero onto sets of measure zero. For instance, the CANTOR singular function is continuous and maps the CANTOR ternary set on a set of measure 1 [1)].

The following definition excludes this situation.

Let $E \subseteq \mathbb{R}^m$ be a measurable set.

1.1 Definition *A mapping* $\phi : E \to \mathbb{R}^n$ *is said to satisfy* LUSIN's (N)-*condition* [*briefly*: (N)-condition] *if*

$$A \subseteq E,\ \lambda_m(A) = 0 \implies \lambda_n(\phi(A)) = 0.$$

1.2 Remark For a mapping $\phi : E \to \mathbb{R}^n$ the following conditions are equivalent:

1° ϕ *satisfies the* (N)-*condition*;

2° $A \subset E,\ \lambda_m(A) = 0 \implies \phi(A)$ *is measurable.*

Indeed, implication 1° $\Longrightarrow$ 2° is obvious.

To prove implication 2° $\Longrightarrow$ 1°, assume there exists $A \subseteq E$, $\lambda_m(A) = 0$ such that $\lambda_n(\phi(A)) > 0$. Then there exists a non-measurable subset $B_0 \subset \phi(A)$. Define

$$A_0 := \{x \in A \mid \phi(x) \in B_0\}.$$

The LEBESGUE measure being complete, it follows that A_0 is measurable, and thus $\lambda_n(A_0) = 0$. By 2°, $B_0 = \phi(A_0)$ is measurable, a contradiction. ■

1.3 Theorem *Let* $\phi : E \to \mathbb{R}^n$ *be a mapping.*

1. *Let* ϕ *be continuous in* E *and satisfy the (N)-condition. Then*

$$F \subseteq E \text{ measurable } \implies \phi(F) \text{ measurable.}$$

2. *Assume*

1) the set $\phi(E)$ *is measurable,*

2) ϕ *is injective,*

3) $\phi^{-1} : \phi(E) \to E$ *is continuous in* $\phi(E)$ *and satisfies the (N)-condition.*

[1)] A brief discussion of the CANTOR ternary set and the CANTOR singular function can be found in `http://www.math.hu-berlin.de/~jnaumann/cantor.ps`

Let $u : \phi(E) \to \bar{\mathbb{R}}$ be measurable. Then

$$u \circ \phi : E \to \bar{\mathbb{R}}$$

is measurable.

Proof $\boxed{\mathbf{1}}$ The measurability of F is equivalent to the existence of sets F_i $(i \in \mathbb{N})$ and A such that

$$F_i \text{ is closed } (i \in \mathbb{N}), \quad \lambda_n(A) = 0, \quad F = \Big(\bigcup_{i=1}^{\infty} F_i\Big) \cup A.$$

The sets $F_i \cap \overline{B_k(0)}$ $(k \in \mathbb{N})$ are compact. Hence $\phi(F_i \cap \overline{B_k(0)})$ is compact and thus Borel. On the other hand, the (N)-condition implies that $\phi(A)$ is a Lebesgue null set. Therefore

$$\phi(F) = \Big(\bigcup_{i=1}^{\infty} \phi(F_i)\Big) \cup \phi(A)$$

is measurable.

$\boxed{\mathbf{2}}$ The measurability of $u : \phi(E) \to \bar{\mathbb{R}}$ means that for every $a \in \mathbb{R}$ the set

$$\{y \in \phi(E) \mid u(y) > a\}$$

is measurable. Observing 3), from **1** it follows that the sets

$$\phi^{-1}(\{y \in \phi(E) \mid u(y) > a\}), \quad a \in \mathbb{R}$$

are measurable.

Finally, it readily seen that, for any $a \in \mathbb{R}$,

$$\{x \in E \mid (u \circ \phi)(x) > a\} = \phi^{-1}(\{y \in \phi(E) \mid u(y) > a\}).$$

Hence the set on the left hand side is measurable. ∎

2. Lipschitz Mappings

2.1 Definition

1. *Let $E \subseteq \mathbb{R}^m$ be a set, let $\phi : E \to \mathbb{R}^n$ be a mapping.*

1.1 *ϕ is called* Lipschitz **continuous** *[briefly:* Lipschitz*] if there exists $L = const < +\infty$ such that*

$$|\phi(x) - \phi(x')| \le L|x - x'| \quad \forall x, x' \in E \text{ [2)]}$$

[2)] We denote the Euclidean norm in $\mathbb{R}^m$ and $\mathbb{R}^n$ by the same symbol $|\cdot|$. An analogous remark refers to the norm $|\cdot|_\infty$.

1.2 *Let $m = n$. ϕ is called* **bi-**LIPSCHITZ **continuous** *[briefly: bi-*LIPSCHITZ*] if there exists $L = const < +\infty$ such that*

$$\frac{1}{L}|x - x'| \le |\phi(x) - \phi(x')| \le L|x - x'| \quad \forall x, x' \in E.$$

2. *Let $E \subseteq \mathbb{R}^m$ be open. A mapping $\phi : E \to \mathbb{R}^n$ is called* **locally** LIPSCHITZ **continuous** *[briefly:* **locally** LIPSCHITZ*] if*

$$\begin{cases} \textit{for every compact } K \subset E \textit{ there exists } L_K = const \ < +\infty \textit{ such that} \\ \qquad\qquad |\phi(x) - \phi(x')| \le L_K|x - x'| \quad \forall x, x' \in K. \end{cases}$$

2.2 Remark The constant L in Definition 2.1/1 is called a LIPSCHITZ constant for ϕ with respect to the norm $|\cdot|$. An analogous notation is used for locally LIPSCHITZ mappings.

Since all norms on $\mathbb{R}^n$ are equivalent to each other, the passage from $|\cdot|$ to any other norm on $\mathbb{R}^n$ only amounts to a different numerical value of a LIPSCHITZ constant. ■

We now show that LIPSCHITZ mappings transform measurable sets into measurable sets. More precisely, we have

2.3 Theorem *Let $n \ge m$. Assume*

$$E \subseteq \mathbb{R}^m \textit{ measurable}, \ \phi : E \to \mathbb{R}^n \text{ LIPSCHITZ},$$

or

$$E \subseteq \mathbb{R}^m \textit{ open}, \ \phi : E \to \mathbb{R}^n \textit{ locally } \text{LIPSCHITZ}.$$

Then

$$F \subseteq E \textit{ measurable} \implies \phi(F) \textit{ measurable}.$$

Proof It suffices to prove that ϕ satisfies the (N)-condition. Then the claim follows from Theorem 1.3/1.

Let $E \subseteq \mathbb{R}^m$ be measurable. By assumption,

$$(2.1) \qquad |\phi(x) - \phi(x')|_\infty \le L|x - x'|_\infty \qquad \forall x, x' \in E, \quad (L = \text{const} < +\infty).$$

Let $A \subseteq E$ with $\lambda_m(A) = 0$. Given any $\epsilon > 0$, there exist cubes $Q^{(k)} \subset \mathbb{R}^m$ such that

$$(2.2) \qquad A \subseteq \bigcup_{k=1}^{\infty} Q^{(k)}, \qquad \sum_{k=1}^{\infty} \lambda_m(Q^{(k)}) \le \epsilon.$$

We may write

$$Q^{(k)} = Q_{r_k}(\xi_k) := \Big\{ x \in \mathbb{R}^m \ \Big| \ |x - \xi_k|_\infty < r_k \Big\}, \quad k \in \mathbb{N}.$$

From (2.1) it follows that

$$|\phi(x)-\phi(\xi_k)|_\infty \le L|x-\xi_k|_\infty < Lr_k \qquad \forall x \in Q^{(k)}$$

i.e. $\phi(Q^{(k)}) \subseteq Q_{Lr_k}(\phi(\xi_k))$ [cube in $\mathbb{R}^n$] for all $k \in \mathbb{N}$. Now (2.2) implies

$$\phi(A) \subseteq \bigcup_{k=1}^{\infty} \phi(Q^{(k)}) \subseteq \bigcup_{k=1}^{\infty} Q_{Lr_k}(\phi(\xi_k)).$$

Clearly, $\lambda_n(Q_{Lr_k}) = (2Lr_k)^n = L^n(2r_k)^{n-m}\lambda_m(Q^{(k)})$. Observing that $2r_k \le \epsilon^{\frac{1}{m}}$ for all $k \in \mathbb{N}$ (cf. (2.2)), we obtain

$$\sum_{k=1}^{\infty} \lambda_n(Q_{Lr_k}) = L^n \sum_{k=1}^{\infty}(2r_k)^{n-m}\lambda_m(Q^{(k)}) \le L^n\epsilon^{\frac{n}{m}-1}\sum_{k=1}^{\infty}\lambda_m(Q^{(k)}) \le L^n\epsilon^{\frac{n}{m}}.$$

Thus, $\lambda_n(\phi(A)) = 0$.

To prove the second claim, let $E \subseteq \mathbb{R}^m$ be an open set. By assumption, for every compact $K \subset E$ there exists $L_K = \text{const} < +\infty$ such that

$$|\phi(x)-\phi(x')|_\infty \le L_K|x-x'|_\infty \qquad \forall x, x' \in K.$$

Let (E_i) $(i \in \mathbb{N})$ be a sequence of open bounded subsets of $\mathbb{R}^m$ such that

$$E_i \subset \bar{E}_{i+1} \subset \ (i \in \mathbb{N}), \quad E = \bigcup_{i=1}^{\infty} E_i.$$

Given any $A \subset E$ with $\lambda_m(A) = 0$, we have $\lambda_m(A \cap E_i) = 0$ for all $i \in \mathbb{N}$. The mapping

$$\phi|_{\bar{E}_i} : \bar{E}_i \to \mathbb{R}^n$$

being Lipschitz, from the preceding part it follows that $\lambda_n(\phi(A \cap E_i)) = 0$ for all $i \in \mathbb{N}$.

Observing that $A = \bigcup\limits_{i=1}^{\infty}(A \cap E_i)$, we obtain

$$\lambda_n(\phi(A)) \le \sum_{i=1}^{\infty} \lambda_n(\phi(A \cap E_i)) = 0.$$

■

Throughout the remainder of the paper, let $m = n$.

Let $E \subseteq \mathbb{R}^n$ be open. The mapping $\phi : E \to \mathbb{R}^n$ [3] is said to be differentiable at $x_0 \in E$ if there exists an $(n \times n)$-matrix $A(x_0)$ such that

$$\phi(x_0+h) = \phi(x_0) + A(x_0)h + \omega(x_0;h) \qquad \forall h \in \mathbb{R}^n,\ (x_0+h) \in E,$$

[3] We write

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}.$$

where

$$A(x_0) = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial x_1}(x_0) & \cdots & \dfrac{\partial \phi_1}{\partial x_n}(x_0) \\ \dots\dots\dots & \dots & \dots\dots\dots \\ \dfrac{\partial \phi_n}{\partial x_1}(x_0) & \cdots & \dfrac{\partial \phi_n}{\partial x_n}(x_0) \end{pmatrix},$$

and

$$\lim_{h \to 0} \frac{\omega(x_0; h)}{|h|} = 0.$$

$A(x_0)$ is called the JACOBI matrix and usually denoted by $\phi'(x_0)$.

The following result is fundamental to our approach to the change of variables formula.

2.4 Theorem *Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be locally* LIPSCHITZ *continuous. If ϕ is differentiable at $x_0 \in E$ then*

$$\lim_{r \to 0} \frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} = \Big| \det \phi'(x_0) \Big|.$$

Proof First, by the continuity of ϕ, the set $\phi(B_r(x_0))$ is BOREL and thus measurable.

We fix a ball $\overline{B_{R_0}(x_0)} \subset E$. Let $L_0 := L_{\overline{B_{R_0}(x_0)}}$ denote a LIPSCHITZ constant for ϕ with respect to $\overline{B_{R_0}(x_0)}$:

$$|\phi(x) - \phi(x')| \leq L_0 |x - x'| \quad \forall x, x' \in \overline{B_{R_0}(x_0)}.$$

We consider the two cases

$$\det \phi'(x_0) = 0 \quad \text{and} \quad \det \phi'(x_0) \neq 0$$

separately.

$\boxed{\det \phi'(x_0) = 0}$ We prove:

$$\limsup_{r \to 0} \frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} = 0. \tag{2.3}$$

This implies the claim.

Without any loss of generality, we may assume that

$$(2.4) \qquad \phi'(x_0)\xi \in \mathbb{R}^{n-1} \times \{0\} \quad \forall \xi \in \mathbb{R}^n$$

Indeed, $\det \phi'(x_0) = 0$ is equivalent to the linear dependence of the rows of $\phi'(x_0)$. Without any loss of generality, we may assume that

$$\sum_{i=1}^{n-1} \alpha_i \frac{\partial \phi_i}{\partial x_j}(x_0) = \frac{\partial \phi_n}{\partial x_j}(x_0) \qquad (j = 1, \dots, n)$$

for reals $\alpha_1, \dots, \alpha_{n-1}$. Define

$$\mathcal{M} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & 1 & 0 \\ -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & 1 \end{pmatrix},$$

and

$$\Psi(x) := \mathcal{M}\phi(x), \quad x \in E.$$

Clearly, Ψ is locally LIPSCHITZ in E, and

$$\lambda_n(\Psi(F)) = |\det \mathcal{M}\,|\,\lambda_n(\phi(F)) = \lambda_n(\phi(F))$$

for every measurable set $F \subseteq E$. Next, $\Psi'(x_0) = \mathcal{M}\phi'(x_0)$, where

$$\mathcal{M}\phi'(x_0) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(x_0) & \cdots & \frac{\partial \phi_1}{\partial x_n}(x_0) \\ \dots & \dots & \dots \\ \frac{\partial \phi_{n-1}}{\partial x_1}(x_0) & \cdots & \frac{\partial \phi_{n-1}}{\partial x_n}(x_0) \\ 0 & \cdots & 0 \end{pmatrix}.$$

Thus

$$(\Psi'(x_0)\xi)_n \quad [= \text{ the n-th component}] = 0 \quad \forall \xi \in \mathbb{R}^n.$$

Let $\epsilon > 0$. By the differentiability of ϕ at x_0, there exists an $R_1 = R_1(\epsilon) > 0$ such that

$$|\phi(x) - \phi(x_0) - \phi'(x_0)(x - x_0)| \le \epsilon |x - x_0| \quad \forall x \in B_{R_1}(x_0)$$

(without loss of generality, we may assume that $R_1 \le R_0$). By (2.4), $(\phi'(x_0)(x - x_0))_n = 0$ for all $x \in \mathbb{R}^n$.

Let $0 < r \le R_1$. We obtain

$$
\begin{aligned}
|\phi_n(x) - \phi_n(x_0)| &= |\phi_n(x) - \phi_n(x_0) - (\phi'(x_0)(x-x_0))_n| \\
&\le |\phi(x) - \phi(x_0) - \phi'(x_0)(x-x_0)| \\
&\le \epsilon |x - x_0| \\
&< \epsilon\, r \qquad (2.5)
\end{aligned}
$$

for all $x \in B_r(x_0)$. On the other hand, from

$$|\phi(x) - \phi(x_0)| \le L_0|x - x_0| < L_0 r \quad \forall x \in B_r(x_0)$$

it follows that

$$\phi(B_r(x_0)) \subset B_{L_0 r}(\phi(x_0)). \tag{2.6}$$

Observing that $B_{L_0 r}(\phi(x_0)) \subset Q_{L_0 r}(\phi(x_0))$, from (2.5) and (2.6) we obtain

$$\phi\Big(B_r(x_0)\Big) \subset \Big\{ y \in \mathbb{R}^n \Big| |y_i - \phi_i(x_0)| < L_0 r \quad (i = 1, \ldots, n-1),\ |y_n - \phi_n(x_0)| < \epsilon r \Big\}.$$

Hence

$$\lambda_n(\phi(B_r(x_0))) \le (2L_0 r)^{n-1} \cdot 2\epsilon r \;=\; 2^n L_0^{n-1} r^n \epsilon,$$

i.e.

$$\frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} \le \frac{2^n L_0^{n-1}}{\lambda_n(B_1)} \cdot \epsilon \quad \forall\, 0 < r \le R_1.$$

Whence (2.3).

$\boxed{\det \phi'(x_0) \neq 0}$ We prove

$$\liminf_{r \to 0} \frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} \ge |\det \phi'(x_0)|; \tag{2.7$_1$}$$

$$\limsup_{r \to 0} \frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} \le |\det \phi'(x_0)|. \tag{2.7$_2$}$$

These inequalities imply the claim.

To begin with, we note that the matrix $\phi'(x_0)$ is invertible. Then

$$C_0 := \sup_{|\xi|=1} \Big| [\phi'(x_0)]^{-1} \xi \Big|$$

is a positive real number.

Proof of (2.7₁) Let $0 < \epsilon < 1$. There exists $R_2 = R_2(\epsilon) > 0$ such that

$$|\phi(x_0 + h) - \phi(x_0) - \phi'(x_0)h| \leq \frac{\epsilon}{C_0}|h| \quad \forall\, h \in B_{R_2}(0)$$

(as above, we may assume that $R_2 \leq R_0$).

Let $0 < r \leq R_2$. Given $y \in B_{(1-\epsilon)r}(0)$, for $h \in \overline{B_r(0)}$ define

$$T_y(h) := y - [\phi'(x_0)]^{-1}(\phi(x_0 + h) - \phi(x_0) - \phi'(x_0)h).$$

Clearly, T_y is continuous. Moreover, for any $h \in \overline{B_r(0)}$,

$$\begin{aligned} |T_y(h)| &\leq |y| + C_0|\phi(x_0 + h) - \phi(x_0) - \phi'(x_0)h| \\ &< (1-\epsilon)r + \epsilon|h| \\ &\leq r, \end{aligned}$$

i.e. $T_y(h) \in B_r(0)$.

By the Brouwer fixed point [4)] theorem, there exists $h^* \in \overline{B_r(0)}$ such that

$$\begin{aligned} h^* = T_y(h^*) &= -\Big[\phi'(x_0)\Big]^{-1}\Big(\phi(x_0 + h^*) - \phi(x_0) - \phi'(x_0)h^*\Big) + y \\ &= -\Big[\phi'(x_0)\Big]^{-1}\Big(\phi(x_0 + h^*) - \phi(x_0) - \phi'(x_0)y\Big) + h^*, \end{aligned}$$

i.e.

$$\phi(x_0 + h^*) - \phi(x_0) - \phi'(x_0)y = 0.$$

Since $|h^*| = |T_y(h^*)| < r$, we obtain

$$\phi(x_0) + \phi'(x_0)y = \phi(x_0 + h^*) \in \phi\Big(B_r(x_0)\Big). \tag{2.8}$$

To conclude the proof of (2.7₁), define

$$E := \Big\{\phi(x_0) + \phi'(x_0)y \;\Big|\; y \in B_{(1-\epsilon)r}(0)\Big\}.$$

Then (2.8) means $E \subseteq \phi(B_r(x_0))$. It follows

$$\begin{aligned} \lambda_n(\phi(B_r(x_0))) &\geq \lambda_n(E) \\ &= \lambda_n[\phi'(x_0)(B_{(1-\epsilon)r}(0))] \\ &= |\det \phi'(x_0)|\, \lambda_n(B_{(1-\epsilon)r}) \\ &= |\det \phi'(x_0)|\, (1-\epsilon)^n \lambda_n(B_r). \end{aligned}$$

[4)] Proofs of the Brouwer fixed point theorem can be found, for instance, in Deimling [2; p.17], Fonseca/Gangbo [4; p. 51] and Traynor [23].

Thus

$$\frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} \geq |\det \phi'(x_0)|(1-\epsilon)^n \qquad \forall\, 0 < r \leq R_1.$$

Whence (2.7_1).

Proof of (2.7₂). Define the affine-linear mapping

$$\mathcal{T}x := \phi(x_0) + \phi'(x_0)(x - x_0), \quad x \in \mathbb{R}^n.$$

For $0 < r \leq R_1$ and $x \in B_r(x_0)$ consider

$$\tilde{x} := x + \Big[\phi'(x_0)\Big]^{-1}\Big(\phi(x) - \phi(x_0) - \phi'(x_0)(x - x_0)\Big).$$

We obtain

$$\begin{aligned} |\tilde{x} - x_0| &\leq |x - x_0| + C_0|\phi(x) - \phi(x_0) - \phi'(x_0)(x - x_0)| \\ &\leq |x - x_0| + \epsilon|x - x_0| \\ &< (1+\epsilon)r, \end{aligned}$$

i.e. $\tilde{x} \in B_{(1+\epsilon)r}(x_0)$. By the definition of $\mathcal{T}$ and $\tilde{x}$,

$$\begin{aligned} \phi(x) &= \phi(x_0) + (\phi(x) - \phi(x_0)) \\ &= \phi(x_0) + \phi'(x_0)(\tilde{x} - x_0) \\ &= \mathcal{T}\tilde{x}. \end{aligned}$$

Hence

$$\phi\Big(B_r(x_0)\Big) \subseteq \mathcal{T}\Big(B_{(1+\epsilon)r}(x_0)\Big).$$

This inclusion implies

$$\begin{aligned} \lambda_n\Big(\phi(B_r(x_0))\Big) &\leq \lambda_n\Big(\mathcal{T}(B_{(1+\epsilon)r}(x_0))\Big) \\ &= \lambda_n\Big(\phi'(x_0)(B_{(1+\epsilon)r}(x_0))\Big) \\ &= |\det \phi'(x_0)|\,(1+\epsilon)^n \lambda_n(B_r), \end{aligned}$$

i.e.

$$\frac{\lambda_n(\phi(B_r(x_0)))}{\lambda_n(B_r)} \leq |\det \phi'(x_0)|\,(1+\epsilon)^n \quad \forall\, 0 < r \leq R_1.$$

Now (2.7_2) follows. ∎

Remark 2.5 The above argument for proving the inclusion

$$\{\phi(x_0) + \phi'(x_0)y \;\Big|\; y \in B_{(1-\varepsilon)r}(0)\} \subseteq \phi(B_r(x_0))$$

(cf. (2.8)) via a fixed point of T_y, has been submitted to the author by J. Malý. In a technically slightly different way, this inclusion is also established in Rudin [19; p. 152]. See also Reshetnyak [18; pp. 98-99]. ∎

3. The Change of Variables Formula

Preliminaries

We begin with stating three theorems which are well-known from real analysis. Together with Theorem 2.4 they form the basis of the present approach to the change of variables formula.

Theorem I (Rademacher) *Let $E \subseteq \mathbb{R}^n$ be open. For every* Lipschitz *function $f : E \to \mathbb{R}$ there exist a set $A \subset E$ and measurable functions $g_i : E \setminus A$ $(i = 1, \dots, n)$, such that*

$$\lambda_n(A) = 0,$$

$$f(x+h) = f(x) + \sum_{i=1}^{n} g_i(x)h_i + \sigma(x;h) \quad \forall\, x \in E \setminus A, \quad \forall h \in \mathbb{R}^n,\ (x+h) \in E,$$

where

$$\lim_{h\to 0} \frac{\sigma(x;h)}{|h|} = 0 \qquad \forall\, x \in E \setminus A.$$

The functions g_i are the partial derivatives of f with respect to x_i in $E \setminus A$; clearly,

$$|g_i(x)| \leq L \qquad \forall\, x \in E \setminus A,$$

where L is a Lipschitz constant for f $(i = 1, \dots, n)$.

Theorem II (Lebesgue) *Let $E \subseteq \mathbb{R}^n$ be open. For every locally integrable function $f : E \to \mathbb{R}$ there holds*

$$\lim_{r\to 0} \frac{1}{\lambda_n(B_r)} \int\limits_{B_r(x)} f(y)d\lambda_n = f(x) \quad \textit{for a. e. } x \in E. \tag{3.1}$$

Those $x \in E$ for which (3.1) holds, are called LEBESGUE **points of** f.

Let $E \subseteq \mathbb{R}^n$ be a (fixed) measurable set. Define

$$\mathcal{A} := \{F \subseteq E \mid F \text{ measurable }\}.$$

The family of sets $\mathcal{A}$ is a σ-algebra. We note a special case of the RADON-NIDODYM theorem which is suitable for our purposes.

Theorem III *Let $\mu : \mathcal{A} \to [0, +\infty]$ be a measure. Assume*

1) $A \in \mathcal{A}, \lambda_n(A) = 0 \quad \Rightarrow \quad \mu(A) = 0$;

2) there exist $K_i \in \mathcal{A}$ $(i \in \mathbb{N})$ such that:

$$K_i \subset K_{i+1} \subset E, \quad \mu(K_i) < +\infty \quad (i \in \mathbb{N}), \quad E = \bigcup_{i=1}^{\infty} K_i.$$

Then there exists a measurable function $D : E \to [0, +\infty]$ such that

$$\mu(F) = \int_F D(x) d\lambda_n \qquad \forall\, F \in \mathcal{A}.$$

The function D is called the RADON-NIKODYM **derivative of μ with respect to λ_n** and denoted by $\dfrac{d\mu}{d\lambda_n}$:

$$\frac{d\mu}{d\lambda_n} = D.$$

■

Change of Variables: Special Case

The following result forms the basis for the proof of the Change of Variables Formula below.

3.1 Theorem *Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be injective and locally* LIPSCHITZ. *Then*

$$\lambda_n(\phi(F)) = \int_F |\det \phi'(x)| d\lambda_n \tag{3.2}$$

for all measurable subsets $F \subseteq E$.

By Theorem 2.3, $\phi(F)$ is measurable whenever $F \subseteq E$ is measurable.

Proof We divide the proof into three steps.

Step 1: Application of Theorem III Define

$$\begin{aligned} \mathcal{A} &:= \Big\{ F \subseteq E \;\Big|\; F \text{ measurable} \Big\}, \\ \mu(F) &:= \lambda_n(\phi(F)), \quad F \in \mathcal{A}. \end{aligned}$$

Clearly, $\mu(F) \geq 0$ for all $F \in \mathcal{A}$. Let $F = \bigcup_{i=1}^{\infty} F_i$ (disjoint union), $F_i \in \mathcal{A}$ ($i \in \mathbb{N}$). The mapping ϕ being injective, we have

$$\phi(F) = \bigcup_{i=1}^{\infty} \phi(F_i) \quad \text{disjoint.}$$

It follows

$$\mu(F) = \lambda_n(\phi(F)) = \sum_{i=1}^{\infty} \lambda_n(\phi(F_i)) = \sum_{i=1}^{\infty} \mu(F_i).$$

Thus, μ is a measure on the σ-algeba $\mathcal{A}$.

We verify conditions 1) and 2) of Theorem III. First, let $A \in \mathcal{A}$, $\lambda_n(A) = 0$. Since ϕ satisfies the (N)-condition, we obtain $\lambda_n(\phi(A)) = 0$ [5)] Second, let K_i ($i \in \mathbb{N}$) be compact subsets of E such that

$$K_i \subset K_{i+1} \; (i \in \mathbb{N}), \quad E = \bigcup_{i=1}^{\infty} K_i.$$

The continuity of ϕ implies the compactness of $\phi(K_i)$. Hence

$$\mu(K_i) = \lambda_n(\phi(K_i)) < +\infty \quad \forall i \in \mathbb{N}.$$

We now apply Theorem III to obtain a measurable function $D : E \to [0, +\infty]$ such that

$$\mu(F) = \int_F D(x) d\lambda_n \quad \forall F \in \mathcal{A}. \tag{3.3}$$

The function D is locally integrable in E. Indeed, for any compact set $K \subset E$ there holds

$$\int_K D(x) d\lambda_n = \mu(K) = \lambda_n(\phi(K)) < +\infty.$$

Step 2: Proof of $D = |\det \phi'|$. Define

[5)] Combine Remark 1.2 and theorem 2.3, or look at the proof of Theorem 2.3.

$$
\begin{aligned}
E_1 &:= \Big\{ x \in E \;\Big|\; \phi \text{ is differentiable at } x \Big\}, \\
E_2 &:= \Big\{ x \in E \;\Big|\; \lim_{r \to 0} \frac{1}{\lambda_n(B_r)} \int\limits_{B_r(x)} D(y) d\lambda_n = D(x) \Big\}.
\end{aligned}
$$

Then

$$\lambda_n(E \setminus E_1) = 0 \qquad \text{(Theorem I)} \ ,$$

$$\lambda_n(E \setminus E_2) = 0 \qquad \text{(Theorem II)} \ .$$

Now, from Theorem 2.4 and (3.3) it follows that

$$\lim_{r \to 0} \frac{1}{\lambda_n(B_r)} \int\limits_{B_r(x)} D(y) d\lambda_n = \lim_{r \to 0} \frac{\lambda_n(\phi(B_r(x)))}{\lambda_n(B_r)} = |\det \phi'(x)|$$

for all $x \in E_1$. Thus

$$D(x) = |\det \phi'(x)| \quad \forall x \in E_1 \cap E_2.$$

Step 3: Proof of (3.2) Let $F \subseteq E$ be any measurable set. We have

$$
\begin{aligned}
F &= \Big(F \cap (E_1 \cap E_2) \Big) \cup \Big(F \setminus (F \cap (E_1 \cap E_2)) \Big), \\
F &\setminus \Big(F \cap (E_1 \cap E_2) \Big) \subseteq E \setminus (E_1 \cap E_2).
\end{aligned}
$$

Clearly, $\lambda_n\Big(F \setminus (F \cap (E_1 \cap E_2))\Big) = 0$. Finally, with $D = |\det \phi'|$ at hand, (3.3) gives

$$
\begin{aligned}
\lambda_n(\phi(F)) &= \int\limits_{F \cap (E_1 \cap E_2)} D(x) d\lambda_n = \int\limits_{F \cap (E_1 \cap E_2)} |\det \phi'(x)| d\lambda_n \\
&= \int\limits_F |\det \phi'(x)| d\lambda_n.
\end{aligned}
$$

■

Change of Variables: General Case

From Theorem 3.1 we now derive

3.2 Theorem (**Change of Variables Formula**) *Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be injective and locally* LIPSCHITZ.

1. *Let $u : E \to [0, +\infty]$ be measurable. Then $u \circ \phi^{-1}$ is measurable, and*

$$\int_{\phi(F)} u(\phi^{-1}(y))d\lambda_n = \int_F u(x)|\det \phi'(x)|d\lambda_n \tag{3.4}$$

for all measurable subsets $F \subseteq E$.

2. *Let $v : E \to \bar{\mathbb{R}}$ be measurable. Then $v \circ \phi^{-1}$ is integrable over $\phi(E)$ if and only if $v(\cdot)|\det \phi'(\cdot)|$ is integrable over E. In either case,*

$$\int_{\phi(F)} v(\phi^{-1}(y))d\lambda_n = \int_F v(x)|\det \phi'(x)|d\lambda_n \tag{3.5}$$

for all measurable subsets $F \subseteq E$.

Proof $\boxed{\mathbf{1}}$ Define $E_1 := \phi(E)$, $\phi_1 := \phi^{-1}$. By Theorem 1.3/1, E_1 is measurable. Next, we have

1) $\phi_1(E_1)$ is measurable,

2) ϕ_1 is injective,

3) $\phi_1^{-1} : \phi_1(E_1) \to E_1$ is locally LIPSCHITZ.

Then from Theorem 1.3/2 it follows that

$$u \circ \phi_1 : E_1 \to [0, +\infty]$$

is measurable.

We prove (3.4) for $F = E$. Given any measurable function $u : E \to [0, +\infty]$, there exist functions $u_k : E \to [0, +\infty]$ $(k \in \mathbb{N})$ such that

$$u_k(x) = \sum_{l=1}^{m_k} a_l^{(k)} \chi_{E_l^{(k)}}(x), \quad x \in E,$$

where $a_l^{(k)} \in [0, +\infty[$, $E_l^{(k)} \subset E$ measurable $(l = 1, \ldots, m_k)$ and $E = \bigcup_{l=1}^{m_k} E_l^{(k)}$ disjoint, and

$$u_k(x) \le u_{k+1}(x) \le \ldots \le u(x) \quad (k \in \mathbb{N}), \quad \lim_{k\to\infty} u_k(x) = u(x) \quad \forall x \in E.$$

Observing that $\chi_F \circ \phi^{-1} = \chi_{\phi(F)}$ for any $F \subseteq E$, we find

$$\begin{aligned}
\int\limits_{\phi(E)} u_k\Big(\phi^{-1}(y)\Big) d\lambda_n &= \sum_{l=1}^{m_k} a_l^{(k)} \int\limits_{\phi(E)} \chi_{E_l^{(k)}}\Big(\phi^{-1}(y)\Big) d\lambda_n \\
&= \sum_{l=1}^{m_k} a_l^{(k)} \int\limits_{\phi(E)} \chi_{\phi(E_l^{(k)})}(y) d\lambda_n \\
&= \sum_{l=1}^{m_k} a_l^{(k)} \lambda_n\Big(\phi(E_l^{(k)})\Big) \\
&= \sum_{l=1}^{m_k} a_l^{(k)} \int\limits_{E_l^{(k)}} |\det \phi'(x)| d\lambda_n \quad \text{[by Theorem 3.1]} \\
&= \int\limits_E u_k(x) |\det \phi'(x)| d\lambda_n \quad (k \in \mathbb{N}).
\end{aligned}$$

Applying the Monotone Convergence Theorem to both sides we obtain

$$\int\limits_{\phi(E)} u\Big(\phi^{-1}(y)\Big) d\lambda_n = \int\limits_E u(x) \mid \det \phi'(x)| d\lambda_n.$$

To prove (3.4) for any measurable subset $F \subseteq E$, we note that the product $u \cdot \chi_F$ is a non-negative measurable function. Again using that $\chi_{\phi(F)} = \chi_F \circ \phi^{-1}$, we obtain

$$\begin{aligned}
\int\limits_{\phi(F)} u\Big(\phi^{-1}(y)\Big) d\lambda_n &= \int\limits_{\phi(E)} u\Big(\phi^{-1}(y)\Big) \chi_{\phi(F)}(y) d\lambda_n \\
&= \int\limits_{\phi(E)} u\Big(\phi^{-1}(y)\Big) \chi_F\Big(\phi^{-1}(y)\Big) d\lambda_n \\
&= \int\limits_E u(x) \chi_F(x) |\det \phi'(x)| d\lambda_n \\
&= \int\limits_F u(x) |\det \phi'(x)| d\lambda_n.
\end{aligned}$$

$\boxed{\mathbf{2}}$ To begin with, we note that the measurability of v is equivalent to the measurability of both v^+ and v^- [6]. Analogously as in part 1, we see that the functions $v \circ \phi^{-1}$ and $v^{\pm} \circ \phi^{-1}$ are measurable.

[6] $t^+ := \max\{t, 0\}$, $t^- := \max\{-t, 0\}$, $t \in \mathbb{R}$.

Applying (3.4) to v^+ and v^- gives

$$(3.5') \qquad \int_{\phi(E)} v^{\pm}\Big(\phi^{-1}(y)\Big) d\lambda_n = \int_E v^{\pm}(x) |\det \phi'(x)| d\lambda_n.$$

We obtain: the functions $v^+ \circ \phi^{-1}$ are integrable over $\phi(E)$ if and only if the function $v^+(\cdot)|\det \phi'(\cdot)|$ and $v^-(\cdot)|\det \phi'(\cdot)|$ are integrable over E. Thus, if either of these conditions is satisfied, (3.5) follows from (3.5'). ■

3.3 Corollary: *Let $E \subseteq \mathbb{R}^n$ be open and let $\phi : E \to \mathbb{R}^n$ be injective and locally* LIPSCHITZ. *Let*

$$F_0 := \{x \in E \mid \det\ \phi'(x) = 0\}.$$

Then

$$\lambda_n(\phi(F_0)) = 0.$$

Proof The function $x \mapsto \det \phi'(x)$ is defined for a. e. $x \in E$ (cf. Theorem I above). It can be extended to a measurable function on all of E. Hence F_0 is measurable.

The claim follows from Theorem 3.2/1. ■

Remarks 1. Our approach to the Change of Variables Formula (Theorem 3.2) is similar to that of ŁOJASIEWICZ [11; pp. 199-201] [7]. The same idea of proof of that theorem is developed in RUDIN [19; pp. 153-155].

Entirely different proofs of Theorem 3.2 which do not make use of the RADON/NIKODYM theorem, can be found in GARIEPY/ZIEMER [5; pp. 326-335] and LEINFELDER/SIMADER [10], SIMADER [21] (ϕ injective and locally bi-LIPSCHITZ).

2. The Change of Variables Formula continues to hold when ϕ is only approximately totally differentiable a. e. in E. Then the BANACH indicatrix of ϕ (= the counting function of ϕ over E) is involved in this formula (cf. HAJŁASZ [8], GIAQUINTA/MODICA/SOUČEK [6; pp. 75-79, 215-216, 219-220]).

3. The statement of Corollary 3.3 is a variant of SARD'S theorem [8]. This theorem is true without the injectivity of the mapping ϕ (cf. e. g. GARIEPY/ZIEMER [5; pp. 210-211] ($n = 1$) and LUKEŠ/MALÝ [12; p. 116]).

The following result has been proved by VARBERG [24] (cf. also the literature quoted therein).

---

[7] Essentially the same approach to this theorem within the realm of C^1-mappings is presented in GÜNTHER, P. ; BEYER, K.; GOTTWALD, S.; WÜNSCH, V.: **Grundkurs Analysis, Teil 3**. BSB B.G. Teubner Verlagsgesellschaft, Leipzig 1973; pp. 96-99.

[8] SARD, A.: *The measure of the critical values of differentiable maps.* Bull. Amer. Math. Soc. 48 (1942), 883-890.

Let $E \subseteq \mathbb{R}^n$ and let $\phi : E \to \mathbb{R}^n$ be a mapping. Let $F \subseteq E$ be any measurable subset where ϕ is differentiable. Then the set $\phi(F)$ is measurable and

$$\lambda_n(\phi(F)) \leq \int_F |\det\phi'(x)\ |\ d\lambda_n.$$

■

This result has been established for locally LIPSCHITZ mappings in CLAESSON/HÖRMANDER [1; p. 60].

4. A presentation of the change of voriables formula within the framework of area and coarea formulas can be found in EVANS/GARIEPY [3]. ■

4. Transformation of L^p Functions

Throughout this section, let $1 \leq p < +\infty$.

4.1 Theorem
1. *Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be injective and locally* LIPSCHITZ. *Then, for every $u \in L^p(E)$,*

$$u \circ \phi^{-1} : \phi(E) \to \bar{\mathbb{R}} \quad \text{is measurable,}$$

$$\|u \circ \phi^{-1}\|_{L^p(\phi(E))} \quad \leq \quad \Big(\operatorname*{ess\,sup}_E |\det \phi'| \Big)^{\frac{1}{p}} \|u\|_{L^p(E)}.$$

2. *Let $E \subseteq \mathbb{R}^n$. Suppose that $\phi : E \to \mathbb{R}^n$ satisfies the following conditions:*

1) $\phi(E)$ is open,

2) ϕ is injective,

3) ϕ^{-1} is locally LIPSCHITZ [9].

Then, for every $v \in L^p(\phi(E))$,

$$v \circ \phi : E \to \bar{\mathbb{R}} \quad \textit{is measurable},$$

$$\|v \circ \phi\|_{L^p(E)} \quad \leq \quad \Big(\operatorname*{ess\,sup}_{\phi(E)} |\det (\phi^{-1})'| \Big)^{\frac{1}{p}} \|v\|_{L^p(\phi(E))}$$

Proof $\boxed{1}$ Let $u \in L^p(E)$. By Theorem 3.2/1, the functions $u \circ \phi^{-1}$ and $|u \circ \phi^{-1}|^p$ are measurable. From (3.4) it follows that

[9] From 1), 2), 3) it follows that E is BOREL.

$$\begin{aligned}\int\limits_{\phi(E)} |u\Big(\phi^{-1}(y)\Big)|^p d\lambda_n &= \int\limits_E |u(x)|^p |\det \phi'(x)| d\lambda_n \\ &\le \operatorname*{ess\,sup}_E |\det \phi'| \int\limits_E |u(x)|^p d\lambda_n.\end{aligned}$$

Whence the claim.

$\boxed{\mathbf{2}}$ Define

$$E_1 := \phi(E),\ \phi_1 := \phi^{-1}.$$

Then E_1 is open, and ϕ_1 is injective and locally LIPSCHITZ.

Let $v \in L^p(E_1)$. By **1**,

$$v \circ \phi_1^{-1} : \phi_1(E_1) \to \bar{\mathbb{R}} \text{ is measurable,}$$

$$\|v \circ \phi_1^{-1}\|_{L^p(\phi_1(E_1))} \le \Big(\operatorname*{ess\,sup}_{E_1} |\det \phi_1'|\Big)^{\frac{1}{p}} \|v\|_{L^p(E_1)}.$$

The second part of Theorem 4.1 is proved. ∎

4.2 Corollary *Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be bi-*LIPSCHITZ*. Then:*

$$u \in L^p(\phi(E)) \iff u \circ \phi \in L^p(E);$$

there holds

$$c_1 \|u\|_{L^p(\phi(E))} \le \|u \circ \phi\|_{L^p(E)} \le c_2 \|u\|_{L^p(\phi(E))},$$

where

$$\begin{aligned} c_1 &= \Big(\operatorname*{ess\,sup}_E |\det \phi'|\Big)^{\frac{1}{p}} \\ c_2 &= \Big(\operatorname*{ess\,sup}_E \frac{1}{|\det \phi'|}\Big)^{\frac{1}{p}} \end{aligned}$$

Remarks 1. The following conditions on E and ϕ are sufficient for $\phi(E)$ to be open.

Let $E \subseteq \mathbb{R}^n$ be open. Let $\phi : E \to \mathbb{R}^n$ be injective and continuous. Then $\phi(E)$ is open.

This statement is the well-known BROUWER **Open Mapping Theorem**. Proofs of this theorem can be found in DEIMLING [2; p. 23], GREENBERG/HARPER [7; p. 110] and

SIMADER [21] (ϕ injective and locally LIPSCHITZ).

2. *Transformation of L^p-functions* The following result is a variant of Theorem 4.1/2.

Let $E, F \subseteq \mathbb{R}^n$ be open sets. Let $\phi : E \to F$ be a bijective mapping such that

1) ϕ is continuous on E;

2) $|\phi^{-1}(y) - \phi^{-1}(\bar{y})|_\infty \le L_1|y - \bar{y}|_\infty \quad \forall y, \bar{y} \in F$ (L_1 =const).

Then, for every $v \in L^p(F)$ $(1 \le p < +\infty)$,

$$v \circ \phi : E \to \bar{\mathbb{R}} \quad \textit{is measurable},$$

$$\|v \circ \phi\|_{L^p(E)} \le \left(\frac{(2L_1)^n}{\lambda_n(B_1)} \right)^{1/p} \|v\|_{L^p(F)}.$$

This result has been proved in NEČAS [15; pp. 65-66] by using the mollification v_ρ of v, estimating the RIEMANN sums for the integral $\int\limits_E |v_\rho \circ \phi|^p dx$ and then carrying out the passage to limit $\rho \to 0$. ■

Appendix: More about LIPSCHITZ Mappings

A.1 Extension of LIPSCHITZ Mappings

Let $E \subset \mathbb{R}^m$ be a set.

1.1 Let $\|\cdot\|$ be any norm on $\mathbb{R}^m$. Let $\phi : E \to \mathbb{R}$ be LIPSCHITZ, i.e.

$$|\phi(x) - \phi(y)| \le L\|x - y\| \quad \forall x, y \in E \quad (L = \text{const}).$$

Define

$$\tilde{\phi}(x) := \sup_{\xi \in E} (\phi(\xi) - L\|x - \xi\|), \quad x \in \mathbb{R}^m.$$

Then

$$\tilde{\phi}(x) = \phi(x) \quad \forall x \in E, |\tilde{\phi}(x) - \tilde{\phi}(x)| \le L\|x - y\| \quad \forall x, y \in \mathbb{R}^m.$$

Indeed, the first property of $\tilde{\phi}$ follows immediately from its definition. To establish the second one, let $x, y \in \mathbb{R}^m$. For any $\xi \in E$,

$$
\begin{aligned}
\phi(\xi) - L\|x-\xi\| &= (\phi(\xi) - L\|y-\xi\|) + L\|y-\xi\| - L\|x-\xi\| \\
&\le \tilde{\phi}(y) + L\|x-y\|.
\end{aligned}
$$

Thus

$$\tilde{\phi}(x) \le \tilde{\phi}(y) + L\|x-y\|.$$

Interchanging the role of x and y, the second property of $\tilde{\phi}$ is easily seen.

Hence, without any loss of generality, **real-valued** LIPSCHITZ **mappings can be assumed to be defined on the whole $\mathbb{R}^m$.**

Next, let $\phi : \mathbb{R}^m \to \mathbb{R}$ be LIPSCHITZ with constant L. The mapping ϕ can be "‘cut off"' as follows:

$$\bar{\phi}(x) := \begin{cases} -L & \text{if } \phi(x) \le L, \\ \phi(x) & \text{if } -L < \phi(x) < L, \\ L & \text{if } \phi(x) \ge L. \end{cases}$$

Then

$$|\bar{\phi}(x)| \le L \quad \forall x \in \mathbb{R}^m, \; |\bar{\phi}(x) - \bar{\phi}(y)| \le L\|x-y\| \quad \forall x, y \in \mathbb{R}^m.$$

■

1.2 Let both $\mathbb{R}^m$ and $\mathbb{R}^n$ $(n > 1)$ be furnished with the norm $|\cdot|_\infty$ [10]. Let $\phi : E \to \mathbb{R}^n$ be LIPSCHITZ, i.e.

$$|\phi(x) - \phi(y)|_\infty \le L|x-y|_\infty \quad \forall x, y \in E \quad (L = \text{const} < +\infty).$$

Writing $\phi = (\phi_1, \dots, \phi_n)$, it follows that

$$|\phi_i(x) - \phi_i(y)| \le L|x-y|_\infty \quad \forall x, y \in E \quad (i = 1, \dots, n).$$

Let $\tilde{\phi}_i : \mathbb{R}^m \to \mathbb{R} \quad (i = 1, \dots, n)$ denote the extension of ϕ_i according to 1.1. We obtain a mapping $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ with the following properties: $\tilde{\phi}(x) = \phi(x)$ for all $x \in E$, and

$$|\tilde{\phi}(x) - \tilde{\phi}(y)|_\infty = \max\{|\tilde{\phi}_1(x) - \tilde{\phi}_1(y)|, \dots, |\tilde{\phi}_n(x) - \tilde{\phi}_n(y)|\} \le L|x-y|_\infty$$

for all $x, y \in \mathbb{R}^m$.

1.3 Let now both $\mathbb{R}^m$ and $\mathbb{R}^n$ $(n > 1)$ be furnished with the (EUCLIDean) norm $|\cdot|$. Let $\phi : E \to \mathbb{R}^n$ be LIPSCHITZ with constant L. Again writing $\phi = (\phi_1, \dots, \phi_n)$ we obtain

[10] Recall that $|x|_\infty := \max\{|x_1|, \dots, |x_k|\}$, $x = (x_1, \dots, x_k) \in \mathbb{R}^k$

$$|\phi_i(x) - \phi_i(y)| \le |\phi(x) - \phi(y)| \le L|x - y| \quad \forall\, x, y \in E \quad (i = 1, \ldots, n).$$

By 1.1, each component ϕ_i can be extended to a LIPSCHITZ mapping $\tilde{\phi}_i : \mathbb{R}^m \to \mathbb{R}$ with the same constant L ($i = 1, \ldots, n$). It follows

$$|\tilde{\phi}(x) - \tilde{\phi}(y)| = \left(\sum_{i=1}^{n} (\phi_i(x) - \phi_i(y))^2 \right)^{1/2} \le \sqrt{n} L|x - y|$$

for all $x, y \in \mathbb{R}^m$. ■

We note a sharper extension result. To this end, for a LIPSCHITZ mapping $\phi : E \to \mathbb{R}^n$, define

$$\mathrm{Lip}(\phi) := \sup \left\{ \frac{|\phi(x) - \phi(y)|}{|x - y|} \;\Big|\; x, y \in E,\ x \ne y \right\}.$$

The following result holds.

A1. Theorem (KIRSZBRAUN[11)]) *Let $\phi : E \to \mathbb{R}^n$ be* LIPSCHITZ. *Then there exists a mapping $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\hat{\phi}(x) = \phi(x) \quad \forall x \in E, \quad \mathrm{Lip}(\hat{\phi}) = \mathrm{Lip}(\phi).$$

■

A.2 Equivalent Characterization of the Local LIPSCHITZ Continuity

We have the following

A2. Theorem *Let $E \subseteq \mathbb{R}^m$ be open. For a mapping $\phi : E \to \mathbb{R}^n$ the following two statements are equivalent:*

1° *ϕ is locally* LIPSCHITZ;

[11)] See KIRSZBRAUN, M. D.: *Über die zusammenziehende und Lipschitzsche Transformationen.* - Fund. Math. 22(1934), 77-108. This result is also proved in FEDERER, H.: **Geometric measure theory.** - Springer-Verlag, Berlin 1969 (p. 201). We note that the example on p. 202 of this book shows that KIRSZBRAUN'S theorem fails when $\mathbb{R}^m$ is furnished with the norm $|\cdot|_\infty$, while $\mathbb{R}^n$ is furnished with the EUCLIDean norm $|\cdot|$.

An extension theorem for real-valued, uniformly continuous functions with a rather general modulus of continuity is proved in DIBENEDETTO, E.: **Real analysis**. Birkhäuser, Boston, Basel 2002 (pp. 197-198).

2° *for every $x \in E$ there exists a ball $B_r(x)$ such that*

$$\overline{B_r(x)} \subset E, \qquad \sup_{\substack{y,y' \in B_r(x) \\ y \neq y'}} \frac{|\phi(y) - \phi(y')|}{|y - y'|} < +\infty.$$

■

A.3 Lipschitz Continuity and Differentiability a. e.

Theorem I (Chap. 3) has been proved by Rademacher [17] under conditions on ϕ which are slightly weaker than the Lipschitz continuity [12)].

There are several different proofs of Rademacher's theorem. Krushkov [9] and Morrey [14; p. 65] proved this theorem by using the notion of weak derivative and the mollification of integrable functions. By using only techniques from Lebesgue measure and integration theory, Rademacher's theorem has been proved by Saint-Pierre [20] and Nekvinda/Zajíček [16] (cf. also Zajíček [25]).

Rademacher's theorem has been sharpened by Stepanov [22]:

Let $E \subseteq \mathbb{R}^m$ be open. Let the mapping $\phi : E \to \mathbb{R}$ satisfy

$$E_0 := \left\{ x \in E \;\middle|\; \limsup_{y \in E, y \to x} \frac{|\phi(y) - \phi(x)|}{|y - x|} < +\infty \right\} \neq \emptyset.$$

Then ϕ is differentiable a. e. in E_0.

Proofs of Stepanov's theorem may be also found in Łojasiewicz [11; pp. 208-209] and Malý [13]. The following weaker version of this theorem has been proved by Väisälä (*Lectures on n-dimensional quasiconformal mappings.* Lecture Notes Math. 229, Springer-Verlag 1971; pp. 97-99):

Let $E \subseteq \mathbb{R}^m$ be open. Let $\phi : E \to \mathbb{R}$. Assume

1) *ϕ is continuous on E;*

2) *the partial derivatives $\dfrac{\partial \phi}{\partial x_i}$ exist a. e. on E ($i = 1, \ldots, m$);*

3) $\displaystyle\limsup_{y \in E, y \to x} \frac{|\phi(y) - \phi(x)|}{|y - x|} < +\infty$ *for a. e. $x \in E$.*

Then ϕ is differentiable a. e. on E. ■

12) In [17; p. 340], Rademacher remarked that C. Carathéodory obtained a certain version of this differentiability result. Cf. Carathéodory, C.: **Vorlesungen über reelle Funktionen**. B. G. Teubner, Leipzig, Berlin 1917; 2. Aufl. 1927; reprinted: Chelsea Publ. Comp., New York 1968.

Remark Let $E \subseteq \mathbb{R}^n$ be open. Let $u \in C^1(E)$. Given any $x \in E$, we fix a cube $Q_r(x)$ such that $\overline{Q_r(x)} \subset E$. Then, for all $y, y' \in Q_r(x)$,

$$\begin{aligned} |u(y) - u(y')| &= \Big| \sum_{i=1}^{n} \int\limits_0^1 D_i u(y' + t(y - y'))dt(y_i - y_i') \Big| \\ &\leq \max_{z \in \overline{Q_r(x)}} |\nabla u(z)||y - y'|. \end{aligned}$$

Thus, u is locally LIPSCHITZ in E (cf. Theorem A.2).

We finally note that a C^1-function with uniformly bounded gradient can fail to be (globally) LIPSCHITZ. Indeed, there exist a domain $E \subset \mathbb{R}^2$ and a function $u \in C^1(E)$ such that

- $\sup\limits_{x \in E} |\nabla u(x)| < +\infty$,
- *there exist* $x_k, \hat{x}_k \in E$ $(k \in \mathbb{N})$ *such that*:

$$u(x_k) - u(\hat{x}_k) = \text{const } \neq 0 \;\; \forall\, k \in \mathbb{N},\; |x_k - \hat{x}_k| \to 0 \textit{ as } k \to \infty.$$

■

Whence

$$\frac{|u(x_k) - u(\hat{x}_k)|}{|x_k - \hat{x}_k|} \to +\infty \;\text{ as }\; k \to \infty.$$

References [13)]

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