# A Projector Based Representation of the Strangeness Index Concept

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**Abstract**

The strangeness index concept is generalized and represented by a matrix chain similar to the structure of the tractability index. The properties of the related projectors are proven. A decoupling of the DAE and a representation of a solution is given.

Keywords: Strangeness Index, matrix chain, projector.

## 1 Introduction

The strangeness index introduced by Kunkel and Mehrmann (see [KM06]) was defined in a constructive way.
Here we will use a more general matrix chain based concept, which contains the index definition given by Kunkel and Mehrmann as a special case. We will restrict ourselves to the square case, i.e., we will consider DAEs with as many equations as variables in the system.
After a motivation, which shows the first steps of the strangeness index algorithm from a different view, we form a matrix chain using projectors onto the related nullspace or image spaces of the involved matrices. The properties of these projectors are summarized and a definition of a generalized strangeness index is given, which is independent of the chosen projectors. The introduced projectors allow us also a decoupling of a DAE and a representation of its solution. At the end of the paper we will use the classical strangeness concept for DAEs with properly stated leading term (see [Meh03]) to find out which projectors are used.

## 2 Motivation

We consider a linear DAE with properly stated leading term

$$A(Dx)' + Bx = q \tag{2.1}$$

with $A(t) \in \mathbb{R}^{m\times n}$, $D(t) \in \mathbb{R}^{n\times m}$, $B(t) \in \mathbb{R}^{m\times m}$ and $t \in I$ (interval of interest). Properly stated leading term means (see also [Mär02]) that $\ker A \oplus \operatorname{im} D = \mathbb{R}^n$ and the projector $R$, which realizes this splitting, belongs to $C^1$. We choose $Q_0$ as a projector onto $\ker D$, and because of the properly stated leading term it holds that $\ker AD = \ker D$. If we introduce the complementary projector $P_0 := I - Q_0$, we can determine a generalized reflexive inverse $D^-$ with $D^-DD^- = D^-$, $DD^-D = D$, $DD^- = R$ and $D^-D = P_0$. Because of $D = DP_0$ only the $P_0x$ part of $x$ influences the derivative $Dx$. The idea is to extract at least a part of $P_0x$ from the algebraic equations and to use its derivative to reduce the dimension of the derived part $Dx$ of the unknown function. From (2.1) we derive

$$A(DP_0x)' + B(P_0 + Q_0)x = q \tag{2.2}$$

and by reordering we obtain

$$\underbrace{(AD + BQ_0)}_{=:\hat{G}_1}(D^-(DP_0x)' + Q_0x) + BP_0x = q. \tag{2.3}$$

Let $\bar{G}_0 := AD$ and $\hat{G}_1 := \bar{G}_0 + BQ_0$. We can extract the interesting part multiplying (2.3) by a projector along $\operatorname{im} \hat{G}_1$. According to the tractability index world we call that projector $\hat{W}_1$. We obtain

$$\hat{W}_1BP_0x = \hat{W}_1q. \tag{2.4}$$

Let $Z_0$ be a projector onto the nullspace of $\hat{W}_1BP_0$. We represent $Z_0$ by $Z_0 = I - (\hat{W}_1BP_0)^-\hat{W}_1BP_0$ with a reflexive generalized inverse $(\hat{W}_1BP_0)^-$. If we multiply (2.4) by $(\hat{W}_1BP_0)^-$, we obtain

$$(I - Z_0)P_0x = (\hat{W}_1BP_0)^-\hat{W}_1q,$$

which represents that part of $P_0x$ we are looking for. Under the assumption that $\operatorname{rank} \hat{W}_1BP_0 = \mathit{const} =: s_0$ and $D(\hat{W}_1BP_0)^-\hat{W}_1q \in C^1$ we convert (2.2) into

$$A(DZ_0x)' + Bx = q - A(D(I - Z_0)x)' =: \bar{q}. \tag{2.5}$$

The DAE (2.5) does not have a properly stated leading term, but using the image projector $R_{Z_0} := DZ_0(DZ_0)^-$ we form, under the assumption that $R_{Z_0} \in C^1$,

$$A(DZ_0x)' = A(R_{Z_0}DZ_0x)' = AR_{Z_0}(DZ_0x)' + AR'_{Z_0}DZ_0,$$

and using this relation we obtain a new DAE with properly stated leading term

$$AR_{Z_0}(DZ_0x)' + (A(R_{Z_0})'DZ_0 + B)x = \bar{q}. \tag{2.6}$$

Now we could apply the same procedure to (2.6).

# 3 A Matrix Chain

Let us consider a regular DAE defined by the three matrices $A_0, D_0$ and $\bar{B}_0$. We calculate $\bar{G}_0 := A_0D_0$ and let $\bar{Q}_0$ be a projector onto $\ker \bar{G}_0$. We define the following matrix chain

$$\hat{G}_{i+1} := \bar{G}_i + \bar{B}_i\bar{Q}_i \quad \text{with}$$

$$\begin{aligned} &\text{a projector } \bar{Q}_i \text{ onto } \ker \bar{G}_i,\\ &\text{a projector } \hat{W}_{i+1} \text{ along } \operatorname{im} \hat{G}_{i+1} \text{ and}\\ &\text{a projector } Z_i \text{ onto the nullspace of } \ker \hat{W}_{i+1}\bar{B}_i. \end{aligned} \tag{3.1}$$

and assume that $\bar{r}_i := \operatorname{rank} \bar{G}_i$ and $s_i := \operatorname{rank} \hat{W}_{i+1}\bar{B}_i$ are constant $\forall t \in I$. We define

$$D_{i+1} = D_iZ_i,\ A_{i+1} := A_iR_{Z_i} \text{ with a projector } R_{Z_i} \in C^1 \text{ onto } \operatorname{im} D_{i+1}$$

and

$$\bar{G}_{i+1} := A_{i+1}D_{i+1} = \bar{G}_iZ_i \text{ and } \bar{B}_{i+1} := A_iR'_{Z_i}D_{i+1} + \bar{B}_i. \tag{3.2}$$

In every chain step, projectors $\bar{Q}_i$, $\hat{W}_{i+1}$ and $Z_i$ are defined. What are their properties and relations ?

**Lemma 1** *The projector $\bar{P}_i$ $(= I - \bar{Q}_i)$ has the structure $\bar{P}_i := P_0Z_0 \dots Z_{i-1}$, $i \geq 1$, $(\bar{P}_0 := P_0)$ built by the projectors $P_0$ and $Z_0, \dots, Z_i$ defined by (3.1). It holds*

(a) $\hat{W}_{i+1}\bar{B}_i = \hat{W}_{i+1}\bar{B}_i\bar{P}_i$,

(b) $\bar{P}_i\bar{P}_j = \bar{P}_{\max(i,j)}$, *and for*

(c) $X_0 := Q_0, X_{j+1} := \bar{P}_j(I - Z_j),\ 0 \le j \le i-1$
*we obtain that $X_j$ are again projectors, with*

(d) $\sum\limits_{k=0}^{i} X_k = I - \bar{P}_i$,

(e) $X_k X_j = 0,\ k \neq j$ *and*

(f) $X_k \bar{P}_i = \bar{P}_i X_k = 0$ *for* $0 \le k \le i$.

Proof: Let $\hat{W}_{i+1}$ be a projector along $\operatorname{im} \hat{G}_{i+1}$. From $\hat{G}_{i+1} := \bar{G}_i + \bar{B}_i \bar{Q}_i$ we have the relation

$$\hat{W}_{i+1} \bar{B}_i \bar{Q}_i = 0, \tag{3.3}$$

i.e., (a) is valid.
$Z_i$ projects onto $\ker \hat{W}_{i+1} \bar{B}_i$, i.e., $\hat{W}_{i+1} \bar{B}_i Z_i = 0$, and $Z_i$ can be represented by $Z_i = I - (\hat{W}_{i+1} \bar{B}_i)^{-} \hat{W}_{i+1} \bar{B}_i$ with an arbitrary generalized reflexive inverse $(\hat{W}_{i+1} \bar{B}_i)^{-}$.
From (3.3) it follows that

$$Z_i \bar{Q}_i = \bar{Q}_i. \tag{3.4}$$

Thus, with $Z_i$ also $\bar{P}_{i+1}$ is a projector because of $(\bar{P}_{i+1})^2 = \bar{P}_i Z_i \bar{P}_i Z_i = \bar{P}_i Z_i Z_i = \bar{P}_{i+1}$.
For a fixed $i$ we consider $\bar{P}_i$ and define

$$X_0 := Q_0,\ X_{j+1} := P_0 Z_0 \dots Z_{j-1}(I - Z_j) = \bar{P}_j(I - Z_j),\ j = 0, \dots, i-1.$$

(d) holds by construction.
From (3.4) we have the relation

$$Z_i(I - \bar{P}_i) = I - \bar{P}_i. \tag{3.5}$$

For $i = 0$ (3.4) means $Z_0 Q_0 = Q_0$ or $(I - Z_0) Q_0 = 0$. Therefore, $X_1 X_0 = 0$ ($X_0 X_1 = 0$ holds trivially) and

$$X_1^2 = P_0(I - Z_0) P_0(I - Z_0) = P_0(I - Z_0)(I - Z_0) = X_1$$

is a projector, too.
For $i = j$ let $X_0, \dots, X_j$ be projectors with $X_k X_l = 0$,
$k, l = 0, \dots, j,\ k \neq l$. From (3.4) the relation $Z_j \sum\limits_{k=0}^{j} X_k = \sum\limits_{k=0}^{j} X_k$ holds and it follows that

$$(I - Z_j) X_l = 0,\ l = 0, \dots, j. \tag{3.6}$$

Because of (d) also $X_l \bar{P}_j = \bar{P}_j X_l = 0$ is valid for $l = 0, \ldots, j$.
For $i = j+1$ we get $X_{j+1} = \bar{P}_j(I - Z_j)$ and with (3.6) we obtain (e)

$$\begin{aligned} X_{j+1}X_l = \bar{P}_j(I - Z_j)X_l &= 0, \\ X_l X_{j+1} = \underbrace{X_l \bar{P}_j}_{=0}(I - Z_j) &= 0, \quad l = 0, \ldots, j. \end{aligned}$$

To show (b) we consider the product of $\bar{P}_l$ and $\bar{P}_r$. It holds for $r > l$

$$\begin{aligned} \bar{P}_r \bar{P}_l = \bar{P}_l Z_l \cdots Z_{r-1} \bar{P}_l &= \bar{P}_l Z_l \cdots Z_{r-1}(I - \sum_{k=0}^{l} X_k) \\ &= \bar{P}_l(Z_l \cdots Z_{r-1} - \sum_{k=0}^{l} X_k)) = \bar{P}_r \end{aligned}$$

and for $r < l$

$$\bar{P}_r \bar{P}_l = \bar{P}_r \bar{P}_r Z_r \cdots Z_{l-1} = \bar{P}_l.$$

To show (c) that $X_{j+1}$ itself is a projector we consider

$$X_{j+1}^2 = \bar{P}_j(I - Z_j)\bar{P}_j(I - Z_j) = X_{j+1}$$

and, additionally with $X_{j+1}\bar{P}_{j+1} = \bar{P}_j(I - Z_j)\bar{P}_j Z_j = 0$ and $\bar{P}_{j+1}X_{j+1} = \bar{P}_{j+1}\bar{P}_j(I - Z_j) = \bar{P}_{j+1}(I - Z_j) = \bar{P}_j Z_j(I - Z_j) = 0$, (f) of Lemma 1 holds. □

**Lemma 2** *For the projectors $\hat{W}_{i+1}$ along* $\operatorname{im} \hat{G}_{i+1}$, *$Z_i$ onto* $\ker \hat{W}_{i+1}\bar{B}_i$ *and for $X_k,\ k = 0, \ldots, i$ it holds for $l = 0, \ldots, i$, that*

(a) $\hat{W}_{i+1}\bar{B}_k X_l = 0,\ 0 \le l - 1 \le k \le i,$

(b) $\hat{W}_{i+1}\bar{B}_l \bar{Q}_l = 0,\ l \leqslant i$

(c) $\hat{W}_{i+1}\bar{B}_l(I - Z_l) = 0,\ 0 \le l < i$

Proof: From the relations

$$\hat{W}_{i+1}\bar{B}_i = \hat{W}_{i+1}\bar{B}_i\bar{P}_i \text{ and } \bar{P}_i X_l = 0,\ l \le i$$

(cf. Lemma 1 (1), (6)) it follows that

$$\hat{W}_{i+1}\bar{B}_i X_l = 0,\ l = 0, \ldots, i.$$

With the structure of

$$\bar{B}_i = \bar{B}_{i-1} + A_{i-1}R'_{Z_{i-1}}D_i, \text{ and } D_i = D_i\bar{P}_i,$$

and using Lemma 1 (6) we obtain

$$\bar{B}_i X_l = \bar{B}_k X_l \text{ with } 0 \le l-1 \le k \le i, \text{ i.e.}$$

$$\hat{W}_{i+1}\bar{B}_i X_l = \hat{W}_{i+1}\bar{B}_k X_l = 0. \tag{3.7}$$

By summation over $l$ we obtain from (3.7)

$$\hat{W}_{i+1}\bar{B}_k \sum_{l=0}^{k} X_l = \hat{W}_{i+1}\bar{B}_k \bar{Q}_k = 0, \; k \leqslant i.$$

It holds now that

$$\begin{aligned} 0 = \hat{W}_{i+1}\bar{B}_{l-1} X_l &= \hat{W}_{i+1}\bar{B}_{l-1}\bar{P}_{l-1}(I - Z_{l-1}) && (3.8)\\ &= \hat{W}_{i+1}\bar{B}_{l-1}(I - Z_{l-1}), \; l = 1, \dots, i. && (3.9) \end{aligned}$$

□

**Corollary 3** *For two projectors $Z_i$ and $\tilde{Z}_i$ onto* $\ker \hat{W}_{i+1}\bar{B}_i$ *it holds that*

$$\operatorname{im} \bar{B}_{i-1} Z_{i-1} \tilde{Z}_{i-1}(I - Z_{i-1}) \subseteq \operatorname{im} \hat{G}_{i+1}.$$

Proof: From Lemma 2(c) we obtain $\hat{W}_{i+1}\bar{B}_{i-1}Z_{i-1} = \hat{W}_{i+1}\bar{B}_{i-1}$, therefore,

$$\begin{aligned} \hat{W}_{i+1}\bar{B}_{i-1}Z_{i-1}\tilde{Z}_{i-1}(I - Z_{i-1}) &= \hat{W}_{i+1}\bar{B}_{i-1}\tilde{Z}_{i-1}(I - Z_{i-1}) \\ &= \hat{W}_{i+1}\bar{B}_{i-1}(I - Z_{i-1}) = 0, \end{aligned}$$

which means $\operatorname{im} \bar{B}_{i-1} Z_{i-1} \tilde{Z}_{i-1}(I - Z_{i-1}) \subseteq \operatorname{im} \hat{G}_{i+1}$. □

**Lemma 4** *The nonsingularity of $\hat{G}_{i+1}$ makes the chain stationary.*

Proof: If $\hat{G}_{i+1}$ is nonsingular, $\hat{W}_{i+1}$ becomes zero and $Z_i = I$. Therefore, $\bar{G}_{i+1} = \bar{G}_i$, and $D_{i+1} = D_i = D_i\bar{P}_i$ leads to $\hat{G}_{i+2} = \bar{G}_{i+1} + B_{i+1}\bar{Q}_{i+1} = \bar{G}_i + (A_i R'_{Z_i} \underbrace{D_{i+1}}_{=D_i\bar{P}_i} + B_i)\bar{Q}_i = \hat{G}_{i+1}$.

□

**Remark 3.1** *For $R_{Z_i}$ we can use the representation $R_{Z_i} = D\bar{P}_{i-1}Z_i(D\bar{P}_{i-1}Z_i)^-$. Using Lemma 1, a special generalized inverse is given by $(D\bar{P}_{i-1}Z_i)^- = \bar{P}_i Z_i D^-$ and a suitable projector by $R_{Z_i} = DZ_0 \dots Z_i D^-$.*

To characterize the different parts of the splitting at each level $i$ we introduce the dimensions of the dynamical part $\bar{r}_i$, the algebraic part $a_i$, and the part we have to differentiate, i.e. $s_i$. It is valid that

$$\bar{r}_i + a_i + s_i = m, \quad \forall i.$$

By construction $\bar{r}_{i+1} = \bar{r}_i - s_i$ and, hence, for reasons of dimension, $s_i$ has to reach $s_i = 0$ for a finite $i$. The relation between the three quantities shows that $\bar{r}_i$ itself describes $s_i$ and $a_i$. We may identify

$$\begin{aligned} \bar{r}_i &:= \operatorname{rank} \bar{G}_i = \operatorname{rank} P_0 Z_0 \dots Z_{i-1}, \\ s_i &:= \operatorname{rank} \hat{W}_{i+1} \bar{B}_i = \operatorname{rank} P_0 Z_0 \dots Z_{i-1}(I - Z_i) = \operatorname{rank} X_{i+1}. \end{aligned}$$

**Definition 3.2** *Let the chain be realizable up to $\mu$, $\hat{G}_i$ for $i = 1, \dots, \mu - 1$ be singular and let $\hat{G}_\mu$ become nonsingular. The numbers*

$$\bar{r}_0 > \bar{r}_1 > \dots > \bar{r}_{\mu-1}$$

*are constant for $t \in I$, then we call the DAE a regular DAE with strangeness index $\mu - 1$.*

To illustrate Definition 3.2 we give two examples.

**Example 3.3** *For* $\begin{pmatrix} 1 \\ 0 \end{pmatrix} (\begin{pmatrix} 0 & 1 \end{pmatrix} x)' + x = q$ *we have*

$$\bar{G}_0 = AD = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \bar{Q}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \bar{B}_0 = I$$

$$\hat{G}_1 = \bar{G}_0 + \bar{B}_0 \bar{Q}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ \hat{W}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\hat{W}_1 \bar{B}_0 = \hat{W}_1$, *which means that* $Z_0 = I - \hat{W}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$\bar{G}_1 = \bar{G}_0 Z_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\bar{Q}_0 = I$, $\bar{B}_1 = \bar{B}_0$, *and with* $\hat{G}_2 = I$ *we obtain that this DAE has strangeness index 1 as expected.*

**Example 3.4** *The second example is not a regular DAE with strangeness index.*

*For* $\begin{pmatrix} 1 \\ 1 \end{pmatrix} (\begin{pmatrix} 1 & 0 \end{pmatrix} x)' + \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = q$ *we have*

$\bar{G}_0 = AD = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\bar{Q}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{B}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

$\hat{G}_1 = \bar{G}_0 + \bar{B}_0\bar{Q}_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\hat{W}_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

$\hat{W}_1\bar{B}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *which means that* $Z_0 = I \Rightarrow \bar{G}_1 = \bar{G}_0 Z_0 = \bar{G}_0$, *and for* $\hat{G}_2 = \hat{G}_1$, *i.e. that the chain ends but* $\hat{G}_2$ *does not become nonsingular. This DAE does not have regular strangeness index.*

As we saw in the definition and, in a more illustrative way in the examples, the determination of the strangeness index of a DAE requires the computation of different projectors. The choice of these projectors is not unique. Therefore it is important to check whether the index depends on the choice of the projectors at the different levels or not.

Before we prove the independence of the choice of the projectors we repeat some properties of projectors. Let $Z$ and $\tilde{Z}$ be two projectors onto the same subspace and $W$ and $\tilde{W}$ two projectors along the same subspace. Then the following relations are valid:

$$\begin{aligned} Z\tilde{Z} &= \tilde{Z}, \quad Z = \tilde{Z}Z, \\ \tilde{Z} &= Z\underbrace{(I + Z\tilde{Z}(I - Z))}_{nonsingular}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} W\tilde{W} &= W, \quad \tilde{W}W = \tilde{W}, \\ \tilde{W} &= \underbrace{(I + (I - W)\tilde{W}W)}_{nonsingular}W. \end{aligned} \tag{3.11}$$

The first step of the matrix chain contains the choice of the nullspace projector $\bar{Q}_0$. Let us assume that we choose two projectors $\bar{Q}_0$ and $\tilde{\bar{Q}}_0$, then

$$\tilde{\hat{G}}_1 = \bar{G}_0 + \bar{B}_0\tilde{\bar{Q}}_0 = (\bar{G}_0 + \bar{B}_0\bar{Q}_0)(I + \bar{Q}_0\tilde{\bar{Q}}_0(I - \bar{Q}_0)) = \hat{G}_1\underbrace{(I + \bar{Q}_0\tilde{\bar{Q}}_0(I - \bar{Q}_0))}_{nonsingular}.$$

We obtain that $\operatorname{im}\tilde{\hat{G}}_1 = \operatorname{im}\hat{G}_1$. Let us assume that we are now at level $i$. We have to choose the projector $\hat{W}_i$ along $\operatorname{im}\hat{G}_i$ and we choose a different $\tilde{\hat{W}}_i$, too. Because of (3.11), the different choice of the projectors does not influence the nullspace of $\hat{W}_i\bar{B}_{i-1}$. The next projector to be chosen is $Z_{i-1}$ with $\hat{W}_i\bar{B}_{i-1}Z_{i-1} = 0$. Here too, we select a distinct $\tilde{Z}_{i-1}$. We compute $\tilde{\hat{G}}_{i+1}$

and will show that $\operatorname{im}\tilde{\hat{G}}_{i+1} = \operatorname{im}\hat{G}_{i+1}$.

$$\begin{aligned}\tilde{\hat{G}}_{i+1} &= \tilde{\bar{G}}_i + \tilde{\bar{B}}_i\tilde{\bar{Q}}_i, \\ &= \tilde{\bar{G}}_i + \bar{B}_{i-1}\tilde{\bar{Q}}_i, \\ &= \bar{G}_{i-1}\tilde{Z}_{i-1} + \bar{B}_{i-1}(I - \bar{P}_{i-1}\tilde{Z}_{i-1}).\end{aligned}$$

Because of (3.10), $\tilde{Z}_{i-1} = Z_{i-1}(I + Z_{i-1}\tilde{Z}_{i-1}(I - Z_{i-1})) =: Z_{i-1}M_{i-1}$, and we obtain

$$\begin{aligned}\tilde{\hat{G}}_{i+1} &= \bar{G}_{i-1}Z_{i-1}M_{i-1} + \bar{B}_{i-1}(I - \bar{P}_{i-1}Z_{i-1}M_{i-1}), \\ &= (\bar{G}_{i-1}Z_{i-1} + \bar{B}_{i-1}M_{i-1}^{-1}(I - M_{i-1}\bar{P}_{i-1}Z_{i-1}))M_{i-1}.\end{aligned}$$

Using the relations given in Lemma 1 we see from (3.5) that

$$M_{i-1}\bar{P}_{i-1}Z_{i-1} = (I + Z_{i-1}\tilde{Z}_{i-1}\underbrace{(I - Z_{i-1}))\bar{P}_{i-1}Z_{i-1}}_{=0} = \bar{P}_{i-1}Z_{i-1}.$$

Now we can represent

$$\tilde{\hat{G}}_{i+1} = \hat{G}_{i+1}M_{i-1} + \bar{B}_{i-1}Z_{i-1}\tilde{Z}_{i-1}(I - Z_{i-1})\bar{Q}_iM_{i-1}$$

and because of Corollary 3 it is obvious that $\tilde{\hat{G}}_{i+1}$ and $\hat{G}_{i+1}$ have the same image, and a different choice of the projectors does not change rank $\bar{G}_i$ and, consequently, the index definition does not depend on the choice of the projector.
This proves the following:

**Lemma 5** *The definition of the regular strangeness index given by Definition 3.2 is independent of the choice of the projectors.*

# 4 Decoupling of a DAE and Representation of a Solution

Let us assume that the DAE has regular strangeness index $\mu - 1$. Then, at each step, the matrix chain forms a DAE

$$A_i(D_ix)' + \bar{B}_ix = q_i \text{ with } q_i := q_{i-1} - A_{i-1}(DX_ix)' \text{ for } i = 1, \ldots, \mu - 1,$$

and the index reduces by one at each step . This index reduction is realized by the differentiation of $DX_ix$. By construction of the matrix chain we can compute (at least theoretically) this part of the solution by

$$X_ix = \bar{P}_{i-1}(I - Z_{i-1})x = (\hat{W}_i\bar{B}_{i-1})^{-}\hat{W}_iq_{i-1},$$

where the generalized inverse $(\hat{W}_i\bar{B}_{i-1})^-$ is exactly the one that forms the chosen $Z_{i-1} = I - (\hat{W}_i\bar{B}_{i-1})^-\hat{W}_i\bar{B}_{i-1}$. $X_i x$ is given by a part of the right hand side $q_{i-1}$, which may contain derivatives of $q$ up to the $(i-1)$-th order. Using the special image projector $R_{Z_i}$ defined by Remark 3.1 $A_i = \bar{G}_i D^-$ is valid and the last DAE for $i = \mu - 1$ reads

$$\bar{G}_{\mu-1}D^-(D\bar{P}_{\mu-1}x)' + \bar{B}_{\mu-1}x = q_{\mu-1}. \tag{4.1}$$

We reformulate (4.1) by

$$\underbrace{(\bar{G}_{\mu-1} + \bar{B}_{\mu-1}\bar{Q}_{\mu-1})}_{\hat{G}_\mu}(\bar{P}_{\mu-1}D^-(D\bar{P}_{\mu-1}x)' + \bar{Q}_{\mu-1}x) + \bar{B}_{\mu-1}\bar{P}_{\mu-1}x = q_{\mu-1}.$$

Using the nonsingularity of $\hat{G}_\mu$ we obtain

$$\bar{P}_{\mu-1}D^-(D\bar{P}_{\mu-1}x)' + \bar{Q}_{\mu-1}x + \hat{G}_\mu^{-1}\bar{B}_{\mu-1}\bar{P}_{\mu-1}x = \hat{G}_\mu^{-1}q_{\mu-1}. \tag{4.2}$$

Multiplying (4.2) by $D\bar{P}_{\mu-1}$ and $\bar{Q}_{\mu-1}$, respectivelly, we obtain

$$D\bar{P}_{\mu-1}D^-(D\bar{P}_{\mu-1}x)' + D\bar{P}_{\mu-1}\hat{G}_\mu^{-1}\bar{B}_{\mu-1}\bar{P}_{\mu-1}x = D\bar{P}_{\mu-1}\hat{G}_\mu^{-1}q_{\mu-1} \tag{4.3}$$

and

$$\bar{Q}_{\mu-1}x + \bar{Q}_{\mu-1}\hat{G}_\mu^{-1}\bar{B}_{\mu-1}\bar{P}_{\mu-1}x = \bar{Q}_{\mu-1}\hat{G}_\mu^{-1}q_{\mu-1}. \tag{4.4}$$

(4.3) leads to an ODE to determine $u := D\bar{P}_{\mu-1}x$ as

$$u - (D\bar{P}_{\mu-1}D^-)'u + D\bar{P}_{\mu-1}\hat{G}_\mu^{-1}\bar{B}_{\mu-1}D^-u = D\bar{P}_{\mu-1}\hat{G}_\mu^{-1}q_{\mu-1}.$$

Using the relation $\bar{Q}_{\mu-1} = \sum_{i=0}^{\mu-1} X_i$ we can compute $\bar{Q}_0 x = X_0 x$ from (4.4), which may contain derivatives of $(\mu-1)$-th order of $q$.

Because of $\bar{Q}_{\mu-1} = \hat{G}_\mu^{-1}\bar{B}_{\mu-1}\bar{Q}_{\mu-1}$ also $\bar{Q}_{\mu-1,s} = \bar{Q}_{\mu-1}\hat{G}_\mu^{-1}\bar{B}_{\mu-1}$ represent a projector onto $\ker \bar{G}_{\mu-1}$. Using this projector (4.3) and (4.4) are decoupled into the dynamical and the algebraic part.
The solution of the DAE is given by

$$x = \bar{P}_{\mu-1}x + \bar{Q}_{\mu-1}x = D^-u + \bar{Q}_{\mu-1}x = D^-u + \sum_{i=0}^{\mu-1} X_i x.$$

# 5 Application to classical Strangeness Index Concept

We apply (3.1), (3.2) to (2.1). There exist orthogonal matrices $\mathcal{P}_1$, $\mathcal{U}_1$ and $\mathcal{Q}_1$ with $\mathcal{P}_1^* A \mathcal{U}_1 = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\mathcal{U}_1^* D \mathcal{Q}_1 = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & 0 \end{pmatrix}$ (see [Meh03]). Using this relation we transform (2.1) into

$$A(Dx)' + Bx = \mathcal{P}_1 \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix} (\begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{Q}_1^* x)' + \underbrace{(B + A\mathcal{U}_1' \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{Q}_1^*)}_{\mathcal{B}} x = q. \tag{5.1}$$

We write

$$\mathcal{B} = \mathcal{P}_1 \left( \mathcal{P}_1^* B \mathcal{Q}_1 + \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U}_1^* \mathcal{U}_1' \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & 0 \end{pmatrix} \right) \mathcal{Q}_1^*. \tag{5.2}$$

Computing the first chain elements we have for $\bar{G}_0 = \mathcal{P}_1 \begin{pmatrix} \mathcal{A}_1 \mathcal{D}_1 & 0 \\ & 0 \end{pmatrix} \mathcal{Q}_1^*$ and $\bar{Q}_0 = \mathcal{Q}_1 \begin{pmatrix} 0 & \\ & I \end{pmatrix} \mathcal{Q}_1^*$, $(\bar{P}_0 := I - \bar{Q}_0)$.

With $\mathcal{P}_1^* \mathcal{B} \mathcal{Q}_1 =: \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{pmatrix}$ we obtain $\hat{G}_1 = \mathcal{P}_1 \begin{pmatrix} \mathcal{A}_1 \mathcal{D}_1 & B_{12} \\ & B_{22} \end{pmatrix} \mathcal{Q}_1^*$, but $\mathcal{P}_1^* \mathcal{B} \mathcal{Q}_1 =: \begin{pmatrix} \mathcal{B}_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ if $\mathcal{P}_1^* B \mathcal{Q}_1 =: \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, because of the structure of the second term of (5.2) (Note the difference between $\mathcal{B}$ and $B$). This means that we apply (3.1), (3.2) to the original data $A, D$, and $B$ of the DAE.

There exist orthogonal matrices such that $B_{22} = \tilde{\mathcal{P}}_2 \begin{pmatrix} \tilde{B}_{22} & \\ & 0 \end{pmatrix} \tilde{\mathcal{Q}}_2^*$. We can choose an orthogonal projector along the image of $\hat{G}_1$ as

$$\hat{W}_1 = \mathcal{P}_1 \underbrace{\begin{pmatrix} I & \\ & \tilde{\mathcal{P}}_2 \end{pmatrix}}_{=:\mathcal{P}_2} \left( \begin{array}{cc:c} 0 & & \\ \hdashline & & \\ & 0 & \\ & & I \end{array} \right) \begin{pmatrix} I & \\ & \tilde{\mathcal{P}}_2^* \end{pmatrix} \mathcal{P}_1^*.$$

If we introduce the relation $\tilde{\mathcal{P}}_2^* B_{21} = \begin{pmatrix} \tilde{B}_{21} \\ \tilde{B}_{31} \end{pmatrix}$, we obtain

$$\hat{W}_1 B \bar{P}_0 = \mathcal{P}_1 \begin{pmatrix} & 0 & \\ \tilde{\mathcal{P}}_2 \begin{pmatrix} 0 \\ \tilde{B}_{31} \end{pmatrix} & & 0 \end{pmatrix} \underbrace{\begin{pmatrix} T & \\ & \tilde{\mathcal{Q}}_2^* \end{pmatrix}}_{\mathcal{Q}_2^*} \mathcal{Q}_1^*,$$

and with $\tilde{B}_{31} = \tilde{\mathcal{P}}_3 \begin{pmatrix} 0 & \hat{B}_{42} \end{pmatrix} \tilde{\mathcal{Q}}_3^*$ with orthogonal matrices $\tilde{\mathcal{P}}_3$ and $\tilde{\mathcal{Q}}_3^*$, and a full rank matrix $\hat{B}_{42}$ it results that

$$\hat{W}_1 B \bar{P}_0 = \mathcal{P}_1 \mathcal{P}_2 \underbrace{\begin{pmatrix} I & & \\ & I & \\ & & \tilde{\mathcal{P}}_3 \end{pmatrix}}_{\mathcal{P}_3} \left(\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & \hat{B}_{42} & 0 & 0 \end{array}\right) \underbrace{\begin{pmatrix} \tilde{\mathcal{Q}}_3^* & \\ & I \end{pmatrix}}_{=:\mathcal{Q}_3^*} \mathcal{Q}_2^* \mathcal{Q}_1^*. \quad (5.3)$$

Now we are looking for a nullspace projector of $\hat{W}_1 B P_0$. The structure given by (5.3) leads to $Z_0 = \mathcal{Q}_1 \mathcal{Q}_2 \begin{pmatrix} \tilde{\mathcal{Q}}_3 & \\ & I \end{pmatrix} \left(\begin{array}{cc:cc} I & & & \\ & 0 & & \\ \hdashline & & I & \\ & & & I \end{array}\right) \begin{pmatrix} \tilde{\mathcal{Q}}_3^* & \\ & I \end{pmatrix} \mathcal{Q}_2^* \mathcal{Q}_1^*$. With $Z_0$ we obtain for $DZ_0 = \mathcal{U}_1 \left(\begin{array}{c:c} \mathcal{D}_1 \tilde{\mathcal{Q}}_3 \begin{pmatrix} I & \\ & 0 \end{pmatrix} & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right) \mathcal{Q}_3^* \mathcal{Q}_2^* \mathcal{Q}_1^*$. We set $\mathcal{D}_1 \tilde{\mathcal{Q}}_3 =: \begin{pmatrix} \hat{D}_1 & \hat{D}_2 \end{pmatrix}$ and there exists an orthogonal matrix with $\tilde{\mathcal{U}}_4^* \hat{D}_1 = \begin{pmatrix} \bar{D}_1 \\ 0 \end{pmatrix}$.

With $\mathcal{U}_4 = \begin{pmatrix} \tilde{\mathcal{U}}_4 & \\ & I \end{pmatrix}$ this leads to

$$D_{new} = DZ_0 = \mathcal{U}_1 \mathcal{U}_4 \left(\begin{array}{cc:c} \bar{D}_1 & 0 & \\ 0 & 0 & \\ \hdashline & & 0 \end{array}\right) \mathcal{Q}_3^* \mathcal{Q}_2^* \mathcal{Q}_1^*$$

and a reflexive inverse is given by

$$(DZ_0)^- = \mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3 \begin{pmatrix} \bar{D}_1^{-1} & 0 & \vdots & \\ 0 & 0 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & & \vdots & 0 \end{pmatrix} \mathcal{U}_4^*\mathcal{U}_1^*.$$

Applying the image projector

$$R_{Z_0} = DZ_0(DZ_0)^- = \mathcal{U}_1\mathcal{U}_4 \begin{pmatrix} I & & \vdots & \\ & 0 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & & \vdots & 0 \end{pmatrix} \mathcal{U}_4^*\mathcal{U}_1^*$$

to $A$ leads to

$$A_{new} = AR_{Z_0} = \mathcal{P}_1 \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U}_1^*\mathcal{U}_1\mathcal{U}_4 \begin{pmatrix} I & & \vdots & \\ & 0 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & & \vdots & 0 \end{pmatrix} \mathcal{U}_4^*\mathcal{U}_1^*.$$

With $\mathcal{A}_1\tilde{\mathcal{U}}_4 =: \begin{pmatrix} \hat{A}_1 & \hat{A}_2 \end{pmatrix}$ and a $\tilde{\mathcal{P}}_4$ exists with $\hat{A}_1 = \tilde{\mathcal{P}}_4 \begin{pmatrix} \bar{A}_1 \\ 0 \end{pmatrix}$ we obtain for

$$A_{new} = \mathcal{P}_1 \underbrace{\begin{pmatrix} \tilde{\mathcal{P}}_4 & \\ & I \end{pmatrix}}_{\mathcal{P}_4} \begin{pmatrix} \bar{A}_1 & 0 & \vdots & \\ 0 & 0 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & & \vdots & 0 \end{pmatrix} \mathcal{U}_4^*\mathcal{U}_1^*$$

and

$$B_{new} = A(R_{Z_0})'DZ_0 + \mathcal{B} =$$

$$\mathcal{P}_1 \begin{pmatrix} \mathcal{A}_1 & 0 \\ & 0 \end{pmatrix} \mathcal{U}_1^* \left\{ (\mathcal{U}_1\mathcal{U}_4)' \begin{pmatrix} I & & \\ & 0 & \\ & & 0 \end{pmatrix} \mathcal{U}_4^*\mathcal{U}_1^* + \underline{\mathcal{U}_1\mathcal{U}_4 \begin{pmatrix} I & & \\ & 0 & \\ & & 0 \end{pmatrix} (\mathcal{U}_4^*\mathcal{U}_1^*)'} \right\} *$$

$$* \, \mathcal{U}_1\mathcal{U}_4 \begin{pmatrix} \bar{D}_1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix} \mathcal{Q}_3^*\mathcal{Q}_2^*\mathcal{Q}_1^* + \mathcal{B}. \tag{5.4}$$

Now, one step of the chain (3.2) is finished. The new DAE is given by

$$A_{new}(D_{new}x)' + B_{new}x = \bar{q}.$$

If we combine the underlined term of (5.4) with $A_{new}(D_{new}x)'$ we obtain the DAE

$$\mathcal{P}_1\mathcal{P}_4 \begin{pmatrix} \bar{A}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left( \begin{pmatrix} \bar{D}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_{new} \right)' \\ + (A(\mathcal{U}_1\mathcal{U}_4)' \begin{pmatrix} \bar{D}_1 & 0 & \\ 0 & 0 & \\ & & 0 \end{pmatrix} + \mathcal{B}\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3)x_{new} = \bar{q}.$$

with $x_{new} := \mathcal{Q}_3^*\mathcal{Q}_2^*\mathcal{Q}_1^*x$, and this DAE is identical with the result of one "strangeness step".

# References


[KM06] Peter Kunkel and Volker Mehrmann. *Differential-Algebraic Equations: Analysis and Numerical Solution.* EMS Textbook in Mathematics, 2006.

[Mär02] R. März. The index of linear differential algebraic equations with properly stated leading terms. 42(3-4):308–338, 2002.

[Meh03] Volker Mehrmann. Weak formulation of linear differential-algebraic systems with variable coefficients. Lecture at GAMM FA: Dynamik und Regelung, 2003.