ERROR REDUCTION AND CONVERGENCE FOR AN ADAPTIVE MIXED FINITE ELEMENT METHOD

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ABSTRACT. An adaptive mixed finite element method (AMFEM) is designed to guarantee an error reduction, also known as saturation property: After each refinement step, the error for the fine mesh is strictly smaller than the error for the coarse mesh up to oscillation terms. This error reduction property is established here for the Raviart-Thomas finite element method with a reduction factor $\rho < 1$ uniformly for the L^2 norm of the flux errors. Our result allows for linear convergence of a proper adaptive mixed finite element algorithm with respect to the number of refinement levels. The adaptive algorithm does surprisingly not require any particular mesh design unlike the conforming finite element method. The new arguments are a discrete local efficiency and a quasi-orthogonality estimate. The proof does neither rely on duality nor on regularity.

1. INTRODUCTION

An adaptive finite element method consists of successive loops of the following sequence

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}. \tag{1.1}$$

The a posteriori error control in the step ESTIMATE has been developed over the last decades (cf. [1, 3, 6, 12, 17] and the references therein). The convergence analysis of the full algorithm (1.1), however, is restricted to the conforming finite element method [15, 16].

This paper investigates convergence properties of such a loop for the mixed finite element method (MFEM) in a 2D model Poisson problem

$$f + \Delta u = 0 \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega. \tag{1.2}$$

Given a (coarse) mesh $\mathcal{T}_H$, a shape-regular triangulation of Ω into triangles, p_H and u_H approximate the exact flux $p := \nabla u \in H(\mathrm{div}, \Omega)$ and the exact displacement field $u \in H^1_0(\Omega)$ of (1.2). In step SOLVE one computes $(p_H, u_H) \in RT_0(\mathcal{T}_H) \times P_0(\mathcal{T}_H)$ that satisfies the *discrete problem* [$(\bullet, \bullet)_{L^2}$ abbreviates the L^2 scalar product]

$$\begin{aligned} (p_H, q_H)_{L^2(\Omega)} + (u_H, \mathrm{div}\, q_H)_{L^2(\Omega)} &= 0 \quad \text{for all } q_H \in RT_0(\mathcal{T}_H), \\ (\mathrm{div}\, p_H, v_H)_{L^2(\Omega)} &= -(f, v_H)_{L^2(\Omega)} \quad \text{for all } v_H \in P_0(\mathcal{T}_H). \end{aligned} \tag{1.3}$$

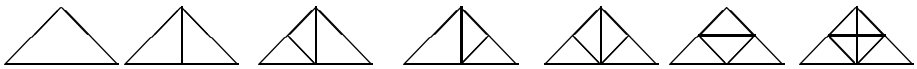

FIGURE 1. Possible refinements of one triangle T in the step REFINE.

Details on the lowest-order Raviart-Thomas finite element space $RT_0(\mathcal{T}_H)$ [8] can be found below in Section 2; $P_0(\mathcal{T}_H)$ denotes the piecewise constants. MATLAB implementations and documentations of the step SOLVE are provided in [5]. In this paper, for the ease of the discussion, the step ESTIMATE is the postprocessing to compute the *residual-based explicit error estimator* [2, 9, 18]

$$\eta_H := \Big(\sum_{E\in\mathcal{E}_H} \eta_E^2\Big)^{1/2} \quad \text{with } \eta_E^2 := h_E \|[p_H]_E\|^2_{L^2(E)}. \tag{1.4}$$

Here and throughout, $[p_h]$ denotes the jump $[p_H] := p_H|_{T_+} - p_H|_{T_-}$ of the discrete flux over an interior edge $E := T_+ \cap T_-$ of length $h_E := \mathrm{diam}(E)$ shared by the two neighboring (closed) triangles $T_\pm \in \mathcal{T}_H$. Furthermore let $f_{\omega_E} := |\omega_E|^{-1}\int_{\omega_E} f(x)\,dx$ denote the integral mean of f over the patch $\omega_E := \mathrm{int}(T_+ \cup T_-)$ of area $|\omega_E| = |T_+| + |T_-|$ and let $\mathcal{E}_H$ denote the set of all interior edges in $\mathcal{T}_H$.

The bulk criterion in the step MARK was introduced and analyzed in [7, 11, 15] for displacement-based AFEMs. Here, it leads to a selection of a subset $\mathcal{M}$ of edges $\mathcal{E}_H$ such that

$$\theta\eta_H^2 \le \sum_{E\in\mathcal{M}} \eta_E^2 \tag{1.5}$$

for some universal constant $0 < \theta < 1$. It came much as a surprise to the authors that the step REFINE does not need any further specification or restriction. It suffices when the output of REFINE satisfies that, for each marked edge $E \in \mathcal{M}$, its midpoint $\mathrm{mid}(E)$ is a new node in the new triangulation $\mathcal{T}_h$.

Typical refinements of one triangle $T \in \mathcal{T}_H$ are displayed in Figure 1.

We further set $h_T := \mathrm{diam}(T)$ and refer to $\|Hf_H\|_{L^2(\Omega)}$ as the first-order term given by

$$\|Hf_H\|_{L^2(\Omega)} := \Big(\sum_{T\in\mathcal{T}_H} h_T^2|T|^{-1}\Big|\int_T f(x)dx\Big|^2\Big)^{1/2} \tag{1.6}$$

while the data oscillations read

$$\operatorname{osc}_H := \Big(\sum_{E\in\mathcal{E}_H} h_E^2 \|f - f_{\omega_E}\|^2_{L^2(\omega_E)}\Big)^{1/2}. \tag{1.7}$$

It is the milestone of this paper to prove the following error reduction property (1.8).

Theorem 1.1 (error reduction property)**.** *Let p_h and p_H be the MFEM flux approximations to p with respect to $\mathcal{T}_h$ and $\mathcal{T}_H$. Then, there exist positive constants $\rho < 1$ and C depending only on θ and on the shape regularity of $\mathcal{T}_h$ and $\mathcal{T}_H$ such that*

$$\|p-p_h\|^2_{L^2(\Omega)} \le \rho\|p-p_H\|^2_{L^2(\Omega)} + C\big(\|H\,f_H\|_{L^2(\Omega)} + \operatorname{osc}_H\big)\operatorname{osc}_H \quad . \tag{1.8}$$

The remaining part of this paper is organized as follows. Section 2 discusses several aspects of AMFEM as well as particularities and generalizations of our analysis. Section 3 presents the necessary details on the notation. The key ingredients of the proof are the strict discrete local efficiency, the quasi-orthogonality, and an estimate for the fluxes, of Section 4 and 5. The proof of the error reduction property (1.8) concludes the paper in Section 6.

2. Comments

Some remarks are given before the subsequent sections are devoted to the technical details of the proof of Theorem 1.1.

2.1. **Data oscillations.** For $f \in H^1(\Omega)$, we note that the data oscillation (1.7) is of quadratic order and so of higher order when compared to the first-order errors $\|p-p_H\|_{H(\mathrm{div})}$ or $\|u-u_H\|_{L^2(\Omega)}$ or the first-order data term $\|H\,f_H\|_{L^2(\Omega)}$.

Hence, Theorem 1.1 asserts that the error on the fine mesh is bounded by a factor $\rho^{1/2}$ times the error on the coarse mesh plus higher-order terms.

We also point out that the oscillations (1.7) of f are patch-oriented while those in the reliability and efficiency estimate of Theorem 3.2 below are element-oriented (and so possibly smaller than (1.7)).

It is an important property of the data oscillation that the mesh-sizes enter explicitly. Given $0 < \vartheta < 1$ and a coarse mesh $\mathcal{T}_H$, it is therefore easy to design a fine mesh $\mathcal{T}_h$ with oscillations $\operatorname{osc}_h \le \vartheta \operatorname{osc}_H$ where osc_h and osc_H denote the data oscillation of the fine and coarse mesh, respectively. The same remark applies to $\|H\,f_H\|_{L^2(\Omega)}$.

2.2. **A convergent AMFEM.** In order to guarantee linear convergence in terms of the refinement levels, suppose that (1.1) is employed successively. At the refinement level j, there is an MFEM solution p_j with error $e_j := \|p - p_j\|_{L^2(\Omega)}$ with respect to a mesh $\mathcal{T}_j$ and and a data oscillation osc_j such that (1.8) reads

$$e_{j+1}^2 \leq \rho\, e_j^2 + C\, d_j \qquad \text{for } j = 0, 1, 2, \ldots \tag{2.1}$$

where d_j abbreviates the data term $(\|H_j f_{H_j}\|_{L^2(\Omega)} + \mathrm{osc}_j)\,\mathrm{osc}_j$ with respect to $\mathcal{T}_j$. Moreover, suppose that MARK provides (1.5) plus (possibly) additional refinements to guarantee

$$d_{j+1} \leq \varrho\, d_j \qquad \text{for } j = 0, 1, 2, \ldots \tag{2.2}$$

with some universal constant $0 < \varrho < 1$ (this is always possible as indicated at the end of the previous subsection).

Mathematical induction proves that (2.1)-(2.2) imply

$$e_j^2 \leq \rho^j\, e_0^2 + C d_0 \sum_{k=0}^{j-1} \rho^k\, \varrho^{j-1-k} \quad \text{and} \quad d_j \leq d_0\, \varrho^j$$

and so R-linear convergence (with any reduction factor between $\max\{\rho, \varrho\}^{1/2}$ and 1):

$$e_j^2 \leq \rho^j\, e_0^2 + C d_0\, j \max\{\rho, \varrho\}^{j-1} \qquad \text{for } j = 1, 2, \ldots \tag{2.3}$$

2.3. **Numerical Experiments.** Numerical experiments throughout the literature are frequently based on the element-oriented maximum criterion in the step MARK, i.e., one marks an element T if the estimator η_T associated with T satisfies $\mathrm{Tol} \leq \eta_T$ and Tol is θ times the largest of such contributions. In the context of AMFEM, data oscillations have not been involved so far. We refer to [5] for algorithmic details and MATLAB routines and to [2, 10, 18, 13] for empirical examples.

It is the authors' overall impression that the AMFEM is very robust in changing algorithmic details in practice. The numerical experiments in [15, 16] with a realization of (2.1)-(2.2) from the previous subsection for conforming AFEM anticipate that the new algorithms perform as optimal as the frequently employed ones. But there is no mathematical justification for that.

2.4. **Optimal Complexity.** The adaptive algorithm is linear convergent with respect to the number of refinement steps. This does not imply any control of the number of degrees of freedom. Based on additional coarsening steps, there exists an algorithm of optimal complexity for the conforming AFEM [7]. The authors anticipate that their results

carry over to the present situation, because it is the universal coarsening step that yields the control of the degrees of freedom. Numerical wisdom, however, tells that coarsening is not needed in practice leaving an open gap between theory and practice.

2.5. **Generalizations.** The arguments below are illustrated by a simple 2D model example only, but they apply to more general boundary value problems as well. In the presence of Neumann boundary data or for non-constant coefficients, the data oscillations apply to such terms as well. The arguments are not restricted to 2D; for instance, Lemma 3.1 also holds true in 3D [5].

The use of alternative refinement indicators [10, 18] is also possible as long as they are globally reliable and locally controlled by the residual-based estimators.

2.6. **Uzawa Algorithms.** The well-established Uzawa algorithm for the iterative solution of the mixed problem on the continuous level consists of two steps: a Poisson solve and and update formula. The substitute of the Poisson solve by some AFEM allows a perturbation of the convergence on the continuous level [4]. The advantage is that even unstable finite element schemes can be employed. The disadvantage is the possibly slow convergence of the Uzawa algorithm relative to multilevel solver [13].

3. Notation and Preliminaries

Throughout this paper suppose that $\mathcal{T}_H$ and $\mathcal{T}_h$ are two shape regular triangulations of the planar Lipschitz domain Ω with polygonal boundary $\partial\Omega$ into triangles where $\mathcal{T}_h$ is some refinement of $\mathcal{T}_H$ such that the refinement $T|_{\mathcal{T}_h} := \{K \in \mathcal{T}_h : K \subseteq T\}$ of each element T in $\mathcal{T}_H$ is depicted in Figure 1. Moreover, let $p_H \in RT_0(\mathcal{T}_H)$ denote the discrete MFEM solution on the coarse triangulation $\mathcal{T}_H$. A regular triangulation $\mathcal{T}$ in triangles, $d = 2$, is a set of closed triangles T of positive area $|T|$ such that any two distinct triangles T_1 and T_2 are either disjoint $T_1 \cap T_2 = \emptyset$ or share exactly one vertex z, $T_1 \cap T_2 = \{z\}$, or have one edge $E = T_1 \cap T_2$ in common. The set of all edges is denoted by $\mathcal{E}$, the set of nodes is denoted by $\mathcal{N}$. Each edge is associated to a length $h_E := \operatorname{diam}(E)$ and a unit normal and unit tangential vector ν_E and τ_E. The subindices H and h refer to the coarse and fine triangulation $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. The words mesh and triangulation are used as synonyms of each other.

The Raviart-Thomas MFEM space and the piecewise constant space read

$$\begin{aligned} RT_0(\mathcal{T}_H) &:= \{q_H \in H(\mathrm{div},\Omega) : \forall T \in \mathcal{T}_H\ \exists a \in \mathbb{R}^2\ \exists b \in \mathbb{R} \forall x \in T, \\ &\qquad q_H(x) = a + b\,x\}, \\ P_0(\mathcal{T}_H) &:= \{v_H \in L^\infty(\Omega) : \forall T \in \mathcal{T}_H\ \exists a \in \mathbb{R} \forall x \in T,\ v_H(x) = a\}. \end{aligned}$$

[Analogous notation for $\mathcal{T}_h$ is not displayed]. The Crouzeix-Raviart FEM space on $\mathcal{T}_H$ reads

$$\begin{aligned} V_H^N := \{&v_H \in P_1(\mathcal{T}_H) : v_H \text{ continuous at } \mathrm{mid}(E) \text{ for } E \in \mathcal{E}_H \\ &\text{and } v_H(\mathrm{mid}(E)) = 0 \text{ for } E \in \mathcal{E} \text{ with } E \subseteq \partial\Omega\}. \end{aligned}$$

Since $V_H^N \not\subset H^1(\Omega)$, the distributional gradient of $v_h \in V_H^N$ is different from its elementwise gradient $D_H v_H \in P_0(\mathcal{T}_H)^d$.

Let u_H^N denote the *Crouzeix-Raviart* FEM solution of

$$(D_H u_H^N, D_H v_H^N)_{L^2(\Omega)} = (f_H, v_H^N)_{L^2(\Omega)} \quad \text{for all } v_H^N \in V_H^N.$$

The discrete fluxes $p_H^N := D_H u_H^N$ and p_H from (1.3) are related.

Lemma 3.1 ([14, 5])**.** Let $f_{T_\pm} := \int_{T_\pm} f(x)\,dx/|T_\pm|$ and let $x_{T_\pm} := \mathrm{mid}(T_\pm)$ denote the barycenter of $T_\pm$. Then there holds

$$p_H|_{T_\pm}(x) = D_H u_H^N|_{T_\pm} - \frac{1}{2} f_{T_\pm}(x - x_{T_\pm}) \quad \text{for } x \in T_\pm. \quad \square$$

In this context, $f_H \in P_0(\mathcal{T}_H)$ and $f_h \in P_0(\mathcal{T}_h)$ denote the piecewise integral means, e.g., $f_H|_T := f_T := \int_T f(x)dx/|T|$ for $T \in \mathcal{T}_H$.

Theorem 3.2 (reliability and efficiency [2, 9])**.** *With* (1.4) *and* (1.7), *there holds*

$$\eta_H \lesssim \|p - p_H\|_{L^2(\Omega)} \lesssim \eta_H + \mathrm{osc}_H \quad .$$

Here and throughout this paper, $A \lesssim B$ abbreviates $A \le CB$ with a mesh-size independent, generic constant $C > 0$. Finally, $A \approx B$ abbreviates $A \lesssim B \lesssim A$. The paper adopts standard notation for Lebesgue and Sobolev spaces and norms.

4. Discrete Local Efficiency

This section provides the first of two main arguments for error reduction. Unlike for conforming AFEM, there is no request on further restriction in REFINE.

Theorem 4.1 (Strict discrete local efficiency)**.** *Suppose that* $E = \partial T_+ \cap \partial T_- \in \mathcal{E}_H$ *is an edge in* $\mathcal{T}_H$ *[shared by the triangles* $T_+, T_- \in \mathcal{T}_H$*] and bisected in the refinement, i.e.* $E = E_1 \cup E_2 \notin \mathcal{E}_h$ *and* $\mathrm{mid}(E) = E_1 \cap E_2 \in \mathcal{N}_h$ *for two distinct* $E_1, E_2 \in \mathcal{E}_h$*. Then there holds*

$$h_E^{1/2} \|[p_H]\|_{L^2(E)} \lesssim \|p_h - p_H\|_{L^2(\omega_E)} + h_E \|f - f_{\omega_E}\|_{L^2(\omega_E)}.$$

The remaining part of this section is devoted to the proof of Theorem 4.1. Observe that $[p_H] \cdot \nu_E = 0$ for the unit normal vector $\nu_E \perp E$ since $p_H \in H(\mathrm{div}, \Omega)$. Therefore, denoting by $\tau_E \perp \nu_E$ the tangential vector, the jump

$$[p_H] := (p_H|_{T_+} - p_H|_{T_-}) \quad \text{along } E = T_+ \cap T_-$$

(and formally $[p_H] := 0$ along $E \subset \partial\Omega$) satisfies

$$\|[p_H]\|_{L^2(E)} = \|[p_H] \cdot \tau_E\|_{L^2(E)}.$$

Taking into account that $[p_H] \cdot \tau_E$ is an affine function along the edge E, we have

$$([p_H] \cdot \tau_E)(x) = \alpha + \beta \cdot (x - \mathrm{mid}(E)) \quad \text{for all } x \in E$$

with fixed $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^2$.

Lemma 4.2. There holds

$$h_E^{1/2} \|\alpha\|_{L^2(E)} \lesssim \|p_h - p_H\|_{L^2(\omega_E)}.$$

Proof. Let φ_E denote the nodal basis function in the conforming P_1 FEM space with respect to the node $\mathrm{mid}(E)$ and with respect to the fine mesh $\mathcal{T}_h$. Then, $q_h := \mathrm{Curl}\, \varphi_E$ belongs to $P_0(\mathcal{T}_h) \cap H(\mathrm{div}, \Omega)$ with $\mathrm{div}\, q_h \equiv 0$. Since $\alpha = \int_E [p_H] \cdot \tau_E \, ds / h_E$, one deduces

$$\int_E \alpha \varphi_E \, ds = \int_E [p_H] \cdot \tau_E \varphi_E \, ds = (p_H, q_h)_{L^2(\Omega)}$$

with an elementwise integration by parts. Since $q_h = \mathrm{Curl}\, \varphi_E \in RT_0(\mathcal{T}_h)$ is an admissible test function, the discrete MFEM problem with respect to the fine mesh $\mathcal{T}_h$ reduces to

$$(p_h, q_h)_{L^2(\Omega)} = 0.$$

Altogether, one obtains the key identity

$$\alpha \int_E \varphi_E \, ds = (p_H - p_h, q_h)_{L^2(\Omega)}.$$

The shape regularity allows the estimates

$$h_E \lesssim \int_E \varphi_E \, ds \quad \text{and} \quad \|q_h\|_{L^2(\omega_E)} \lesssim 1.$$

The foregoing key identity therefore leads to the assertion:

$$\begin{aligned} h_E\|\alpha\|^2_{L^2(E)} &= h_E^2(\int_E \varphi_E\,ds)^{-2}(\alpha\int_E \varphi_E\,ds)^2 \\ &\lesssim (p_H - p_h, q_h)^2_{L^2(\Omega)} \\ &\le \|q_h\|^2_{L^2(\omega_E)}\|p_h - p_H\|^2_{L^2(\omega_E)} \\ &\lesssim \|p_h - p_H\|^2_{L^2(\omega_E)}. \quad \square \end{aligned}$$

Lemma 4.3. There holds

$$|\beta| \le \frac{1}{2}(|T_+|^{-1} + |T_-|^{-1})^{1/2}\|f - f_{\omega_E}\|_{L^2(\omega_E)}.$$

Proof. The differences of the representation formula of Lemma 3.1 for $x \in E$ lead to

$$\beta = \frac{1}{2}(f_{T_-} - f_{T_+})\,\tau_E \in \mathbb{R}^2.$$

Consider the piecewise constant function

$$g(x) := \begin{cases} -|T_+|^{-1} & \text{for } x \in T_+, \\ +|T_-|^{-1} & \text{for } x \in T_-, \\ 0 & \text{for } x \notin \overline{\omega_E} \end{cases}$$

and notice $\int_{\omega_E} g(x)\,dx = 0$. The definition of the piecewise integral means $f_{T_\pm} := \int_{T_\pm} f(x)\,dx/|T_\pm|$ then implies the identity

$$f_{T_-} - f_{T_+} = (g, f)_{L^2(\omega_E)}.$$

Since $(g, 1)_{L^2(\Omega)} = 0$ and f_{ω_E} is constant on ω_E,

$$f_{T_-} - f_{T_+} = (g, f - f_{\omega_E})_{L^2(\omega_E)}.$$

Cauchy's inequality and $\|g\|^2_{L^2(\omega_E)} = |T_+|^{-1} + |T_-|^{-1}$ conclude the proof:

$$2|\beta| \le (|T_+|^{-1} + |T_-|^{-1})^{1/2}\|f - f_{\omega_E}\|_{L^2(\omega_E)}. \quad \square$$

The proof of Theorem 4.1 immediately follows from Lemma 4.2 and 4.3: Since α and $\beta \cdot (\bullet - \mathrm{mid}(E))$ are $L^2(E)$ orthogonal, there holds

$$\begin{aligned} h_E\|[p_H]\|^2_{L^2(E)} &= h_E\|[p_H]\cdot\tau_E\|^2_{L^2(E)} \\ &= h_E\|\alpha\|^2_{L^2(E)} + h_E\|\beta\cdot(\bullet - \mathrm{mid}(E))\|^2_{L^2(E)} \\ &\lesssim \|p_h - p_H\|^2_{L^2(\omega_E)} + h_E^{-1}\|\bullet - \mathrm{mid}(E))\|^2_{L^2(E)}\|f - f_{\omega_E}\|^2_{L^2(\omega_E)}. \quad \square \end{aligned}$$

5. Quasi-Orthogonality

The second main argument for error reduction is a generalization of the Galerkin orthogonality in the conforming AFEM [11, 15, 16].

Theorem 5.1 (Quasi-orthogonality)**.** *There holds*

$$|(p-p_h, p_H-p_h)_{L^2(\Omega)}| \lesssim \|H(f_h-f_H)\|_{L^2(\Omega)} \times \big(\|p-p_h\|_{L^2(\Omega)} + \|p-p_H\|_{L^2(\Omega)} + \|Hf_H\|\big)_{L^2(\Omega)}.$$

Theorem 5.1 is an immediate consequence of Lemma 5.4 and 5.5 below. Throughout the rest of this section set $p_h^N := D_h u_h^N$ for the *Crouzeix-Raviart* FEM solution u_h^N in V_h^N with respect to $\mathcal{T}_h$.

Lemma 5.2. There holds

$$(p-p_h^N, p_H-p_h)_{L^2(\Omega)} = (u-u_h^N, f_H-f_h)_{L^2(\Omega)}.$$

Proof. Since $p = Du$, $-\operatorname{div} p_H = f_H$, and $-\operatorname{div} p_h = f_h$, the assertion follows from an elementwise integration by parts. The edge contributions vanish indeed: Given any $E \in \mathcal{E}_h$ the resulting boundary term over E reads

$$\int_E [u-u_h^N](p_H-p_h)\cdot\nu_E\, ds.$$

This is zero because of $\int_E [u-u_h^N]\, ds = 0$ by construction of V_h^N and since $p_H\cdot\nu_E$ and $p_h\cdot\nu_E$ are continuous from both sides of E and constant along E. □

Lemma 5.3. There holds

$$|(u-u_h^N, f_H-f_h)_{L^2(\Omega)}| \lesssim \|H(f_h-f_H)\|_{L^2(\Omega)} \times \Big(\|p-p_H^N\|_{L^2(\Omega)} + \|p_h^N-p_H^N\|_{L^2(\Omega)}\Big).$$

Proof. To estimate $(u-u_h^N, f_H-f_h)_{L^2(\Omega)}$ notice that $\int_T (f_H-f_h)\, dx = 0$ for any $T \in \mathcal{T}_H$. Hence, for some $\overline{e}_H^N \in P_0(\mathcal{T}_H)$ with

$$\overline{e}_H^N|_T := \int_T (u(x)-u_H^N(x))\, dx/|T|$$

and $e_H^N := u - u_H^N$ a Poincaré inequality on T shows in total

$$\begin{aligned}|(u-u_H^N, f_H-f_h)_{L^2(\Omega)}| &= |(e_H^N-\overline{e}_H^N, f_H-f_h)_{L^2(\Omega)}| \\ &\le 1/\pi\, \|p-p_H^N\|_{L^2(\Omega)}\|H(f_h-f_H)\|_{L^2(\Omega)}.\end{aligned}$$

The remaining term reads $(u_H^N-u_h^N, f_H-f_h)_{L^2(\Omega)}$ and is analyzed separately for each $T \in \mathcal{T}_H$. In fact, let $V_h^N(T) := \{v_h|_T : v_h \in P_1(\mathcal{T}_h|_T)$

continuous at mid(E) for all $E \in \mathcal{E}_h\}$ and notice $u_H^N - u_h^N \in V_h^N(T)$. Moreover, for any $v_h \in V_h^N(T)$ set

$$\varrho_1(v_h) := \min_{w\in\mathbb{R}} \|v_h - w\|_{L^2(T)} \quad \text{and} \quad \varrho_2(v_h) := h_T \|D_h v_h\|_{L^2(T)}.$$

This defines two semi-norms ϱ_1, ϱ_2 on the finite-dimensional space $V_h^N(T)$. Consequently, $\varrho_1 \approx \varrho_2$. Therein, the equivalence constants are independent of h_T according to a scaling argument (transform to a reference triangle $T_{\rm ref}$ first and notice that there exists a finite number of possible refinements only, compute the constants, and transform back). In particular, for some average $c := \int_T (u_H^N - u_h^N)\, dx/|T|$,

$$\begin{aligned} |(u_h^N - u_H^N, f_h - f_H)_{L^2(T)}| &= |(u_H^N - u_h^N - c, f_H - f_h)_{L^2(T)}| \\ &\le \varrho_1(u_H^N - u_h^N) \|f_h - f_H\|_{L^2(T)} \\ &\lesssim \|D_h(u_H^N - u_h^N)\|_{L^2(T)} \|h_T(f_h - f_H)\|_{L^2(T)}. \end{aligned}$$

The sum over all $T \in \mathcal{T}_H$ shows that

$$|(u_H^N - u_h^N, f_H - f_h)_{L^2(\Omega)}| \lesssim \|p_h^N - p_H^N\|_{L^2(\Omega)}\, \|H(f_h - f_H)\|_{L^2(\Omega)}. \quad \square$$

Lemma 5.4. There holds

$$\begin{aligned} |(p - p_h^N, p_H - p_h)_{L^2(\Omega)}| &\lesssim \|H(f_h - f_H)\|_{L^2(\Omega)} \\ &\quad \times \Big(\|p - p_h\|_{L^2(\Omega)} + \|p - p_H\|_{L^2(\Omega)} + \|H f_H\|_{L^2(\Omega)} + \|h f_h\|_{L^2(\Omega)} \Big). \end{aligned}$$

Proof. The combination of Lemma 5.2—5.3 readily gives

$$\begin{aligned} |(p - p_h^N, p_H - p_h)_{L^2(\Omega)}| &\lesssim \|H(f_h - f_H)\|_{L^2(\Omega)} \\ &\quad \times \Big(\|p - p_H^N\|_{L^2(\Omega)} + \|p_h^N - p_H^N\|_{L^2(\Omega)} \Big). \end{aligned}$$

An immediate consequence of Lemma 3.1 is that

$$\begin{aligned} \Big| \, \|p - p_H^N\|_{L^2(\Omega)} - \|p - p_H\|_{L^2(\Omega)} \, \Big|^2 &\le \sum_{T\in\mathcal{T}_H} |f_T|^2 \| \bullet - x_T\|_{L^2(T)}^2 \\ &\le \|H f_H\|_{L^2(\Omega)}^2. \end{aligned}$$

A similar estimate also holds true with H replaced by h. The combination of those two estimates with a triangular inequality concludes the proof of the lemma. $\square$

Lemma 5.5. There holds

$$|(p_h^N - p_h, p_H - p_h)_{L^2(\Omega)}| \lesssim \|h f_h\|_{L^2(\Omega)}\, \|h(f_h - f_H)\|_{L^2(\Omega)}.$$

Proof. Let $x_H \in P_0(\mathcal{T}_H;\mathbb{R}^2)$ and $x_h \in P_0(\mathcal{T}_h;\mathbb{R}^2)$ denote the piecewise center of inertia, e.g. $x_H|_T := \mathrm{mid}(T)$ for $T \in \mathcal{T}_H$. Then, Lemma 3.1 results in

$$p_H(x) - p_H^N(x) = -\frac{1}{2} f_H\,(x - x_H) \quad \text{for } x \in \Omega$$

plus a corresponding equation with H replaced by h. Then,

$$\begin{aligned}(p_h^N - p_h, p_H - p_h)_{L^2(\Omega)} &= \frac{1}{2}(f_h(\bullet - x_h), p_H^N - p_h^N)_{L^2(\Omega)} \\ &\quad + \frac{1}{4}(f_h(\bullet - x_h), f_h(\bullet - x_h) - f_H(\bullet - x_H))_{L^2(\Omega)}.\end{aligned}$$

The first term on the right-hand side vanishes because $p_H^N - p_h^N$ is constant and $\int_T (x - x_T)\,dx = 0$ for each $T \in \mathcal{T}_h$. The same argument shows $(f_h(\bullet - x_h), x_H - x_h)_{L^2(\Omega)} = 0$. There remains

$$4(p_h^N - p_h, p_H - p_h)_{L^2(\Omega)} = (f_h(\bullet - x_h), (f_h - f_H)(\bullet - x_h))_{L^2(\Omega)}.$$

An elementwise Cauchy inequality in the previous identity concludes the proof. □

6. Proof of Error Reduction Property

This section is devoted to the proof of the error reduction property (1.8) in Theorem 1.1.

The proof starts with the reliability from Theorem 3.2 and continues with the bulk criterion (1.5), i.e.,

$$\eta^2 := \sum_{E \in \mathcal{E}_H} h_E \|[p_H]\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{M}} h_E \|[p_H]\|_{L^2(E)}^2 \tag{6.1}$$

for the set $\mathcal{M}$ of marked edges. This leads to

$$\|p - p_H\|_{L^2(\Omega)}^2 \lesssim \eta^2 + \mathrm{osc}_H^2 \lesssim \sum_{E \in \mathcal{M}} h_E \|[p_H]\|_{L^2(E)}^2 + \mathrm{osc}_H^2\,.$$

The discrete local efficiency of Theorem 4.1 plus the finite overlap of the edge-patches $(\omega_E : E \in \mathcal{E}_H)$ show

$$\|p - p_H\|_{L^2(\Omega)}^2 \lesssim \sum_{E \in \mathcal{M}} \|p_h - p_H\|_{L^2(\omega_E)}^2 + \mathrm{osc}_H^2 \le \|p_h - p_H\|_{L^2(\Omega)}^2 + \mathrm{osc}_H^2\,.$$

With some constant c_1, this reads

$$\|p - p_H\|_{L^2(\Omega)}^2 \le c_1 \|p_h - p_H\|_{L^2(\Omega)}^2 + c_1\,\mathrm{osc}_H^2$$

On the other hand,

$$\|p_h - p_H\|_{L^2(\Omega)}^2 = \|p - p_H\|_{L^2(\Omega)}^2 - \|p - p_h\|_{L^2(\Omega)}^2 - 2(p - p_h, p_h - p_H)_{L^2(\Omega)}$$

and the last term can be bounded with the quasi-orthogonality. With some constant c_2, Theorem 5.1 leads to

$$\|p_h - p_H\|^2_{L^2(\Omega)} \le \|p - p_H\|^2_{L^2(\Omega)} - \|p - p_h\|^2_{L^2(\Omega)} + c_2(\|p - p_h\|_{L^2(\Omega)} + \|p - p_H\|_{L^2(\Omega)} + \|H f_H\|_{L^2(\Omega)}) \operatorname{osc}_H .$$

The combination with the preceding inequality plus a Young inequality yield

$$\begin{aligned} c_1\|p - p_h\|^2_{L^2(\Omega)} &\le (c_1 - 1)\|p - p_H\|^2_{L^2(\Omega)} + c_1 \operatorname{osc}_H^2 \\ &\quad + c_2 c_1 \Big(\|p - p_h\|_{L^2(\Omega)} + \|p - p_H\|_{L^2(\Omega)} + \|H f_H\|_{L^2(\Omega)}\Big) \operatorname{osc}_H \\ &\le \frac{1}{4}\|p - p_h\|^2_{L^2(\Omega)} + (c_1 - 1/2)\|p - p_H\|^2_{L^2(\Omega)} \\ &\quad + c_4 \Big(\|H f_h\|_{L^2(\Omega)} + \operatorname{osc}_H\Big) \operatorname{osc}_H . \end{aligned}$$

This proves

$$(c_1 - 1/4)\|p - p_h\|^2_{L^2(\Omega)} \le (c_1 - 1/2)\|p - p_H\|^2_{L^2(\Omega)} + c_4(\|H f_h\|_{L^2(\Omega)} + \operatorname{osc}_H) \operatorname{osc}_H$$

and so the theorem with $\rho = (c_1 - 1/2)/(c_1 - 1/4)$ and $C = c_4/(c_1 - 1/4)$. □

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