On parameter estimation of stochastic delay differential equations with guaranteed accuracy by noisy observations*

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Abstract

Let $(X(t), t \geq -1)$ and $(Y(t), t \geq 0)$ be stochastic processes satisfying

$$dX(t) = aX(t)dt + bX(t-1)dt + dW(t)$$

and

$$dY(t) = X(t)dt + dV(t),$$

respectively. Here $(W(t), t \geq 0)$ and $(V(t), t \geq 0)$ are independent standard Wiener processes and $\vartheta = (a, b)'$ is assumed to be an unknown parameter from some subset Θ of $\mathcal{R}^2$.

The aim here is to estimate the parameter ϑ based on continuous observation of $(Y(t), t \geq 0)$.

Sequential estimation plans for ϑ with preassigned mean square accuracy ε are constructed using the so-called correlation method. The limit behaviour of the duration of the estimation procedure is studied if ε tends to zero.

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1 Introduction

Assume $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ is a given filtered probability space and the processes $W = (W(t),\ t \geq 0)$ and $V = (V(t),\ t \geq 0)$ are real-valued standard Wiener processes on $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$, adopted to $(\mathcal{F}(t))$ and mutually independent. Furtherer assume that $X_0 = (X_0(t),\ t \in [-1, 0])$ and Y_0 are a real-valued cadlag process and a real-valued random variable, respectively, on $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), P)$ with

$$E \int_{-1}^{0} X_0^2(s) ds < \infty \quad \text{and} \quad EY_0^2 < \infty.$$

Assume that Y_0 and $X_0(s)$ are $\mathcal{F}_0-$measurable for every s from $[-1, 0]$ and that the quantities $W,\ V,\ X_0$ and Y_0 are mutually independent.

Consider a two–dimensional random process $(X, Y) = (X(t), Y(t))$ described by the system of stochastic differential equations

$$dX(t) = aX(t)dt + bX(t-1)dt + dW(t),\ t \geq 0, \tag{1}$$

$$dY(t) = X(t)dt + dV(t),\ t \geq 0 \tag{2}$$

with the initial conditions $X(t) = X_0(t),\ t \in [-1, 0]$ and $Y(0) = Y_0$. The process X is supposed to be hidden, i.e. unobservable, and the process Y is observed. Such models are used in applied problems connected with control, filtering and prediction of stochastic processes (see, for example, [1], [7]).

The parameter $\vartheta = (a, b)'$ with $a, b \in \mathcal{R}^1$ is assumed to be unknown and shall be estimated by using the observation of Y.

Equations (1) and (2) together with the initial values $X_0(\cdot)$ and Y_0 respectively have uniquely solutions $X(\cdot)$ and $Y(\cdot)$, for details see [9].

Equation (1) is a very special case of stochastic differential equations with time delay, see [3] and [10] for examples.

To estimate the true parameter ϑ with a preassigned least square accuracy ε we shall construct sequential plans $(T_\varepsilon, \vartheta_\varepsilon^*)$. Moreover, we will derive asymptotic properties of the duration T_ε of these plans for ε tending to zero.

The method used below is to transform the equations (1) and (2) to a single equation (see (4) below) for the process $(Y(t), t \geq 0)$, which can be treated by modifying a method from [11]. The construction of $(T_\varepsilon, \vartheta_\varepsilon^*)$ may depend on the asymptotic behaviour of the correlation function of the solution of (1) and their estimators if the observation time is increasing unboundedly. These asymptotic properties vary if ϑ runs through $\mathcal{R}^2$. Our construction does not seem to work for all ϑ in $\mathcal{R}^2$. Therefore we restrict the discussion to two sets Θ_1 and Θ_2 of parameters, for which we are able to derive the desired properties.

The organization of this paper is as follows. In Section 2 we summarize some known properties of equation (1) needed in the sequel. The two mentioned cases for Θ, namely Θ_1 and Θ_2, are presented and equations (1), (2) are transformed into a new one for the one-dimensional observed process $(Y(t), t \geq 0)$ (see (4)). In Section 3 the two sequential plans are constructed and the assertions are formulated. Section 4 contains the proofs.

2 Preliminaries

First we summarize some known facts about equation (1). For details the reader is refer to [2]. Together with the described initial condition equation (1) has a uniquely determined solution X which can be represented as follows for $t \geq 0$:

$$X(t) = x_0(t)X_0(t) + b\int_{-1}^{0} x_0(t-s-1)X_0(s)ds + \int_0^t x_0(t-s)dW(s),\ t \geq 0.$$

Here $x_0 = (x_0(t), t \geq -1)$ denotes the so-called fundamental solution of the deterministic equation

$$x_0(t) = 1 + \int_0^t (\vartheta_0 x_0(s) + \vartheta_1 x_0(s-1))ds, \qquad t \geq 0,$$

corresponding to (1) with $x_0(t) = 0,\ \ t \in [-1, 0),\ \ x_0(0) = 1.$

The solution X has the property $E\int_0^T X^2(s)ds < \infty$ for every $T > 0$.

The limit behavior of $x_0(t)$ and therefore also of $X(t)$ for t tending to infinity is closely connected with the properties of the set $\Lambda = \{\lambda \in \mathbb{C} | \lambda = a + be^{-\lambda}\}$ ($\mathbb{C}$ denotes the set of complex numbers). The set Λ is countable infinite (if $b \neq 0$), and for every real c the set $\Lambda_c = \Lambda \cap \{\lambda \in \mathbb{C} |\ Re\lambda \geq c\}$ is finite. In particular, $v_0 := v_0(\vartheta) = \sup\{Re\lambda | \lambda \in \Lambda\} < \infty,\ \ \sup\{\emptyset\} = -\infty$. Define $v_1(\vartheta) =: \sup\{Re\lambda | \lambda \in \Lambda,\ Re\lambda < v_0(\vartheta)\}$.

The values $v_0(\vartheta)$ and $v_1(\vartheta)$ determine the asymptotic behaviour of $x_0(t)$ as $t \to \infty$. Indeed, it exist a real γ less than v_1 and a polynomial $\Psi_1(\cdot)$ of degree less than or equal one, being specified in the proof of Theorem 3.1 (Section 4 below), such that

$$x_0(t) = \frac{1}{v_0 - a + 1}e^{v_0 t} + \Psi_1(t)e^{v_1 t} + o(e^{\gamma t}) \ \text{ as } \ t \to \infty.$$

Now we define a subset Θ of $\mathcal{R}^2$ consisting of two disjoint sets Θ_1 and Θ_2. First fix a positive real $\overline{\vartheta}$.

Case I. The set Θ_1 : Assume L is an arbitrary line in the plane $\mathcal{R}^2$:

$$L = L(\alpha, \beta, \omega) = \{\tilde{\vartheta} = (\tilde{a},\ \tilde{b})' |\ \alpha\tilde{a} + \beta\tilde{b} = c\}.$$

Let $\tilde{\Theta}$ be the segment $L \cap \{||\tilde{\vartheta}|| \leq \overline{\vartheta}\}$ (it is no restriction of generality to assume that $\tilde{\Theta}$ is non-void), $|| \cdot ||$ denotes the Euclidean norm.

Now we introduce the set S by

$$S = \{\vartheta = (a, b)' \in \tilde{\Theta} |\ v_0(\vartheta) \cdot v_1(\vartheta) = 0 \text{ or } (a > 1, b = -e^{(a-1)})\}$$

and put $\Theta_1 = \tilde{\Theta} \setminus S$.

Case II. The set Θ_2 : Define

$$\Theta_2 = \{\vartheta \in \mathcal{R}^2 |\ ||\vartheta|| \leq \overline{\vartheta},\ v_0(\vartheta) < 0 \text{ or } (v_0(\vartheta) > 0 \text{ and } v_0(\vartheta) \notin \Lambda)\}.$$

The definition of the two sets Θ_1 and Θ_2 looks quite complicate. But they are distinguished by the property, that for all of their elements ϑ the correlation function of $X(\cdot)$ has an asymptotic property which is analogous to (16), (17), (41) and (42) below.

In particular, in Case I the partly observable two-dimensional process $(X(t), Y(t))$ will be reduced to a scalar observable linear process with a scalar function in the dynamic part. The asymptotic properties of this function are given in (16) and (17).

In Case II the information matrix $G_X(T)$ given by

$$G_X(T) = \begin{pmatrix} \int\limits_0^T X^2(t)dt & \int\limits_0^T X(t)X(t-1)dt \\ \int\limits_0^T X(t)X(t-1)dt & \int\limits_0^T X^2(t-1)dt \end{pmatrix}$$

has the asymptotic property (see [2] and [5, 6] for details)

$$\lim_{T\to\infty} |\varphi^{-1}(T)G_X(T) - I_\infty(T)| = 0 \quad P_\vartheta - \text{a.s.}, \tag{3}$$

where

$$\varphi(T) = \begin{cases} T, & if\ v_0 < 0, \\ e^{2v_0 T}, & if\ v_0 > 0,\ v_0 \notin \Lambda. \end{cases}$$

If $v_0 < 0$ then (1) admits a stationary solution and $I_\infty(T) \equiv I_\infty$ is a constant positive definite $2 \times 2-$matrix (in the sequel we shall call this case the stationary case); if $v_0 > 0$ and $v_0(\vartheta) \notin \Lambda$, then $I_\infty(T)$ is nondeterministic periodic with the period $\Delta = \pi/Im\lambda_0$, where λ_0 is the unique element of Λ with $Re\lambda_0 = v_0(\vartheta)$ and $Im\lambda_0 > 0$ (below we refer to this case as the periodic case).

The problem of sequential estimation of ϑ by observation without noise under the condition (3) was considered in [5, 6].

To construct a sequential plan for estimating ϑ based on the observation of $Y(\cdot)$ we shall apply the idea of a method first used in [11]. To this end we shall reduce equations (1) and (2) to a single one for Y.

Using the integrated form of equations (1) and (2) we can get the following equation for the observed process Y

$$\begin{aligned} dY(t) &= [aY(t) + bY(t-1)]dt + [X(0) - aY(0) - bY(0) + b\int_{-1}^{0} X_0(s)ds \\ &- aV(t) - bV(t-1) + W(t)]dt + dV(t),\ t \geq 1. \end{aligned}$$

Thus we have reduced the system (1), (2) to the form

$$dY(t) = \vartheta' A(t)dt + \xi(t)dt + dV(t), \tag{4}$$

with

$$A(t) = (Y(t), Y(t-1))',$$

$$\xi(t) = X(0) - aY(0) - bY(0) + b\int_{-1}^{0} X_0(s)ds - aV(t) - bV(t-1) + W(t),$$

where the observable process $(A(t),\ t \geq 0)$ and the noise $\xi = (\xi(t),\ t \geq 0)$ are some $(\mathcal{F}(t))$–adapted processes. The problem of estimation of ϑ with guaranteed accuracy in models of the type (4) was considered in [11].

The functions $A(t)$ and $\xi(t)$ are $\mathcal{F}(t)$-measurable for every $t \geq 1$ and a short calculating shows that all conditions of type (3) in [11], consisting of

$$E \int_1^T (||A(t)||_1 + |\xi(t)|)dt < \infty \quad \text{for all} \quad T > 1,$$

$$E[\tilde{\Delta}\xi(t)|\mathcal{F}(t-2)] = 0,\ E[(\tilde{\Delta}\xi(t))^2|\mathcal{F}(t-2)] \leq \overline{s}^2,\ t \geq 2, \tag{5}$$

$$\overline{s}^2 = 1 + \overline{\vartheta}^2,\ ||A||_1 = \sum_i |A_i|$$

hold in our case. Here $\tilde{\Delta}$ denotes the difference operator defined by $\tilde{\Delta}f(t) = f(t) - f(t-1)$.

Using this operator and the definition of ξ we obtain the following equation:

$$\begin{aligned} d\tilde{\Delta}Y(t) &= a\tilde{\Delta}Y(t)dt + b\tilde{\Delta}Y(t-1)dt + \tilde{\Delta}\xi(t)dt \\ &+ dV(t) - dV(t-1),\ t \geq 2 \end{aligned} \tag{6}$$

with initial condition $\tilde{\Delta}Y(1) = Y(1) - Y_0$.

We have reduced the system (1)–(2) to a single differential equation (6) for the observed process $(\tilde{\Delta}Y(t), t \geq 2)$ depending on the unknown parameters a and b. The term $\tilde{\Delta}\xi(t)$ also contains a and b, but its variance is controllable in certain sense (see formula (5)).

Nevertheless, a and b can not be estimated from (6) by the maximum likelihood or sequential maximum likelihood method given in [2] or [5, 6] respectively, because of the appearance of the terms $\tilde{\Delta}\xi(t)dt$ and $dV(t-1)$. Below we shall propose another way following an idea taken from [11].

3 Results

3.1 Sequential estimation procedure I

Consider the estimation problem of a linear combination $\theta = l'\vartheta,\ \vartheta \in \Theta_1$, where $l = (l_1, l_2)'$ is some known constant vector such that $\sigma = l_1\beta - l_2\alpha \neq 0$. Here α and β are the constants from the definition of the line L, defined in Section 2.

We introduce processes $Z_1,\ Z_2$ and Ψ by the formulae

$$dZ_1(t) = \sigma^{-1}(\beta d\tilde{\Delta}Y(t) - c\tilde{\Delta}Y(t-1)dt),\ t \geq 2,$$

$$dZ_2(t) = -\sigma^{-1}(\alpha d\tilde{\Delta}Y(t) - c\tilde{\Delta}Y(t)dt),\ t \geq 2,$$

$$\Psi(t) = \begin{cases} \sigma^{-1}(\beta\tilde{\Delta}Y(t) - \alpha\tilde{\Delta}Y(t-1)), & t \geq 2, \\ 0, & t < 2. \end{cases}$$

From (6) and from the definition of Θ_1 we get for $t \geq 2$ the system of equations

$$dZ_1(t) = a\Psi(t)dt + \beta\sigma^{-1}(\tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t)),$$

$$dZ_2(t) = b\Psi(t)dt - \alpha\sigma^{-1}(\tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t)),\ t \geq 2.$$

Now we obtain an equation for the observable scalar process $Z(t) = l_1 Z_1(t) + l_2 Z_2(t)$:

$$dZ(t) = \theta\Psi(t)dt + \tilde{\Delta}\xi(t)dt + d\tilde{\Delta}V(t),\ t \geq 2 \tag{7}$$

with unknown parameter θ. For $t < 2$ we set $Z(t) = 0$.

In a similar way as in [11] we can define a sequential plan for the estimation of θ from $\{l`\vartheta|\ \vartheta \in \Theta_1\}$ with mean square deviation less than a given positive ε. The sequential estimation plans for θ have been constructed in [11] based on so-called correlation estimators which are generalized least squares estimators. Here we use an analogous definition as follows:

$$\theta^*(T) = G^{-1}(T,u)\Phi(T,u), \tag{8}$$

$$G(T,u) = \int_0^T \Psi(t-u)\Psi(t)dt,\quad \Phi(T,u) = \int_0^T \Psi(t-u)dZ(t),\quad T > 2,\quad u \geq 2.$$

Under the condition $u \geq 2$ the function $\Psi(t-u)$ in equation (7) is uncorrelated with respect to the noise $\tilde{\Delta}\xi(t)$ as well as to $\tilde{\Delta}V(t)$.

From (7) and (8) we find the deviation of the estimator $\theta^*(T)$:

$$\theta^*(T) - \theta = G^{-1}(T,u)\zeta(T,u), \tag{9}$$

where

$$\zeta(T,u) = \zeta(T,u,1) + \zeta(T,u,2) + \zeta(T,u,3)$$

with

$$\zeta(T,u,1) = \int_0^T \Psi(t-u)\tilde{\Delta}\xi(t)dt,\quad \zeta(T,u,2) = \int_0^T \Psi(t-u)dV(t)$$

and

$$\zeta(T,u,3) = -\int_0^T \Psi(t-u)dV(t-1).$$

As we will see from the proof of Theorem 3.1 (Section 4 below), there exist increasing functions $\varphi(T)$ corresponding to the various regions for the parameter ϑ from Θ_1 and Θ_2 such that for every $u \geq 2$ the function $g(T,u) = \varphi^{-1}(T)G(T,u)$ has one of the following properties:
either

a) the limit $g(u) = \lim\limits_{T\to\infty} g(T,u)$ exists $P-$a.s. and is deterministic with $meas\{u \in (2,3] : \ g(u) = 0\} = 0$ ($meas\{B\}$ is the Lebesgue measure of the set B) and $g(0) > 0$; or

b) the limit $g(u) = \lim\limits_{T\to\infty} g(T,u)$ exists $P-$ a.s. and is non-deterministic, it holds $P\{g(u) = 0\} = 0, \ u \geq 0$;
or

c) there exists a random periodic function $\tilde{g}(T,u), \ T > 0$, periodic with respect to T and with period $\Delta > 1$, such that

$$P\{\lim_{T\to\infty} |g(T,u) - \tilde{g}(T,u)| = 0\} = 1, \ u \geq 0$$

holds (see the formulae (16) and (17) below).

It will be clear from the proofs in Section 4 below that in the periodic case c) the function $\tilde{g}(T,u)$ has for every $u \geq 0$ two roots as a maximum on every interval of the unknown period length Δ. Then the function $\varphi(T)G^{-1}(T,u)$ and consequently the deviation $\theta^*(T) - \theta$ may be unbounded.

Remark 1 *Properties a) and c) do not exclude that the limit functions $g(u)$ and $\tilde{g}(T,u)$ may be equal to zero for some u and (T,u) respectively. A similar picture arises in Case II (see proof of Theorem 3.2 below). Due to this fact the estimation procedure, used in [11] can not be applied in the cases considered above.*

To exclude this effect we introduce a discretization of the time of observations. Note that in the case of observations without noise we also need a similar discretization (by using Δ) for the investigation of asymptotic properties of maximum likelihood estimators [2]. The procedure which we construct here is non-asymptotic and we can not use the unknown value Δ in the construction of estimators.

For some $h \in (0, 1/3]$ put

$$r_n = \arg\max_{k=\overline{1,3}} |G(nh - kh, 2 + 3h)|.$$

Such a choice of the value of h implies that for every $n \geq 1$ and $T > 0$ there are one or more values $nh - kh, \ k = \overline{1,3}$, with $\tilde{g}(nh - kh, T) \neq 0$. In such a way (see the proof of Theorem 3.1) the sequence $\{g(nh - r_n h, 2 + 3h), \ n \geq 1\}$ is non-degenerate in the case c) for any $h \in (0, 1/3]$ asymptotically as $n \to \infty$.

To construct the estimators with preassigned accuracy we first change first the value nh in the argument of G (see the definition of r_n just given) to stopping times. As we will see later (inequalities (11)) this substitution gives us the possibility to control the second moments of the noise ζ.

Let $(c_n, \ n \geq 1)$ be some unboundedly increasing sequence of positive numbers. We shall define the stopping times $(\tau_\varepsilon(n), n \geq 1)$ from the discrete sequence $\{kh, \ k \geq 1\}$ with an arbitrary but fixed step size h by formula

$$\tau_\varepsilon(n) = h \inf\{k \geq 1 : \ \int_0^{kh} \Psi^2(t - 2 - 3h)dt \geq \varepsilon^{-1} c_n\}, \ n \geq 1. \qquad (10)$$

Using formulae (16) and (17) below it is easy to see that $P(\tau_\varepsilon(n) < \infty) = 1$ for any $\varepsilon > 0$ and every $n \geq 1$.

For $k=\overline{1,3},\ n\geq 1$ we put

$$G_\varepsilon(n,k)=G(\tau_\varepsilon(n)-kh,2+3h),\quad \Phi_\varepsilon(n,k)=\Phi_\varepsilon(\tau_\varepsilon(n)-kh,2+3h),$$

$$\zeta_\varepsilon(n,k)=\zeta(\tau_\varepsilon(n)-kh,2+3h);$$

$$k_n=\arg\max_{k=\overline{1,3}}\{|G_\varepsilon(n,k)|\},\ n\geq 1.$$

Now we introduce the sequence of estimators

$$\theta_\varepsilon(n)=G_\varepsilon^{-1}(n)\Phi_\varepsilon(n)$$

with

$$G_\varepsilon(n)=G_\varepsilon(n,k_n),\quad \Phi_\varepsilon(n)=\Phi(n,k_n),\ n\geq 1.$$

They have the deviation

$$\theta_\varepsilon(n)-\theta=G_\varepsilon^{-1}(n)\zeta_\varepsilon(n),\quad \zeta_\varepsilon(n)=\zeta_\varepsilon(n,k_n),\ n\geq 1.$$

Fix an h_0 from $(0,1/3)$ and choose an arbitrary random variable h being $\mathcal{F}(0)$-measurable and having a continuous distribution concentrated on the interval $[h_0,1/3]$. We need such randomization of the discretization step h in the case a) for the almost surely non-degeneration of the limit $g(2+3h)=\lim_{n\to\infty}\varphi^{-1}(\tau_\varepsilon(n)-k_nh)G_\varepsilon(n,k_n)$.

We will show that the second moments of the noise ζ calculated at times $\tau_\varepsilon(n)-k_nh,\ n\geq 1$ have known upper bounds. Note that the processes $(\zeta(T,2+3h,i),\mathcal{F}(T))$, $i=\overline{1,3}$ are square integrable martingales and the times $\tau_\varepsilon(n)-kh,\ n\geq 1,\ k=\overline{1,3}$, are Markovian with respect to the system $(\mathcal{F}(T-2))$. From the theory of martingales (see e.g. [8]) and from the definition of $\tau_\varepsilon(n)$ we obtain for all $\vartheta\in\mathcal{R}^2,\ k=\overline{1,3}$ and $n\geq 1$ the inequalities

$$E_\vartheta\zeta^2(\tau_\varepsilon(n)-kh,2+3h,1)\leq \overline{s}^2E_\vartheta\int_0^{\tau_\varepsilon(n)-kh}\Psi^2(t-2-3h)dt<\overline{s}^2\varepsilon^{-1}c_n,$$

$$E_\vartheta\zeta^2(\tau_\varepsilon(n)-kh,2+3h,i)\leq\varepsilon^{-1}c_n,\ i=2,3.$$

Thus for all $\varepsilon>0$ and $n\geq 1$ the sequence $(\zeta_\varepsilon(n),\ n\geq 1)$ satisfies the inequalities

$$\begin{aligned}E_\vartheta\zeta_\varepsilon^2(n) &\leq \sum_{k=1}^3 E_\vartheta\zeta^2(\tau_\varepsilon(n)-kh,2+3h)\\ &\leq 3\sum_{k=1}^3\sum_{i=1}^3 E_\vartheta\zeta^2(\tau_\varepsilon(n)-kh,2+3h,i)\leq 9(2+\overline{s}^2)\varepsilon^{-1}c_n. \qquad (11)\end{aligned}$$

The asymptotic properties of the sequence $(G_\varepsilon(n),n\geq 1)$ and the inequalities (11) imply that the estimation of the parameter θ should be performed at the times $\tau_\varepsilon(n)-k_nh,\ n\geq 1$. Note that the estimators $\theta_\varepsilon(n)$ are strongly consistent (see

Theorem 3.1).

We want obtain estimators with fixed mean square deviation. Therefore, taking into account the representation for the deviation of estimators $\theta_\varepsilon(n)$, one has to control the behaviour of the sequence of random variables $G_\varepsilon(n),\ n \geq 1$. This can be achieved by observations up to the time $\tau_\varepsilon(n) - k_n h$ with a specially chosen number n.

Let $(\kappa_n,\ n \geq 1)$ be some unboundedly increasing sequence of positive numbers. Introduce the stopping time

$$\nu_\varepsilon = \inf\{n \geq 1 : |G_\varepsilon(n)| \geq \rho^{1/2}\varepsilon^{-1}\kappa_n\},$$

where

$$\rho = 9(2 + \bar{s}^2) \sum_{n \geq 1} c_n/\kappa_n^2.$$

We define the sequential plan $(T(\varepsilon), \theta_\varepsilon^*)$ for the estimation of θ as

$$T(\varepsilon) = \tau_\varepsilon(\nu_\varepsilon),\ \theta_\varepsilon^* = \theta_\varepsilon(\nu_\varepsilon) = G_\varepsilon^{-1}(\nu_\varepsilon)\Phi_\varepsilon(\nu_\varepsilon). \tag{12}$$

It should be pointed out that the estimator (12) coincides with the sequential estimator which is obtained from general least squares criteria [11].

The following theorem presents the conditions under which $T(\varepsilon)$ and θ_ε^* are well-defined and have the desired property of preassigned mean square accuracy.

First we divide the parameter set Θ_1 into nine subsets, according to the definitions of Section I.

Define the functions $u(a),\ a < 1$, and $w(a), a \in \mathcal{R}^1$, as in [2]: consider a parametric curve $(a(\xi), b(\xi)),\ \xi > 0,\ \xi \neq \pi, 2\pi, \ldots$, in $\mathcal{R}^2$ by

$$a(\xi) = \xi \cot \xi,\ b(\xi) = -\xi/\sin \xi,$$

then functions $b = u(a)$ and $b = w(a)$ are defined to be the branches of this curve corresponding to $\xi \in (0, \pi)$ and $\xi \in (\pi, 2\pi)$ respectively. Put also $v(a) = -e^{a-1}, a \in \mathcal{R}^1$, and introduce the indices

$$i = \begin{cases} 0, & \text{if } \ \alpha \neq \beta e^{v_0}, \\ 1, & \text{if } \ \alpha = \beta e^{v_0}, \end{cases}$$

$$j = \begin{cases} 1, & \text{if } \ a < 1,\ u(a) < b < -a, \\ 2, & \text{if } \ -a < b < w(a), \\ 3, & \text{if } \ a > 1,\ v(a) < b < -a, \\ 4, & \text{if } \ a > 1,\ b = v(a), \\ 5, & \text{if } \ b > w(a), \\ 6, & \text{if } \ a < 1,\ b < u(a) \text{ or } a \geq 1,\ b < v(a), \\ 7, & \text{if } \ a < 1,\ b = -a,\ a \neq 0, \\ 8, & \text{if } \ a > 1,\ b = -a, \\ 9, & \text{if } \ b = w(a). \end{cases}$$

Note that the sets corresponding to different values of j are disjoint and the union of all the cases corresponding to $j = \overline{1,9}$ is the whole plane $\mathcal{R}^2$ exept for some one-dimensional smooth curve. We know that $v_0 < 0$ if $j = 1$; $v_0 = 0$ if $j = 7$ and $v_0 > 0$ in all other cases. Moreover we have $v_1 < 0$ if $j = 1, 2, 7$; $v_1 = 0$ if $j = 8, 9$ and $v_1 > 0$ if $j = 3, 5$ [2].

Introduce the sets

$$I_1 = \{(0,1), (1,1), (1,2), (1,7)\},$$

$$I_2 = \{(0,2), (0,3), (0,5), (0,8), (0,9), (1,4)\},$$

$$I_3 = \{(1,3)\}, \quad I_4 = \{(0,4)\}, \quad I_5 = \{(0,6), (1,5), (1,6)\},$$

$$I_6 = I_2 \cup I_5 \setminus \{(1,5)\}, \quad I_7 = I_3 \cup \{(1,5)\}.$$

Theorem 3.1 *Assume that the sequences (c_n) and (κ_n) defined above satisfy the conditions*

$$\sum_{n \geq 1} \frac{c_n}{\kappa_n^2} < \infty \tag{13}$$

and

$$\lim_{n \to \infty} \kappa_n / c_n = 0 \tag{14}$$

Then we obtain the following result:

I. For any $\varepsilon > 0$ and every $\theta \in \Theta_1$ the sequential plan $(T(\varepsilon), \theta_\varepsilon^*)$ defined by (12) *is closed (i.e. $T(\varepsilon) < \infty$ $P - a.s.$) and has the following properties:*

$$1^\circ. \quad \sup_{\Theta_1} E_\vartheta (\theta_\varepsilon^* - \theta)^2 \leq \varepsilon \quad \textit{for every} \quad \varepsilon > 0,$$

2°. *for every $\theta \in \Theta_1$ the following relations hold:*

- *if $(i,j) \in I_1$ then*

$$0 < \varliminf_{\varepsilon \to 0} \ \varepsilon \cdot T(\varepsilon) \ \leq \ \varlimsup_{\varepsilon \to 0} \varepsilon \cdot T(\varepsilon) \ < \ \infty \quad P - a.s.,$$

- *if $(i,j) \in I_2 \cup I_3 \cup I_5$ then*

$$0 < \varliminf_{\varepsilon \to 0} \ [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \ \leq \ \varlimsup_{\varepsilon \to 0} \ [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \ < \ \infty \quad P - a.s.,$$

- *if $(i,j) \in I_4$ then*

$$0 < \varliminf_{\varepsilon \to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} \leq \varlimsup_{\varepsilon \to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} < \infty \quad P - a.s.$$

II. For any $\varepsilon > 0$ and every $\theta \in \Theta_1$ the estimator $\theta_\varepsilon(n)$ is strongly consistent:

$$\lim_{n \to \infty} \theta_\varepsilon(n) = \theta \quad P - a.s.$$

The proofs of this and the next theorem are given in Section 4.

Remark 2 *Consider the special case of the system* (1), (2), *when the parameter b equals zero, which means that $X(\cdot)$ is an Ornstein-Uhlenbeck process. Then the assertions of Theorem 3.1 are true if in equation* (1) *we have $a \neq 0$. Note, that in* [11] *only the case $(a < 0)$ has been considered.*

3.2 Sequential estimation procedure II

Consider the problem of estimating $\vartheta \in \Theta_2$. Based on equation (6) we define the estimation procedure analogously to the one given in Section 3.1. Assume $\tilde{h}_0$ is a real number in $(0, 1/5)$ and $\tilde{h}$ is a random variable with values in $[\tilde{h}_0, 1/5]$ only, $\mathcal{F}(0)$-measurable and having a known continuous distribution function.

We introduce several quantities:

– the functions

$$\tilde{\Psi}_s(t) = \begin{cases} (\tilde{\Delta}Y(t), \tilde{\Delta}Y(t-s))' & \text{for} \quad t \geq 1+s, \\ (0,0)' \quad \text{for} \quad t < 1+s; \end{cases}$$

– the sequence of stopping times

$$\tilde{\tau}_\varepsilon(n) = \tilde{h} \inf\{k \geq 1 : \int_0^{k\tilde{h}} ||\tilde{\Psi}_{\tilde{h}}(t-2-5\tilde{h})||^2 dt \geq \varepsilon^{-1} c_n\} \quad \text{for} \quad n \geq 1;$$

– the matrices

$$G^*(T,s) = \int_0^T \tilde{\Psi}_s(t-2-5s)\tilde{\Psi}_1'(t)dt,$$

$$\Phi^*(T,s) = \int_0^T \tilde{\Psi}_s(t-2-5s)d\tilde{\Delta}Y(t),$$

$$\tilde{G}_\varepsilon(n,k) = G^*(\tilde{\tau}_\varepsilon(n) - k\tilde{h}, \tilde{h}), \quad \tilde{\Phi}_\varepsilon(n,k) = \Phi^*(\tilde{\tau}_\varepsilon(n) - k\tilde{h}, \tilde{h});$$

– the times

$$\tilde{k}_n = \arg\min_{k=\overline{1,5}} ||\tilde{G}_\varepsilon^{-1}(n,k)||, \; n \geq 1;$$

– the estimators

$$\tilde{\vartheta}_\varepsilon(n) = \tilde{G}_\varepsilon^{-1}(n)\tilde{\Phi}_\varepsilon(n), \; n \geq 1, \quad \text{where}$$

$$\tilde{G}_\varepsilon(n) = \tilde{G}_\varepsilon(n, \tilde{k}_n), \; \tilde{\Phi}_\varepsilon(n) = \tilde{\Phi}_\varepsilon(n, \tilde{k}_n);$$

– the stopping time

$$\tilde{\nu}_\varepsilon = \inf\{n \geq 1 : ||\tilde{G}_\varepsilon^{-1}(n)|| \leq \varepsilon(\tilde{\rho}^{1/2}\kappa_n)^{-1}\}, \quad \text{where}$$

$$\tilde{\rho} = 15(2+\overline{s}^2)\sum_{n\geq 1} c_n/\kappa_n^2.$$

Define the sequential estimation plan of ϑ by

$$\tilde{T}(\varepsilon) = \tilde{\tau}_\varepsilon(\tilde{\nu}_\varepsilon), \quad \tilde{\vartheta}(\varepsilon) = \tilde{\vartheta}_\varepsilon(\tilde{\nu}_\varepsilon) = \tilde{G}_\varepsilon^{-1}(\tilde{\nu}_\varepsilon)\tilde{\Phi}_\varepsilon(\tilde{\nu}_\varepsilon). \tag{15}$$

We can see that the construction of the sequential estimator $\tilde{\vartheta}(\varepsilon)$ bases on the family of estimators $\vartheta^*(T,s) = (G^*(T,s))^{-1}\Phi^*(T,s),\ \ s \ge 0$. We have taken the discretization step $\tilde{h}$ as above, because from (49) below it follows that the functions

$$\tilde{f}(T,s) = \frac{1}{e^{2v_0T}}\ G^*(T,s)$$

for every $s \ge 0$ have some periodic matrix functions as a limit almost surely. These limiting matrix functions are finite and may be degenerate only for four values of their argument T on every interval of periodicity of length $\Delta > 1$ (see proof of Theorem 3.2 below).

We state the results concerning the estimation of the parameter $\vartheta \in \Theta_2$ in the following theorem.

Theorem 3.2 *Assume that the conditions* (13) *and* (14) *on the sequences* (c_n) *and* (κ_n) *hold and let the parameter* $\vartheta = (a,b)'$ *in* (1) *is such that* $\vartheta \in \Theta_2$. *Then we obtain:*

I. For any $\varepsilon > 0$ *and every* $\vartheta \in \Theta_2$ *the sequential plan* $(\tilde{T}(\varepsilon), \tilde{\vartheta}(\varepsilon))$ *defined by* (15) *is closed and possesses the following properties:*

$$1^o. \quad \sup_{\Theta_2} E_\vartheta ||\tilde{\vartheta}(\varepsilon) - \vartheta||^2 \le \varepsilon \quad for\ every \ \ \varepsilon > 0,$$

2°. *for every* $\theta \in \Theta_2$ *one of the inequalities below is valid:*
- in the stationary case $(v_0 < 0)$

$$0 < \underline{\lim}_{\varepsilon \to 0}\ \varepsilon \cdot \tilde{T}(\varepsilon) \ \le \ \overline{\lim}_{\varepsilon \to 0}\ \varepsilon \cdot \tilde{T}(\varepsilon) \ < \ \infty \quad P - a.s.,$$

- in the periodic case $(v_0 > 0,\ \ v_0 \notin \Lambda)$

$$0 < \underline{\lim}_{\varepsilon \to 0}\ [\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln \varepsilon^{-1}] \ \le \ \overline{\lim}_{\varepsilon \to 0}\ [\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln \varepsilon^{-1}] \ < \ \infty \quad P - a.s.$$

II. For any $\varepsilon > 0$ *and every* $\vartheta \in \Theta_2$ *the estimator* $\vartheta_\varepsilon(n)$ *is strongly consistent:*

$$\lim_{n \to \infty} \tilde{\vartheta}_\varepsilon(n) = \vartheta \quad P - a.s.$$

Remark 3 *Property I in Theorems 3.1 and 3.2 yields the rates of convergence of the considered sequential plans. These depend on the region to which the parameter* ϑ *belongs to. They have the same rate of convergence as the maximum likelihood estimator of* ϑ, *see* [2], *constructed directly from the observations of the process* $X(\cdot)$.

4 Proofs

4.1 Proof of Theorem 3.1

At first we prove the finiteness of the stopping times $T(\varepsilon)$.

To this aim we put

$$\varphi_{ij}(T) = \begin{cases} T, & (i,j) \in I_1, \\ e^{2v_iT}, & (i,j) \in I_2 \cup I_3 \cup I_5, \\ T^2 e^{2v_0T}, & (i,j) \in I_4. \end{cases}$$

and prove the following auxiliary results: Fix $u = 0$ or $u \in [2, \infty)$. Then
– for $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4$ it holds

$$\lim_{T\to\infty} \frac{1}{\varphi_{ij}(T)} \int_0^T \Psi(t-u)\Psi(t)dt = f_{iju} \quad P-\text{a.s.}, \tag{16}$$

where f_{iju} are some constants or random variables;
– for $(i,j) \in I_5$ we have

$$\lim_{T\to\infty} \left|\frac{1}{\varphi_{ij}(T)} \int_0^T \Psi(t-u)\Psi(t)dt - f_{iju}(T)\right| = 0 \quad P-\text{a.s.}, \tag{17}$$

where $f_{iju}(T)$ are periodic random functions of T with the period $\Delta = 2\pi/\xi_0,\ \xi_0 \in (0,\pi)$ if $(i,j) = \{(0,6),(1,6)\}$ and $\Delta = 2\pi/\xi_1,\ \xi_1 \in (\pi, 2\pi)$ if $(i,j) = (1,5)$.

Proof of (16) and (17). Now we establish the equalities (16) in the cases I_1 for $u = 0,\ u \geq 2$ and the other equalities in (16) and (17) for $u \geq 0$. According to [2] for $\vartheta \in \Theta_1$ the solution $X(t)$ of (1) has the representation

$$\begin{aligned} X(t) &= x_0(t)X_0(0) + b\int_{-1}^{0} x_0(t-s-1)X_0(s)ds \\ &+ \int_0^t x_0(t-s)dW(s),\ t \geq 0, \end{aligned} \tag{18}$$

where $x_0(\cdot)$ is the so called fundamental solution of (1). It has the properties $x_0(t) = 0,\ t \in [-1,0),\ x_0(0) = 1$ and satisfies for $t \to \infty$

$$x_0(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{1}{v_0-a+1}e^{v_0t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{1}{v_0-a+1}e^{v_0t} + \frac{1}{a-v_1-1}e^{v_1t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 3, \\ (2t + \frac{2}{3})e^{v_0t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 4, \\ \frac{1}{v_0-a+1}e^{v_0t} + \phi_1(t)e^{v_1t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 5, \\ \phi_0(t)e^{v_0t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 6, \\ \frac{1}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{1}{v_0-a+1}e^{v_0t} - \frac{1}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{1}{v_0-a+1}e^{v_0t} + \phi_1(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9, \end{cases}$$

for all $\gamma, \gamma_0, \gamma_1$ satisfying the mentioned inequalities respectively and may be different in different lines,

$$\phi_i(t) = A_i \cos \xi_i t + B_i \sin \xi_i t \ \text{ with}$$

$$A_i = \frac{2(v_i - a + 1)}{(v_i - a + 1)^2 + \xi_i^2},\ B_i = \frac{2\xi_i}{(v_i - a + 1)^2 + \xi_i^2},\ i = 0, 1.$$

By the definition of Ψ we have

$$\Psi(t) = \tilde{\Psi}(t) + \tilde{V}(t),\ t \geq -1, \tag{19}$$

$$\tilde{\Psi}(t) = \begin{cases} \sigma^{-1}(\beta\tilde{X}(t) - \alpha\tilde{X}(t-1)), & t \geq 2, \\ 0, & t \in [-1, 2], \end{cases}$$

$$\tilde{X}(t) = \int_{t-1}^{t} X(s)ds,$$

$$\tilde{V}(t) = \begin{cases} \sigma^{-1}(\beta\tilde{\Delta}V(t) - \alpha\tilde{\Delta}V(t-1)), & t \geq 2, \\ 0, & t \in [-1, 2]. \end{cases}$$

It is easy to show that the process $(\tilde{X}(\cdot))$ has the following representation:

$$\tilde{X}(t) = \sigma^{-1}(\tilde{x}_0(t)X_0(0) + b\int_{-1}^{0} \tilde{x}_0(t-s-1)X_0(s)ds + \int_0^t \tilde{x}_0(t-s)dW(s))$$

for $t \geq 1$, $\tilde{X}(t) = \int_{t-1}^{0} X_0(s)ds + \int_0^t X(s)ds$ for $t \in [0,1)$ and $\tilde{X}(t) = 0$ for $t \in [-1,0)$. Based on (18) and the subsequent properties of $x_0(t)$ the function $\tilde{x}_0(t) = \int_{t-1}^{t} x_0(s)ds$ can easily be shown to fulfill $\tilde{x}_0(t) = 0,\ t \in [-1,0]$ and as $t \to \infty$

$$\tilde{x}_0(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \frac{1-e^{-v_1}}{v_1(a-v_1-1)}e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 3, \\ \frac{2}{v_0}[(1-e^{-v_0})t + e^{-v_0} - \frac{1-e^{-v_0}}{v_0}]e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 4, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \tilde{\phi}_1(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1,\ j = 5, \\ \tilde{\phi}_0(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0,\ j = 6, \\ \frac{1}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} - \frac{1}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{1-e^{-v_0}}{v_0(v_0-a+1)}e^{v_0 t} + \tilde{\phi}_1(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9, \end{cases}$$

where

$$\tilde{\phi}_i(t) = \tilde{A}_i \cos\xi_i t + \tilde{B}_i \sin\xi_i t,$$

$$\begin{aligned} \tilde{A}_i &= \frac{1}{v_i^2 + \xi_i^2}[\xi_i e^{-v_i}\sin\xi_i - v_i e^{-v_i}\cos\xi_i + v_i]A_i \\ &+ \frac{1}{v_i^2 - \xi_i^2}[v_i e^{-v_i}\sin\xi_i + v_i e^{-v_i}\cos\xi_i - \xi_i]B_i, \end{aligned}$$

$$\begin{aligned} \tilde{B}_i &= \frac{1}{v_i^2 + \xi_i^2}[\xi_i - v_i e^{-v_i}\sin\xi_i - \xi_i e^{-v_i}\cos\xi_i]A_i \\ &+ \frac{1}{v_i^2 - \xi_i^2}[\xi_i e^{-v_i}\sin\xi_i - v_i e^{-v_i}\cos\xi_i + v_i]B_i. \end{aligned}$$

Analogously we can get the following representation for the process $\tilde{\Psi}(t)$ with $x_\Psi(t) = \beta\tilde{x}_0(t) - \alpha\tilde{x}_0(t-1)$:

$$\begin{aligned}\tilde{\Psi}(t) &= \sigma^{-1}(x_\Psi(t)X_0(0) + b\int_{-1}^{0} x_\Psi(t-s-1)X_0(s)ds \\ &+ \int_0^t x_\Psi(t-s)dW(s))\end{aligned} \tag{20}$$

for $t \geq 2$; and x_φ has the properties $x_\Psi(t) = 0$ for $t \in [-1, 0]$; and for $t \to \infty$ it holds

$$x_\Psi(t) = \begin{cases} o(e^{\gamma t}), & \gamma < 0, \quad j = 1, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)} e^{v_0 t} + o(e^{\gamma t}), & \gamma < 0, \quad j = 2, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)} e^{v_0 t} + \frac{(1-e^{-v_1})(\beta-\alpha e^{-v_1})}{v_1(a-v_1-1)} e^{v_1 t} & \\ +o(e^{\gamma_1 t}), & \gamma_1 < v_1, \; j = 3, \\ \frac{2}{v_0}\{[(1-e^{-v_0})t + e^{-v_0} - \frac{1-e^{-v_0}}{v_0}](\beta - \alpha e^{-v_0}) & \\ +\alpha e^{-v_0}(1-e^{-v_0})\}e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0, \; j = 4, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)} e^{v_0 t} + \phi_1^*(t)e^{v_1 t} + o(e^{\gamma_1 t}), & \gamma_1 < v_1, \; j = 5, \\ \phi_0^*(t)e^{v_0 t} + o(e^{\gamma_0 t}), & \gamma_0 < v_0, \; j = 6. \\ \frac{\beta-\alpha}{1-a} + o(e^{\gamma t}), & \gamma < 0, \quad j = 7, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)} e^{v_0 t} - \frac{\beta-\alpha}{a-1} + o(e^{\gamma t}), & \gamma < 0, \quad j = 8, \\ \frac{(1-e^{-v_0})(\beta-\alpha e^{-v_0})}{v_0(v_0-a+1)} e^{v_0 t} + \phi_1^*(t) + o(e^{\gamma t}), & \gamma < 0, \quad j = 9. \end{cases}$$

Here

$$\phi_i^*(t) = A_i^* \cos\xi_i t + B_i^* \sin\xi_i t,$$

$$A_i^* = \beta\tilde{A}_i - \alpha\tilde{A}_i e^{-v_i}\cos\xi_i - \alpha\tilde{B}_i e^{-v_i}\sin\xi_i,$$

$$B_i^* = \beta\tilde{B}_i - \alpha\tilde{A}_i e^{-v_i}\sin\xi_i - \alpha\tilde{B}_i e^{-v_i}\cos\xi_i, \; i = 0, 1.$$

The processes $\tilde{\Psi}(t)$ and $\tilde{V}(t)$ are mutually independent (by assumption, W, V and X_0 are independent), and the process $\tilde{\Psi}(t)$ has a representation similar to (18). This is a consequence of the definition of $\tilde{\Psi}$ and the preceding calculations.
Then, after a series of calculations similar to those in [2] and [5, 6] we get the following limits:
– for $(i,j) \in I_1$

$$f_{iju} = \begin{cases} \sigma^{-2}(\int_0^\infty x_\Psi^2(t)dt + 1), & u = 0, \\ \sigma^{-2}\int_0^\infty x_\Psi(t+u)x_\Psi(t)dt, & u \geq 2; \end{cases}$$

– for $(i,j) \in I_2 \cup I_3$

$$\lim_{t\to\infty} e^{-v_i t}\tilde{\Psi}(t) = \tilde{c}_{ij}U_i \quad P-\text{a.s.},$$

$$U_i = X_0(0) + b\int_{-1}^{0} e^{-v_i(s+1)}X_0(s)ds + \int_0^\infty e^{-v_i s}dW(s),$$

$$\tilde{c}_{0j} = \frac{(1-e^{-v_0})(\beta - \alpha e^{-v_0})}{v_0(v_0 - a + 1)}\sigma^{-1}, \ \tilde{c}_{13} = \frac{(1-e^{-v_1})(1-e^{v_0 - v_1})}{v_1(a - v_1 - 1)}\beta\sigma^{-1},$$

$$\tilde{c}_{14} = \frac{2(1-e^{-v_0})}{v_0}\beta\sigma^{-1}$$

and as follows

$$f_{iju} = \frac{\tilde{c}_{ij}^2 U_i^2}{2v_i} e^{-v_i u}, \ u \geq 0;$$

– for $(i,j) \in I_4$

$$\lim_{t\to\infty} t^{-1} e^{-v_0 t}\tilde{\Psi}(t) = \tilde{c}_0 U_0 \quad P - \text{a.s.},$$

$$\tilde{c}_0 = \frac{2(1-e^{-v_0})}{v_0}(\beta - \alpha e^{-v_0})\sigma^{-1}$$

and

$$f_{iju} = \frac{\tilde{c}_0^2 U_0^2}{4v_0} e^{-v_i u}, \ u \geq 0;$$

– for $(i,j) \in I_5$

$$\lim_{t\to\infty} |e^{-v_i t}\tilde{\Psi}(t) - U_{ij}(t)| = 0 \quad P - \text{a.s.},$$

where for $(i,j) \in I_5 \setminus \{(1,6)\}$

$$\begin{aligned} U_{ij}(t) &= \sigma^{-1}(X_0(0)\phi_i^*(t) + b\int_{-1}^0 \phi_i^*(t-s-1)e^{-v_i(s+1)}X_0(s)ds \\ &+ \int_0^\infty \phi_i^*(t-s)e^{-v_i s}dW(s)), \end{aligned}$$

$$\begin{aligned} U_{16}(t) &= \sigma^{-1}(X_0(0)\phi_0^*(t) + b\int_{-1}^0 \phi_0^*(t-s-1)e^{-v_i(s+1)}X_0(s)ds \\ &+ \int_0^\infty \phi_0^*(t-s)e^{-v_i s}dW(s)) \end{aligned}$$

and

$$f_{iju}(T) = \sigma^{-2}e^{v_i u}\int_0^\infty e^{-2v_i t}U_i(T-t)\hat{U}_i(T-t)dt, \ u \geq 0,$$

$$\begin{aligned} \hat{U}_i(t) &= X_0(0)\hat{\phi}_i(t) + b\int_{-1}^0 \hat{\phi}_i(t-s-1)e^{-v_i(s+1)}X_0(s)ds \\ &+ \int_0^\infty \hat{\phi}_i(t-s)e^{-v_i s}dW(s), \end{aligned}$$

$$\hat{\phi}_i(t) = \hat{A}_i \cos \xi_i t + \hat{B}_i \sin \xi_i t,$$

$$\hat{A}_i = A_i^* \cos \xi_i u - B_i^* \sin \xi_i u,\ \hat{B}_i = -A_i^* \sin \xi_i u + B_i^* \cos \xi_i u,\ i = 0, 1.$$

Here $U_i(t) \equiv \hat{U}_i(t)$ by $u = 0$.

The relations (16) and (17) are proved. We continue to show the finiteness of $T(\varepsilon)$.

Because the function $x_\Psi(t)$ is defined similar to the function $x_0(t)$ (its structure and properties have been investigated, for example, in [2]), we can see that $meas\{u \in (2,3] :\ f_{iju} = 0\} = 0$ in the cases $(i,j) \in I_1$ and it is obviously that $f_{iju} \neq 0\ P-$a.s. for $(i,j) \in I_2 \cup I_3 \cup I_4$.

Define for $(i,j) \in I_5$

$$\overline{f}_{iju} = \sup_{t \in (0,\infty)} |f_{iju}(t)|,\ \ \underline{f}_{ij0} = \inf_{t \in (0,\infty)} |f_{ij0}(t)|.$$

It is clear that for $u = 0$ and $u \geq 2$ respectively these values are positive and finite. From here and (16), (17) it follows, in particular, the finiteness of the stopping times $\tau_\varepsilon(n),\ n \geq 1$ defined by (10), because for all $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4$ the limits f_{ij0} are positive $P - a.s.$

By using (16) and the definition of $\tau_\varepsilon(n)$ we have the next limiting equalities:

– for $(i,j) \in I_1$

$$\lim_{n\to\infty} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1} c_n} = \lim_{\varepsilon\to 0} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1} c_n} = f_{ij0}^{-1}\ P_\vartheta - \text{a.s.} \tag{21}$$

Taking into account the inequalities

$$\int_0^{\tau_\varepsilon(n)-2-4h} \Psi^2(t)dt < \varepsilon^{-1} c_n \leq \int_0^{\tau_\varepsilon(n)-2-3h} \Psi^2(t)dt,$$

we obtain:

– for $(i,j) \in I_2 \cup I_3$

$$e^{2v_i(2+3h)} f_{ij0}^{-1} \leq \underline{\lim}_{n\to\infty} \frac{e^{2v_i \tau_\varepsilon(n)}}{\varepsilon^{-1} c_n} \leq \overline{\lim}_{n\to\infty} \frac{e^{2v_i \tau_\varepsilon(n)}}{\varepsilon^{-1} c_n} \leq e^{4v_i(1+2h)} f_{ij0}^{-1}\ P - \text{a.s.}, \tag{22}$$

$$e^{2v_i(2+3h)} f_{ij0}^{-1} \leq \underline{\lim}_{\varepsilon\to 0} \frac{e^{2v_i \tau_\varepsilon(n)}}{\varepsilon^{-1} c_n} \leq \overline{\lim}_{\varepsilon\to 0} \frac{e^{2v_i \tau_\varepsilon(n)}}{\varepsilon^{-1} c_n} \leq e^{4v_i(1+2h)} f_{ij0}^{-1}\ P - \text{a.s.} \tag{23}$$

and as follows

$$2 + 3h - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln \varepsilon^{-1} \leq \underline{\lim}_{n\to\infty} [\tau_\varepsilon(n) - \frac{1}{2v_i} \ln c_n] \leq \overline{\lim}_{n\to\infty} [\tau_\varepsilon(n)$$

$$-\frac{1}{2v_i} \ln c_n] \leq 2(1+2h) - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln \varepsilon^{-1}\ P - \text{a.s.}, \tag{24}$$

$$2+3h-\frac{1}{2v_i}\ln f_{ij0}+\frac{1}{2v_i}\ln c_n\le \varliminf_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le \varlimsup_{\varepsilon\to 0}[\tau_\varepsilon(n)$$

$$-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le 2(1+2h)-\frac{1}{2v_i}\ln f_{ij0}+\frac{1}{2v_i}\ln c_n\ \ P-\text{a.s.}; \tag{25}$$

– for $(i,j)\in I_4$

$$e^{2v_i(2+3h)}f_{ij0}^{-1}\le \varliminf_{n\to\infty}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le \varlimsup_{n\to\infty}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}$$

$$\le e^{4v_i(1+2h)}f_{ij0}^{-1}\ \ P-\text{a.s.}, \tag{26}$$

$$e^{2v_i(2+3h)}f_{ij0}^{-1}\le \varliminf_{\varepsilon\to 0}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le \varlimsup_{\varepsilon\to 0}\frac{\tau_\varepsilon^2(n)e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}$$

$$\le e^{4v_i(1+2h)}f_{ij0}^{-1}\ \ P-\text{a.s.} \tag{27}$$

From (17) and by the definition (10) of $\tau_\varepsilon(n)$ for all $(i,j)\in I_5$ we have

$$e^{2v_i(2+3h)}\overline{f}_{ij0}^{-1}\le \varliminf_{n\to\infty}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le \varlimsup_{n\to\infty}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}\underline{f}_{ij0}^{-1}\ \ P-\text{a.s.} \tag{28}$$

and

$$e^{2v_i(2+3h)}\overline{f}_{ij0}^{-1}\le \varliminf_{\varepsilon\to 0}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le \varlimsup_{\varepsilon\to 0}\frac{e^{2v_i\tau_\varepsilon(n)}}{\varepsilon^{-1}c_n}\le e^{4v_i(1+2h)}\underline{f}_{ij0}^{-1}\ \ P-\text{a.s.} \tag{29}$$

From (28) we obtain for every $\varepsilon>0$

$$2+3h-\frac{1}{2v_i}\ln\overline{f}_{ij0}+\frac{1}{2v_i}\ln\varepsilon^{-1}\le \varliminf_{n\to\infty}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln c_n]\le \varlimsup_{n\to\infty}[\tau_\varepsilon(n)$$

$$-\frac{1}{2v_i}\ln c_n]\le 2(1+2h)-\frac{1}{2v_i}\ln\underline{f}_{ij0}+\frac{1}{2v_i}\ln\varepsilon^{-1}\ \ P-\text{a.s.} \tag{30}$$

and from (29) for $n\ge 1$ if follows

$$2+3h-\frac{1}{2v_i}\ln\overline{f}_{ij0}+\frac{1}{2v_i}\ln c_n\le \varliminf_{\varepsilon\to 0}[\tau_\varepsilon(n)-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le \varlimsup_{\varepsilon\to 0}[\tau_\varepsilon(n)$$

$$-\frac{1}{2v_i}\ln\varepsilon^{-1}]\le 2(1+2h)-\frac{1}{2v_i}\ln\underline{f}_{ij0}+\frac{1}{2v_i}\ln c_n\ \ P-\text{a.s.} \tag{31}$$

Note that in the cases $I_2\cup I_3\cup I_5$ we have

$$\lim_{n\to\infty}\frac{\tau_\varepsilon(n)}{\ln c_n}=\lim_{\varepsilon\to 0}\frac{\tau_\varepsilon(n)}{\ln\varepsilon^{-1}}=\frac{1}{2v_i}\ \ P-\text{a.s.} \tag{32}$$

Put $\delta_\varepsilon(n) = \tau_\varepsilon(n) k_n h$.

Now we are able to show the finiteness of the stopping time ν_ε. From (16), (21), (22) and (26) with P–probability one we have the relations:
– for $(i,j) \in I_1$

$$\lim_{n\to\infty} \frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt = (\varepsilon f_{ij0})^{-1} f_{ij(2+3h)}; \tag{33}$$

– for $(i,j) \in I_2 \cup I_3 \cup I_4$

$$e^{4v_i}(\varepsilon f_{ij0})^{-1}|f_{ij(2+3h)}| \le \underline{\lim}_{n\to\infty} |\frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt|$$

$$\le \overline{\lim_{n\to\infty}} |\frac{1}{c_n} \int_2^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt| \le e^{2v_i(2+3h)}(\varepsilon f_{ij0})^{-1}|f_{ij(2+3h)}|. \tag{34}$$

Consider the cases $(i,j) \in I_5$. For all $u \ge 2$ and $(i,j) \in I_5$ the functions $f_{iju}(T)$ of T are periodic with corresponding periods $\Delta > 1$ and each of them has at most two roots on every interval of the lengths Δ. Denote these roots for $u = 2+3h$ as $t_m(i,j),\ m \le 2$ on the set $(0,\Delta]$. Then define V_{ij} to be the union of open disjoint neighborhoods with the radius less then $1/6$ for all roots $t_m(i,j) + N\Delta,\ m = 1,2,\ N \ge 0$ and put

$$\mathcal{R}_{ij}^+ = (0,\infty) \setminus V_{ij}.$$

Define

$$f^*_{iju} = \inf_{t\in\mathcal{R}_{ij}^+} |f_{iju}(t)|,$$

$$Q_n(i,j) = \{k = \overline{1,3} : nh - kh \in \mathcal{R}_{ij}^+\},$$

$$r_{ij}(n) = \arg\max_{k\in Q_n(ij)} |f_{ij(2+3h)}(nh-kh)|.$$

By the continuity of $f_{iju}(\cdot)$ we have $f^*_{iju} > 0$ for $u = 0$ and $u \ge 2$. Note that for any $h \in (0,1/3]$ and $(i,j) \in I_5$ the sets $Q_n(i,j)$ are non-empty and for n large enough from (17) we have

$$r_{ij}(n) = \arg\max_{k\in Q_n(i,j)} |e^{-2v_i(nh-kh)} \int_0^{nh-kh} \Psi(t-2-3h)\Psi(t)dt| \quad P-\text{a.s.},$$

besides by the definition of $Q_n(i,j)$ for n large enough with P–probability one

$$f^*_{ij(2+3h)} \le |f_{ij(2+3h)}(nh - r_{ij}(n)h)| \le \overline{f}_{ij(2+3h)}$$

and

$$f^*_{ij(2+3h)} \le |e^{-2v_i(nh-r_{ij}(n)h)} \int_0^{nh-r_{ij}(n)h} \Psi(t-2-3h)\Psi(t)dt| \le \overline{f}_{ij(2+3h)}.$$

Then for $(i,j) \in I_5$ with P–probability one we obtain the following relations

$$\overline{\lim_{n\to\infty}} |e^{-2v_i nh} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt|$$

$$= \overline{\lim_{n\to\infty}} e^{-2v_i r_n h} |e^{-2v_i(nh-r_n h)} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt| \le e^{-2v_i h} \overline{f}_{ij(2+3h)},$$

$$\underline{\lim_{n\to\infty}} |e^{-2v_i nh} \int_0^{nh-r_n h} \Psi(t-2-3h)\Psi(t)dt| \ge \underline{\lim_{n\to\infty}} e^{-2v_i r_{ij}(n)h}$$

$$\cdot |e^{-2v_i(nh-r_{ij}(n)h)} \int_0^{nh-r_{ij}(n)h} \Psi(t-2-3h)\Psi(t)dt| \ge e^{-6v_i h} f^*_{ij(2+3h)}$$

and as follows for all $\varepsilon > 0$

$$e^{-6v_i h} f^*_{ij(2+3h)} \le \underline{\lim_{n\to\infty}} |e^{-2v_i \tau_\varepsilon(n)} \int_0^{\tau_\varepsilon(n)-k_n h} \Psi(t-2-3h)\Psi(t)dt|$$

$$\le \overline{\lim_{n\to\infty}} |e^{-2v_i \tau_\varepsilon(n)} \int_0^{\tau_\varepsilon(n)-k_n h} \Psi(t-2-3h)\Psi(t)dt| \le e^{-2v_i h} \overline{f}_{ij(2+3h)}. \tag{35}$$

In such a way for the cases $(i,j) \in I_5$ from (28) and (35) with P–probability one we have

$$e^{4v_i} (\varepsilon \overline{f}_{ij0})^{-1} f^*_{ij(2+3h)} \le \underline{\lim_{n\to\infty}} \frac{1}{c_n} |\int_2^{\delta_\varepsilon(n)} \tilde{\Psi}(t-2-3h)\tilde{\Psi}(t)dt|$$

$$\le \overline{\lim_{n\to\infty}} \frac{1}{c_n} |\int_2^{\delta_\varepsilon(n)} \tilde{\Psi}(t-2-3h)\tilde{\Psi}(t)dt| \le e^{2v_i(2+3h)} (\varepsilon \underline{f}_{ij0})^{-1} \overline{f}_{ij(2+3h)}. \tag{36}$$

The finiteness of ν_ε follows from the definition ν_ε, (33), (34), (36) and the condition (14) on the sequences (c_n) and (κ_n).

Thus the finiteness of the stopping times $T(\varepsilon)$ is established.

Let us estimate the mean square deviation of θ^*_ε. From (11) and by definitions of the stopping time ν_ε and ρ it follows that for all $\vartheta \in \mathcal{R}^2$

$$E_\vartheta(\theta^*_\varepsilon - \theta)^2 = E_\vartheta G_\varepsilon^{-2}(\nu_\varepsilon)\zeta_\varepsilon^2(\nu_\varepsilon) \le \frac{\varepsilon^2}{\rho} E_\vartheta \frac{1}{\kappa^2_{\nu_\varepsilon}} \zeta^2_\varepsilon(\nu_\varepsilon)$$

$$\le \frac{\varepsilon^2}{\rho} \sum_{n\ge1} \frac{1}{\kappa_n^2} E_\vartheta \zeta^2_\varepsilon(n) \le \frac{9(2+\overline{s}^2)\varepsilon}{\rho} \sum_{n\ge1} \frac{c_n}{\kappa_n^2} = \varepsilon.$$

Thus the first property I.1^o of the sequential plans $(T(\varepsilon), \theta^*_\varepsilon)$ in Theorem 3.1 is proved.

In order to establish the second property note that similar to (33), (34), (36) for all $n \geq 1$ we can prove P-a.s.
– for $(i,j) \in I_1$

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt = f_{ij0}^{-1} f_{ij(2+3h)} c_n; \tag{37}$$

– for $(i,j) \in I_2 \cup I_3 \cup I_4$

$$e^{4v_i} f_{ij0}^{-1} |f_{ij(2+3h)}| c_n \leq \varliminf_{\varepsilon\to 0} \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right|$$

$$\leq \varlimsup_{\varepsilon\to 0} \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \leq e^{2v_i(2+3h)} f_{ij0}^{-1} |f_{ij(2+3h)}| c_n; \tag{38}$$

– for $(i,j) \in I_5$

$$e^{4v_i} \overline{f}_{ij0}^{-1} f_{ij(2+3h)}^* c_n \leq \varliminf_{\varepsilon\to 0} \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right|$$

$$\leq \varlimsup_{\varepsilon\to 0} \varepsilon \left| \int_0^{\delta_\varepsilon(n)} \Psi(t-2-3h)\Psi(t)dt \right| \leq e^{2v_i(2+3h)} \underline{f}_{ij0}^{-1} \overline{f}_{ij(2+3h)} c_n. \tag{39}$$

Analogously to [11] from the definition of ν_ε and from (37)-(39) we can see that for ε small enough and $(i,j) \in I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$

$$\nu'_{ij} \leq \nu_\varepsilon \leq \nu''_{ij} \quad P_\vartheta - \text{a.s.}, \tag{40}$$

where

$$\nu'_{ij} = \max\{\inf\{n \geq 1 : \; c_n/\kappa_n > g'_{ij}\} - 1, \; 1\},$$

$$\nu''_{ij} = \inf\{n \geq 1 : \; c_n/\kappa_n > g''_{ij}\},$$

$$g'_{ij} = \begin{cases} \rho^{1/2} f_{ij0} |f_{ij(2+3h)}^{-1}|, & (i,j) \in I_1, \\ \rho^{1/2} e^{-2v_i(2+3h)} f_{ij0} |f_{ij(2+3h)}^{-1}|, & (i,j) \in I_2 \cup I_3 \cup I_4, \\ \rho^{1/2} e^{-2v_i(2+3h)} \underline{f}_{ij0} \overline{f}_{ij(2+3h)}^{-1}, & (i,j) \in I_5, \end{cases}$$

$$g''_{ij} = \begin{cases} g'_{ij}, & (i,j) \in I_1, \\ \rho^{1/2} e^{-4v_i} f_{ij0} |f_{ij(2+3h)}^{-1}|, & (i,j) \in I_2 \cup I_3 \cup I_4, \\ \rho^{1/2} e^{-4v_i} \overline{f}_{ij0} (f_{ij(2+3h)}^*)^{-1}, & (i,j) \in I_5. \end{cases}$$

Now from (12), (21), (25), (27), (31) and (40) the second I.2^o assertion of Theorem 3.1 follows:

– for $(i,j) \in I_1$ by

$$f_{ij0}^{-1} c_{\nu'_{ij}} \leq \underline{\lim}_{\varepsilon \to 0} \ \varepsilon T(\varepsilon) \leq \overline{\lim}_{\varepsilon \to 0} \ \varepsilon T(\varepsilon) \leq f_{ij0}^{-1} c_{\nu''_{ij}} \ P - \text{a.s.};$$

– for $(i,j) \in I_2 \cup I_3$ by

$$2 + 3h - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln c_{\nu'_{ij}} \leq \underline{\lim}_{\varepsilon \to 0} \ [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}]$$

$$\leq \overline{\lim}_{\varepsilon \to 0} \ [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \leq 2(1 + 2h) - \frac{1}{2v_i} \ln f_{ij0} + \frac{1}{2v_i} \ln c_{\nu''_{ij}} \ P - \text{a.s.};$$

– for $(i,j) \in I_4$ by

$$e^{2v_0(2+3h)} f_{ij0}^{-1} c_{\nu'_{ij}} \leq \underline{\lim}_{\varepsilon \to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)} \leq \overline{\lim}_{\varepsilon \to 0} \ \varepsilon T^2(\varepsilon) e^{2v_0 T(\varepsilon)}$$

$$\leq e^{4v_0(1+2h)} f_{ij0}^{-1} c_{\nu''_{ij}} \ P - \text{a.s.};$$

– for $(i,j) \in I_5$ by

$$2 + 3h - \frac{1}{2v_i} \ln \overline{f}_{ij0} + \frac{1}{2v_i} \ln c_{\nu'_{ij}} \leq \underline{\lim}_{\varepsilon \to 0} [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}]$$

$$\leq \overline{\lim}_{\varepsilon \to 0} [T(\varepsilon) - \frac{1}{2v_i} \ln \varepsilon^{-1}] \leq 2(1 + 2h) - \frac{1}{2v_i} \ln \underline{f}_{ij0} + \frac{1}{2v_i} \ln c_{\nu''_{ij}} \ P - \text{a.s.}$$

Thus the proof of part I of Theorem 3.1 is finished.

In order to prove the second assertion II of Theorem 3.1 note that according to (33), (34) and (36)

$$\overline{\lim}_{n \to \infty} c_n |G_\varepsilon^{-1}(n)| < \infty \ P - \text{a.s.}$$

and from (13), (14) it follows that

$$\sum_{n \geq 1} \frac{1}{c_n} < \infty.$$

In view of the form for the deviation of the estimators $\theta_\varepsilon(n)$ from ϑ it suffices to establish the next limiting equality

$$\lim_{n \to \infty} \frac{1}{c_n} \zeta_\varepsilon(n) = 0 \ P - \text{a.s.},$$

which follows from (11), as well as Chebychev's inequality and by the Borel–Cantelli lemma.

Therefore strong consistency of the estimators $\theta_\varepsilon(n), \ \varepsilon > 0$ is obtained. $\square$

4.2 Proof of Theorem 3.2

Firstly we show the finiteness of the stopping times $\tilde{T}(\varepsilon)$.

We start by calculating for $u = 0$ and $u \geq 1$ the limits

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \tilde{\Delta}Y(t)\tilde{\Delta}Y(t-u)dt = f(u) \ P - \text{a.s.} \tag{41}$$

in the stationary case and

$$\lim_{T\to\infty} \left| \frac{1}{e^{2v_0 T}} \int_1^T \tilde{\Delta}Y(t)\tilde{\Delta}Y(t-u)dt - f_u(T) \right| = 0 \ P - \text{a.s.} \tag{42}$$

in the periodic case, where $f(u)$ is random function of u and $f_u(T)$ are periodic functions of T for all $u \geq 0$.

From (2) we have

$$\tilde{\Delta}Y(t) = \tilde{X}(t) + \tilde{\Delta}V(t), \ t \geq 1.$$

By assumption the processes $\tilde{X}(t)$ and $\tilde{\Delta}Y(t)$ are mutually independent. Similar to the proof of Theorem 3.1 we can get the following limiting relations using the definition of the process $\tilde{X}(t)$:

– in the stationary case

$$f(u) = \begin{cases} \int_0^\infty \tilde{x}_0^2 dt + 1, & u = 0, \\ \int_0^\infty \tilde{x}_0(t+u)\tilde{x}_0(t)dt, & u \geq 1; \end{cases}$$

– in the periodic case

$$f_u(T) = e^{-v_0 u} \int_0^\infty e^{-2v_0 t} U_0^*(T-t)U_u^*(T-t)dt, \ u \geq 0,$$

$$\begin{aligned} U_u^*(t) &= X_0(0)\tilde{\phi}_u^*(t) + b\int_{-1}^0 \tilde{\phi}_u^*(t-s-1)e^{-v_0(s+1)}X_0(s)ds \\ &+ \int_0^\infty \tilde{\phi}_u^*(t-s)e^{-v_0 s}dW(s), \end{aligned}$$

$$\tilde{\phi}_u^*(t) = \tilde{A}_u^* \cos\xi_0 t + \tilde{B}_u^* \sin\xi_0 t,$$

$$\tilde{A}_u^* = \tilde{A}_0 \cos\xi_0 u - \tilde{B}_0 \sin\xi_0 u, \ \tilde{B}_u^* = \tilde{B}_0 \cos\xi_0 u - \tilde{A}_0 \sin\xi_0 u.$$

By the definition of ξ_0 we can see that functions $f_u(T)$ are periodic with the period $\Delta > 1$. Note that $f(0) > 0$ and $0 < \underline{f}_0 = \inf_T f_0(T) < \sup_T f_0(T) = \overline{f}_0 < \infty$.

The relations (41), (42) and therefore the finiteness of the times $\tilde{\tau}_\varepsilon(n), \ n \geq 1, \ \varepsilon > 0$ are established.

From (41), (42) and by the definition of the stopping times $\tilde{\tau}_\varepsilon(n)$ we have the next limiting relations:

– in the stationary case

$$\lim_{n\to\infty}\frac{\tilde\tau_\varepsilon(n)}{\varepsilon^{-1}c_n}=\lim_{\varepsilon\to 0}\frac{\tilde\tau_\varepsilon(n)}{\varepsilon^{-1}c_n}=(2f(0))^{-1}\ P-\text{a.s.};\tag{43}$$

– in the periodic case for any $\varepsilon>0$

$$e^{2v_0(2+5\tilde h)}[\varepsilon(1+e^{2v_0\tilde h})\overline{f}_0]^{-1}\le\varliminf_{n\to\infty}c_n^{-1}e^{2v_0\tilde\tau_\varepsilon(n)}$$

$$\le\varlimsup_{n\to\infty}c_n^{-1}e^{2v_0\tilde\tau_\varepsilon(n)}\le e^{4v_0(1+3\tilde h)}[\varepsilon(1+e^{2v_0\tilde h})\underline{f}_0]^{-1}\ P-\text{a.s.}\tag{44}$$

and for $n\ge 1$

$$e^{4v_0(1+3\tilde h)}[(1+e^{2v_0\tilde h})\overline{f}_0]^{-1}c_n\le\varliminf_{\varepsilon\to 0}\varepsilon e^{2v_0\tilde\tau_\varepsilon(n)}\le\varlimsup_{\varepsilon\to 0}\varepsilon e^{2v_0\tilde\tau_\varepsilon(n)}$$

$$\le e^{2v_0(2+7\tilde h)}[(1+e^{2v_0\tilde h})\underline{f}_0]^{-1}c_n\ P-\text{a.s.}\tag{45}$$

From (43), (44) in the periodic case for $\varepsilon>0$

$$2(1+3\tilde h)-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde h})-\frac{1}{2v_0}\ln\overline{f}_0+\frac{1}{2v_0}\ln\varepsilon^{-1}$$

$$\le\varliminf_{n\to\infty}[\tilde\tau_\varepsilon(n)-\frac{1}{2v_0}\ln c_n]\le\varlimsup_{n\to\infty}[\tilde\tau_\varepsilon(n)-\frac{1}{2v_0}\ln c_n]\le 2+7\tilde h$$

$$-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde h})-\frac{1}{2v_0}\ln\underline{f}_0+\frac{1}{2v_0}\ln\varepsilon^{-1}\ P-\text{a.s.}\tag{46}$$

and for $n\ge 1$

$$2(1+3\tilde h)-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde h})-\frac{1}{2v_0}\ln\overline{f}_0+\frac{1}{2v_0}\ln c_n$$

$$\le\varliminf_{\varepsilon\to 0}[\tilde\tau_\varepsilon(n)-\frac{1}{2v_0}\ln\varepsilon^{-1}]\le\varlimsup_{\varepsilon\to 0}[\tilde\tau_\varepsilon(n)-\frac{1}{2v_0}\ln\varepsilon^{-1}]$$

$$\le 2+7\tilde h-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde h})-\frac{1}{2v_0}\ln\underline{f}_0+\frac{1}{2v_0}\ln c_n\ P-\text{a.s.}\tag{47}$$

From (41), (43) we can obtain in the stationary case

$$\lim_{n\to\infty}\frac{\varepsilon}{c_n}\int_1^{\tilde\tau_\varepsilon(n)}\tilde\Delta Y(t-u)\tilde\Delta Y(t)dt=\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}\int_1^{\tilde\tau_\varepsilon(n)}\tilde\Delta Y(t-u)\tilde\Delta Y(t)dt$$

$$=(2f(0))^{-1}f(u),\ u\ge 1\ \ P-\text{a.s.}$$

and

$$\lim_{n\to\infty}\frac{\varepsilon}{c_n}\tilde G_\varepsilon(n)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}\tilde G_\varepsilon(n)=\tilde G(\tilde h)\ P-\text{a.s.},\tag{48}$$

$$\tilde G(\tilde h)=(2f(0))^{-1}\begin{pmatrix} f(2+5\tilde h) & f(1+\tilde h)\\ f(2+6\tilde h) & f(1+6\tilde h)\end{pmatrix}.$$

Similar to the Case I we can see that $meas\{u\in[\tilde h_0,1/5]:\ f(u)=0\}=0$ and $meas\{u\in[\tilde h_0,1/5]:\ \det\tilde G(u)=0\}=0$. As follows $\det\tilde G(\tilde h)\neq 0\ P-$a.s. From here, (14), (15) and (48) we have the finiteness of the times $\tilde\nu_\varepsilon$ in the stationary case.

Put

$$\tilde G(T,\tilde h)=\begin{pmatrix} f_{2+5\tilde h}(T) & e^{-2v_0}f_{1+5\tilde h}(T)\\ f_{2+6\tilde h}(T) & e^{-2v_0}f_{1+6\tilde h}(T)\end{pmatrix}.$$

From (42) in the periodic case it follows that the matrices $\tilde G(T,s)$ are the limits of the matrix functions $\tilde f(T,s)=\dfrac{1}{e^{2v_0T}}G^*(T,s)$ in the almost surely sense:

$$\lim_{T\to\infty}|\tilde f(T,s)-\tilde G(T,s)|=0,\ \ s\geq 0\ P-\text{a.s.}\tag{49}$$

The matrix functions $\tilde G(T,s)$ are periodic with the period $\Delta>1$ and according to the definition of functions $f_u(T),\ u\geq 0$ the equation

$$\det\tilde G(T,s)=0$$

has at most four roots $\tilde t_m,\ m=\overline{1,4}$ on the set $(0,\Delta]$ for any s. Put $\tilde\delta_\varepsilon(n)=\tilde\tau_\varepsilon(n)-\tilde k_n\tilde h$. Note that in the periodic case by the definition of $\tilde G_\varepsilon(n)$ (15) and from (42) analogously to the proof of Theorem 3.1 we can get the following relations

$$\lim_{n\to\infty}\left|\frac{1}{e^{2v_0\tilde\delta_\varepsilon(n)}}\tilde G_\varepsilon(n)-\tilde G(\tilde\delta_\varepsilon(n),\tilde h)\right|$$

$$=\lim_{\varepsilon\to 0}\left|\frac{1}{e^{2v_0\tilde\delta_\varepsilon(n)}}\tilde G_\varepsilon(n)-\tilde G(\tilde\delta_\varepsilon(n),\tilde h)\right|=0\ P-\text{a.s.}\tag{50}$$

and for some constants $\tilde g_1,\ \tilde g_2$

$$0<\tilde g_1=\underline{\lim}_{n\to\infty}||\tilde G^{-1}(\tilde\delta_\varepsilon(n),\tilde h)||\leq\overline{\lim}_{n\to\infty}||\tilde G^{-1}(\tilde\delta_\varepsilon(n),\tilde h)||=\tilde g_2<\infty,\tag{51}$$

$$0<\tilde g_1=\underline{\lim}_{\varepsilon\to 0}||\tilde G^{-1}(\tilde\delta_\varepsilon(n),\tilde h)||\leq\overline{\lim}_{\varepsilon\to 0}||\tilde G^{-1}(\tilde\delta_\varepsilon(n),\tilde h)||=\tilde g_2<\infty.\tag{52}$$

From (14), (15), (44), (50) and (51) the finiteness of times $\tilde\nu_\varepsilon$ in the periodic case follows.

Thus the finiteness of the stopping times $\tilde T(\varepsilon)$ is established.

The property I.1^o of the sequential estimators $(\tilde{T}(\varepsilon), \tilde{\vartheta}(\varepsilon))$ and the strong consistency of the estimators $\tilde{\vartheta}_\varepsilon(n)$ may be proved similar to the proof of Theorem 3.1.

Now we find the limiting low and upper bounds for the duration time $\tilde{T}(\varepsilon)$ of our sequential estimation. Put for $k = 1, 2$

$$\tilde{\nu}(k) = \inf\{n \geq 1 : \ c_n/\kappa_n > \tilde{g}(k)\} - 1,$$

$$\nu^*(k) = \inf\{n \geq 1 : \ c_n/\kappa_n > g^*(k)\},$$

$$\tilde{g}(1) = g^*(1) = 2f(0)\tilde{\rho}^{1/2}||\tilde{G}^{-1}(\tilde{h})||,$$

$$\tilde{g}(2) = \tilde{\rho}^{1/2}\tilde{g}_1 e^{-4v_0(1+3\tilde{h})}(1 + e^{2v_0\tilde{h}})\underline{f}_0,$$

$$g^*(2) = \tilde{\rho}^{1/2}\tilde{g}_2 e^{-2v_0(2+\tilde{h})}(1 + e^{2v_0\tilde{h}})\overline{f}_0.$$

By the definition of $\tilde{\nu}_\varepsilon$ and from (43), (45), (48), (50), (52) it follows that for ε small enough
– in the stationary case

$$\tilde{\nu}(1) \leq \tilde{\nu}_\varepsilon \leq \nu^*(1); \tag{53}$$

– in the periodic case

$$\tilde{\nu}(2) \leq \tilde{\nu}_\varepsilon \leq \nu^*(2). \tag{54}$$

From (15), (43), (47), (53) and (54) the assertion I.2^o of Theorem 3.2 follows:
–in the stationary case

$$(2f(0))^{-1}c_{\tilde{\nu}(1)} \leq \varliminf_{\varepsilon\to 0} \varepsilon\tilde{T}(\varepsilon) \leq \varlimsup_{\varepsilon\to 0} \varepsilon\tilde{T}(\varepsilon) \leq (2f(0))^{-1}c_{\nu^*(1)} \ P - \text{a.s.};$$

– in the periodic case

$$2(1+3\tilde{h}) - \frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\overline{f}_0 + \frac{1}{2v_0}\ln c_{\tilde{\nu}(2)}$$

$$\leq \varliminf_{\varepsilon\to 0}[\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \leq \varlimsup_{\varepsilon\to 0}[\tilde{T}(\varepsilon) - \frac{1}{2v_0}\ln\varepsilon^{-1}] \leq 2 + 7\tilde{h}$$

$$-\frac{1}{2v_0}\ln(1+e^{2v_0\tilde{h}}) - \frac{1}{2v_0}\ln\underline{f}_0 + \frac{1}{2v_0}\ln c_{\nu^*(2)} \ P - \text{a.s.} \ \square$$

Remark 4 *It should be pointed out that one could obtain the following limiting equalities for $(i,j) \in I_1$ in Problem I*

$$\lim_{\varepsilon\to 0}\varepsilon\tilde{T}(\varepsilon) = f_{ij0}^{-1}c_{\nu^*_{ij}} \ P - a.s.$$

and in stationary case in Problem II

$$\lim_{\varepsilon\to 0}\varepsilon\tilde{T}(\varepsilon) = (2f(0))^{-1}c_{\nu^*(1)} \ P - a.s.$$

if the magnitudes $\rho^{1/2}\varepsilon^{-1}c_n|G_\varepsilon^{-1}(n)|$ and $\tilde{\rho}^{1/2}\varepsilon^{-1}c_n|\tilde{G}_\varepsilon^{-1}(n)|$ in the definitions of ν_ε and $\tilde{\nu}_\varepsilon$ respectively were replaced by the nearest integer from above and the sequences (c_n) and (κ_n) were chosen in such a way that the relation c_n/κ_n were fractional for all $n \geq 1$.

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