A Fourth Order Split Scheme for Elastic Wave Propagation

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April 17, 2007

Abstract

Motivated by seismological problems we have studied a 4th order split scheme for the elastic wave equation. We split in the spatial directions and obtain locally one-dimensional systems to be solved. We have analyzed the new scheme and obtained results showing consistency and stability. We have used the split scheme to solve problems in two and three dimensions. We have also looked at the influence of singular forcing terms on the convergence properties of the scheme.

Keywords: Partial Differential Equations, Operator-Splitting Methods, Seismology, Singular Sources, Consistency Analysis.

1 Introduction

Inspired by work for the scalar wave equation presented in [2], we devise a fourth order split scheme for the elastic wave equation.

We consider the initial-value problem for the elastic wave equation for constant coefficients, given as

$$\rho \partial_{tt} \mathbf{U} = \mu \nabla^{\mathbf{2}} \mathbf{U} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{U}) + \mathbf{f}, \tag{1}$$

$$\mathbf{U}(t = 0, \mathbf{x}) = \mathbf{g}_0(\mathbf{x}), \tag{2}$$

$$\partial_t \mathbf{U}(t = 0, \mathbf{x}) = \mathbf{g}_1(\mathbf{x}), \tag{3}$$

where $\mathbf{U} \equiv \mathbf{U}(\mathbf{x}, t)$ is the displacement vector with components $(u, v)^T$ or $(u, v, w)^T$ in two and three dimensions, $\mathbf{f}, \mathbf{g}_0$, and $\mathbf{g}_1$ are known initial functions, and finally $\mathbf{x} = (x, y, z)^T$. This equation is commonly used to simulate seismic events.

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In seismology it is common to use spatially singular forcing terms which can look like

$$\mathbf{f} = \mathbf{F}\delta(\mathbf{x})g(t), \tag{4}$$

where $\mathbf{F}$ is a constant vector. A numeric method for (1) needs to approximate the Dirac function correctly in order to achieve full convergence.

We will first introduce some explicit and implicit time stepping schemes of 2nd and 4th order accuracy. Then we will describe operator split versions of the 4th order implicit scheme and also analyze the new split scheme. In the last sections some benchmark problems are solved using the different schemes and the results compared, the effect of different approximations of the Dirac function are also shown.

2 Basic numerical methods

One of the first practical difference scheme with central differences used everywhere was introduced in [1]. To save space we exemplify it and some newer schemes in two dimensions first.

If we discretize uniformly in space and time on the unit square we get a grid with grid points $x_j = jh, y_k = kh, t_n = n\Delta t$ where $h > 0$ is the spatial grid size and Δt the time step. Defining the grid function $\mathbf{U}_{j,k}^n = \mathbf{U}(x_j, y_k, t_n)$ the basic explicit scheme is

$$\rho\frac{\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}}{\Delta t^2} = \mathcal{M}_2\mathbf{U}_{j,k}^n + \mathbf{f}_{j,k}^n, \tag{5}$$

where $\mathcal{M}_2$ is a difference operator

$$\mathcal{M}_2 = \begin{pmatrix} (\lambda + 2\mu)D^{x2} + \mu D^{y2} & (\lambda + \mu)D_0^x D_0^y v_{j,k} \\ (\lambda + \mu)D_0^x D_0^y v_{j,k} & (\lambda + 2\mu)D^{y2} + \mu D^{x2} \end{pmatrix} \tag{6}$$

and we use the standard difference operator notation:

$$D_+^x v_{j,k} = \frac{1}{h}(v_{j+1,k} - v_{j,k}), \quad D_-^x v_{j,k} = D_+^x v_{j-1,k}, \quad D_0^x = \frac{1}{2}\left(D_+^x + D_-^x\right),$$

and

$$D^{x2} = D_+^x D_-^x.$$

$\mathcal{M}_2$ is a second order difference approximation of the right-hand side operator of equation (1). This explicit scheme is stable for time steps satisfying [3]

$$\Delta t < \frac{h}{\sqrt{\lambda + 3\mu}}. \tag{7}$$

Replacing $\mathcal{M}_2$ with $\mathcal{M}_4$, a 4th order difference operator given by

$$\mathcal{M}_4 = \begin{pmatrix} (\lambda + 2\mu)\left(1 - \frac{h^2}{12}D^{x2}\right)D^{x2} + \mu\left(1 - \frac{h^2}{12}D^{y2}\right)D^{y2} \\ (\lambda + \mu)\left(1 - \frac{h^2}{6}D^{x2}\right)D_0^x\left(1 - \frac{h^2}{6}D^{y2}\right)D_0^y \end{pmatrix}$$

$$\begin{pmatrix} (\lambda + \mu) \left(1 - \frac{h^2}{6} D^{x2}\right) D_0^x \left(1 - \frac{h^2}{6} D^{y2}\right) D_0^y \\ (\lambda + 2\mu) \left(1 - \frac{h^2}{12} D^{y2}\right) D^{y2} + \mu \left(1 - \frac{h^2}{12} D^{x2}\right) D^{x2} \end{pmatrix}, \tag{8}$$

and using the modified equation approach to eliminate the lower order error terms in the time difference [3], we obtain the explicit fourth order scheme

$$\rho \frac{\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}}{\Delta t^2} = \mathcal{M}_4 \mathbf{U}_{j,k}^n + \mathbf{f}_{j,k}^n +$$
$$\frac{\Delta t^2}{12} (\mathcal{M}_2^2 \mathbf{U}_{j,k}^n + \mathcal{M}_2 \mathbf{f}_{i,j}^n + \partial_{tt} \mathbf{f}_{i,j}^n), \tag{9}$$

where $\mathcal{M}_2^2$ is a 2nd order approximation to the squared right-hand side operator in equation (1). As it only needs to be second order accurate, $\mathcal{M}_2^2$ has the same extent in space as $\mathcal{M}_4$ and no more grid points are used. This scheme has the same time step restriction as (7).

In [2] the following implicit scheme for the scalar wave equation was introduced:

$$\rho \frac{\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}}{\Delta t^2} = \mathcal{M}_4 \left(\theta \mathbf{U}_{j,k}^{n+1} + (1 - 2\theta)\mathbf{U}_{j,k}^n + \theta \mathbf{U}_{j,k}^{n-1}\right)$$
$$+ \theta \mathbf{f}_{j,k}^{n+1} + (1 - 2\theta)\mathbf{f}_{j,k}^n + \theta \mathbf{f}_{j,k}^{n-1}. \tag{10}$$

When $\theta = 1/12$ the error of this scheme is fourth order in time and space. For this θ value it is however only conditionally stable, allowing a time step approximately 45% larger than (7) (for $\theta \in [0.25, 0.5]$ it is unconditionally stable).

In order to make it competitive with the explicit scheme (9) we provide an operator split version of the implicit scheme (10). This is made complicated by the presence of the mixed derivative terms that couple different coordinate directions.

3 Fourth order splitting method

In the following we present a fourth order splitting method based on the basic scheme (10). We split the operator $\mathcal{M}_4$ into three parts: $\mathcal{M}_{xx}$, $\mathcal{M}_{yy}$, and $\mathcal{M}_{xy}$ where we have

$$\mathcal{M}_{xx} = \begin{pmatrix} (\lambda + 2\mu) \left(1 - \frac{h^2}{12} D^{x2}\right) D^{x2} & 0 \\ 0 & \mu \left(1 - \frac{h^2}{12} D^{x2}\right) D^{x2} \end{pmatrix},$$

$$\mathcal{M}_{yy} = \begin{pmatrix} \mu \left(1 - \frac{h^2}{12} D^{y2}\right) D^{y2} & 0 \\ 0 & (\lambda + 2\mu) \left(1 - \frac{h^2}{12} D^{y2}\right) D^{y2} \end{pmatrix},$$

$$\mathcal{M}_{xy} = \mathcal{M}_4 - \mathcal{M}_{xx} - \mathcal{M}_{yy}.$$

Our proposed split method has the following steps:

1. $\rho \dfrac{\mathbf{U}^*_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}}{\Delta t^2} = \mathcal{M}_4 \mathbf{U}^n_{j,k} + \theta \mathbf{f}^{n+1}_{j,k} + (1 - 2\theta)\mathbf{f}^n_{j,k} + \theta \mathbf{f}^{n-1}_{j,k}$ (11)

2. $\rho \dfrac{\mathbf{U}^{**}_{j,k} - \mathbf{U}^*_{j,k}}{\Delta t^2} = \theta \mathcal{M}_{xx}\left(\mathbf{U}^{**}_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}\right)$

$\qquad\qquad\qquad + \dfrac{\theta}{2}\mathcal{M}_{xy}\left(\mathbf{U}^*_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}\right)$ (12)

3. $\rho \dfrac{\mathbf{U}^{n+1}_{j,k} - \mathbf{U}^{**}_{j,k}}{\Delta t^2} = \theta \mathcal{M}_{yy}\left(\mathbf{U}^{n+1}_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}\right)$

$\qquad\qquad\qquad + \dfrac{\theta}{2}\mathcal{M}_{xy}\left(\mathbf{U}^{**}_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}\right).$ (13)

Here the first step is explicit, while the second and third steps treat the derivatives along the coordinate axes implicitly and the mixed derivatives explicitly. This is similarly to how the mixed case is handled for parabolic problems [4].

Notice that each of the equation systems that needs to be solved in step 2. and 3. are actually two decoupled tri-diagonal systems that can be solved independently.

3.1   Stability and Consistency of the fourth order splitting method

The consistency of the fourth order splitting method is given in the following theorem.

We have for all sufficiently smooth functions $\mathbf{U}(\mathbf{x}, t)$ the following discretization order :

$$\mathcal{M}_4 \mathbf{U} = \mu \nabla^2 \mathbf{U} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{U}) + \mathcal{O}(h^4). \qquad (14)$$

Furthermore the split operators are also discretized with the same order of accuracy.

The we obtain the following consistency result for the split method (11)-(13) :

Theorem 3.1 *The split method has a splitting error which for smooth solutions* **U** *is* $\mathcal{O}(\Delta t^4)$, *where we assume* $\Delta t = \mathcal{O}(h)$.

Proof 3.2 *We assume in the following* $\mathbf{f} = (0,0)^T$. *We add the equations (11)-(13) and obtain, like in the scalar case see [2], the following result for the discretized equations :*

$$D^t_+ D^t_- \mathbf{U}^n_{j,k} - \mathcal{M}_4(\theta \mathbf{U}^{n+1}_{j,k} + (1 - 2\theta)\mathbf{U}^n_{j,k} + \theta \mathbf{U}^{n-1}_{j,k})$$
$$- \mathcal{N}_{4,\theta}(\mathbf{U}^{n+1}_{j,k} - 2\mathbf{U}^n_{j,k} + \mathbf{U}^{n-1}_{j,k}) = \mathbf{0} \qquad (15)$$

where
$\mathcal{N}_{4,\theta} = \theta^2 \Delta t^2 (\mathcal{M}_{xx}\mathcal{M}_{yy} + \mathcal{M}_{xx}\mathcal{M}_{xy} + \mathcal{M}_{xy}\mathcal{M}_{yy}) + \theta^3 \Delta t^4 \mathcal{M}_{xx}\mathcal{M}_{yy}\mathcal{M}_{xy}$ *We therefore obtain a splitting error of* $\mathcal{N}_{4,\theta}(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^{n} + \mathbf{U}_{j,k}^{n-1})$.

For sufficient smoothness we have $(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^{n} + \mathbf{U}_{j,k}^{n-1}) = \mathcal{O}(\Delta t^2)$ *and we obtain* $\mathcal{N}_{4,\theta}(\mathbf{U}^{n+1} - 2\mathbf{U}^{n} + \mathbf{U}^{n-1}) = \mathcal{O}(\Delta t^4)$.

Important is that the influence of the mixed terms can be also be discretized as 4th order method and therefore the terms are canceled in the proof.

For the stability we have to denote an appropriate norm, which is in our case the $L_2(\Omega)$.

In the following we introduce the notation of the norms.

Remark 3.3 *For our system we extend the L_2-norm as*

$$||\mathbf{U}||_{L_2}^2 = (\mathbf{U}, \mathbf{U})_{L_2} = \int_{\Omega} \mathbf{U}^2 \, dx \qquad (16)$$

where $\mathbf{U}^2 = u^2 + v^2$ *or* $\mathbf{U}^2 = u^2 + v^2 + w^2$ *in two and three dimensions.*

Remark 3.4 *The matrix*

$$\begin{aligned}\mathcal{N}_{4,\theta} &= \theta^2 \Delta t^2 (\mathcal{M}_{xx}\mathcal{M}_{yy} + \mathcal{M}_{xx}\mathcal{M}_{xy} + \mathcal{M}_{xy}\mathcal{M}_{yy}) \\ &\quad + \theta^3 \Delta t^4 \mathcal{M}_{xx}\mathcal{M}_{yy}\mathcal{M}_{xy} ,\end{aligned} \qquad (17)$$

where $\mathcal{M}_{xx}$, $\mathcal{M}_{yy}$ *and* $\mathcal{M}_{xy}$ *are symmetrical and positive define matrices, therefore the matrix* $\mathcal{N}_{4,\theta}$ *is also symmetrical and positive definite.*

Furthermore we can estimates for the norms and define a weighted norm, see [9] and [11].

Remark 3.5 *The energy norm is given as*

$$(\mathcal{N}_{4,\theta}\mathbf{U}, \mathbf{U})_{L_2} = \int_{\Omega} (\mathcal{N}_{4,\theta}\mathbf{U} \ \mathbf{U}) \, dx \qquad (18)$$

Consequently we can denote

$$||\mathcal{N}_{4,\theta}\mathbf{U}|| \leq \omega ||\mathbf{U}|| , \ \forall \mathbf{U} \in H^d \qquad (19)$$

where $\omega \in \mathbb{R}^+$ *is the weight and* $\mathcal{N}_{4,\theta}$ *is bounded. d is the dimension and H is Sobolev-space, see [5].*

The stability of the fourth order splitting method is given in the following theorem.

Theorem 3.6 *Let* $\theta \in [0.25, 0.5]$, *then the implicit time-stepping algorithm, see (5), and the split procedure, see (11) – (13) are unconditionally stable. We can estimate the split procedure iteratively as*

$$\begin{aligned}||(1 - \Delta t^2 \tilde{\omega})^{1/2} D_+^t \mathbf{U}_{j,k}^{n}||^2 + \mathcal{P}^+(\mathbf{U}_{j,k}^{n}, \theta) &\leq ||(1 - \Delta t^2 \tilde{\omega})^{1/2} D_+^t \mathbf{U}_{j,k}^{0}||^2 \\ &\quad + \ \mathcal{P}^+(\mathbf{U}_{j,k}^{0}, \theta)\end{aligned} \qquad (20)$$

where we have $\mathcal{P}^{\pm}(\mathbf{U}_{j,k}^n, \theta) := \theta(\mathcal{M}_4\mathbf{U}_{j,k}^n, \mathbf{U}_{j,k}^n) + \theta(\mathcal{M}_4\mathbf{U}_{j,k}^{n\pm1}, \mathbf{U}_{j,k}^{n\pm1}) + (1 - 2\theta)(\mathcal{M}_4\mathbf{U}_{j,k}^n, \mathbf{U}_{j,k}^{n\pm1})$ *and* $\mathcal{P}^{\pm} \geq 0$ *for* $\theta \in [0.25, 0.5]$. *Further* $1 - \Delta t^2\tilde{\omega} \in \mathbb{R}^+$ *is the factor for the weighted norm* $(\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})\mathbf{U} \leq \tilde{\omega}\mathbf{U}$, $\forall \mathbf{U} \in H^d$.

We have to prove the iterative estimate for the split procedure and the proof is given as follows

Proof 3.7 *To obtain an energy estimate for the scheme we multiply with a test-function* $D_0^t\mathbf{U}_{j,k}^n$.

The following result is given for the discretized equations, see also equation (15)

$$(\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})D_+^tD_-^t\mathbf{U}_{j,k}^n$$
$$-\mathcal{M}_4(\theta\mathbf{U}_{j,k}^{n+1} + (1-2\theta)\mathbf{U}_{j,k}^n + \theta\mathbf{U}_{j,k}^{n-1}) = \mathbf{0} \qquad (21)$$

So for $n \geq 1$ *we can rewrite the equation (21) for the stability proof as*

$$((\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})D_+^tD_-^t\mathbf{U}_{j,k}^n, D_0^t\mathbf{U}_{j,k}^n)$$
$$-(\mathcal{M}_4(\theta\mathbf{U}_{j,k}^{n+1} + (1-2\theta)\mathbf{U}_{j,k}^n + \theta\mathbf{U}_{j,k}^{n-1}), D_0^t\mathbf{U}_{j,k}^n) = \mathbf{0} \qquad (22)$$

Multiplying with Δt *and summarizing over the time levels we obtain :*

$$\sum_n ((\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})D_+^tD_-^t\mathbf{U}_{j,k}^n, D_0^t\mathbf{U}_{j,k}^n)\Delta t$$
$$-\sum_n (\mathcal{M}_4(\theta\mathbf{U}_{j,k}^{n+1} + (1-2\theta)\mathbf{U}_{j,k}^n + \theta\mathbf{U}_{j,k}^{n-1}), D_0^t\mathbf{U}_{j,k}^n)\Delta t = \mathbf{0}, \qquad (23)$$

for each term of the sum one can derive the following identities. So for $\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta}$ *we have*

$$((\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})D_+^tD_-^t\mathbf{U}_{j,k}^n, D_0^t\mathbf{U}_{j,k}^n)\Delta t$$
$$= \frac{1}{2}((\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})(D_+^t - D_-^t)\mathbf{U}_{j,k}^n, (D_+^t + D_-^t)\mathbf{U}_{j,k}^n)$$
$$= \int_\Omega \left((\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta})(D_+^t - D_-^t)\right)^T (D_+^t + D_-^t)\mathbf{U}_{j,k}^n \, dx$$
$$\leq (1 - \Delta t^2\tilde{\omega}) \int_\Omega (D_+^t\mathbf{U}_{j,k}^n)^2(D_-^t\mathbf{U}_{j,k}^n)^2 \, dx \qquad (24)$$

where the operator $\mathcal{I} - \Delta t^2\mathcal{N}_{4,\theta}$ *is symmetric and positive definite and we can apply the weighted norm, see Remark 3.5 and [5].*

We obtain the following result :

$$(1 - \Delta t^2\tilde{\omega}) \int_\Omega (D_+^t\mathbf{U}_{j,k}^n)^2(D_-^t\mathbf{U}_{j,k}^n)^2 \, dx \qquad (25)$$
$$= 1/2||(1 - \Delta t^2\tilde{\omega})^{1/2}D_+^t\mathbf{U}_{j,k}^n||^2$$
$$-1/2||(1 - \Delta t^2\tilde{\omega})^{1/2}D_-^t\mathbf{U}_{j,k}^n||^2 , \qquad (26)$$

Further we have for $-\mathcal{M}_4$ we have

$$(-\mathcal{M}_4(\theta \mathbf{U}_{j,k}^{n+1} - (1 - 2\theta)\mathbf{U}_{j,k}^n + \theta \mathbf{U}_{j,k}^{n-1}), D_0^t \mathbf{U}_{j,k}^n)\Delta t$$
$$= 1/2(\mathcal{P}^+(\mathbf{U}_{j,k}^n, \theta) - \mathcal{P}^-(\mathbf{U}_{j,k}^n, \theta)) \,. \tag{27}$$

Due to the result of the operators :
$\mathcal{P}^-(\mathbf{U}_{j,k}^n, \theta) = \mathcal{P}^+(\mathbf{U}_{j,k}^{n-1}, \theta)$ and $D_-^t \mathbf{U}_{j,k}^n = D_+^t \mathbf{U}_{j,k}^{n-1}$,
we can recursively derive the following result

$$||(1 - \Delta t^2 \tilde{\omega})^{1/2} D_+^t \mathbf{U}_{j,k}^n||^2 + \mathcal{P}^+(\mathbf{U}_{j,k}^n, \theta) \quad \leq \quad ||(1 - \Delta t^2 \tilde{\omega})^{1/2} D_+^t \mathbf{U}_{j,k}^0||^2$$
$$+ \quad \mathcal{P}^+(\mathbf{U}_{j,k}^0, \theta) \tag{28}$$

where for $\theta \in [0.25, 0.5]$ we have $\mathcal{P}^+(\mathbf{U}_{j,k}^n, \theta) \geq 0$ for all $n \in \mathbb{N}^+$ and therefore we have the unconditional stability. The scalar proof is also presented in the work of [2].

Remark 3.8 For $\theta = \frac{1}{12}$ the split method is 4th order accurate in time and space.

See the following theorem.

Theorem 3.9 We obtain a 4th order accurate scheme in time and space for the split method, see (11)-(13), when $\theta = 1/12$. That reads

$$D_+^t D_-^t \mathbf{U}_{j,k}^n - 1/12\mathcal{M}_4(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) + \mathcal{M}_4 \mathbf{U}_{j,k}^n$$
$$+ \mathcal{N}_{4,\theta}(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) = \mathbf{0} \tag{29}$$

where $\mathcal{M}_4$ is a fourth order discretization scheme in space.

The proof is given as

Proof 3.10 We consider the following Taylor-expansion :

$$\partial_{tt} \mathbf{U}_{j,k}^n = D_+^t D_-^t \mathbf{U}_{j,k}^n - \frac{\Delta t^2}{12} \partial_{tttt} \mathbf{U}_{j,k}^n + \mathcal{O}(\Delta t^4) \,, \tag{30}$$

Furthermore we have :

$$\partial_{tttt} \mathbf{U}_{j,k}^n \approx \mathcal{M}_4 \partial_{tt} \mathbf{U}_{j,k}^n \,, \tag{31}$$

and we can rewrite (30) as

$$\partial_{tt} \mathbf{U}_{j,k}^n \quad \approx \quad D_+^t D_-^t \mathbf{U}_{j,k}^n - \frac{\Delta t^2}{12} \mathcal{M}_4 \partial_{tt} \mathbf{U}_{j,k}^n + \mathcal{O}(\Delta t^4)$$
$$\approx \quad D_+^t D_-^t \mathbf{U}_{j,k}^n - \frac{\Delta t^2}{12} \mathcal{M}_4(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) + \mathcal{O}(\Delta t^4) \,, \tag{32}$$

So the fourth order time-stepping algorithm can be formulated as

$$D_+^t D_-^t \mathbf{U}_{j,k}^n - \frac{1}{12}\mathcal{M}_4(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) - \mathcal{M}_4 \mathbf{U}_{j,k}^n = \mathbf{0}. \tag{33}$$

The split method, (11)-(13) becomes

$$D_+^t D_-^t \mathbf{U}_{j,k}^n - \frac{1}{12}\mathcal{M}_4(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) - \mathcal{M}_4\mathbf{U}_{j,k}^n$$
$$-\mathcal{N}_{4,\frac{1}{12}}(\mathbf{U}_{j,k}^{n+1} - 2\mathbf{U}_{j,k}^n + \mathbf{U}_{j,k}^{n-1}) = \mathbf{0}, \tag{34}$$

and we obtain an 4th order split scheme, cf. the scalar case [2].

Remark 3.11 *As follows form the theorem 3.9 we obtain 4th order in time for $\theta = 1/12$. For the stability analysis the method is conditional stable for $\theta \in (0, 0.25)$. So the splitting method will not restrict our stability condition for the fourth order method with $\theta = 1/12$.*

Our theoretical results are verified by the following numerical examples.

4 Numerical tests of the split method

To test the 4th order split method we have done grid convergence studies on two types of problems. For the first we impose a smooth solution of (1) using a specific form of the forcing function **f** and check the error of the numerical solution against the known solution as the grid is refined.

For the second problem we use a singular forcing function (4), and compare the numerical solution to a solution computed using the Green's function for the free space elastodynamic problem. The convergence for this case is dependent not only on the approximations of time and space derivatives, but also on how the Dirac function is approximated.

During the numerical testing we have observed a need to reduce the allowable time step when the ration of λ over μ became to large. This is likely from the influence of the explicitly treated mixed derivative. For really high ratios (¿ 20) a reduction of 35% was necessary to avoid numerical instabilities.

4.1 Initial values and boundary conditions

In order to start the time stepping scheme we need to know the values at two earlier time levels. Starting at time $t = 0$ we know the value at level $n = 0$ as $\mathbf{U}^0 = \mathbf{g}_0$. The value at level $n = -1$ can be obtained by Taylor expansion as

$$\mathbf{U}^{-1} = \mathbf{U}^0 - \Delta t \partial_t \mathbf{U}^0 + \frac{\Delta t^2}{2}\partial_{tt}\mathbf{U}^0 - \frac{\Delta t^3}{6}\partial_{ttt}\mathbf{U}^0 + \frac{\Delta t^4}{24}\partial_{tttt}\mathbf{U}^0 + O(\Delta t^5), \tag{35}$$

where we use

$$\partial_t \mathbf{U}_{j,k}^0 = \mathbf{g}_{1\,j,k}, \tag{36}$$

$$\partial_{tt}\mathbf{U}_{j,k}^0 \approx \frac{1}{\rho}\left(\mathcal{M}_4\mathbf{g}_{0\,j,k}\right) + \mathbf{f}_{j,k}\right), \tag{37}$$

$$\partial_{ttt}\mathbf{U}_{j,k}^0 \approx \frac{1}{\rho}\left(\mathcal{M}_4\mathbf{g}_{1\,j,k}\right) + \partial_t\mathbf{f}^0{}_{j,k}\right), \tag{38}$$

$$\partial_{tttt}\mathbf{U}^0{}_{j,k} \approx \frac{1}{\rho}\left(\mathcal{M}_2^2\mathbf{g}_{0\,j,k}\right) + \mathcal{M}_4\mathbf{f}^0{}_{j,k} + \partial_{tt}\mathbf{f}^0{}_{j,k}\right), \tag{39}$$

and also for (38) and (39)

$$\partial_t \mathbf{f}^0_{\ j,k} \approx \frac{\mathbf{f}^1_{\ j,k} - \mathbf{f}^{-1}_{\ j,k}}{2\Delta t} \tag{40}$$

$$\partial_{tt} \mathbf{f}^0_{\ j,k} \approx \frac{\mathbf{f}^1_{\ j,k} - 2\mathbf{f}^0_{\ j,k} + \mathbf{f}^{-1}_{\ j,k}}{\Delta t^2}. \tag{41}$$

We are not considering the boundary value problem in this paper and so will not be concerned with constructing proper difference stencils at grid points close to the boundaries of the computational domain. We have simply added a two point thick layer of extra grid points at the boundaries of the domain and assigned the correct analytical solution at all points in the layer every time step.

Remark 4.1 *For the Dirichlet boundary conditions the splitting method, see (11)-(13), conserve also the conditions. We can use for the 3 equations, see (11)-(13), so for* $\mathbf{U}^*$, $\mathbf{U}^{**}$ *and for* $\mathbf{U}^{n+1}$ *the same conditions.*

For the Neumann boundary conditions and other boundary conditions of higher order we have also to split the boundary conditions with respect to the split operators, see [10].

4.2 Test example

For the first test case we use a forcing function

$$\begin{aligned}
\mathbf{f} = (\sin(t-x)\sin(y) - 2\mu\sin(t-x)\sin(y) \\
-(\lambda + \mu)(\cos(x)\cos(t-y) + \sin(t-x)\sin(y)), \\
\sin(t-y)\sin(x) - 2Vs^2\sin(x)\sin(t-y) \\
-(\lambda + \mu)(\cos(t-x)\cos(y) + \sin(y)\sin(t-y)))^T,
\end{aligned} \tag{42}$$

giving the analytical solution

$$\mathbf{U}^{\text{true}} = (\sin(x-t)\sin(y), \sin(y-t)\sin(x))^T. \tag{43}$$

Using the split method we solved (1) on a domain $x \times y = [-11] \times [-11]$ up to $t = 2$. We used two sets of material parameters; for the first case ρ, λ, and μ were all equal to 1, for the second case ρ and μ were 1 and λ was set to 14. Solving on four different grids with a refinement factor of two in each direction between the successive grids we obtained the results shown in table 1. The errors are measured in the ∞-norm defined as $||\mathbf{U}_{j,k}|| = \max(\max_{j,k}|u_{j,k}|, \max_{j,k}|v_{j,k}|)$. As can be seen we get the expected 4th order convergence for problems with smooth solutions.

To check the influence of the splitting error $\mathcal{N}_{4,\theta}$ on the error we solved the same problems using the non-split scheme (10). The results are shown in table 2. The errors are only marginally smaller than for the split scheme.

	$t = 2$		$e_h = \|\mathbf{U}^n - \mathbf{U}^{\text{true}}\|_\infty$	
h	case 1	$\log_2(\frac{e_{2h}}{e_h})$	case 2	$\log_2(\frac{e_{2h}}{e_h})$
0.05	1.7683e-07		2.5403e-07	
0.025	1.2220e-08	3.855	2.1104e-08	3.589
0.0125	7.9018e-10	3.951	1.4376e-09	3.876
0.006125	5.0013e-11	3.982	9.2727e-11	3.955

Table 1: Errors in max-norm for decreasing h and smooth analytical solution $\mathbf{U}^{\text{true}}$. Convergence rate indicates 4th order convergence for the split scheme.

	$t = 2$, $e_h = \|\mathbf{U}^n - \mathbf{U}^{\text{true}}\|_\infty$	
h	case 1	case 2
0.05	1.6878e-07	2.4593e-07
0.025	1.1561e-08	2.0682e-08
0.0125	7.4757e-10	1.4205e-09
0.006125	4.8112e-11	9.2573e-11

Table 2: Errors in max-norm for decreasing h and smooth analytical solution $\mathbf{U}^{\text{true}}$ and using the non-split scheme. Comparing with table 1 we see that the splitting error is very small for this case.

4.3 Singular forcing terms

In seismology and acoustics it is common to use spatially singular forcing terms which can look like

$$\mathbf{f} = \mathbf{F}\delta(\mathbf{x})g(t), \tag{44}$$

where $\mathbf{F}$ is a constant direction vector. A numeric method for (1) needs to approximate the Dirac function correctly in order to achieve full convergence. Obviously we cannot expect convergence close to the source as the solution will be singular for two and three dimensional domains.

The analyzes in [6] and [7] demonstrate that it is possible to derive regularized approximations of the Dirac function, which result in point wise convergence of the solution away from the sources. Based on these analyzes, we define one 2nd (δ_{h^2}) and one 4th (δ_{h^4}) order regularized approximations of the one dimensional Dirac function,

$$\delta_{h^2}(\tilde{x}) = \frac{1}{h} \begin{cases} 1 + \tilde{x}, & -h \leq \tilde{x} < 0, \\ 1 - \tilde{x}, & 0 \leq \tilde{x} < h, \\ 0, & \text{elsewhere,} \end{cases} \tag{45}$$

$$\delta_{h^4}(\tilde{x}) = \frac{1}{h} \begin{cases} 1 + \frac{11}{6}\tilde{x} + \frac{5}{8}\tilde{x}^2 + \frac{1}{6}\tilde{x}^3, & -2h \leq \tilde{x} < -h, \\ 1 + \frac{1}{2}\tilde{x} - \tilde{x}^2 - \frac{1}{2}\tilde{x}^3, & -h \leq \tilde{x} < 0, \\ 1 - \frac{1}{2}\tilde{x} - \tilde{x}^2 + \frac{1}{2}\tilde{x}^3, & 0 \leq \tilde{x} < h, \\ 1 - \frac{11}{6}\tilde{x} + \tilde{x}^2 - \frac{1}{6}\tilde{x}^3, & h \leq \tilde{x} < 2h, \\ 0, & \text{elsewhere,} \end{cases} \tag{46}$$
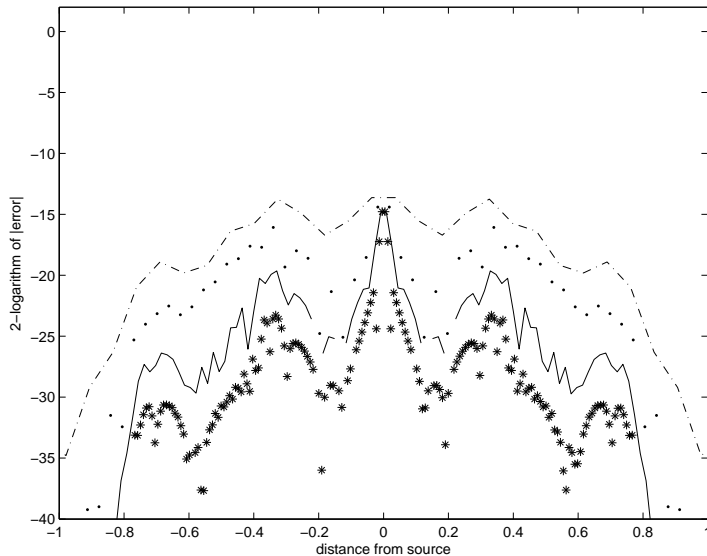
Figure 1: The 2−logarithm of the error along a line going through the source point for a point force located at $x = 0, y = 0$, and approximated in space by 46. Note that the error decays as $\mathcal{O}(h^4)$ away from the source, but not near it. The grid sizes were $h = 0.05\,(-\cdot)$, $0.025\,(\cdot)$, $0.0125\,(-)$, $0.006125\,(*)$. The numerical quadrature had an absolute error of approximately $10^{-11} \approx 2^{-36}$, so the error cannot be resolved beneath that limit.

where in the above $\tilde{x} = x/h$. The two and three dimensional Dirac functions are then approximated as $\delta_{h^{2,4}}(\tilde{x})\delta_{h^{2,4}}(\tilde{y})$ and $\delta_{h^{2,4}}(\tilde{x})\delta_{h^{2,4}}(\tilde{y})\delta_{h^{2,4}}(\tilde{z})$. The chosen time dependence was a smooth function given by

$$g(t) = \begin{cases} \exp(-1/(t(1-t))), & 0 \leq t < 1, \\ 0, & \text{elsewhere,} \end{cases} \tag{47}$$

which is C^∞. Using this forcing function we can compute the analytical solution by integrating the Green's function given in [8] in time. The integration was done using numerical quadrature routines from Matlab. Figures 1 and 2 shows examples of what the errors look like on a radius passing through the singular source at time $t = 0.8$ for different grid sizes h and the two approximations δ_{h^2} and δ_{h^4}. As can be seen the error is smooth and converges a small distance away from the source. However, using δ_{h^2} limits the convergence to 2nd order, while using δ_{h^4} gives the full 4th order convergence away from the singular source. When $t > 1$ the forcing goes to zero and the solution will be smooth everywhere. Table 3 shows the convergence behavior at time $t = 1.1$ for four different grids. Note that the full convergence is achieved even if the lower order δ_{h^2} is used as an approximation for the Dirac function. The convergence rate approaches 4 as we refine the grids, even though the solution was singular up to time $t = 1$.
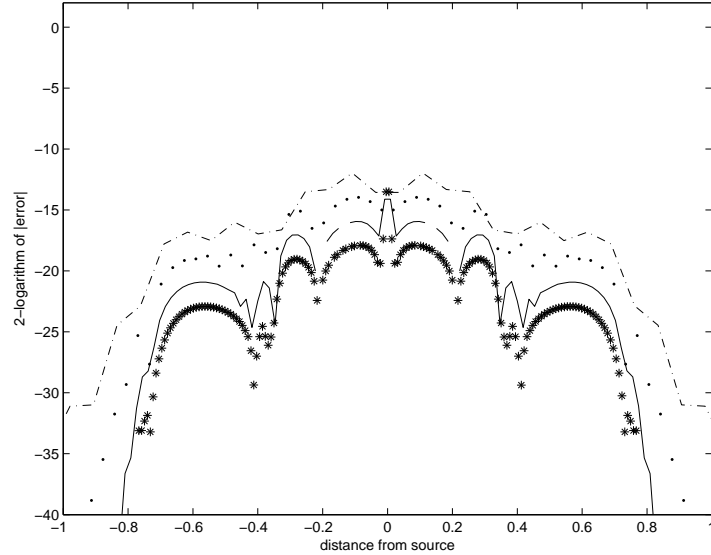
Figure 2: The 2−logarithm of the error along a line going through the source point for a point force located at $x = 0, y = 0$, and approximated in space by 45. Note that the error only decays as $\mathcal{O}(h^2)$ away from the source. The grid sizes were $h = 0.05\,(-\cdot)$, $0.025\,(\cdot)$, $0.0125\,(-)$, $0.006125\,(*)$.

h	$t = 1.1$, $e_h = \|\mathbf{U}^n - \mathbf{U}^{\text{true}}\|_\infty$	
		$\log_2\!\left(\frac{e_{2h}}{e_h}\right)$
0.05	1.1788e-04	
0.025	1.4146e-05	3.0588
0.0125	1.3554e-06	3.3836
0.00625	1.0718e-07	3.6606
0.003125	7.1890e-09	3.8981

Table 3: Errors in max-norm for decreasing h and analytical solution $\mathbf{U}^{\text{true}}$. Convergence rate approaches 4th order after the singular forcing term goes to zero.

4.4 Computational cost of the split method

For a two dimensional problem the 4th order explicit method (9) can be implemented using approximately 160 *floating point operations* (flops) per grid point.

The split method requires approximately 120 flops (first step) plus 2 times 68 flops (second and third step) for a total of 256 flops. This increase of ca 60% in the number of flops is somewhat offset by the larger time steps allowed by the split method, especially for "nice" material properties, making the two methods roughly comparable in computational cost.

5 A Three-dimensional Split Method

In three dimensions a 4th order difference approximation of the right hand side operator becomes

$$
\mathcal{M}_4 = \left(
\begin{array}{c}
(\lambda + 2\mu)\left(1 - \frac{h^2}{12}D^{x2}\right)D^{x2} + \mu\left(1 - \frac{h^2}{12}D^{y2}D^{y2} + 1 - \frac{h^2}{12}D^{z2}\right)D^{z2} \\
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{x2}\right)D_0^x\left(1 - \frac{h^2}{6}D^{y2}\right)D_0^y \\
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{x2}\right)D_0^x\left(1 - \frac{h^2}{6}D^{z2}\right)D_0^z
\end{array}
\right.
$$

$$
\begin{array}{c}
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{x2}\right)D_0^x\left(1 - \frac{h^2}{6}D^{y2}\right)D_0^y \\
(\lambda + 2\mu)\left(1 - \frac{h^2}{12}D^{y2}\right)D^{y2} + \mu\left(1 - \frac{h^2}{12}D^{x2}\right)D^{x2} + \mu\left(1 - \frac{h^2}{12}D^{z2}\right)D^{z2} \\
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{z2}\right)D_0^z\left(1 - \frac{h^2}{6}D^{y2}\right)D_0^y
\end{array}
$$

$$
\left.
\begin{array}{c}
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{x2}\right)D_0^x\left(1 - \frac{h^2}{6}D^{z2}\right)D_0^z \\
(\lambda + \mu)\left(1 - \frac{h^2}{6}D^{y2}\right)D_0^y\left(1 - \frac{h^2}{6}D^{z2}\right)D_0^z \\
(\lambda + 2\mu)\left(1 - \frac{h^2}{12}D^{z2}D^{z2} + \mu\left(1 - \frac{h^2}{12}D^{x2}\right)D^{x2} + 1 - \frac{h^2}{12}D^{y2}\right)D^{y2}
\end{array}
\right),
$$

operating on grid functions $\mathbf{U}_{j,k,l}^n$ defined at grid points x_j, y_k, z_l, t_n similarly to the two dimensional case. We can split $\mathcal{M}_4$ into six parts; $\mathcal{M}_{xx}, \mathcal{M}_{yy}, \mathcal{M}_{zz}$ containing the three second order directional difference operators, and $\mathcal{M}_{xy}, \mathcal{M}_{yz}, \mathcal{M}_{xz}$ containing the mixed difference operators.

There are a number of different ways we could split this scheme, depending on how we treat the mixed derivative terms. We have chosen to implement the

following split scheme in three dimensions:

1. $\rho\dfrac{\mathbf{U}_{j,k,l}^{*} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}}{\Delta t^2} = \mathcal{M}_4\mathbf{U}_{j,k,l}^{n} + \theta\mathbf{f}_{j,k,l}^{n+1} + (1 - 2\theta)\mathbf{f}_{j,k,l}^{n} + \theta\mathbf{f}_{j,k,l}^{n-1}$

2. $\rho\dfrac{\mathbf{U}_{j,k,l}^{**} - \mathbf{U}_{j,k,l}^{*}}{\Delta t^2} = \theta\mathcal{M}_{xx}\left(\mathbf{U}_{j,k,l}^{**} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right)$
$$+\dfrac{\theta}{2}\left(\mathcal{M}_{xy} + \mathcal{M}_{xz}\right)\left(\mathbf{U}_{j,k,l}^{*} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right)$$

3. $\rho\dfrac{\mathbf{U}_{j,k,l}^{***} - \mathbf{U}_{j,k,l}^{**}}{\Delta t^2} = \theta\mathcal{M}_{xx}\left(\mathbf{U}_{j,k,l}^{***} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right)$
$$+\dfrac{\theta}{2}\left(\mathcal{M}_{xy} + \mathcal{M}_{yz}\right)\left(\mathbf{U}_{j,k,l}^{**} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right)$$

4. $\rho\dfrac{\mathbf{U}_{j,k,l}^{n+1} - \mathbf{U}_{j,k,l}^{***}}{\Delta t^2} = \theta\mathcal{M}_{xx}\left(\mathbf{U}_{j,k,l}^{n+1} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right)$
$$+\dfrac{\theta}{2}\left(\mathcal{M}_{xz} + \mathcal{M}_{yz}\right)\left(\mathbf{U}_{j,k,l}^{***} - 2\mathbf{U}_{j,k,l}^{n} + \mathbf{U}_{j,k,l}^{n-1}\right).$$

The properties such as splitting error, accuracy, stability, etc., for the three dimensional case are similar to the two dimensional case treated in the earlier sections.

5.1 Testing the three dimensional scheme

We have done some numerical experiments with the three dimensional scheme in order to test the convergence and stability. We used a forcing

$$\mathbf{f} = -(-1 + \lambda + 4\mu)\sin(t - x)\sin(y)\sin(z) -$$
$$(\lambda + \mu)\cos(x)(2\sin(t)\sin(y)\sin(z) + \cos(t)\sin(y + z)),$$
$$-(-1 + \lambda + 4\mu)\sin(x)\sin(t - y)\sin(z) -$$
$$(\lambda + \mu)\cos(y)(2\sin(t)\sin(x)\sin(z) + \cos(t)\sin(x + z)),$$
$$-(\lambda + \mu)\cos(t - y)\cos(z)\sin(x) - \sin(y)((\lambda + \mu)\cos(t - x)\cos(z) +$$
$$(-1 + \lambda + 4\mu)\sin(x)\sin(t - z)))^T, \quad (48)$$

giving the analytical solution

$$\mathbf{U}^{\text{true}} = (\sin(x - t)\sin(y)\sin(z)\,,$$
$$\sin(y - t)\sin(x)\sin(z),$$
$$\sin(z - t)\sin(x)\sin(y))^T\,. \quad (49)$$

As earlier we tested this for a number of different grid sizes. Using the same two sets of material parameters as for the two dimensional case we ran up until $t = 2$ and checked the max error for all components of the solution. The results are given in table 4. We have also tested the three dimensional scheme using singular forcing functions approximated using (45) and (46). The results are very similar to the two dimensional case and we have therefore omitted them here.

	$t = 2$		$e_h = \|\mathbf{U}^n - \mathbf{U}^{\text{true}}\|_\infty$	
h	case 1	$\log_2(\frac{e_{2h}}{e_h})$	case 2	$\log_2(\frac{e_{2h}}{e_h})$
0.1	4.2986e-07		1.8542e-06	
0.05	3.5215e-08	3.61	1.3605e-07	3.77
0.025	3.0489e-09	3.53	8.0969e-09	4.07
0.0125	2.0428e-10	3.90	4.7053e-10	4.10

Table 4: Errors in max-norm for decreasing h and smooth analytical solution $\mathbf{U}^{\text{true}}$. Convergence rate indicates 4th order convergence for the three dimensional split scheme.

6   Conclusion

The split scheme has been proven to work well in practice for different types of material properties. It is comparable to the fully explicit 4th order scheme (9) in terms of computational cost, but should be easier to implement, as no difference approximations of higher order operators are needed.

A vital component of a model in seismology is stable higher order approximations of the boundary conditions, something we have saved for future paper.

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