Motivic integration: seminar talks

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In this text elements of motivic integration are discussed, mostly with full proofs. The main sources are original papers by Denef-Loeser [2, 3].

There exist several survey articles on motivic integration: [10], [12], [11], [9], [13], [14].

Throughout the text k is any field.

Let X, Y be varieties over k; let A, B be constructible subsets of X, Y respectively. A map $\pi : A \to B$ is said to be a *piecewise fibration with fiber* F if there exists a finite partition of B into subsets B_i such that all $\pi^{-1}(B_i)$ are locally closed in X and isomorphic to $B_i \times F$ over B_i.

1 Arc space

Denote $\mathrm{D} = \operatorname{Spec} k[[t]]$, $\mathrm{D}_n = \operatorname{Spec} k[[t]]/(t^{n+1})$ and $\hat{\mathrm{D}}$ the formal scheme

$$\operatorname{Spf}(k[[t]]) = (\mathbb{A}^1_k)^\wedge_0.$$

For an arbitrary scheme Z we denote by $\hat{\mathrm{D}}_Z$ the formal scheme

$$(Z \times \mathbb{A}^1_k)^\wedge_{Z\times 0} = \varinjlim_n (Z \times \mathrm{D}_n)$$

($\varinjlim$ in the category of locally ringed spaces). For any algebraic k-scheme X we get cofunctors on the category of k-schemes, denoted by $\mathcal{L}(X)$, $\mathcal{L}_n(X)$, and natural transformations

$$\begin{aligned}
&\mathcal{L}(X) \xrightarrow{\pi_n} \mathcal{L}_n(X) \xrightarrow{\pi^n_m} \mathcal{L}_n(X), \quad m \le n,\\
&\mathcal{L}(X)(Z) = \operatorname{Hom}(\hat{\mathrm{D}}_Z, X)\\
&\mathcal{L}_n(X)(Z) = \operatorname{Hom}(Z \times \mathrm{D}_n, X)
\end{aligned}$$

(Hom in the category of formal schemes, or locally ringed spaces over k). Obviously

$$\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X).$$

1.1 Example. $\mathcal{L}_n(\mathbb{A}^q) = \mathbb{A}^q \times \ldots \times \mathbb{A}^q$ ($n+1$ factors) since $\mathcal{L}_n(\mathbb{A}^q)(Z)$ can be identified with the set of q-tuples of truncated power series $[\varphi_Z[[t]]/(t^{n+1})]^q$. Introducing indeterminates

$$x_1, \ldots, x_q, w_{11}, \ldots, w_{q1}, \ldots, w_{1n}, \ldots, w_{qn}$$

we get

$$\mathcal{L}_n(\mathbb{A}^q) = \operatorname{Spec} k[x_1, \ldots, x_q, w_{11}, \ldots, w_{q1}, \ldots, w_{qn}]$$

and

$$\mathcal{L}(\mathbb{A}^q) = \operatorname{Spec} k[x_1, \ldots, x_1, w_{11}, \ldots, w_{q1}, \ldots, w_{1n}, \ldots, w_{qn}, \ldots,].$$

If $X \subset \mathbb{A}^q$ is defined by $F_1(x) = \ldots = F_\nu(x) = 0$ and if $w(t) = x + w_1 t + w_2 t^2 + \ldots$ ($w_j = (w_{1j}, \ldots, w_{qj})$) and

$$F(w(t)) = (F_1(w(t)), \ldots, F_\nu(w(t))) = F(x) + G_1(x, w_1)t + G_2(x, w_1, w_2)t^2 + \ldots$$

is the expansion in $k[x,w][[t]]$, then $\mathcal{L}_n(X) \subset (\mathbb{A}^q)^{n+1}$ defined by $F = G_1 = \ldots = G_n = 0$ and $\mathcal{L}(X) \subset \operatorname{Spec} k[x,w]$ defined by $F = G_1 = G_2 = \ldots = 0$.

The space $\mathcal{L}(X)$ is called the *space of formal arcs on* X, the spaces $\mathcal{L}_n(X)$ are called the *spaces of truncated arcs*.

Special cases: $\mathcal{L}_0(X) = X$, $\mathcal{L}_1(X) = T(X) = \mathbb{V}(\Omega^1_{X/k})$.

1.2 Theorem. (1) *The cofunctors $\mathcal{L}(X)$ resp. $\mathcal{L}_n(X)$ are represented by an affine X-scheme resp. by an affine algebraic X-scheme, which we denote by the same letters.*

(2) *The following properties are equivalent:*

(i) *For any n, $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is surjective.*

(ii) *For any n, $\mathcal{L}(X) \to \mathcal{L}_n(X)$ is surjective.*

(*"surjective" means: on geometric points, i.e. $\bar{k}$-valued points.*)

(iii) *X is smooth.*

(3) *Each morphism $X \to Y$ induces commutative diagrams of morphisms ($m < n$):*

$$\begin{array}{ccc} \mathcal{L}(X) & \longrightarrow & \mathcal{L}(Y) \\ \downarrow & & \downarrow \\ \mathcal{L}_n(X) & \longrightarrow & \mathcal{L}_n(Y) \\ \downarrow & & \downarrow \\ \mathcal{L}_m(X) & \longrightarrow & \mathcal{L}_m(Y) \end{array}$$

If $X \to Y$ is a closed (resp. open) embedding, the induced horicontal morphisms have the same property.

1.3 Remark. If $Z = \operatorname{Spec}(R)$ is an affine Noetherian scheme we have a natural morphism

$$\hat{\mathrm{D}}_Z \longrightarrow \operatorname{Spec} R[[t]].$$

For any algebraic k-scheme X it induces a natural transformation

$$X(\operatorname{Spec} R[[t]]) \longrightarrow \mathcal{L}(X)(\operatorname{Spec}(R)). \qquad (1)$$

The following is true:

a) This is always injective.

b) In many cases (1) is bijective. A sufficient condition for this is:
X is quasiprojective or X admits a birational morphism to a quasiprojective variety.

c) We do not know any example where (1) is not bijective.

Injectivity: Given $\varphi, \psi : \operatorname{Spec} R[[t]] \to X$ which coincide on $\operatorname{Spf}(R[[t]]) = \hat{\mathrm{D}}_Z$. So, for any prime ideal $\wp > tR[[t]]$ and the corresponding point x (under φ and ψ), the comorphisms

$$\mathcal{O}_{X,x} \underset{\psi^*}{\overset{\varphi^*}{\rightrightarrows}} R[[t]]_\wp \longrightarrow \mathcal{O}_{\operatorname{Spf} R[[t]],\wp}$$

into $\mathcal{O}_{\operatorname{Spf} R[[t]],\wp}$ coincide. But $R[[t]]_\wp \longrightarrow \mathcal{O}_{\operatorname{Spf} R[[t]],\wp}$ is faithfully flat, so $\varphi^* = \psi^*$ and φ, ψ coincide on $\operatorname{Spec}(R[[t]]_\wp) \subset \operatorname{Spec}(R[[t]])$. Since $\operatorname{Spec}(R[[t]])$ is covered by such subschemes we infer $\psi = \varphi$.

Bijectivity in the quasiprojective case: If $X \subset \mathbb{P}^N$ is given by homogenous equations $F = (F_1, \ldots, F_q) = 0$ and inequalities $(G_1, \ldots, G_p) \neq (0, \ldots, 0)$, a morphism $\operatorname{Spf}(R[[t]]) \xrightarrow{\varphi} X$ is given by a line bundle $\mathcal{L}$ $(= \varphi^*\mathcal{O}(1))$ and global sections $\xi_0, \ldots, \xi_N$ generating $\mathcal{L}$, such that $F(\xi) = 0$ and $(G_1(\xi), \ldots, G_p(\xi))$ have no common zero.
$\mathcal{L}$ corresponds to a rank 1 locally free $R[[t]]$-module L with generators $\xi_0, \ldots, \xi_N$. Now $(L, \xi_0, \ldots, \xi_N)$ define an extension $\operatorname{Spec}(R[[t]]) \longrightarrow \mathbb{P}^N$ of $\operatorname{Spf}(R[[t]]) \longrightarrow X \subset \mathbb{P}^N$ and this extension factors through X.

Bijectivity in the case X admits a birational morphism to a quasiprojective variety: To prove the bijectivity it suffice to prove the following equivalence:

a morphism $\phi : \hat{\mathrm{D}}_Z \to X$ factors through $\operatorname{Spec} R[[t]]$ if and only if for every element $x \in \mathcal{O}_{X,\eta}$, where η is a point of X, the image $\phi^*(x) \in (R[[t]])_{g_x}$ for some element $g_x \in R[[t]]$.

Indeed, assuming that the equivalence is true, we need to prove that for any morphism $\phi : \hat{\mathrm{D}}_Z \to X$ the image $\phi^*(x) \in (R[[t]])_{g_x}$. Clearly, it suffice to check it only for a general point η of X. Since X admits a birational morphism to a quasiprojective variety, say Y, we have $\mathcal{O}_{X,\eta} \simeq \mathcal{O}_{Y,\phi(\eta)}$ for general points $\eta, \phi(\eta)$. Now the rest of the proof follows from the equivalence applied to the morphism $\hat{\mathrm{D}}_Z \to Y$ and the proof of bijectivity of (1) for quasiprojective varieties given above.

One direction of the equivalence is obvious. To prove the other one let's note that the condition on the image of elements x together with the condition on X (X is of finite type over k) imply that for any given morphism ϕ and a finite affine covering $\{U_i\}$ of X there are finite number of morphisms $(D(g_{ij}), \mathcal{O}_{\operatorname{Spec} R[[t]]}|_{D(g_{ij})}) \to (U_i, \mathcal{O}_X|_{U_i})$ induced by the morphism ϕ. We have $\cup D(g_{ij}) = \underline{\operatorname{Spec}} R[[t]]$ due to the following simple fact:

if for given open sets $\{V_i\} \subset \underline{\operatorname{Spec}} R[[t]]$ the condition $\cup(V_i \cap \underline{\operatorname{Spec}} R) = \underline{\operatorname{Spec}} R$ holds, then $\cup V_i = \underline{\operatorname{Spec}} R[[t]]$ (here $\underline{\operatorname{Spec}} R$ is embedded into $\underline{\operatorname{Spec}} R[[t]]$ due to the morphism $R[[t]] \to R$, $t \mapsto 0$).

Now the glueing conditions of the morphism ϕ for the covering $\{U_i\}$ guarantee that the morphisms above are glued well and form a morphism $\operatorname{Spec} R[[t]] \to X$, which is a factor of ϕ. $\square$

Proof of representability

Given k-algebras A, B, a Hasse-Schmidt derivation of A with values in B is a k-algebra-homomorphism

$$\begin{array}{rcl} A & \longrightarrow & B[[u]] \\ a & \longmapsto & D_0(a) + D_1(a)u + D_2(a)u^2 + \dots \end{array}$$

(resp. a truncated HS-derivation is a k-algebra-homomorphism $A \to B[[u]]/(u^{n+1})$.

1.4 Proposition. *There exists a universal HS-derivation*

$$\begin{array}{rcl} A & \longrightarrow & \mathrm{HS}_{A/k}[[u]] \\ a & \longmapsto & a + d_1(a)u + d_2(a)u^2 + \dots \end{array}$$

resp.

$$\begin{array}{rcl} A & \longrightarrow & \mathrm{HS}^n_{A/k}[[u]]/(u^{n+1}) \\ a & \longmapsto & a + d_1(a)u + d_2(a)u^2 + \dots \end{array}$$

(*for each* $a \in A$ *take indeterminates* $u_{a1}, u_{a2}, \dots$, *define* $\tilde{H} = A[(u_{av})]$, $a \in A$, $v = 1, 2, \dots$, $\tilde{d}(a) = a + u_{a1}t + u_{a2}t^2 + \dots$ *and divide out by universal relations to make* $\tilde{d}$ *to a* k*-algebra homomorphism*). *Similar for truncated HS-derivations*

$$A \longrightarrow \mathrm{HS}^n_{A/k}[[u]]/(u^{n+1}). \qquad \Box$$

These constructions are compatible with localization on A and yield a sheaf of universal HS-derivations

$$\mathcal{H}_{X/k} \quad \text{resp.} \quad \mathcal{H}^n_{X/k}$$

which are $\mathcal{O}_X$-algebras, quasicoherent as $\mathcal{O}_X$-modules.

An element of $\mathcal{L}(X)(Z)$ is given by a morphism $Z \xrightarrow{\varphi} X$ and a ring homomorphism extending $\varphi^* : \mathcal{O}_X \to \varphi_*\mathcal{O}_Z$

$$D : \mathcal{O}_X \longrightarrow \varphi_*\mathcal{O}_Z[[t]].$$

D is a HS-derivation, and so we get

$$\mathcal{L}(X)(Z) \xrightarrow{\sim} \mathrm{Spec}_X(\mathcal{H}_{X/k})$$

resp.

$$\mathcal{L}_n(X)(Z) \xrightarrow{\sim} \mathrm{Spec}_X(\mathcal{H}^n_{X/k}).$$

Obviously $\mathcal{L}_n(X)$ is of finite type over X, as explained before.

Implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious and we have to show (i) $\Rightarrow$ (ii). A morphism $\mathrm{D}_n \to X$ is given by a point $x \in X$ and a morphism $\hat{\mathcal{O}}_{X,x} \to k[[t]]/t^{n+1}$. We can assume $k = \bar{k}$ and if X is not smooth in x we choose a minimal presentation

$$\mathcal{O}_{X,x} = k[[z_1, \dots, z_n]]/(f_1, \dots, f_q)$$

such that f_1 is of minimal order, say n (≥ 2), and by Weierstraß preparation theorem of the form

$$f_1 = z_n^m + a_1(z')z_n^{m-1} + \dots + a_m(z'), \quad z' = (z_1, \dots, z_{n-1}), \quad \mathrm{ord}(a_j(z')) \geq j.$$

Then $\gamma(t) = (0, \dots, 0, t)$ is a point of $\mathcal{L}_{m-1}(X)$ which has no lift to $\mathcal{L}_m(X)$.

1.5 Proposition. *Let X be smooth. Then*

$$\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \longrightarrow \mathcal{L}_n(X)$$

is an affine bundle (torsor) with structure group $\mathrm{Aff}(n)$*, the associated vector bundle is* $\pi_n^* T(X)$*.*

Sketch of proof: If $q = \dim X$ consider two systems of local coordinates (in étale sense) $(x_1, \ldots, x_q)$, $(x'_1, \ldots, x'_q)$. Each of these étale morphisms

$$U \underset{X}{\longrightarrow} \mathbb{A}^q \quad \text{resp.} \quad U \underset{X'}{\longrightarrow} \mathbb{A}^q$$

yield an isomorphism between the bundles $\mathcal{L}_{n+1}(X)/U \xrightarrow{\pi_n^{n+1}} \mathcal{L}_n(X)/U$ and

$$\begin{array}{ccc} U \times (\mathbb{A}^q)^{n+1} & \xrightarrow{p_n^{n+1}} & U \times (\mathbb{A}^q)^n \\ (u, \xi^{(1)}, \ldots, \xi^{(n+1)}) & \longrightarrow & (u, \xi^{(1)}, \ldots, \xi^{(n)}), \end{array}$$

more precisely $x_* \circ \pi_n^{n+1} = p_n^{n+1} \circ x_*$, $x'_* \circ \pi_n^{n+1} = p_n^{n+1} \circ x'_*$.
We can write in a given point p with $x(p) = c$, $x'(p) = c'$

$$x' = \phi(x) \bmod (x_1 - c_1, \ldots, x_q - c_q)^{n+2}$$

with a polynomial $\phi(x_1, \ldots, x_q)$ of degree $\leq n+1$. Given an arc corresponding to $c + \xi^{(1)} t + \ldots + \xi^{(n+1)} t^{n+1}$ in x-coordinates resp. $c' + \xi'^{(1)} t + \ldots + \xi'^{(n+1)} t^{n+1} = \xi'(t)$ if we write $\tilde{\xi} = c + \xi^{(1)} t + \ldots + \xi^{(n)} t^n$ then

$$\xi'(t) = \phi(\tilde{\xi} + \xi^{(n+1)} t^{n+1}) \bmod t^{n+2} = \phi(\tilde{\xi}) + \sum_{j=1}^{q} \frac{\partial \phi}{\partial x_j}(c) \xi_j^{(n+1)} t^{n+2}.$$

If $A(c, \xi^{(1)}, \ldots, \xi^{(n)}) t^{n+1}$ is the part of degree t^{n+1} in $\phi(\tilde{\xi}(t))$ we get

$$\xi'^{(n+1)} = A(c, \xi^{(1)}, \ldots, \xi^{(n)}) + \sum_j \frac{\partial \phi}{\partial x_j}(c) \xi_j^{(n+1)}$$

so $\xi'^{(n+1)}$, $\xi^{(n+1)}$ are related by an affine transformation (over each point of $\mathcal{L}_n(X)$).

□

$\mathcal{L}_n(X)$ can be non-reduced, *but we shall always assume throughout this text that $\mathcal{L}_n(X)$ are endowed with the reduced scheme structure.*

Singular case Here one applies Strong Approximation Property (SAP):

Given a algebraic scheme X, one can find $m_0 > 0$ such that for any $\overline{w} \in \mathcal{L}_{m_0(n+1)}(X)$ there exists $w \in \mathcal{L}(X)$ with

$$\pi_n(w) = \pi_n^{m_0(n+1)}(\overline{w}).$$

(For the case of DVR, proved by Greenberg or by Lang in his thesis.)

This is true also for arbitrary complete local Noetherian rings (instead of DVR), maybe with "some function" instead of "linear function": a result by Pfister-Popescu (Inv.Math.)

Let us give more details.

1.6 Definition. Let X be an algebraic scheme over k, R a local k-algebra. A function $s : \mathbb{N} \to \mathbb{N}$ is called an *approximation function* for (X, R) if for any $c \in \mathbb{N}$ and any $\overline{\xi} \in X(R/\mathfrak{m}^{c+1+s(c)})$ there exists a solution $\xi \in X(R)$ with $\overline{\xi} \equiv \xi \mod \mathfrak{m}^{c+1}$.

1.7 Definition. R has *strong approximation property*(SAP) if for any algebraic scheme X there exists an approximation function for (X, R).

1.8 Theorem. *(Greenberg, S. Lang) If R is a complete discrete valuation ring, then R has SAP with linear approximation function.*

Proof. Without loss of generality, we may assume that X is affine and irreducible; $R = k[[t]]$. For simplicity, assume $k = \overline{k}$.

Apply induction by $\dim X$. The case $\dim X = 0$ is trivial.

Let $X \subset \mathbb{A}^q$ be of codimension r; let it be defined by a prime ideal $\mathfrak{p}$.

Since X is generically smooth, there exist $f_1, \dots, f_r \in \mathfrak{p}$ such that a certain $r \times r$ minor δ of $\left(\frac{\partial f_i}{\partial X_j}\right)$ is not identically 0 on X. It follows that $X \cap V(\delta)$ is of dimension $< \dim X$. Let $s_1(c)$ be an approximation function for $X \cap V(\delta)$ (hypothesis of induction).

1.9 Lemma. *Let $I = \mathfrak{p} + ((f_1, \dots, f_r) : \mathfrak{p})$. Then there exists l with $\delta^l \in I$ (i. e., δ vanishes on $V(I)$).*

Proof. Let $\xi \in V(I) \subset V(\mathfrak{p}) = X$, and $\delta(\xi) \neq 0$. Then ξ is a non-singular point of X (since one minor does not vanish), whence

$$\mathcal{O}_{X,\xi} = \mathcal{O}_{\mathbb{A}^q,\xi}/(f_1, \dots, f_r)\mathcal{O}_{\mathbb{A}^q,\xi},$$

i. e., $[(f_1, \dots, f_r) : \mathfrak{p}]\mathcal{O}_{\mathbb{A}^q,\xi} = \mathcal{O}_{\mathbb{A}^q,\xi}$, and $\xi \notin V(I)$, a contradiction. $\square$

Using this l from Lemma, we introduce

$$s(c) := (l + 1)(c + 1) + (l + 2)s_1(c).$$

We shall prove that s is an approximation function for X.

Consider any $\overline{\xi} \in X(R/t^{c+1+s(c)})$.

Case 1: $\delta(\overline{\xi}) \equiv 0 \mod t^{c+1+s_1(c)}$. Apply hypothesis of induction to the scheme $X \cap V(\delta)$ (which is defined by $(\mathfrak{p}, \delta)$). We obtain that there exists $\xi \equiv \overline{\xi} \mod t^{c+1}$ with $f(\xi) = 0$ for all $f \in \mathfrak{p}$ and $\delta(\xi) = 0$.

Case 2: $\delta(\overline{\xi}) = t^m\varepsilon$, $\varepsilon \in R^*$, $m \leq c + s_1(c)$. The condition $f(\overline{\xi}) \equiv 0 \mod t^{c+1+s(c)}$ (where $\overline{\xi}$ is any lifting of the original $\overline{\xi}$) can be written also as

$$f(\overline{\xi}) \equiv 0 \mod t^{c+1+s(c)-2m} \cdot \delta(\overline{\xi})^2,$$

whence

$$f_j(\overline{\xi}) \equiv 0 \mod \delta(\overline{\xi})^2 t^{l(c+1+s_1(c))+2}, \quad j = 1, \dots, r. \tag{2}$$

There exists Newton's lemma: let a ring R be Henselian along an ideal $\mathfrak{m}$; $f = (f_1, \dots, f_r) \in R[X_1, \dots, X_q]^r$, $r \leq q$; $\overline{\xi} \in R^q$ with $f(\overline{\xi}) \equiv 0 \mod \Delta^2\mathfrak{m}^k$, where Δ is an ideal generated by $r \times r$-minors of $\left(\frac{\partial f_i}{\partial X_j}(\overline{\xi})\right)$. Then there exists $\xi \in R^q$, $\xi \equiv \overline{\xi} \mod \Delta \cdot \mathfrak{m}^k$, such that $f(\xi) = 0$.

Applying Newton's Lemma to the system (2), we obtain: there exists $\xi \in \mathbb{A}^q(R)$ with $f_j(\xi) = 0$, $j = 1, \dots, r$, $\xi \equiv \overline{\xi} \mod t^{l(c+1+s_1(c))+2+m}$.

It remains to prove that $f(\xi) = 0$ for all $f \in \mathfrak{p}$. Recall that $\delta^l \in \mathfrak{p} + ((f_1, \dots, f_r) : \mathfrak{p})$, i. e., $\delta^l = f_0 + g_0$, $f_0 \in \mathfrak{p}$, $g_0 \in ((f_1, \dots, f_r) : \mathfrak{p})$.

Since $\delta(\overline{\xi})$ is of order m, and $\operatorname{ord}_t(\xi - \overline{\xi}) \geq l(c+1+s_1(c)) + 2 + m > m$, we have $\operatorname{ord}_t \delta(\xi) = m$. Similarly,

$$\operatorname{ord}_t(f_0(\xi)) \geq \min(c+1+s(c), l(c+1+s_1(c)) + 2 + m) > lm,$$

whence $\operatorname{ord}(g_0(\xi)) = \operatorname{ord}(\delta^l(\xi) - f_0(\xi)) = lm$. In particular, $g_0(\xi) \neq 0$. However, $g_0 f \in (f_1, \dots, f_r)$, whence $g_0(\xi) f_0(\xi) = 0$, and $f(\xi) = 0$. $\square$

1.10 Corollary. $\operatorname{Im} \pi_n = \operatorname{Im} \pi_n^{n+N}$ *for* $n + N \geq m_0(n+1)$*. In particular,* $\operatorname{Im} \pi_n$ *is a constructible set.* $\square$

2 Grothendieck ring of algebraic varieties

2.1 Definition

We recall that the Euler characteristic $e(X)$ of an algebraic variety X has the following properties:

1) $e(X) = e(X \setminus Y) + e(Y)$ for a closed subvariety $Y \subset X$;

2) $e(X \times Y) = e(X)e(Y)$.

Functions $\mathbf{Var}_k \to A$ with such properties, where A is a commutative ring, are called generalized Euler characteristics.

Important example: Hodge polynomial. Each algebraic variety over $\mathbb{C}$ can be endowed with a mixed Hodge structure in a canonical way. Let W be the weight filtration. Then gr_i^W has a pure Hodge structure of weight i; denote by h_i^{pq} the corresponding Hodge numbers. We define Hodge polynomial $\chi_{hp} : \mathbf{Var}_k \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ as

$$\chi_{hp}(X) = \sum_{i,p,q} (-1)^i h_i^{pq} u^p v^q.$$

Define $K_0(\mathbf{Var}_k)$ as the free abelian group generated by isomorphism classes of k-varieties divided by relations $[X] - ([X \setminus Y] + [Y])$, where $Y \subset X$ is a closed subvariety. It is a ring with multiplication $[X] \cdot [Y] = [X \times Y]$.

Next, define

$$M_k = K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}],$$

where $\mathbb{L} = [\mathbb{A}_k^1] \in K_0(\mathbf{Var}_k)$.

2.1 Remark. We can define $[C] \in K_0(X)$ for any constructible $C \subset X$.

We define $F_n K_0(\mathbf{Var}_k)$ as the subgroup generated by all $[X]$ with $\dim X \leq n$. This filtration yields a function

$$\deg : K_0(\mathbf{Var}_k) \to \mathbb{Z}$$

that can be prolonged to M_k ($\deg \mathbb{L}^{-1} := -1$). We define a non-archimedian semi-norm on M_k:

$$\|\alpha\| = e^{-\deg(\alpha)}.$$

The ring $\widehat{M_k}$ is defined as the completion of M_k with respect to this semi-norm.

2.2 Summary about Hodge structures (HS), mixed Hodge structures (MHS)

There are 3 equivalent definitions of HS.

Real HS of weight m is a real vector space H with...

(i) ...rational representation of

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\} = G$$

such that $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mapsto a^m \mathrm{id}$.

(ii) ... $H_{\mathbb{C}} = \bigoplus H^{pq}$; $H^{qp} = \overline{H^{pq}}$. $(p+q=m)$

(iii) ...filtration on $H_{\mathbb{C}}$: $\cdots \supseteq F^p \supseteq F^{p+1} \supseteq \ldots$ such that $F^p \oplus \overline{F_{m-p+1}} = H_{\mathbb{C}}$.

From (i) to (ii): H^{pq} are eigenspaces of

$$\begin{gathered} G \to \mathbb{C}^* \\ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto (a+ib)^p (a-ib)^q \end{gathered}$$

From (ii) to (iii): $F^p = \bigoplus_{r \geq p} H^{rs}$.

HS over $A \subset \mathbb{R}$ ($A = \mathbb{Z}$, $\mathbb{Q}$, ...) is a real HS with submodule $H_A \subset H$ such that $H_A \otimes_A \mathbb{R} = H$.

MHS is a vector space over $\mathbb{Q}$ with 2 filtrations:

$$\subseteq W_m \subseteq W_{m+1} \subseteq \cdots \subset H$$

defined over $\mathbb{Q}$ (weight filtration), $W_m = 0$ for $m \ll 0$, $W_m = H$ for $m \gg 0$, and

$$\cdots \supseteq F_p \supseteq F_{p+1} \supseteq \cdots \subset H_{\mathbb{C}}$$

with the property: $\mathrm{gr}_m^W(H) = W_m / W_{m-1}$ is a pure HS of weight m with the corresponding Hodge filtration induced by F.

MHS were introduced by Deligne.

MHS *is defined over* $\mathbb{Z}$, if it is endowed with a lattice $H_{\mathbb{Z}} \subset H$, $H_{\mathbb{Z}} \otimes \mathbb{Q} = H$.

MHS form an abelian category: a morphism $H \to H'$ is a $\mathbb{Q}$-linear (or $\mathbb{Z}$-linear) map, respecting both filtrations. The category has internal $\otimes$ and internal Hom.

For example,

$$\begin{aligned} &W_m \operatorname{Hom}(H, H') = \{ f : H \to H' \ \mathbb{Q}\text{-linear} | f W_r(H) \subseteq W_{r+m}(H') \}, \\ &F^p \operatorname{Hom}(H, H')_{\mathbb{C}} = \{ f : H_{\mathbb{C}} \to H'_{\mathbb{C}} \ \mathbb{C}\text{-linear} | f F^r(H_{\mathbb{C}}) \subseteq W^{r+p}(H'_{\mathbb{C}}) \}. \end{aligned}$$

We have a functor

$$\begin{gathered} \mathbf{Var}^2_{\mathbb{C}} \to \mathbf{MHS} \\ (X, Y) \mapsto H_m(X, Y, \mathbb{Z}) / \mathrm{torsion} \end{gathered}$$

where $\mathbf{Var}^2_{\mathbb{C}}$ is the category of pairs $X \supset Y$, Y a closed subvariety in X. The functor has following nice properties.

(i) Let X be smooth proper, $Y = \emptyset$. Then $H^m(X, \mathbb{Z})/\text{torsion}$ is a pure Hodge structure of weight m given by Hodge decomposition. Namely,

$$H^m(X, \mathbb{C}) = H^m_{DR}(X, \mathbb{C}) = \mathbb{H}^m(X, \Omega^*_{X/\mathbb{C}}),$$

where $\Omega^*_{X/\mathbb{C}}$ is either analytic or rational de Rham complex, and

$$F^p H^m = \mathrm{Im}(\mathbb{H}^m(X, F_p\Omega^*) \to \mathbb{H}^m(X, \Omega^*)),$$

where $F^p\Omega^*$ is the subcomplex

$$\dots \to 0 \to 0 \to \Omega^p \to \Omega^{p+1} \to \dots$$

(ii) The cup product

$$H^m(X, Y) \otimes H^{m'}(X', Y') \xrightarrow{\cup} H^{m+m'}(X \times X', X \times Y' \cup X' \times Y)$$

is a morphism of MHS.

(iii) The long exact sequence

$$\dots \to H^m(Y) \xrightarrow{\delta} H^{m+1}(X, Y) \to H^m(X) \to \dots$$

is exact in the category of MHS. (The only statement requiring proof is that δ is a morphism of MHS.)

(iv) Let $\dim X = d$. Then

$$\begin{aligned} W_{-1}H^m(X, Y) &= 0, \\ W_{2d}H^m(X, Y) &= H^m(X, Y), \\ F^0 H^m(X, Y)_{\mathbb{C}} &= H^m(X, Y)_{\mathbb{C}}, \\ F^{d+1}H^m(X, Y)_{\mathbb{C}} &= 0. \end{aligned}$$

For example, $\mathbb{Z}(-m) \overset{\text{def}}{=} H^{2m}(\mathbb{P}^n)$ is of Hodge type (m, m).

For any variety X cohomology with compact support $H^m_c(X)$ is defined as $H^m(\overline{X}, \partial\overline{X})$, where $X \subset \overline{X}$ is any compactification, $\partial\overline{X} = \overline{X} \setminus X$; it does not depend on the choice of compactification.

Then for any pair (X, Y), where Y is a closed subvariety in X we have an exact sequence of MHS:

$$\dots \to H^m_c(X \setminus Y) \to H^m_c(X) \to H^m_c(Y) \to H^{m+1}_c(X \setminus Y) \to \dots$$

Since **MHS** is an abelian category with internal Hom and $\otimes$, we have the Grothendieck ring $K_0(\mathbf{MHS})$.

It is easy to see that $K_0(\mathbf{MHS})$ is generated (as abelian group) by elements $[H]$, where H is a pure HS. A pure HS has Hodge numbers $h^{pq} = \dim H^{pq}$; dim is an additive functor; thus, we get a homomorphism

$$\begin{aligned} K_0(\mathbf{MHS}) &\to \mathbb{Z}[u, v, u^{-1}, v^{-1}] \\ H &\mapsto \sum_{p+q=m} h^{pq}u^p v^q, \end{aligned} \tag{3}$$

where H is a pure HS of weight m.

Associating to algebraic variety $X/\mathbb{C}$ the alternating sum

$$\sum_m (-1)^m [H^c_m(X)] \in K_0(\mathbf{MHS}),$$

we get a natural homomorphism (*Hodge characteristic*)

$$K_0(\mathbf{Var}_{\mathbb{C}}) \to K_0(\mathbf{MHS}). \tag{4}$$

Composing (3) and (4), we get the Hodge polynomial χ_{hp} discussed earlier.

Note that the Hodge characteristic (4) factors through $M_{\mathbb{C}}$.

2.2 Example. The MHS of $\mathbb{A}^m$. Consider embedding $\mathbb{A}^m \subset \mathbb{P}^m$ and use exact sequence for H^*_c:

$$H^{2m-1}(\mathbb{P}^{m-1})[=0] \to H^{2m}_c(\mathbb{A}^m) \to H^{2m}(\mathbb{P}^m) \to H^{2m}(\mathbb{P}^{m-1})[=0]$$

It follows

$$H^l_c(\mathbb{A}^m) = \begin{cases} \mathbb{Z}(-m), & l = 2m \\ 0, & l \neq 2m \end{cases}$$

Thus, the image of $\mathbb{L}$ in $K_0(\mathbf{MHS})$ is $[\mathbb{Z}(-1)] = [\mathbb{Z}(1)]^{-1}$ (Hodge structure of type $(-1,-1)$), and $\chi_{hp}(\mathbb{L}) = uv$.

3 Measurable sets and measure

3.1 Definition. Let X be an algebraic variety over k, $d = \dim X$, $A \subset \mathcal{L}(X)$ a subset. A is *stable* if for some $n \in \mathbb{N}$:

(i) A is a *cylinder* at level n, i. e., $A = \pi_n^{-1}\pi_n(A)$ and $\pi_n(A)$ is constructible;

(ii) for any $i \geq n$: $\pi_{i+1}(A) \to \pi_i(A)$ is a piecewise trivial fibration with fiber $\mathbb{A}^d_k$.

3.2 Remark. In the smooth case (i) implies (ii).

3.3 Definition. Let A be stable. Then

$$\mu(A) := [\pi_n(A)]\mathbb{L}^{-nd}$$

for $n \gg 0$.

3.4 Definition. $A \subset \mathcal{L}(X)$ is a *measurable set*, if for any $m \in \mathbb{N}$ there exists a stable A_m such that

$$A \triangle A_m \subset \bigcup_{j \in \mathbb{N}} C_{mj},$$

where C_{mj} are stable, $\forall j : \dim C_{mj} < m$, and $\dim C_{mj} \underset{j\to\infty}{\to} -\infty$, assuming $\dim C_{mj} := \dim \mu(C_{mj})$.

In this situation we put

$$\mu(A) := \lim_{m \to -\infty} \mu(A_m).$$

To prove that μ is well defined, we need

3.5 Lemma. *Let $C_1 \supseteq C_2 \supseteq \dots$ be non-empty stable subsets in $\mathcal{L}(X)$. Then $\bigcap C_i \neq \emptyset$.*

In the case of a variety over an uncountable field of characteristic 0 there exists a simple proof based on:

3.6 Lemma. *(Bair property) Let X be a variety over k, where k is uncountable, and $\operatorname{char} k = 0$. Let $K_1 \supseteq K_2 \supseteq \dots$ be non-empty constructible subsets in X. Then $\bigcap K_i \neq \emptyset$.*

In the general case we need Compactness Theorem 3.11.

3.7 Corollary. *Let C, C_i be stable sets, $C \subset \bigcup C_i$. Then $C \subset \bigcup_{finite} C_i$.*

3.8 Proposition. *μ is well defined.*

Proof. Assume that for A we have also suitable A'_m and $C'_{m,j}$. We have to show:

$$\dim(\mu(A_i) - \mu(A'_j) \leq i$$

for all $j \leq i \in -\mathbb{N}$. Indeed,

$$A_i \setminus A'_j \subseteq (A \triangle A_i) \cup (A \triangle A'_j) \subseteq \bigcup_m C_{i,m} \cup \bigcup_m C'_{j,m}.$$

By Corollary, $A_i \setminus A'_j$ is covered by finite number of stable sets of dimension $\leq i$. It follows $\dim \mu(A_i \setminus A'_j) \leq i$, and similarly $\dim \mu(A'_j \setminus A_i) \leq i$. Next, $A_i = (A_i \cap A'_j) \cup (A_i \setminus A'_j)$ and $A'_j = (A_i \cap A'_j) \cup (A'_j \setminus A_i)$ imply

$$\begin{aligned}\dim(\mu(A_i) - \mu(A'_j)) &= \dim(\mu(A_i \setminus A'_j) - \mu(A'_j \setminus A_i)) \\ &\leq \max(\dim \mu(A_i \setminus A'_j), \dim \mu(A'_j \setminus A_i)) \\ &\leq i\end{aligned}$$

We see that $\mu(A_i)$ is a Cauchy sequence; thus, its limit exists and is the same as $\lim \mu(A'_j)$. □

3.9 Proposition. *All measurable sets form a boolean algebra.*

Proof. The proof of the fact that it is a ring is easy, the proof that it is an algebra follows from below.

3.10 Example. Let X be smooth, Y a smooth subvariety. Then the function

$$\begin{aligned}\operatorname{ord}_Y : \mathcal{L}(X) &\to \mathbb{N} \cup \{\infty\} \\ \gamma &\mapsto \sup_e \{\gamma(I_Y) \subset (t^e)\} = \sup_e \{\pi_{e-1}(\gamma) \in \mathcal{L}_{e-1}(Y)\}\end{aligned}$$

is measurable.

Proof. Indeed, for $s \geq 1$:

$$\begin{aligned}\operatorname{ord}_Y(\gamma) \geq s &\iff \pi_{s-1}(\gamma) \in \mathcal{L}_{s-1}(Y), \\ \operatorname{ord}_Y(\gamma) = \infty &\iff \gamma \in \mathcal{L}(Y), \\ \operatorname{ord}_Y^{-1}(\geq s) &= \pi_{s-1}^{-1}\mathcal{L}_{s-1}(Y), \\ \operatorname{ord}_Y^{-1}(s) &= \pi_{s-1}^{-1}\mathcal{L}_{s-1}(Y) \setminus \pi_s^{-1}\mathcal{L}_s(Y)\end{aligned}$$

is a cylinder. Next,

$$\mathrm{ord}_Y^{-1}(0) = \mathcal{L}(X) \setminus \pi^{-1}(Y).$$

Finally,

$$\mathrm{ord}_Y^{-1}(\infty) = \bigcap_s \mathrm{ord}_Y^{-1}(\geq s) = \mathcal{L}(Y)$$

is not a cylinder. However, it is measurable, and $\mu(\mathcal{L}(Y)) = 0$, since

$$\dim(\mu(\mathrm{ord}_Y^{-1}(\geq s))) \leq \dim \mathcal{L}_{s-1}(Y) - sd = (s-1)(d-c) = n - c - cs \to -\infty,$$

where $c = \operatorname{codim} Y$.

3.1 Some words about Model Theory

In this section, we will introduce informally some of the concepts occurring in Model Theory that will be needed for later proofs in this work.

Model Theory studies definable sets in specific structures appearing in Mathematics, always concerned within a given language. It has two main directions: first, classifying such *real* mathematical structures in terms of their definable sets, and second, exhibiting algebraic or analytical properties arising from their model theoretical behaviour. These two aspects are intrinsically related, and there is a constant back and forth between these two points of view when using model theoretical techniques.

A *language* L is a collection of symbols, in which we can distinguish certain kinds: variable symbols, constant symbols, function symbols (of different aruities), and relations symbols. Moreover, we also have in L the logical connectors such as negation, conjunction, disjunction, and implication, and quantifiers $\forall$ and $\exists$.

We can now build up *formulae* in L, which typically are of the following form:

$$\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \left(\bigwedge_{i=1}^{p} \bigvee_{j=1}^{q} \varphi(\vec{z}, \vec{y}, \vec{x}) \right),$$

where φ is an atom (involves only equality and/or negation of the relation symbols on *terms*, that is, basic pieces we can consider via constants, function symbols and variables) in which only the variables $\vec{z}$,$\vec{y}$ and $\vec{x}$ occur (at most). We say that the variables $\vec{z}$ are *free* and that $\vec{y}$ and $\vec{x}$ are *bounded.* A formula with no free variables is a *sentence.*

An L-*structure* $\mathcal{M}$ is a set (denoted usually as M) in which we give an interpretation of the constants, relations and functions appearing in L.

An example of this is the archetypical example in geometric model theory: let $\mathrm{L}_{\mathrm{rings}} = \{+, -, \cdot, 0, 1\}$. Examples of $\mathrm{L}_{\mathrm{rings}}$-structure are rings, fields,...

A theory T is a collection of L-sentences which allows a *model*, that is, there is some L-structure $\mathcal{M}$ in which each sentence in T holds.

Let us first see an example of this: Consider now T to be the collection of $\mathrm{L}_{\mathrm{rings}}$-sentences expressing that the universe is a field, and the following sentences for each n in $\mathbb{N}$:

$$\forall x_0 \dots \forall x_n \exists y (x_n \neq 0 \to x_o + x_1 y + \cdots + x_n y^n = 0)$$

$$\underbrace{1+\cdots+1}_{n} \neq 0$$

It is easy to see that a model of T is a field of characteristic 0 in which every nonzero polynomial has a root, that is, an algebraically closed field.

Given an L-structure $\mathcal{M}$, there is a theory attached to it, denoted by $Th(\mathcal{M})$, which is the collection of all L-sentences that hold in $\mathcal{M}$.

We recall now the following theorem, which is one of the main achievements in first-order logic.

3.11 Theorem. (Compactness Theorem) *A collection Σ of formulae has a model if and only if each finite subcollection $\Sigma_0 \subset \Sigma$ does.*

The reason for the name is not accidental. It is strongly related to a certain topological space being compact. Hence, from any open cover we can extract a finite open cover. An open cover is related to a collection of formulae being inconsistent (*i.e.* without a model).

There are two relevant proofs for this theorem: The first one, due to Henkin, is a syntactical proof, in which we can obtain a model as some equivalence classes of certain symbols, all witnessed within Σ. Hence, the model obtained does not have a nice description in terms of the models regarding the finite subsets of Σ. The other proof, using *ultraproducts*, is a more sematical one, which exhibits a model in terms of the models for the finite subsets, in which we later need to quotient out by an equivalence relation according to what these finite sets induce asymptotically. This is done via *ultrafilters*, which give some kind of probability measure with values on $\{0, 1\}$. Both proofs can be found in [5] and [6].

3.2 Explicit description of measurable sets

Let $\overline{k}$ denote some fixed algebraic closure of k.

We will throughout this section work in a 2-sorted language as follows:

$$\mathrm{L} = \{+, -, \cdot, 0, 1, +_{\text{group}}, \mathrm{ord}, \overline{ac}, \leq, \equiv_d, c\}_{\substack{d\in\mathbb{Z}\\ c\in k}}$$

where we can distinguish variables from each sort. One sort will be a valued field and the other the valuation group. Moreover, $\overline{k}$ will be the residue field. The function ord will denote the valuation. In a natural fashion, $(\overline{k}((t)), \mathbb{Z})$ is an L-structure in which we interpret $\mathrm{ord} = \mathrm{ord}_t$ as the valuation in $\overline{k}((t))$ and $\overline{ac}(x)$ is the leading coefficient of $x \in \overline{k}((t))$, if $x \neq 0$, and 0, if $x = 0$. We also have names for each element in k.

3.12 Definition. Let $x_1, \ldots, x_m$ be variables from the sort $\overline{k}((t))$, and $e_1, \ldots, e_r$ in the sort $\mathbb{Z}$. We say that the formula $\theta(x_1, \ldots, x_m; e_1, \ldots, e_r)$ defines a semi-algebraic condition if it is quantifier-free, *i.e.* a finite boolean combination of conditions of the following 3 types:

(i) $\mathrm{ord}_t\, f_1(x_1, \ldots, x_m) \geq \mathrm{ord}_t\, f_2(x_1, \ldots, x_m) + L(e_1, \ldots, e_r)$;
(ii) $\mathrm{ord}_t\, f_1(x_1, \ldots, x_m) \equiv L(e_1, \ldots, e_r) \mod d$;
(iii) $h(\overline{ac}(f_1(x_1, \ldots, x_m)), \ldots, f_{m'}(x_1, \ldots, x_m))) = 0$,
where

- f_i, h are polynomials over k;

- L is a linear polynomial over $\mathbb{Z}$;
- $d \in \mathbb{N}$;

3.13 Definition. A set $A \subset \overline{k}((t))^m \times \mathbb{Z}^r$ is *semi-algebraic*, if $A = \{(x_1, \ldots, x_m; l_1, \ldots, l_r) \in \overline{k}((t))^m \times \mathbb{Z}^r \mid \theta(x_1, \ldots, x_m; l_1, \ldots, l_r)\}$, where θ is a semi-algebraic condition.

3.14 Theorem. *(Pas) Let θ be a semi-algebraic condition. Then the condition $\exists x_1 \in \overline{k}((t)) : \theta(x_1, \ldots, x_m; l_1, \ldots, l_r)$ is also semi-algebraic.*
Equivalently, $Th(\overline{k}((t)), \mathbb{Z})$ has Quantifier Elimination.

See [7].

3.15 Remark. The proof of the above theorem follows from cell decomposition where $char(k) = 0$. It is not known in general for prime characteristic. Removing quantifiers bounding variables from the valuation group is well-known for the sublanguage in $\mathbb{Z}$ that we are considering. Hence, the relevance of this theorem is that it allows us to eliminate quantifiers bounding variables from the valued field.

3.16 Definition. A family of subsets $A_l \subset \mathcal{L}(X)$, $l \in \mathbb{N}^n$ is *semi-algebraic family of semi-algebraic subsets* if there exists affine covering of X by open subsets U such that for each l in $\mathbb{N}$:

$$A_l \cap \mathcal{L}(U) = \{x \in \mathcal{L}(U) | \theta(h_1(\tilde{x}), \ldots, h_m(\tilde{x}); l)\},$$

where $\tilde{x} \in \mathcal{L}(X)(k_x)$ corresponds to x, $h_1, \ldots, h_m$ are regular function on U, θ is a semi-algebraic condition.

We say that $A \subset \mathcal{L}(X)$ is semi-algebraic if the family $\{A\}$ is semi-algebraic.

3.17 Definition. A function $\alpha : A \times \mathbb{Z}^m \to \mathbb{Z} \cup \{\infty\}$ is *simple* if the family of sets

$$\{x \in A | \alpha(x, l) = l_{m+1}\}_{(l, l_{m+1}) \in \mathbb{N}^{m+1}}$$

is a semi-algebraic family of semi-algebraic subsets.

3.18 Corollary. *(Pas' theorem) Let $A \subset \mathcal{L}(X)$ be a semi-algebraic set. Then*
(i) $\pi_n(A)$ is constructible for any n;
(ii) $\pi_n^{-1}\pi_n(A)$ is semi-algebraic for any n;
(iii) if $f : X \to Y$ is a morphism of algebraic varieties, then $f(A)$ is semi-algebraic.

Proof. This is a trivial application of Theorem 3.14. We will include the proof nevertheless for completeness of this work.

We may assume that X is affine (by considering the affine covering from the definition of semi-algebraic sets).

For *i*), we have that y is in $\pi_n(A)$ if and only if $\exists x \in \overline{k}((t))\,(\mathrm{ord}_t(x) \geq 0 \wedge \mathrm{ord}_t(x - y) \geq n + 1 \wedge x \in A)$. Clearly, this is a semi-algebraic condition. Now, since we work up to order n, any instance appearing in a semi-algebraic condition can be reduced to a polynomial equation by comparing coefficients, since congruence is only relevant up to n and likewise for the order. Hence, $\pi_n(A)$ is constructible.

For *ii*), note that $x \in \pi_n^{-1}\pi_n(A)$ if and only if

$$\exists y \in \pi_n(A)\,(\mathrm{ord}_t(y - x) \geq n + 1)\,.$$

iii) is also easy by considering affine pieces of X so that f is a given quotient of two polynomials (which can always be done). $\square$

3.19 Definition. Let X be an algebraic variety over k, $d = \dim X$, $A \subset \mathcal{L}(X)$ a subset. A is *weakly stable* if it is semi-algebraic, and for some $n \in \mathbb{N}$: $A = \pi_n^{-1}\pi_n(A)$.

3.20 Lemma. *Let $A = \bigcup_{i\in\mathbb{N}} A_i$; A, A_i be weakly stable sets. Then $A = \bigcup_{finite} A_i$.*

Proof. Recall that A being weakly stable of level n means that $x \in A$ is first-order expressable, since $x \in A$ if and only if $\exists y \in \pi_n(A)\,(\mathrm{ord}_t(x-y) \geq n+1)$. That is, although for semi-algebraic sets there need not be a bound on the extension on the degree of k_x for points $\tilde{x} \in \mathcal{L}(X)(k_x)$, we know that any constructible set is a finite boolean combination of zero sets of polynomials, and hence, use this to define A.

We will write $x \in A$ to refer to the formula defining A.

Let α be a new constant symbol not in L. Consider now the following collection of sentences:

$$Th(\overline{k}((t)), \mathbb{Z}) \cup \{\alpha \in A\} \cup \{\alpha \notin A_i\}_{i\in\mathbb{N}}$$

It is inconsistent by hypothesis. By Theorem 3.11, there is a finite subset that is inconsistent, that is, inconsistency is witnessed for $\{\alpha \in A\} \cup \{\alpha \notin A_i\}_{i\leq m}$ for some $m \in \mathbb{N}$. Since α is not in L, this is equivalent to the following:

Any model of $Th(\overline{k}((t)), \mathbb{Z})$ satisfies that $\forall x(x \in A \Rightarrow \bigcup\limits_{i\leq m} x \in A_i)$.

In particular, $(\overline{k}((t)), \mathbb{Z})$ does.

3.21 Definition. $\mathcal{L}^{(e)}(X) = \mathcal{L}(X) \setminus \pi_e^{-1}\mathcal{L}_e(X_{\mathrm{sing}})$.

3.22 Proposition. *If A is a weakly stable subset such that $A \cap \mathcal{L}(X_{sing}) = \emptyset$, then A is stable.*

Proof. Apply Lemma 3.20 and 3.29 below to

$$A = \bigcup_e (A \cap \mathcal{L}^{(e)}(X)).$$

$\square$

3.23 Remark. The set $\mathbb{B}$ of all semi-algebraic sets is exactly the set of all measurable sets.

3.24 Proposition. *Let X be an algebraic variety over k of dimension d. Then there exists a unique function $\mu : \mathbb{B} \to \widehat{M_k}$ such that:*

(i) if A is stable, then $\mu(A) = [\pi_n(A)]\mathbb{L}^{-nd}$, $n \gg 0$;

(ii) if S is a closed subset of x, $\dim S < d$, then $\mu(\mathcal{L}(S)) = 0$;

(iii) if $A = \bigcup_{i\in\mathbb{N}} A_i$, where A_i are disjoint elements of $\mathbb{B}$, then $\sum \mu(A_i)$ converges to $\mu(A)$ in $\widehat{M_k}$;

(iv) if $A \subset B$ are elements of $\mathbb{B}$, and $\mu(B) \subset F^m\widehat{M_k}$, then $\mu(A) \subset F^m\widehat{M_k}$.

Proof. The key ingredients in the proof are lemmas 3.32, 3.30 and 3.20. The uniqueness of μ follows directly from lemma 3.32 (see also proposition 3.8), so it only remains to prove the existence of a map $\mu : \mathbb{B} \to \widehat{M_k}$ satisfying (i) up to (iv).

Let $\mathbb{B}_0$ denote the set of all A in $\mathbb{B}$ which are stable. Thus $\mathbb{B}_0$ is closed under finite unions and finite intersections. Clearly, there exists a map $\mu_0 : \mathbb{B}_0 \to \widehat{M_k}$ satisfying (i) and (iv) with μ and $\mathbb{B}$ replaced by μ_0 and $\mathbb{B}_0$. Obviously μ_0 is additive, hence lemma 3.20 yields (iii) with μ and $\mathbb{B}$ replaced by μ_0 and $\mathbb{B}_0$. Next let $\mathbb{B}_1$ be the set of all A in $\mathbb{B}$ which can be written as $A = \bigcup_{i\in\mathbb{N}} A_i$ with the A_i's in $\mathbb{B}_0$ mutually disjoint and $\lim_{i\to\infty} \mu_0(A_i) = 0$. For A in $\mathbb{B}_1$ we set $\mu_1(A) = \sum_{i=0}^{\infty} \mu_0(A_i)$. This is independent of the choice of the A_i's. Indeed, suppose that also $A = \bigcup_{i\in\mathbb{N}} A'_i$ with the A'_i's in $\mathbb{B}_0$ mutually disjoint and $\lim_{i\to\infty} \mu_0(A'_i) = 0$. Then

$$\sum_{i=0}^{\infty} \mu_0(A_i) = \sum_{i=0}^{\infty} \mu_0\Big(\bigcup_{j\in\mathbb{N}} (A_i \cap A'_j)\Big) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mu_0(A_i \cap A'_j) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \mu_0(A_i \cap A'_j) = \sum_{j=0}^{\infty} \mu_0(A'_j)$$

because (iii) and (iv) hold for μ and $\mathbb{B}$ replaced by μ_0 and $\mathbb{B}_0$. One verifies that (i) and (iv) are true for μ and $\mathbb{B}$ replaced by μ_1 and $\mathbb{B}_1$. From lemma 3.30 one easily deduces the following

(*) If S is a closed subvariety of X with $\dim S < \dim X$ and if A belongs to $\mathbb{B}_1$, then $A \backslash \mathcal{L}(S)$ belongs also to $\mathbb{B}_1$ and $\mu_1(A\backslash\mathcal{L}(S)) = \mu_1(A)$.

Indeed, we may assume A belongs to $\mathbb{B}_0$ and consider the following partitions by elements of $\mathbb{B}_0$:

$$A\backslash\mathcal{L}(S) = (A\backslash\pi_m^{-1}\pi_m(\mathcal{L}(S))) \cup \bigcup_{n\geq m} ((\pi_n^{-1}\pi_n(\mathcal{L}(S))\backslash\pi_{n+1}^{-1}\pi_{n+1}(\mathcal{L}(S))) \cap A),$$

$$A = (A\backslash\pi_m^{-1}\pi_m(\mathcal{L}(S))) \cup (\pi_m^{-1}\pi_m(\mathcal{L}(S)) \cap A),$$

for $m \in \mathbb{N}$ large enough.

Next let A be any element of $\mathbb{B}$. Then, by lemma 3.32, there exists a closed subvariety S of X with $\dim S < \dim X$ such that $A\backslash\mathcal{L}(S)$ belongs to $\mathbb{B}_1$. Define μ by $\mu(A) = \mu_1(A\backslash\mathcal{L}(S))$. By (*), this definition is indepenedent of the choice of S. Indeed, if S' is another closed subvariety of X with $\dim S' < \dim X$ such that $A\backslash\mathcal{L}(S')$ belongs to $\mathbb{B}_1$, then

$$\mu_1(A\backslash\mathcal{L}(S')) = \mu_1((A\backslash\mathcal{L}(S'))\backslash\mathcal{L}(S)) = \mu_1((A\backslash\mathcal{L}(S))\backslash\mathcal{L}(S')) = \mu_1(A\backslash\mathcal{L}(S)).$$

Clearly (i), (ii) and (iv) are satisfied and μ is additive on finite disjoint unions. It remains to prove (iii). Let A and the A_i's be elements of $\mathbb{B}$ as in (iii) and let m be in $\mathbb{N}$. Then there exist *weakly stable* A'_m and A'_{im}'s in $\mathbb{B}$ such that $A \subset A'_m$, $A_i \subset A'_{im}$ and $\mu(A) - \mu(A'_m)$ and $\mu(A_i) - \mu(A'_{im})$ belong to $F^m\widehat{M_k}$. Namely, we can take $A'_m = \pi_m^{-1}\pi_m(A)$, $A'_{im} = \pi_m^{-1}\pi_m(A_i)$. The properties $\mu(A) - \mu(A'_m)$ and $\mu(A_i) - \mu(A'_{im})$ are satisfied because of lemma 3.32: namely, as it is shown in the proof of this lemma, the elements of the partition for A and A_i constructed there coinside with the elements of the partition for A'_m and A'_{im} up to the elements of volume belonging to $F^m\widehat{M_k}$.

Now we have $A'_m = \bigcup_{i\in\mathbb{N}} A'_{im}$, and by lemma 3.20 A'_m is the union of a finite number of the sets A'_{im}, say $A'_m = \bigcup_{i=1}^{e} A'_{im}$ whenever e is large enough. Since

$$A'_m = (\bigcup_{i=1}^{e} A_i) \cup (\bigcup_{i=1}^{e} (A'_{im} \backslash A_i)),$$

we get

$$\mu(A) = \mu(A'_m) = \sum_{i=1}^{e} \mu(A_i) \mod F^m\widehat{M_k}.$$

Since this holds for all $m \in \mathbb{N}$, we obtain (iii).

$\square$

3.25 Definition. The function μ from Proposition is *motivic volume* on $\mathcal{L}(X)$; it will be always denoted by μ. If $\alpha : A \to \mathbb{Z}$ is a simple function, put

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{s\in\mathbb{Z}} \mu(A \cap \alpha^{-1}(s))\mathbb{L}^{-sd},$$

if the sum converges in $\widehat{M_k}$. In this case we say that α is *integrable.* If the function α is bounded from below, then $\mathbb{L}^{-\alpha}$ is integrable on A because of (iv) from Proposition.

3.26 Lemma. *Let $A = \bigcup_{i\in\mathbb{N}} A_i$, where A_i are disjoint semi-algebraic sets. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \sum_{i\in\mathbb{N}} \int_{A_i} \mathbb{L}^{-\alpha} d\mu.$$

Proof. It is enough to see that the double sum

$$\sum_{i\in\mathbb{N}} \sum_{s\in\mathbb{Z}} \mu(A_i \cap \alpha^{-1}(s))\mathbb{L}^{-sd}$$

converges. To do this, it is sufficient to note: given r, there exists only finitely many s with $\dim(\mu(A \cap \alpha^{-1}(s))\mathbb{L}^{-sd}) \geq r$, and for any such s there exists only finitely many i with $\dim(\mu(A_i \cap \alpha^{-1}(s))\mathbb{L}^{-sd}) \geq r$ by property (iii) in Proposition 3.24. Note also that for any other s we have $\dim(\mu(A_i \cap \alpha^{-1}(s))\mathbb{L}^{-sd}) < r$ for all i by property (iv) in Proposition 3.24 $\square$

3.27 Example. ord_Y is a measurable function.

1. If $Y = \emptyset$, then $\mathrm{ord}_Y = 0$. For smooth X we obtain

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_Y} d\mu = [X].$$

2. Let Y be a smooth divisor, X smooth. Since

$$\mu(\mathrm{ord}_Y^{-1}(s)) = [\mathcal{L}_{s-1}(Y)]\mathbb{L}^{-d(s-1)} - [\mathcal{L}_s(Y)]\mathbb{L}^{-ds} = [Y](\mathbb{L} - 1)\mathbb{L}^{-s},$$

we obtain

$$\begin{aligned}
\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_Y} d\mu &= [X \setminus Y] + \sum_{s=1}^{\infty} [Y](\mathbb{L}-1)\mathbb{L}^{-s}\mathbb{L}^{-s} \\
&= [X \setminus Y] + [Y](\mathbb{L}-1)\mathbb{L}^{-2} \sum_{s=0}^{\infty} \mathbb{L}^{-2s} \\
&= [X \setminus Y] + [Y](\mathbb{L}-1) \cdot \frac{1}{\mathbb{L}^{-2}(1-\mathbb{L}^{-2})} \\
&= [X \setminus Y] + \frac{[Y]}{\mathbb{L}+1} \\
&= [X \setminus Y] + \frac{[Y]}{[\mathbb{P}^1]}.
\end{aligned}$$

3. Let Y be a smooth subscheme of codimension c. Then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_Y} d\mu = [X \setminus Y] + [Y] \frac{\mathbb{L}^c - 1}{\mathbb{L}^{c+1} - 1}.$$

3.28 Example. We can generalize the function from example 3.10 as follows. Let $\mathcal{I}$ be a sheaf of ideals on Y. Then the function

$$\operatorname{ord}_{\mathcal{I}} : \mathcal{L}(X) \to \mathbb{N} \cap \{\infty\}$$

maps any arc γ to $\min_g \operatorname{ord}_t \gamma^*(g)$, where g runs over sections of $\mathcal{I}$ in a neighborhood of $\pi_0(\gamma)$.

In particular, if $\mathcal{I}$ is the sheaf of ideals of a smooth subscheme Y, we have $\operatorname{ord}_{\mathcal{I}} = \operatorname{ord}_Y$.

An important case is that of a locally principal sheaf of ideals $\mathcal{I}$. Such $\mathcal{I}$ corresponds to an effective Cartier divisor D, and we write $\operatorname{ord}_{\mathcal{I}} = \operatorname{ord}_D$.

Let $D = \sum r_i D_i$ be a divisor with normal crossings, $r_i \geq 0$, D_i smooth. Then (see Craw, Th.1.13, for $k = \mathbb{C}$)

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D} d\mu = \sum_{J \subset \{1,\ldots,s\}} [D_J^\circ] \prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{r_j+1} - 1},$$

where $D_J = \bigcap_{j \in J} D_j$, $D_J^\circ = D_J \setminus \bigcup_{j \notin J} D_J$.

3.29 Lemma. *Let X be an algebraic variety over k of pure dimension* $\dim X = d$*, where k is perfect. There exists a positive integer c such that for any $e, n \in \mathbb{N}$ with $n \geq ce$:*

a) $\theta_n : \pi_{n+1}(\mathcal{L}(X)) \to \pi_n(\mathcal{L}(X))$ is a piecewise trivial fibration over $\pi_n(\mathcal{L}^{(e)}(X))$ with fiber $\mathbb{A}_k^d$.

b) $[\pi_n(\mathcal{L}^{(e)}(X))] = [\pi_{ce}(\mathcal{L}^{(e)}(X))]\mathbb{L}^{d(n-ce)}$.

Proof. Since b) is a direct consequence of a), we only have to prove a). We may assume X is affine, say $X = \operatorname{Spec}([X_1, \ldots, X_N]/I) \subset \mathbb{A}_k^N$. Using lemmas 4.8 and 4.9 below, we reduce the proof to the case of intersection

$$\mathcal{L}(X) \cap A_i = \mathcal{L}(\operatorname{Spec}([X_1, \ldots, X_N]/(f_1, \ldots, f_{N-d}))) \cap A_i,$$

i.e. we may assume $I = (f_1, \ldots, f_{N-d})$. By lemma 4.9

$$\mathcal{L}^{(e)}(X) \subset \bigcup_{i=1}^{m} A_i \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{l} A_{ij},$$

where $A_{ij} = \{\varphi \in k[[X]]^N | s_i(\pi_{ce}\varphi)\delta_{ij}(\pi_{ce}\varphi) \neq 0\}$ and δ_{ij} run over all minors of order $N-d$ including δ_i in the notation of lemma 4.9. Note that

$$\bigcup_{i=1}^{m} \bigcup_{j=1}^{l} A_{ij} \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{l} \bigcup_{e'=0}^{ce} A_{ije'},$$

where

$$A_{ije'} = \{\varphi \in A_{ij} |\, \mathrm{ord}_t\, \delta_{ij}(\tilde{\varphi}) = e' \quad \text{and} \quad \mathrm{ord}_t\, \delta'_{ij}(\tilde{\varphi}) \geq e' \\ \text{for all other minors } \delta'_{ij} \quad \text{of order } N-d\}$$

So, it is sufficient to prove that the map θ_n is a piecewise trivial fibration over $\pi_n(\mathcal{L}(X) \cap A_{ije'})$. Without loss of generality we may assume δ_{ij} is the minor of the first $N-d$ columns. Let $s : (\bar{k}[t]/t^{n+1})^N \to \bar{k}[t]^N$ be the $\bar{k}$-linear map given by $t^l \mod t^{n+1} \mapsto t^l$ for $l = 0, 1, \ldots, n$. Let $\bar{\varphi} \in (\bar{k}[t]/t^{n+1})^N$ be any $\bar{k}$-rational point of $\pi_n(\mathcal{L}(X) \cap A_{ije'}$. We have

$$\theta_n^{-1}(\bar{\varphi}) = \{s(\bar{\varphi}) + t^{n+1}y \mod t^{n+2} | y \in \bar{k}[t]^N, \quad f(s(\bar{\varphi}) + t^{n+1}y) = 0\},$$

where f is the column with entries $f_1, \ldots, f_{N-d}$. By Taylor expansion, the condition $f(s(\bar{\varphi}) + t^{n+1}y) = 0$ can be rewritten as

$$f(s(\bar{\varphi})) + t^{n+1}\Delta(s(\bar{\varphi}))y + t^{2(n+1)}(\ldots) = 0, \tag{5}$$

where $\Delta = \frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_1,\ldots,X_N)}$. There exists an $N-d$ by $N-d$ matrix M over $k[x_1, \ldots x_n]$, independent of the choice of $\bar{\varphi}$, such that

$$M\Delta = (\delta_{ij} I_{N-d}, W),$$

where I_{N-d} is the identity matrix with $N-d$ columns and W is a $N-d$ by d matrix such that $W(s(\bar{\varphi})) = 0 \mod t^{e'}$. To check the last congruence one expresses the last d columns of Δ in terms of the first $N-d$ columns by Cramer's rule and then one uses the definition of $A_{ije'}$.

Condition (5) is equivalent to

$$t^{-e'-n-1}(Mf)(s(\bar{\varphi})) + t^{-e'}(M\Delta)(s(\bar{\varphi}))y + t^{n+1-e'}(\ldots) = 0. \tag{6}$$

Note that $t^{-e'}(M\Delta)(s(\bar{\varphi}))$ is a matrix over $\bar{k}[[t]]$, whose minor determined by the first $N-d$ columns is not divisible by t, because $\mathrm{ord}_t\, \delta_{ij}(\tilde{\varphi}) = e'$. Moreover, $n+1-e' \geq 1$. Since $\bar{\varphi}$ is liftable to $\mathcal{L}(X)$ (i.e. belongs to $\pi_n(\mathcal{L}(X))$), equation (6) has a solution y in $\bar{k}[[t]]^N$, and thus $t^{-e'-n-1}(Mf)(s(\bar{\varphi}))$ is a column matrix over $\bar{k}[[t]]$. By Newton's lemma we deduce that $\theta_n^{-1}(\bar{\varphi})$ is equal to the set of all $s(\bar{\varphi}) + t^{n+1}y_0$ with $y_0 \in \bar{k}^N$ such that

$$t^{-e'-n-1}(Mf)(s(\bar{\varphi})) + t^{-e'}(M\Delta)(s(\bar{\varphi}))y_0 = 0 \mod t.$$

Thus the fiber $\theta_n^{-1}(\bar{\varphi})$ is a d-dimensional affine subspace of $\mathbb{A}_k^N$, given by linear equations which express the first $N-d$ coordinates in terms of linear combinations of the last d coordinates, with coefficients which are regular functions on each locally closed subset of $\mathcal{L}_n(X)$ contained in $\pi_n(\mathcal{L}(X) \cap A_{ije'})$. Now, since the map θ_n is locally of finite type, the set of points $y \in \pi_{n+1}(\mathcal{L}(X) \cap A_{ije'})$ such that θ_n is flat is open in $\pi_{n+1}(\mathcal{L}(X) \cap A_{ije'})$. Since the schemes $\mathcal{L}_n(X)$ are assumed to be reduced, this set is nonempty on each irreducible component of $\pi_{n+1}(\mathcal{L}(X) \cap A_{ije'})$, so it is nonempty (see, for example [4], Chapter VIII). Then by one of the equivalent smoothness condition (see SGA. 1,II), the morphism θ_n is smooth on it and is therefore locally an affine d-space, that is a piecewise trivial fibration. Repeating this argument for the restriction of θ_n on the complement to the open set in $\pi_{n+1}(\mathcal{L}(X) \cap A_{ije'})$, we get that θ_n is a piecewise trivial fibration over $\pi_n(\mathcal{L}(X) \cap A_{ije'})$ with fiber $\mathbb{A}_k^d$.

□

3.30 Lemma. *Let X be an algebraic variety over k, $\dim X = d$.*

(1) For any $n \in \mathbb{N}$, $\dim \pi_n(\mathcal{L}(X)) \leq (n+1)d$

(2) For any $n, m \in \mathbb{N}$ such that $m \geq n$, the fibers of

$$\pi_m(\mathcal{L}(X)) \to \pi_n(\mathcal{L}(X))$$

are of dimension $\leq (m-n)d$.

Proof. Assertion (1) follows from assertion (2). Moreover, it suffices to prove (2) for $m = n+1$, and we may assume that X is affine and that $k = \bar{k}$. As we have seen in section 1, each fiber of $\pi_{n+1}(\mathcal{L}(X)) \to \pi_n(\mathcal{L}(X))$ is contained in some reduction modulo t^l of the affine scheme over $\operatorname{Spec}(k[t])$, which is given in $\operatorname{Spec}(k[t](x_1, \ldots, x_N))$ by the equations $f_i(x_1, \ldots, x_N)$, $i = 1, \ldots, r$, where f_i are the polynomials defining X over k (namely, points of the fiber have coordinates $a_i + t^{n+1}x_i$, where a_i are coordinates of a point in $\pi_n(\mathcal{L}(X))$ and $f_i(a_i + t^{n+1}x_i) = 0 \mod t^{n+2}$). Obviously, the scheme $\operatorname{Spec}(k[t](x_1, \ldots, x_N)/(f_1, \ldots, f_r))$ is of finite type over $\operatorname{Spec}(k[t])$. Moreover, it is flat: to prove the flatness, it is sufficient to prove that $t-a$ is not a zero divisor in the ring $(k[[t]](x_1, \ldots, x_N)/(f_1, \ldots, f_r))$ for all $a \in k$, because $k[t]$ is a principal ideal domain and k is algebraically closed. But it is easy to check because the polynomials f_i have coefficients belonging to k. The generic fiber of this scheme is equal to $X \otimes_k k(t)$, so each fiber has dimension d.

The assertion of lemma now follows from the observation that

$$\dim \operatorname{Spec}(k[[t]](x_1, \ldots, x_N)/(f_1, \ldots, f_r, t^l)) = \dim \operatorname{Spec}(k[[t]](x_1, \ldots, x_N)/(f_1, \ldots, f_r, t)) = \dim X.$$

□

3.31 Lemma. *Let X be an algebraic variety over k, $\dim X = d$, S a closed subvariety of dimension $< d$, γ_S a Greenberg's function for S. Then for any positive integers n, i, e with $n \geq i \geq \gamma(e)$ we have*

$$\dim \pi_{n,X}(\pi_{i,X}^{-1}\mathcal{L}_i(S)) \leq (n+1)d - e - 1.$$

Proof. Clearly we may assume $i = \gamma(e)$. By lemma 3.30 (2) applied to the projection

$$\pi_{n,X}(\pi_{\gamma(e),X}^{-1}\mathcal{L}_{\gamma(e)}(S)) \to \pi_{e,X}(\pi_{\gamma(e),X}^{-1}\mathcal{L}_{\gamma(e)}(S))$$

we obtain

$$\dim \pi_{n,X}(\pi^{-1}_{\gamma(e),X}\mathcal{L}_{\gamma(e)}(S)) \leq (n-e)d + \dim \pi_{e,X}(\pi^{-1}_{\gamma(e),X}\mathcal{L}_{\gamma(e)}(S)).$$

Since, by definition of the Greenberg function, $\pi_{e,X}(\pi^{-1}_{\gamma(e),X}\mathcal{L}_{\gamma(e)}(S)) = \pi^{\gamma(e)}_{e,X}\mathcal{L}_{\gamma(e)}(S) = \pi^{\gamma(e)}_{e,S}\mathcal{L}_{\gamma(e)}(S) = \pi_{e,S}(\mathcal{L}(S))$, the result follows because, by lemma 3.30 (1), $\dim \pi_{e,S}(\mathcal{L}(S)) \leq (e+1)(d-1)$. □

3.32 Lemma. *Let X be an algebraic variety over k of pure dimension d, $A \in \mathbb{B}$. Then there exists closed $S \subset X$, $\dim S < d$, and stable A_i of level n_i, $i \in \mathbb{N}$, such that*

$$A = \bigcup_{i\in\mathbb{N}} A_i \cup (A \cap \mathcal{L}(S))$$

is a partition (disjoint union), and

$$\dim(\pi_{n_i}(A_i)) - (n_i+1)d \underset{i\to\infty}{\to} -\infty.$$

Also, if $\alpha : A \to \mathbb{Z}$ is a simple function, then we can find A_i such that $\alpha(A_i)$ is constant for any i.

Proof. We may assume that X is affine and irreducible and that A is given by a semi-algebraic condition. Let g be a nonzero regular function on X which vanishes on the singular locus of X. Let F be the product of g and all the functions f_i (assumed to be regular and not identically zero on X) appearing in the conditions of the semi-algebraic condition that defines A. Then we can take S to be the locus of $F = 0$ and

$$A_i = \{x \in A\backslash\mathcal{L}(S) | \operatorname{ord}_t F(x) = i\}.$$

It's clear that A_i are semi-algebraic, because $A, \mathcal{L}(S)$ are semi-algebraic and so is its difference and the condition $\operatorname{ord}_t F(x) = i$ is a difference of two standart conditions $\operatorname{ord}_t F(x) \geq i$ and $\operatorname{ord}_t F(x) \geq i+1$.

Every A_i is weakly stable of level i, because $\pi_i^{-1}(\pi_i(A_i))$ consists exactly of arcs $x \in \mathcal{L}(X)$ satisfying the condition $\operatorname{ord}_t F(x) = i$. Since F has a factor g, these arcs do not belong to $\mathcal{L}(S)$. Since F has as factors all functions from the semi-algebraic condition, we conclude that $\operatorname{ord}_t f_j(x) \leq i$ for all j, therefore, our semi-algebraic condition is preserved under the composition $\pi_i^{-1}\pi_i$ and all such arcs must belong to the set A.

Next, it's obvious that $\{A_i\}$, $A \cap \mathcal{L}(S)$ form a partition of A. By proposition 3.22 all A_i are stable. Let's prove that

$$\dim(\pi_i(A_i)) - (i+1)d \underset{i\to\infty}{\to} -\infty.$$

Since $\operatorname{ord}_t F(x) = i > 0$ for all $x \in A_i$, we obtain $\pi_{i-1}(A_i) \subset \mathcal{L}_{i-1}(S)$. Therefore, using the same arguments as in lemma 3.31 we have

$$\begin{aligned}\dim(\pi_i(A_i)) - (i+1)d \leq (i+1) \leq \dim(\pi_i(\pi^{-1}_{i-1}(\mathcal{L}_{i-1}(S))) - (i+1)d \leq \\ d + \dim \mathcal{L}_{i-1}(S) - (i+1)d \leq d + i(d-1) - (i+1)d = -i \underset{i\to\infty}{\to} -\infty.\end{aligned}$$

The proof of the last assertion is quite similar.

□

4 Transformation rule

Throughout this section X and Y are k-varieties of pure dimension d, Y is smooth, and $h : Y \to X$ is a proper birational morphism. Then h induces a proper morphism $h_n : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$.

Note that Ω_Y^d is an invertible sheaf, and $h^*(\Omega_X^d)$ is a subsheaf of it. We define a sheaf of ideals Jac_h on Y from the formula

$$h^*(\Omega_X^d) = Jac_h \cdot \Omega_Y^d.$$

If X is also non-singular, Jac_h is locally generated by the Jacobian of h. The corresponding Cartier divisor on Y is called the relative canonical divisor $K_{Y/X}$; we have $K_{Y/X} = K_Y - h^*K_X$.

4.1 Lemma. *1. Let $A \subset \mathcal{L}(X)$ be a semi-algebraic set, m an integer. Let $\alpha, \beta : A \times \mathbb{Z}^m \to \mathbb{Z} \cup \{\infty\}$ be simple functions. Then $\alpha + \beta$ is a simple function as well.*

2. Let α, β be integrable. Then $\alpha + \beta$ is also integrable.

Proof. The affine coverings for of α and β (from the definition of simple function) can be refined to an affine covering $\mathcal{U}$ of A with the following property. For any $U \in \mathcal{U}$ there exist positive integers a, b, semi-algebraic conditions θ and ψ, and regular functions g_i $(i = 1, \ldots, a)$ and h_i $(i = 1, \ldots, b)$ on U such that for any $l \in \mathbb{Z}^{m+1}$ we have

$$\{x \in \mathcal{L}(U) | \alpha(x, l_1, \ldots, l_m) = l_{m+1}\}_{l \in \mathbb{N}^{m+1}} = \{x \in \mathcal{L}(U) | \theta(g_1(\tilde{x}), \ldots, g_a(\tilde{x}); l)\},$$
$$\{x \in \mathcal{L}(U) | \beta(x, l_1, \ldots, l_m) = l_{m+1}\}_{l \in \mathbb{N}^{m+1}} = \{x \in \mathcal{L}(U) | \psi(h_1(\tilde{x}), \ldots, h_b(\tilde{x}); l)\}.$$

It follows

$$\{x \in \mathcal{L}(U) | \alpha(x, l_1, \ldots, l_m) + \beta(x, l_1, \ldots, l_m) = l_{m+1}\}_{l \in \mathbb{N}^{m+1}}$$
$$= \{x \in \mathcal{L}(U) | \exists \lambda : \theta(g_1(\tilde{x}), \ldots, g_a(\tilde{x}); l_1, \ldots, l_m, \lambda) \& \psi(h_1(\tilde{x}), \ldots, h_b(\tilde{x}); l_1, \ldots, l_m, l_{m+1} - \lambda)\}.$$

Note that the condition

$$\theta(x_1, \ldots, x_a; e_1, \ldots, e_m, e_{m+2}) \& \theta(x_{a+1}, \ldots, x_{a+b}; e_1, \ldots, e_m, e_{m+1} - e_{m+2})$$

is semi-algebraic, and by Presburger's Theorem the condition

$$\exists e_{m+1} : \theta(x_1, \ldots, x_a; e_1, \ldots, e_m, e_{m+2}) \& \theta(x_{a+1}, \ldots, x_{a+b}; e_1, \ldots, e_m, e_{m+1} - e_{m+2})$$

is semi-algebraic as well.

The second statement is easy. $\square$

4.2 Theorem. *Let $A \subset \mathcal{L}(X)$ be a constructible set, and let $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ be a simple function. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}A} \mathbb{L}^{-(\alpha \circ h) - \mathrm{ord}_{Jac_h}} d\mu. \tag{7}$$

This is the transformation rule for motivic measure. If X is also smooth, and $\alpha = \mathrm{ord}_D$ for some effective Cartier divisor D, this can be restated as follows:

$$\int_A \mathbb{L}^{-\mathrm{ord}_D} d\mu = \int_{h^{-1}A} \mathbb{L}^{-\mathrm{ord}_{h*D+K_{Y/X}}} d\mu.$$

4.3 Example. One can directly verify the transformation rule, if X is a smooth surface, and $h : Y \to X$ is the blowing-up of one closed point on X.

Proof. We deduce Theorem from Proposition 4.15 below.

By Lemma 4.6, we may assume that $h : h^{-1}A \to A$ is bijective. (Indeed, $\mathcal{L}(E)$ and $\mathcal{L}(h(E))$ have measure 0.)

Applying Lemma 3.32 to the function $\mathrm{ord}_{X_{\mathrm{sing}}} \circ h$, we can write

$$h^{-1}A = \bigcup_{i \in \mathbb{N}} B_i \cup (h^{-1}A \cap \mathcal{L}(S)),$$

where S is a closed subset of Y of dimension $< d$, and $\mathrm{ord}_{X_{\mathrm{sing}}} \circ h$ is constant on each B_i. Since both $h^{-1}A \cap \mathcal{L}(S)$ and $h(h^{-1}A \cap \mathcal{L}(S))$ are of measure 0, Lemma 3.26 reduces Theorem to the case when $h^{-1}A$ is stable (of some level n), and $\mathrm{ord}_{X_{\mathrm{sing}}}$ is constant on A. It follows that $A \subset \mathcal{L}^{(e')}(X)$ for suitable e'.

Applying Lemma 3.26 once more, we reduce Theorem to the case when both α and $\mathrm{ord}_{\mathrm{Jac}_h}$ are constant on $h^{-1}A$. Let a and e be their values. Then (7) is

$$\mu(A)\mathbb{L}^{-a} = \mu(h^{-1}A)\mathbb{L}^{-a-e},$$

i. e., $\mu(h^{-1}A) = \mathbb{L}^e \mu(A)$. This follows immediately from Proposition 4.15 and its Corollary since $\pi_n(h^{-1}A) \subset \Delta_{e,e',n}$. $\square$

4.4 Theorem. *(Kontsevich) Let X_1 and X_2 be birationally equivalent Calabi-Yau manifolds. Then $[X_1] = [X_2]$ in $\widehat{M_{\mathbb{C}}}$.*

4.5 Remark. Here a Calabi-Yau manifold M is a nonsingular complete algebraic variety over $\mathbb{C}$ with $K_M = 0$.

Proof. Since X_1 and X_2 are birationally equivalent there exists a nonsingular complete algebraic variety Y and proper birational morphisms $Y \to X_1$ and $Y \to X_2$. Then for $i = 1, 2$ the transformation rule yields

$$[X_i] = \mu(\mathcal{L}(X_i)) = \int_{\mathcal{L}(X_i)} 1 d\mu = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_{K_{Y/X_i}}} d\mu = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\mathrm{ord}_{K_Y}} d\mu.$$

$\square$

4.6 Lemma. *Let E be the exceptional locus of h. Then*

$$h : \mathcal{L}(Y) \setminus \mathcal{L}(E) \to \mathcal{L}(X) \setminus \mathcal{L}(h(E))$$

is a bijection.

Proof. By the valuative criterion of properness. $\square$

4.7 Lemma. *Let $f : V \to W$ be a bijective separable morphism of varieties over a field k. Assume that W is normal. Then f is an isomorphism.*

Proof. By Zariski's main theorem in Grothendieck's form, $f = g \circ v$, where $v : V \to C$ is an open immersion, and $g : C \to W$ is a finite morphism. Since $\#g^{-1}(x) = 1$ for $x \notin g(C \setminus v(V))$, we have $\deg g = 1$, and $v(V) = C$, i. e., f is a finite. Since $\deg g = 1$, and g is separable, g is a birational morphism. Finally, by Zariski's main theorem in original form, a finite birational morphism into a normal variety is an isomorphism. $\square$

4.8 Lemma. *Let* $X = \operatorname{Spec}([X_1, \ldots, X_N]/I) \subset \mathbb{A}_k^N$ *be a d-dimensional affine variety. Then* X_{sing} *is the intersection of* X *and all hypersurfaces* $s\delta = 0$*, where* δ *is some minor of order* $N - d$ *in the matrix* $\frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_1,\ldots,X_N)}$ *for some* $f_1, \ldots, f_{N-d} \in I$*, and* $s \in k[X_1, \ldots, X_N]$ *is such that* $sI \subset (f_1, \ldots, f_{N-d})$*.*

Proof. Clearly, X_{sing} lies in any hypersurface $\delta = 0$ and *a fortiori* $s\delta = 0$. Conversely, let $x \in X \setminus X_{\text{sing}}$ be a point that belongs to all hypersurfaces $s\delta = 0$, where δ and s are as above. The topological closure V of x lies in all these hypersurfaces; X_{sing} cannot contain all closed points of V because X_{sing} is closed. Thus, we may assume that x is closed.

Now, x is a regular closed point of X. One can find $f_1, \ldots, f_{N-d} \in k[X_1, \ldots, X_N]$ such that $(f_1, \ldots, f_{N-d})\mathcal{O}_{\mathbb{A}_k^N,x} = I\mathcal{O}_{\mathbb{A}_k^N,x}$. Since I is finitely generated, there exists $s \in k[X_1, \ldots, X_N]$ such that $sI \subset (f_1, \ldots, f_{N-d})$, and s does not vanish at x. Since $\mathcal{O}_{\mathbb{A}_k^N,x}/(f_1, \ldots, f_{N-d})$ is a d-dimensional regular local ring, some minor δ of order $N-d$ in the matrix $\frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_1,\ldots,X_N)}$ does not vanish at x. Then x does not belong to the hypersurface $s\delta = 0$, a contradiction. □

4.9 Lemma. *Let* $X = \operatorname{Spec}([X_1, \ldots, X_N]/I) \subset \mathbb{A}_k^N$ *be a d-dimensional affine variety. Then there exist positive integers* $c = c_X$ *and* m *with the following property. For any positive integer* e*, there exist semi-algebraic subsets* $A_1, \ldots, A_m$ *such that* $\mathcal{L}^{(e)}(X) \subset \bigcup_{i=1}^m A_i$*;* A_i *are weakly stable at level* ce*; for any* i *we have*

$$\mathcal{L}(X) \cap A_i = \mathcal{L}(\operatorname{Spec}([X_1, \ldots, X_N]/(f_1, \ldots, f_{N-d}))) \cap A_i \tag{8}$$

for some $f_1, \ldots, f_{N-d} \in I$*, and for some minor* δ *of order* $N - d$ *in* $\frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_1,\ldots,X_N)}$ *we have* $\operatorname{ord}_t \delta(x) \le ce$ *for all* $x \in A_i$*.*

Proof. Let J_0 be the ideal of $k[X_1, \ldots, X_n]$ generated by all $s\delta$ as in Lemma 4.8. Then set-theoretically $X_{\text{sing}} = X \cap V(J_0)$. Let $J = Z(X_{\text{sing}}) \subset k[X_1, \ldots, X_n]$. By Nullstellensatz, there exists a positive integer c such that $J^c \subset I + J_0$. We can also fix some generators $s_i\delta_i$ of J_0, $i = 1, \ldots, m$. By definition,

$$\begin{aligned}
\mathcal{L}^{(e)}(X)(\overline{k}) &= \{\varphi \in \mathcal{L}(X)(\overline{k}) \subset k[[X]]^N | \pi_e f(\varphi) \neq 0 \text{ for some } f \in J\} \\
&= \{\varphi \in \mathcal{L}(X)(\overline{k}) | \pi_{ce} f(\varphi) \neq 0 \text{ for some } f \in I + J_0\} \\
&= \{\varphi \in \mathcal{L}(X)(\overline{k}) | \pi_{ce} f(\varphi) \neq 0 \text{ for some } f \in J_0\} \\
&= \{\varphi \in \mathcal{L}(X)(\overline{k}) | \pi_{ce}(s_i(\varphi)\delta_i(\varphi)) \neq 0 \text{ for some } i\} \\
&= \bigcup_{i=1}^m (\mathcal{L}(X) \cap A_i),
\end{aligned}$$

where $A_i = \{\varphi \in k[[X]]^N | s_i(\pi_{ce}\varphi)\delta_i(\pi_{ce}\varphi) \neq 0\}$.

The minor δ_i is in fact the determinant of $\frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_{j_1},\ldots,X_{j_{N-d}})}$ for some $f_1, \ldots, f_{N-d} \in I$ with $s_iI \subset (f_1, \ldots, f_{N-d})$ and some $\{j_1, \ldots, j_{N-d}\} \subset \{1, \ldots, N\}$. Since $s_i(\varphi) \neq 0$ for any $\varphi \in A_i$, we obtain (8). □

4.10 Lemma. *Let* $X = \operatorname{Spec}([X_1, \ldots, X_N]/I) \subset \mathbb{A}_k^N$ *be a d-dimensional affine variety, and let* $I = (f_1, \ldots, f_{N-d})$*. Let* $p : X \to \mathbb{A}_k^d$ *be induced by the projection of* $\mathbb{A}_k^N$ *onto the last* d *coordinates.*

Let $\varphi, \varphi' \in \mathcal{L}(X)(\overline{k}) \subset \overline{k}[[t]]^N$ *be such that* $\varphi' \equiv \varphi \mod t^{e+1}\overline{k}[[t]]$*, and* $p(\varphi) = p(\varphi')$*. Assume that for the* $(N-d)\times(N-d)$ *minor* δ *of* $\Delta = \frac{\partial(f_1,\ldots,f_{N-d})}{\partial(X_1,\ldots,X_N)}$ *formed by the first* $N - d$ *columns we have* $\operatorname{ord}_t(\delta(\varphi)) \le e$*. Then* $\varphi = \varphi'$*.*

Proof. Let $\varphi \neq \varphi'$, and let E be maximal such that $\varphi' \equiv \varphi \mod t^E\overline{k}[[t]]$; we have $E \geq e+1$. Let $\varphi' = \varphi + \lambda$; then $\lambda = (\lambda_i)_{i=1}^N \in t^E\overline{k}[[t]]$, and $\lambda_{N-d+1} = \cdots = \lambda_N = 0$. Denote the $(N-d)$-tuple $(f_1, \ldots, f_{N-d})$ by f; then

$$0 = f(\varphi') \equiv f(\varphi) + \Delta\lambda = \Delta\lambda \mod t^{2E}\overline{k}[[t]].$$

Let B be the inverse matrix to the matrix formed by the first $N-d$ columns of Δ; then t^eB has elements in $k[[t]]$, and

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{N-d} \end{pmatrix} = B\Delta\lambda \in t^{E+1}\overline{k}[[t]]^{N-d},$$

a contradiction. □

For $e, e' \in \mathbb{N}$, introduce a semi-algebraic subset

$$\Delta_{e,e'} := \{\varphi \in \mathcal{L}(Y) |\, \mathrm{ord}_t(\mathrm{Jac}_h)(\varphi) = e;\ h(\varphi) \in \mathcal{L}^{(e')}(X)\},$$

and denote by $\Delta_{e,e',n}$ the image of $\Delta_{e,e'}$ in $\mathcal{L}_n(Y)$.

4.11 Remark. $\Delta_{e,e',n}$ is locally closed.

4.12 Conjecture. *Let B be a locally closed subset of $\mathcal{L}_n(X)$ such that $B \subset h_n(\Delta_{e,e',n})$. Then the morphism of varieties $h_n^{-1}(B) \to B$ induced by h_n is separable.*

4.13 Lemma. *Let A (resp., B) be an $a \times (a+b)$ (resp., $(a+b) \times b$) matrix over a ring of discrete valuation $\mathcal{O}$ such that $AB = 0$. Assume that the first a columns of A form the minor α of minimal valuation among all minors of A of order A. Then the last b rows of B form a minor β of minimal valuation among all minors of B of order b.*

Proof. Columns of B are solutions of the homogeneous linear system with matrix A. The first a components of any such solution are fixed linear combinations of the remaining b components, where the coefficients are ratios of certain minors of order a by Cramer's rule with the minor α in denominator. By the minimality condition, these coefficients are in $\mathcal{O}$. This means that the first a rows of B are $\mathcal{O}$-linear combinations of the last b rows. It follows that any minor of B order b is the product of β and the determinant of a certain $b \times b$ matrix over $\mathcal{O}$. □

4.14 Lemma. *Let A be an $n \times n$ matrix over $k[[t]]$. Let α be the map of $(k[[t]]/t^{n+1})^d$ to itself corresponding to multiplication by A. Then* $\dim \mathrm{Ker}\, \alpha = \mathrm{ord}_t(\det \alpha)$ *provided that $n \geq v(\det \alpha)$.*

Proof. According to elementary divisors theorem, $A = UA'V$, where U and V are invertible matrices over $k[[t]]$, and A' is a diagonal matrix. This reduces Lemma to the trivial case $A = A'$. □

4.15 Proposition. *Let $e, e', n \in \mathbb{N}$, $n \geq e + ce'$, where $c = c_X$. Then:*

(a) for all $\overline{\varphi} \in \Delta_{e,e',n}$ we have

$$h_n^{-1}(h_n(\overline{\varphi})) \subset \{\overline{y} \in \mathcal{L}(Y) | \pi_{n-e}(\overline{\varphi}) = \pi_{n-e}(\overline{y})\};$$

(b) the restriction of h_n on $\Delta_{e,e',n}$ is a piecewise trivial fibration onto its image with fiber $\mathbb{A}^e_k$.

Proof. The statement (a) is equivalent to the following

(a') For any $\varphi \in \Delta_{e,e'}$, and any $x \in \mathcal{L}(X)$ such that $\pi_n(\varphi) = \pi_n(x)$ there exists $y \in \mathcal{L}(Y)$ such that $h(y) = x$, and $\pi_{n-e}(\varphi) = \pi_{n-e}(y)$.

Indeed, $h : \Delta_{e,e'} \to h(\Delta_{e,e'})$ is a bijection.

1. The case $Y = X = \mathbb{A}^d_k$. Then h can be identified with an element of $k[X_1, \dots, X_d]^d$. In this case $\mathcal{L}_n(Y) = \mathcal{L}_n(X)$ is a $(n+1)d$-dimensional affine space, and

$$\mathcal{L}_n(Y)(\overline{k}) = \mathcal{L}_n(X)(\overline{k}) = (\overline{k}[[t]]/t^{n+1})^d.$$

To prove (a'), it is sufficient to check that for any $v \in \overline{k}[[t]]^d$ there exists $u \in \overline{k}[[t]]^d$ such that

$$h(\varphi + t^{n+1-e}u) = h(\varphi) + t^{n+1}v.$$

This can be rewritten as

$$h(\varphi) + J_h(\varphi) \cdot t^{n+1-e}u + t^{2(n+1-e)}\Lambda(u) = h(\varphi) + t^{n+1}v,$$

where J_h is the Jacobi matrix of h, and $\Lambda \in k[[t]][Z_1, \dots, Z_d]^d$, $\Lambda \equiv 0 \mod \deg 2$. This is equivalent to

$$J_h(\varphi) \cdot t^{-e}u + t^{n+1-2e}\Lambda(u) = v,$$

or

$$t^{-e}J_h(\varphi) \cdot (u + t^{n+1-e}\Lambda(u)) = v.$$

Since $\varphi \in \Delta_{e,e'}$, we have $v(\det J_h(\varphi)) = e$; therefore, $t^e(J_h(\varphi))^{-1}$ is a matrix with elements in $\overline{k}[[t]]$. It remains to notice that the d-tuple of formal power series $(Z_1, \dots, Z_d)+t^{n+1-e}\Lambda(Z_1, \dots, Z_d)$ is invertible with respect to composition. This proves (a').

We have

$$h((\sum_{j=0}^{n} a_{ij}t^j)_{i=1}^d) \equiv (\sum_{j=0}^{n} H_{ij}((a_{\alpha\beta})_{\alpha=1}^{d}{}_{\beta=0}^{j})t^j)_{i=1}^d \mod t^{n+1}$$

for some polynomials $H_{ij} \in k[(X_{\alpha\beta})_{\alpha=1}^{d}{}_{\beta=0}^{j}]$.

We have

$$(H_{ij})_{i=1}^{d}{}_{j=n-e+1}^{n} = H^{(0)} + \big(J \cdot ((X_{\alpha\beta})_{\alpha=1}^{d}{}_{\beta=n-e+1}^{n})\big)_{i=1}^{d}{}_{j=n-e+1}^{n},$$

where $H^{(0)}$ (resp. J) is a $1 \times de$ (resp. $de \times de$) matrix whose elements are polynomials in X_{ij} with $1 \le i \le d$, $0 \le j \le n-e$. The columns and rows are numbered by the elements of $I = \{(i,j) | 1 \le i \le d,\, n-e+1 \le j \le n\}$. Namely,

$$H^{(0)}_{(i,j)} = H_{ij}|_{X_{\alpha\beta}:=0,\, \beta \ge n-e+1},$$

and J is defined from

$$J_h\Bigg(\Big(\sum_{j=0}^{n-e} X_{ij}t^j\Big)_{i=1}^{d}\Bigg)\Big(\sum_{j=n-e+1}^{n} X_{ij}t^j\Big)_{i=1}^{d} \equiv \Bigg(\sum_{j=n-e+1}^{n} \big(J \cdot ((X_{\alpha\beta})_{\alpha=1}^{d}{}_{\beta=n-e+1}^{n})\big)_{(i,j)} t^j\Bigg)_{i=1}^{d} \quad (9)$$

The definition of $\Delta_{e,e'}$ implies that for any $\varphi \in \Delta_{e,e'}$ we have $\operatorname{rk} J_h(\varphi) = e$. Therefore, by Lemma 4.14, the rank of $J((x_{ij}))$ is $de - e$ for all $(x_{ij}) \in \Delta_{e,e',n-e}$.

Let $Z = \{(i_1, j_1), \dots, (i_e, j_e)\}$ be any e-element subset of I. Denote by Δ_Z preimage in $\Delta_{e,e',n}$ of the subset of $\Delta_{e,e',n-e}$ where the submatrix of J after striking out the columns numbered by elements of Z is of rank $de - d$. Clearly, Δ_Z is open in $\Delta_{e,e',n}$. Consider the morphism $(h_Z, \pi) : \Delta_Z \to h(\Delta_Z) \times \mathbb{A}^e$, where h_Z is induced by h, and π is induced by the projection of $\mathbb{A}^{(n+1)d}$ onto the coordinates numbered by elements of Z. By construction of Δ_Z, $(h_Z, \pi)(\overline{k})$ is a bijection.

Since $h(\Delta_Z)$ is a constructible, it can be written as a union of subsets B_i, which are locally closed in $\mathbb{A}^{(n+1)d}$ and smooth. By Lemmas 4.7 in the characteristic 0 case the morphisms $h^{-1}(B_i) \to B_i \times \mathbb{A}^e$ induced by (h_Z, π) are isomorphisms. In the prime characteristic case we need also Conjecture 4.12.

2. The case $Y = \mathbb{A}^d_k$, $X = \operatorname{Spec}(k[X_1, \dots, X_n]/I) \subset \mathbb{A}^k_N$. Let $c = c_X$.

In view of Lemma 4.9, we may assume that $h(\Delta_{e,e'}) \subset \bigcup_{i=1}^m A_i$, where each $A = A_i$ is a semi-algebraic subset of $\mathcal{L}(X)$, weakly stable at level e', with the following property:

$$\mathcal{L}(X) \cap A = \mathcal{L}(\operatorname{Spec}(k[X_1, \dots, X_N]/(f_1, \dots, f_{N-d})))] \cap A,$$

and for some $(N-d) \times (N-d)$ minor δ of $\Delta = \frac{\partial(f_1, \dots, f_{N-d})}{\partial(X_1, \dots, X_N)}$ we have $\operatorname{ord}_t \delta(x) \leq ce'$ for any $x \in A(\overline{k})$.

After renumbering coordinates and taking smaller A, we may assume that:

(i) δ is formed by the first $N - d$ columns;

(ii) $\operatorname{ord}_t \delta(x)$ has the same value e'' for all $x \in A(\overline{k})$;

(iii) for any other $(N-d) \times (N-d)$ minor δ' of Δ, $\operatorname{ord}_t \delta'(x)$ has the value $\geq e''$ for all $x \in A(\overline{k})$.

Take $\varphi \in \Delta_{e,e'}(\overline{k}) \cap h^{-1}(A)$. Denote by J_h the Jacobi matrix of $h : Y \to X \hookrightarrow \mathbb{A}^N_k$. We have $\Delta(h(\varphi))J_h(\varphi) = 0$ because it is the Jacobi matrix of zero map. By the definition of $\Delta_{e,e'}$ and by Lemma 4.13, the minimal value of the function ord_t on the $d \times d$ minors of $J_h(\varphi)$ is e, and this minimum is realized on the minor formed by the last d rows of $J_h(\varphi)$.

Let $p : X \to \mathbb{A}^d_k$ be the projection $(x_1, \dots, x_N) \mapsto (x_{N-d+1}, \dots, x_N)$. Denote by $J_{p\circ h}$ the Jacobi matrix of $p \circ h$. It consists of the last d rows of J_h, whence

$$\operatorname{ord}_t \det J_{p\circ h}(\varphi) = e.$$

With the use of Lemma 4.10, the proof of Proposition proceeds as in the case $X = \mathbb{A}^d_k$. One has only to replace J_h by $J_{p\circ h}$, $\Delta_{e,e'}$ by $\Delta_{e,e'} \cap h^{-1}(A)$, $\Delta_{e,e',n}$ by the image of $\Delta_{e,e'} \cap h^{-1}(A)$ in $\mathcal{L}_n(Y)$.

3. The general case. We may assume that Y and X are affine, and that there exist functions $y_1, \dots, y_d$ on Y such that $(y_1, \dots, y_d) : Y \to \mathbb{A}^d_k$ is an étale map. Then the above argument works, with $y_1, \dots, y_d$ instead of coordinates $a_{10}, \dots, a_{d0}$. We use also that the natural map $\mathcal{L}_n(Y) \to Y \times_{\mathbb{A}^d_k} \mathcal{L}_n(\mathbb{A}^d_k)$ is an isomorphism. □

4.16 Corollary. *Let $e, e', n \in \mathbb{N}$, $n \geq 2e$, $n \geq e + c_X e'$. Then $\Delta_{e,e',n}$ is a union of fibers of h_n.*

Proof. $\{\overline{y} \in \mathcal{L}_n(Y) | \overline{\varphi} \equiv \overline{y} \mod \mathcal{L}_{n-e}(Y)\} \subset \Delta_{e,e',n}$. □

5 Motivic Zeta function

5.1 Summary about the Monodromy Zeta function and the Hodge Spectrum

Below we shall always assume that $X \xrightarrow{f} \mathbb{A}^1$ is a non-constant morphism of a smooth variety to $\mathbb{A}^1$; $X_0 = f^{-1}(0) \subset X$.

Assume $k = \mathbb{C}$ and $x \in X_0$ is an isolated critical point of f. Then there exists a neighborhood U of x in X, such that for every sufficiently small neighborhood Δ of 0 in $\mathbb{A}^1$ the map

$$U \cap f^{-1}(\Delta \setminus 0) \xrightarrow{f} \Delta \setminus 0 = \Delta^*$$

is a $\mathcal{C}^\infty$-locally trivial fibration.

So, $R^* f_*(\mathbb{Q}_{U \cap f^{-1}(\Delta^*)}) = \mathcal{H}^*$ is a locally free sheaf on Δ^* with fibers $\mathcal{H}^*_s = H^*(F_s, \mathbb{Q})$ (F_s is a fiber of $U \cap f^{-1}(\Delta^*) \to \Delta^*$ over s).

(Rem: F_s is homotopically equivalent to a bouquet of $\mu = \mu(f)$ $(d-1)$-Spheres, where $d = \dim X$, $\mu = \dim \left(\mathcal{O}_{X,x} / \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right)\right)$. This is a so called "Milnor-fibration.")

The group $\pi_1(\Delta^*, s)$ ($\cong \mathbb{Z}$) acts on $H^*(F_s, \mathbb{Q})$, and the image of a generator (of a counterclockwise loop around 0 in Δ^*) defines an automorphism $M_x : H^*(F_s, \mathbb{Q}) \to H^*(F_s, \mathbb{Q})$ (*the MonodromyOperator*).

(Rem:
1. By the Mondromy Theorem M_x is quasi-unipotent, namely, $(M_x^d - \mathrm{Id})^\mu = 0$.

2. Since $\tilde{\Delta}^*$ (the universal covering) is contractible, we have $\tilde{U} = [U \cap f^{-1}(\Delta^*)] \times_{\Delta^*} \tilde{\Delta}^* \simeq F_s \times \tilde{\Delta}^*$ as $\mathcal{C}^\infty$-manifold, and from this we get the *geometric monodromy*, a diffeomorphism defined uniquely up to isotypy: $M_x^{\mathrm{geom}} : F_s \to F_s$ with $M_x = H^*(M_x^{\mathrm{geom}})$ und $H^*(F_s) \cong H^*(\tilde{U}, \mathbb{Q})$).

$$Z(T, M_x) = \frac{\prod\limits_{q \text{ even}} \det((\mathrm{Id} - TM_x)/H^q(F_s))}{\prod\limits_{q \text{ odd}} \det((\mathrm{Id} - TM_x)/H^q(F_s))}$$

is called *monodromy zeta function*.

$\Lambda(M_x^n) = \Sigma(-1)^q \mathrm{Tr}(M_x/H^q(F_s))$ is *Lefschetz number* (geometrically: "number" of fixed points of $(M_x^{\mathrm{geom}})^n$).

We have (exercise in linear algebra): $-T\frac{d}{dT} \log Z(T_j M_x) = \sum\limits_{n \geq 1} \Lambda(M_x^n) T^n$.

Hodge spectrum

$H_c^*(F_s, \mathbb{Q})$ possessrs a natural mixed Hodge structure (Steenbrink, Saito), and M_x is compatible with it. If HS^{mon} is a category of HS with quasi-unipotent endomorphisms, we get from here elements

$$\begin{aligned}
\chi_h(F_s) &= \sum_q (-1)^q [H^q(F_x, Q)] \in K_0(HS^{\mathrm{mon}}) \\
\mathrm{hsp} &: K_0(HS^{\mathrm{mon}}) \longrightarrow \mathbb{Z}\Big[(t^{\frac{1}{N}})_{N\in\mathbb{N}}, \frac{1}{t}\Big] \\
\mathrm{hsp}([H]) &= \sum_{\substack{0\le\alpha<1 \\ \alpha\in\mathbb{Q}}} t^\alpha \Big(\sum_{p,q\in\mathbb{Z}} \dim(H_\alpha^{pq}) \Big) t^p \qquad \text{(Hodge spectrum)}
\end{aligned}$$

(H_α^{pq} is the root space with eigenvalue $\exp(2\pi i\alpha)$).

5.2 Relativ and equivariant versions of Grothendieck ring of varieties

Var_S = the category of varieties over S (a base variety)
(Objects: $X \to S = X/S$, X a variety, morphisms are S-morphisms.
Fiber product: $(X \times_S Y)_{\mathrm{red}}$, where $X \times_S Y$ is the fiber product in the categore of schemes.
However, we shall denote it simply $X \times_S Y$.)

Definition $K_0(\mathrm{Var}_S)$ is the following ring (Grothendieck ring).
Elements are $[X/S]$ (isomorphism classes) with *relations*

$$Y \subset X \text{ closed: } [X/S] = [X \setminus Y/S] + [Y/S]$$

$$\textit{Product} \qquad [X/S] \cdot [Y/S] = [X \times_S Y/S]$$

$$M_S = K_0(\mathrm{Var}_S)[\mathbb{L}^{-1}] \quad \text{with } \mathbb{L} = [\mathbb{A}^1 \times S/S]$$

$$\widehat{M}_S = \varprojlim_{n\to\infty} (M_S/F_{-n}M_S)$$

where $F_{-n}(M_S)$ = subgroup generated by $\dfrac{[X/S]}{\mathbb{L}^j}$, $j \ge \dim X - \dim S + n$ (not used in what follows, since we work with "naïve" motivic measure only, with values in M_S).

Equivariant version

G is a finite (or profinite) group.
Var_S^G
is the category of S-varieties with equivariant (and continuous in the profinite case) G-action on X (trivial action on S) such that each orbit has an affine neighborhood.

We denote the isomorphism classes by $[X/S, G]$. $K_0^G(\mathrm{Var}_S^G)$ is defined as above with an additional relation:

$$[V \times S/S, G] = \mathbb{L}^d,$$

where V is a d-dimensional representation of G, $\mathbb{L} = (\mathbb{A}^1 \times S, G)$ with trivial G-action, $M_S^G = K_0^G(\mathrm{Var}_S^G)[\mathbb{L}^{-1}]$, $\widehat{M}_S^G$ as above.

Motivation

The Euler characteristic yields the ring homomorphism

$$\begin{array}{ccc} K_0(\mathrm{Var}_{\mathbb{C}}) & \xrightarrow{e} & K_0(\mathrm{Vect}_{\mathbb{Q}}) \\ \downarrow & & || \\ M_{\mathbb{C}} & \xrightarrow{e} & \mathbb{Z} \end{array}$$

Analog: equivariant Euler characteristic

$$\begin{array}{ccc} K_0^G(\mathrm{Var}_{\mathbb{C}}^G) & \xrightarrow{e^G} & K_0(\mathbb{C}[G]) \\ \downarrow & & || \\ M_{\mathbb{C}}^G & \xrightarrow{e^G} & \text{representation ring of } G \end{array}$$

$$X \longmapsto \sum_q (-1)^q [H^q(X, \mathbb{C})] \qquad \text{in representation ring of } G$$

For a representation V of G (considered as a variety) we have

$$e^G(V) = e^G(\mathbb{C}^d) = [\mathbb{C}], \qquad d = \dim(V) \qquad \text{(triv. representation)}$$

Another example

HS is the category of Hodge structures

$K_0(HS) \xrightarrow{\chi_h} \mathbb{Z}[t, \frac{1}{t}], \quad \chi_h([H]) = \sum_{p,q \in \mathbb{Z}} \dim(H^{pq}) t^p$

$X \to H_c^q(X, \mathbb{Q})$ yields functors

$\mathrm{Var}_{\mathbb{C}} \to MHS$

$$\begin{array}{ccc} K_0(\mathrm{Var}_{\mathbb{C}}) & \longrightarrow & K_0(MHS) \\ \downarrow & & || \\ M_{\mathbb{C}} & \longrightarrow & K_0(HS) \end{array}$$

$\chi_h(X) = \sum (-1)^i [H_c^i(X, \mathbb{Q})]$

Fiber functor

$s \in S$ is a k-rational point.

$X/S \longmapsto X_s = X \times_s \{s\}$ yields the ring homomorphisms

$$\begin{array}{ccc} K^G(\mathrm{Var}_S^G) & \longrightarrow & K^G(\mathrm{Var}_k^G) \\ \downarrow & & \downarrow \\ M_S^G & \longrightarrow & M_k^G \\ \downarrow & & \downarrow \\ \widehat{M}_S^G & \longrightarrow & \widehat{M}_k^G \end{array}$$

5.1 Remark. $K^G(\mathrm{Var}_S^G)$ (resp. M_S^G, resp. $\widehat{M}_S^G$) is an augmented $K(\mathrm{Var}_S)$- (resp. M_S-, resp. $\widehat{M}_S$-) algebra.

$$[X/S] \longrightarrow [X/S, G] \qquad \text{with triv. } G\text{-action}$$

Augmentation (at least if G is abelian): $[X/S, G] \mapsto [\bar{X}/S]$ with $\bar{X} = G \setminus X$.

5.3 Motivic zeta function

$$\begin{aligned}&\mathcal{X}_n \subset \mathcal{L}_n(X) = \underline{\mathrm{Hom}}(\mathrm{Spec}\ k||t||/t^{n+1}, X)\\ &\ \ ||\\ &\{\gamma \in \mathcal{L}_n(X) \mid f(\gamma) = ct^n,\ c \neq 0\}\\ &\mathcal{X}_{n,1} = \{\gamma \in \mathcal{X}_n \mid f(\gamma) = t^n\}\end{aligned}$$

For $n \geq 1$ the sets $\mathcal{X}_n$, $\mathcal{X}_{n,1}$ are naturally X_0-varieties via

$$\mathcal{X}_{n,1} \subset \mathcal{X}_n \longrightarrow X_0, \quad \gamma \longmapsto \gamma(0).$$

Let $G = \varprojlim_{n \in \mathbb{Z}} \mu_n = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu)$ ($\mu =$ the group of all roots of unity).
Then $\mathcal{X}_{n,1}$ are G-varieties via $g(\gamma) = \gamma(g(\frac{1}{n})t)$ ($g(\frac{1}{n}) \in \mu_n$!)

5.2 Definition. $Z^{\mathrm{mot}}(T) = \sum_{n \geq 1} [\mathcal{X}_{n,1}/X_0, G]\mathbb{L}^{-nd}T^n \in M_{X_0}^G[[T]]$

$$Z^{\mathrm{naive}}(T) = \sum_{n \geq 1} [\mathcal{X}_n/X_0]\mathbb{L}^{-nd}T^n \in M_{X_0}[[T]]$$

5.3 Proposition. *Let $Y \xrightarrow{\sigma} X$ be proper, $Y \setminus \sigma^{-1}(X_0) \simeq X \setminus X_0$, Y smooth, $\sigma^{-1}(X_0)$ a divisor with strict normal crossings with components $(E_j)_{j \in J}$. Let $N_j = \mathrm{ord}_{E_j}(f \circ \sigma)$, and let $\omega_Y = \sigma^*\omega_X(\sum_j(\nu_j - 1)E_j)$, $(\nu_j \geq 1)$. For $\emptyset \neq I \subset J$ let*

$$\overset{\circ}{E}_I = \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j.$$

Then

$$Z^{\mathrm{naive}}(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L}-1)^{|I|}[\overset{\circ}{E}_I/X_0]P_I \qquad \textit{with}\ \ P_I = \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i}T^{N_i}}{1 - \mathbb{L}^{-\nu_i}T^{N_i}}$$

and

$$Z^{\mathrm{mot}}(T) = \sum_{\emptyset \neq I \subseteq J} (\mathbb{L}-1)^{|I|-1}[\widetilde{\overset{\circ}{E}}_I/X_0, G]P_I$$

(*where $\widetilde{\overset{\circ}{E}}_I \to \overset{\circ}{E}_I$ are certain cyclic coverings described below*).

For the proof we need only "naïve" motivic measure $\tilde{\mu}$ and integral, resp. a relative and equivariant version of it.

Erinnerung: $\tilde{\mu}(A)$ is described for subsets $A \subset \mathcal{L}_\infty(X)$ with the properties of ("*stable sets*")

- $\pi_n^{-1}(\pi_n(A)) = A$ for large n ($n \geq n_0$).
- $\pi_n(A)$ is constructible for $n \geq n_0$.
- $\pi_{n+1}(A) \to \pi_n(A)$ is peicewise trivial (i. e., $\pi_n(A) = \coprod C_j$, C_j constructible, $\pi_{n+1}(A)/C_j = C_j \times \mathbb{A}^d$, $d = \dim X$). (The last condition is always satisfied for smooth X.)

Then $\tilde{\mu}(A) \underset{\text{def}}{=} \dfrac{[\pi_n(A)/S, G]}{\mathbb{L}^{(n+1)d}} \in M_S^G$ (independent of n for large n).

5.4 Remark. (general remark) If G acts on $\mathbb{D} = \operatorname{Spec} k[[t]]$ so that $\mathbb{D}_n = \operatorname{Spec}(k[[t]]/t^{n+1})$ are invariant and the induced action on $\mathbb{D}_n$ is continuous (i. e., factors through a finite quotient), then $\mathcal{L}_n(X)$ resp. $\mathcal{L}_\infty(X)$ are G-varieties via $g(\gamma)(t) = g\gamma(g^{-1}t)$. The equivariant measure is defined with respect to *this* G-action.

For computation of $[\mathcal{X}_{n,1}/X_0, G]$ we need G-action on quotients $\mu \to \mu_n$, which is trivial on X, Y and is determined by $f(t) \mapsto f(\zeta t)$, $\zeta \in \mu_n$, on D. Thus,

$$|\mathcal{X}_{n,1}/X_0, G] = \mathbb{L}^{(n+1)d}\tilde{\mu}(\pi_n^{-1}(\mathcal{X}_{n,1})),$$

and

$$\pi_n^{-1}(\mathcal{X}_{n,1}) = \{\gamma \in \mathcal{L}_\infty(X) \mid f(\gamma) \equiv t^n \bmod t^{n+1}\}.$$

For $\tilde{\mu}(\pi_n^{-1}(\mathcal{X}_{n,1}))$ we need (equivariant, relative version of) *transformation rule*.

$Y \xrightarrow{\sigma} X$ is a proper, birational (S-morphism)

$\mathcal{J}_\sigma = F_0(\Omega^1_{Y/X})$ is the 0th Fitting ideal ("Jacobi ideal")

$A \subset \mathcal{L}_\infty(X)$ stable, then

$$\tilde{\mu}_X(A) = \int\limits_{\sigma^{-1}(A)} \mathbb{L}^{-\operatorname{ord}_{\mathcal{J}_\sigma}} d\tilde{\mu}_Y \ (= \sum_\ell \mathbb{L}^{-\ell}\tilde{\mu}_Y(\{\gamma \mid \sigma(\gamma) \in A,\ \operatorname{ord}_{\mathcal{J}_\sigma}(\gamma) = \ell\}))$$

Erinnerung: $\operatorname{ord}_{\mathcal{J}}(\gamma) = \min\{\operatorname{ord}_t(f \circ \gamma) \mid f \in \mathcal{J}_{\gamma(0)} \subset \mathcal{O}_{Y,\gamma(0)}\}$.

In our case $\mathcal{J}_\sigma = \mathcal{O}(-\sum\limits_{i\in J}(\nu_i - 1)E_i)$.

Step 1 Decomposition of $\pi_n^{-1}(\mathcal{X}_{n,1})$.

For $m \in \mathbb{N}^J$ with $\sum\limits_{i \in J} m_i N_i = n > 0$ let

$$\mathcal{L}(Y)(m) = \{\gamma \in \mathcal{L}_\infty(Y) \mid \operatorname{ord}_{E_i}(\gamma) = m_i\},$$
$$\mathcal{L}(Y)(m)_1 = \{\gamma \in \mathcal{L}(Y)(m) \mid \frac{f \circ \sigma(\gamma)}{t^n}|_0 = 1\}.$$

If $(\mathbb{N}^J)_n = \{m | \sum m_j N_j = n\}$, then $\pi_n^{-1}(\mathcal{X}_{n,1}) = \coprod\limits_{m \in (\mathbb{N}^J)_n} \mathcal{L}(Y)(m)_1$.

On $\mathcal{L}(Y)(m)$ we have $\operatorname{ord}_{J_\sigma}(\gamma) = \sum m_i(\nu_i - 1)$.

Step 2 Fibration of $\mathcal{L}(Y)(m)_1$.

Let $U_i \to E_i$ be the principal bundle that belongs to the normal bundle of E_i (Normal bundle $\setminus$ zero section). For each $i \in \operatorname{supp}(m) = I$ there is a natural map

$$\begin{array}{ccc} \mathcal{L}(Y)(m)_1 & \xrightarrow[e_i]{} & U_i \\ \downarrow & & \downarrow \\ X_0 & \longleftarrow & E_i \end{array}$$

If g_i is a local equation of E_i, then the normal bundle is generated by the local section $[\frac{1}{g_i}]$, and $e_i(\gamma) = \frac{g_i(\gamma)}{t^{m_i}}|_{t=0}[\frac{1}{g_i}](\gamma(0))$

If U_I is the fiber product of $U_i/\overset{\circ}{E}_I$, then one gets a natural map

$$\begin{array}{ccc} \mathcal{L}(Y)(m)_1 & \underset{e_I}{\longrightarrow} & (U_I)_1 \subset U_I \\ \downarrow & & \downarrow \\ X_0 & \longleftarrow & \overset{\circ}{E}_I \end{array}$$

$(U_I)_1$ is defined as follows. On $Y \setminus \bigcup_{j \notin I} E_j$ we have

$$\mathrm{div}(f \circ \sigma) = \sum_{i \in I} N_i E_i; \text{ thus, } \bigotimes_{i \in I} U_i^{\otimes N_i} \simeq \overset{\circ}{E}_I \times \mathbb{G}_m$$

(canonically) on $\overset{\circ}{E}_I$. Then $(U_I)_1$ is the set of all I-tuples (s_i) with $\bigotimes_{i \in I} s_i^{\otimes N_i} = 1$ by this isomorphism.

$U_I \to \overset{\circ}{E}_I$ is a $\mathbb{G}_m^I$-principal bundle. Let $m_I = \mathrm{ggT}(N_i, i \in I)$, and $N_i = m_I N_i'$ $(i \in I)$. The subgroups

$$\begin{aligned} H &= \Big\{(\lambda_i) \in \mathbb{G}_m^I, \prod_{i \in I} \lambda_i^{N_i} = 1\Big\} \\ &\bigcup \\ H' &= \Big\{(\lambda_i) \in H, \prod_{i \in I} \lambda_i^{N_i'} = 1\Big\} \end{aligned}$$

act on $(U_I)_1$; in view of $\gcd(N_i', i \in I) = 1$ we have $H' \simeq \mathbb{G}_m^{|I|-1}$ and $H \simeq \mu_{m_I} \times H'$.

Definition of $\widetilde{\overset{\circ}{E}}_I \to \overset{\circ}{E}_I$:

$$\widetilde{\overset{\circ}{E}}_I = (U_I)_1/H' \longrightarrow \overset{\circ}{E}_I = (U_I)_1/H.$$

This is an unramified μ_{m_I}-covering.

Then $(U_I)_1 \to \widetilde{\overset{\circ}{E}}_I$ is a $\mathbb{G}_m^{|I|-1}$-principal bundle; thus,

$$[U(m)_1/X_0, G] = [\mathbb{G}_m \times X_0/X_0, G]^{|I|-1} G][\widetilde{\overset{\circ}{E}}_I/X_0, G] = (\mathbb{L} - 1)^{|I|-1}[\widetilde{\overset{\circ}{E}}_I/X_0, G]$$

Step 3 The fibration $\mathcal{L}(Y)(m)_1 \longrightarrow (U_I)_1$

If $|I| = k$, there exists (locally) an (étale) coordinate system $x_1, \ldots, x_d$ on Y with $E_i = \mathrm{div}(x_i)$, $i = 1, \ldots, k$,

$$\overset{\circ}{E}_I = [x_1 = \ldots = x_k = 0].$$

If $(y, c_1, \ldots, c_k) \in (U_I)_1$ ($\Leftrightarrow c_1^{N_1} \ldots c_k^{N_k} u(y) = 1$, if $f \circ \sigma = u x_1^{N_1} \ldots x_k^{N_k}$), then the fiber

$$F = \{\gamma = (c, t^{m_1} + t^{m_1+1}\psi_1, \ldots, c_k t^{m_k} + t^{m_{k+1}}\psi_k, x_{k+1}(y) + t\psi_{k+1}, \ldots), \psi_j \in k[[T]]\}.$$

For $\ell > \max(m_i, i \in I)$ we have therefore

$$F = \pi_\ell^{-1}(\pi_\ell(F)) \qquad \text{und} \qquad [\pi_\ell(F)] = \mathbb{L}^{\ell d - \sum m_i}.$$

Thus (by Step 2),

$$\tilde{\mu}(\mathcal{L}_\infty(Y)(m)_1) = \frac{(\mathbb{L}-1)^{|I|-1}}{\mathbb{L}^{d+\Sigma m_i}} [\widetilde{\overset{\circ}{E}}_I / X_q G],$$

and

$$\begin{aligned}
[\mathcal{X}_{n,1}/X_0, G] &= \mathbb{L}^{nd} \sum_{m \in (\mathbb{N}^J)_n} \frac{(\mathbb{L}-1)^{|\mathrm{supp}(m)|-1}}{\mathbb{L}^{\Sigma m_i \nu_i}} [\widetilde{\overset{\circ}{E}}_{\mathrm{supp}(m)}/X_0, G] \\
&= \mathbb{L}^{nd} \sum_{\emptyset \neq I \subset J} \sum_{\substack{m \in (\mathbb{N}^J)_n \\ \mathrm{supp}(m) = I}} \frac{(\mathbb{L}-1)^{|I|-1}}{\mathbb{L}^{\Sigma m_i \nu_i}} [\widetilde{\overset{\circ}{E}}_I / X_0, G] \\
Z^{\mathrm{mot}}(T) &= \sum_{\emptyset \neq I \subseteq J} (\mathbb{L}-1)^{|I|-1} [\widetilde{\overset{\circ}{E}}_I / X_0, G] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}
\end{aligned}$$

For $Z^{\mathrm{naive}}(T)$ the proof is analogous, with fiberings $\mathcal{L}(Y)(m) \to U_I \to \overset{\circ}{E}_I$. The former has the same fibers as above, and $[U_i/X_0] = (\mathbb{L}-1)^{|I|}[\overset{\circ}{E}_I/X_0]$. $\square$

The motivic zeta function yields the monodromy zeta function as a special case.
First, the fiber functor

$$\mathrm{Var}_S^G \longrightarrow \mathrm{Var}_k^G, \qquad X \mapsto X_s$$

is a applied to a k-rational point $s \in S$, $X_S = X \times_S \{s\}$. Dieser induziert Ringhomomorphismen

$$\begin{aligned}
&K_0^G(\mathrm{Var}_S) \longrightarrow K_0^G(\mathrm{Var}_k) \\
&M_S^G \longrightarrow M_k^G \\
&\widehat{M}_S^G \longrightarrow \widehat{M}_k^G.
\end{aligned}$$

Further, we have a ring homomorphism

$$K_0(\mathbb{C}[G]) \longrightarrow \mathbb{Z} = K_0(\mathbb{C})$$

induced by dimension. This homomorphism maps the equivariant Euler characteristic $\widehat{M}_{\mathbb{C}}^G \to K_0(\mathbb{C}[G])$ into the Euler characteristic. Both are applied to monodromy zeta function; this gives at a point $x \in f^{-1}(0) = X_0$

$$\sum_{n \geq 1} e(\mathcal{X}_{n,1,x}) T^n = \sum_{i \in J} N_i e(\overset{\circ}{E}_i \cap \sigma^{-1}(0)) p_i, \quad p_i = \frac{T^{N_i}}{1 - T^{N_i}}.$$

Thus,

$$e(\mathcal{X}_{n,1,x}) = \sum_{N_i/n} N_i e(\overset{\circ}{E}_i \cap \sigma^{-1}(0)).$$

According to a result of A'Campo [1], the right hand side is equal to the Lefschetz number $\Lambda(M_x^n)$ of the monodromy. Thus, by specialization the motivic zeta function yields the logarithmic derivation of the monodromy zeta function.

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