

# Boundary Layer Solutions to Singularly Perturbed Problems via the Implicit Function Theorem

Oleh Omel'chenko<sup>1,2</sup> and Lutz Recke<sup>1</sup>

<sup>1</sup> *Department of Mathematics, Humboldt University of Berlin,  
Unter den Linden 6, D-10099 Berlin, Germany*

<sup>2</sup> *Institute of Mathematics, National Academy of Sciences of Ukraine,  
Tereshchenkivska Str. 3, 01601 Kyiv-4, Ukraine*

## Abstract.

We prove existence, local uniqueness and asymptotic estimates for boundary layer solutions to singularly perturbed problems of the type  $\varepsilon^2 u'' = f(x, u, \varepsilon u', \varepsilon)$ ,  $0 < x < 1$ , with Dirichlet and Neumann boundary conditions. For that we assume that there is given a family of approximate solutions which satisfy the differential equation and the boundary conditions with certain low accuracy. Moreover, we show that, if this accuracy is high, then the closeness of the approximate solution to the exact solution is correspondingly high. The main tool of the proofs is a modification of an Implicit Function Theorem of R. MAGNUS. Finally we show how to construct approximate solutions under certain natural conditions.

**Keywords:** singular perturbation, asymptotic approximation, boundary layer, implicit function theorem.

## 1 Introduction

This paper is concerned with boundary value problems for second order semilinear ODEs of the type

$$\left. \begin{aligned} \varepsilon^2 u''(x) &= f(x, u(x), \varepsilon u'(x), \varepsilon), & 0 < x < 1, \\ u(0) &= \beta_0, \quad u'(1) = \beta_1, \end{aligned} \right\} \quad (1.1)$$

where the function  $f : [0, 1] \times \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}$  is of class  $C^2$ , the boundary data  $\beta_0, \beta_1 \in \mathbb{R}$  are fixed numbers, and  $\varepsilon$  is a small positive parameter. We suppose that

$$f(x, 0, 0, 0) = 0 \quad \text{and} \quad \partial_2 f(x, 0, 0, 0) > 0 \quad \text{for all} \quad x \in [0, 1], \quad (1.2)$$

and that there exist  $C^2$ -functions  $w_0, w_1 : [0, \infty) \rightarrow \mathbb{R}$  such that

$$\left. \begin{aligned} w_0''(y) &= f(0, w_0(y), w_0'(y), 0), & y > 0, \\ w_0(0) &= \beta_0, \quad w_0(\infty) = 0, \quad w_0'(0) \neq 0, \end{aligned} \right\} \quad (1.3)$$

and

$$\left. \begin{aligned} w_1''(y) &= f(1, w_1(y), -w_1'(y), 0), & y > 0, \\ w_1'(0) &= 0, \quad w_1(\infty) = 0. \end{aligned} \right\} \quad (1.4)$$

Here  $\partial_2 f$  denotes the partial derivative of the function  $f$  with respect to its second variable. Similar notation will be used later on.

Our goal here is to describe existence, local uniqueness and asymptotic behavior for  $\varepsilon \rightarrow 0$  of boundary layer solutions to (1.1), i.e. of solutions  $u$  with

$$u(x) \approx \mathcal{W}_\varepsilon(x) := w_0 \left( \frac{x}{\varepsilon} \right) + w_1 \left( \frac{1-x}{\varepsilon} \right). \quad (1.5)$$

In particular, we consider approximate solutions of the general form

$$\mathcal{U}_\varepsilon(x) := \mathcal{W}_\varepsilon(x) + r_\varepsilon(x) = w_0 \left( \frac{x}{\varepsilon} \right) + w_1 \left( \frac{1-x}{\varepsilon} \right) + r_\varepsilon(x), \quad (1.6)$$

where  $r_\varepsilon \in W^{2,2}(0,1)$  is small in a certain sense, and derive an explicit estimate for the distance between  $\mathcal{U}_\varepsilon(x)$  and the exact solution to (1.1). This distance will be measured by the norms  $\|u\|_\infty := \max \{|u(x)| : x \in [0,1]\}$  and  $\|u\|_{2,\varepsilon}$ , where

$$\|u\|_{n,\varepsilon}^2 := \frac{1}{\varepsilon} \int_0^1 \left( \sum_{k=0}^n |\varepsilon^k u^{(k)}(x)|^2 \right) dx. \quad (1.7)$$

Our main result is the following

**Theorem 1.1** *Let  $f \in C^2([0,1] \times \mathbb{R}^2 \times [0,\infty))$  satisfy (1.2)–(1.4) and let  $r_\varepsilon \in W^{2,2}(0,1)$  obey*

$$\|r_\varepsilon\|_{2,\varepsilon} \rightarrow 0 \quad \text{for } \varepsilon \rightarrow +0. \quad (1.8)$$

*Then there exist  $\varepsilon_0 > 0$  and  $\delta > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there exists exactly one solution  $u = u_\varepsilon$  to (1.1) such that  $\|u - \mathcal{U}_\varepsilon\|_{2,\varepsilon} < \delta$ . Moreover, there exists  $c > 0$  such that*

$$\|u_\varepsilon - \mathcal{U}_\varepsilon\|_{2,\varepsilon} \leq c\omega(\varepsilon), \quad (1.9)$$

where

$$\begin{aligned} \omega(\varepsilon) := & \sqrt{\frac{1}{\varepsilon} \int_0^1 \left( \varepsilon^2 \mathcal{U}_\varepsilon'''(x) - f(x, \mathcal{U}_\varepsilon(x), \varepsilon \mathcal{U}_\varepsilon'(x), \varepsilon) \right)^2 dx +} \\ & + |\mathcal{U}_\varepsilon(0) - \beta_0| + |\varepsilon \mathcal{U}_\varepsilon'(1) - \varepsilon \beta_1|. \end{aligned} \quad (1.10)$$

**Remark 1.2** *It is easy to see that  $\|u\|_{L^\infty(0,1/\varepsilon)} \leq \text{const} \cdot \|u\|_{W^{2,2}(0,1/\varepsilon)}$  uniformly with respect to  $\varepsilon \in (0,1]$ . Hence, there exists a constant  $c_0 > 0$  such that*

$$\|u\|_\infty \leq c_0 \|u\|_{2,\varepsilon} \quad \text{and} \quad \|\varepsilon u'\|_\infty \leq c_0 \|u\|_{2,\varepsilon}$$

*for all  $\varepsilon \in (0,1]$  and all  $u \in W^{2,2}(0,1)$ . Therefore, the conclusion of Theorem 1.1 implies that*

$$\|u_\varepsilon - \mathcal{U}_\varepsilon\|_\infty \leq \text{const} \cdot \omega(\varepsilon) \quad \text{and} \quad \|\varepsilon u'_\varepsilon - \varepsilon \mathcal{U}'_\varepsilon\|_\infty \leq \text{const} \cdot \omega(\varepsilon).$$

**Remark 1.3** Assumption (1.2) implies that the origin of the phase plane  $(w, w')$  is a hyperbolic saddle point for each of the two-dimensional autonomous systems

$$w''(y) = f(i, w(y), w'(y), 0), \quad i = 0, 1. \quad (1.11)$$

Indeed, the corresponding linearized equations have two nonzero eigenvalues with opposite signs

$$\lambda_i^\pm = \frac{1}{2} \left( \partial_3 f(i, 0, 0, 0) \pm \sqrt{[\partial_3 f(i, 0, 0, 0)]^2 + 4\partial_2 f(i, 0, 0, 0)} \right). \quad (1.12)$$

Therefore, on the phase plane  $(w, w')$  there exist one-dimensional stable manifolds corresponding to the systems  $(1.11)_0$  and  $(1.11)_1$ , which contain solutions of the boundary value problems (1.3) and (1.4), respectively [1]. Moreover, every solution to (1.3) or (1.4) satisfies exponential estimates

$$|w_i(y)|, |w'_i(y)| \leq b_i e^{\lambda_i^- y} \quad \text{for all } y \geq 0, \quad i = 0, 1, \quad (1.13)$$

where  $b_i > 0$  are some fixed constants (cf. Lemma 5.1). Note, we do not impose any restrictions on the form of functions  $w_0$  and  $w_1$ . In particular, they may be non-monotone and even partly oscillating.

**Remark 1.4** Using Theorem 1.1 one can justify formal solution asymptotics to problem (1.1) provided they are close to  $\mathcal{W}_\varepsilon$  in the  $\|\cdot\|_{2,\varepsilon}$  norm. In particular, due to this theorem one obtain different accuracy estimates depending on the structure of function  $f$  and boundary data of problem (1.1) (compare, for example, Remark 3.3 and Theorem 4.2).

**Remark 1.5** If the function  $f$  doesn't depend neither on  $u'(x)$  nor on  $\varepsilon$ , then (1.1) reads as

$$\left. \begin{aligned} \varepsilon^2 u''(x) &= f(x, u(x)), \quad 0 < x < 1, \\ u(0) &= \beta_0, \quad u'(1) = \beta_1. \end{aligned} \right\} \quad (1.14)$$

For those problems J. HALE and D. SALAZAR showed in [2] existence and asymptotic behavior for  $\varepsilon \rightarrow 0$  of solutions with boundary and interior layers. Their existence proofs were based on a combination of the Liapunov-Schmidt procedure and the implicit function theorem. For that they needed eigenvalue estimates for the differential operator

$$\varepsilon^2 \frac{d^2}{dx^2} + \partial_2 f(x, \mathcal{U}(x, \varepsilon))$$

with corresponding homogeneous boundary conditions, where  $\mathcal{U}(x, \varepsilon)$  is a family of approximate solutions to (1.14).

The proof of our Theorem 1.1 is also based on the implicit function theorem, but we need neither the Liapunov-Schmidt procedure nor eigenvalue estimates. Instead we use a lemma of R. MAGNUS [8, Lemma 1.3] which helps to verify the assumptions of a quite general implicit function theorem (see our Section 2). Moreover, our results work already in the case when just leading terms describing the behavior of the boundary

layers are known (Remark 3.2), while the Hale and Salazar's approach requires an approximation of a higher order.

Remark that existence and asymptotic behavior for  $\varepsilon \rightarrow 0$  of solutions to (1.14) with monotone Dirichlet boundary layers and with trivial Neumann boundary layers is proved also by upper and lower solution techniques, see, for example, [3, 4].

**Remark 1.6** We deal here with a special choice of boundary conditions. But our results and the technique of proof remains valid with slight changes for the case of general linear boundary conditions:

$$\alpha_0 u(0) - (1 - \alpha_0)u'(0) = \beta_0, \quad \alpha_1 u(1) + (1 - \alpha_1)u'(1) = \beta_1,$$

where  $0 \leq \alpha_i \leq 1$ ,  $i = 0, 1$ .

## 2 Modified Implicit Function Theorem

In this section we formulate and prove an implicit function theorem with minimal assumptions concerning continuity with respect to the control parameter. This is just what we need for the proof of our Theorem 1.1.

Our implicit function theorem is very close to that of R. MAGNUS [8, Theorem 1.2]. The difference is that we work in bundles of Banach spaces, while MAGNUS works with a fixed pair of Banach spaces. For other implicit function theorems with weak assumptions concerning continuity with respect to the control parameter see [5, Theorem 7], [6, Theorem 3.4] and [7, Theorem 4.1].

**Theorem 2.1** Let for any  $\varepsilon \in (0, \varepsilon_0)$  be given Banach spaces  $U_\varepsilon$  and  $V_\varepsilon$  and maps  $F_\varepsilon \in C^1(U_\varepsilon, V_\varepsilon)$  such that

$$\|F_\varepsilon(0)\| \rightarrow 0 \text{ for } \varepsilon \rightarrow +0, \quad (2.1)$$

$$\|F'_\varepsilon(u) - F'_\varepsilon(0)\| \rightarrow 0 \text{ for } |\varepsilon| + \|u\| \rightarrow 0 \quad (2.2)$$

and

$$\left. \begin{array}{l} \text{there exist } \varepsilon_1 \in (0, \varepsilon_0] \text{ and } c > 0 \text{ such that for all } \varepsilon \in (0, \varepsilon_1) \\ \text{the operators } F'_\varepsilon(0) \text{ are invertible and } \|F'_\varepsilon(0)^{-1}\| \leq c. \end{array} \right\} \quad (2.3)$$

Then there exist  $\varepsilon_2 \in (0, \varepsilon_1)$  and  $\delta > 0$  such that for all  $\varepsilon \in (0, \varepsilon_2)$  there exists exactly one  $u = u_\varepsilon$  with  $\|u\| < \delta$  and  $F_\varepsilon(u) = 0$ . Moreover,

$$\|u_\varepsilon\| \leq 2c \|F_\varepsilon(0)\|. \quad (2.4)$$

**Proof** For  $\varepsilon \in (0, \varepsilon_1)$  we have  $F_\varepsilon(u) = 0$  if and only if

$$G_\varepsilon(u) := u - F'_\varepsilon(0)^{-1} F_\varepsilon(u) = 0. \quad (2.5)$$

Moreover, for such  $\varepsilon$  and all  $u, v \in U_\varepsilon$  we have

$$\begin{aligned} G_\varepsilon(u) - G_\varepsilon(v) &= \int_0^1 G'_\varepsilon(su + (1-s)v)(u-v) ds = \\ &= F'_\varepsilon(0)^{-1} \int_0^1 (F'_\varepsilon(su + (1-s)v) - F'_\varepsilon(0))(u-v) ds. \end{aligned}$$

Hence, assumptions (2.2) and (2.3) imply that there exist  $\varepsilon_2 \in (0, \varepsilon_1)$  and  $\delta > 0$  such that for all  $\varepsilon \in (0, \varepsilon_2)$

$$\|G_\varepsilon(u) - G_\varepsilon(v)\| \leq \frac{1}{2}\|u - v\| \text{ for all } u, v \in K_\varepsilon^\delta := \{w \in U_\varepsilon : \|w\| \leq \delta\}.$$

Using this and (2.3) again, for all  $\varepsilon \in (0, \varepsilon_2)$  we get

$$\|G_\varepsilon(u)\| \leq \|G_\varepsilon(u) - G_\varepsilon(0)\| + \|G_\varepsilon(0)\| \leq \frac{1}{2}\|u\| + c\|F_\varepsilon(0)\|. \quad (2.6)$$

Hence, assumption (2.1) yields that  $G_\varepsilon$  maps  $K_\varepsilon^\delta$  into  $K_\varepsilon^\delta$  for all  $\varepsilon \in (0, \varepsilon_2)$ , if  $\varepsilon_2$  is chosen sufficiently small. Now, Banach's fixed point theorem gives a unique in  $K_\varepsilon^\delta$  solution  $u = u_\varepsilon$  to (2.5) for all  $\varepsilon \in (0, \varepsilon_2)$ . Moreover, (2.6) yields  $\|u_\varepsilon\| \leq 1/2\|u_\varepsilon\| + c\|F_\varepsilon(0)\|$ , i.e. (2.4).  $\blacksquare$

The following lemma is [8, Lemma 1.3], translated to our setting. It gives a criterion how to verify the key assumption (2.3) of Theorem 2.1:

**Lemma 2.2** *Let  $F'_\varepsilon(0)$  be Fredholm of index zero for all  $\varepsilon \in (0, \varepsilon_0)$ . Suppose that there do not exist sequences  $\varepsilon_1, \varepsilon_2 \dots \in (0, \varepsilon_0)$  and  $u_1 \in U_{\varepsilon_1}, u_2 \in U_{\varepsilon_2} \dots$  with  $\|u_n\| = 1$  for all  $n \in \mathbb{N}$  and  $|\varepsilon_n| + \|F'_{\varepsilon_n}(0)u_n\| \rightarrow 0$  for  $n \rightarrow \infty$ . Then (2.3) is satisfied.*

**Proof** Suppose that (2.3) is not true. Then there exist a sequence  $\varepsilon_1, \varepsilon_2 \dots \in (0, \varepsilon_0)$  with  $|\varepsilon_n| \rightarrow 0$  for  $n \rightarrow \infty$  such that either  $F'_{\varepsilon_n}(0)$  is not invertible or it is but  $\|F'_{\varepsilon_n}(0)^{-1}\| \geq n$  for all  $n \in \mathbb{N}$ . In the first case there exist  $u_n \in U_{\varepsilon_n}$  with  $\|u_n\| = 1$  and  $F'_{\varepsilon_n}(0)u_n = 0$  (because  $F'_{\varepsilon_n}(0)$  is Fredholm of index zero). In the second case there exist  $v_n \in V_{\varepsilon_n}$  with  $\|v_n\| = 1$  and  $\|F'_{\varepsilon_n}(0)^{-1}v_n\| \geq n$ , i.e.

$$\|F'_{\varepsilon_n}(0)u_n\| \leq \frac{1}{n} \text{ with } u_n := \frac{F'_{\varepsilon_n}(0)^{-1}v_n}{\|F'_{\varepsilon_n}(0)^{-1}v_n\|}.$$

But this contradicts to the assumptions of the lemma.  $\blacksquare$

### 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Hence, we always suppose the assumptions of Theorem 1.1 to be satisfied. In particular, we use the functions  $w_0$  and  $w_1$ , which are introduced in (1.3) and (1.4), respectively.

We are going to apply Theorem 2.1. Thus, from the beginning we introduce the necessary setting. First, for every  $\varepsilon \in (0, 1]$  we set

$$U_\varepsilon := W^{2,2}(0, 1) \text{ with norm } \|\cdot\|_{2,\varepsilon},$$

$$V_\varepsilon := L^2(0, 1) \times \mathbb{R}^2 \text{ with norm } \|\cdot\|_{0,\varepsilon} + |\cdot| + |\cdot|.$$

Then we define  $F_\varepsilon \in C^1(U_\varepsilon, V_\varepsilon)$  as follows

$$F_\varepsilon(v) := \begin{pmatrix} \varepsilon^2 v''(x) + \varepsilon^2 \mathcal{U}_\varepsilon''(x) - f(x, v(x) + \mathcal{U}_\varepsilon(x), \varepsilon v'(x) + \varepsilon \mathcal{U}'_\varepsilon(x), \varepsilon) \\ v(0) + \mathcal{U}_\varepsilon(0) - \beta_0 \\ \varepsilon v'(1) + \varepsilon \mathcal{U}'_\varepsilon(1) - \varepsilon \beta_1 \end{pmatrix}.$$

Obviously, we have  $F_\varepsilon(v) = 0$  if and only if  $v + \mathcal{U}_\varepsilon$  is a solution to boundary value problem (1.1). Besides, we remark that  $\omega(\varepsilon) = \|F_\varepsilon(0)\|_{V_\varepsilon}$ .

In what follows we often use a change of variables  $x \rightarrow x/\varepsilon$  and  $x \rightarrow (1-x)/\varepsilon$ . Expressions obtained in result of this transformation can be estimated with a help of the next

**Remark 3.1** *Let  $g : [0, 1] \times [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$  have uniformly bounded partial derivatives  $\partial_1 g(x, y, \varepsilon)$  and  $\partial_3 g(x, y, \varepsilon)$ . Then there exists a constant  $C > 0$  such that*

$$|g(\varepsilon y, y, \varepsilon) - g(0, y, 0)| + |g(1 - \varepsilon y, y, \varepsilon) - g(1, y, 0)| \leq \varepsilon \cdot C(1 + y)$$

for all  $y \in [0, 1/\varepsilon]$  and  $\varepsilon \in (0, 1]$ .

Now we start to verify assumptions of Theorem 2.1. In a first step we consider condition (2.1). Taking into account assumptions (1.3), (1.4) and definition (1.5) we find that the first component of  $F_\varepsilon(0)$  equals

$$\begin{aligned} \mathcal{F}_\varepsilon(x) &:= \varepsilon^2 r_\varepsilon''(x) + f\left(0, w_0\left(\frac{x}{\varepsilon}\right), w_0'\left(\frac{x}{\varepsilon}\right), 0\right) + \\ &+ f\left(1, w_1\left(\frac{1-x}{\varepsilon}\right), -w_1'\left(\frac{1-x}{\varepsilon}\right), 0\right) - f(x, \mathcal{U}_\varepsilon(x), \varepsilon \mathcal{U}'_\varepsilon(x), \varepsilon). \end{aligned} \quad (3.1)$$

Obviously, the latter term in (3.1) can be rewritten as follows

$$\begin{aligned} f(x, \mathcal{U}_\varepsilon(x), \varepsilon \mathcal{U}'_\varepsilon(x), \varepsilon) &= f(x, \mathcal{W}_\varepsilon(x), \varepsilon \mathcal{W}'_\varepsilon(x), \varepsilon) + \\ &+ \int_0^1 \partial_2 f(x, \mathcal{U}_\varepsilon(x) - sr_\varepsilon(x), \varepsilon \mathcal{U}'_\varepsilon(x) - \varepsilon sr'_\varepsilon(x), \varepsilon) r_\varepsilon(x) ds + \\ &+ \int_0^1 \partial_3 f(x, \mathcal{U}_\varepsilon(x) - sr_\varepsilon(x), \varepsilon \mathcal{U}'_\varepsilon(x) - \varepsilon sr'_\varepsilon(x), \varepsilon) \varepsilon r'_\varepsilon(x) ds. \end{aligned} \quad (3.2)$$

Next, we transform the first term in the right hand side of (3.2) and obtain

$$\begin{aligned} f(x, \mathcal{W}_\varepsilon(x), \varepsilon \mathcal{W}'_\varepsilon(x), \varepsilon) &= f(x, 0, 0, \varepsilon) + \\ &+ \left[ f\left(x, w_0\left(\frac{x}{\varepsilon}\right), w_0'\left(\frac{x}{\varepsilon}\right), \varepsilon\right) - f(x, 0, 0, \varepsilon) \right] + \\ &+ \left[ f\left(x, w_1\left(\frac{1-x}{\varepsilon}\right), -w_1'\left(\frac{1-x}{\varepsilon}\right), \varepsilon\right) - f(x, 0, 0, \varepsilon) \right] + \\ &+ \int_0^1 \int_0^1 \partial_2^2 f\left(x, sw_0\left(\frac{x}{\varepsilon}\right) + tw_1\left(\frac{1-x}{\varepsilon}\right), sw_0'\left(\frac{x}{\varepsilon}\right) - tw_1'\left(\frac{1-x}{\varepsilon}\right), \varepsilon\right) \times \\ &\quad \times w_0\left(\frac{x}{\varepsilon}\right) w_1\left(\frac{1-x}{\varepsilon}\right) ds dt + \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \int_0^1 \partial_2 \partial_3 f \left( x, sw_0 \left( \frac{x}{\varepsilon} \right) + tw_1 \left( \frac{1-x}{\varepsilon} \right), sw'_0 \left( \frac{x}{\varepsilon} \right) - tw'_1 \left( \frac{1-x}{\varepsilon} \right), \varepsilon \right) \times \\
& \quad \times \left[ w'_0 \left( \frac{x}{\varepsilon} \right) w_1 \left( \frac{1-x}{\varepsilon} \right) + w_0 \left( \frac{x}{\varepsilon} \right) w'_1 \left( \frac{1-x}{\varepsilon} \right) \right] ds dt + \\
& + \int_0^1 \int_0^1 \partial_3^2 f \left( x, sw_0 \left( \frac{x}{\varepsilon} \right) + tw_1 \left( \frac{1-x}{\varepsilon} \right), sw'_0 \left( \frac{x}{\varepsilon} \right) - tw'_1 \left( \frac{1-x}{\varepsilon} \right), \varepsilon \right) \times \\
& \quad \times w'_0 \left( \frac{x}{\varepsilon} \right) w'_1 \left( \frac{1-x}{\varepsilon} \right) ds dt. \tag{3.3}
\end{aligned}$$

Inequalities (1.13) imply that the absolute values of the three last integrals in (3.3) can be estimated with  $\text{const} \cdot e^{\lambda_0^- x/\varepsilon} \cdot e^{\lambda_1^- (1-x)/\varepsilon} = \text{const} \cdot e^{\lambda_m/\varepsilon}$ , where  $\lambda_m := \max\{\lambda_0^-, \lambda_1^-\}$ . Thus, substituting (3.3) into (3.2) and the latter one into (3.1), we obtain

$$\begin{aligned}
\|\mathcal{F}_\varepsilon\|_{0,\varepsilon} & \leq \|f(x, 0, 0, \varepsilon)\|_{0,\varepsilon} + \left\| f \left( x, w_0 \left( \frac{x}{\varepsilon} \right), w'_0 \left( \frac{x}{\varepsilon} \right), \varepsilon \right) - f(x, 0, 0, \varepsilon) - \right. \\
& \quad \left. - f \left( 0, w_0 \left( \frac{x}{\varepsilon} \right), w'_0 \left( \frac{x}{\varepsilon} \right), 0 \right) \right\|_{0,\varepsilon} + \\
& + \left\| f \left( x, w_1 \left( \frac{1-x}{\varepsilon} \right), -w'_1 \left( \frac{1-x}{\varepsilon} \right), \varepsilon \right) - f(x, 0, 0, \varepsilon) - \right. \\
& \quad \left. - f \left( 1, w_1 \left( \frac{1-x}{\varepsilon} \right), -w'_1 \left( \frac{1-x}{\varepsilon} \right), 0 \right) \right\|_{0,\varepsilon} + O(\|r_\varepsilon\|_{2,\varepsilon} + e^{\lambda_m/\varepsilon}). \tag{3.4}
\end{aligned}$$

Now assumption (1.2) and Remark 3.1 imply that  $\|f(x, 0, 0, \varepsilon)\|_{0,\varepsilon} \leq \text{const} \cdot \sqrt{\varepsilon}$ ,

$$\begin{aligned}
& \left\| f \left( x, w_0 \left( \frac{x}{\varepsilon} \right), w'_0 \left( \frac{x}{\varepsilon} \right), \varepsilon \right) - f(x, 0, 0, \varepsilon) - f \left( 0, w_0 \left( \frac{x}{\varepsilon} \right), w'_0 \left( \frac{x}{\varepsilon} \right), 0 \right) \right\|_{0,\varepsilon}^2 = \\
& = \int_0^{1/\varepsilon} |f(\varepsilon y, w_0(y), w'_0(y), \varepsilon) - f(\varepsilon y, 0, 0, \varepsilon) - f(0, w_0(y), w'_0(y), 0)|^2 dy = \\
& = \int_0^{1/\varepsilon} \left[ \int_0^1 [\partial_2 f(\varepsilon y, sw_0(y), w'_0(y), \varepsilon) ds - \partial_2 f(0, sw_0(y), sw'_0(y), 0)] w_0(y) ds + \right. \\
& \quad \left. + \int_0^1 [\partial_3 f(\varepsilon y, sw_0(y), w'_0(y), \varepsilon) ds - \partial_3 f(0, sw_0(y), sw'_0(y), 0)] w'_0(y) ds \right]^2 dy \leq \\
& \leq \text{const} \cdot \varepsilon^2 \int_0^{1/\varepsilon} (1+y)^2 w_0(y)^2 dy \leq \text{const} \cdot \varepsilon^2 \int_0^\infty (1+y)^2 w_0(y)^2 dy
\end{aligned}$$

and analogous estimate for the third term in (3.4). Thus, taking into account inequalities (1.13) we finally obtain  $\|\mathcal{F}_\varepsilon\|_{0,\varepsilon} = O(\sqrt{\varepsilon} + \|r_\varepsilon\|_{2,\varepsilon})$ .

The second and the third components of  $F_\varepsilon(0)$  equal  $w_1(1/\varepsilon) + r_\varepsilon(0)$  and  $w'_0(1/\varepsilon) + \varepsilon r'_\varepsilon(1) - \varepsilon\beta_1$ , respectively, and they can be estimated due to inequalities (1.13) as  $O(\varepsilon + \|r_\varepsilon\|_{2,\varepsilon})$ . Consequently,

$$\|F_\varepsilon(0)\|_{V_\varepsilon} = O(\sqrt{\varepsilon} + \|r_\varepsilon\|_{2,\varepsilon}), \quad (3.5)$$

i.e. (2.1) is satisfied. Remark that, if the (2.2) and (2.3) are also satisfied and, hence, Theorem 2.1 works, then its assertion (2.4) together with (3.5) imply the claimed asymptotic estimate (1.9), (1.10).

In a second step we verify assumption (2.2) of Theorem 2.1. Obviously, the second and the third components of  $F'_\varepsilon(v) - F'_\varepsilon(0)$  vanish. The square of the  $\|\cdot\|_{0,\varepsilon}$  norm of the first component of  $(F'_\varepsilon(v) - F'_\varepsilon(0))\bar{v}$  is

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^1 |[\partial_2 f(x, v + \mathcal{U}_\varepsilon, \varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) - \partial_2 f(x, \mathcal{U}_\varepsilon, \varepsilon \mathcal{U}'_\varepsilon, \varepsilon)]\bar{v}(x) + \\ & \quad + [\partial_3 f(x, v + \mathcal{U}_\varepsilon, \varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) - \partial_3 f(x, \mathcal{U}_\varepsilon, \varepsilon \mathcal{U}'_\varepsilon, \varepsilon)]\varepsilon \bar{v}'(x)|^2 dx = \\ & = \frac{1}{\varepsilon} \int_0^1 \left| \int_0^1 \partial_2^2 f(x, sv + \mathcal{U}_\varepsilon, s\varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) ds \cdot \bar{v}(x)v(x) + \right. \\ & \quad + \int_0^1 \partial_2 \partial_3 f(x, sv + \mathcal{U}_\varepsilon, s\varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) ds \cdot \varepsilon \bar{v}(x)v'(x) + \\ & \quad + \int_0^1 \partial_3 \partial_2 f(x, sv + \mathcal{U}_\varepsilon, s\varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) ds \cdot \varepsilon \bar{v}'(x)v(x) + \\ & \quad \left. + \int_0^1 \partial_3^2 f(x, sv + \mathcal{U}_\varepsilon, s\varepsilon v' + \varepsilon \mathcal{U}'_\varepsilon, \varepsilon) ds \cdot \varepsilon^2 \bar{v}'(x)v'(x) \right|^2 dx \leq \\ & \leq \text{const} \cdot \max_{0 \leq x \leq 1} \{|\bar{v}(x)|^2 + |\varepsilon \bar{v}'(x)|^2\} \cdot \frac{1}{\varepsilon} \int_0^1 (v(x)^2 + \varepsilon^2 v'(x)^2) dx \leq \text{const} \|\bar{v}\|_{U_\varepsilon}^2 \|v\|_{U_\varepsilon}^2, \end{aligned}$$

i.e. (2.2) is also satisfied.

In the third and last step we verify assumption (2.3) of Theorem 2.1. For that we use Lemma 2.2. It is well-known that linear differential operators of the type

$$v \in W^{2,2}(a, b) \mapsto (v'' + c(y)v' + d(y)v, v(a), v'(b)) \in L^2(a, b) \times \mathbb{R}^2$$

with continuous coefficients  $c$  and  $d$  are Fredholm of index zero. Hence, it remains to verify the second assumption of Lemma 2.2.

Let  $\varepsilon_n \in (0, 1)$  and  $v_n \in W^{2,2}(0, 1)$  be sequences with

$$\frac{1}{\varepsilon_n} \int_0^1 (v_n(x)^2 + \varepsilon_n^2 v'_n(x)^2 + \varepsilon_n^4 v''_n(x)^2) dx = 1 \quad (3.6)$$



and

$$\begin{aligned} & \varepsilon_n^2 + |v_n(0)|^2 + |\varepsilon_n v_n'(1)|^2 + \\ & + \frac{1}{\varepsilon_n} \int_0^1 [\varepsilon_n^2 v_n''(x) - \partial_2 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) v_n(x) - \\ & - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \varepsilon_n v_n'(x)]^2 dx \rightarrow 0. \end{aligned} \quad (3.7)$$

We introduce auxiliary functions  $\tilde{v}_n$  in the next way:

$$\tilde{v}_n(y) := \begin{cases} v_n(\varepsilon_n y) & \text{for } 0 \leq y \leq 1/(2\varepsilon_n), \\ v_n\left(\frac{1}{2}\right) \exp\left[\lambda_0^- \left(y - \frac{1}{2\varepsilon_n}\right)\right] & \text{for } y \geq 1/(2\varepsilon_n), \end{cases}$$

where  $\lambda_0^-$  is defined in (1.12). Then it follows from (3.6) that  $\tilde{v}_n$  is a bounded sequence in the Hilbert space  $W^{1,2}(0, \infty)$  with its usual norm. Hence, without loss of generality we can assume that there exists  $v_* \in W^{1,2}(0, \infty)$  such that

$$\tilde{v}_n \rightharpoonup v_* \quad \text{in } W^{1,2}(0, \infty) \quad \text{for } n \rightarrow \infty. \quad (3.8)$$

Remark that, because of the continuous embedding  $W^{1,2}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ ,  $\tilde{v}_n$  is a bounded sequence also in  $L^\infty(0, \infty)$ . Moreover, assumption (3.6) and the continuous embedding  $W^{2,2}(0, 1) \hookrightarrow C^1([0, 1])$  imply that the sequence  $\tilde{v}_n'(0) = \varepsilon_n v_n'(0)$  is bounded too. These two facts will be used in the following.

We are going to show that  $v_* = 0$ . For that reason we derive a variational equation for  $v_*$ . Let us take a test function  $\eta \in W^{1,2}(0, \infty)$  with  $\eta(0) = 0$  and consider an integral

$$\begin{aligned} & \int_0^\infty [\tilde{v}_n'(y) \eta'(y) + \partial_2 f(0, w_0(y), w_0'(y), 0) \tilde{v}_n(y) \eta(y) + \\ & + \partial_3 f(0, w_0(y), w_0'(y), 0) \tilde{v}_n'(y) \eta(y)] dy = \int_0^{1/(2\varepsilon_n)} [\dots] dy + \int_{1/(2\varepsilon_n)}^\infty [\dots] dy. \end{aligned} \quad (3.9)$$

Under the above formulated assumptions, both terms in the right hand side of (3.9) vanish for  $n \rightarrow \infty$ . Indeed, the latter term of (3.9) can be rewritten as a sum of tending to zero sequences

$$\begin{aligned} & \int_{1/(2\varepsilon_n)}^\infty [\tilde{v}_n'(y) \eta'(y) + \partial_2 f(0, w_0(y), w_0'(y), 0) \tilde{v}_n(y) \eta(y) + \\ & + \partial_3 f(0, w_0(y), w_0'(y), 0) \tilde{v}_n'(y) \eta(y)] dy = \\ & = \int_{1/(2\varepsilon_n)}^\infty [-\tilde{v}_n''(y) + \partial_2 f(0, 0, 0, 0) \tilde{v}_n(y) + \partial_3 f(0, 0, 0, 0) \tilde{v}_n'(y)] \eta(y) dy + \end{aligned}$$

$$\begin{aligned}
& + \tilde{v}'_n \left( \frac{1}{2\varepsilon_n} + 0 \right) \eta \left( \frac{1}{2\varepsilon_n} \right) + \\
& + \int_{1/(2\varepsilon_n)}^{\infty} [\partial_2 f(0, w_0(y), w'_0(y), 0) - \partial_2 f(0, 0, 0, 0)] \tilde{v}_n(y) \eta(y) dy + \\
& + \int_{1/(2\varepsilon_n)}^{\infty} [\partial_3 f(0, w_0(y), w'_0(y), 0) - \partial_3 f(0, 0, 0, 0)] \tilde{v}'_n(y) \eta(y) dy. \tag{3.10}
\end{aligned}$$

The first term in the right hand side of (3.10) tends to zero by definition of function  $\tilde{v}_n$ . The second one vanishes since the sequence  $\tilde{v}'_n(1/(2\varepsilon_n) + 0) = \lambda_0^- v_n(1/2)$  is bounded and the sequence  $\eta(1/(2\varepsilon_n))$  tends to zero. Finally, the rest two terms of (3.10) tend to zero because of inequalities (1.13).

In a similar way we treat the first term in the right hand side of (3.9). Namely, we rewrite it as follows:

$$\begin{aligned}
& \int_0^{1/(2\varepsilon_n)} [\tilde{v}'_n(y) \eta'(y) + \partial_2 f(0, w_0(y), w'_0(y), 0) \tilde{v}_n(y) \eta(y) + \\
& \quad + \partial_3 f(0, w_0(y), w'_0(y), 0) \tilde{v}'_n(y) \eta(y)] dy = \\
& = \int_0^{1/(2\varepsilon_n)} [-\tilde{v}''_n(y) + \partial_2 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n) \tilde{v}_n(y) + \\
& \quad + \partial_3 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n) \tilde{v}'_n(y)] \eta(y) dy + \\
& + \tilde{v}'_n \left( \frac{1}{2\varepsilon_n} - 0 \right) \eta \left( \frac{1}{2\varepsilon_n} \right) + \\
& + \int_0^{1/(2\varepsilon_n)} [\partial_2 f(0, w_0(y), w'_0(y), 0) - \\
& \quad - \partial_2 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n)] \tilde{v}_n(y) \eta(y) dy + \\
& + \int_0^{1/(2\varepsilon_n)} [\partial_3 f(0, w_0(y), w'_0(y), 0) - \\
& \quad - \partial_3 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy. \tag{3.11}
\end{aligned}$$

Then the first term in (3.11) tends to zero due to (3.7), and the second one vanishes as a product of bounded sequence  $\tilde{v}'_n(1/(2\varepsilon_n) - 0) = \varepsilon_n v'_n(1)$  (see again (3.7)) and tending to zero sequence  $\eta(1/(2\varepsilon_n))$ . To prove that the rest two terms in (3.11) vanish too, we need a more sophisticated analysis. Since the arguments which we use below are the same for both terms, we consider only one of them, namely, the last term of (3.11). For

that we rewrite it as follows

$$\begin{aligned}
& \int_0^{1/(2\varepsilon_n)} [\partial_3 f(0, w_0(y), w'_0(y), 0) - \partial_3 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy = \\
& = \int_0^{1/(2\varepsilon_n)} [\partial_3 f(0, w_0(y), w'_0(y), 0) - \partial_3 f(\varepsilon_n y, w_0(y), w'_0(y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy + \\
& + \int_0^{1/(2\varepsilon_n)} [\partial_3 f(\varepsilon_n y, w_0(y), w'_0(y), \varepsilon_n) - \\
& \quad - \partial_3 f(\varepsilon_n y, \mathcal{W}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy + \\
& + \int_0^{1/(2\varepsilon_n)} [\partial_3 f(\varepsilon_n y, \mathcal{W}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n) - \\
& \quad - \partial_3 f(\varepsilon_n y, \mathcal{U}_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(\varepsilon_n y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy. \tag{3.12}
\end{aligned}$$

The absolute value of the second term in the right hand side of (3.12) is bounded by a constant times

$$\max_{0 \leq y \leq 1/(2\varepsilon_n)} \left\{ \left| w_1 \left( \frac{1}{\varepsilon_n} - y \right) \right| + \left| w'_1 \left( \frac{1}{\varepsilon_n} - y \right) \right| \right\} \int_0^{1/(2\varepsilon_n)} |\tilde{v}'_n(y)| \cdot |\eta(y)| dy,$$

and this tends to zero due to inequalities (1.13), Cauchy inequality and assumption (3.6). Analogously, an absolute value of the last term of (3.12) is bounded by a constant times

$$\|\eta\|_{L^\infty(0, \infty)} \int_0^{1/(2\varepsilon_n)} \{ |r_{\varepsilon_n}(\varepsilon_n y)| + |\varepsilon_n r'_{\varepsilon_n}(\varepsilon_n y)| \} \cdot |\tilde{v}'_n(y)| dy,$$

and this tends to zero due to Cauchy inequality and assumptions (1.8), (3.6). Finally, given arbitrary  $R \in (0, 1/(2\varepsilon_n))$ , the first term in the right hand side of (3.12) can be estimated by

$$\begin{aligned}
& \left| \int_0^R [\partial_3 f(0, w_0(y), w'_0(y), 0) - \partial_3 f(\varepsilon_n y, w_0(y), w'_0(y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy \right| + \\
& + \left| \int_R^{1/(2\varepsilon_n)} [\partial_3 f(0, w_0(y), w'_0(y), 0) - \partial_3 f(\varepsilon_n y, w_0(y), w'_0(y), \varepsilon_n)] \tilde{v}'_n(y) \eta(y) dy \right| \leq \\
& \leq \text{const} \cdot \left( \varepsilon_n R(1 + R) + \int_R^\infty \eta(y)^2 dy \right),
\end{aligned}$$

where we used Remark 3.1, Cauchy inequality and assumption (3.6) to write the right hand side of this inequality. Now taking first  $R$  sufficiently large such that  $\int_R^\infty \eta(y)^2 dy$  is small, and then, fixing such  $R$ , take  $n$  sufficiently large such that  $\varepsilon_n R(1+R)$  is small, we see that the first term in the right hand side of (3.12) also tends to zero for  $n \rightarrow \infty$ .

Thus, using (3.8) and taking the limit  $n \rightarrow \infty$  in (3.9), we finally get

$$\int_0^\infty [v'_*(y)\eta'(y) + \partial_2 f(0, w_0(y), w'_0(y), 0)v_*(y)\eta(y) + \partial_3 f(0, w_0(y), w'_0(y), 0)v'_*(y)\eta(y)] dy = 0$$

for all  $\eta \in W^{1,2}(0, \infty)$  with  $\eta(0) = 0$ . Therefore  $v_*$  is  $C^2$ -smooth and satisfies

$$v''_*(y) = \partial_2 f(0, w_0(y), w'_0(y), 0)v_*(y) + \partial_3 f(0, w_0(y), w'_0(y), 0)v'_*(y)$$

for all  $y > 0$ . The function  $w'_0$  together with an exponentially growing function constitutes a fundamental system for this linear homogeneous ODE (see Lemma 5.1), hence  $v_* = \text{const} \cdot w'_0$ . Moreover, (3.7), (3.8) and the compact embedding  $W^{1,2}(0, 1) \hookrightarrow C([0, 1])$  yield  $v_*(0) = 0$ , hence  $v_* = 0$ .

In a similar way one can consider another auxiliary sequence of functions:

$$\hat{v}_n(y) := \begin{cases} v_n(1 - \varepsilon_n y) & \text{for } 0 \leq y \leq 1/(2\varepsilon_n), \\ v_n\left(\frac{1}{2}\right) \exp\left[\lambda_1^- \left(y - \frac{1}{2\varepsilon_n}\right)\right] & \text{for } y \geq 1/(2\varepsilon_n), \end{cases}$$

and demonstrate that it converges weakly in  $W^{1,2}(0, \infty)$  to zero too. Remark, that for this consideration one should take test functions  $\eta \in W^{1,2}(0, \infty)$  with arbitrary values at zero, since the initial problem (1.1) has Neumann boundary condition at  $x = 1$ .

It is well-known that the compact embedding  $W^{1,2}(0, R) \hookrightarrow C([0, R])$  holds for any fixed  $R > 0$ . Thus, the above obtained weak limits for  $\tilde{v}_n$  and  $\hat{v}_n$  imply

$$\begin{aligned} \max\{|\tilde{v}_n(y)| : y \in [0, R]\} &\rightarrow 0 \quad \text{for } n \rightarrow \infty, \\ \max\{|\hat{v}_n(y)| : y \in [0, R]\} &\rightarrow 0 \quad \text{for } n \rightarrow \infty. \end{aligned} \tag{3.13}$$

Now we are going to show that

$$\frac{1}{\varepsilon_n} \int_0^1 (v_n(x)^2 + \varepsilon_n^2 v'_n(x)^2 + \varepsilon_n^4 v''_n(x)^2) dx \rightarrow 0 \quad \text{for } n \rightarrow \infty, \tag{3.14}$$

which is the needed contradiction to (3.6). Obviously, assumption (1.2) implies that there exists a constant  $c > 0$  such that

$$\frac{c}{\varepsilon_n} \int_0^1 [v_n(x)^2 + \varepsilon_n^2 v'_n(x)^2] dx \leq \frac{1}{\varepsilon_n} \int_0^1 [\varepsilon_n^2 v'_n(x)^2 + \partial_2 f(x, 0, 0, 0)v_n(x)^2] dx =$$

$$\begin{aligned}
&= \frac{1}{\varepsilon_n} \int_0^1 \left[ \varepsilon_n^2 v_n'(x)^2 + \partial_2 f(x, 0, 0, 0) v_n(x)^2 + \partial_3 f(x, 0, 0, 0) \varepsilon_n v_n'(x) v_n(x) \right] dx - \\
&\quad - \frac{1}{\varepsilon_n} \int_0^1 \partial_3 f(x, 0, 0, 0) \varepsilon_n v_n'(x) v_n(x) dx. \tag{3.15}
\end{aligned}$$

Therefore, we shall obtain the needed contradiction, if we prove that both terms in the right hand side of (3.15) tends to zero for  $n \rightarrow \infty$ . Integrating the latter term of (3.15) by parts and taking into account assumption (3.6) and relations 3.13 we see

$$\begin{aligned}
\frac{1}{\varepsilon_n} \int_0^1 \partial_3 f(x, 0, 0, 0) \varepsilon_n v_n'(x) v_n(x) dx &= \partial_3 f(x, 0, 0, 0) \frac{v_n(x)^2}{2} \Big|_{x=0}^{x=1} - \\
&\quad - \int_0^1 \partial_1 \partial_3 f(x, 0, 0, 0) v_n(x)^2 dx \rightarrow 0 \quad \text{for } n \rightarrow \infty.
\end{aligned}$$

Thus, it remains just to consider the first term in the right hand side of (3.15). For that, we rewrite it as follows

$$\begin{aligned}
&\frac{1}{\varepsilon_n} \int_0^1 \left[ \varepsilon_n^2 v_n'(x)^2 + \partial_2 f(x, 0, 0, 0) v_n(x)^2 + \partial_3 f(x, 0, 0, 0) \varepsilon_n v_n'(x) v_n(x) \right] dx = \\
&= \frac{1}{\varepsilon_n} \int_0^1 \left[ -\varepsilon_n^2 v_n''(x) + \partial_2 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) v_n(x) + \right. \\
&\quad \left. + \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \varepsilon_n v_n'(x) \right] v_n(x) dx + \\
&\quad + \varepsilon_n v_n'(1) v_n(1) - \varepsilon_n v_n'(0) v_n(0) + \\
&+ \frac{1}{\varepsilon_n} \int_0^1 \left[ \partial_2 f(x, 0, 0, 0) - \partial_2 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \right] v_n(x)^2 dx + \\
&+ \frac{1}{\varepsilon_n} \int_0^1 \left[ \partial_3 f(x, 0, 0, 0) - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \right] \varepsilon_n v_n'(x) v_n(x) dx. \tag{3.16}
\end{aligned}$$

Then, the first three terms in the right hand side of (3.16) vanish because of (3.7) (recall, that sequences  $\varepsilon_n v_n'(0)$  and  $v_n(1)$  were shown to be bounded). Concerning the rest two terms, we remark that they have a similar structure and therefore it is enough to consider one of them only, the another one can be estimated analogously. For that reason, below we continue with analysis of the latter term of (3.16). First, we rewrite it in the next way

$$\frac{1}{\varepsilon_n} \int_0^1 \left[ \partial_3 f(x, 0, 0, 0) - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \right] \varepsilon_n v_n'(x) v_n(x) dx =$$

$$\begin{aligned}
&= \frac{1}{\varepsilon_n} \int_0^1 [\partial_3 f(x, 0, 0, 0) - \partial_3 f(x, \mathcal{W}_{\varepsilon_n}(x), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(x), 0)] \varepsilon_n v'_n(x) v_n(x) dx + \\
&+ \frac{1}{\varepsilon_n} \int_0^1 [\partial_3 f(x, \mathcal{W}_{\varepsilon_n}(x), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(x), 0) - \\
&\quad - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), 0)] \varepsilon_n v'_n(x) v_n(x) dx + \\
&+ \frac{1}{\varepsilon_n} \int_0^1 [\partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), 0) - \\
&\quad - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n)] \varepsilon_n v'_n(x) v_n(x) dx. \tag{3.17}
\end{aligned}$$

Absolute values of the last two terms in the right hand side of (3.17) can be estimated by a constant times

$$\frac{1}{\varepsilon_n} \int_0^1 \{|r_\varepsilon(x)| + |\varepsilon_n r'_\varepsilon(x)|\} \cdot |\varepsilon_n v'_n(x) v_n(x)| dx \quad \text{and} \quad \int_0^1 |\varepsilon_n v'_n(x) v_n(x)| dx,$$

respectively. Hence, Cauchy inequality and assumptions (1.8), (3.6), (3.7) imply that these terms tend to zero. The first term in the right hand side of (3.17) we first rewrite as a sum of two integrals

$$\begin{aligned}
&\frac{1}{\varepsilon_n} \int_0^1 [\partial_3 f(x, 0, 0, 0) - \partial_3 f(x, \mathcal{W}_{\varepsilon_n}(x), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(x), 0)] \varepsilon_n v'_n(x) v_n(x) dx = \\
&= \frac{1}{\varepsilon_n} \int_0^{1/2} [\dots] dx + \frac{1}{\varepsilon_n} \int_{1/2}^1 [\dots] dx \tag{3.18}
\end{aligned}$$

and then consider each of them independently. So the first integral in the right hand side of (3.18) can be rewritten as follows

$$\begin{aligned}
&\frac{1}{\varepsilon_n} \int_0^{1/2} [\partial_3 f(x, 0, 0, 0) - \partial_3 f(x, \mathcal{W}_{\varepsilon_n}(x), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(x), 0)] \varepsilon_n v'_n(x) v_n(x) dx = \\
&= \frac{1}{\varepsilon_n} \int_0^{1/2} \left[ \partial_3 f(x, 0, 0, 0) - \partial_3 f\left(x, w_0\left(\frac{x}{\varepsilon_n}\right), w'_0\left(\frac{x}{\varepsilon_n}\right), 0\right) \right] \varepsilon_n v'_n(x) v_n(x) dx + \\
&+ \frac{1}{\varepsilon_n} \int_0^{1/2} \left[ \partial_3 f\left(x, w_0\left(\frac{x}{\varepsilon_n}\right), w'_0\left(\frac{x}{\varepsilon_n}\right), 0\right) - \right. \\
&\quad \left. - \partial_3 f(x, \mathcal{W}_{\varepsilon_n}(x), \varepsilon_n \mathcal{W}'_{\varepsilon_n}(x), 0) \right] \varepsilon_n v'_n(x) v_n(x) dx. \tag{3.19}
\end{aligned}$$

Now, the latter term of (3.19) can be estimated by a constant times

$$\frac{1}{\varepsilon_n} \int_0^{1/2} \left\{ \left| w_1 \left( \frac{1-x}{\varepsilon_n} \right) \right| + \left| w_1' \left( \frac{1-x}{\varepsilon_n} \right) \right| \right\} |\varepsilon_n v_n'(x) v_n(x)| dx.$$

Hence, inequalities (1.13) and assumption (3.6) imply that this tends to zero for  $n \rightarrow \infty$ . Analogously, one can estimate the first term in the right hand side of (3.19) with a constant times

$$\frac{1}{\varepsilon_n} \int_0^{1/2} \left\{ \left| w_0 \left( \frac{x}{\varepsilon_n} \right) \right| + \left| w_0' \left( \frac{x}{\varepsilon_n} \right) \right| \right\} |\varepsilon_n v_n'(x) v_n(x)| dx,$$

and prove that the latter expression vanishes too. Indeed, using the Cauchy inequality, assumption (3.6) and inequalities (1.13) we obtain

$$\begin{aligned} & \frac{1}{\varepsilon_n} \int_0^{1/2} \left\{ \left| w_0 \left( \frac{x}{\varepsilon_n} \right) \right| + \left| w_0' \left( \frac{x}{\varepsilon_n} \right) \right| \right\} |\varepsilon_n v_n'(x) v_n(x)| dx \leq \\ & \leq \sqrt{\frac{1}{\varepsilon_n} \int_0^{1/2} \left\{ \left| w_0 \left( \frac{x}{\varepsilon_n} \right) \right| + \left| w_0' \left( \frac{x}{\varepsilon_n} \right) \right| \right\}^2 v_n(x)^2 dx} \cdot \sqrt{\frac{1}{\varepsilon_n} \int_0^{1/2} \varepsilon_n^2 v_n'(x)^2 dx} \leq \\ & \leq \sqrt{\frac{1}{\varepsilon_n} \int_0^{1/2} \left\{ \left| w_0 \left( \frac{x}{\varepsilon_n} \right) \right| + \left| w_0' \left( \frac{x}{\varepsilon_n} \right) \right| \right\}^2 v_n(x)^2 dx} = \\ & = \sqrt{\int_0^{1/(2\varepsilon_n)} \{ |w_0(y)| + |w_0'(y)| \}^2 \tilde{v}_n(y)^2 dy} \leq \\ & \leq \text{const} \cdot \sqrt{\int_0^R \tilde{v}_n^2(y) dy + \int_R^\infty \{ |w_0(y)| + |w_0'(y)| \}^2 dy}, \end{aligned} \tag{3.20}$$

where  $R > 0$  is arbitrary. Now we proceed as above. First take  $R$  sufficiently large such that the integral  $\int_R^\infty \{ |w_0(y)| + |w_0'(y)| \}^2 dy$  is small. Then fix this  $R$ , use relations 3.13

and take  $n$  sufficiently large, such that the integral  $\int_0^R \tilde{v}_n^2(y) dy$  is small. Thus, we have verified that the first integral in the right hand side of (3.18) vanishes. In the similar way as above one can consider the second integral from (3.18) and verify that it tends to zero too. The only difference of this consideration consists of that one should use functions  $\hat{v}_n(y)$  instead of  $\tilde{v}_n(y)$  in a final estimate analogous to (3.20).

So, for (3.14) it remains to show that  $\frac{1}{\varepsilon_n} \int_0^1 \varepsilon_n^4 v_n''(x)^2 dx \rightarrow 0$  for  $n \rightarrow \infty$ . But this follows from

$$\begin{aligned} \left\| \varepsilon_n^2 v_n''(x) \right\|_{0, \varepsilon_n} &\leq \left\| \varepsilon_n^2 v_n''(x) - \partial_2 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) v_n(x) - \right. \\ &\quad \left. - \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \varepsilon_n v_n'(x) \right\|_{0, \varepsilon_n} + \\ &+ \left\| \partial_2 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) v_n(x) + \right. \\ &\quad \left. + \partial_3 f(x, \mathcal{U}_{\varepsilon_n}(x), \varepsilon_n \mathcal{U}'_{\varepsilon_n}(x), \varepsilon_n) \varepsilon_n v_n'(x) \right\|_{0, \varepsilon_n}. \end{aligned}$$

The first term in the right hand side tends to zero because of (3.7), and the second one because of  $\frac{1}{\varepsilon_n} \int_0^1 [\varepsilon_n^2 v_n'(x)^2 + v_n(x)^2] dx \rightarrow 0$  (which was shown above).  $\blacksquare$

**Remark 3.2** *If  $r_\varepsilon = 0$ , then Theorem 1.1 and estimate (3.5) imply that for sufficiently small  $\varepsilon$  problem (1.1) has a locally unique solution  $u_\varepsilon$  satisfying estimate  $\|u_\varepsilon - \mathcal{W}_\varepsilon\|_{2, \varepsilon} = O(\sqrt{\varepsilon})$ . Note, the proof of this fact does not require constructing of other asymptotic approximations which are more accurate than  $\mathcal{W}_\varepsilon(x)$ . From the other side, under assumptions of Theorem 1.1 one can always construct a first order formal solution asymptotics to problem (1.1) (see Theorem 4.2 with  $n = 1$ ), therefore Remark 4.3 implies that the above  $O(\sqrt{\varepsilon})$ -estimate could be replaced with a more accurate one  $\|u_\varepsilon - \mathcal{W}_\varepsilon\|_{2, \varepsilon} = O(\varepsilon)$ .*

**Remark 3.3** *Suppose that the function  $f$  depends neither on  $x$  nor on  $\varepsilon$  and, moreover, that  $\beta_1 = 0$ , then Theorem 1.1 provides us a significantly better estimate than that mentioned in Remark 3.2. Indeed, taking into account estimate (3.4) with  $r_\varepsilon = 0$  and corresponding estimates for boundary conditions,  $\mathcal{W}_\varepsilon(0) - \beta_0 = O(e^{\lambda_1^-/\varepsilon})$  and  $\varepsilon \mathcal{W}'_\varepsilon(1) = O(e^{\lambda_0^-/\varepsilon})$ , we obtain that the difference between the exact solution  $u_\varepsilon$  to problem (1.1) and asymptotic approximation  $\mathcal{W}_\varepsilon(x)$  satisfies  $\|u_\varepsilon - \mathcal{W}_\varepsilon\|_{2, \varepsilon} = O(e^{\lambda_m/\varepsilon})$ , where  $\lambda_m := \max\{\lambda_0^-, \lambda_1^-\}$ .*

## 4 Asymptotics of the higher orders

If the function  $f$  satisfies additional smoothness properties, then based on the leading term asymptotics  $\mathcal{W}_\varepsilon(x)$ , one can construct a more precise formal asymptotic approximations of the boundary layer solution to problem (1.1). For this purpose one can use, for example, the boundary function method (see [9, 10]). In present section we demonstrate that the above-proved results (cf. Theorem 1.1) justify any such formal approximation without involving any additional assumptions.

Recall, that the ansatz of the boundary function method usually reads

$$u(x, \varepsilon) = \bar{\mathcal{U}}_\varepsilon(x) + \mathcal{P}_\varepsilon\left(\frac{x}{\varepsilon}\right) + \mathcal{R}_\varepsilon\left(\frac{1-x}{\varepsilon}\right), \quad (4.1)$$

where

$$\bar{\mathcal{U}}_\varepsilon(x) = \sum_{k=0}^{\infty} \varepsilon^k \bar{u}_k(x), \quad \mathcal{P}_\varepsilon(y) = \sum_{k=0}^{\infty} \varepsilon^k P_k(y), \quad \mathcal{R}_\varepsilon(y) = \sum_{k=0}^{\infty} \varepsilon^k R_k(y).$$



Substituting first the regular part  $\bar{U}_\varepsilon(x)$  into the differential equation of problem (1.1) and equating formally powers of  $\varepsilon$ , we obtain a sequence of algebraic equations

$$\begin{aligned} f(x, \bar{u}_0(x), 0, 0) &= 0, \\ \partial_2 f(x, \bar{u}_0(x), 0, 0) \cdot \bar{u}_1(x) + \partial_3 f(x, \bar{u}_0(x), 0, 0) \cdot \bar{u}'_0(x) + \partial_4 f(x, \bar{u}_0(x), 0, 0) &= 0, \\ \dots & \\ \partial_2 f(x, \bar{u}_0(x), 0, 0) \cdot \bar{u}_k(x) + \{\text{terms depending on } \bar{u}_0(x), \dots, \bar{u}_{k-1}(x) \text{ only}\} &= 0. \end{aligned}$$

Now, if we take  $\bar{u}_0(x) = 0$ , then assumption (1.2) allows us to define uniquely from the above equations all terms  $\bar{u}_k(x)$  in a recurrent way.

When the terms of the regular part are known, we can continue with definition of the boundary functions  $\mathcal{P}_\varepsilon(y)$  and  $\mathcal{R}_\varepsilon(y)$ . For that we write boundary value problems

$$\begin{aligned} \mathcal{P}_\varepsilon''(y) &= f(\varepsilon y, \bar{U}_\varepsilon(\varepsilon y) + \mathcal{P}_\varepsilon(y), \varepsilon \bar{U}'_\varepsilon(\varepsilon y) + \mathcal{P}'_\varepsilon(y), \varepsilon) - \\ &\quad - f(\varepsilon y, \bar{U}_\varepsilon(\varepsilon y), \varepsilon \bar{U}'_\varepsilon(\varepsilon y), \varepsilon), \quad y > 0, \\ \mathcal{P}_\varepsilon(0) + \bar{U}_\varepsilon(0) &= \beta_0, \quad \mathcal{P}_\varepsilon(\infty) = 0, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \mathcal{R}_\varepsilon''(y) &= f(1 - \varepsilon y, \bar{U}_\varepsilon(1 - \varepsilon y) + \mathcal{R}_\varepsilon(y), \varepsilon \bar{U}'_\varepsilon(1 - \varepsilon y) - \mathcal{R}'_\varepsilon(y), \varepsilon) - \\ &\quad - f(1 - \varepsilon y, \bar{U}_\varepsilon(1 - \varepsilon y), \varepsilon \bar{U}'_\varepsilon(1 - \varepsilon y), \varepsilon), \quad y > 0, \\ \mathcal{R}'_\varepsilon(0) + \varepsilon \bar{U}'_\varepsilon(1) &= \varepsilon \beta_1, \quad \mathcal{R}_\varepsilon(\infty) = 0, \end{aligned} \tag{4.3}$$

and equate formally powers of  $\varepsilon$  in equations and boundary conditions. In result we obtain a sequence of  $\varepsilon$ -independent problems for definition of all terms in series  $\mathcal{P}_\varepsilon(y)$  and  $\mathcal{R}_\varepsilon(y)$ . In particular, the zeroth order terms give problems (1.3) and (1.4). Consequently, we may assume that  $P_0(y) = w_0(y)$  and  $R_0(y) = w_1(y)$ . Further, the problems involving higher order terms are linear and read

$$\begin{aligned} P_k''(y) &= \partial_2 f(0, P_0(y), P'_0(y), 0) \cdot P_k(y) + \\ &\quad + \partial_3 f(0, P_0(y), P'_0(y), 0) \cdot P'_k(y) + F_k(y), \quad y > 0, \\ P_k(0) + \bar{u}_k(0) &= 0, \quad P_k(\infty) = 0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} R_k''(y) &= \partial_2 f(1, R_0(y), -R'_0(y), 0) \cdot R_k(y) - \\ &\quad - \partial_3 f(1, R_0(y), -R'_0(y), 0) \cdot R'_k(y) + G_k(y), \quad y > 0, \\ R'_k(0) + \bar{u}'_{k-1}(1) &= \beta_1 \cdot \delta_{k1}, \quad R_k(\infty) = 0, \end{aligned} \tag{4.5}$$

respectively. Here the function  $F_k(y)$  is expressed recursively through the  $P_i(y)$  with  $i < k$ . In particular,

$$\begin{aligned} F_1(y) &= [\partial_1 f(0, P_0(y), P'_0(y), 0) - \partial_1 f(0, 0, 0, 0)] y + \\ &\quad + [\partial_2 f(0, P_0(y), P'_0(y), 0) - \partial_2 f(0, 0, 0, 0)] \bar{u}_1(0) + \\ &\quad + [\partial_4 f(0, P_0(y), P'_0(y), 0) - \partial_4 f(0, 0, 0, 0)]. \end{aligned}$$

Analogously, the function  $G_k(y)$  is expressed recursively through the  $R_i(y)$  with  $i < k$ .

Function  $w'_0(y)$  and the linearly independent solution  $\bar{w}_0(y)$  of equation (5.2) with  $\bar{w}_0(0) = 0$  and  $\bar{w}'_0(0) = 1$  constitute a fundamental system of the linear homogeneous equation corresponding to the equation of problem (4.4) (see Lemma 5.1). Thus, there exists a Green's function of the problem (4.4)

$$G(y, z) := \begin{cases} \frac{\bar{w}_0(y)w'_0(z)}{w'_0(0)} \exp \left[ - \int_0^z \partial_2 f(0, P_0(s), P'_0(s), 0) ds \right] & \text{for } 0 \leq y \leq z, \\ \frac{w'_0(y)\bar{w}_0(z)}{w'_0(0)} \exp \left[ - \int_0^z \partial_2 f(0, P_0(s), P'_0(s), 0) ds \right] & \text{for } z < y, \end{cases}$$

such that, given a bounded function  $F_k(y)$ , this problem has a unique solution which can be written in the integral form

$$P_k(y) = -\bar{u}_k(0) \frac{w'_0(y)}{w'_0(0)} + \int_0^\infty G(y, z) F_k(z) dz. \quad (4.6)$$

Suppose that  $|F_k(y)| \leq ce^{-\alpha y}$  for all  $y \geq 0$ , where  $c, \alpha > 0$  are some constants, then using the formula (4.6) it is easy to verify that there exists a constant  $\tilde{c} > 0$  such that a solution to problem (4.4) satisfies an exponential estimate  $|P_k(y)| \leq \tilde{c}e^{-\alpha y}$  for all  $y \geq 0$ . From the other side, if all functions  $P_i(y)$  with  $i < k$  satisfy exponential estimates similar to (1.13) then due to the construction of function  $F_k(y)$  it obeys analogous exponential estimate at infinity. Therefore solving step by step problems (4.4) one can define any number of terms  $P_k(y)$ . Obviously, the same statement is valid concerning problems (4.5) too. Thus, following the above describe algorithm one can easily verify the next

**Lemma 4.1** *Let  $f \in C^{(n+1)}([0, 1] \times \mathbb{R}^2 \times [0, \infty))$ ,  $n \geq 1$ , satisfy (1.2)–(1.4). Then there exists a unique  $n$ -th order formal asymptotics*

$$\begin{aligned} \mathcal{U}_{\varepsilon, n}(x) = & w_0\left(\frac{x}{\varepsilon}\right) + w_1\left(\frac{1-x}{\varepsilon}\right) + \\ & + \sum_{k=1}^n \varepsilon^k \left[ \bar{u}_k(x) + P_k\left(\frac{x}{\varepsilon}\right) + R_k\left(\frac{1-x}{\varepsilon}\right) \right] \end{aligned} \quad (4.7)$$

defined by boundary function method. Moreover, there exist constants  $c_n > 0$  and  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0]$  hold

$$\begin{aligned} \|\varepsilon^2 \mathcal{U}_{\varepsilon, n}''(x) - f(x, \mathcal{U}_{\varepsilon, n}(x), \varepsilon \mathcal{U}'_{\varepsilon, n}(x), \varepsilon)\|_{C[0,1]} &\leq c_n \varepsilon^{n+1}, \\ |\mathcal{U}_{\varepsilon, n}(0) - \beta_0| \leq c_n \varepsilon^{n+1}, \quad |\varepsilon \mathcal{U}'_{\varepsilon, n}(1) - \varepsilon \beta_1| &\leq c_n \varepsilon^{n+1}. \end{aligned} \quad (4.8)$$

Since asymptotics (4.7) satisfies by construction assumptions of Theorem 1.1, the latter implies

**Theorem 4.2** *Let  $f \in C^{(n+1)}([0, 1] \times \mathbb{R}^2 \times [0, \infty))$ ,  $n \geq 1$ , satisfy (1.2)–(1.4). Then there exist  $\varepsilon_0 > 0$  and  $\delta > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  there exists exactly one solution  $u = u_\varepsilon$  to (1.1) such that  $\|u - \mathcal{U}_{\varepsilon, n}\|_{2, \varepsilon} < \delta$ . Moreover, there exists  $c_n > 0$  such that*

$$\|u_\varepsilon - \mathcal{U}_{\varepsilon, n}\|_{2, \varepsilon} \leq c_n \varepsilon^{n+1/2}.$$

**Remark 4.3** *Suppose that function  $f$  is of class  $C^k$ ,  $k \geq 2$ . Then one can apply Theorem 4.2 with  $n = 1$ ,  $n = 2, \dots$ , and  $n = k$  and obtain an array of solutions  $u_{\varepsilon, n}$  to problem (1.1), each of which is unique in the corresponding ball*

$$B_n := \{u \in W^{2,2}(0, 1) : \|u - \mathcal{U}_{\varepsilon, n}\|_{2, \varepsilon} < \delta_n\}.$$

*Since  $\min\{\delta_n : n \leq k\} > 0$  and the centers of these balls converge to each other as  $\varepsilon \rightarrow +0$ , one can choose  $\varepsilon_0 > 0$  such that for  $\varepsilon \in (0, \varepsilon_0)$  all solution  $u_{\varepsilon, n}$  should coincide. In other words, for sufficiently small  $\varepsilon$  Theorem 4.2 provides different asymptotics for one and the same solution to problem (1.1) which is unique in  $\cup_{n=0}^k B_n$ .*

## 5 Appendix: Exponential Decay and Growth of Solutions to Second Order ODEs

The following lemma collects well-known facts about the behavior of solutions to some auxiliary second order ODEs.

**Lemma 5.1** *Let  $f \in C^2(\mathbb{R}^2)$  and  $w_0 \in C^2([0, \infty))$  be such that  $f(0, 0) = 0$ ,  $\partial_1 f(0, 0) > 0$ ,  $w_0(0) \neq 0$ ,  $w_0(\infty) = 0$  and*

$$w_0''(y) = f(w_0(y), w_0'(y)) \text{ for } y > 0. \quad (5.1)$$

*Then the following holds:*

(i) *There exist  $a, b, y_0 > 0$  such that*

$$\begin{aligned} ae^{\lambda y} &\leq |w_0(y)| \leq be^{\lambda y} && \text{for all } y \geq y_0, \\ ae^{\lambda y} &\leq |w_0'(y)| \leq be^{\lambda y} && \text{for all } y \geq y_0, \end{aligned}$$

*where  $\lambda = \{\partial_2 f(0, 0) - \sqrt{[\partial_2 f(0, 0)]^2 + 4\partial_1 f(0, 0)}\}/2$ .*

(ii) *The function  $w_0'$  together with an exponentially growing function constitutes a fundamental system for the linear homogeneous ODE*

$$w''(y) = \partial_1 f(w_0(y), w_0'(y))w(y) + \partial_2 f(w_0(y), w_0'(y))w'(y). \quad (5.2)$$

**Acknowledgements.** The first author gratefully acknowledges support from the Alexander von Humboldt Foundation during his stay at the Humboldt University of Berlin.

## References

- [1] E.A. CODDINGTON and N. LEVINSON. Theory of ordinary differential equations. McGraw-Hill. New York Toronto London, 1955.
- [2] J. HALE and D. SALAZAR. Boundary layers in a semilinear parabolic problem. *Tohoku Math. J.* 51 (1999), 421–432.
- [3] K.W. CHANG and F.A. HOWES. Nonlinear singular perturbation phenomena. Springer-Verlag. New York Berlin Heidelberg, 1984.
- [4] N.N. NEFEDOV. The method of differential inequalities for some singularly perturbed partial differential equations. *Diff. Eq.* 31 (1995), 668–671.
- [5] J. APPELL, A. VIGNOLI, and P.P. ZABREJKO. Implicit function theorems and nonlinear integral equations. *Expo. Math.* 14 (1996), 384–424.
- [6] P.C. FIFE. Boundary and interior transition layer phenomena for pairs of second-order differential equations. *J. Math. Anal. Appl.* 54 (1976), 497–521.
- [7] P.C. FIFE and W.M. GREENLEE. Transition layers for elliptic boundary value problems with small parameters. *Uspechi Mat. Nauk* 24 (1974), 103–130.
- [8] R. MAGNUS. The implicit function theorem and multi-bump solutions of periodic partial differential equations. *Proc. Royal Soc. Edinburgh* 136A (2006), 559–583.
- [9] A.B. VASIL'EVA, V.F. BUTUZOV and L.V. KALACHEV. Boundary function method for singular perturbation problems. SIAM. Philadelphia, 1995.
- [10] R.E. O'MALLEY. Singular perturbation methods for ordinary differential equations. Springer-Verlag. New York Berlin Heidelberg, 1991.