Linear differential-algebraic equations with properly stated leading term: B-critical points*

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Abstract

We examine in this paper so-called B-critical points of linear, time-varying differential-algebraic equations (DAEs) of the form $A(t)(D(t)x(t))' + B(t)x(t) = q(t)$. These critical or singular points, which cannot be handled by classical projector methods, require adapting a recently introduced framework based on Π-projectors. Via a continuation of certain invariant spaces through the singularity, we arrive at an scenario which accommodates both A- and B-critical DAEs. The working hypotheses apply in particular to standard-form analytic systems although, in contrast to other approaches to critical problems, the scope of our approach extends beyond the analytic setting. Some examples illustrate the results.

Keywords: differential-algebraic equation, index, projector, critical point, singularity.

AMS subject classification: 34A09, 34A30.

1 Introduction

The present paper extends our investigation of critical points arising in linear, time-varying differential-algebraic equations (DAEs) of the form

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{J}, \tag{1}$$

where $\mathcal{J} \subseteq \mathbb{R}$ is an interval, and the matrix coefficients $A(t) \in L(\mathbb{R}^n, \mathbb{R}^m)$, $D(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$, $B(t) \in L(\mathbb{R}^m)$ depend continuously on t. The leading term $A(t)(D(t)x(t))'$ arises in that form in different application fields and provides several analytical and numerical advantages, see [1, 4, 8] and references therein. Additionally, "classical" or standard form linear DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{J}, \tag{2}$$

with $E(t),\ F(t) \in L(\mathbb{R}^m)$, are comprised in the setting defined by (1) under the mild assumption that there exists a C^1 projector $P(t)$ such that $E(t)P(t) = E(t)$, since we may rewrite (2) as

$$E(t)(P(t)x(t))' + [F(t) - E(t)P'(t)]x(t) = q(t), \tag{3}$$

and the equation takes the form (1). Such a projector P exists in particular if E is C^1 with constant rank, but it may also exist even if the rank varies.

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Most frameworks, in particular [2, 7, 8, 13, 14], for the analysis of linear DAEs, either in the form (1) or in the classical setting (2), focus on problems with a well-defined (differentiation, strangeness, tractability, or geometrical) index on a given interval. The different index notions are meant to support solvability results on that interval using different approaches. In particular, recent works [8, 9, 10] introduce a tractability chain $\{G_i\}$ leading to solvability results and a canonical form for linear DAEs (1).

In contrast, less attention has been paid to so-called *critical* or *singular* problems, which can be roughly defined as those where the assumptions supporting an index notion fail. In [11, 12], extending previous results from [15], a framework for the local analysis of (1) is introduced, and an invariant taxonomy of critical points is presented. Apart from the latter references, the literature on singularities of linear DAEs virtually amounts to [5, 13], the scope in both cases being limited to analytic problems.

Broadly speaking, the above-mentioned taxonomy distinguishes type-A and type-B critical points. A-critical points are defined by rank deficiencies in the matrix sequence $\{G_i\}$. A working scenario to analyze type-A critical points is developed in [12]; the decoupling procedure ends up with a scalarly implicit inherent ODE together with several possibly singular relations for the remaining solution components.

For sufficiently smooth (C^{m-1}) problems , critical points which do not fall in the A-category above must belong to the so-called type-B class, as shown in [12, Theorem 3.5]. These critical points are somehow more subtle than A-points, being defined by the loss of transversality of $N_i := \ker G_i$, for some $i \geq 1$, with respect to the characteristic (i.e. independent of projectors) time-dependent space $K_{i-1} := N_0 \oplus \ldots \oplus N_{i-1}$. As in the A-case, the definition of a B-critical point is independent of projectors and invariant with respect to premultiplication and linear time-varying coordinate changes [12, Theorem 3.3]. Type-B critical points can be displayed in the constant coefficient context: these cases necessarily correspond to a singular matrix pencil. Also, based on [11, Corollary 3], B-critical points can be shown to yield non-regular transposed matrix pairs in the reduction framework of Rabier and Rheinboldt [13], meaning that related phenomena should be expected within the reduction approach.

The scope of the analysis of [12] explicitly excludes type-B critical points (see specifically Proposition 4.2 there): their study is the main goal of the present work. A rough picture of the reason for the above-mentioned framework to exclude B-critical points is the following: the matrix sequence $\{G_i\}$ as constructed in [12] is based on the choice of so-called admissible projectors Q_i onto the spaces $N_i = \ker G_i$; the admissibility condition reads $Q_i Q_j = 0$ for $j < i$, and relies upon the transversality condition $K_{i-1} \cap N_i = \{0\}$. Trying to force $Q_i Q_j = 0$ through a type-B critical point yields an unbounded projector and therefore rules out the construction of the matrix chain beyond that step. A different framework is necessary in order to analyze these problems.

This new framework has been recently introduced by the authors, cf. [16]. Writing $P_i = I - Q_i$, the main idea is that the matrix chain construction can be carried out using only certain products of the form $P_0 \cdots P_i$ and $P_0 \cdots P_{i-1} Q_i$. These products can be replaced by certain alternative projectors Π_i, M_i which capture their essential properties and yield an equivalent, technically simpler index definition. Several advantages of this approach for regular DAEs are discussed in the above-mentioned reference [16]. In the present work we adapt this framework for it to accommodate B-critical points, inspired in a property depicted by the circuit example discussed in 2.3: in this example, the Π-projectors can be continuously extended through certain type-B critical points, even though some of the Q_i's become unbounded there.

The paper is structured as follows: Section 2 compiles some previous results concerning linear DAEs of the form (1). It includes in particular a summary of the Π-projectors chain construction from [16], and also background on A-critical points from [12]. In Section 3 we introduce a working scenario which is able to include B-critical points besides A-critical ones. This setting is proved in Proposition 2 and Theorem 1 to hold for analytic problems in the standard form (2), showing that our approach covers the analytic context as a particular case. The solutions of the critical DAE are then unveiled through the scalarly implicit decoupling presented in Theorem 2. Some examples in Section 4 illustrate the discussion, whereas concluding remarks can be found in Section 5.

2 Background: Regular and critical points of linear DAEs

2.1 Regular problems

Definition 1. *The leading term of the DAE (1) is properly stated on the interval $\mathcal{J}$ if the coefficients A and D satisfy*

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \quad t \in \mathcal{J}, \tag{4}$$

and both subspaces are spanned by continuously differentiable basis functions.

For $i \in \mathbb{N} \cup \{0\}$, a time-varying subspace $L(t) \subseteq \mathbb{R}^l$, $t \in \mathcal{J}$, which has constant dimension and is spanned by basis functions in $C^i(\mathcal{J}, \mathbb{R}^l)$ will be said to be a C^i-subspace on $\mathcal{J}$.

In the sequel we compile from [8] the matrix chain construction supporting the tractability index notion for the DAE (1). Assuming the leading term of (1) to be stated properly on the interval $\mathcal{J}$, we denote by $R(t)$ the C^1 projector function which realizes the decomposition (4) with $\operatorname{im} R(t) = \operatorname{im} D(t)$, $\ker R(t) = \ker A(t)$, $t \in \mathcal{J}$.

Hereafter we mostly drop the argument t, the relations being meant pointwise. Set $G_0 := AD$, $B_0 := B$. If the leading term is properly stated, then G_0 has constant rank r_0 on $\mathcal{J}$. Defining $N_0 := \ker G_0$, we let P_0 be any continuous projector along N_0 and take $Q_0 := I - P_0$.

Now, we denote as $D^-(t)$ the continuous in t generalized inverse of $D(t)$ uniquely defined by the four conditions

$$DD^-D = D, \quad D^-DD^- = D^-, \quad DD^- = R, \quad D^-D = P_0, \tag{5}$$

for all $t \in \mathcal{J}$. For $i \geq 1$, define

$$G_i := G_{i-1} + B_{i-1}Q_{i-1}. \tag{6}$$

If G_i has constant rank r_i, let $N_i := \ker G_i$, and choose a continuous projector P_i along N_i. Write $Q_i = I - P_i$ and

$$B_i := B_{i-1}P_{i-1} - G_i D^-(DP_0 \cdots P_i D^-)' DP_0 \cdots P_{i-1}. \tag{7}$$

The sequence is then continued by defining G_{i+1} and so on. As detailed below, meeting a non-singular (on $\mathcal{J}$) matrix function G_μ will define the problem as regular with index μ.

To build the matrix chain (6)-(7), we assume the products $DP_0 \cdots P_i D^-$ to be C^1. The projectors Q_i, $i \geq 1$, are additionally required to satisfy $Q_iQ_j = 0$, for all $0 \leq j < i$ and all $t \in \mathcal{J}$. A sequence $Q_0, Q_1, \ldots, Q_i$ (or, respectively, $P_0, P_1, \ldots, P_i$) satisfying those requirements is called *admissible* up to level i; if such a sequence exists then the DAE is said to be *nice* at level i, and this notion can be proved independent of the actual choice of admissible sequences [11, Corollary 1]. The characteristic values $r_i = \operatorname{rk} G_i$ and the spaces $N_0 \oplus \ldots \oplus N_i$ are also independent of the choice of admissible projectors [9].

For later use we emphasize that, given a DAE nice at level $i-1$, the existence of a P_i (resp. Q_i) continuing the sequence in an admissible manner relies on the following properties:

(a) G_i has constant rank r_i on $\mathcal{J}$, for some (hence any) admissible up to level $i-1$ sequence $Q_0, \ldots, Q_{i-1}$;

(b) $(N_0 \oplus \cdots \oplus N_{i-1}) \cap N_i = \{0\}$ on $\mathcal{J}$,

(c) $DP_0 \cdots P_i D^- \in C^1(\mathcal{J}, L(\mathbb{R}^n))$.

If the coefficients A, D, B in (1) are C^r, $r \geq 1$, and the DAE satisfies the conditions (a), (b) above up to a given level $k-1$ with $k \leq r$, then it is nice at level k, that is, the projector Q_k can be taken in a way such that (c) holds [11, Proposition 3].

Assume the DAE (1) to have a properly stated leading term on $\mathcal{J}$. If both A and D are invertible on $\mathcal{J}$, then (1) is said to be a regular DAE with tractability index zero (on $\mathcal{J}$). The DAE (1) is said to be regular with tractability index $\mu \in \mathbb{N}$ on $\mathcal{J}$ if there exists an admissible projector sequence $Q_0, \ldots, Q_{\mu-1}$, and $r_{\mu-1} < r_\mu = m$ (that is, G_μ is non-singular on $\mathcal{J}$).

An equivalent, simpler construction of this matrix chain, displaying several advantages for regular DAEs, has been recently introduced [16]. Denoting $\Pi_0 = P_0$, $M_0 = Q_0 = I - \Pi_0$, $K_0 := N_0$, $H_0 := B$, for $i \geq 1$ we replace (6) and (7) by

$$\begin{aligned} G_i &:= G_{i-1} + H_{i-1} M_{i-1} && (8) \\ H_i &:= H_{i-1} - G_i D^-(D\Pi_i D^-)' D, && (9) \end{aligned}$$

where Π_i is a continuous projector along $K_i := K_{i-1} \oplus N_i$, with $\operatorname{im} \Pi_i \subseteq \operatorname{im} \Pi_{i-1}$, and $M_i := \Pi_{i-1} - \Pi_i$. The constant rank condition (a) above reads the same within this framework, in the understanding that G_i is now constructed according to (8)-(9), whereas (b) and (c) can now be stated as (b) $K_{i-1} \cap N_i = \{0\}$ and (c) $D\Pi_i D^-$ being in C^1.

In regular contexts this construction is proved in [16] (see specifically Theorem 1 and Corollary 1 there) to be equivalent to the previous one in the sense that the DAE (1) has tractability index μ on $\mathcal{J}$ if and only if there exists a Π-sequence satisfying the above-mentioned requirements at every step, for which G_i is singular if $i < \mu$ and G_μ is non-singular on $\mathcal{J}$. The key aspect is that the projectors Π_i and M_i replace the products $P_0 \cdots P_i$ and $P_0 \cdots P_{i-1} Q_i$ arising in the previous framework, but now the construction relies on the spaces $K_i = N_0 \oplus \ldots \oplus N_i$, which are independent of the choice of projectors, and not on the individual ones N_i which certainly depend on this choice. Besides the advantages for regular problems discussed in [16], this construction will allow for the analysis of B-critical points, as discussed in Section 3 below.

Decoupling. The significance of the frameworks summarized above is supported on the fact that solutions of the DAE can be computed explicitly in the original setting of the problem. Indeed, for the below-depicted continuous coefficients $\mathcal{K}_k$, $\mathcal{L}_k$, $\mathcal{N}_{kj}$, $\mathcal{M}_{kj}$, solutions of a regular index μ can be written as

$$x = D^- u + v_{\mu-1} + \cdots + v_1 + v_0, \tag{10}$$

where $u = \Pi_{\mu-1}x \in \operatorname{im} D\Pi_{\mu-1}$ and the components $v_k = M_k x,\ k = \mu-1,\ \ldots,\ 0$, satisfy

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}BD^-u = D\Pi_{\mu-1}G_\mu^{-1}q, \tag{11a}$$

$$v_{\mu-1} = -\mathcal{K}_{\mu-1}D^-u + \mathcal{L}_{\mu-1}q, \tag{11b}$$

$$v_k = -\mathcal{K}_k D^-u + \mathcal{L}_k q + \sum_{j=k+1}^{\mu-1} \mathcal{N}_{kj}(Dv_j)' + \sum_{j=k+2}^{\mu-1} \mathcal{M}_{kj}v_j, \quad k = \mu-2,\ \ldots,\ 1,\ 0. \tag{11c}$$

The coefficients $\mathcal{K}_k,\ \mathcal{L}_k,\ \mathcal{N}_{kj},\ \mathcal{M}_{kj}$ as computed in [9] are immediately shown to read, in terms of the $\Pi_i,\ M_i$ projectors:

$$\mathcal{K}_k = M_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1} + M_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}, \tag{12}$$

$$\mathcal{L}_k = M_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}, \tag{13}$$

$$\mathcal{N}_{kj} = M_kP_{k+1}\cdots P_{j-1}Q_jD^-, \quad k+1 \leq j \leq \mu-1, \tag{14}$$

$$\begin{aligned}\mathcal{M}_{kj} = & -M_k\{Q_{k+1}D^-(DM_{k+1}D^-)' + P_{k+1}Q_{k+2}D^-(DM_{k+2}D^-)' + \\ & + \cdots + P_{k+1}\cdots P_{\mu-2}Q_{\mu-1}D^-(DM_{\mu-1}D^-)'\}DM_j - \\ & - \sum_{i=1}^{j} M_kP_{k+1}\cdots P_{\mu-1}\cdots P_iD^-(D\Pi_iD^-)'DM_j, \quad k+2 \leq j \leq \mu-1,\end{aligned}$$

with $k = 0, \ldots, \mu-1$. Here, P_i and Q_i can be computed as $Q_i = G_\mu^{-1}H_iM_i$, $P_i = I - Q_i$ [16].

We remark that further simplification of the $\mathcal{M}_{kj}$ coefficients is possible. Indeed, using the properties $P_{k+1}\cdots P_{\mu-1}P_r = P_{k+1}\cdots P_{\mu-1} - Q_r$ if $r \leq k$, $M_iM_j = 0$ if $i \neq j$ and $\Pi_iM_j = M_j$ if $i < j$, via some technical computations one can derive

$$\begin{aligned}\mathcal{M}_{kj} = & M_k(I - P_{k+1}\cdots P_{\mu-1} - P_{k+1}\cdots P_{j-1}Q_j)D^-(DM_jD^-)'D + \\ & + M_k(P_{k+1}\cdots P_{\mu-1} - \sum_{i=k+1}^{j} P_{k+1}\cdots P_i)D^-(DM_jD^-)'D,\end{aligned}$$

which can be rewritten as

$$\mathcal{M}_{kj} = M_k\left(Q_{k+1} - P_{k+1} + \sum_{i=k+2}^{j-1} P_{k+1}\cdots P_{i-1}(Q_i - P_i)\right)D^-(DM_jD^-)'D, \quad k+2 \leq j \leq \mu-1. \tag{15}$$

2.2 Critical problems

Using a local version of the P-framework summarized in 2.1, a point $t^* \in \mathcal{J}$ is called regular in [11] if there exists an open interval $\mathcal{I}$ with $t^* \in \mathcal{I} \subseteq \mathcal{J}$ where the DAE is regular. The (open) set of regular points $\mathcal{J}_{\text{reg}}$ is well-defined independently of projectors; points in $\mathcal{J} - \mathcal{J}_{\text{reg}}$ are called *critical*. These notions can be equivalently defined in terms of Π-projectors.

In order to be able to handle critical points arising in the initial formulation of the problem, we will relax the proper formulation allowing for rank deficiencies in A, as follows:

Definition 2. *The DAE (1) is* quasi-properly stated *on $\mathcal{J}$ if there exists a projector function $R \in C^1(\mathcal{J}, L(\mathbb{R}^n))$ satisfying* $\operatorname{im} R = \operatorname{im} D$ *and* $\ker R \subseteq \ker A$, *for all $t \in \mathcal{J}$.*

Note that the definition above implies that $D(t)$ has constant rank, $\operatorname{im} D$ is C^1, and there exists a C^1 subspace of $\ker A(t)$ transversal to $\operatorname{im} D(t)$. The constant rank assumption on D is reasonable since this matrix is intended to capture the derivatives actually involved in the problem; in many practical cases D will be a constant matrix. Analytic, standard form linear DAEs always admit a quasi-proper formulation, as will be shown in Proposition 2.

For linear DAEs with C^{m-1} coefficients, critical points belong to one of the types A and B below, as shown in [12, Theorem 3.5]. This means that the smoothness assumption (c) in the admissibility notion can be met in sufficiently smooth cases.

Definition 3. *Assume the DAE (1) to be quasi-properly stated with continuous coefficients; $t_* \in \mathcal{J}$ is said to be a critical point of*

(i) type 0 if G_0 has a rank drop at t_*;

(ii) type k-A, $k \geq 1$, if there exists a neighborhood $\mathcal{I} \subseteq \mathcal{J}$ of t_* where the DAE is nice up to level $k-1$, but G_k has a rank drop at t_* for some (hence any) admissible sequence $Q_0, \ldots, Q_{k-1}$.

(iii) type k-B, $k \geq 1$, if there exists a neighborhood $\mathcal{I} \subseteq \mathcal{J}$ of t_* where the DAE is nice up to level $k-1$ and G_k has constant rank for some (hence any) admissible sequence $Q_0, \ldots, Q_{k-1}$, but the intersection $N_k(t_*) \cap \{N_0(t_*) \oplus \cdots \oplus N_{k-1}(t_*)\}$ is nontrivial, for these (hence any other) projectors and G_k.

As indicated in Section 1, a framework for the projector-based analysis of A-critical points is discussed in [12]. Roughly speaking, even in the presence of a rank deficiency in some matrix G_k (defining a k-A critical point), if we assume that the kernel N_k admits a smooth continuation through the critical point then it is possible to extend the projector Q_k continuously to the whole working interval. This way the chain construction can be still performed and we are led to a decoupling of the form

$$\omega_\mu u' - \omega_\mu (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}BD^-u = D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}q, \tag{16a}$$

$$\omega_\mu v_{\mu-1} = -\mathcal{K}_{\mu-1}^{\mathrm{adj}}D^-u + \mathcal{L}_{\mu-1}^{\mathrm{adj}}q, \tag{16b}$$

$$\omega_\mu v_k = -\mathcal{K}_k^{\mathrm{adj}}D^-u + \mathcal{L}_k^{\mathrm{adj}}q + \omega_\mu \sum_{j=k+1}^{\mu-1} \mathcal{N}_{kj}(Dv_j)' + \omega_\mu \sum_{j=k+2}^{\mu-1} \mathcal{M}_{kj}v_j, \quad k = \mu-2, \ \ldots, \ 0, \tag{16c}$$

where all the coefficients are continuous and the leading scalar coefficient $\omega_\mu = \det G_\mu$ typically vanishes at critical points. The analysis hence leads to a singular ODE setting. See details in [12].

However, those working assumptions do not accommodate B-critical points, as shown in [12, Proposition 4.2]. The obstruction is actually an important one since it owes to the fact that the requirement $Q_iQ_j = 0$ yields unbounded projectors as a B-critical point (where the transversality of the N_i spaces is lost) is approached.

In contrast, using the K_i-spaces and Π_i, M_i projectors this difficulty can be overcome. In order to handle critical points of type-B, we will use in Section 3 the reformulation of the matrix chain construction in terms of (8)-(9) introduced in [16]. This approach will yield a decoupling similar to (16) but with increasing exponents in the leading singular coefficients ω_μ.

2.3 Example 1

The circuit displayed in Figure 2.1 is taken from [12]. Besides an independent voltage source, a capacitor and an inductor, the circuit includes a current-controlled current source (CCCS) with a continuously time-varying controlling parameter $\gamma(t)$: when $\gamma(t) < 1$ the CCCS behaves as an attenuator, whereas $\gamma(t) > 1$ makes the source behave as an amplifier. In the transition between both regimes, that is, at time values t for which $\gamma(t) = 1$, critical points will be displayed, as discussed below.

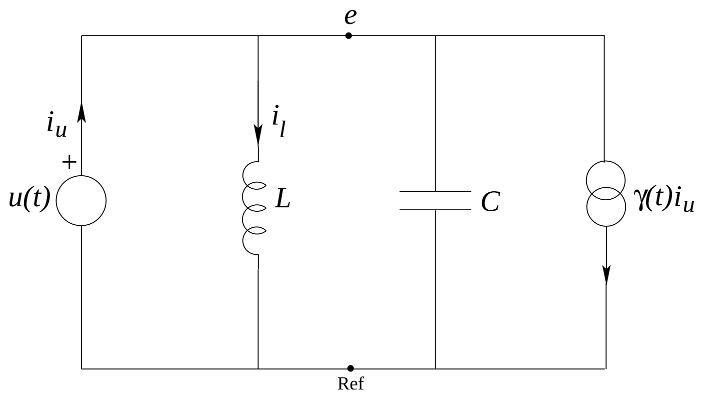

Figure 2.1: Example 1. A linear time-varying circuit.

The interest of this circuit stems from the fact that it includes a loop defined by a capacitor and a voltage source and, in addition, a CCCS for which the controlling current is the one of a voltage source within a *C*-*V* loop: this puts the circuit beyond the scope of the structural analysis of [3] (see item 4 of Table V there). The model provided by Modified Node Analysis (MNA) [3] reads

$$\begin{aligned} (Ce)' + i_l + (\gamma(t) - 1)i_u &= 0, && (17a)\\ (Li_l)' - e &= 0, && (17b)\\ e &= u(t). && (17c) \end{aligned}$$

Normalizing $C = L = 1$ and letting $\alpha(t) = \gamma(t) - 1$, we may consider (17) as the particular case with $q_1 = q_2 = 0$, $q_3(t) = u(t)$ of the Hessenberg DAE

$$\begin{aligned} x' + y + \alpha(t)z &= q_1(t) && (18a)\\ y' - x &= q_2(t) && (18b)\\ x &= q_3(t). && (18c) \end{aligned}$$

Points where $\gamma(t) = 1$ yield zeros of $\alpha(t)$. With the notation depicted in (18), this system can be written as the following DAE with properly stated leading term:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & \alpha(t) \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Obviously, $N_0 = \ker G_0 = \text{span}[(0, 0, 1)]$. Set

$$D^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so that } G_1 = \begin{bmatrix} 1 & 0 & \alpha(t) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $G_1(t)$ has constant rank $r_1 = 2$, and the nullspace $N_1 = \ker G_1 = \text{span}[(-\alpha(t), 0, 1)]$ is continuous.

Now, the intersection $N_0(t) \cap N_1(t)$ is defined by the conditions $x = y = 0, \alpha(t)z = 0$; therefore, at zeros t_* of α we get $N_0(t_*) \cap N_1(t_*) = N_0(t_*) \neq 0$. Hence, points t_* where $\alpha(t_*) = 0$ are critical points of type 1-B.

It can be checked that at points where $\alpha(t_*) \neq 0$, the DAE is regular with index-2: within the P-framework summarized in 2.1, the projector $Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{\alpha} & 0 & 0 \end{bmatrix}$, yields

$$P_0P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ P_0Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B_1 = BP_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ G_2 = \begin{bmatrix} 1 & 0 & \alpha \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix $G_2(t)$ is indeed nonsingular, under the assumed condition $\alpha(t_*) \neq 0$.

In contrast, remark that the chosen projector Q_1 (as it would happen with any other choice of Q_1 onto N_1 satisfying $N_0 \subseteq \ker Q_1$) is not defined at critical points where $\alpha(t_*) = 0$, and is unbounded in any punctured neighborhood of t^*. Nevertheless, the *products* $\Pi_1 = P_0P_1$, $M_1 = P_0Q_1$ can be smoothly extended through critical points, yielding a well-defined matrix G_2 which becomes singular at points where $\alpha(t) = 0$. The key idea is that even though the spaces N_1 and N_0 are not transversal at B-critical points, the direct sum $N_0 \oplus N_1$ (well-defined at regular points) will admit a smooth continuation on the whole working interval. This property can be used to accommodate a broad family of critical problems, as discussed in the next Section.

3 Type-B critical points

As indicated in 2.1, at regular points the setting described by the Π-projectors is equivalent to the one defined by P-projectors; nevertheless, besides the simplification provided for regular problems, the former presents an important advantage regarding type-B critical problems. Namely, if the intersection $K_{i-1} \cap N_i$ becomes non-trivial at a given point, there is no way to extend continuously through it any projector Q_i onto N_i satisfying the admissibility condition $K_{i-1} \subseteq \ker Q_i$: see specifically Example 1 above, where the transversality of $K_0 = N_0$ and N_1 is lost at critical points. In contrast, it may well happen (as it is the case in the above-mentioned circuit example) that the space K_i, well-defined as $K_{i-1} \oplus N_i$ at regular points, could be continuously extended through the critical point, and therefore the Π-projectors could be extended through the critical point in a way such that the matrix chain construction can be pursued one-step further. This rough picture is made precise in the present Section.

3.1 The setting for critical problems

The setting for our analysis of critical problems will go beyond just isolated critical points:

Assumption 1. *The set $\mathcal{J}_{\text{reg}}$ of regular points is dense in $\mathcal{J}$.*

We will restrict the attention to problems with *almost uniform characteristic values*, defined by Assumption 2 below (see Proposition 1). In virtue of the quasi-proper statement of the problem, the

C^1 projector R is well-defined in the whole of $\mathcal{J}$, and there exists a continuous projector along $\ker D$, to be denoted by $\Pi_0 = P_0$. Note however that Π_0 does not need to project along $K_0 = \ker G_0$ since there may be rank deficiencies in G_0 coming from A. The existence of a continuous D^- satisfying the four properties displayed in (5) is also guaranteed.

Assumption 2. *The DAE is quasi-properly stated, it holds that* $\ker \Pi_0(t) = K_0(t)$ *for* $t \in \mathcal{J}_{\text{reg}}$*, and there exist projector functions* $\Pi_1, \ldots, \Pi_{m-1}$ *continuous on* $\mathcal{J}$*, with* $D\Pi_i D^-$ *continuously differentiable on* $\mathcal{J}$*, which satisfy* $\operatorname{im} \Pi_i \subseteq \operatorname{im} \Pi_{i-1}$ *and such that, for* $t \in \mathcal{J}_{\text{reg}}$*,* $\ker \Pi_i(t) = K_i(t)$*.*

This assumption mainly expresses a geometrical property, namely, the existence of continuous extensions of the "sum" spaces K_i preserving the transversality property depicted in (b) (page 4). Its importance relies on the fact that the construction of the chain $\{G_i\}$ according to the rules specified in (8)-(9) is feasible also in this setting, and particularizes to a tractability chain at regular points.

Proposition 1. *Under assumptions 1-2, the DAE has the same characteristic values* $r_0, \ldots, r_{\mu-1}$ *and the same index* μ *in the whole* $\mathcal{J}_{\text{reg}}$*.*

Proof. Note that the characteristic values can be computed at regular points as $r_0 = \operatorname{rk} G_0$, $r_i = (i+1)m - \dim \ker \Pi_i - \sum_{j=0}^{i-1} r_j$, $i = 1, \ldots, \mu-1$. From the assumed continuity of Π_i, it follows that $\dim \ker \Pi_i$ is constant and therefore the expressions defining r_i are constant on $\mathcal{J}_{\text{reg}}$; since μ is defined also in terms of r_i, it also follows that all regular points have index μ. □

This setting allows for the existence of not only type-A critical points but also B-points, in contrast to the framework presented in [12]. Actually, the decoupling arrangement (11) discussed in Section 2 for regular problems can be extended to the context defined by Assumptions 1-2, as will be shown later (cf. Theorem 2).

It is worth remarking that these working assumptions hold in particular for analytic problems: this follows from Proposition 2 and Theorem 1 below. Note, in this direction, that analytic problems fill the scope of other approaches to singular linear time-varying DAEs [5, 13]. In the proof of these results we will make repeated use of the following property: an analytic matrix function has constant rank except at an isolated set $\mathcal{S}$; additionally, the orthogonal projectors along its kernel and along its image are analytic on $\mathcal{J} - \mathcal{S}$ and can be extended as an analytic function on the whole interval $\mathcal{J}$ [13, Lema 2.2]. We will also use the property $\ker A \cap \ker B = \ker (A^T A + B^T B)$, for any two matrices A, B having the same order.

Proposition 2. *Let* $E(t)$*,* $F(t)$ *in the standard form DAE (2) be analytic. Then (2) admits a quasi-proper statement of the form (1) with analytic coefficients and analytic* R*.*

Proof. Let P be the analytic extension to $\mathcal{J}$ of the orthogonal projector along $\ker E$ at maximal rank points. The reformulation (3) is supported on the fact that $E = EP$ (or, equivalently, $E(I - P) = 0$) holds on a dense subset of $\mathcal{J}$ and, therefore, it must hold on the whole $\mathcal{J}$. The requirements in Definition 2 are easily checked to be satisfied with $A := E$, $D = R := P$, and $B := F - EP'$. □

Theorem 1. *Let* A, D, B *in (1) be analytic, and assume that the DAE is quasi-properly stated in* $\mathcal{J}$*. If the regular set is non-empty, then critical points are isolated, and Assumptions 1 and 2 hold.*

Proof. The analytic product $G_0 = AD$ has constant rank except on a set of isolated points (type-0 critical points). Let Π_0 be the analytic extension on $\mathcal{J}$ of the orthogonal projector along $N_0 = \ker G_0$

at nice at level 0 points; we remark for later use that $\ker \Pi_0 = N_0$ except at type-0 critical points, and that $I - \Pi_0$ is, at nice at level 0 points, the orthogonal projector onto N_0. We will also make use of the fact that orthogonal projectors are symmetric and therefore $\Pi_0^T \Pi_0 = \Pi_0 \Pi_0 = \Pi_0$.

The analytic matrix function $G_1 = G_0 + H_0 M_0$ meets rank deficiencies on a set of isolated points; its intersection with $\mathcal{J} - \mathcal{J}_{\text{crit0}}$ defines the set of type 1-A critical points. Write $N_1 = \ker G_1$.

Now, $\ker(\Pi_0^T \Pi_0 + G_1^T G_1) = \ker(\Pi_0 + G_1^T G_1)$ equals the intersection $N_0 \cap N_1$ except maybe at type 0 and type 1-A critical points; therefore, excluding types 0 and 1-A, the intersection $N_0 \cap N_1$ is trivial if and only the analytic product $\Pi_0 + G_1^T G_1$ has maximal rank m: since the regular set is non-empty, this maximal rank is met at some point and hence on the whole interval except on a set of isolated points. The intersection of the latter with $\mathcal{J} - (\mathcal{J}_{\text{crit0}} \cup \mathcal{J}_{\text{crit1}A})$ defines the set of critical points of type 1-B.

Additionally, note that the orthogonal projector $Q_1^\perp$ onto N_1 at nice at level 1 points can be extended as an analytic function on the whole interval. We then express at nice at level 1 points:

$$N_0 \oplus N_1 = (N_0^\perp \cap N_1^\perp)^\perp = (\ker((I - \Pi_0)(I - \Pi_0) + Q_1^\perp Q_1^\perp))^\perp = (\ker(I - \Pi_0 + Q_1^\perp))^\perp = \operatorname{im}(I - \Pi_0 + Q_1^\perp),$$

where again we have used the fact that orthogonal projectors are symmetric. This means that, at nice at level 1 points, the direct sum $N_0 \oplus N_1$ can be expressed as the image space of the analytic matrix function $I - \Pi_0 + Q_1^\perp$; therefore, there exists an analytic matrix function Π_1 which yields the orthogonal projector along $N_0 \oplus N_1$ at nice at level 1 points.

Define M_1 and H_1 as indicated in 2.1. Note that D^- in (9) is well-defined and analytic in virtue of the quasi-proper assumption and Proposition 2. We let $G_2 = G_1 + H_1 M_1$ and proceed analogously in order to show that G_2 has rank deficiencies only on a set of isolated points; that critical points defined by the condition $\ker(\Pi_1 + G_2^T G_2) \neq \{0\}$ are also isolated; and that $(N_0 \oplus N_1) \oplus N_2 = \operatorname{im}(I - \Pi_1 + Q_2^\perp)$ admits an analytic extension of the orthogonal projector Π_2 along it.

The proof is completed by repeating the procedure up to the step in which a non-singular G_μ is met. This will happen on the whole interval except, again, on a set of isolated points; therefore on a dense set. □

3.2 Scalarly implicit decoupling

The setting defined by Assumptions 1 and 2 allows one to unravel the behavior of critical DAEs through a scalarly implicit decoupling, as detailed below. Note, with respect to the analogous result for A-critical points presented in [12], that the broader generality of the current one yields increasing exponents $\mu - k$ in the leading coefficient of the solution components v_k in (22).

Theorem 2. *Denote $\omega_\mu = \det G_\mu$, and let G_μ^{adj} be the transposed matrix of cofactors of G_μ. Under Assumptions 1-2, $x \in C_D^1(\mathcal{I}, \mathbb{R}^m) := \{x \in C(\mathcal{I}, \mathbb{R}^m) : Dx \in C^1(\mathcal{I}, \mathbb{R}^n)\}$ solves (1) in a given subinterval $\mathcal{I} \subseteq \mathcal{J}$ if and only if it can be written as*

$$x = D^- u + v_{\mu-1} + \cdots + v_1 + v_0, \tag{19}$$

where $u \in C^1(\mathcal{I}, \mathbb{R}^n)$ is a solution of the scalarly implicit ODE

$$\omega_\mu u' - \omega_\mu (D\Pi_{\mu-1} D^-)' u + D\Pi_{\mu-1} G_\mu^{\text{adj}} B D^- u = D\Pi_{\mu-1} G_\mu^{\text{adj}} q, \tag{20}$$

on the locally invariant space $\operatorname{im} D\Pi_{\mu-1}D^-$*, whereas the solution components* $v_k \in C^1_D(\mathcal{I},\mathbb{R}^m)$*,* $k = \mu-1,\ldots,1$ *and* $v_0 \in C(\mathcal{I},\mathbb{R}^m)$ *verify*

$$\omega_\mu v_{\mu-1} \;=\; -\tilde{\mathcal{K}}_{\mu-1}D^-u + \tilde{\mathcal{L}}_{\mu-1}q, \tag{21}$$

$$\omega_\mu^{\mu-k} v_k \;=\; -\tilde{\mathcal{K}}_k D^-u + \tilde{\mathcal{L}}_k q + \sum_{j=k+1}^{\mu-1} \tilde{\mathcal{N}}_{kj}(Dv_j)' + \sum_{j=k+2}^{\mu-1} \tilde{\mathcal{M}}_{kj}v_j, \quad k = \mu-2,\ \ldots,\ 0. \tag{22}$$

Setting $\tilde{Q}_k = G_\mu^{\mathrm{adj}} H_k M_k$*,* $\tilde{P}_k = \omega_\mu I - \tilde{Q}_k$*, the coefficients of (20)-(22) read*

$$\tilde{\mathcal{K}}_k = M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}(G_\mu^{\mathrm{adj}}B\Pi_{\mu-1} + \tilde{P}_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}), \tag{23}$$

$$\tilde{\mathcal{L}}_k = M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\mathrm{adj}}, \tag{24}$$

$$\tilde{\mathcal{N}}_{kj} = \omega_\mu^{\mu-j}M_k\tilde{P}_{k+1}\cdots\tilde{P}_{j-1}\tilde{Q}_jD^-, \tag{25}$$

$$\tilde{\mathcal{M}}_{kj} = M_k\left(\omega_\mu^{\mu-k-1}(\tilde{Q}_{k+1}-\tilde{P}_{k+1}) + \sum_{i=k+2}^{j-1}\omega_\mu^{\mu-i}\tilde{P}_{k+1}\cdots\tilde{P}_{i-1}(\tilde{Q}_i-\tilde{P}_i)\right)D^-(DM_jD^-)'D. \tag{26}$$

Proof. Assume that $x \in C^1_D(\mathcal{I},\mathbb{R}^m)$ is a solution of (1) in $\mathcal{I} \subseteq \mathcal{J}$. Remark that the operators $\tilde{Q}_k = G_\mu^{\mathrm{adj}}H_kM_k$, $\tilde{P}_k = \omega_\mu I - \tilde{Q}_k$ are continuous in the whole $\mathcal{J}$ and, at regular points, the projectors Q_k, P_k are well-defined and verify $\omega_\mu Q_k = \tilde{Q}_k$, $\omega_\mu P_k = \tilde{P}_k$. Note also that $AR = A$ holds in the whole $\mathcal{I}$; therefore, the leading coefficient A in (1) can be written as $AR = ADD^- = G_0D^-$ and the equation reads

$$G_0D^-(Dx)' + Bx = q. \tag{27}$$

The decoupling strategy extends the one introduced for the regular case in terms of P-projectors in [8, 9], and is based on the projection of the equation (27) onto certain subspaces, together with the decomposition of the solution vector via

$$\Pi_{\mu-1} + M_{\mu-1} + \ldots + M_1 + M_0 = I, \tag{28}$$

which follows from $M_0 = I - \Pi_0$, $M_i = \Pi_{i-1} - \Pi_i$.

The relation (28) makes it possible to decompose in turn

$$B = B\Pi_{\mu-1} + BM_{\mu-1} + \ldots + BM_1 + BM_0. \tag{29}$$

Via the identities $D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}G_0 = \omega_\mu D\Pi_{\mu-1}$, $D\Pi_{\mu-1}D^-(Dx)' = (D\Pi_{\mu-1}x)' - (D\Pi_{\mu-1}D^-)'Dx$, $x = \Pi_{\mu-1}x + (I-\Pi_{\mu-1})x$ and the decomposition (29), premultiplying (27) by $D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}$ we arrive, making use of the property $\omega_\mu(D\Pi_{\mu-1}D^-)'D(I-\Pi_{\mu-1})x = D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}B(I-\Pi_{\mu-1})x$, at

$$\omega_\mu(D\Pi_{\mu-1}x)' - \omega_\mu(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}x + D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}BD^-D\Pi_{\mu-1}x \;=\; D\Pi_{\mu-1}G_\mu^{\mathrm{adj}}q,$$

which is the scalarly implicit inherent ODE (20) with $u = D\Pi_{\mu-1}x$.

Note that the space $\operatorname{im} D\Pi_{\mu-1}$ is invariant for this ODE since $y = (I - D\Pi_{\mu-1}D^-)u$ satisfies the homogeneous equation

$$\omega_\mu[y' + (D\Pi_{\mu-1}D^-)'y] = 0$$

on $\mathcal{I}$. Again, taking into account that $\omega_\mu \neq 0$ on a dense set, we get $y' + (D\Pi_{\mu-1}D^-)'y = 0$ on $\mathcal{I}$, and therefore a vanishing initial condition for y, which owes to $u = D\Pi_{\mu-1}D^-u$, has as unique solution the trivial one on the whole $\mathcal{I}$.

The relation depicted in (21) is obtained by multiplying the DAE (27) by $M_{\mu-1}G_\mu^{\text{adj}}$. This is based on the identities $M_{\mu-1}G_\mu^{\text{adj}}G_0 = 0$, $M_{\mu-1}G_\mu^{\text{adj}}BM_{\mu-1}x = \omega_\mu M_{\mu-1}x$ and $M_{\mu-1}G_\mu^{\text{adj}}BM_i x = 0$ for $0 \le i \le \mu-2$, which follow from the matrix chain construction. This yields the scalarly implicit equation (21) for the component $v_{\mu-1} = M_{\mu-1}x$. Note that in this case the coefficient $\tilde{\mathcal{K}}_{\mu-1}$ amounts to $M_{\mu-1}G_\mu^{\text{adj}}B\Pi_{\mu-1}$.

Equation (22), yielding the solution components $v_{\mu-2}, \dots, v_0$, is obtained analogously by multiplying (27) by $M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\text{adj}}$. The key step here is the decomposition of the terms

$$M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\text{adj}}G_0D^-(Dx)' = M_k(\omega_\mu\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1} - \omega_\mu^{\mu-k}I)D^-(Dx)'$$

and

$$M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\text{adj}}Bx = M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\text{adj}}B\Pi_{\mu-1}x + \sum_{j=0}^{\mu-1} M_k\tilde{P}_{k+1}\cdots\tilde{P}_{\mu-1}G_\mu^{\text{adj}}BM_jx \quad (30)$$

which result from the above-indicated multiplication. After some computations, and denoting $v_k = M_kx$ for $k = \mu-2, \dots, 0$, we obtain the expression displayed in (22) with the coefficients (23)-(26). Although we omit some technical details for the sake of simplicity, it is worth emphasizing that the last term of (30) has the form $\omega_\mu^{\mu-k}M_kx + \dots$, yielding the expression $\omega_\mu^{\mu-k}v_k$ in the leading term of (22).

On the other hand, assuming that $u \in C^1(\mathcal{I}, \mathbb{R}^n)$, $v_{\mu-1}, \dots, v_1 \in C^1_D(\mathcal{I}, \mathbb{R}^m)$ and $v_0 \in C(\mathcal{I}, \mathbb{R}^m)$ satisfy (20)-(22), it follows that $x = D^-u + v_{\mu-1} + \cdots + v_0 \in C^1_D(\mathcal{I}, \mathbb{R}^m)$. Additionally, the identity $A(Dx)' + Bx - q = 0$ holds on the dense (in $\mathcal{I}$) set $\mathcal{I} \cap \mathcal{J}_{\text{reg}}$: since the map $A(Dx)' + Bx - q$ is continuous, $A(Dx)' + Bx - q = 0$ remains true on $\mathcal{I}$, and therefore $x = D^-u + v_{\mu-1} + \cdots + v_0$ is a solution of the properly stated DAE (1) in $C^1_D(\mathcal{I}, \mathbb{R}^m)$. □

It is worth remarking that Assumption 2 holds in particular in the setting considered in [12]; namely, if the individual projectors Q_i (resp. P_i) admit continuous extensions through critical points preserving the transversality property (b), then Π_i can be constructed as the product $P_0\cdots P_i$. The important aspect is that the converse is not true: the Π_i projectors may have a continuous extension through the critical points without the P-projectors admitting it, at least under the transversality requirement (b). This is the case at B-critical points. Thereby the current scenario accommodates both A and B critical problems. Of course, in the narrower A-setting of [12] the advantage is that there are no increasing exponents for ω_μ in the leading terms of the decoupling (compare (16) with (20)-(22)).

4 Examples

4.1 Example 1 revisited

Consider again the DAE (18), coming from the circuit example discussed in 2.3. Let A, D, G_0, B, D^-, $Q_0 = M_0$, $P_0 = \Pi_0 = I - M_0$ and G_1 be given as in 2.3. As detailed there, $N_0 = \ker G_0 = \text{span}[(0,0,1)]$, and $N_1 = \ker G_1 = \text{span}[(-\alpha(t), 0, 1)]$.

The spaces N_0, N_1 have a trivial intersection only at regular points, where $\alpha(t) \neq 0$. But the important point is that the corresponding direct sum at regular points

$$N_0 \oplus N_1 = \text{span}[(0,0,1), (-\alpha(t), 0, 1)] = \text{span}[(0,0,1), (1,0,0)]$$

can be smoothly extended through the critical points. Therefore, Assumption 2 is satisfied with

$$\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We may therefore take

$$M_1 = \Pi_0 - \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and perform the required computations in order to obtain the scalarly implicit decoupling given in Theorem 2. The key aspect is that this construction in the Π-framework is possible without any need to refer to individual projectors Q_1, P_1. Some simple computations lead to the expected solution description given by $x(t) = q_3(t)$, $y(t) = y(0) + \int_0^t (q_2+q_3)$, $\alpha(t)z(t) = q_1(t) - y(0) - \int_0^t (q_2+q_3) - q_3'(t)$.

4.2 Example 2

In Example 1 above, only the variable z (corresponding to the $v_{\mu-2}$-component) is affected by the scalar coefficient $\alpha(t) = -\det G_2(t)$ vanishing at critical points, in contrast to what happens with the state variable y and the variable x (which corresponds to $v_{\mu-1}$). This raises the question of whether the exponents in the scalar coefficient ω_μ of the decoupling (20)-(22) can be simplified further; in particular, inspired in the behavior of the previous example, one might conjecture that the coefficient ω_μ in the leading coefficient of the inherent ODE (20) and in the $v_{\mu-1}$-component in (21) may be canceled in general. This is not the case, as the following example shows.

Consider a DAE of the form (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & t \\ 1 & 0 & 0 \\ t & 0 & 0 \end{bmatrix}. \tag{31}$$

As in Example 1 above, we have $G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, with $N_0 = \ker G_0 = \operatorname{span}[(0,0,1)]$. Letting again

$$D^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_0 = Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we get

$$G_1 = G_0 + BM_0 = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $G_1(t)$ has constant rank $r_1 = 2$ as before, with $N_1 = \ker G_1 = \operatorname{span}[(-t,0,1)]$.

At $t = 0$ a critical point of type 1-B is met, since $N_0(0) \cap N_1(0) = N_0(0) \neq 0$. In any case, the space

$$N_0 \oplus N_1 = \operatorname{span}[(0,0,1),(1,0,0)]$$

can be again extended through the critical point, and therefore we may define

$$\Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; M_1 = \Pi_1 - \Pi_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding

$$G_2 = \begin{bmatrix} 1 & 0 & t \\ 1 & 1 & 0 \\ t & 0 & 0 \end{bmatrix}.$$

Indeed, G_2 becomes singular at the critical point $t = 0$, with $\det G_2 = -t^2$. But in this case, some computations show that the inherent ODE is indeed a singular one, being described in the basic invariant space as $ty' = tq_2 - q_3$. The other solution components are given by $tx = q_3$, $tz = -x' + q_1$ and hence are also affected by a coefficient which vanishes at the critical point.

5 Concluding remarks

In this paper we have presented a framework for the analysis of linear DAEs which accommodates both A- and B-critical points, by using a recently introduced construction based on Π-projectors. Our approach relies on the structure of the characteristic spaces $N_0 \oplus \ldots \oplus N_i$, which are independent of the choice of projectors, and not on the individual ones N_i arising in the matrix chain construction. This way we extend the scope of the analysis of critical points in linear DAEs beyond the A-setting of [12] and also beyond the analytic context of [5, 13].

Concerning future work, it is worth emphasizing that this framework does not accommodate index changes. In the context here discussed, further simplification of the exponents of ω_μ in the leading terms of the decoupling (20)-(22) might be possible; this is in turn related to the way in which the different solutions components are affected by the singularity. Finally, analogous phenomena may be expected and are still unexplored within other approaches to DAE analysis, notably in the reduction framework of [13, 14].

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