Existence of turbulent weak solutions to the generalized Navier-Stokes equations in exterior domains and large time behaviour

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Abstract. Let Ω be an exterior domain in $\mathbb{R}^n$ $(n = 2, 3, 4)$, with boundary being not necessarily smooth. For any initial velocity $\mathbf{u}_0 \in L^2(\Omega)^n$ such that $\nabla \cdot \mathbf{u}_0 = 0$ (in sense of distribution) and external forces

$$\mathbf{F} \in L^1(0,\infty; L^2(\Omega)^n) + L^2(0,\infty; W^{-1,2}(\Omega)^n)$$

we are able to construct a turbulent weak solution $\mathbf{u} \in C_w([0,\infty); L^2(\Omega)^n) \cap L^2(0,\infty; W_0^{1,2}(\Omega)^n)$ to the equations of motion of a non-Newtonian fluid. Simultaneously, we prove that this solution fulfils the non-uniform decay condition

$$\|\mathbf{u}(t)\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

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1. Introduction. Statement of the Main Result

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3, 4)$ be an exterior domain. We set $Q := \Omega \times (0, \infty)$. In the present paper we consider the generalized Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0, \text{ [1]} \tag{1.1}$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbf{S} + p\mathbf{I}) = \mathbf{f} - \nabla \cdot \mathbf{g} \quad \text{in} \quad Q, \tag{1.2}$$

where

$$\begin{aligned} p \,&=\, \text{pressure}, \\ \mathbf{u} \,&=\, \{u^1, \ldots, u^n\} \,=\, \text{velocity}, \\ \mathbf{S} \,&=\, \{S_{ij} \,|\, i,j = 1, \ldots, n\} \,=\, \text{deviatoric stress tensor}, \\ \mathbf{f} - \nabla \cdot \mathbf{g} \,&=\, \text{external force}. \end{aligned}$$

Boundary and initial conditions. On the boundary of Ω we assume the following condition of adherence

$$\mathbf{u} \,=\, 0 \quad \text{on} \quad \partial\Omega \times (0, \infty). \tag{1.3}$$

At the initial time $t = 0$ we assume

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{1.4}$$

where $\mathbf{u}_0$ denotes a given initial velocity field, with $\nabla \cdot \mathbf{u}_0 = 0$.

Constitutive law. For the usual Navier-Stokes equations the stress $\mathbf{S}$ is given by

$$\mathbf{S} \,=\, \mathbf{S}(D) \,=\, \nu_0 D, ^{2}$$

where $\nu_0 = \text{const} > 0$ denotes the viscosity, while $D = \{D_{ij}\}$ denotes the "rate of strain tensor", which is defined by

$$D(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}).$$

However in many applications one often considers fluids where the viscosity is not constant but ranges between two positive constants. For instance considering a fluid where the mechanical energy is transferred into heat the viscosity may depend on the temperature. In this case the stress is given by the law

$$\mathbf{S} \,=\, \mathbf{S}(x, t, D) \,=\, \mu(\theta(x, t))D, \quad (x, t) \in Q \tag{1.5}$$

with

$$\begin{gathered} \mu \in C(\mathbb{R}), \quad 0 < \mu_1 \le \mu \le \mu_2 < \infty \quad \text{in } \mathbb{R}, \\ \theta \,=\, \text{temperature}. \end{gathered}$$

(concerning the continuum mechanical background we refer to [2], [11]).

Having in mind (1.5) as special case we impose the following conditions on the components of the deviatoric stress tensor $\mathbf{S}$.

(I) $\qquad \mathbf{S} : Q \times \mathbb{M}^{n^2}_{\mathrm{sym}} \to \mathbb{M}^{n^2}_{\mathrm{sym}}$ [3] $\quad$ is a Carathéodory function;

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[1] Here, $\nabla \cdot \mathbf{v} := \partial_{x_i} v^i$, where $\partial_{x_i} \,=\, \dfrac{\partial}{\partial x_i}$ $(i = 1, \ldots, n)$. Throughout repeated subscripts imply summation over 1 to n.

[2] Clearly, $\nabla \cdot D = \Delta$ on the space of all divergence free function.

[3] $\mathbb{M}^{n^2}_{\mathrm{sym}} \,=\,$ vector space of all symmetric $n \times n$ matrices $\boldsymbol{\xi} = \{\xi_{ij}\}$. We equip $\mathbb{M}^{n^2}_{\mathrm{sym}}$ with scalar product $\boldsymbol{\xi} : \boldsymbol{\eta} = \xi_{ij}\eta_{ij}$ and norm $\|\boldsymbol{\xi}\| := (\boldsymbol{\xi} : \boldsymbol{\xi})^{1/2}$. - By $\mathbf{a} \cdot \mathbf{b}$ we denote the usual scalar product in $\mathbb{R}^n$ and by $|\mathbf{a}|$ we denote the Euclidean norm.

growth condition:

(II) $\qquad \|\mathbf{S}(x,t,\boldsymbol{\xi})\| \le c_0\|\boldsymbol{\xi}\| + \kappa_1 \quad \forall\, \boldsymbol{\xi} \in \mathbb{M}^{n^2}_{\mathrm{sym}},\ \text{f.a.a.}\,(x,t) \in Q$

$(c_0 > 0, \kappa_1 \in L^2(Q), \kappa_1 \ge 0)$;

coercivity:

(III) $\qquad \mathbf{S}(x,t,\boldsymbol{\xi}) : \boldsymbol{\xi} \ge \nu_0\|\boldsymbol{\xi}\|^2 - \kappa_2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{M}^{n^2}_{\mathrm{sym}},\ \text{f.a.a.}\,(x,t) \in Q$

$(\nu_0 > 0, \kappa_2 \in L^1(Q), \kappa_2 \ge 0)$;

strict monotonicity:

$$\text{(IV)} \qquad \begin{cases} (\mathbf{S}(x,t,\boldsymbol{\xi}) - \mathbf{S}(x,t,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0 \\ \forall\, \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{M}^{n^2}_{\mathrm{sym}}\, (\boldsymbol{\xi} \ne \boldsymbol{\eta}),\ \text{f.a.a.}\,(x,t) \in Q. \end{cases}$$

Weak solution to (1.1)*,* (1.2). Before we introduce the notion of a weak solution to (1.1)-(1.4) let us provide some notations and function spaces which will be used in sequence of the paper. By $W^{k,q}(\Omega), W^{k,q}_0(\Omega)$ $(k \in \mathbb{N}; 1 \le q \le \infty)$ we denote the usual Sobolev spaces. Let $(X, \|\cdot\|_X)$ be a normed space. By $L^q(0,T;X)\,(0 < T \le \infty)$ we denote the space of all Bochner measurable functions $\varphi : (0,T) \to X$, such that

$$\begin{cases} \|\varphi\|_{L^q(0,T;X)} := \left(\displaystyle\int_0^T \|\varphi(t)\|_X^q \,\mathrm{d}t \right)^{\frac{1}{q}} < \infty & \text{if} \quad 1 \le q < \infty, \\ \|\varphi\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|\varphi(t)\|_X < \infty & \text{if} \quad q = \infty. \end{cases}$$

By $\mathcal{D}_\sigma(\Omega)$ we denote set of all $\boldsymbol{\varphi} \in C^\infty_0(\Omega)$ with $\nabla \cdot \varphi = 0$. Then we set

$$\mathbf{H} := \textit{ closure of } \ \mathcal{D}_\sigma(\Omega) \ \textit{ in } \ L^2(\Omega)^n,$$
$$\mathbf{V} := \textit{ closure of } \ \mathcal{D}_\sigma(\Omega) \ \textit{ in } \ W^{1,2}(\Omega)^n.$$

Definition 1.1. Let $\mathbf{u}_0 \in \mathbf{H}$. Let $\mathbf{f} \in L^1(0,\infty; L^2(\Omega)^n)$ and $\mathbf{g} \in L^2(Q)^{n^2}$ be given forces. A vector-valued function $\mathbf{u} \in L^2(0,\infty;\mathbf{V}) \cap L^\infty(0,\infty;\mathbf{H})$ is called a weak solution to (1.1)-(1.4) if the following identity

$$\int_Q -\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} - \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} + \mathbf{S}(x,t,D(\mathbf{u})) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_Q \mathbf{f} \cdot \boldsymbol{\varphi} + \mathbf{g} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0)\,\mathrm{d}x \qquad (1.6)$$

holds for all $\boldsymbol{\varphi} \in C^\infty(Q)^n$ with $\nabla \cdot \boldsymbol{\varphi} = 0$ and $\mathrm{supp}(\boldsymbol{\varphi}) \subset\subset \Omega \times [0,\infty)$. [4]

[4] Here $A \subset\subset B$ means A, B are open subsets of $\mathbb{R}^n$, A is bounded and $\bar{A} \subset B$.

In addition, a weak solution $\mathbf{u}$ to (1.1)-(1.4) is said to be **turbulent** if for almost all $0 < s < t < \infty$

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{L^2(\Omega)} + \int_s^t \int_\Omega \mathbf{S}(x,\tau,D(\mathbf{u})) : D(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}\tau \\ \leq \frac{1}{2}\|\mathbf{u}(s)\|^2_{L^2(\Omega)} + \int_s^t \int_\Omega \mathbf{f}\cdot\mathbf{u} + \mathbf{g} : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{1.7}$$

Remark 1.2. By virtue of Sobolev's imbedding theorem using multiplicative inequality one easily verifies

$$L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H}) \hookrightarrow L^r(0,T,L^\rho(\Omega)^n) \tag{1.8}$$

for all $\rho \in [2, 2^*]$ and all $r \in [1,\infty]$ such that

$$\frac{2}{r} + \frac{n}{\rho} \geq \frac{n}{2},$$

where

$$2^* := \begin{cases} \dfrac{2n}{n-2} & \text{if} \quad n \geq 3, \\ q \in [1,\infty) \text{ arbitrarily} & \text{if} \quad n = 2. \end{cases}$$

Statement of the Main Result. The aim of the present paper is to prove the existence of a turbulent weak solution to (1.1)-(1.4). That is

Theorem 1.3 (Main Theorem). *For every given initial velocity $\mathbf{u}_0 \in \mathbf{H}$ and forces $\mathbf{f} \in L^1(0,\infty;L^2(\Omega)^n)$, $\mathbf{g} \in L^2(Q)^{n^2}$ there exists a turbulent weak solution $\mathbf{u} \in L^2(0,\infty;\mathbf{V}) \cap C_w([0,\infty);\mathbf{H})$ to (1.1)-(1.4) with non-uniform decay*

$$\|\mathbf{u}(t)\|_{L^2(\Omega)} \to 0 \quad \textit{as} \quad t \to \infty. \tag{1.9}$$

Remark 1.4. The first result on the existence of turbulent weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$ is due to Leray. In his pioneering paper [12] he also pointed out the importance of the strong energy inequality (1.7) for the decay problem of the energy $\|\mathbf{u}(t)\|_{L^2}$. Later Masuda [15] provided the property

$$\int_{t-1}^t \|\mathbf{u}(\tau)\|_{L^2}\,\mathrm{d}\tau \to 0 \quad \text{as} \quad t \to \infty, \tag{1.10}$$

and was able to get weak solutions with non-uniform decay in the energy $\|\mathbf{u}(t)\|_{L^2}$ in a general domain if $n = 2, \mathbf{f} \in L^1(0,\infty;L^2)$ or if $n \geq 3, \mathbf{f} = 0$ (see also Schonbeck [18]). Using Masuda's estimate Sohr, von Wahl and Wiegner [22] proved the existence of a turbulent weak solution in a three dimensional exterior domain satisfying the non-uniform decay in the energy norm for initial datas $\mathbf{u}_0 \in \mathbf{H}$ satisfying an additional decay as $|x| \to \infty$. Later Miyakawa and Sohr [16] achieved a similar result for all initial data $\mathbf{u}_0 \in \mathbf{H}$ in case $n = 3, 4$. In particular they extended Masuda's result for external forces $\mathbf{f} \in L^1(0,\infty;L^2)$. These results then were generalized by Kozono, Ogawa and Sohr in [10] where the authors studied L^q-decay for weak solution of the Navier-Stokes equations in an exterior domain satisfying the strong energy inequality. Concerning further results in other unbounded domains we refer to Heywood [7], Borchers and Miyakawa [3], Ukai [24], Maremonti [14] and Kozono and Ogawa [9]. However the existence of a turbulent weak solution in a general unbounded domain with uniform C^2-boundary was open for a long time and has been achieved by Farwig, Kozono and Sohr in the recent paper [5].

Notice that the results stated above were obtained by using functional analytic arguments such as Fourier analysis, semi group theory and well-known properties for the Stokes operator in appropriate function spaces. Since these methods make essential use of the special structure of the Navier-Stokes equations and the regularity of the boundary $\partial\Omega$ they are not suitable for proving similar results in the case of the generalized Navier-Stokes equations in exterior domains with non-smooth boundary. Nevertheless, using an alternative local method we are able to prove the existence of turbulent weak solutions to the generalized Navier-Stokes equations with non-uniform decay in the energy norm. The key of this method, which has been applied also in [26] for proving the existence of weak solutions to a general non-Newtonian fluid, lies in the special local pressure representation based on Simader's decomposition of $L^r = A^r \oplus B^r$ $(A^r = \{\Delta\phi \,|\, \phi \in W_0^{2,\,r}\}, B^r = \{v \in L^r | \Delta v = 0\})$ $\quad (1 < r < \infty)$. By this method we achieve the necessary estimates of the energy outside a large ball.

The paper is organized as follows. In Section 2 based on a variational estimate which is due to R. Müller (cf. Lemma 2.1) we study the spaces A^r and B^r. In particular, we mention the important decomposition $L^r = A^r \oplus B^r$ by Simader. Using these properties we finally show the existence of the local pressure together with optimal estimates on the sets $\{R < |x| < 16R\}$ $(0 < R < \infty)$.

The aim of Section 3 will be the proof of Theorem 1.3, which will be divided into four main steps. Firstly, by truncating the non-linear term we establish an approximate weak solution $\mathbf{u}_m$ which tends to a weak solution $\mathbf{u}$ of (1.1)-(1.4) as $m \to \infty$. Secondly verifying that $\mathbf{u}_m$ fulfils a local energy identity on the sets $\{2^{\ell-1} < |x| < 2^{\ell+3}\}$ $(\ell \in \mathbb{N})$ we obtain an appropriate estimate of the energy norm outside of the ball B_{2^k} $(k \in \mathbb{N})$. By the aid of these estimates we get the strong convergence of $\mathbf{u}_m(T)$ to $\mathbf{u}(T)$ in L^2 for almost all $T > 0$. Thirdly, we complete the proof of the strong energy inequality taking into account the monotonicity of $\mathbf{S}$. Finally, by virtue of the global bound $\|\mathbf{u}\|_{L^{2\frac{n+2}{n}}} < \infty$ verifying a similar estimate for $\mathbf{u}$ as we have found in the second step for $\mathbf{u}_m$ we get the non-uniform decay $\|\mathbf{u}(t)\|_{L^2} \to 0$ as $t \to \infty$.

2. Harmonic decomposition

The purpose of this section is to state a few lemmas which form the base to construct an appropriate pressure function for weak solutions to the problem (1.1)-(1.4).

During the whole section let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^2$. By $W_0^{2,r}(G)\,(1 < r < \infty)$ we denote the closure of $C_0^\infty(G)$ with respect to the norm

$$\|\phi\|_{W_0^{2,r}(G)} := \left(\int_G |\Delta\phi|^r \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

Clearly, by the well-known Calderón-Zygmond's estimate we have

$$\|\nabla^2\phi\|_{L^r(G)} \;\le\; C_{CZ}\|\phi\|_{W_0^{2,r}(G)}, \tag{2.1}$$

where $C_{CZ} = \mathrm{const} > 0$ depending on n only.

We begin our discussion by stating a variational estimate which is due to R. Müller [17].

Lemma 2.1 (Variational estimate in $W_0^{2,r}(G)$). *Let $1 < r < \infty$. Then there exists a constant $C_r = C_r(r, n, G) > 0$, such that*

$$\|u\|_{W_0^{2,r}(G)} \;\le\; C_r \sup_{0\neq\phi\in W_0^{2,r'}(G)} \frac{\int_G \Delta u\Delta\phi\,\mathrm{d}x}{\|\phi\|_{W_0^{2,r'}(G)}} \quad \forall\, u \in W_0^{2,r}(G). \tag{2.2}$$

□

Next, we introduce the following subspaces of $L^r(G)\,(1 < r < \infty)$, which will be used in the sequel

$$A^r(G) := \overline{\{\Delta\phi \,|\, \phi \in C_0^\infty(G)\}}^{L^r(G)},$$
$$B^r(G) := \{\varphi \in L^r(G) \,|\, \Delta\varphi = 0 \text{ in } G\}.$$

Remark 2.2. Owing to the reflexivity of $W_0^{2,r}(G)$ the space $A^r(G)$ introduced above is given by

$$A^r(G) \;=\; \{\Delta u \,|\, u \in W_0^{2,r}(G)\}. \tag{2.3}$$

With help of Lemma 2.1 one obtains the following estimates which play an essential role in the proof of the main theorem (cf. Section 3).

Lemma 2.3. *Let $p \in A^r(G)$ and $\mathbf{h} = \{h_{ij}\} \in L^r(G)^{n^2}$ $(1 < r < \infty)$, such that*

$$\int_G p\Delta\phi\,\mathrm{d}x \;=\; \int_G \mathbf{h} : \nabla^2\phi\,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(G). \tag{2.4}$$

Then

$$\|p\|_{L^r(G)} \;\le\; C_r\, C_{CZ}\, \|\mathbf{h}\|_{L^r(G)}. \tag{2.5}$$

Lemma 2.4. *Let $1 < r < \infty$. Then for every $v^* \in (W_0^{2,r'}(G))^*$ there exists a unique $u \in W_0^{2,r}(G)$ such that*

$$\int_G \Delta u\Delta\phi\,\mathrm{d}x \;=\; \langle v^*, \phi\rangle \quad \forall\, \phi \in C_0^\infty(G). \tag{2.6}$$

(For the proof of this lemma see in [26]).

As an immediate consequence of Lemma 2.4 we get

Corollary 2.5 (C. G. Simader). *For every* $p \in L^r(G)\,(1 < r < \infty)$ *there exist* $p_0 \in A^r(G)$ *and* $p_h \in B^r(G)$ *such that*

$$p \,=\, p_0 \,+\, p_h. \tag{2.7}$$

In addition, there exists a constant $C'_r = C'_r(r,n,G)$ *such that*

$$\|p_0\|_{L^r(G)} \,+\, \|p_h\|_{L^r(G)} \,\le\, C'_r\|p\|_{L^r(G)}, \tag{2.8}$$

i.e. the sum $A^r(G) + B^r(G)$ *is direct.*

Proof. Let $p \in L^r(G)$. According to Lemma 2.4 there exists $u \in W^{2,\,r}_0(G)$ satisfying

$$\int_G \Delta u \Delta\phi \,\mathrm{d}x \,=\, \int_G p\Delta\phi \,\mathrm{d}x \quad \forall\, \phi \in C^\infty_0(G). \tag{2.9}$$

In particular, by (2.3) we have $\Delta u \in A^r(G)$. On the other hand, as one can easily check (2.9) is equivalent to

$$\int_G (\Delta u - p)\Delta\phi \,\mathrm{d}x \,=\, 0 \quad \forall\, \phi \in C^\infty_0(G).$$

Using Weyl's Lemma we have $\Delta(p - \Delta u) = 0$ in G. This shows that the function $p - \Delta u$ belongs to $B^r(G)$. Finally, setting $p_0 := \Delta u$ and $p_h := p - \Delta u$ gives (2.7).

Now, it only remains to prove (2.8). For, let $p = p_0 + p_h$ with $p_0 \in A^r(G)$ and $p_h \in B^r(G)$. It is readily seen that

$$\int_G p_0\Delta\phi \,\mathrm{d}x \,=\, \int_G p\Delta\phi \,\mathrm{d}x \quad \forall\, \phi \in C^\infty_0(G).$$

Thus, applying Lemma 2.3 with $\mathbf{h} = p\,\mathbf{I}$ shows that

$$\|p_0\|_{L^r(G)} \,\le\, C_r C_{CZ}\|p\|_{L^r(G)}.$$

Whence, (2.8). □

Next, let us introduce the following subspace of $B^r(G)$

$$\dot{B}^r(G) := \Big\{ p \in B^r(G) \,\Big|\, \int_G p \,\mathrm{d}x \,=\, 0 \Big\}.$$

Now we present the following result on the local pressure decomposition (cf. also in [26]).

Theorem 2.6. *Let* $\mathbf{u} \in C_w([0,\infty); L^2(G)^n)$ *with* $\nabla\cdot\mathbf{u} = 0$ *(in the sense of distributions) and let* $\mathbf{h} \in L^1_{\mathrm{loc}}([0,\infty); L^r(G)^{n^2})$ $(1 < r \le 2)$. *Suppose that*

$$\int_0^\infty \int_G -\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} \,+\, \mathbf{h} : \nabla\boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \,=\, 0 \tag{2.10}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(G\times(0,\infty))^n$ *with* $\nabla\cdot\boldsymbol{\varphi} = 0$. *Then there exist unique functions*

$$\begin{aligned} &p_0 \in L^1_{\mathrm{loc}}([0,\infty); A^r(G)),\\ &\tilde{p}_h \in C_w([0,\infty); \dot{B}^r(G)) \quad \textit{with} \quad \tilde{p}_h(0) = 0, \end{aligned}$$

such that

$$\int_0^\infty\int_G -\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{h} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty\int_G p_0\nabla\cdot\boldsymbol{\varphi} + \nabla\tilde{p}_h\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \tag{2.11}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(G\times(0,\infty))$. *Furthermore, we have the a-priori estimates*

$$\|p_0(t)\|_{L^r(G)} \le c\,\|\mathbf{h}(t)\|_{L^r(G)}, \tag{2.12}$$

$$\|\tilde{p}_h(t)\|_{L^r(G))} \le c\,\|\mathbf{u}(t) - \mathbf{u}(0)\|_{L^2(G)} + c\left\|\int_0^t \mathbf{h}(s)\,\mathrm{d}s\right\|_{L^r(G)} \tag{2.13}$$

for almost all $t \in (0,\infty)$, *where* $c = \mathrm{const} > 0$ *depending only on* r, n *and* G.

Proof. Let $\boldsymbol{\psi} \in C_0^\infty(G)^n$ with $\nabla\cdot\boldsymbol{\psi} = 0$ and let $\eta \in C_0^\infty(0,\infty)$. Into (2.10) inserting $\boldsymbol{\varphi}(x,t) = \eta(t)\boldsymbol{\psi}(x)$ using Fubini's theorem yields

$$-\int_0^\infty \alpha\,\eta'\,\mathrm{d}t = \int_0^\infty \beta\,\eta\,\mathrm{d}t,$$

where

$$\begin{aligned} \alpha(t) &:= \int_G \mathbf{u}(t)\cdot\boldsymbol{\psi}\,\mathrm{d}x,\\ \beta(t) &:= -\int_G \mathbf{h}(t) : \nabla\boldsymbol{\psi}\,\mathrm{d}x, \quad t\in(0,\infty). \end{aligned}$$

By the assumptions of the theorem we have $\beta \in L^1_{\mathrm{loc}}([0,\infty))$. Therefore $\alpha \in W^{1,1}([0,\infty))$ with $\alpha' = \beta$. In particular, α is represented by an absolutely continuous function, which will be denoted also by α. Using integration by parts one calculates

$$\alpha(t) = \alpha(0) + \int_0^t \beta(s)\,\mathrm{d}s \quad \forall\, t\in(0,\infty). \tag{2.14}$$

Define

$$\tilde{\mathbf{h}}(t) := \int_0^t \mathbf{h}(s)\,\mathrm{d}s, \quad t\in[0,\infty).$$

Let $t \in (0,\infty)$ be fixed. Using Fubini's theorem the identity (2.14) reads

$$\int_G (\mathbf{u}(t) - \mathbf{u}(0))\cdot\boldsymbol{\psi} + \tilde{\mathbf{h}}(t) : \nabla\boldsymbol{\psi}\,\mathrm{d}x = 0.$$

Thus, according to [6] (Th. III. 3.1, Th. III. 5.2) there exists a unique function $\tilde{p}(t) \in L^r(G)$ with $\int_G \tilde{p}(t)\,\mathrm{d}x = 0$ such that for all $\boldsymbol{\psi} \in W_0^{1,r'}(G)^n$

$$\int_G (\mathbf{u}(t) - \mathbf{u}(0))\cdot\boldsymbol{\psi} + \tilde{\mathbf{h}}(t) : \nabla\boldsymbol{\psi}\,\mathrm{d}x = \int_G \tilde{p}(t)\nabla\cdot\boldsymbol{\psi}\,\mathrm{d}x. \tag{2.15}$$

In addition, there exists a constant $c>0$ depending only on r,n and G such that

$$\|\tilde{p}(t)\|_{L^r(G)} \,\leq\, c\,(\|\mathbf{u}(t)-\mathbf{u}(0)\|_{L^r(G)} \,+\, \|\tilde{\mathbf{h}}(t)\|_{L^r(G)}). \tag{2.16}$$

Next, we are going to prove that $t\mapsto \tilde{p}(t)$ is a Bochner measurable function. For, let $v\in L^{r'}(G)$ be arbitrarily chosen. From [6] (Th. III. 3.4) we get the existence of a function $\boldsymbol{\psi}\in W^{1,\,r'}_0(G)^n$ such that $\nabla\cdot\boldsymbol{\psi}=v-v_G$ [5] (cf. also [23] (Lemma 2.1, p. 252)). Thus, from (2.15) we infer that

$$\begin{aligned}\int_G \tilde{p}(t)\,v\,\mathrm{d}x \;&=\; \int_G \tilde{p}(t)\,(v-v_G)\,\mathrm{d}x\\ &=\;\int_G (\mathbf{u}(t)-\mathbf{u}(0))\cdot\boldsymbol{\psi}+\tilde{\mathbf{h}}(t):\nabla\boldsymbol{\psi}\,\mathrm{d}x.\end{aligned}$$

By the assumption of the theorem the function on the right is continuous, so also the function on the left. Consequently, $\tilde{p}\in C_w([0,\infty);L^r(G))$, and by a well-known theorem of Pettis this shows that $\tilde{p}$ is Bochner measurable.

Let $\boldsymbol{\varphi}\in C^\infty_0(G\times(0,\infty))^n$ be a given test function. Into (2.15) putting $\boldsymbol{\psi}=\boldsymbol{\varphi}(\cdot,t)$ integrating both sides of this identity over the interval $(0,\infty)$ yields

$$\int_0^\infty\int_G (\mathbf{u}-\mathbf{u}(0))\cdot\boldsymbol{\varphi}+\tilde{\mathbf{h}}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_0^\infty\int_G \tilde{p}\,\nabla\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.17}$$

Now, applying Corollary 2.5 one finds unique functions $\tilde{p}_0\in C_w([0,\infty);A^r(G))$ and $\tilde{p}_h\in C_w([0,\infty);\dot{B}^r(G))$ such that

$$\tilde{p}=\tilde{p}_0+\tilde{p}_h \quad \text{in} \quad G\times(0,\infty).$$ [6]

As one may easily check (2.15) implies $\tilde{p}(0)=0$. Thus, by Corollary 2.5 we deduce $\tilde{p}_h(0)=0$. Moreover, observing (2.8) using (2.16) gives (2.13).

Into (2.17) inserting $\boldsymbol{\psi}=\nabla\phi$ for $\phi\in C^\infty_0(G)$ recalling $\nabla\cdot\mathbf{u}=0$ and $\Delta\tilde{p}_h=0$ using integration by parts one obtains

$$\int_G \tilde{\mathbf{h}}(t):\nabla^2\phi\,\mathrm{d}x \;=\; \int_G \tilde{p}_0(t)\Delta\phi\,\mathrm{d}x \quad \forall\,\phi\in C^\infty_0(G).$$

Let $t\in(0,\infty)$ and let $0<\rho<1$. Then from the above identity we derive

$$\int_G \frac{\tilde{\mathbf{h}}(t+\rho)-\tilde{\mathbf{h}}(t)}{\rho}:\nabla^2\phi\,\mathrm{d}x \;=\; \int_G \frac{\tilde{p}_0(t+\rho)-\tilde{p}_0(t)}{\rho}\Delta\phi\,\mathrm{d}x \tag{2.18}$$

---

[5] Here v_G denotes the mean value $\dfrac{1}{\mathcal{L}_n(G)}\displaystyle\int_G v(x)\,\mathrm{d}x$.

[6] Notice, that $\int_G \tilde{p}_0(t)\,\mathrm{d}x=0$ and $\int_G \tilde{p}(t)\,\mathrm{d}x=0$ implies $\int_G \tilde{p}_h(t)\,\mathrm{d}x=0$.

for all $\phi \in C_0^\infty(G)$. With help of Lemma 2.3 one gets the estimate

$$\left\| \frac{\tilde{p}_0(t+\rho) - \tilde{p}_0(t)}{\rho} \right\|_{L^r(G)} \leq c \left\| \frac{\tilde{\mathbf{h}}(t+\rho) - \tilde{\mathbf{h}}(t)}{\rho} \right\|_{L^r(G)} = c \left\| \frac{1}{\rho} \int_t^{t+\rho} \mathbf{h}(s) \, \mathrm{d}s \right\|_{L^r(G)}.$$

Then integrating both sides of the above estimate over the interval $(0,T)$ $(0 < T < \infty)$ using Minkowski's inequality and Fubini's theorem gives

$$\left\| \frac{\tilde{p}_0(\cdot+\rho) - \tilde{p}_0}{\rho} \right\|_{L^1(0,T;L^r(G))} \leq c \, \|\mathbf{h}\|_{L^1(0,T+\rho;L^r(G))}, \tag{2.19}$$

where $c = \text{const} > 0$ depending only on r, n and G. This shows that $\tilde{p}_0 \in W^{1,1}_{\mathrm{loc}}([0,\infty); A^r(G))$. Thus, setting $p_0 := \partial_t \tilde{p}_0$ from (2.19) we deduce

$$\|p_0\|_{L^1(0,T;L^r(G))} \leq c \, \|\mathbf{h}\|_{L^1(0,T;L^r(G))}.$$

Moreover, on both sides of (2.18) passing to the limit $\rho \to 0$ using Riesz-Fischer's theorem one gets

$$\int_G \mathbf{h}(t) : \nabla^2 \phi \, \mathrm{d}x = \int_G p_0(t) \Delta \phi \, \mathrm{d}x \quad \forall \, \phi \in C_0^\infty(G)$$

for almost all $t \in (0,\infty)$. Then applying (2.5) (cf. Lemma 2.3) gives (2.12).

On the other hand, the identity (2.11) easily follows from (2.17) replacing $\boldsymbol{\varphi}$ by $\partial_t \boldsymbol{\varphi}$ therein and applying integration by parts. The uniqueness of p_0 and $\tilde{p}_h$ follows directly from (2.12) and (2.13). □

Next, by the aid of Theorem 2.6 we will establish optimal estimates for the special case $G = D_R$, where

$$D_R := \{x \in \mathbb{R}^n \mid R < |x| < 16R\}.$$

In the proof of the main result we will make extensive use of the following

Corollary 2.7. *Let $0 < R < \infty$. Let $\mathbf{u} \in C_w([0,\infty); L^2(D_R)^n)$ with $\nabla \cdot \mathbf{u} = 0$ (in sense of distributions), such that*

$$-\int_0^\infty \int_{D_R} \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_0^\infty \int_{D_R} (\mathbf{h}_1 + \mathbf{h}_2) : \nabla \boldsymbol{\varphi} - \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.20}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(D_R \times (0,\infty))^n$ with $\nabla \cdot \boldsymbol{\varphi} = 0$, where

$$\mathbf{h}_1 \in L^{\frac{n+2}{n}}(D_R \times (0,\infty))^{n^2},$$
$$\mathbf{h}_2 \in L^2(D_R \times (0,\infty))^{n^2},$$
$$\mathbf{f} \in L^1(0,\infty; L^2(D_R)^n)$$

are given functions. Then there exist unique functions

$$
\begin{aligned}
&p_1 \in L^{\frac{n+2}{n}}(0,\infty; A^{\frac{n+2}{n}}(D_R)),\\
&p_2 \in L^2(0,\infty; A^2(D_R)),\\
&p_3 \in L^1(0,\infty; A^{2^*}(D_R)),\\
&\tilde p_h \in C_w([0,\infty); \dot B^{\frac{n+2}{n}}(D_R)),
\end{aligned}
$$

such that

$$
\begin{aligned}
-\int_0^\infty \int_{D_R} \mathbf{u}\cdot \partial_t \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{D_R} (\mathbf{h}_1+\mathbf{h}_2) : \nabla \boldsymbol{\varphi} - \mathbf{f}\cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t&\\
= \int_0^\infty \int_{D_R} (p_1+p_2+p_3)\nabla\cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{D_R} \nabla \tilde p_h \cdot \partial_t \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t& \qquad (2.21)
\end{aligned}
$$

for all $\boldsymbol{\varphi} \in C_0^\infty(D_R\times(0,\infty))^n$ and

$$
\begin{aligned}
\int_{D_R} p_1(t)\Delta\phi\,\mathrm{d}x &= \int_{D_R} \mathbf{h}_1(t) : \nabla^2\phi\,\mathrm{d}x, && (2.22)\\
\int_{D_R} p_2(t)\Delta\phi\,\mathrm{d}x &= \int_{D_R} \mathbf{h}_2(t) : \nabla^2\phi\,\mathrm{d}x, && (2.23)\\
\int_{D_R} p_3(t)\Delta\phi\,\mathrm{d}x &= \int_{D_R} \mathbf{f}(t) \cdot \nabla\phi\,\mathrm{d}x \quad \forall\,\phi\in C_0^\infty(D_R) && (2.24)
\end{aligned}
$$

for almost all $t\in(0,\infty)$. In addition, we have the a-priori estimates

$$
\begin{aligned}
\|p_1(t)\|_{L^{\frac{n+2}{n}}(D_R)} &\le c\,\|\mathbf{h}_1(t)\|_{L^{\frac{n+2}{n}}(D_R)}, && (2.25)\\
\|p_2(t)\|_{L^2(D_R)} &\le c\,\|\mathbf{h}_2(t)\|_{L^2(D_R)}, && (2.26)\\
\|p_3(t)\|_{L^2(D_R)} &\le c\,R\|\mathbf{f}(t)\|_{L^2(D_R)} && (2.27)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde p_h(t)\|^2_{L^{\frac{n+2}{n}}(D_R)} \le{}& c\,R^{\frac{n^2+4}{n+2}}\|\mathbf{u}(t)-\mathbf{u}(0)\|^2_{L^2(D_R)} + c\|\mathbf{h}_1\|^2_{L^{\frac{n+2}{n}}(D_R\times(0,\infty))}\\
&+ cR^{n\frac{n-2}{n+2}}\|\mathbf{h}_2\|^2_{L^2(D_R\times(0,\infty))} + c\,R^{\frac{n^2+4}{n+2}}\left\|\int_0^t \mathbf{f}(s)\,\mathrm{d}s\right\|_{L^2(D_R)} \qquad (2.28)
\end{aligned}
$$

for almost all $t\in(0,\infty)$, where $c=\mathrm{const}>0$ depending only on n.

Proof. First, let us prove the assertion for $R=1$. To begin with we shall write $\mathbf{f} = -\nabla\cdot\mathbf{h}_3$ for an appropriate $\mathbf{h}_3 : D_1 \to \mathbb{R}^{n^2}$. For this purpose with help of [6] (Th. III. 3.4) we introduce a linear operator $\mathcal{B} : L^2(D_1) \to W^{1,2}(D_1)^n$ fulfilling the following properties

(i) $\nabla\cdot\mathcal{B}g = g - g_{D_1} \quad \forall\, g\in L^2(D_1)$;

(ii) $\|\mathcal{B}g\|_{W^{1,2}(D_1)} \le c\,\|g\|_{L^2(D_1)} \quad \forall\, g\in L^2(D_1)$.

Then we set

$$\underline{\mathbf{h}}_3(t) := \mathcal{B}\mathbf{f}(t) + \frac{1}{n}(\mathbf{f}(t))_{D_1} \otimes x, \quad t \in (0,\infty).$$

Clearly, $\underline{\mathbf{h}}_3 \in L^1(0,\infty; W^{1,2}(D_1))$. Next, defining

$$\mathbf{h}_3 := \underline{\mathbf{h}}_3 - (\underline{\mathbf{h}}_3)_{D_1} \quad \text{a.e. in} \quad D_1 \times (0,\infty)$$

it results

$$\nabla \cdot \mathbf{h}_3 = \mathbf{f} \quad \text{a.e. in} \quad D_1 \times (0,\infty). \tag{2.29}$$

In addition, using Poincaré's inequality and Hölder's inequality combined with the property (ii) of $\mathcal{B}$ one gets

$$\|\mathbf{h}_3(t)\|_{L^{2^*}(D_1)} \le c\,\|\nabla \underline{\mathbf{h}}_3(t)\|_{L^2(D_1)} \le c\,\|\mathbf{f}(t)\|_{L^2(D_1)} \tag{2.30}$$

for almost all $t \in (0,\infty)$.

Now, set $r_1 = \dfrac{n+2}{n}$, $r_2 = 2$ and $r_3 = 2^*$. Let $k \in \{1,2,3\}$. According to Lemma 2.4 for almost every $t \in (0,\infty)$ there exist unique functions

$$p_k(t) \in A^{r_k}(D_1),$$

such that

$$\int_{D_1} p_k(t)\Delta\phi \,\mathrm{d}x = \int_{D_1} \mathbf{h}_k(t) : \nabla^2\phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(D_1). \tag{2.31}$$

We claim that $t \mapsto p_k(t)$ is Bochner measurable. Indeed, letting $v \in L^{r_k'}(D_1)$ be a arbitrarily chosen by Corollary 2.5 there exist unique functions $\phi \in W_0^{2,r_k'}(D_1)$ and $p \in B^{r_k'}(D_1)$ such that $v = \Delta\phi + p$. Thus from (2.31) it follows that

$$\int_{D_1} p_k(t) v \,\mathrm{d}x = \int_{D_1} p_k(t)\Delta\phi \,\mathrm{d}x = \int_{D_1} \mathbf{h}_k(t) : \nabla^2\phi \,\mathrm{d}x. \tag{2.32}$$

Since $t \mapsto \mathbf{h}_k(t)$ is Bochner measurable the function on the left of (2.32) is Lebesgue measurable. Thus, by means of Petti's theorem $t \mapsto p_k(t)$ is Bochner measurable.

Next, let $p_0 \in L^1_{\mathrm{loc}}([0,\infty); A^{\frac{n+2}{n}}(D_1))$ and $\tilde{p}_h \in C_w([0,\infty); \dot{B}^{\frac{n+2}{n}}(D_1))$ denote the function obtained by Theorem 2.6. By an inspection of the proof therein one finds

$$\int_{D_1} p_0(t)\Delta\phi \,\mathrm{d}x = \int_{D_1} (\mathbf{h}_1(t) + \mathbf{h}_2(t) + \mathbf{h}_3(t)) : \nabla^2\phi \,\mathrm{d}x \quad \forall\, \phi \in C_0^\infty(D_1)$$

for almost all $t \in (0,\infty)$. On the other hand, summing up (2.31) from $k = 1$ to 3 one sees that for almost all $t \in (0,\infty)$ the function $p_1(t) + p_2(t) + p_3(t)$ satisfies the above identity too. This implies

$$p_0 = p_1 + p_2 + p_3 \quad \text{a.e. in} \quad D_1 \times (0,\infty)$$

which proves (2.21), (2.22), (2.23) and (2.24).

The a-priori estimate (2.25) ((2.26) resp.) immediately follows from (2.22) ((2.23) resp.) using Lemma 2.3, while (2.27) follows from (2.31) (with $k = 3$) using Lemma 2.3 along with (2.30) and Hölder's inequality. On the other hand, the a-priori estimate (2.28) can verified easily from (2.13) applying Minkowski's inequality.

Next, let $0 < R < \infty$ be arbitrarily chosen. Defining

$$\begin{aligned}
\underline{\mathbf{u}}(y,t) &:= R\mathbf{u}(Ry,t),\\
\underline{\mathbf{h}}_1(y,t) &:= \mathbf{h}_1(Ry,t),\\
\underline{\mathbf{h}}_2(y,t) &:= \mathbf{h}_2(Ry,t),\\
\underline{\mathbf{f}}(y,t) &:= R\mathbf{f}(Ry,t), \quad (y,t) \in D_1 \times (0,\infty)
\end{aligned}$$

using the transformation formula of the Lebesgue integral the identity (2.20) turns into

$$-\int_0^\infty \int_{D_1} \underline{\mathbf{u}} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{D_1} (\underline{\mathbf{h}}_1 + \underline{\mathbf{h}}_2) : \nabla \boldsymbol{\varphi} - \underline{\mathbf{f}} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = 0 \qquad (2.33)$$

for all $\boldsymbol{\varphi} \in C_0^\infty(D_1 \times (0,\infty))^n$ with $\nabla \cdot \boldsymbol{\varphi} = 0$. From the first part of the proof we get functions

$$\begin{aligned}
&\underline{p}_1 \in L^{\frac{n+2}{n}}(0,\infty; A^{\frac{n+2}{n}}(D_1)),\\
&\underline{p}_2 \in L^2(0,\infty; A^2(D_1)),\\
&\underline{p}_3 \in L^1(0,\infty; A^{2^*}(D_1)),\\
&\underline{\tilde{p}}_h \in C_w([0,\infty); \dot{B}^{\frac{n+2}{n}}(D_1)),
\end{aligned}$$

satisfying the identities (2.21), (2.22), (2.23) and (2.24) and the a priori estimates (2.25), (2.26) , (2.27) and (2.28) (with $R = 1$). Finally setting

$$\begin{aligned}
p_k(x,t) &:= \underline{p}_k(R^{-1}x,t) \quad (k = 1,2,3)\\
\tilde{p}_h(x,t) &:= \underline{p}_h(R^{-1}x,t), \quad (x,t) := D_R \times (0,\infty)
\end{aligned}$$

once more applying the transformation formula of the Lebesgue integral shows that these functions fulfil (2.21), (2.22), (2.23) and (2.24) together with the a priori estimates (2.25), (2.26) , (2.27) and (2.28). This completes the proof of the corollary. □

3. Proof of the Main Theorem

We divide the proof into four steps. At first we provide a sequence of weak solutions $\{\mathbf{u}_m\}$ to the corresponding approximate system, which converge to a weak solution $\mathbf{u}$ to (1.1)-(1.4). Secondly, for almost all $0 < T < \infty$ we prove the strong convergence of $\mathbf{u}_m(T)$ to $\mathbf{u}(T)$ in $L^2(\Omega)^n$. Thirdly, based on the monotonicity condition of $\mathbf{S}$ and the L^2-convergence we have achieved in step 2 we deduce that $\mathbf{u}$ fulfils the strong energy inequality (1.7). Finally, in the fourth step we verify the decay condition (1.9).

Existence of a weak solution to the approximate system

Let $\Phi \in C^\infty([0,\infty))$ be a non-increasing function, such that $\Phi \equiv 1$ on $[0,1]$, $\Phi \equiv 0$ in $[2,\infty)$ and $0 \le -\Phi' \le 2$. For $\varepsilon > 0$ we set

$$\Phi_\varepsilon(\tau) := \Phi(\varepsilon\tau), \quad \tau \in [0,\infty).$$

Let $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{f} \in L^1(0,\infty; L^2(\Omega)^n)$ and $\mathbf{g} \in L^2(Q)^{n^2}$. Using the method developed in [26] one gets a unique weak solution $\mathbf{u}_\varepsilon \in L^2(0,\infty;\mathbf{V}) \cap BC(0,\infty;\mathbf{H})$ to the system

$$\nabla \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in} \quad Q, \tag{3.1}$$

$$\partial_t \mathbf{u}_\varepsilon + \nabla \cdot (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \Phi_\varepsilon(|\mathbf{u}_\varepsilon|) - \mathbf{S}(x,t,D(\mathbf{u}_\varepsilon))) \tag{3.2}$$

$$= -\nabla p_\varepsilon + \mathbf{f} - \nabla \cdot \mathbf{g} \quad \text{in} \quad Q, \tag{3.3}$$

$$\mathbf{u}_\varepsilon\big|_{\partial\Omega\times(0,T)} = 0, \tag{3.4}$$

$$\mathbf{u}_\varepsilon(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{3.5}$$

i.e. the following identity

$$\int_Q -\mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + (\mathbf{S}(x,t;D(\mathbf{u}_\varepsilon) - \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \Phi_\varepsilon(|\mathbf{u}_\varepsilon|) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_Q \mathbf{f} \cdot \boldsymbol{\varphi} + \mathbf{g} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0)\,\mathrm{d}x \tag{3.6}$$

holds for every $\boldsymbol{\varphi} \in C^\infty(Q)^n$ with $\nabla \cdot \boldsymbol{\varphi} = 0$ and $\mathrm{supp}(\boldsymbol{\varphi}) \subset\subset \Omega \times [0,\infty)$. Furthermore, using integration by parts from (3.6) it follows

$$\frac{1}{2}\|\mathbf{u}_\varepsilon(t)\|_H^2 + \int_s^t \int_\Omega \mathbf{S}(x,\tau,D(\mathbf{u}_\varepsilon)) : D(\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\tau$$

$$= \int_s^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_\varepsilon + \mathbf{g} : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\|\mathbf{u}_\varepsilon(s)\|_H^2 \tag{3.7}$$

for all $0 \le s < t < \infty$.

In what follows let K denote a positive constant, whose numerical value may vary from line to line and its dependence will be specified if necessary. From (3.7) observing (III) together with Korn's inequality and Young's inequality one easily gets

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,\infty;H)} + \|\nabla \mathbf{u}_\varepsilon\|_{L^2(Q)} \le K. \tag{3.8}$$

Furthermore observing (II) using (3.8) yields

$$\|\mathbf{S}(\cdot, D(\mathbf{u}_\varepsilon)\|_{L^2(Q)} \le K. \tag{3.9}$$

By virtue of Sobolev's imbedding theorem using multiplicative inequalities from (3.8) one deduces

$$\|\mathbf{u}_\varepsilon\|_{L^r(0,\infty;L^\rho(\Omega))} \le K \tag{3.10}$$

for all $\rho \in [2, 2^*]$ and all $1 \le r \le \infty$ such that

$$\frac{2}{r} + \frac{n}{\rho} = \frac{n}{2}. \tag{3.11}$$

By means of reflexivity of the spaces under consideration there exist a sequence of positive numbers $\{\varepsilon_m\}$ with $\varepsilon_m \to 0$ as $m \to \infty$ and functions $\mathbf{u} \in L^2(0, \infty; \mathbf{V}) \cap L^\infty(0, \infty; \mathbf{H})$ and $\tilde{\mathbf{S}} \in L^2(Q)^{n^2}$ such that

$$\mathbf{u}_{\varepsilon_m} \to \mathbf{u} \quad \text{weakly in} \quad L^{2\frac{n+2}{n}}(Q)^n, \tag{3.12}$$

$$\nabla \mathbf{u}_{\varepsilon_m} \to \nabla \mathbf{u} \quad \text{weakly in} \quad L^2(Q)^{n^2}, \tag{3.13}$$

$$\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) \to \tilde{\mathbf{S}} \quad \text{weakly in} \quad L^2(Q)^{n^2} \quad \text{as} \quad m \to \infty. \tag{3.14}$$

As it has been proved in [26] we have $\mathbf{u} \in C_w([0, \infty); \mathbf{H}) \cap L^2(0, \infty; \mathbf{V})$ and

$$\tilde{\mathbf{S}} = \mathbf{S}(x, t, D(\mathbf{u})) \quad \text{a.e. in} \quad Q.$$

Thus, $\mathbf{u}$ is a weak solution to (1.1)-(1.4). Simultaneously, using Lions' compactness argument one has for every $G \subset\subset \bar{\Omega}$ and $0 < T < \infty$

$$\mathbf{u}_{\varepsilon_m}\big|_{G\times(0,T)} \to \mathbf{u}\big|_{G\times(0,T)} \ \text{strongly in} \quad L^2(G \times (0, T))^n \quad \text{as} \quad m \to \infty. \tag{3.15}$$

In addition, using Riesz-Fischer's theorem, from (3.15) (eventually passing to a subsequence) one may assume that for almost all $T \in (0, \infty)$ and for every $G \subset\subset \bar{\Omega}$ there holds

$$\mathbf{u}_{\varepsilon_m}(T)\big|_G \to \mathbf{u}(T)\big|_G \quad \text{in} \quad L^2(G)^n \quad \text{as} \quad m \to \infty. \tag{3.16}$$

Strong convergence of $\mathbf{u}_{\varepsilon_m}(T)$ in $L^2(\Omega)$

To begin with, let us define the following subsets of $\mathbb{R}^n$

$$G_\ell := \{x \in \mathbb{R}^n \mid 2^{\ell-1} < |x| < 2^{\ell+3}\},$$
$$\widehat{G}_\ell := \{x \in \mathbb{R}^n \mid 2^{\ell} < |x| < 2^{\ell+2}\}, \quad \ell \in \mathbb{N}.$$

Let $N \in \mathbb{N}$, such that $G_\ell \subset\subset \Omega$ for all $\ell \ge N$. Obviously, the identity (3.6) implies

$$- \int_0^\infty \int_{G_\ell} \mathbf{u}_{\varepsilon_m} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{G_\ell} (\mathbf{h}_{1,m,\ell} + \mathbf{h}_{2,m,\ell}) : \nabla \boldsymbol{\varphi} - \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.17}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(G_\ell \times (0, \infty))^n$ with $\nabla \cdot \boldsymbol{\varphi} = 0$, where

$$\mathbf{h}_{1,m,\ell} := -\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m} \Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|),$$
$$\mathbf{h}_{2,m,\ell} := \mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) - \mathbf{g} \quad \text{a.e. in} \quad G_\ell \times (0, \infty).$$

Hence, we are in a position to apply Corollary 2.7 with $\mathbf{h}_1 = \mathbf{h}_{1,m,\ell}$ and $\mathbf{h}_2 = \mathbf{h}_{2,m,\ell}$, which provides unique functions

$$
\begin{aligned}
p_{1,m,\ell} &\in L^{\frac{n+2}{n}}(0,\infty; A^{\frac{n+2}{n}}(G_\ell)),\\
p_{2,m,\ell} &\in L^2(0,\infty; A^2(G_\ell)),\\
p_{3,\ell} &\in L^1(0,\infty; A^{2^*}(G_\ell)),\\
\tilde{p}_{h,m,\ell} &\in C_w(0,\infty; \dot{B}^{\frac{n+2}{n}}(G_\ell)) \quad \text{with} \quad \tilde{p}_{h,m,\ell}(0) = 0,
\end{aligned}
$$

such that

$$
\begin{aligned}
-\int_0^\infty \int_{G_\ell} &\mathbf{u}_{\varepsilon_m} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\infty \int_{G_\ell} (\mathbf{h}_{1,m,\ell} + \mathbf{h}_{2,m,\ell}) : \nabla \boldsymbol{\varphi} - \mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^\infty \int_{G_\ell} (p_{1,m,\ell} + p_{2,m,\ell} + p_{3,\ell}) \nabla\cdot\boldsymbol{\varphi} + \nabla \tilde{p}_{h,m,\ell} \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{3.18}
$$

for all $\boldsymbol{\varphi} \in C_0^\infty(G_\ell \times (0,\infty))^n$ and

$$
\int_{G_\ell} p_{1,m,\ell}(t) \Delta\phi \,\mathrm{d}x = \int_{G_\ell} \mathbf{h}_{1,m,\ell}(t) : \nabla^2 \phi \,\mathrm{d}x, \tag{3.19}
$$

$$
\int_{G_\ell} p_{2,m,\ell}(t) \Delta\phi \,\mathrm{d}x = \int_{G_\ell} \mathbf{h}_{2,m,\ell}(t) : \nabla^2 \phi \,\mathrm{d}x, \tag{3.20}
$$

$$
\int_{G_\ell} p_{3,\ell}(t) \Delta\phi \,\mathrm{d}x = \int_{G_\ell} \mathbf{f}(t) \cdot \nabla \phi \,\mathrm{d}x, \quad \forall\, \phi \in C_0^\infty(G_\ell) \tag{3.21}
$$

for almost all $t \in (0,\infty)$.

It is readily seen that (2.25), (2.26), (2.27) and (2.28) imply

$$
\|p_{1,m,\ell}(t)\|_{L^{\frac{n+2}{n}}(G_\ell)} \le K \|\mathbf{u}_{\varepsilon_m}(t)\|^2_{L^{2\frac{n+2}{n}}(G_\ell)}, \tag{3.22}
$$

$$
\|p_{2,m,\ell}(t)\|_{L^2(G_\ell)} \le K \|\mathbf{S}(\cdot, t, D(\mathbf{u}_{\varepsilon_m}(t))) - \mathbf{g}(t)\|_{L^2(G_\ell)}, \tag{3.23}
$$

$$
\|p_{3,\ell}(t)\|_{L^2(G_\ell)} \le K 2^\ell \|\mathbf{f}(t)\|_{L^2(G_\ell)}, \tag{3.24}
$$

and

$$
\begin{aligned}
\|\tilde{p}_{h,m,\ell}(t)\|^2_{L^{\frac{n+2}{n}}(G_\ell)} \le{}& K 2^{\ell \frac{n^2+4}{n+2}} \|\mathbf{u}_{\varepsilon_m}(t) - \mathbf{u}_0\|^2_{L^2(G_\ell)}\\
&+ K \|\mathbf{h}_{1,m,\ell}\|^2_{L^{\frac{n+2}{n}}(G_\ell \times (0,\infty))}\\
&+ K 2^{\ell n \frac{n-2}{n+2}} \|\mathbf{h}_{2,m,\ell}\|^2_{L^2(G_\ell \times (0,\infty))}\\
&+ K 2^{\ell \frac{n^2+4}{n+2}} \left\| \int_0^t \mathbf{f}(s)\,\mathrm{d}s \right\|^2_{L^2(G_\ell)}
\end{aligned}
\tag{3.25}
$$

for almost all $t \in (0,\infty)$ [7]. In particular, observing (II) using (3.8) and (3.10) from (3.22), (3.23) and (3.24) one deduces

$$\|p_{1,m,\ell}(t)\|_{L^{\frac{n+2}{n}}(G_\ell)} + \|p_{2,m,\ell}(t)\|_{L^2(G_\ell)} \le K, \tag{3.26}$$

$$\|p_{3,\ell}(t)\|_{L^2(G_\ell)} \le K2^\ell \tag{3.27}$$

for almost all $t \in (0,\infty)$, while from (3.25) one infers

$$\begin{aligned}&\|\tilde{p}_{h,m,\ell}(t)\|^2_{L^{\frac{n+2}{n}}(G_\ell)}\\ &\qquad \le K2^{\ell\frac{n^2+4}{n+2}}\Big(\|\mathbf{u}_{\varepsilon_m}(t)-\mathbf{u}_0\|^2_{L^2(G_\ell)} + \|\tilde{\mathbf{f}}(t)\|^2_{L^2(G_\ell)} + 2^{-2\ell}\Big)\end{aligned} \tag{3.28}$$

for all $t \in (0,\infty)$, where

$$\tilde{\mathbf{f}}(t) := \int_0^t \mathbf{f}(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

We wish to mention that in all these estimates the constant K depends neither on m nor on ℓ and also not on $t \in (0,\infty)$.

Next setting

$$\mathbf{v}_{m,\ell} := \mathbf{u}_{\varepsilon_m} + \nabla\tilde{p}_{h,m,\ell} \quad \text{in} \quad G_\ell \times (0,\infty),$$

from (3.18) we deduce

$$\begin{aligned}&\mathbf{v}'_{m,\ell} + \nabla\cdot(\mathbf{u}_{\varepsilon_m}\otimes\mathbf{u}_{\varepsilon_m}\varPhi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|) - \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) + \mathbf{g})\\ &\qquad = \mathbf{f} - \nabla(p_{1,m,\ell}+p_{2,m,\ell}+p_{3,\ell}) \quad \text{in} \quad G_\ell \times (0,\infty).\end{aligned} \tag{3.29}$$

Here $\mathbf{v}'_{m,\ell} \in L^2(0,\infty;W^{1,2}(G_\ell)^n) + L^1(0,T;\mathbf{X}(G_\ell)^*)$ stands for the distributive time derivative of $\mathbf{v}$, where

$$\mathbf{X}(G_\ell) := \{\boldsymbol{\eta} \in L^2(G_\ell)^n \,|\, \nabla\cdot\boldsymbol{\eta} \in L^2(G_\ell)\}\,^{8}.$$

To be more precise, for every $T \in (0,\infty)$ we have the following identity

---

[7] Recall that $G_\ell = D_{2^{\ell-1}}$ (cf. Section 2).

[8] Here $\nabla\cdot\boldsymbol{\eta} \in L^2(G_\ell)$ means there exists $w \in L^2(G_\ell)$, such that

$$\int_{G_\ell} w\phi\,\mathrm{d}x = -\int_{G_\ell} \boldsymbol{\eta}\cdot\nabla\phi\,\mathrm{d}x \quad \forall\,\phi \in C_0^\infty(G_\ell).$$

$$
\begin{aligned}
\int_0^T \langle \mathbf{v}'_{m,\ell}(t), \boldsymbol{\varphi}(t)\rangle \,\mathrm{d}s \,+\, &\int_0^T \int_{G_\ell} \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{G_\ell} (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m} \Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|) + \mathbf{g}) : D(\boldsymbol{\varphi}) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{G_\ell} (p_{1,m,\ell} + p_{2,m,\ell} + p_{3,\ell}) \nabla \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{G_\ell} \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.30}
$$

for all $\boldsymbol{\varphi} \in L^2(0,\infty; W_0^{1,\,2}(G_\ell)^n) \cap L^\infty(0,\infty; \mathbf{X}(G_\ell))$.

Clearly, for every $\psi \in C_0^\infty(G_\ell)$ the function $\mathbf{v}_{m,\ell}\psi$ appears to be an appropriate test function in (3.30). However having in mind a-priori estimates for $\mathbf{v}$ on the complement of a large ball we are going to define an appropriate partition of unity subordinated to the covering $\{G_\ell\}_{\ell \geq N}$. That is a family of smooth functions $\{\psi_\ell\}_{\ell \geq N}$, such that

(1) $\operatorname{supp}(\psi_\ell) \subset \widehat{G}_\ell$,

(2) $\sum\limits_{\ell=N}^{\infty} \psi_\ell \equiv 1 \quad \text{in} \quad \{x \in \mathbb{R}^3 \,|\, |x| > 2^{N+1}\}$,

(3) $|\nabla \psi_\ell| \,\leq\, K 2^{-\ell}$.

Obviously, as $\operatorname{supp}(\psi_\ell) \cap \operatorname{supp}(\psi_{\ell+2}) = \emptyset$ for all $\ell \geq N$ besides (2) there holds

(2′) $\sum\limits_{\ell=k}^{\nu} \psi_\ell \equiv 1 \quad \text{in} \quad \{x \in \mathbb{R}^3 \,|\, 2^{k+1} < |x| < 2^{\nu+1}\} \quad \forall\, N \leq k < \nu \leq \infty.$

Now, choose $T \in (0,\infty)$ such that (3.16) is valid. Then into (3.30) inserting $\boldsymbol{\varphi} = \mathbf{v}_{m,\ell}\psi_\ell$ using integration by parts yields

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\mathbf{v}_{m,\ell}(T)|^2 \psi_\ell \,\mathrm{d}x \,\leq\, &\frac{1}{2}\int_\Omega |\mathbf{u}_0|^2 \psi_\ell \,\mathrm{d}x \\
&- \int_0^T \int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : D(\mathbf{v}_{m,\ell}\psi_\ell) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m} \Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \nabla(\mathbf{v}_{m,\ell}\psi_\ell) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega (p_{1,m,\ell} + p_{2,m,\ell} + p_{3,\ell}) \mathbf{v}_{m,\ell} \cdot \nabla \psi_\ell \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{v}_{m,\ell}\psi_\ell \,\mathrm{d}x\,\mathrm{d}t \,+\, \int_0^T \int_\Omega \mathbf{g} : \nabla(\mathbf{v}_{m,\ell}\psi_\ell) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.31}
$$

Observing (III) using the product rule one estimates

$$\begin{aligned}
&\int_0^T\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : D(\mathbf{v}_{m,\ell}\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\ge -\int_0^T\int_\Omega \kappa_2\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : (\mathbf{v}_{m,\ell}\otimes\nabla\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : \nabla^2\tilde{p}_{h,m,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Thus, from (3.31) it follows that

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\mathbf{v}_{m,\ell}(T)|^2\psi_\ell\,\mathrm{d}x \;\le\; &\frac{1}{2}\int_\Omega |\mathbf{u}_0|^2\psi_\ell\,\mathrm{d}x.\\
&+\int_0^T\int_\Omega \kappa_2\psi_\ell\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{v}_{m,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\\
&+\mathbf{I}_{m,\ell}+\mathbf{II}_{m,\ell}+\mathbf{III}_{m,\ell}+\mathbf{IV}_{m,\ell}+\mathbf{V}_{m,\ell},
\end{aligned}\tag{3.32}$$

where

$$\begin{aligned}
\mathbf{I}_{m,\ell} &:= -\int_0^T\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : (\mathbf{v}_{m,\ell}\otimes\nabla\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{II}_{m,\ell} &:= -\int_0^T\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}_{\varepsilon_m})) : \nabla^2\tilde{p}_{h,m,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{III}_{m,\ell} &:= \int_0^T\int_\Omega \mathbf{g} : \nabla(\mathbf{v}_{m,\ell}\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{IV}_{m,\ell} &:= \int_0^T\int_\Omega (\mathbf{u}_{\varepsilon_m}\otimes\mathbf{u}_{\varepsilon_m}\varPhi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \nabla(\mathbf{v}_{m,\ell}\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{V}_{m,\ell} &:= \int_0^T\int_\Omega (p_{1,m,\ell}+p_{2,m,\ell}+p_{3,\ell})\mathbf{v}_{m,\ell}\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$(m,\ell\in\mathbb{N};\ell\ge N)$.

In order to estimate integrals $\mathbf{I}_{m,\ell}$-$\mathbf{V}_{m,\ell}$ we first state the following Lemma, which for reader's convenience will be proved in the appendix of this paper.

Lemma 3.1. *For every $1<r<\infty, 1\le q\le\infty$ and $d\in\mathbb{N}\cup\{0\}$ there exists a positive constant γ_0, which depends only on r,q,d and n such that*

$$\|\nabla^d w\|_{L^q(\widehat{G}_\ell)} \;\le\; \gamma_0\{2^\ell\}^{-d-n/r+n/q}\|w\|_{L^r(G_\ell)} \quad \forall\, w\in B^r(G_\ell).\tag{3.33}$$

Proof of Theorem 1.3 continued. Combining (3.33) (with $r = \dfrac{n+2}{n}$) and (3.28) implies

$$\|\nabla^d \tilde{p}_{h,m,\ell}(t)\|_{L^q(\widehat{G}_\ell)} \le K\{2^\ell\}^{-d+1-n/2+n/q} \times \\ \times \left(\|\mathbf{u}_{\varepsilon_m}(t) - \mathbf{u}_0\|_{L^2(G_\ell)} + \|\tilde{\mathbf{f}}(t)\|_{L^2(G_\ell)} + 2^{-\ell}\right) \tag{3.34}$$

for almost all $t \in (0, \infty)$. Thus, integrating both sides of the last inequality over $(0,T)$ taking into account (3.8) applying Minkowski's inequality shows that

$$\|\nabla^d \tilde{p}_{h,m,\ell}\|_{L^\infty(0,\infty;L^q(\widehat{G}_\ell))} \le K\{2^\ell\}^{-d+1-n/2+n/q}. \tag{3.35}$$

(i) Using Hölder's inequality together with the property (3) of the family $\{\psi_\ell\}_{\ell \ge N}$ one finds

$$|\mathbf{I}_{m,\ell}| \le K2^{-\ell}\sqrt{T}\|\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m}))\|_{L^2(Q)}\|\mathbf{u}_{\varepsilon_m}\|_{L^\infty(0,\infty;L^2(\Omega))} \\ + K2^{-\ell}\sqrt{T}\|\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m}))\|_{L^2(Q)}\|\nabla \tilde{p}_{h,m,\ell}\|_{L^\infty(0,T;L^2(\widehat{G}_\ell))}.$$

Making use of the a-priori estimates (3.9), (3.10) and applying (3.35) (with $d = 1$ and $q = 2$) yields

$$|\mathbf{I}_{m,\ell}| \le K2^{-\ell}\sqrt{T}.$$

(ii) Similarly, estimating

$$|\mathbf{II}_{m,\ell}| \le \sqrt{T}\|\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m}))\|_{L^2(Q)}\|\nabla^2 \tilde{p}_{h,m,\ell}\|_{L^\infty(0,T;L^2(\widehat{G}_\ell))}$$

using (3.35) (with $d = 2$ and $q = 2$) gives

$$|\mathbf{II}_{m,\ell}| \le K2^{-\ell}\sqrt{T}.$$

(iii) Arguing as in (i) and (ii) one gets

$$|\mathbf{III}_{m,\ell}| \le K2^{-\ell}\sqrt{T} + \int_0^T\int_\Omega |\mathbf{g}|\,|\nabla \mathbf{u}_{\varepsilon_m}|\psi_\ell \,\mathrm{d}x\,\mathrm{d}t.$$

(iv) With help of the product rule one easily calculates

$$\begin{aligned} \mathbf{IV}_{m,\ell} &= \int_0^T\int_\Omega (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m}\Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \nabla \mathbf{u}_{\varepsilon_m}\psi_\ell \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m}\Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \nabla^2 \tilde{p}_{h,m,\ell}\psi_\ell \,\mathrm{d}x\,\mathrm{d}t \qquad (3.36) \\ &\quad + \int_0^T\int_\Omega (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m}\Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \mathbf{v}_{m,\ell} \otimes \nabla\psi_\ell \,\mathrm{d}x\,\mathrm{d}t \\ &= \mathbf{IV}^{(1)}_{m,\ell} + \mathbf{IV}^{(2)}_{m,\ell} + \mathbf{IV}^{(3)}_{m,\ell}. \end{aligned}$$

Firstly, applying integration by parts one finds

$$\mathbf{IV}^{(1)}_{m,\ell} = -\int_0^T\int_\Omega (\mathbf{u}_{\varepsilon_m} \otimes \mathbf{u}_{\varepsilon_m}\Phi_{\varepsilon_m}(|\mathbf{u}_{\varepsilon_m}|)) : \mathbf{u}_{\varepsilon_m} \otimes \nabla\psi_\ell \,\mathrm{d}x\,\mathrm{d}t.$$

Then, observing property (3) of the covering $\{\psi_\ell\}_{\ell\geq N}$ applying Hölder's inequality along with (3.10) gives

$$|\mathbf{IV}^{(1)}_{m,\ell}| \leq K2^{-\ell}\|\mathbf{u}_{\varepsilon_m}\|^3_{L^3(Q)} \leq K2^{-\ell}T^{1-n/4}.$$

Secondly, by the aid of Hölder's inequality one gets

$$|\mathbf{IV}^{(2)}_{m,\ell}| \leq KT^{1/3}\|\mathbf{u}_{\varepsilon_m}\|^2_{L^3(Q)}\|\nabla^2\tilde{p}_{h,m,\ell}\|_{L^\infty(0,T;L^3(\widehat{G}_\ell))}.$$

Applying (3.35) (with $d=2$ and $q=3$) using (3.10) implies

$$|\mathbf{IV}^{(2)}_{m,\ell}| \leq K2^{-\ell(1+n/6)}T^{1-n/6}.$$

Thirdly, using Hölder's inequality together with property (3) of the covering $\{\psi_\ell\}_{\ell\geq N}$ one estimates

$$|\mathbf{IV}^{(3)}_{m,\ell}| \leq K2^{-\ell}(\|\mathbf{u}_{\varepsilon_m}\|^3_{L^3(Q)} + T^{1/3}\|\mathbf{u}_{\varepsilon_m}\|^2_{L^3(Q)}\|\nabla\tilde{p}_{h,m,\ell}\|_{L^\infty(0,T;L^3(\widehat{G}_\ell))}).$$

Once more appealing to (3.35) (with $d=1$ and $q=3$) taking into account (3.10) gives

$$|\mathbf{IV}^{(3)}_{m,\ell}| \leq K(2^{-\ell}T^{1-n/4} + 2^{-\ell(1+n/6)}T^{1-n/6}).$$

Thus, inserting the above estimates of $\mathbf{IV}^{(1)}_{m,\ell}, \mathbf{IV}^{(2)}_{m,\ell}$ and $\mathbf{IV}^{(3)}_{m,\ell}$ into (3.36) taking into account $T^{1-n/4} \leq \sqrt{T+1}$ for $n=2,3,4$ shows that

$$|\mathbf{IV}_{m,\ell}| \leq K(2^{-\ell}\sqrt{T+1} + 2^{-\ell(1+n/6)}T^{1-n/6}).$$

(v) With help of Hölder's inequality one easily estimates

$$\begin{aligned}|\mathbf{V}_{m,\ell}| &\leq c\,2^{-\ell}\|p_{1,m,\ell}\|_{L^{\frac{n+2}{n}}(G_\ell\times(0,T))}\|\mathbf{v}_{m,\ell}\|_{L^{\frac{n+2}{2}}(\widehat{G}_\ell\times(0,T))}\\ &\quad + c\,2^{-\ell}\|p_{2,m,\ell}\|_{L^2(G_\ell\times(0,T))}\|\mathbf{v}_{m,\ell}\|_{L^2(\widehat{G}_\ell\times(0,T))}\\ &\quad + \int_0^T\int_\Omega p_{3,\ell}\mathbf{v}_m\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using (3.22), (3.23) and the a priori estimate (3.10) along with (3.35) gives

$$|\mathbf{V}_{m,\ell}| \leq K2^{-\ell}\sqrt{T+1} + \int_0^T\int_\Omega p_{3,\ell}\mathbf{v}_m\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t.$$

Let $k\in\mathbb{N}$ with $k\geq N$ be arbitrarily chosen. Inserting the estimates for $\mathbf{I}_{m,\ell} - \mathbf{V}_{m,\ell}$ into (3.32) summing up the resulting inequalities from $\ell=k$ to ∞

making use of the property $(2')$ of the partition of unity $\{\psi_\ell\}_{\ell\ge N}$ gives

$$
\begin{aligned}
&\sum_{\ell=k}^{\infty}\int_\Omega |\mathbf{v}_{m,\ell}(T)|^2\psi_\ell \,\mathrm{d}x\\
&\qquad\le K(2^{-k}\sqrt{T+1} + 2^{-k(1+n/6)}T^{1-n/6}) + \int_{\{|x|>2^k\}}|\mathbf{u}_0|^2\,\mathrm{d}x\\
&\qquad\quad \int_0^T\int_{\{|x|\ge 2^k\}}\kappa_2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\{|x|\ge 2^k\}}|\mathbf{g}|\,|\nabla\mathbf{u}_{\varepsilon_m}|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\qquad + \sum_{\ell=k}^{\infty}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{v}_{m,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\qquad\quad + \sum_{\ell=k}^{\infty}\int_0^T\int_\Omega p_{3,\ell}\mathbf{v}_{m,\ell}\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.37)
\end{aligned}
$$

1. *Estimation of the right hand side of* (3.37) *from above.* a) With help of Cauchy-Schwarz's inequality using (3.8) results

$$
\int_0^T\int_{\{|x|\ge 2^k\}}|\mathbf{g}|\,|\nabla\mathbf{u}_{\varepsilon_m}|\,\mathrm{d}x\,\mathrm{d}t \le K\|\mathbf{g}\|_{L^2(0,T;L^2(\mathbb{R}^n\setminus B_{2^k}))}. \qquad (3.38)
$$

b) Next, we claim that the two sums on the right of (3.37) are convergent. To prove this fact we proceed as follows. From (3.34) with $d=1, q=2$ it follows

$$
\int_{\widehat{G}_\ell}|\nabla\tilde{p}_{h,m,\ell}(t)|^2\,\mathrm{d}x \le K(\|\mathbf{u}_{\varepsilon_m}(t)-\mathbf{u}_0\|^2_{L^2(G_\ell)} + \|\tilde{\mathbf{f}}(t)\|^2_{L^2(G_\ell)} + 2^{-2\ell}) \qquad (3.39)
$$

for almost all $t\in(0,\infty)$. Taking the sum on both sides of (3.34) from $\ell=k$ to ∞ recalling that $G_\ell\cap G_{\ell+4}=\emptyset$ for every $\ell\ge N$ using Minkowski's inequality together with (3.8) yields

$$
\sum_{\ell=k}^{\infty}\int_{\widehat{G}_\ell}|\nabla\tilde{p}_{h,m,\ell}(t)|^2\,\mathrm{d}x \le K \qquad (3.40)
$$

for almost all $t\in(0,\infty)$, where $K=\text{const}>0$ depending neither on m nor on $t\in(0,\infty)$.

Now, define

$$
\Psi_k := \sum_{\ell=k}^{\infty}\psi_\ell \quad \text{in} \quad \mathbb{R}^n.
$$

Recalling the definition of $\mathbf{v}_{m,\ell}$ one calculates

$$
\begin{aligned}
&\sum_{\ell=k}^{\infty}\int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{v}_{m,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{u}_{\varepsilon_m}\Psi_k\,\mathrm{d}x\,\mathrm{d}t + \sum_{\ell=k}^{\infty}\int_0^T\int_\Omega \mathbf{f}\cdot\nabla\tilde{p}_{h,m,\ell}\,\psi_\ell\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.41)
\end{aligned}
$$

Firstly, by means of Cauchy-Schwarz's inequality one estimates

$$\left| \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{u}_{\varepsilon_m} \Psi_k \,\mathrm{d}x \,\mathrm{d}t \right| \;\leq\; K \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^k}))}. \tag{3.42}$$

Secondly, applying Cauchy-Schwarz's inequality for sequences in l_2 and recalling that $\operatorname{supp}(\psi_\ell) \cap \operatorname{supp}(\psi_{\ell+2}) = \emptyset$ for all $\ell \geq N$ using (3.40) yields

$$\begin{aligned}
&\sum_{\ell=k}^{\infty} \left| \int_\Omega \mathbf{f}(t) \cdot \nabla \tilde{p}_{h,m,\ell}(t)\, \psi_\ell \,\mathrm{d}x \right| \\
&\qquad \leq\; 2\|\mathbf{f}(t)\|_{L^2(\mathbb{R}^n \setminus B_{2^k})} \left(\sum_{\ell=k}^{\infty} \int_{\widehat{G}_\ell} |\nabla \tilde{p}_{h,m,\ell}(t)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \leq\; K \|\mathbf{f}(t)\|_{L^2(\mathbb{R}^n \setminus B_{2^k})}
\end{aligned}$$

for almost all $t \in (0,\infty)$. Integrating both sides of the last estimate over the interval $(0,T)$ with respect to t one arrives at

$$\left| \sum_{\ell=k}^{\infty} \int_0^T \int_\Omega \mathbf{f} \cdot \nabla \tilde{p}_{h,m,\ell}\, \psi_\ell \,\mathrm{d}x \,\mathrm{d}t \right| \;\leq\; K \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^k}))}. \tag{3.43}$$

Estimating right hand side of (3.41) by (3.42) and (3.43) gives

$$\left| \sum_{\ell=k}^{\infty} \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{v}_{m,\ell} \psi_\ell \,\mathrm{d}x \,\mathrm{d}t \right| \;\leq\; K \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^k}))}. \tag{3.44}$$

c) To estimate the second sum on the right hand side of (3.37) we argue as follows. Clearly, by the definition of $\mathbf{v}_{m,\ell}$ we have

$$\begin{aligned}
&\sum_{\ell=k}^{\infty} \int_0^T \int_\Omega p_{3,\ell} \mathbf{v}_{m,\ell} \cdot \nabla \psi_\ell \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega p_{3,k} \mathbf{u}_{\varepsilon_m} \cdot \nabla \Psi_k \,\mathrm{d}x \,\mathrm{d}t \;+\; \sum_{\ell=k}^{\infty} \int_0^T \int_\Omega p_{3,\ell} \nabla \tilde{p}_{h,m,\ell} \cdot \nabla \psi_\ell \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{3.45}$$

Noticing that $|\nabla \Psi_k| \;\leq\; K 2^{-k}$ in $\mathbb{R}^n$ and $|\nabla \Psi_k| \equiv 0$ in $\mathbb{R}^n \setminus \widehat{G}_k$ applying Cauchy-Schwarz's inequality together with the a-priori estimate (3.24) yields

$$\left| \int_0^T \int_\Omega p_{3,k} \mathbf{u}_{\varepsilon_m} \cdot \nabla \Psi_k \,\mathrm{d}x \,\mathrm{d}t \right| \;\leq\; K \|\mathbf{f}\|^2_{L^1(0,\infty;L^2(\widehat{G}_k))}. \tag{3.46}$$

Moreover, applying Cauchy-Schwarz's inequality for sequences in l_2 using (3.40) and (3.24) one finds

$$\Big|\sum_{\ell=k}^{\infty}\int_0^T\int_\Omega p_{3,\ell}(t)\nabla\tilde{p}_{h,m,\ell}(t)\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\Big|$$

$$\le K\int_0^T\bigg(\sum_{\ell=k}^{\infty}2^{-2\ell}\|p_{3,\ell}(t)\|^2_{L^2(G_\ell)}\bigg)^{\frac12}\mathrm{d}t.$$

$$\le K\|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n\setminus B_{2^{k-1}}))}. \qquad (3.47)$$

Combining (3.45), (3.46) and (3.47) implies

$$\sum_{\ell=k}^{\infty}\int_0^T\int_\Omega p_{3,\ell}\mathbf{v}_{m,\ell}\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t\le K\|\mathbf{f}\|^2_{L^1(0,\infty;L^2(\mathbb{R}^n\setminus B_{2^{k-1}}))}. \qquad (3.48)$$

2. *Estimation of the left hand side of* (3.37) *from below.* Calculating

$$\sum_{\ell=k}^{\infty}|\mathbf{v}_{m,\ell}(T)|^2\psi_\ell = |\mathbf{u}_{\varepsilon_m}(T)|^2\Psi_k + \sum_{\ell=k}^{\infty}|\nabla\tilde{p}_{h,m,\ell}(T)|^2\psi_\ell + 2\sum_{\ell=k}^{\infty}\mathbf{u}_{\varepsilon_m}(T)\cdot\nabla\tilde{p}_{h,m,\ell}(T)\psi_\ell$$

almost everywhere in Ω making use of the property ($2'$) of the family $\{\psi_\ell\}_{\ell\ge N}$ it follows that

$$\sum_{\ell=k}^{\infty}\int_\Omega|\mathbf{v}_{m,\ell}(T)|^2\psi_\ell\,\mathrm{d}x \ge \int_{|x|\ge 2^{k+1}}|\mathbf{u}_{\varepsilon_m}(T)|^2\,\mathrm{d}x + 2\sum_{\ell=k}^{\infty}\int_\Omega\mathbf{u}_{\varepsilon_m}(T)\cdot\nabla\tilde{p}_{h,m,\ell}(T)\psi_\ell\,\mathrm{d}x. \qquad (3.49)$$

Estimating the left hand side of (3.37) from below by (3.49) and the right hand side of (3.37) from above by (3.44) and (3.48) yields

$$\begin{aligned}\int_{\{|x|>2^{k+1}\}}|\mathbf{u}_{\varepsilon_m}(T)|^2\,\mathrm{d}x \le{}& \int_{\{|x|>2^k\}}|\mathbf{u}_0|^2\,\mathrm{d}x + K(2^{-k}\sqrt{T+1}+2^{-k(1+n/6)}T^{1-n/6})\\ &+K\|\kappa_2\|_{L^2(\mathbb{R}^n\setminus B_{2^{k-1}}\times(0,\infty))}\\ &+K\|\mathbf{g}\|_{L^2(\mathbb{R}^n\setminus B_{2^{k-1}}\times(0,\infty))}+K\|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n\setminus B_{2^{k-1}}))}\\ &-2\sum_{\ell=k}^{\infty}\int_\Omega\mathbf{u}_{\varepsilon_m}(T)\cdot\nabla\tilde{p}_{h,m,\ell}(T)\psi_\ell\,\mathrm{d}x. \end{aligned}\qquad (3.50)$$

Now, it only remains to estimate the last term on the right of (3.50). For, take a sequence $\{\boldsymbol{\eta}_\nu\}$ in $\mathcal{D}_\sigma(\Omega)$ which converges to $\mathbf{u}_{\varepsilon_m}(T)$ in $L^2(\Omega)^n$ as $\nu\to\infty$.

Using (3.40) applying Cauchy-Schwarz's inequality in l_2 recalling $G_\ell \cap G_{\ell+4} = \emptyset$ for all $\ell \geq N$ one easily deduces

$$\Big\| \sum_{\ell=k}^{\infty} (\mathbf{u}_{\varepsilon_m}(T) - \boldsymbol{\eta}_\nu) \cdot \nabla \tilde{p}_{h,m,\ell}(T) \psi_\ell \Big\|_{L^2(\Omega)} \leq K \|\mathbf{u}_{\varepsilon_m}(T) - \boldsymbol{\eta}_\nu\|_{L^2(\Omega)}.$$

Hence,

$$\sum_{\ell=k}^{\infty} \boldsymbol{\eta}_\nu \cdot \nabla \tilde{p}_{h,m,\ell}(T) \psi_\ell \to \sum_{\ell=k}^{\infty} \mathbf{u}_{\varepsilon_m}(T) \cdot \nabla \tilde{p}_{h,m,\ell}(T) \psi_\ell \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \nu \to \infty.$$

Next, define

$$\begin{aligned}
\tilde{p}_{1,m,\ell}(x) &:= \int_0^T p_{1,m,\ell}(x,t)\, \mathrm{d}t, \\
\tilde{p}_{2,m,\ell}(x) &:= \int_0^T p_{2,m,\ell}(x,t)\, \mathrm{d}t, \\
\tilde{p}_{3,\ell}(x) &:= \int_0^T p_{3,\ell}(x,t)\, \mathrm{d}t, \\
\tilde{\mathbf{h}}_{1,m,\ell}(x) &:= \int_0^T \mathbf{h}_{1,m,\ell}(x,t)\, \mathrm{d}t, \\
\tilde{\mathbf{h}}_{2,m,\ell}(x) &:= \int_0^T \mathbf{h}_{2,m,\ell}(x,t)\, \mathrm{d}t, \quad x \in G_\ell \quad \ell \geq N.
\end{aligned}$$

As an immediate consequence of (3.19), (3.20) and (3.21) we have

$$\int_{G_\ell} \tilde{p}_{1,m,\ell} \Delta\phi \, \mathrm{d}x = \int_{G_\ell} \tilde{\mathbf{h}}_{1,m,\ell} : \nabla^2 \phi \, \mathrm{d}x, \tag{3.51}$$

$$\int_{G_\ell} \tilde{p}_{2,m,\ell} \Delta\phi \, \mathrm{d}x = \int_{G_\ell} \tilde{\mathbf{h}}_{2,m,\ell} : \nabla^2 \phi \, \mathrm{d}x, \tag{3.52}$$

$$\int_{G_\ell} \tilde{p}_{3,\ell} \Delta\phi \, \mathrm{d}x = \int_{G_\ell} \tilde{\mathbf{f}}(T) \cdot \nabla \phi \, \mathrm{d}x \quad \forall\, \phi \in C_0^\infty(G_\ell). \tag{3.53}$$

With help of Minkowski's inequality and Hölder's inequality making use of (3.10) yields

$$\begin{aligned}
\|\tilde{\mathbf{h}}_{1,m,\ell}\|_{L^2(G_\ell)} &\leq \|\mathbf{h}_{1,m,\ell}\|_{L^1(0,T;L^2(G_\ell))} \\
&\leq \|\mathbf{u}_{\varepsilon_m}\|^2_{L^2(0,T;L^4(G_\ell))} \\
&\leq T^{1-n/4} \|\mathbf{u}_{\varepsilon_m}\|^2_{L^{8/n}(0,T;L^4(G_\ell))} \leq K T^{1-n/4} \text{ [9]}.
\end{aligned}$$

Using Hölder's inequality taking into account (3.9) gives

$$\|\tilde{\mathbf{h}}_{2,m,\ell}\|_{L^2(G_\ell)} \leq \|\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) + \mathbf{g}\|_{L^2(G_\ell \times (0,T))} \leq K.$$

[9] Notice that $\dfrac{2n}{8} + \dfrac{n}{4} = \dfrac{n}{2}$.

Thus, owing to Lemma 2.3 it is readily seen that

$$\tilde{p}_{1,m,\ell}, \tilde{p}_{2,m,\ell}, \tilde{p}_{3,\ell} \in L^2(G_\ell).$$

Furthermore, we have the following a-priori estimates

$$\|\tilde{p}_{1,m,\ell}\|_{L^2(G_\ell)} + \|\tilde{p}_{2,m,\ell}\|_{L^2(G_\ell)} \le K\sqrt{T+1}, \tag{3.54}$$

$$\|\tilde{p}_{3,\ell}\|_{L^2(G_\ell)} \le K2^\ell \|\mathbf{f}\|_{L^1(0,\infty;L^2(G_\ell))}. \tag{3.55}$$

By an inspection of the proof of Theorem 2.6 one gets a function $\tilde{p}_m \in L^2_{\text{loc}}(\Omega \setminus B_{2^{N-1}})$, such that for all $\ell \in \mathbb{N}, \ell \ge N$ there holds

$$\tilde{p}_{1,m,\ell} + \tilde{p}_{2,m,\ell} + \tilde{p}_{3,\ell} + \tilde{p}_{h,m,\ell}(T) = \tilde{p}_m - (\tilde{p}_m)_{G_\ell} \quad \text{a.e. in} \quad G_\ell. \tag{3.56}$$

In addition, using (3.54), (3.55) and (3.34) (with $d = 0, q = r = 2$) we get the a-priori estimate

$$\begin{aligned} &\|\tilde{p}_m - (\tilde{p}_m)_{G_k}\|_{L^2(G_k)} \\ &\le K2^k(\|\mathbf{u}_{\varepsilon_m}(T) - \mathbf{u}_0\|_{L^2(G_k)} + \|\mathbf{f}\|_{L^1(0,\infty;L^2(G_k))} + 2^{-k}\sqrt{T+1}). \end{aligned} \tag{3.57}$$

Applying integration by parts using (3.56) one calculates

$$\begin{aligned} \sum_{\ell=k}^{\infty} \int_\Omega \boldsymbol{\eta}_\nu \cdot \nabla \tilde{p}_{h,m,\ell}(T)\psi_\ell \,\mathrm{d}x &= -\int_\Omega \tilde{p}_m \boldsymbol{\eta}_\nu \cdot \sum_{\ell=k}^{\infty} \nabla\psi_\ell \,\mathrm{d}x \\ &\quad + \sum_{\ell=k}^{\infty} \int_\Omega (\tilde{p}_{1,m,\ell} + \tilde{p}_{2,m,\ell} + \tilde{p}_{3,\ell})\boldsymbol{\eta}_\nu \cdot \nabla\psi_\ell \,\mathrm{d}x. \end{aligned} \tag{3.58}$$

Thanks to (1) and (2′) verifying that $\sum\limits_{\ell=k}^{\infty} \nabla\psi_\ell \equiv 0$ in $\{|x| > 2^{k+1}\})$ yields

$$-\int_\Omega \tilde{p}_m \boldsymbol{\eta}_\nu \cdot \sum_{\ell=k}^{\infty} \nabla\psi_\ell \,\mathrm{d}x = -\int_{\widehat{G}_k} (\tilde{p}_m - (\tilde{p}_m)_{G_k})\boldsymbol{\eta}_\nu \cdot \nabla(\psi_k + \psi_{k+1}) \,\mathrm{d}x.$$

Observing property (3) of the partition of unity $\{\psi_\ell\}_{\ell \ge N}$ using Cauchy-Schwarz's inequality and applying (3.57) one obtains

$$\begin{aligned} &\Big| \int_\Omega \tilde{p}_m \boldsymbol{\eta}_\nu \cdot \sum_{\ell=k}^{\infty} \nabla\psi_\ell \,\mathrm{d}x \Big| \\ &\le K\|\boldsymbol{\eta}_\nu\|_{L^2(G_k)}(\|\mathbf{u}_{\varepsilon_m}(T) - \mathbf{u}_0\|_{L^2(G_k)} + \|\mathbf{f}\|_{L^1(0,\infty;L^2(G_k))} + 2^{-k}\sqrt{T+1}). \end{aligned}$$

On the other hand, using Cauchy-Schwarz's inequality for sequences in l_2 together with (3.54) and (3.55) it follows that

$$\begin{aligned}
&\Big| \sum_{\ell=k}^{\infty} \int_{\Omega} (\tilde{p}_{1,m,\ell} + \tilde{p}_{2,m,\ell} + \tilde{p}_{3,\ell}) \boldsymbol{\eta}_\nu \cdot \nabla \psi_\ell \, \mathrm{d}x \Big| \\
&\qquad \le \|\boldsymbol{\eta}_\nu\|_{L^2(\Omega)} \Big(\sum_{\ell=k}^{\infty} 2^{-2\ell} \|\tilde{p}_{1,m,\ell} + \tilde{p}_{2,m,\ell} + \tilde{p}_{3,\ell})\|^2_{L^2(G_\ell)} \Big)^{\frac{1}{2}} \\
&\qquad \le K \|\boldsymbol{\eta}_\nu\|_{L^2(\Omega)} (2^{-k}\sqrt{T+1} + \|\mathbf{f}\|_{L^1(0,\infty;L^2(G_k))}).
\end{aligned}$$

Now, estimating the right hand side of (3.58) by the aid of the two estimates we have just obtained and afterwards carrying out the passage to the limit $\nu \to \infty$ leads to

$$\begin{aligned}
&\Big| \sum_{\ell=k}^{\infty} \int_{\Omega} \mathbf{u}_{\varepsilon_m}(T) \cdot \nabla \tilde{p}_{h,m,\ell}(T) \psi_\ell \, \mathrm{d}x \Big| \\
&\quad \le K(\|\mathbf{u}_{\varepsilon_m}(T)\|^2_{L^2(G_k)} + \|\mathbf{u}_0\|^2_{L^2(G_k)} + \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^{k-1}}))}) \\
&\qquad\qquad + K 2^{-k} \sqrt{T+1}.
\end{aligned}$$

Hence using the estimate above from (3.50) it follows

$$\begin{aligned}
&\int_{\{|x|>2^{k+1}\}} |\mathbf{u}_{\varepsilon_m}(T)|^2 \, \mathrm{d}x \\
&\quad \le K \int_{G_k} |\mathbf{u}_{\varepsilon_m}(T)|^2 \, \mathrm{d}x + K \int_{\{|x|>2^{k-1}\}} |\mathbf{u}_0|^2 \, \mathrm{d}x \\
&\qquad + K \|\kappa_2\|_{L^1(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad + K \|\mathbf{g}\|_{L^2(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad + K \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^{k-1}}))} \\
&\qquad + K(2^{-k}\sqrt{T+1} + 2^{-k(1+n/6)} T^{1-n/6}).
\end{aligned} \tag{3.59}$$

Next, let $\alpha > 0$ be arbitrarily chosen. Clearly, one can select $k \ge N$ such that

$$\begin{aligned}
&\int_{G_k} |\mathbf{u}(T)|^2 \, \mathrm{d}x + \int_{\{|x|>2^{k-1}\}} |\mathbf{u}_0|^2 \, \mathrm{d}x \\
&\qquad + \|\kappa_2\|_{L^1(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad + \|\mathbf{g}\|_{L^2(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad + \|\mathbf{f}\|_{L^1(0,T;L^2(\mathbb{R}^n \setminus B_{2^{k-1}}))} \\
&\qquad + (2^{-k}\sqrt{T+1} + 2^{-k(1+n/6)} T^{1-n/6}) \le \frac{\alpha}{4K}
\end{aligned} \tag{3.60}$$

and

$$\int_{\{|x|>2^{k-1}\}} |\mathbf{u}(T)|^2 \, \mathrm{d}x \le \frac{\alpha}{4(K+1)}. \tag{3.61}$$

By the aid of (3.16) one can choose $m_0 \in \mathbb{N}$ such that

$$\int_{G_k} |\mathbf{u}_{\varepsilon_m}(T)|^2 \,\mathrm{d}x - \int_{G_k} |\mathbf{u}(T)|^2 \,\mathrm{d}x \le \frac{\alpha}{4K}, \tag{3.62}$$

$$\left| \int_{\Omega\cap B_{2^{k+1}}} |\mathbf{u}_{\varepsilon_m}(T)|^2 \,\mathrm{d}x - \int_{\Omega\cap B_{2^{k+1}}} |\mathbf{u}(T)|^2 \,\mathrm{d}x \right| \le \frac{\alpha}{4} \tag{3.63}$$

for all $m \ge m_0$.

Let $m \ge m_0$. With help of the triangular inequality making use of (3.59) along with (3.60), (3.61), (3.62) and (3.63) one obtains

$$\begin{aligned}\Big| \|\mathbf{u}_{\varepsilon_m}(T)\|^2_{L^2(\Omega)} - \|\mathbf{u}(T)\|^2_{L^2(\Omega)} \Big| &\le \Big| \|\mathbf{u}_{\varepsilon_m}(T)\|^2_{L^2(\Omega\cap B_{2^{k+1}})} - \|\mathbf{u}(T)\|^2_{L^2(\Omega\cap B_{2^{k+1}})} \Big| \\ &\quad + \|\mathbf{u}_{\varepsilon_m}(T)\|^2_{L^2(\mathbb{R}^3\setminus B_{2^{k+1}})} + \|\mathbf{u}(T)\|^2_{L^2(\mathbb{R}^3\setminus B_{2^{k+1}})} \\ &\le \frac{3}{4}\alpha + (K+1)\|\mathbf{u}(T)\|^2_{L^2(\mathbb{R}^3\setminus B_{2^{k-1}})} \le \alpha.\end{aligned}$$

Thus,

$$\|\mathbf{u}_{\varepsilon_m}(T)\|^2_{L^2(\Omega)} \to \|\mathbf{u}(T)\|^2_{L^2(\Omega)} \quad \text{as} \quad m \to \infty.$$

Moreover recalling that $\mathbf{u}_{\varepsilon_m}(T)$ converges weakly to $\mathbf{u}(T)$ in $L^2(\Omega)^n$ as $m \to \infty$ one infers

$$\mathbf{u}_{\varepsilon_m}(T) \to \mathbf{u}(T) \quad \text{strongly in} \quad L^2(\Omega)^n \quad \text{as} \quad m \to \infty.$$

The Strong Energy Inequality

Let us begin by defining the set $\mathcal{J} \subset [0,\infty)$ of all real numbers for which (3.16) is true. Clearly $\mathcal{L}_1([0,\infty)\setminus\mathcal{J}) = 0$. As it has been proved in the previous step there holds

$$\|\mathbf{u}_{\varepsilon_m}(t)\|_H \to \|\mathbf{u}(t)\|_H \quad \text{as} \quad m \to \infty \quad \forall\, t \in \mathcal{J}. \tag{3.64}$$

Now, let $s, t \in \mathcal{J}$ with $0 < s < t < \infty$ be fixed. Owing to (III) one can easily check that

$$\begin{aligned}&\int_s^t \int_\Omega \mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) : D(\mathbf{u}_{\varepsilon_m}) \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad = \int_s^t \int_\Omega \mathbf{S}(\cdot, D(\mathbf{u})) : D(\mathbf{u}_{\varepsilon_m}) \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad + \int_s^t \int_\Omega (\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) - \mathbf{S}(\cdot, D(\mathbf{u}))) : D(\mathbf{u}_{\varepsilon_m}) \,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad \ge \int_s^t \int_\Omega \mathbf{S}(\cdot, D(\mathbf{u})) : D(\mathbf{u}_{\varepsilon_m}) \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad + \int_s^t \int_\Omega (\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) - \mathbf{S}(\cdot, D(\mathbf{u}))) : D(\mathbf{u}) \,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Thus, from (3.7) one deduces that

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_{\varepsilon_m}(t)\|_H^2 \;+\; &\int_s^t\int_\Omega \mathbf{S}(\cdot, D(\mathbf{u})) : D(\mathbf{u}_{\varepsilon_m})\,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_{\varepsilon_m} + \mathbf{g} : \nabla\mathbf{u}_{\varepsilon_m}\,\mathrm{d}x\,\mathrm{d}\tau \;+\; \frac{1}{2}\|\mathbf{u}_{\varepsilon_m}(s)\|_H^2 \\ &\qquad + \int_s^t (\mathbf{S}(\cdot, D(\mathbf{u}_{\varepsilon_m})) - \mathbf{S}(\cdot, D(\mathbf{u}))) : D(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (3.65)\end{aligned}$$

By means of (3.14), (3.13) and (3.64) we are in a position to carry out the passage to the limit $m\to\infty$ on both sides of (3.65). This implies

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}(t)\|_{\mathbf{H}}^2 \;+\; &\int_s^t\int_\Omega \mathbf{S}(\cdot, D(\mathbf{u})) : D(\mathbf{u})\,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \int_s^t\int_\Omega \mathbf{f}\cdot\mathbf{u} + \mathbf{g} : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau \;+\; \frac{1}{2}\|\mathbf{u}(s)\|_{\mathbf{H}}^2. \qquad (3.66)\end{aligned}$$

Whence, $\mathbf{u}$ is a turbulent solution.

Decay for $\|\mathbf{u}(T)\|_{L^2(\Omega)}$ as $T\to\infty$

First we define

$$\tilde{\mathcal{J}} := \Big\{T\in\mathcal{J}\;\Big|\;\mathbf{u}(T)\in L^{2\frac{n+2}{n}}(\Omega)^n\Big\}.$$

Recalling that $\mathbf{u}\in L^{2\frac{n+2}{n}}(Q)^n$ shows that $\mathcal{L}_1([0,\infty)\setminus\tilde{\mathcal{J}}) = 0$. Moreover, in $\tilde{\mathcal{J}}$ there exists a sequence of numbers $1 < T_1 < T_2 < \ldots < T_j < \ldots$ with $T_j\to\infty$ as $j\to\infty$ such that

$$\int_\Omega |\mathbf{u}(T_j)|^{2\frac{n+2}{n}}\,\mathrm{d}x \;\le\; \frac{1}{T_j\ln(1+T_j)} \qquad \forall\, j\in\mathbb{N}. \qquad (3.67)$$

Let $0<\delta<1$ be a number which will be specified below. Let $j\in\mathbb{N}$. Clearly, there exists a unique integer $k = k(j)$ fulfilling

$$2^{k-1} \le \delta^{-1}\sqrt{T_j} < 2^k. \qquad (3.68)$$

Without loss of generality we may assume that $k-1>N$, i.e. $G_k\subset\Omega$.

Let $\ell\ge k$. Arguing as in step 2 from Corollary 2.7 one gets unique functions

$$\begin{aligned}&p_{1,\ell}\in L^{\frac{n+2}{n}}(0,\infty; A^{\frac{n+2}{n}}(G_\ell)),\\ &p_{2,\ell}\in L^2(0,\infty; A^2(G_\ell)),\\ &p_{3,\ell}\in L^1(0,\infty; A^{2^*}(G_\ell)),\\ &\tilde{p}_{h,\ell}\in C_w(0,\infty; \dot{B}^{\frac{n+2}{n}}(G_\ell)) \quad\text{with}\quad \tilde{p}_{h,\ell}(0) = 0,\end{aligned}$$

such that

$$
\begin{aligned}
-\int_0^\infty \int_{G_\ell} \mathbf{u}\cdot \partial_t \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t
&\\
+ \int_0^\infty \int_{G_\ell} \mathbf{u}\otimes\mathbf{u} + \mathbf{S}(\cdot, D(\mathbf{u}(t))) - \mathbf{g}) : \nabla\boldsymbol{\varphi} - \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t &\\
= \int_0^\infty \int_{G_\ell} (p_{1,\ell} + p_{2,\ell} + p_{3,\ell})\nabla\cdot\boldsymbol{\varphi} + \nabla\tilde{p}_{h,\ell}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t & \qquad (3.69)
\end{aligned}
$$

for all $\boldsymbol{\varphi}\in C_0^\infty(G_\ell\times(0,\infty))^n$ and

$$
\int_{G_\ell} p_{1,\ell}(t)\Delta\phi\,\mathrm{d}x = \int_{G_\ell} (\mathbf{u}(t)\otimes\mathbf{u}(t)) : \nabla^2\phi\,\mathrm{d}x, \qquad (3.70)
$$

$$
\int_{G_\ell} p_{2,\ell}(t)\Delta\phi\,\mathrm{d}x = \int_{G_\ell} (\mathbf{S}(\cdot, D(\mathbf{u}(t))) + \mathbf{g}(t)) : \nabla^2\phi\,\mathrm{d}x, \qquad (3.71)
$$

$$
\int_{G_\ell} p_{3,\ell}(t)\Delta\phi\,\mathrm{d}x = \int_{G_\ell} \mathbf{f}(t)\cdot\nabla\phi\,\mathrm{d}x \qquad (3.72)
$$

for almost all $t\in(0,\infty)$, for all $\phi\in C_0^\infty(G_\ell)$.

Observing, (III) taking into account (3.16), (3.14) (3.12) and the fact

$$
p_{h,m,\ell} \to p_{h,\ell} \quad \text{in} \quad L^n(0,T;W^{2,n}(\widehat{G}_\ell)) \quad \text{as} \quad m\to\infty \qquad (3.73)
$$

carrying out the passage to the limit on both sides of (3.32) as $m\to\infty$ (replacing T by T_j therein) one arrives at

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\mathbf{v}_\ell(T_j)|^2\psi_\ell\,\mathrm{d}x \le \frac{1}{2}\int_\Omega |\mathbf{u}_0|^2\psi_\ell\,\mathrm{d}x. &\\
+ \int_0^{T_j}\int_\Omega \kappa_2\psi_\ell\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_j}\int_\Omega \mathbf{f}\cdot\mathbf{v}_\ell\psi_\ell\,\mathrm{d}x\,\mathrm{d}t &\\
+ \mathbf{I}_\ell + \mathbf{II}_\ell + \mathbf{III}_\ell + \mathbf{IV}_\ell + \mathbf{V}_\ell, & \qquad (3.74)
\end{aligned}
$$

where $\mathbf{v}_\ell := \mathbf{u} + \nabla\tilde{p}_{h,\ell}$ and

$$
\begin{aligned}
\mathbf{I}_\ell &:= -\int_0^{T_j}\int_\Omega \mathbf{S}(x,t,D(\mathbf{u}))) : (\mathbf{v}_\ell\otimes\nabla\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{II}_\ell &:= -\int_0^{T_j}\int_\Omega \mathbf{S}(x,t,D(\mathbf{u})) : \nabla^2\tilde{p}_{h,\ell}\psi_\ell\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{III}_\ell &:= \int_0^{T_j}\int_\Omega \mathbf{g} : \nabla(\mathbf{v}_\ell\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{IV}_\ell &:= \int_0^{T_j}\int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \nabla(\mathbf{v}_\ell\psi_\ell)\,\mathrm{d}x\,\mathrm{d}t,\\
\mathbf{V}_\ell &:= \int_0^{T_j}\int_\Omega (p_{1,\ell} + p_{2,\ell} + p_{3,\ell})\mathbf{v}_\ell\cdot\nabla\psi_\ell\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$(\ell \in \mathbb{N}; \ell \geq N)$.

By an analogous reasoning we have used to get (3.59) one proves the a-priori estimate

$$\begin{aligned}
&\int_{\{|x|>2^{k+1}\}} |\mathbf{u}(T_j)|^2 \, \mathrm{d}x \\
&\qquad \leq K \int_{B_{2^{k+3}} \cap \Omega} |\mathbf{u}(T_j)|^2 \, \mathrm{d}x \, + \, K \int_{\{|x|>2^{k-1}\}} |\mathbf{u}_0|^2 \, \mathrm{d}x \\
&\qquad\quad + \, K \|\kappa_2\|_{L^1(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad\quad + \, K \|\mathbf{g}\|_{L^2(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\qquad\quad + \, K \|\mathbf{f}\|_{L^1(0,\infty; L^2(\mathbb{R}^n \setminus B_{2^{k-1}}))} \\
&\qquad\quad + \, K (2^{-k} \sqrt{T_j} \, + \, 2^{-2k(1-n/6)} T_j^{1-n/6}).
\end{aligned} \tag{3.75}$$

Using Hölder's inequality taking into account (3.68) and (3.67) one estimates

$$\begin{aligned}
\int_{B_{2^{k+3}} \cap \Omega} |\mathbf{u}(T_j)|^2 \, \mathrm{d}x \, &\leq \, K \left(2^{2k} \int_{B_{2^{k+3}} \cap \Omega} |\mathbf{u}(T_j)|^{2 \frac{n+2}{n}} \, \mathrm{d}x \right)^{\frac{n}{n+2}} \\
&\leq \, \frac{K}{\delta^{2n/(n+2)} (\ln(1+T_j))^{2n/(n+2)}}.
\end{aligned} \tag{3.76}$$

Combining (3.75) and (3.76) one gets a positive constant K_1 such that

$$\begin{aligned}
\|\mathbf{u}(T_j)\|^2_{L^2(\Omega)} \, &= \, \|\mathbf{u}(T_j)\|^2_{L^2(\mathbb{R}^n \setminus B_{2^{k+1}})} \, + \, \|\mathbf{u}(T_j)\|^2_{L^2(B_{2^{k+1}} \cap \Omega)} \\
&\leq K_1 \int_{\{|x|>2^{k-1}\}} |\mathbf{u}_0|^2 \, \mathrm{d}x \\
&\quad + \, K_1 \|\kappa_2\|_{L^1(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty))} \\
&\quad + \, K_1 \|\mathbf{g}\|_{L^2(\mathbb{R}^n \setminus B_{2^k} \times (0,\infty)))} \\
&\quad + \, K_1 \|\mathbf{f}\|_{L^1(0,\infty; L^2(\mathbb{R}^n \setminus B_{2^{k-1}}))} \\
&\quad + \, \frac{K_1}{\delta^{2n/(n+2)} (\ln(1+T_j))^{2n/(n+2)}} \, + \, K_1 \delta^{2/3}.
\end{aligned} \tag{3.77}$$

Let $0 < \alpha < 1$ be arbitrarily chosen. Let $\delta > 0$ be fixed such that

$$K_1 \delta^{2/3} \, \leq \, \frac{\alpha}{4}. \tag{3.78}$$

Clearly, there exists $j_1 \in \mathbb{N}$, such that

$$\begin{aligned}
&\int_{\{|x|>2^{k(j)-1}\}} |\mathbf{u}_0|^2 \,\mathrm{d}x \\
&\quad + \|\kappa_2\|_{L^1(0,\infty;L^2(\mathbb{R}^n\setminus B_{2^{k(j)}}))} \\
&\quad + \|\mathbf{g}\|_{L^2(0,\infty;L^2(\mathbb{R}^n\setminus B_{2^{k(j)}}))} \\
&\quad + \|\mathbf{f}\|_{L^1(0,\infty;L^2(\mathbb{R}^n\setminus B_{2^{k(j)-1}}))} \\
&\qquad + \frac{1}{\delta^{2n/(n+2)}(\ln(1+T_j))^{2n/(n+2)}} \;\le\; \frac{\alpha}{4K_1}
\end{aligned} \tag{3.79}$$

for all $j \ge j_1$. Hence, using (3.78) and (3.79) from (3.77) one gets

$$\|\mathbf{u}(T_j)\|^2_{L^2(\Omega)} \;\le\; \frac{\alpha}{2} \quad \forall\, j \in \mathbb{N},\; j \ge j_1. \tag{3.80}$$

Owing to the strong energy inequality we have for all $T \in \mathcal{J}, T > T_j$

$$\begin{aligned}
\|\mathbf{u}(T)\|^2_{L^2(\Omega)} \;&\le\; \|\mathbf{u}(T_j)\|^2_{L^2(\Omega)} \\
&\quad - 2\int_{T_j}^{T}\int_\Omega (\mathbf{S}(x,t,D(\mathbf{u})) - \mathbf{g}) : \nabla\mathbf{u} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + 2\int_{T_j}^{T}\int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&\le\; \|\mathbf{u}(T_j)\|^2_{L^2(\Omega)} + K\|\nabla\mathbf{u}\|_{L^2(\Omega\times(T_j,\infty))} \\
&\qquad + K\|\mathbf{f}\|_{L^1(T_j,\infty;L^2(\Omega))}.
\end{aligned} \tag{3.81}$$

Finally choosing $j_2 \in \mathbb{N}$ such that

$$K\|\nabla\mathbf{u}\|_{L^2(\Omega\times(T_j,\infty))} + K\|\mathbf{f}\|_{L^1(T_j,\infty;L^2(\Omega))} \;\le\; \frac{\alpha}{2}$$

for all $j \ge j_2$ using (3.80) and (3.81) shows that

$$\|\mathbf{u}(T)\|^2_{L^2(\Omega)} \;\le\; \alpha \quad \forall\, T \ge \max\{T_{j_1}, T_{j_2}\}.$$

Hence the assertion of the theorem is completely proved. $\square$

Appendix A. Caccioppoli Inequalities for Harmonic Functions

Proof of Lemma 3.1.

1. L^∞*-estimates for harmonic functions.* Let $V \subset\subset U$ be two open bounded sets with $\rho := \operatorname{dist}(V, \partial U)$. Let $p \in L^1(U)$ being harmonic in U. Then

$$\sup_{x\in V} |w(x)| \;\le\; \frac{1}{\mathcal{L}_n(B_1)\rho^n}\int_U |w|\,\mathrm{d}x. \tag{A.1}$$

Indeed, let $x \in V$. Then $B_\rho(x) \subset U$. By the mean value property of harmonic functions we have

$$|w(x)| = \left| \fint_{B_\rho(x)} w(y)\,\mathrm{d}y \right| \le \frac{1}{\mathcal{L}_n(B_1)\rho^n} \int_U |w|\,\mathrm{d}y.$$

Whence (A.1)

2. *The case* $\ell = 1$. Let $w \in B^r(G_1)$. We define the sets

$$G' := \Big\{ x \in \mathbb{R}^n \,\Big|\, \frac{7}{8} < |x| < 5 \Big\},$$
$$G'' := \Big\{ x \in \mathbb{R}^n \,\Big|\, \frac{5}{8} < |x| < 7 \Big\}.$$

Verifying $\operatorname{dist}(\widehat{G}_1, G') = \dfrac{1}{8}$ using (A.1) together with Caccioppli's inequality yields

$$\sup_{x \in \widehat{G}_1} |\nabla^d w(x)| \le K \|\nabla^d w\|_{L^2(G')} \le K \sup_{x \in G''} |w(x)|.$$

where $K = \mathrm{const} > 0$ depending only on n and d.

Next, noticing that $\operatorname{dist}(G'', G_1) = \dfrac{1}{8}$ once more appealing to (A.1) gives

$$\sup_{x \in G''} |w(x)| \le \frac{8^n}{\mathcal{L}_n(B_1)} \int_{G_1} |w|\,\mathrm{d}x.$$

Thus, combining the last two estimates applying Hölder's inequality implies

$$\left(\fint_{\widehat{G}_1} |\nabla^d w|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \le K \left(\fint_{G_1} |w|^r \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

This completes the proof of (3.33) for $\ell = 1$.

3. *The case* $\ell \in \mathbb{N}$. Let $w \in B^r(G_\ell)$. Using an appropriate changing of coordinates applying the transformation formula of Lebesgue's integral gives

$$\left(\fint_{\widehat{G}_\ell} |\nabla^d w|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \le K 2^{-\ell d} \left(\fint_{G_\ell} |w|^r \,\mathrm{d}x \right)^{\frac{1}{r}}.$$

Whence, (3.33). □

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