Mixed discretisation methods for the Discontinuous Galerkin method with analytical test-functions

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Abstract. The idea to this paper is a framework for modifying a standard Discontinuous Galerkin method for convection-diffusion-reaction-equations. We develop an abstract theory for the stability and error-estimates in L^2-norms for a mixed formulation. For diverse test-functions we apply our abstract theory. We apply the theory for the test-functions coming from analytical solutions of the adjoint problem. With the new test-functions we could improve the approximation. At the end we discuss the application of the new analytical test-functions.

Key words. Discontinuous Galerkin method, Stability, Error estimates, Analytical solutions, Test-functions

AMS subject classifications. 65N12, 65N15, 65N30, 65N60

1. Introduction

Based on the idea for our future work we present a framework for the analytical test-functions based on solving the adjoint-problem for the Discontinuous Galerkin method in a mixed formulation, so called local Discontinuous Galerkin method (LDG-methods), confer [17].

These local Discontinuous Galerkin methods are done with finite element test-functions, confer [16]. We introduce the improved test-functions and derive local analytical solutions. First we derive an abstract theory for the stability and the error-estimates in the L^2-norm for an arbitrary test-function. In a second part we apply our results with respect to the analytical test-functions and derive the improved results for the approximate solutions.

We explain the new test-functions from the adjoint problem of the convection-diffusion-reaction-equation. For these new test-functions we could develop an algebra for calculating the new test-functions for the applications.

The paper is organized as follows. In section 2 we introduce our equation and our underlying model for the equation. In the next section we describe the weak-formulation for the mathematical problems. The variational-formulation is introduced in section 3. In section 4 we introduce the Discontinuous Galerkin method in a mixed form for the discretization. In section 5 we develop an abstract theory for the stability and error-

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estimates. Further we applied the theory for different test-spaces and introduce a local analytical test-space in section 6. We discuss the possible applications and the advantages for the new method.

Finally in section 7 we discuss our future works and the new results.

2. Mathematical model and mathematical equations

The mathematical model is based on a potential waste scenario of radioactive contaminants, which are transported and reacted with flowing groundwater in porous media, confer [5], [6] and applied in our work [21]. The mathematical formulation of such models are convection-diffusion-reaction-equations. We will concentrate us in our analysis of the stability and error-estimates to the following convection-diffusion-reaction-equation with initial- and boundary-values, given as

$$\partial_t R\, u \; + \; \nabla \cdot (\underline{v}\, u - a\, \nabla u) + \lambda\, R\, u \;=\; f \;\text{ in } \Omega_T \;, \quad (1)$$

$$u(0) \;=\; u_0 \;\text{ on } \Omega \;, \quad (2)$$

where the parameter $\underline{v}$ is a smooth velocity, with $\nabla \cdot \underline{v} = 0$, a is the diffusion-term, given as a symmetric positive definite, bounded tensor and $\lambda \geq 0$ is the constant decay-rate, confer [25]. $R \geq 0$ is a constant retardation-factor. The definition for the domains are $\Omega_T = \Omega \times [0, T]$ where $T > 0$ and $\Omega \subset I\!R^d$ and d is the space-dimension. For the boundaries we have $\Gamma_T = \Gamma \times (0, T]$ where $T > 0$, where $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$. The dependent solution is $u(x, t) \in C^2(\Omega_T) \cap C(\Omega_T)$, where $u : \Omega_T \to I\!R$. The initial conditions are given as $u(0) = u_0 \in L^2(\Omega)$, where $u : \Omega \to I\!R$.
The Dirichlet boundary-conditions are given as

$$u \;=\; g_1 \;\text{ on } \Gamma_T^1 \;, \quad (3)$$

where $g_1 : \Gamma_T^1 \to I\!R$.
The Neumann boundary-conditions are given as

$$-a\, \nabla \cdot \underline{n}\, u \;=\; g_2 \;\text{ on } \Gamma_T^2 \;, \quad (4)$$

where $g_2 : \Gamma_T^2 \to I\!R$.
The inflow and outflow conditions are given as :

$$(\underline{v} - a\, \nabla) \cdot \underline{n}\, u \;=\; \underline{v} \cdot \underline{n}\, u_\Gamma = g_3 \;\text{ on } \Gamma_T^{3,b} \;, \quad (5)$$

where $g_3 : \Gamma_T^3 \to I\!R$ and $b = in$, out or no. We define the inflow part $\Gamma^{3,in}$ of the boundary Γ^2 for $\underline{n}(\gamma) \cdot \underline{v}(\gamma) < 0$ for $\gamma \in \Gamma^{3,in} \subset \Gamma$, we have the value $u_\Gamma(\gamma, t)$. The outflow part is $u_\Gamma(\gamma, t) = u(\gamma, t)$ with $\gamma \in \Gamma^{3,out} \subset \Gamma$. We set $g_3 = 0$ for no inflow- and outflow boundary $\Gamma^{3,no}$ and the boundary is $\Gamma^{3,in} \cup \Gamma^{3,out} \cup \Gamma^{3,no} = \Gamma^3$.

In the next section we describe the weak formulation of our equations, confer [9] and [18].

3. Variational formulation for the discretization methods

We use the variational formulation for the discretization methods, confer [18], and describe our weak formulation. We focus us on the mixed method for our Discontinuous Galerkin methods. With the weak-formulation we are flexible to introduce our modified spaces and improve the error-estimates.

3.1. Variational formulation and weak-solutions

For the discretization methods we introduce the variational formulation and derive the weak-solutions for the underlying convection-diffusion-reaction equation. The weak-solutions allow us to decrease the derivation order of the solution and for our discretization methods we could be used less smooth solutions, confer [7].

We applied the weak-formulation for the space variable and multiply with the variable $\phi \in H^1(\Omega)$.

For the notation of the weak formulation we introduce the inner product $L^2(S)$, which is denoted as $(\cdot, \cdot)_S$, and for $S = \Omega$, we skip the S. We denote it for the scalar-functions

$$(u, \phi)_S \;=\; \int_S u \, \phi \, ds \; , \tag{6}$$

and for the vector-functions we have the inner-product :

$$(\underline{p}, \underline{q})_S \;=\; \sum_{i=1}^{d} (p_i, q_i)_S \; , \tag{7}$$

where $\underline{p} = (p_1, \ldots, p_d)^t$ and $\underline{q} = (q_1, \ldots, q_d)^t$ are vectors. For a simpler notation we use for the vector-functions also the same bracelets as for the scalar-functions.

We multiply the equation (1) with the test-function $\phi(x)$ and find $u(x, t) \in H^1(\Omega_T)$ such that

$$\begin{aligned} &\int_\Omega R \, \partial_t u \, \phi \, dx \;+\; \int_\Gamma (\underline{v} \cdot \underline{n} \, u) \, \phi \, ds \;-\; \int_\Omega u \, (\underline{v} \cdot \nabla \phi) \, dx \\ &- \int_\Gamma (\underline{n} \cdot a \nabla u) \, \phi \, ds \;+\; \int_\Omega (a \, \nabla u) \cdot \nabla \phi \, dx \;+\; \int_\Omega R \, \lambda \, u \, \phi \, dx \;=\; \int_\Omega f \, \phi \, dx \; , \end{aligned} \tag{8}$$

for all $\phi(x) \in H^1(\Omega)$.

We substitute the boundary-conditions in the equation (8). We obtain the following formulations for the continuous form, confer [26].

Let u_0 in $H^1(\Omega_T)$ and satisfy $u_0 = g_1$ on $\Gamma^1 \times [0, T]$.

Find $u \in H^1(\Omega_T)$ such that :

$$\begin{aligned} &u - u_0 \in H^1_0(\Omega_T) \; , \\ &(\partial_t u, \phi) - (g_2, \phi)_{\Gamma^2} - (u, \underline{v} \cdot \nabla \phi) + (a \nabla u, \nabla \phi) + (\lambda u, \phi) = (f, \phi) \; , \end{aligned} \tag{9}$$

for all $\phi(x) \in H^1(\Omega)$. The initial conditions $u_0 = u(x, 0)$ on Ω are applied in the time-integration, where we use explicit methods, confer [24]. The right hand side is defined in $f \in L^2(\Omega_T)$, and the boundaries are defined for $g_1 \in L^2(H^{1/2}(\Gamma^1), [0, T])$ and $g_2 \in L^2(H^{-1/2}(\Gamma^2), [0, T])$.

3.2. Weak formulation for a mixed method

We introduce the weak formulation for the mixed methods. This formulation is done in the continuous space, we will apply the mixed method later for the discontinuous space. We use therefore the notation $\underline{p} = a^{1/2}\ \nabla u$ and could reformulate in a mixed method. The diffusion-term is formulated in a mixed method for the further mixed discretization methods. The solution is given by $u(x,t) \in C^2(\Omega) \times C^1([0,T])$ and $\underline{p}(x,t) \in (C^2(\Omega) \times C^1([0,T]))^d$ for the classical formulation

$$\partial_t R u + \nabla \cdot \underline{v}\, u - \nabla \cdot a^{1/2}\ \underline{p} + R\lambda\, c = f \ , \ \text{in}\ \Omega \ , \tag{10}$$
$$-a^{1/2}\ \nabla u + \underline{p} = 0 \ , \ \text{in}\ \Omega \ , \tag{11}$$
$$u = g_1 \ , \ \text{on}\ \Gamma_1 \ ,$$
$$(\underline{v}\, u - a^{1/2}\ \nabla u) \cdot \underline{n} = g_2 \ , \ \text{on}\ \Gamma_2 \ ,$$
$$u(0) = u_0 \ , \ \text{in}\ \Omega \ .$$

We use equation (10) and formulate the weak solutions. We find
$u(x,t) \in L^2(H^1(\Omega),[0,T])$ and $\underline{p}(x,t) \in (L^2(H^1(\Omega),[0,T]))^d$ for the formulation

$$\int_\Omega \partial_t R u\, \phi\, dx \ + \ \int_\Gamma (\underline{v} \cdot \underline{n}\, u)\, \phi\, ds \ - \ \int_\Omega u\, (\underline{v} \cdot \nabla\, \phi)\, dx \tag{12}$$
$$- \int_\Gamma (a^{1/2}\ \underline{p} \cdot \underline{n})\, \phi\, ds \ + \ \int_\Omega a^{1/2}\ \underline{p} \cdot \nabla \phi\, dx \ + \ \int_\Omega R\lambda\, u\, \phi\, dx \ = \ \int_\Omega f\, \phi\, dx \ ,$$
$$- \ \int_\Gamma u\, (\underline{n} \cdot a^{1/2}\ \underline{\chi})\, ds \ + \ \int_\Omega u\, (\nabla \cdot a^{1/2}\ \underline{\chi})\, dx \ + \ \int_\Omega \underline{p} \cdot \underline{\chi}\, dx \ = \ 0 \ , \tag{13}$$
$$\int_\Omega u(0)\, \phi\, dx \ = \ \int_\Omega u_0\, \phi\, dx \ , \tag{14}$$
$$u = g_1 \ , \ \text{on}\ \Gamma_T^1 \ ,$$
$$(\underline{v}\, u - a^{1/2}\ \nabla u) \cdot \underline{n} = g_2 \ , \ \text{on}\ \Gamma_T^2 \ ,$$

for all $\phi \in H^1(\Omega)$ and for all $\underline{\chi} \in (H^1(\Omega))^d$.

The continuous situation is given in equation (12) and (13) we could apply the boundary values for the equations and derive the following formulation

$$(\partial_t R u, \phi) \ - \ (u, (\underline{v} \cdot \nabla\, \phi)) \tag{15}$$
$$+ \ (a^{1/2}\ \underline{p}, \nabla \phi) \ + \ (R\lambda\, u, \phi) \ = \ (f, \phi) \ - \ (g_2, \phi)_{\Gamma^2} \ ,$$
$$-(u, (\nabla \cdot a^{1/2}\ \underline{\chi})) \ + (\underline{p}, \underline{\chi}) \ = \ (g_1, (\underline{n} \cdot \underline{\chi}))_{\Gamma^1} \ , \tag{16}$$
$$(u(0), \phi) = (u_0, \phi) \ . \tag{17}$$

In the next section we describe the weak-formulation with adequate trial- and test-space, with respect to the discrete formulations for Discontinuous Galerkin method.

4. Discretization method with Discontinuous Galerkin methods

4.1. Broken sobolev spaces

In the following notation the multi-dimensional case for the Discontinuous Galerkin methods is introduced.

We use the triangulation $\mathcal{K}_h$ with $h > 0$ for the domain Ω and h is the cell-width. We have for each sub-domain $K \in \mathcal{K}_h$ a Lipschitz boundary. The adjacent elements of $\mathcal{K}_h$ could be lie on an edge or a face. $\mathcal{E}_h^i$ is the set of all interior boundaries e of $\mathcal{K}_h$ and $\mathcal{E}_h^b$ is the set of all exterior boundaries e of $\Gamma = \partial\Omega$, whereby $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. The exterior boundaries could be imposed as Dirichlet-boundary $\mathcal{E}_h^1$ on Γ^1 or both as Neumann-boundary and as inflow- and outflow boundaries $\mathcal{E}_h^2$ on Γ^2 .

We define the broken Sobolev-space by:

$$H^l(\mathcal{K}_h) = \{v \ \in \ L^2(\Omega) \ : \ v|_K \in H^l(K) \ \ \forall K \in \mathcal{K}_h\} \ . \tag{18}$$

where $l \geq 0$ is the order of the Sobolev-space and we have the H^l-norm:

$$||v||_{H^l(\Omega)} = \left(\sum_{K \in \mathcal{K}_h} ||v||^2_{H^l(K)} \right)^{1/2} \ . \tag{19}$$

The function in $H^l(\mathcal{K}_h)$ are piecewise smooth.

We introduce the jumps across the edge $e = \partial K_1 \cap \partial K_2$

$$[v] = (v|_{K_2})|_e - (v|_{K_1})|_e \ , \tag{20}$$

and the averages on the interfaces are introduced as

$$\{v\} = \frac{(v|_{K_2})|_e + (v|_{K_1})|_e}{2} \ . \tag{21}$$

For the boundary Γ we introduce the notation of the jumps and averages, confer [16] and [17]

$$\{v\} = v|_e \ . \tag{22}$$

$$[v] = \begin{cases} 0 \ , & e \in \mathcal{E}_h^D \\ v \ , & e \in \mathcal{E}_h^N \end{cases} \tag{23}$$

For our further proofs we use the following identity for the jumps:

$$[u \ v] = [u]\{v\} + \{u\}[u] \ . \tag{24}$$

We define the orientation for the normal-vector from element K_2 to element K_1, see figure 1.

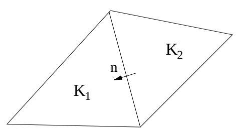

Figure 1. The Orientation of the normal vector for the Element K_1, K_2.

We apply the bilinear-forms for the broken Sobolev-space. We apply the integration over the elements and boundaries and rewrite the bilinear-forms with the following identities.

Lemma 1 *The identities for the bilinear-forms to the broken Sobolev-spaces over the elements and boundaries are given as:*

$$\int_\Omega R\, \partial_t u\, \phi\, dx \;=\; \sum_{K \in \mathcal{K}_h} (R\, \dot{u}, \phi)_K \;, \tag{25}$$

$$\int_\Gamma (\underline{v} \cdot \underline{n}\; u)\; \phi\; ds \;=\; \sum_{e \in \mathcal{E}_h} (\{\underline{v} \cdot \underline{n}\; u\}, [\phi])_e \;, \tag{26}$$

$$\int_\Omega u\; (\underline{v} \cdot \nabla\, \phi)\; dx \;=\; \sum_{K \in \mathcal{K}_h} (u, \underline{v} \cdot \nabla \phi)_K \;, \tag{27}$$

$$\int_\Gamma (a^{1/2}\; \underline{p} \cdot \underline{n})\; \phi\; ds \;=\; \sum_{e \in \mathcal{E}_h} (\{a^{1/2}\; \underline{p} \cdot \underline{n}\}, [\phi])_e \;, \tag{28}$$

$$\int_\Omega a^{1/2}\; \underline{p} \cdot \nabla \phi\; dx \;=\; \sum_{K \in \mathcal{K}_h} (a^{1/2}\; \underline{p}, \nabla \phi)_K \;, \tag{29}$$

$$\int_\Omega R\; \lambda\; u\; \phi\; dx \;=\; \sum_{K \in \mathcal{K}_h} (R\; \lambda u, \phi)_K \;. \tag{30}$$

We derive the identities in the following proof.

Proof. The identities (25), (27), (29) and (30) are trivial, we rewrite the integration over the whole domain in partial integrations over the partitions.

The identity (26) is rewritten by the boundary partitions as

$$\int_\Gamma (\underline{v} \cdot \underline{n}\; u)\; \phi\; ds \;=\; \sum_{K \in \mathcal{K}_h} \sum_{e \in \mathcal{E}_h \cap e \subset \partial K} (\underline{v} \cdot \underline{n}\; u_h, \phi)_e \;, \tag{31}$$

where $e \in \mathcal{E}_h \cap e \subset \partial K$ denote the edges of the element K.

We apply the jump-notation for the boundary-integrals with respect to the outer-normal vector of each K_2 element from the boundary $e = \partial K_2 \cap \partial K_1$ such that

$$\begin{aligned} &\sum_{K \in \mathcal{K}_h} \sum_{e \in \mathcal{E}_h \cap e \subset \partial K} (\underline{v} \cdot \underline{n}\; u, \phi)_e \\ &= \sum_{e \in \mathcal{E}_h} \int_e (\underline{v}_2 \cdot \underline{n}_2\; (u|_{K_2})|_e\; \phi|_{K_2} + \underline{v}_1 \cdot \underline{n}_1\; (u|_{K_1})|_e\; \phi|_{K_1})\; ds \\ &= \sum_{e \in \mathcal{E}_h} ([\underline{v} \cdot \underline{n}\;\; u\; \phi], 1)_e \;, \end{aligned} \tag{32}$$

where the definition of the normal-vector $\underline{n}_1 = -\underline{n}_2$ and $(u|_{K_2})|_e$ is the value for the element K_2 in the edge e.

Further we use the definition of the jumps and get

$$\sum_{e \in \mathcal{E}_h} ([\underline{v} \cdot \underline{n}\; u\; \phi], 1) = \sum_{e \in \mathcal{E}_h} (\{\underline{v} \cdot \underline{n}\; u\}, [\phi])_e + \sum_{e \in \mathcal{E}_h} ([\underline{v} \cdot \underline{n}\; u], \{\phi\})_e \;. \tag{33}$$

We now assume to have a smooth trial functions u and $\underline{p}$, confer [16], [11], because of the next step. We introduce the polynomial space where $\phi \in L_2(\Omega)$ and $\underline{\chi} \in (L_2(\Omega))^d$ and apply the Greens-formulation. For the smooth trial functions we defined for the jumps the zero condition, because of the not defined flux, confer [14].

$$[u] = 0 \ , \ [\underline{p}] = 0 \ . \tag{34}$$

Therefore we rewrite the jumps as

$$\sum_{e \in \mathcal{E}_h} ([\underline{v} \cdot \underline{n} \ u \ \phi], 1)_e = \sum_{e \in \mathcal{E}_h} (\{\underline{v} \cdot \underline{n} \ u\}, [\phi])_e \ . \tag{35}$$

The same result we get with the term $\underline{p}_h$

$$\int_\Gamma (a^{1/2} \ \underline{p} \cdot \underline{n}) \ \phi \ ds \ = \ \sum_{e \in \mathcal{E}_h} (\{a^{1/2} \ \underline{p} \cdot \underline{n}\}, [\phi])_e \ . \tag{36}$$

□

Further we could use the smoothness assumption for the solutions and the result of (34) and so the fluxes are given by

$$h_{conv}(u) = \{u \ \underline{v} \ \underline{n}\} \ , \tag{37}$$

$$h_{diff}(\underline{w}) = (a^{1/2} \ \underline{n} \ \{u\}, \{a^{1/2} \ \underline{p} \cdot \underline{n}\})^t \ , \tag{38}$$

where we have $\underline{w} = (u, \underline{p})^t$ and we use in formulations the central fluxes. We rewrite the formulations in the bilinear-forms.

We have to find $u(t) \in L_2(H^l(K), [0,T])$ and $\underline{p}(t) \in (L_2(H^l(K), [0,T]))^d$.
For $t > 0$

$$\begin{aligned}
&(R \ \partial_t u, \phi) - \sum_{K \in \mathcal{K}_h} (u, \underline{v} \cdot \nabla \phi)_K + \sum_{e \in \mathcal{E}_h} (h_{conv}(u), \phi)_e \\
&+ \sum_{K \in \mathcal{K}_h} (a^{1/2} \ \underline{p}, \nabla \phi)_K - \sum_{e \in \mathcal{E}_h} (h_{diff}(\underline{p}), [\phi])_e + (R \ \lambda \ u, \phi) \\
&= \sum_{e \in \mathcal{E}_h^N} (g_2, \phi)_e \ + \ (f, \phi) \ , \quad \phi \in H^1(\mathcal{K}_h) \ ,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&(\underline{p}, \underline{\chi}) + \sum_{K \in \mathcal{K}_h} (u, \nabla \cdot a^{1/2} \ \underline{\chi})_K - \sum_{e \in \mathcal{E}_h} (h_{diff}(u), [\underline{\chi}])_e \\
&= \sum_{e \in \mathcal{E}_h^D} (g_1, \underline{\chi} \cdot \underline{n})_e \ , \quad \underline{\chi} \in (H^1(\mathcal{K}_h))^d \ ,
\end{aligned} \tag{40}$$

and

$$(u(0), \phi) \ = \ (u_0, \phi) \ , \quad \phi \in H^1(\mathcal{K}_h) \ , \ t = 0. \tag{41}$$

We introduce the bilinear forms

$$A, C : V \times V \to I\!R \ , \ B : W \times V \to I\!R \ , \ D : W \times W \to I\!R \ , \tag{42}$$

$$F : V \to I\!R \ , \ G : W \to I\!R \ , \tag{43}$$

where $V = H^l(K)$ and $W = (H^l(K))^d$.

We have the formulation for the time-derivative

$$(\dot{u}, \phi) \;=\; (\partial_t u, \phi) \;, \tag{44}$$

and the bilinear-forms are given as

$$A(u, \phi) \;=\; - \sum_{K \in \mathcal{K}_h} (u, \underline{v} \cdot \nabla \phi)_K + \sum_{e \in \mathcal{E}_h} (h_{conv}(u), [\phi])_e \;, \tag{45}$$

$$B(\underline{p}, \phi) \;=\; \sum_{K \in \mathcal{K}_h} (a^{1/2} \; \underline{p}, \nabla \phi)_K - \sum_{e \in \mathcal{E}_h} (h_{diff}(\underline{p}), [\phi])_e \;, \tag{46}$$

$$B^T(u, \underline{\chi}) \;=\; \sum_{K \in \mathcal{K}_h} (u, \nabla \cdot a^{1/2} \; \underline{\chi})_K - \sum_{e \in \mathcal{E}_h} (h_{diff}(u), [\underline{\chi}])_e \;, \tag{47}$$

$$C(u, \phi) \;=\; (R \, \lambda \, u, \phi) \;, \tag{48}$$

$$D(\underline{p}, \underline{\chi}) \;=\; (\underline{p}, \underline{\chi}) \;, \tag{49}$$

$$F(\phi) \;=\; \sum_{e \in \mathcal{E}_h^N} (g_2, \phi)_e + (f, \phi) \;, \tag{50}$$

$$G(\underline{\chi}) \;=\; \sum_{e \in \mathcal{E}_h^D} (g_1, \underline{\chi} \cdot \underline{n})_e \;. \tag{51}$$

We formulate the equation (39) and (40) with the bilinear-forms (45) - (51).
Find $u(t) \in V$ and $\underline{p}(t) \in W$ such that,

$$(R \, \dot{u}, \phi) + A(u, \phi) + B(\underline{p}, \phi) + C(u, \phi) \;=\; F(\phi) \;, \; \phi \in V \;, \; t > 0 \;, \tag{52}$$

$$D(\underline{p}, \underline{\chi}) + B^T(u, \underline{\chi}) \;=\; G(\underline{\chi}) \;, \; \underline{\chi} \in W \;, \; t > 0 \;, \tag{53}$$

$$(u(0), \phi) \;=\; (u_0, \phi) \;, \; \phi \in V \;, \; t = 0. \tag{54}$$

4.2. Discrete formulation for local spaces

To apply the results for concrete spaces, we introduce the following local spaces. For the following abstract stability and error-indicator we introduce a local space $\mathcal{Q}(K)$ with arbitrary functions such that

$$\mathcal{Q}(\mathcal{K}_h) = \{ v \;\in\; L^2(\Omega) \;:\; v|_K \in \mathcal{F}(K) \;\; \forall K \in \mathcal{K}_h \} \;, \tag{55}$$

where $\mathcal{F}$ is finite dimensional space (e.g. polynomials, exponential functions, etc.).

We have to find the unknowns $\underline{p}_h(t) \in (L_2(\mathcal{Q}(\mathcal{K}_h), [0, T]))^d$
and $u_h(t) \in L_2(\mathcal{Q}(\mathcal{K}_h), [0, T])$ as follows

$$(R \, \dot{u}_h, \phi) + A(u_h, \phi) + B(\underline{p}_h, \phi) + C(u_h, \phi) \;=\; F(\phi) \;, \; \phi \in \mathcal{Q}(\mathcal{K}_h) \;, \; t > 0 \;,$$

$$D(\underline{p}_h, \underline{\chi}) + B^T(u_h, \underline{\chi}) \;=\; G(\underline{\chi}) \;, \; \underline{\chi} \in (\mathcal{Q}(\mathcal{K}_h))^d \;, \; t > 0 \;,$$

$$(u(0), \phi) \;=\; (u_0, \phi) \;, \; \phi \in \mathcal{Q}(\mathcal{K}_h) \;, \; t = 0 \;,$$

where the fluxes are defined as

$$\hat{h}_{conv}(u_h) = \begin{cases} \{u_h \; \underline{v} \; \underline{n}\} & \text{central differences} \\ \{u_h \; \underline{v} \; \underline{n}\} - \frac{|\underline{v} \; \underline{n}|}{2} [u_h] & \text{upwind} \end{cases} \;, \tag{56}$$

$$\hat{h}_{diff}(\underline{w}_h) = (a^{1/2} \; \underline{n} \; \{u_h\}, \{a^{1/2} \; \underline{p}_h \cdot \underline{n}\})^t + C_{diff}[(u_h, \underline{p}_h)^t] \;, \tag{57}$$

where the flux-matrix C_{diff} is given as

$$C_{diff} = \begin{pmatrix} 0 & -c_{1,2} & \dots & -c_{1,d+1} \\ c_{1,2} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ c_{1,d+1} & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\underline{c}^T \\ \underline{c} & 0 \end{pmatrix} , \tag{58}$$

where $\underline{c} = (c_{1,2}, \dots, c_{1,d+1})^T$, and $c_{1,i} = c_{1,i}((\underline{w}_h|_{K_2})|_e, (\underline{w}_h|_{K_1})|_e)$ is locally Lipschitz, confer [16].
One could rewrite the diffusive-flux

$$\hat{h}_{diff}(\underline{w}_h) = (\, a^{1/2} \, \underline{n} \, \{u_h\} + \underline{c} \, [u_h] \, , \, \{a^{1/2} \, \underline{p}_h \cdot \underline{n}\} - \underline{c}^T \cdot [\underline{p}_h])^T \, . \tag{59}$$

We denote the bilinear forms for the discrete formulation

$$\hat{A}, \hat{C} : V_h \times V_h \to I\!R \, , \; \hat{B} : W_h \times V_h \to I\!R \, , \; \hat{D} : W_h \times W_h \to I\!R \, , \tag{60}$$
$$\hat{F} : V_h \to I\!R \, , \; \hat{G} : W_h \to I\!R \, , \tag{61}$$

with the spaces $V_h = \mathcal{Q}(\mathcal{K}_h)$ and $W_h = (\mathcal{Q}(\mathcal{K}_h))^d$. We could use the bilinear-forms given in (44) - (51). The bilinear-form A is modified with the different convective flux, confer (56).

We have the bilinear-forms

$$\hat{A}(u_h, \phi) = - \sum_{K \in \mathcal{K}_h} (u_h, \underline{v} \cdot \nabla \phi)_K + \sum_{e \in \mathcal{E}_h} (\hat{h}_{conv}(u_h), [\phi])_e \, , \tag{62}$$
$$\hat{B}(\underline{p}_h, \phi) = \sum_{K \in \mathcal{K}_h} (a^{1/2} \, \underline{p}_h, \nabla \phi)_K - \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(\underline{p}_h), [\phi])_e \, , \tag{63}$$
$$\hat{B}^T(u_h, \underline{\chi}) = \sum_{K \in \mathcal{K}_h} (u_h, \nabla \cdot a^{1/2} \, \underline{\chi})_K - \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(u_h), [\underline{\chi}])_e \, , \tag{64}$$
$$\hat{C}(u_h, \phi) = (R \, \lambda u_h, \phi) \, , \tag{65}$$
$$\hat{D}(\underline{p}_h, \underline{\chi}) = (\underline{p}_h, \underline{\chi}) \, , \tag{66}$$
$$\hat{F}(\phi) = \sum_{e \in \mathcal{E}_h^N} (g_2, \phi)_e + (f, \phi) \, , \tag{67}$$
$$\hat{G}(\underline{\chi}) = \sum_{e \in \mathcal{E}_h^D} (g_1, \underline{\chi} \cdot \underline{n})_e \, , \tag{68}$$

where the bilinear-forms $C = \hat{C}$, $D = \hat{D}$, $F = \hat{F}$, $G = \hat{G}$ are equal and the bilinear-form $\hat{A}$, $\hat{B}$ could be different because of using the numerical fluxes, e.g. up-wind.
We define the discrete formulation.
Find $u_h(t) \in V_h$ and $\underline{p}_h(t) \in W_h$ such that

$$(R \, \dot{u}_h, \phi) + \hat{A}(u_h, \phi) + \hat{B}(\underline{p}_h, \phi) + \hat{C}(u_h, \phi) = \hat{F}(\phi) \, , \; \phi \in V_h \, , \; t > 0 \, , \tag{69}$$
$$\hat{D}(\underline{p}_h, \underline{\chi}) + \hat{B}^T(u_h, \underline{\chi}) = \hat{G}(\underline{\chi}) \, , \; \underline{\chi} \in W_h \, , \; t > 0 \, , \tag{70}$$
$$(u(0), \phi) = (u_0, \phi) \, , \; \phi \in V_h \, , \; t = 0 \, . \tag{71}$$

For the stability theorem we apply the next lemma for the proof. This lemma denotes the anti-symmetry for the bilinear-form $\hat{B}$ and this is used for the stability.

Lemma 2 *We have the bilinear-form $\hat{B}(\underline{p}_h, u_h)$ as defined in equation (63) and we have the solutions $u_h(t) \in V_h$, $\underline{p}_h(t) \in W_h$. We assume for the diffusive-flux the central-flux. For such assumption we could proof the anti-symmetry of the bilinear-form $\hat{B}$*

$$\hat{B}(\underline{p}_h, u_h) = -\hat{B}^T(u_h, \underline{p}_h) \; . \tag{72}$$

We proof the lemma 2 in the next step.

Proof. We have the formulation for the bilinear-form $\hat{B}(\underline{p}_h, u_h)$

$$\hat{B}(\underline{p}_h, u_h) \;=\; \sum_{K \in \mathcal{K}_h} (a^{1/2} \; \underline{p}_h, \nabla u_h)_K - \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(\underline{p}_h), [u_h])_e \; ,$$

we apply the Greens-formula and derive the following results

$$\begin{aligned}\hat{B}(\underline{p}_h, u_h) \;=&\; -\sum_{K \in \mathcal{K}_h} (\nabla \cdot a^{1/2} \; \underline{p}_h, u_h)_K + \sum_{K \in \mathcal{K}_h} \sum_{e \in \mathcal{E}_h \cap e \subset \partial K} (a^{1/2} \; \underline{p}_h, u_h)_e \\ &- \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(\underline{p}_h), [u_h])_e \; ,\end{aligned}$$

we use the identity (32) such that

$$\begin{aligned}\hat{B}(\underline{p}_h, u_h) \;=&\; -\sum_{K \in \mathcal{K}_h} (\nabla \cdot a^{1/2} \; \underline{p}_h, u_h)_K + \sum_{e \in \mathcal{E}_h} ([a^{1/2} \underline{p}_h \cdot \underline{n} \; u_h], 1)_e \\ &- \sum_{e \in \mathcal{E}_h} (\hat{h}_{diff}(\underline{p}_h), [u_h])_e \; ,\end{aligned}$$

and we obtain the following equation and we apply the diffusive flux (57),

$$\begin{aligned}\hat{B}(\underline{p}_h, u_h) \;=&\; -\sum_{K \in \mathcal{K}_h} (\nabla \cdot a^{1/2} \; \underline{p}_h, u_h)_K + \sum_{e \in \mathcal{E}_h} (\{a^{1/2} \underline{p}_h \cdot \underline{n}\}, [u_h])_e \\ &+ \sum_{e \in \mathcal{E}_h} ([\underline{p}_h], \{a^{1/2} \underline{n} \; u_h\})_e - \sum_{e \in \mathcal{E}_h} (\{a^{1/2} \; \underline{p}_h \cdot \underline{n}\} - \underline{c}^T \cdot [\underline{p}_h], [u_h])_e \; ,\end{aligned}$$

and we obtain the next equation and multiply and commute the last term.

$$\begin{aligned}\hat{B}(\underline{p}_h, u_h) \;=&\; -\sum_{K \in \mathcal{K}_h} (\nabla \cdot a^{1/2} \; \underline{p}_h, u_h)_K + \sum_{e \in \mathcal{E}_h} (\{a^{1/2} \underline{p}_h \cdot \underline{n}\}, [u_h])_e \\ &+ \sum_{e \in \mathcal{E}_h} ([\underline{p}_h], \{a^{1/2} \underline{n} \; u_h\})_e \\ &- \sum_{e \in \mathcal{E}_h} (\{a^{1/2} \; \underline{p}_h \cdot \underline{n}\}, [u_h])_e + \sum_{e \in \mathcal{E}_h} (\underline{c} \; [u_h], [\underline{p}_h])_e \; .\end{aligned} \tag{73}$$

We skip the equal terms and apply the E-fluxes of equation (57). We then obtain the results for the bilinear-forms

$$\begin{aligned}\hat{B}(\underline{p}_h, u_h) \;=&\; -\sum_{K \in \mathcal{K}_h} (\nabla \cdot a^{1/2} \; \underline{p}_h, u_h)_K \; + \sum_{e \in \mathcal{E}_h} < \hat{h}_{diff}(u_h), [\underline{p}_h] >_e \\ =&\; -\hat{B}^T(u_h, \underline{p}_h) \; ,\end{aligned} \tag{74}$$

this is the result of our lemma 2. □

The next lemma 3 is used for the proof of lemma 2.

Lemma 3 *We use the Greens-Formula for the multi-dimensional case with $u_h \in V_h$ and $\underline{p}_h \in W_h$ and $\underline{n}$ is the outer-normal vector at the edge e. For the formula we have the formulation*

$$\sum_{e \in \mathcal{E}_h} (\underline{p}_h \cdot \underline{n}, u_h)_e = \sum_{K \in \mathcal{K}_h} (\underline{p}_h, \nabla u_h)_K + \sum_{K \in \mathcal{K}_h} (\nabla \cdot \underline{p}_h, u_h)_K . \tag{75}$$

Proof. We use partial integration to proof the result. The proof is done in [8]. □

In the next section we proof the stability and derive the error-estimator for the new test-functions.

5. Stability and error-estimates

We proof the stability and the error-estimates for general broken Sobolev spaces and apply the special test-spaces. We apply the abstract results for different test-spaces, e.g. the standard test-space (polynomial space) or the new test-space (local exponential-space).

We will concentrate us in the next section to the boundary-values with $g_1 = 0$ and $g_2 = 0$, these mean the trivial inflow and outflow boundaries.

The abstract theory is formulate in the following section.

5.1. Stability of the scheme

We will concentrate us to the multi-dimensional case and proof the stability for arbitrary test-functions.

We derive the stability from the given bilinear-forms (69), (70) and (71).

For a simpler notation we define the error bilinear-form E_h for the further assumptions. We add the equations (69) and (70) and obtain the following results

$$\begin{aligned} E_h(\underline{w}_h, \underline{\psi}) \ &:= \ (R\, \dot{u}_h, \phi) + \hat{A}(u_h, \phi) + \hat{B}(\underline{p}_h, \phi) \\ &+ \hat{C}(u_h, \phi) + \hat{D}(\underline{p}_h, \underline{\chi}) + \hat{B}^T(u_h, \underline{\chi}) , \end{aligned} \tag{76}$$

whereby $\underline{w}_h = (u_h, \underline{p}_h)^T$ and $\underline{\psi} = (\phi, \underline{\chi})^T$.

Applying the results (74) we obtain

$$E_h(\underline{w}_h, \underline{w}_h) \ = \ (R\, \dot{u}_h, u_h) + \Theta_C(\underline{w}_h, \underline{w}_h) + (\underline{p}_h, \underline{p}_h) + R\, \lambda\, (u_h, u_h) . \tag{77}$$

whereby $\Theta_C(\underline{w}_h, \underline{w}_h)$ is given as

$$\Theta_C(\underline{w}_h, \underline{w}_h) = \sum_{e \in \mathcal{E}_h} ([\underline{w}_h] \ , \ C\ [\underline{w}_h])_e , \tag{78}$$

for C we have

$$C = \begin{pmatrix} c_{1,1} & -\underline{c}^T \\ & \\ \underline{c} & 0 \end{pmatrix} , \tag{79}$$

where the convective flux is a central flux for $c_{1,1} = 0$ and the upwind scheme is given by $c_{1,1} = \frac{|\underline{v} \cdot \underline{n}|}{2}$. We denote $c_{1,i} = 0$ for a 5 point central difference scheme for the diffusion flux, for $i = 2, \ldots, d+1$ and $c_{1,i} = \frac{a_{i-1}^{1/2}}{2}$ for a upwind scheme for the diffusive flux for $i = 2, \ldots, d+1$.

The flux-matrix fulfills the positivity of the bilinear-form $\Theta_C(\underline{w}_h, \underline{w}_h)$

Lemma 4 *Suppose that C is given in (79) and $\underline{w}_h \in V_h \times W_h$. Then we have the positivity for the bilinear-form Θ_C such that*

$$\Theta_C(\underline{w}_h, \underline{w}_h) = \sum_{e \in \mathcal{E}_h} ([\underline{w}_h] \ , \ C \ [\underline{w}_h])_e \ \geq \ 0 \ . \tag{80}$$

For the proof we did the following transformation.

Proof. We proof the positivity for each e and such that

$$([\underline{w}_h] \ , \ C \ [\underline{w}_h])_e \ \geq \ 0 \ , \tag{81}$$

We therefore apply the matrix C, given in (79), and obtain the results

$$\begin{aligned} & ([\underline{w}_h] \ , \ C \ [\underline{w}_h])_e \\ & = \int_e (\ [u_h] \ c_{1,1} \ [u_h] - u_h \ \underline{c}^T \cdot \underline{p}_h + \ \underline{p}_h^T \cdot \underline{c} \ u_h \ ) \ ds \\ & = \int_e c_{1,1} \ [u_h]^2 \ ds \ \geq \ 0 \ , \end{aligned} \tag{82}$$

where $c_{1,1} \geq 0$. ☐

In the following lemma we present the identity between the derivation-notation and jump-average-notation.

Lemma 5 *We have the divergence free velocity $\nabla \cdot \underline{v} = 0$ and Dirichlet-boundaries. We obtain the identity for the following terms*

$$\sum_{K \in \mathcal{K}_h} (u_h, \underline{v} \cdot \nabla u_h)_K = \sum_{e \in \mathcal{E}_h} (\underline{v} \cdot \underline{n} \ \{u_h\}, [u_h])_e \ . \tag{83}$$

The identity is proofed as follows

Proof. We start with the left-hand side of equation (91) and apply further the Gauss-theorem, confer [26] and rewrite the results with the jump-average-notation by using the equation (33), such that

$$\begin{aligned} \sum_{K \in \mathcal{K}_h} (u_h, \underline{v} \cdot \nabla u_h)_K \ &= \ \sum_{K \in \mathcal{K}_h} (\frac{\underline{v}}{2} \cdot \nabla u_h^2, 1)_K = \sum_{K \in \mathcal{K}_h} \sum_{e \in \mathcal{E}_h \cap e \subset \partial K} (\frac{\underline{v} \cdot \underline{n}}{2} \ u_h^2, 1)_e \\ &= \ \sum_{e \in \mathcal{E}_h} \frac{\underline{v} \cdot \underline{n}}{2} \ ([u_h^2], 1)_e = \sum_{e \in \mathcal{E}_h} (\underline{v} \cdot \underline{n} \ \{u_h\}, [u_h])_e \ . \end{aligned} \tag{84}$$

☐

We follow the result for the stability as

Theorem 6 *Suppose the bilinear-form E_h given in (77), the lemmas 4, 5 and the boundary-conditions are given for $g_1 = 0$ and $g_2 = 0$. Then we have for the stability the inequality*

$$\begin{aligned} E_h(\underline{w}_h, \underline{w}_h) \ &\geq \ R \ \frac{1}{2} \frac{\partial}{\partial t} ||u_h(t)||^2_{L^2(\Omega)} + \Theta_C(\underline{w}_h, \underline{w}_h) \\ &+ \ ||\underline{p}_h||^2_{(L^2(\Omega))^d} + R \ \lambda \ ||u_h||^2_{L^2(\Omega)} \ , \end{aligned} \tag{85}$$

The theorem 6 is proofed in the next step.
Proof. We estimate each term and derive the following error-estimates.
For the time-term we obtain the estimation

$$(R\,\dot{u}_h, u_h) \;=\; \int_\Omega R\,\partial_t u_h\, u_h\, dx = R\,\frac{1}{2}\frac{\partial}{\partial t}||u_h||^2_{L_2} \,. \tag{86}$$

For the convection-term we get the estimation

$$\begin{aligned}
\hat{A}(u_h, u_h) &= -\sum_{K\in\mathcal{K}_h}(u_h, \underline{v}\cdot\nabla u_h)_K + \sum_{e\in\mathcal{E}_h}(h_{conv}(u_h), [u_h])_e \\
&= -\sum_{e\in\mathcal{E}_h}(\{\underline{v}\cdot\underline{n}u_h\}, [u_h])_e + (h_{conv}(u_h), [u_h])_e \\
&= \begin{cases} 0 & \text{for central differences (E-Flux)} \\ \sum_{e\in\mathcal{E}_h}([u_h], \frac{|\underline{v}\cdot\underline{n}|}{2}[u_h])_e & \text{for upwind} \end{cases} \\
&= \sum_{e\in\mathcal{E}_h}([u_h], c_{11}[u_h])_e \;\geq\; 0\,,
\end{aligned}$$

where we use lemma 4.
For the mixed terms $\hat{B}$ for the diffusion we obtain the estimation

$$\begin{aligned}
&\hat{B}(\underline{p}_h, u_h) + \hat{B}^T(u_h, \underline{p}_h) \qquad (87)\\
=\;& \sum_{K\in\mathcal{K}_h}(a^{1/2}\;\underline{p}_h, \nabla u_h)_K - \sum_{e\in\mathcal{E}_h}(h_{diff}(\underline{p}_h), [u_h])_e \\
&+ \sum_{K\in\mathcal{K}_h}(u_h, \nabla\cdot a^{1/2}\;\underline{p}_h)_K - \sum_{e\in\mathcal{E}_h}(h_{diff}(u_h), [\underline{p}_h])_e \\
=\;& \sum_{K\in\mathcal{K}_h}\sum_{e\in\mathcal{E}_h\cap e\subset\partial K}(\nabla\cdot(a^{1/2}\;\underline{p}_h\; u_h), 1)_e \\
&- \sum_{e\in\mathcal{E}_h}(\;(h_{diff}(u_h), [u_h])_e + (h_{diff}(\underline{p}_h), [\underline{p}_h])_e\;) \\
=\;& \sum_{e\in\mathcal{E}_h}(\;([a^{1/2}\;\underline{p}_h\; u_h], 1)_e - (h_{diff}(\underline{w}_h), [\underline{w}_h])_e\;) \\
&= \sum_{e\in\mathcal{E}_h}(\;(\{a^{1/2}\;\underline{n}\cdot\underline{p_h}\}\;,\;[u_h])_e - (\{u_h\}\;,\;[a^{1/2}\;\underline{n}\cdot\underline{p}_h])_e \\
&+(\{a^{1/2}\;\underline{n}\cdot\underline{p_h}\}\;,\;[u_h])_e + <\{a^{1/2}u_h\;\underline{n}\},\;[\underline{p}_h]>_e) + ([\underline{w}_h], C_{diff}\;[\underline{w}_h])_e\;) \\
&= \sum_{e\in\mathcal{E}_h}([\underline{w}_h], C_{diff}[\underline{w}_h])_e \;\;\geq\;0\,,
\end{aligned}$$

For the reaction term we get the estimation

$$C(u_h, u_h) \;=\; R\,\lambda\int_\Omega u_h\; u_h dx \;=\; R\,\lambda||u_h(t)||^2_{L_2(\Omega)}\,, \tag{88}$$

For the mixed term D we get the estimation

$$D(\underline{p}_h, \underline{p}_h) \;=\; \int_\Omega \underline{p}_h\cdot\underline{p}_h dx \;=\; ||p_h(t)||^2_{(L_2(\Omega))^d}\,, \tag{89}$$

We add the terms (86) - (89), such that:

$$
\begin{aligned}
& (R\, \dot{u}_h, u_h) + \hat{A}(u_h, u_h) + \hat{B}(\underline{p}_h, u_h) + \hat{B}^T(u_h, \underline{p}_h) \\
& + C(u_h, u_h) + D(\underline{p}_h, \underline{p}_h) \\
& \geq\ R\, \frac{1}{2}\, \frac{\partial}{\partial t} ||u_h||^2_{L_2}\ +\ \Theta_C(\underline{w_h}, \underline{w_h}) \\
& + R\, \lambda ||u_h(t)||^2_{L_2(\Omega)}\ +\ ||p_h(t)||^2_{(L_2(\Omega))^d}\ \geq\ 0\ ,
\end{aligned} \tag{90}
$$

This is the result for the stability for arbitrary test-functions, confer equation (85) . □
For the stability we estimate the right-hand side as follows

$$
\int_\Omega f\, u_h\, dx \quad \leq \quad ||f||^2_{L_2(\Omega)}\, ||u_h||^2_{L_2(\Omega)}\ \leq\ \frac{1}{2\, R\, \lambda}\, ||f||^2_{L_2(\Omega)}\ +\ \frac{R\, \lambda}{2}\, ||u_h||^2_{L_2(\Omega)}\ ,
$$

where we use the Schwarz and Cronwall's inequality.

To get the full discrete formulation we integrate over the time-interval $(0, T)$ such that

$$
\int_0^T\ E_h(\underline{w}_h, \underline{w}_h)\ dt\ +\ \int_0^T\ F(u_h)\ dt = 0\ , \tag{91}
$$

using the stability result of equation (90) we get the stability result over the time integration. Therefore we obtain the following corollary.

Corollary 7 *We have the stability for the full-discrete form with the solutions* $u_h \in V$ *and* $\underline{p}_h \in W$ *such that*

$$
\begin{aligned}
& R\, \frac{1}{2} ||u_h(T)||^2_{L^2(\Omega)} + \int_0^T\ \Theta_C(\underline{w}_h, \underline{w}_h)\ dt + \int_0^T\ ||\underline{p}_h||^2_{(L^2(\Omega))^d}\ dt \\
& + \int_0^T\ \frac{R\, \lambda}{2}\ ||u_h||^2_{L^2(\Omega)}\ dt \leq \frac{1}{2} ||u_h(0)||^2_{L^2(\Omega)} + \int_0^T\ \frac{1}{2\, R\, \lambda}\ ||f||^2_{L^2(\Omega)}\ dt\ .
\end{aligned} \tag{92}
$$

In the next section we derive the abstract error estimates.

5.2. Abstract error-estimates

The error estimates for the multi-dimensional convection-diffusion-reaction equation is based on our former stability assumptions. We derive an abstract error-estimates for the equation and apply the results in the next section.

The error-estimator for multi-dimensions for the convection-diffusion-reaction term is given in the following theorem

Theorem 8 *The error-estimates is given as follows*
If $u, \underline{p}$ *and* $u_h, \underline{p}_h$ *are respective solutions of (10) and (11) and* $\mathcal{P}_h$ *is the* L^2*-projection*

$\mathcal{P}_h : L^2(V) \rightarrow V_h$ and $\mathcal{P}_h : L^2(W) \rightarrow W_h$. We get the error-estimates such that

$$R\,\frac{1}{2}||u(T) - u_h(T)||^2_{L^2(\Omega)} \; + (1 - \frac{\epsilon}{2}) \int_0^T \; ||\underline{p} - \underline{p}_h||^2_{L^2(\Omega)^d} \; dt \tag{93}$$
$$+(1 - \frac{\epsilon}{2}) \int_0^T \; \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h) \; dt$$
$$+ \, (\frac{R\lambda}{2} - \frac{R\lambda\,\epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon(\underline{v}\cdot\underline{v})}{2}) \int_0^T \; ||u - u_h||^2_{L^2(\Omega)} \; dt$$

$$\leq \; \frac{c}{2\epsilon} \int_0^T \Big(\; ||\mathcal{P}_h(\dot{u}(t)) - \dot{u}(t)||^2_{L^2(\Omega)} \; + \; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) \tag{94}$$
$$+ \sum_{e\in\mathcal{E}_h} \; ||\underline{v}\cdot\underline{n}\;\{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} \; + \sum_{e\in\mathcal{E}_h} ||\{a^{1/2}\underline{n}\cdot\{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)}$$
$$+ \sum_{e\in\mathcal{E}_h} \; ||\{a^{1/2}\;\underline{n}\;(\mathcal{P}_h(u) - u)\}||_{(L^2(e))^d}$$
$$+ \, \alpha_1(h_K)^{-2} \sum_{K\in\mathcal{K}_h} ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \; + \; \alpha_2(h_K)^{-2} \sum_{K\in\mathcal{K}_h} ||\mathcal{P}_h(u) - u||^2_{L^2(K)}$$
$$+ \, \alpha_3(h_K)^{-2} \sum_{K\in\mathcal{K}_h} ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{(L^2(K))^d} \Big) \; dt$$

where c is a constant, independent from t. The functions $\alpha_i(h_K)$ with $i = 1, 2, 3$ and $\gamma(h_e)$ depend from the test-spaces and are specified in the application in the next sections.

The functions $\alpha_i(h_K)$ with $i = 1, 2, 3$ are used to estimate the derivation, such that one could skip them to the left-hand-side. For the new test-functions same functions α_i will be vanish.

Lemma 9 *We have the local inequality*

$$\begin{aligned} \alpha_1(h_K)\;||\underline{v}\cdot\nabla(\mathcal{P}_h(u) - u_h) - R\,\lambda\,(\mathcal{P}_h(u) - u_h)||_{L^2(K)} & \\ \leq \; c\,||\mathcal{P}(u) - u_h||_{L^2(K)} \; , \; u_h \in V_h \; , & \\ \alpha_2(h_K)\;||\nabla(\mathcal{P}_h(u) - u_h)||_{(L^2(K))^d} \; \leq \; c\,||\mathcal{P}(u) - u_h||_{L^2(K)} \; , \; u_h \in V_h \; , & \\ \alpha_3(h_K)\;||a^{1/2}\;\nabla\cdot(\mathcal{P}_h(\underline{p}) - \underline{p}_h)||_{L^2(K)} \; \leq \; c\,||\mathcal{P}(\underline{p}) - \underline{p}_h||_{(L^2(K))^d} \; , \; \underline{p}_h \in W_h \; , & \end{aligned}$$

where c is a constant and h_K is the diameter of the element K.

Proof. We could apply the equation general introduced in [11]. For the special test-functions we apply the functions. □

We proof the theorem 8 in the following section.

Proof. We have the following error-equation from the orthogonality relation, confer [34], such that

$$E_h(\underline{w} - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \; = \; 0 \; , \tag{95}$$

where $\mathcal{P}_h(\underline{w}) - \underline{w}_h \in V_h \times W_h$ and $t \in (0, T)$.

For the application of the error-estimates we reformulate the error-equation by enlarge it with the projection-terms $\mathcal{P}_h$ such that

$$E_h(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) = E_h(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \tag{96}$$

For the left-hand side of equation (96) we use the result of the stability in equation (85) such that

$$\begin{aligned} & E_h(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \\ & \geq R\, \frac{1}{2}\frac{\partial}{\partial t} ||\mathcal{P}_h(u(t)) - u_h(t)||^2_{L^2(\Omega)} + \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \\ & + ||\mathcal{P}_h(\underline{p}) - \underline{p}_h||^2_{L^2(\Omega)^d}\; dt \; + \frac{R\,\lambda}{2} ||\mathcal{P}_h(u) - u_h||^2_{L^2(\Omega)} \;, \end{aligned} \tag{97}$$

For the right-hand side of the equation (96) we get the following formulation using (77)

$$\begin{aligned} & E_h(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \\ = \; & (R\, \mathcal{P}_h(\dot{u}) - \dot{u}, \mathcal{P}_h(u) - u_h) + \hat{A}(\mathcal{P}_h(u) - u, \mathcal{P}_h(u) - u_h) \\ & + \hat{B}(\mathcal{P}_h(\underline{p}) - \underline{p}, \mathcal{P}_h(u) - u_h) + \hat{C}(\mathcal{P}_h(u) - u, \mathcal{P}_h(u) - u_h) \\ & + \hat{D}(\mathcal{P}_h(\underline{p}) - \underline{p}, \mathcal{P}_h(\underline{p}) - \underline{p}_h) + \hat{B}^T(\mathcal{P}_h(u) - u, \mathcal{P}_h(\underline{p}) - \underline{p}_h) \;, \end{aligned} \tag{98}$$

We estimate the special terms in the following step.
For the time-term we get

$$\begin{aligned} & R\; (\mathcal{P}_h(\dot{u}) - \dot{u}, \mathcal{P}_h(u) - u_h) \\ & \leq \; R\; (\; \frac{1}{2\epsilon} \; ||\mathcal{P}_h(\dot{u}) - \dot{u}||^2_{L^2(\Omega)} \; + \; \frac{\epsilon}{2} \; ||\mathcal{P}_h(u) - u_h||^2_{L^2(\Omega)} \;) \end{aligned} \tag{99}$$

where ϵ is constant and independent from time.
We estimate the flux-term with the central-fluxes in the terms $\hat{A}$ and add the term Θ_C for the different up-winding.
For the flux-term we get the estimation

$$\begin{aligned} & \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \\ & \leq \frac{1}{2\epsilon}\; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) + \frac{\epsilon}{2}\; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \end{aligned} \tag{100}$$

For the term $\hat{A}$ and $\hat{C}$ we have :

$$\begin{aligned} & \hat{A}(\mathcal{P}_h(u) - u, \mathcal{P}_h(u) - u_h) + \hat{C}(\mathcal{P}_h(u) - u, \mathcal{P}_h(u) - u_h) \\ = \; & - \sum_{K \in \mathcal{K}_h} (\mathcal{P}_h(u) - u, \underline{v} \cdot \nabla(\mathcal{P}_h(u) - u_h) - R\,\lambda\; (\mathcal{P}_h(\underline{u}) - \underline{u}_h))_K \\ & + \sum_{e \in \mathcal{E}_h} (\underline{v} \cdot \underline{n}\; \{\mathcal{P}_h(u) - u\}, \mathcal{P}_h(u) - u_h])_e \; \leq \; \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_1(h_K)^{-2} ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \\ & + \frac{\epsilon}{2} \sum_{K \in \mathcal{K}_h} \alpha_1(h_K)^2 || - \underline{v} \cdot \nabla(\mathcal{P}_h(u) - u_h) + R\,\lambda\; (\mathcal{P}_h(\underline{u}) - \underline{u}_h)||^2_{L^2(K)} \\ & + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} ||\underline{v} \cdot \underline{n}\; \{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}_h} ||[\mathcal{P}_h(u) - u_h]||^2_{L^2(e)} \end{aligned} \tag{101}$$

For the terms $\hat{B}$ and $\hat{B}^T$ we obtain

$$
\begin{aligned}
& \hat{B}(\mathcal{P}_h(\underline{p}) - \underline{p}, \mathcal{P}_h(u) - u_h) \\
= & \sum_{K \in \mathcal{K}_h} (a^{1/2}\ \mathcal{P}_h(\underline{p}) - \underline{p}, \nabla(\mathcal{P}_h(u) - u_h))_K \\
& - \sum_{e \in \mathcal{E}_h} (\{a^{1/2}\ \underline{n} \cdot \mathcal{P}_h(\underline{p}) - \underline{p}\}, [\mathcal{P}_h(u) - u_h])_e \ , \\
\leq & \ \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{-2} ||a^{1/2}(\mathcal{P}_h(\underline{p}) - \underline{p})||^2_{(L^2(K))^d} \\
& + \frac{\epsilon}{2} \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^2 ||\nabla(\mathcal{P}_h(u) - u_h)||^2_{(L^2(K))^d} \\
& + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} ||\{a^{1/2} \underline{n} \cdot \{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}_h} ||[\mathcal{P}_h(u) - u_h]]||^2_{L^2(e)} \ ,
\end{aligned} \tag{102}
$$

where ϵ is a constant and independent from the time t. We use the Cromwall's Lemma, confer [18].

$$
\begin{aligned}
& \hat{B}^T(\mathcal{P}_h(u) - u, \mathcal{P}_h(\underline{p}) - \underline{p}_h) \\
= & \sum_{K \in \mathcal{K}_h} (\mathcal{P}_h(u) - u, a^{1/2} \nabla \cdot (\mathcal{P}_h(\underline{p}) - \underline{p}_h))_K \\
& - \sum_{e \in \mathcal{E}_h} (\{a^{1/2}\ \underline{n}\ (\mathcal{P}_h(u) - u)\}, [\mathcal{P}_h(\underline{p}) - \underline{p}_h])_e \ , \\
\leq & \ \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^{-2} ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} \\
& + \frac{\epsilon}{2} \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^2 ||a^{1/2} \nabla \cdot (\mathcal{P}_h(\underline{p}_h) - \underline{p}_h)||^2_{L^2(K)} \\
& + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} ||\{a^{1/2}\ \underline{n}\ (\mathcal{P}_h(u) - u)\}||^2_{(L^2(e))^d} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}_h} ||[\mathcal{P}_h(\underline{p}) - \underline{p}_h]||^2_{(L^2(e))^d} \ ,
\end{aligned} \tag{103}
$$

where ϵ is a constant and independent from the time t. We use the Cromwall's Lemma, confer [18].

We estimate the mixed term in the following inequality

$$
\begin{aligned}
& (\mathcal{P}_h(\underline{p}) - \underline{p}, \mathcal{P}_h(\underline{p}) - \underline{p}_h) \\
\leq & \ \frac{1}{2\epsilon} \ ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{(L^2(\Omega))^d} + \frac{\epsilon}{2} \ ||\mathcal{P}_h(\underline{p}) - \underline{p}_h)||^2_{(L^2(\Omega))^d} \ ,
\end{aligned} \tag{104}
$$

where ϵ is a constant.

We get the result for the right-hand-side

$$
\begin{aligned}
& E_h(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \qquad (105) \\
& \leq \; R \; (\; \frac{1}{2\epsilon} \; ||\mathcal{P}_h(\dot{u}(t)) - \dot{u}(t)||^2_{L^2(\Omega)} + \; \frac{\epsilon}{2} \; ||\mathcal{P}_h(u) - u_h||^2_{L^2(\Omega)} \;) \\
& + \; \frac{1}{2\epsilon} \; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) \; + \; \frac{\epsilon}{2} \; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \\
& + \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_1 (h_K)^{-2} ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} \\
& + \frac{\epsilon}{2} \sum_{K \in \mathcal{K}_h} \alpha_1 (h_K)^{2} || - \underline{v} \cdot \nabla(\mathcal{P}_h(u) - u_h) + R\lambda(\mathcal{P}_h(u) - u_h)||^2_{L^2(K)} \\
& + \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{-2} ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} + \frac{\epsilon}{2} \; \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{2} ||a^{1/2} \nabla \cdot (\mathcal{P}_h(\underline{p}) - \underline{p}_h)||_{L^2(K)} \\
& + \frac{1}{2\epsilon} \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^{-2} ||a^{1/2} (\mathcal{P}_h(\underline{p}) - \underline{p})||^2_{(L^2(\Omega))^d} + \frac{\epsilon}{2} \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^{2} ||\nabla(\mathcal{P}_h(u) - u_h)||^2_{(L^2(K))^d} \\
& + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} \; ||\underline{v} \cdot \underline{n} \; \{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} + \frac{\epsilon}{2} \; \sum_{e \in \mathcal{E}_h} \; ||[\mathcal{P}_h(u) - u_h]]||^2_{L^2(e)} \\
& + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} \; ||\{a^{1/2} \underline{n} \cdot \{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}_h} \; ||[\mathcal{P}_h(u) - u_h]]||^2_{L^2(e)} \\
& + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}_h} \; ||\{a^{1/2} \; \underline{n} \; (\mathcal{P}_h(u) - u)\}||^2_{(L^2(e))^d} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}_h} \; ||[\mathcal{P}_h(\underline{p}) - \underline{p}_h]||^2_{(L^2(e))^d} \\
& + \; \frac{1}{2\epsilon} \; ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{(L^2(\Omega))^d} + \; \frac{\epsilon}{2} \; ||\mathcal{P}_h(\underline{p}) - \underline{p}_h)||^2_{(L^2(\Omega))^d} \; .
\end{aligned}
$$

We set the left-hand-side equal to the right-hand-side. We apply the lemma 9 to move the new terms $P_h(\cdot) - (\cdot_h)$ to the left-hand-side such that:

$$
\begin{aligned}
& R \; \frac{1}{2} \frac{\partial}{\partial t} ||\mathcal{P}_h(u(t)) - u_h(t)||^2_{L^2(\Omega)} + (1 - \frac{\epsilon}{2}) \; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}_h, \mathcal{P}_h(\underline{w}) - \underline{w}_h) \qquad (106) \\
& + (1 - \frac{\epsilon}{2}) \; ||\mathcal{P}_h(\underline{p}) - \underline{p}_h||^2_{L^2(\Omega)^d} \; dt \; + (\frac{R\lambda}{2} - \frac{R\lambda \; \epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon(\underline{v} \cdot \underline{v})}{2}) \; ||\mathcal{P}_h(u) - u_h||^2_{L^2(\Omega)} \\
& \leq \frac{1}{2\epsilon} \Big(\; R \; ||\mathcal{P}_h(\dot{u}(t)) - \dot{u}(t)||^2_{L^2(\Omega)} \; + \; \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) \\
& + \; \sum_{K \in \mathcal{K}_h} \alpha_1 (h_K)^{-2} \; ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} \; + \; \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{-2} \; ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} \\
& + \; \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^{-2} ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{(L^2(\Omega))^d} \\
& + \sum_{e \in \mathcal{E}_h} \; ||\underline{v} \cdot \underline{n} \; \{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h} \; ||a^{1/2} \underline{n} \cdot \{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)} \\
& + \sum_{e \in \mathcal{E}_h} \; ||\{a^{1/2} \; \underline{n} \; (\mathcal{P}_h(u) - u)\}||^2_{(L^2(e))^d} \Big) \; .
\end{aligned}
$$

We use the inequality for the left-hand-side

$$||\mathcal{P}_h(u) - u_h||^2_{L^2(\Omega)} \geq ||u - u_h||^2_{L^2(\Omega)}|| - ||u - \mathcal{P}_h(u)||^2_{L^2(\Omega)} ,$$

where we have the similar result for $\underline{p}$ and the bilinear-form Θ_C and get the error-estimates after skip the terms $P_h(\cdot) - (\cdot)$ to the right-hand-side :

$$\begin{aligned}
& R \, \frac{1}{2} \frac{\partial}{\partial t} ||u(t) - u_h(t)||^2_{L^2(\Omega)} + (1 - \frac{\epsilon}{2}) \, \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h) \\
& + (1 - \frac{\epsilon}{2}) \, ||\underline{p} - \underline{p}_h||^2_{L^2(\Omega)^d} \, dt \, + (\frac{R\lambda}{2} - \frac{R\lambda \, \epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon(\underline{v} \cdot \underline{v})}{2}) \, ||u - u_h||^2_{L^2(\Omega)} \\
& \leq \frac{1}{2\epsilon} \Big(\, R \, ||\mathcal{P}_h(\dot{u}(t)) - \dot{u}(t)||^2_{L^2(\Omega)} \, + \, \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) \\
& + \sum_{K \in \mathcal{K}_h} \alpha_1 (h_K)^{-2} \, ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \, + \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{-2} \, ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \\
& + \sum_{K \in \mathcal{K}_h} \alpha_3 (h_K)^{-2} \, ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{L^2(K)} \\
& + \sum_{e \in \mathcal{E}_h} ||\underline{v} \cdot \underline{n} \, \{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h} ||\{a^{1/2} \underline{n} \cdot \{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)} \\
& + \sum_{e \in \mathcal{E}_h} ||\{a^{1/2} \, \underline{n} \, (\mathcal{P}_h(u) - u)\}||^2_{(L^2(e))^d} \Big) \, .
\end{aligned} \tag{107}$$

We integrate the both side over the time and use the projection-result, confer [11], such that

$$u(0) - u_h(0) = 0 \, . \tag{108}$$

We integrate over the time and use the result of equation (108), such that we get the error-estimates for the full discretization

$$\begin{aligned}
& R \, \frac{1}{2} \, ||u(T) - u_h(T)||^2_{L^2(\Omega)} \, + \, (1 - \frac{\epsilon}{2}) \int_0^T \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h) \, dt \\
& + (1 - \frac{\epsilon}{2}) \int_0^T ||\underline{p} - \underline{p}_h||^2_{L^2(\Omega)^d} \, dt \, + \, (\frac{R\lambda}{2} - \frac{R\lambda \, \epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon(\underline{v} \cdot \underline{v})}{2}) \int_0^T ||u - u_h||^2_{L^2(\Omega)} \, dt \\
& \leq \frac{1}{2\epsilon} \Big(\int_0^T ||\mathcal{P}_h(\dot{u}(t)) - \dot{u}(t)||^2_{L^2(\Omega)} \, dt \\
& + \int_0^T \Theta_C(\mathcal{P}_h(\underline{w}) - \underline{w}, \mathcal{P}_h(\underline{w}) - \underline{w}) \, dt \, + \int_0^T R \, \lambda ||\mathcal{P}_h(u) - u||^2_{L^2(\Omega)} \, dt \\
& + \int_0^T \sum_{K \in \mathcal{K}_h} \alpha_1 (h_K)^{-2} ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \, dt \, + \int_0^T \sum_{K \in \mathcal{K}_h} \alpha_2 (h_K)^{-2} ||\mathcal{P}_h(u) - u||_{L^2(K)} \, dt
\end{aligned} \tag{109}$$

$$+ \int_0^T \sum_{K \in \mathcal{K}_h} \alpha(h_K)_3^{-2} ||\mathcal{P}_h(\underline{p}) - \underline{p}||^2_{(L^2(K))^d} \, dt$$
$$+ \int_0^T \sum_{e \in \mathcal{E}_h} ||\underline{v} \cdot \underline{n} \, \{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} \, dt \; + \int_0^T \sum_{e \in \mathcal{E}_h} ||\{a^{1/2} \underline{n} \cdot \{\mathcal{P}_h(\underline{p}) - \underline{p}\}||^2_{L^2(e)} \, dt$$
$$+ \int_0^T \sum_{e \in \mathcal{E}_h} ||\{a^{1/2} \, \underline{n} \, (\mathcal{P}_h(u) - u)\}||^2_{(L^2(e))^d} \, dt \, \Big)$$

where the functions α_i are derived for the special test-spaces. We apply the lemma 9 and integrate over the time t.

□

In the next section we apply the error-estimates for the special test-spaces.

6. Application of the error-estimates

6.1. Application for standard polynomial-space

In the first application for the discontinuous finite element space we apply the polynomial spaces.

Let $\Omega \subset I\!R^d$ be a polynomial domain and let K_h be a regular finite element partition of Ω. We define

$$\mathcal{D}_r(\mathcal{K}_h) = \{v \;\in\; L^2(\Omega) \;:\; v|_K \in \mathcal{P}^r(K) \;\; \forall K \in \mathcal{K}_h\} \, , \tag{110}$$

where $\mathcal{P}^r$ is the set of polynomials of degree at most r on K.

We introduce the following projection as an Galerkin-approximation for the L^2-norm, confer [34]

Lemma 10 *Let the projection $\mathcal{P}_h : L^2 \to \mathcal{P}^r$ be the L^2-orthogonal projection-operator. In the case that Ω is a rectangular domain and T_h is a Cartesian product of uniform grids in each of the coordinate directions and for r even we obtain*

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||_{L^2(\Omega)} \;\leq\; c \, h^{r+1} \, ||u||_{H^{r+1}(\Omega)} \; ,$$
$$\sum_{e \in \mathcal{E}_h} ||[\mathcal{P}_h(u) - u]||_{L^2(e)} \;\leq\; c \, h^{r+1/2} \, ||u||_{H^{r+1}(\Omega)} \; ,$$
$$\sum_{e \in \mathcal{E}_h} ||\{\mathcal{P}_h(u) - u\}||_{L^2(e)} \;\leq\; c \, h^{r+3/2} \, ||u||_{H^{r+2}(\Omega)} \; ,$$

where c depending on r. We have be the similar result for $\underline{p}$.

Proof. The proof is done in [11] and [16]. □

For the polynomial test-space we have the following estimates for the functions α_i.

Lemma 11 *We have the local inequality for the polynomial-space $V_h = \mathcal{P}^r$ and $W_h = (\mathcal{P}^r)^d$:*

$$||\nabla(\mathcal{P}_h(u) - u)||_{(L^2(K))^d} \;\leq\; c \, h^{-1} \, ||\mathcal{P}(u) - u||_{L^2(K)} \; , \; u \in V_h \; , \tag{111}$$
$$||[\mathcal{P}_h(u) - u]||_{L^2(e)} \leq c \, h^{-1/2} ||\mathcal{P}_h(u) - u||_{L^2(K)} \; , \; u \in V_h \; , \tag{112}$$

where we get $\alpha_1(h_K) = \alpha_2(h_K) = \alpha_3(h_K) = h_K$.

Proof. The proof is done in [15]. □

We derive the error-estimates in the following theorem 12.

Theorem 12 *Error-Estimates for the polynomial space, where h is the diameter of the element. We have the estimates for the terms such that*

$$R\,\frac{1}{2}\;||u(T) - u_h(T)||^2_{L^2(\Omega)} \;+\; (1 - \frac{\epsilon}{2})\;\int_0^T\;\Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h)\;dt \tag{113}$$
$$+(1 - \frac{\epsilon}{2})\;\int_0^T\;||\underline{p} - \underline{p}_h||^2_{L^2(\Omega)^d}\;dt \;+\; (\frac{R\lambda}{2} - \frac{R\lambda\;\epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon(\underline{v}\cdot\underline{v})}{2})\;\int_0^T\;||u - u_h||^2_{L^2(\Omega)}\;dt$$
$$\le \frac{1}{2\epsilon}\left(\;h^{2r}\;||u||^2_{H^{r+1}(\Omega)} \;+\; h^{2r}\;||\underline{p}||^2_{(H^{r+1}(\Omega))^d}\right)\,.$$

Proof. We use the lemma 10 and get the equation. □

We have a suboptimal error-estimates because we loose one order of the polynoms. That means we investigate one order more to have the error-estimates of r.

We could improve our results with an approach in the test-spaces, by using analytical solutions, so that we obtain improved error-estimates. In the next section we get results of the analytical test-functions.

6.2. Application for special function spaces with respect to local analytical solutions

In the next subsection we derive the analytical solutions for the new test-functions.

6.2.1. Motivation for the new test-spaces

The motivation for the new test-spaces came from the idea to improve the local behavior of the test-functions. Standard test-functions like polynomes do not respect the local character.

To have a local behavior of the solutions we use the ideas of the adjoint-problems. They are done in the ELLAM-schemes [19]. We use these ideas for the space-terms and solve the locally adjoint problems for the space dimensions.

We concentrate us on the convection-reaction-equation, given as

$$-\sum_{K\in\mathcal{K}_h}(u_h, \underline{v}\cdot\nabla\phi)_K + (u_h, R\;\lambda\phi) = \sum_{K\in\mathcal{K}_h}(u_h, -\underline{v}\cdot\nabla\phi + R\;\lambda\;\phi) = 0\;, \tag{114}$$

where $u_h, \phi \in V_h$ and solve the adjoint local equation for the convection-reaction in space

$$-\underline{v}\cdot\nabla\phi + R\;\lambda\;\phi = 0\;, \tag{115}$$

where the initial condition is $\phi(0) = \phi_0$. We derive the local solution of the equation (115) the next subsection.

6.2.2. Local test-functions in space for one dimension

We derive the one-dimensional solutions for the local convection-reaction equation, given as adjoint problem

$$-v\;\partial_x\phi + R\;\lambda\;\phi = 0\;, \tag{116}$$

where by $\phi(0) = a_0$ is a constant.

The equation (116) is solved exactly and the solution is denoted with respect to the velocity $v \in I\!R$ and $v \neq 0$

$$\phi_{anal,i}(x) = a_0 \begin{cases} \exp(-\beta\ (x_{i+1/2} - x)) & v > 0 \\ \\ \exp(-\beta\ (x - x_{i-1/2})) & v < 0 \end{cases} , \tag{117}$$

where $\beta = \frac{R\,\lambda}{|v|}$,

$$\phi_{new,i} = \phi_{anal,i}(x) , \tag{118}$$

where $x_{i-1/2} < x < x_{i+1/2}$.
The local solution are applied as analytical weight with $0 \leq \phi_{anal,i}(x) \leq 1$.
For this test-function we have one freedom-degree, so that we could use only a constant initial condition.

For linear initial-conditions we use the analytical test-function and multiply it with the standard-test-function of first order.

Such that

$$\phi_{new,i} = \phi_{stand,i}(x)\ \phi_{anal,i}(x) , \tag{119}$$

whereby the standard test-functions are given as polynomial-functions
$\phi_{stand}(x) = \{1, x, x^2, \ldots\}$.

The test-functions are used for the cases of $|v| >> \lambda \geq 0$ for the case $v = 0$ we use the standard test-functions, confer [16].

So we improve it for constant cell impulses. To present the test-functions we get the next figures 2 and 3 for two extreme cases : $\lambda << v$ and for $\lambda = v$.

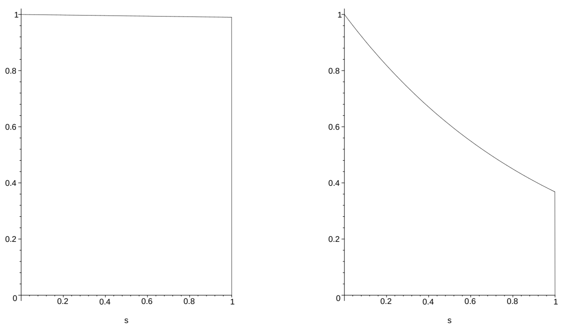

Figure 2. Local test-functions constructed with analytical solution and constant initial condition.

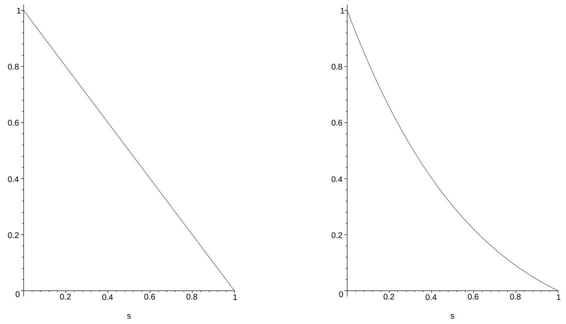

Figure 3. Local test-functions constructed with analytical solution and linear initial condition.

6.2.3. Local test-functions in space for two dimension

For two dimensions we have the possibilities of the operator splitting methods and the coupled solutions with Laplacian-Transformation. The solution is given for a rectangular-grid.

$$-v_1 \partial_x \phi - v_2 \partial_y \phi + R\lambda\phi = 0 \ , \tag{120}$$

where by $\phi(0,0) = a_0$.

We get the solution by operator-splitting, confer [28] ,

$$-v_1 \ \partial_x \phi = -R \ \frac{\lambda}{2} \ \phi \quad , \ \phi(0,y) = a_0^{1/2} \ , \tag{121}$$

$$-v_2 \ \partial_y \phi = -R \ \frac{\lambda}{2} \ \phi \quad , \ \phi(x,0) = a_0^{1/2} \ , \tag{122}$$

where by $\phi(0,0) = a_0$.

We get the solutions :

$$\phi_{anal,i}(x) = \begin{cases} \exp(-\beta_1(x_{i+1/2} - x)) & v_1 > 0 \\ \\ \exp(-\beta_2(x - x_{i-1/2})) & v_1 < 0 \end{cases} \ , \tag{123}$$

$$\phi_{anal,i}(y) = \begin{cases} \exp(-\beta_1(y_{i+1/2} - y)) & v_2 > 0 \\ \\ \exp(-\beta_2(y - y_{i-1/2})) & v_2 < 0 \end{cases} \ , \tag{124}$$

where $\beta_1 = \frac{R\lambda}{2|v_1|}$ and $\beta_2 = \frac{R\lambda}{2|v_2|}$,

$$\phi_{anal,i}(x,y,t) = a_0 \ \phi_{anal,i}(x) \ \phi_{anal,i}(y) \ , \tag{125}$$

where $x_{i-1/2} < x < x_{i+1/2}$ and $y_{i-1/2} < y < y_{i+1/2}$.

For linear initial-conditions we use the analytical test-function and multiply it with the standard-test-function of first order.

Such that

$$\phi_{new,i}(x, y) = \phi_{stand,i}(x, y)\ \phi_{anal,i}(x, y)\ , \tag{126}$$

where the standard test-functions are the polynomial functions.

In the next section we apply the new test-functions which are analytical for the local convection-reaction-equation. For these new functions we could derive a new error-analysis.

6.2.4. Error-estimates for the new test-functions with analytical test-functions in one dimension

We derive in the following section the error-estimates for our analytical solutions.

The idea came from the infimum of the error between the exact and the improved equations.

We use for $\phi \in V_h$ our new test-space $\mathcal{F}$ is as follows

$$\mathcal{F}_i \ = \ \{\exp(-\beta x), x\ \exp(-\beta x), \ldots, x^i \exp(-\beta x)\}\ , \tag{127}$$

where we assume the one-dimensional problem, and $\beta = \frac{R\lambda}{|v|}$ and we transform our local coordinates to $0 < x < h$.

We present the following lemma for the exponential test-functions.

Lemma 13 *Let the projection $\mathcal{P}_h : L^2 \to \mathcal{F}_i$, with all $i \geq 0$, a projection-operator to the finite space of the exponential-functions.*

Such that for $\mathcal{F}_0 = \{\exp(-\beta x)\}$

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||_{L^2(K)} \leq\ c\ h\ ||u||_{H^1(K)}\ , \tag{128}$$

and for $\mathcal{F}_1 = \{\exp(-\beta x), x \exp(-\beta x)\}$

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||_{L^2(K)} \leq\ c\ h^2\ ||u||_{H^2(K)}\ , \tag{129}$$

and for $\mathcal{F}_2 = \{\exp(-\beta x), x\ \exp(-\beta x), x^2\ \exp(-\beta x)\}$

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||_{L^2(K)} \ \leq \ \ c\ h^3\ ||u||_{H^3(K)}\ , \tag{130}$$

and for $\mathcal{F}_i = \{\exp(-\beta x), x\ \exp(-\beta x), \ldots, x^i \exp(-\beta x)\}$

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||_{L^2(K)} \ \leq \ \ c\ h^{i+1}\ ||u||_{H^{i+1}(K)}\ , \tag{131}$$

where $i \geq 2$. For the proof we use the result of lemma 15.
We have also the result with $\underline{p}$.

For the proof we could use the following lemma introduced in [15] for the projection-function. We rewrite the expansion with the new basis-function and apply the Taylor expansion coefficient of the polynomial expansion. Therefore we could estimate our error in the shape of the derivation of the function u.

Lemma 14 *We have the polynomial base* $1, x, x^2, \ldots$ *and the exponential base* $\mathcal{F} = \{1, \exp(-\beta x), x\exp(-\beta x), \ldots$ *The Taylor-expansion based on the polynomial-space is applied for the exponential space as follows :*

The Taylor-expansion with the polynomial-space are given as

$$\begin{aligned} u(x) &= c_0 1 + c_1(x-a) + c_2(x-a)^2 + \ldots, & (132)\\ u(x) &= T_p^i u + R_p^i u & (133)\\ T_p^i u &= c_0 1 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_i(x-a)^i & (134)\\ R_p^i u &= c\, c_{i+1}(x-a)^{i+1} & (135)\\ c_i &= \frac{1}{i!}\frac{\partial^i u}{\partial x^i}(a) & (136) \end{aligned}$$

where c is a constant.

The Taylor-expansion with the exponential-space are given as

$$\begin{aligned} u(x) &= \tilde{c}_0 \exp(-\beta(x-a)) + \tilde{c}_1(x-a)\exp(-\beta(x-a)) & (137)\\ &\quad + \tilde{c}_2(x-a)^2\exp(-\beta(x-a)) + \ldots, \\ u(x) &= T_e^i u + R_e^i u\ , & (138)\\ T_e^i u &= \tilde{c}_0 \exp(-\beta(x-a)) + \tilde{c}_1(x-a)\exp(-\beta(x-a)) & (139)\\ &\quad + \tilde{c}_2(x-a)^2\exp(-\beta(x-a)) + \ldots + \tilde{c}_i(x-a)^i\exp(-\beta(x-a))\ , \\ R_e^i u &= c\, \tilde{c}_{i+1}\exp(-\beta(x-a))(x-a)^{i+1}\ , & (140)\\ \tilde{c}_i &= \frac{1}{i!}(\beta^i \exp(-\beta a)u(a) + \beta^i\exp(-\beta a)\frac{\partial u}{\partial x}(a) + \ldots + \exp(-\beta a)\frac{\partial^i u}{\partial x^i}(a)) \end{aligned}$$

whereby c is a constant.

Proof. We use the following application of the Taylor-expansion

$$\exp(\beta(x-a))u(x) = \tilde{c}_0 1 + \tilde{c}_1 x + \tilde{c}_2 x^2 + \ldots\ , \quad (141)$$

We use the term-wise expansion such that :

$$\begin{aligned} \tilde{c}_0 &= u(a)\ , & (142)\\ \tilde{c}_1 &= \beta u(a) + \frac{\partial u}{\partial x}(a)\ , & (143)\\ &\ldots \\ \tilde{c}_i &= \beta^i u(a) + \beta^{i-1}\frac{\partial u}{\partial x}(a) + \ldots + \frac{\partial^i u}{\partial x^i}(a)\ . & (144) \end{aligned}$$

□

We could proof the lemma 15 by using the idea described in [15].

In the following we use the exponential functions and set $T = T_e$ and $R_e = R$

Lemma 15

$$||\mathcal{P}_h(\underline{u}) - \underline{u}||^2_{L^2(K)} \leq\ c\ ||R(u)||^2_{L^2(K)}, \quad (145)$$

where $R(u)$ *is the remainder of the Taylor-formula for the solution* u *and c is a constant.*

$$||R(u)||^2_{L^2(K)} \leq c\ h^{2(i+1)}\ ||u||^2_{H^{i+1}(K)}\ , \quad (146)$$

where $i \geq 0$ *is the order of the test-space* $\mathcal{F}_i$.

We proof the lemma with introducing the Taylor-expansion.

Proof. We use the Taylor-expansion for the error-estimate to derive the remainder and use the projection result $\mathcal{P}_h(T(u)) = T(u)$ such that

$$||\mathcal{P}_h(u) - u||^2_{L^2(K)} = ||\mathcal{P}_h(T(u)) - T(u) + \mathcal{P}_h(T(R)) - T(R)||^2_{L^2(K)} \quad (147)$$

$$= ||\mathcal{P}_h(T(R)) - T(R)||^2_{L^2(K)} \leq ||R(u)||^2_{L^2(K)} , \quad (148)$$

where we use the projection of the exact solution

The remainder is given as

$$||R(u)||^2_{L^2(K)} = c\, ||\tilde{c}_{i+1}||^2_{L^2(K)} || \exp(-\beta x) x^{i+1}||^2_{L^2(K)} , \quad (149)$$

where c is a constant.

$$||\tilde{c}_{i+1}||^2_{L^2(K)} = ||\tilde{c}_{i+1}(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{i+1} u}{\partial x^{i+1}})||^2_{L^2(K)} = ||u||^2_{H^{i+1}(K)} , \quad (150)$$

$$|| \exp(-\beta x) x^{i+1}||^2_{L^2(K)} \leq h^{2(i+1)} , \quad (151)$$

We use the results such that we have the estimates

$$||\mathcal{P}_h(u) - u||^2_{L^2(K)} \leq c\, h^{2(i+1)} \, ||u||^2_{H^{i+1}(K)} , \quad (152)$$

□

The jumps and averages are estimated with the next lemmas.

We derive the estimate for the jump :

Lemma 16 *Let the projection $\mathcal{P}_h : L^2 \to \mathcal{F}_i$, with $i \geq 0$, a projection-operator to the finite space of the exponential-functions.*

$$||[\mathcal{P}_h(u) - u]||_{L^2(e)} \leq h^{i+1/2} \, ||u||_{H^{i+1}(K)} , \quad (153)$$

Proof. Confer the proof in [15].

We have the estimation between the jump value $[\mathcal{P}_h(u)]$ and the original value $\mathcal{P}_h(u)$ with the equation :

$$||[\mathcal{P}_h(u) - u]||^2_{L^2(e)} = ||\mathcal{P}_h((u|_{K_2})|_e) - (u|_{K_2})|_e - \mathcal{P}_h((u|_{K_1})|_e) + (u|_{K_1})|_e||^2_{L^2(e)}$$

$$\leq c\, h_K^{-1} \int_0^{h_K} ||\mathcal{P}_h((u|_K)|_e) - u||^2_{L^2(K)} \, dx$$

$$\leq c\, h_K^{-1} ||\mathcal{P}_h(u) - u||^2_{L^2(K)} \leq h_K^{2i+1} \, ||u||_{H^{i+1}(K)} \quad (154)$$

where we for the next results $h_K = h$.

We derive the estimate for the average :

Lemma 17 *Let the projection $\mathcal{P}_h : L^2 \to \mathcal{F}_i$, with $i \geq 0$, a projection-operator to the finite space of the exponential-functions.*

$$||\{\mathcal{P}_h(u) - u\}||^2_{L^2(e)} \leq h^{2i+1} ||u||^2_{H^{i+1}(K)} , \quad (155)$$

Proof. Confer the proof-idea in lemma 16. □

For the derivations we could use the results for the analytical test-function to skip the derivation of the convection-term.

Lemma 18 *We use the local solution of the test-function such that*

$$\alpha_1(h_K)\ ||v\ \frac{\partial}{\partial x}(\mathcal{P}_h(u) - u_h) + R\ \lambda\ (\mathcal{P}_h(u) - u)||^2_{L^2(K)}\ =\ 0\ ,\ u_h \in V_h \tag{156}$$

where we have $\alpha_1(h_K) = 0$. We have fulfill the equation for the case $r = 1$.

Proof.

We use our analytical solution of the test-functions ϕ for the solution u and get :

We use the analytical function for the test-functions given in 116 and fulfill our equation

$$-v\ \frac{\partial}{\partial x}\tilde{u} +\ R\ \lambda\ \tilde{u} = 0 \tag{157}$$

where $\tilde{u} = \mathcal{P}_h(u) - u_h$.

Therefore we could skip the derivation such that we have an improved error-estimates for the convection-term. □

We use the inequality or the next terms of coming from the diffusion-term.

Lemma 19 *We apply the inequality for the derivation and get*

$$\alpha_2(h_K)\ ||\frac{\partial}{\partial x}(\mathcal{P}_h(u) - u_h)||^2_{L^2(K)}\ \leq\ ||\mathcal{P}_h(u) - u_h||^2_{L^2(K)}\ ,\ u_h \in V_h \tag{158}$$

where we have $\alpha_2(h_K) = h_k$ and increase the order of the solution p. One could balance the order of the error-estimates between u_h and p_h.

We obtain then the result

$$\alpha_3(h_e)\ ||a^{1/2}\frac{\partial}{\partial x}(\mathcal{P}_h(p) - p_h)||^2_{L^2(K)}\ \leq\ ||\mathcal{P}_h(p) - p_h||^2_{H^1(K)}\ ,\ p_h \in W_h \tag{159}$$

where we have $\alpha_3(h_K) = 1$.

The idea is to increase the order for the p_h solutions to obtain an optimal error-estimate for the u_h solutions.

Proof.

The proof is done in [34].

□

We could the use the results and derive the error-estimates for the new test-space.

The sub-optimal error-estimates is given as: Error-Estimates for the polynomial space, where h is the diameter of the element. We have the estimates for the terms such that

Theorem 20 *We denote the error-estimates for the one-dimensional convection-diffusion-reaction-equation as*

$$R\ \frac{1}{2}\ ||u(T) - u_h(T)||^2_{L^2(\Omega)}\ +\ (1 - \frac{\epsilon}{2})\ \int_0^T\ \Theta_C(\underline{w} - \underline{w}_h, \underline{w} - \underline{w}_h)\ dt \tag{160}$$

$$+(1 - \frac{\epsilon}{2})\ \int_0^T\ ||p - p_h||^2_{L^2(\Omega)}\ dt\ +\ (\frac{R\lambda}{2} - \frac{R\lambda\ \epsilon}{2} - \frac{\epsilon}{2} - \frac{\epsilon\ v^2}{2})\ \int_0^T\ ||u - u_h||^2_{L^2(\Omega)}\ dt$$

$$\leq\ \frac{c}{2\epsilon}\ \int_0^T\ h^{2r+1}\ \big(\ ||u||^2_{H^{r+1}(\Omega)}\ +\ ||p||^2_{H^{r+2}(\Omega)}\ \big)\ dt\ .$$

where c and ϵ are constants independent from h and t. We present the proof for $r = 1$.

Proof.
We use the lemmas 18 and 19 and get the reduced order by balancing between u_h and p_h and skip the inequality of the derivatives u_h.
□

We obtain sub-optimal results for the error-estimates and could improve the order by 1/2. In the application we could balance between the smoothness of u and p and derive sub-optimal error-estimates.

We could reach higher order results with this error-estimates for our applications. Through the adequate test-functions we approximate the different scales for the convection and reaction term. The improvement of this new approach are applications in our large scale problems with less artifical errors.

In the next section we discuss our next works in this context.

7. Conclusions

We discuss a new discretization method based on the local Discontinuous Galerkin method (LDG) with improved test-functions. We derive the stability and the error-estimates for the new discretization method. The new test-functions are derived from the adjoint problem with respect to the standard test-functions. We introduce the error-estimates for the new test-functions and obtain sub-optimal results.

In future works we would generalize our results for the different Discontinuous Galerkin methods and apply our results for different test-cases.

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