Boundary Layer Solutions to Problems with Infinite Dimensional Singular and Regular Perturbations

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Abstract.

We prove existence, local uniqueness and asymptotic estimates for boundary layer solutions to singularly perturbed equations of the type $(\varepsilon(x)^2u'(x))' = f(x,u(x)) + g(x,u(x),\varepsilon(x)u'(x))$, $0 < x < 1$, with Dirichlet and Neumann boundary conditions. Here the functions ε and g are small and, hence, regarded as singular and regular functional perturbation parameters. The main tool of the proofs is a generalization (to Banach space bundles) of an Implicit Function Theorem of R. Magnus.

Keywords: singular perturbation, asymptotic approximation, boundary layer, implicit function theorem, space depending small diffusion coefficient.

1 Introduction and Results

This paper concerns boundary value problems for second order semilinear ODEs of the type

$$\left.\begin{array}{l}(\varepsilon(x)^2u'(x))' = f(x,u(x)) + g(x,u(x),\varepsilon(x)u'(x)),\ 0 < x < 1,\\ u(0) = b_0,\ u'(1) = b_1.\end{array}\right\} \tag{1.1}$$

In (1.1) the functions $\varepsilon : [0,1] \to (0,\infty)$ and $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are close to zero (in the sense of certain function space norms), i.e. ε and g are the infinite dimensional singular and regular perturbation parameters, respectively. The boundary data $b_0, b_1 \in \mathbb{R}$ are fixed as well as the function $f \in C^2([0,1] \times \mathbb{R})$.

We suppose that there exist C^2-functions $u_0 : [0,1] \to \mathbb{R}$ and $v_0, w_0 : [0,\infty) \to \mathbb{R}$ such that

$$f(x,u_0(x)) = 0 \text{ and } \partial_2 f(x,u_0(x)) > 0,\ x \in [0,1] \tag{1.2}$$

and

$$\left.\begin{array}{l}v_0''(y) = f(0,u_0(0)+v_0(y)),\ y > 0,\\ v_0(0) = b_0 - u_0(0), v_0(\infty) = 0, v_0'(0) \neq 0\end{array}\right\} \tag{1.3}$$

and

$$\left.\begin{array}{l}w_0''(y) = f(1,u_0(1)+w_0(y)),\ y > 0,\\ w_0'(0) = w_0(\infty) = 0.\end{array}\right\} \tag{1.4}$$

In (1.2), $\partial_2 f$ denotes the partial derivative of the function f with respect to its second variable. Similar notation will be used later on.

Our goal is to describe existence, local uniqueness and asymptotic behavior for $\varepsilon \to 0$ and $g \to 0$ of boundary layer solutions to (1.1), i.e. of solutions u with

$$u(x) \approx \mathcal{U}_\varepsilon(x) := u_0(x) + v_0\left(\int_0^x \frac{d\xi}{\varepsilon(\xi)}\right) + w_0\left(\int_x^1 \frac{d\xi}{\varepsilon(\xi)}\right). \tag{1.5}$$

The existence and uniqueness part of our main result Theorem 1.1 below has the following structure: For all $\varepsilon \approx 0$ and all $g \approx 0$ there exists exactly one solution $u \approx \mathcal{U}_\varepsilon$ to (1.1). In order to make this statement rigorous we have to introduce norms which measure the distances of the perturbation parameters ε and g from zero and of the solution u from the approximate solution $\mathcal{U}_\varepsilon$. The singular perturbation parameter ε will vary in the set

$$C^1_+([0,1]) := \{\varepsilon \in C^1([0,1]) : \ \varepsilon(x) > 0 \text{ for all } x \in [0,1]\}, \tag{1.6}$$

and its distance from zero will be measured by the norm

$$\|\varepsilon\|_\infty := \max\{|\varepsilon(x)| : \ x \in [0,1]\}$$

as well as by the norm $\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty$. The solutions u will belong to $C^2([0,1])$, and their distance from $\mathcal{U}_\varepsilon$ will be measured by the norm

$$\|u\|_\varepsilon := \sqrt{\int_0^1 \left(|u(x)|^2 + |\varepsilon(x)u'(x)|^2 + |\varepsilon(x)\left(\varepsilon(x)u'(x)\right)'|^2\right)\frac{dx}{\varepsilon(x)}}. \tag{1.7}$$

Remark that there exists a positive constant such that for all $\varepsilon \in C^1_+([0,1])$ with $\|\varepsilon\|_\infty \le 1$ and all $u \in C^2([0,1])$ it holds

$$\|u\|_\infty + \|\varepsilon u'\|_\infty \le \text{const } \|u\|_\varepsilon. \tag{1.8}$$

(cf. (3.16)). Hence, the behavior of the function g for large second and third arguments is not relevant for our results. Therefore, the regular perturbation parameter g will be considered to belong to the space

$$C^{0,1,1}([0,1] \times K^2) := \{g \in C([0,1] \times K^2) : \ \partial_2 g \text{ and } \partial_3 g \text{ exist and are continuous}\}, \tag{1.9}$$

and its distance from zero will be measured by the norm

$$\|g\|_\infty := \max\{|g(x,u,v)| : \ x \in [0,1], u, v \in K\},$$

where $K := [-4k, 4k]$ is a compact interval defined by a parameter $k > 0$ which should be choosen sufficiently large such that $|u_0(x)|, |v_0(y)|, |v_0'(y)|, |w_0(y)|, |w_0'(y)| \le k$ for all $x \in [0,1]$ and $y \in [0,\infty)$.

Our main result is the following

Theorem 1.1 *Let $f \in C^2([0,1] \times \mathbb{R})$ satisfy (1.2)–(1.4). Then there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in C^1_+([0,1])$ and $g \in C^{0,1,1}([0,1] \times K^2)$ with*

$$\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty + \sqrt{\int_0^1 \frac{dx}{\varepsilon(x)}}\ \left(\|g\|_\infty + \|\partial_2 g\|_\infty + \|\partial_3 g\|_\infty\right) < \varepsilon_0 \tag{1.10}$$

there exists exactly one solution $u = u_{\varepsilon,g}$ to (1.1) such that $\|u - \mathcal{U}_\varepsilon\|_\varepsilon < \delta$. Moreover, there exists $c > 0$ such that

$$\|u_{\varepsilon,g} - \mathcal{U}_\varepsilon\|_\varepsilon \le c \left(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty + \sqrt{\int_0^1 \frac{dx}{\varepsilon(x)}}\, \|g\|_\infty \right). \tag{1.11}$$

Our paper is organized as follows:

In Section 2 we formulate and prove a generalization of the Implicit Function Theorem which is perhaps of its own interest. There we use recent results of R. Magnus [2]. In Section 3 we apply this Implicit Function Theorem for proving Theorem 1.1.

Remark 1.2 about the boundary layer property of the solutions $\boldsymbol{u_{\varepsilon,g}}$ *Assumptions (1.2), (1.3) and (1.4) imply that there exists $a, \alpha > 0$ such that $|v_0(y)|, |w_0(y)| \le ae^{-\alpha y}$ for all $y \ge 0$. Hence, the definition (1.5) of $\mathcal{U}_\varepsilon$ and the assertion (1.11) yield that for each $\gamma \in (0, 1/2)$ there exists $c_\gamma > 0$ such that*

$$|u_{\varepsilon,g}(x)| \le c_\gamma \left(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty + \sqrt{\int_0^1 \frac{dy}{\varepsilon(y)}}\, \|g\|_\infty \right) \text{ for all } x \in [\gamma, 1-\gamma].$$

Remark 1.3 about sufficient conditions for (1.3) and (1.4) *Suppose $b_0 - u_0(0) > 0$. Then assumption (1.3) is satisfied if, for example, the conservative system*

$$v'' = f(0, u_0(0) + v)$$

has a homoclinic solution $v_* : \mathbb{R} \to \mathbb{R}$ with $v_*(\pm\infty) = 0$ and $v_*(y) > b_0 - u_0(0)$ for at least one $y \in \mathbb{R}$. In order to show this, without loss of generality we can assume $v_*'(0) = 0$. Then there exist $y_1 < 0 < y_2$ such that $v_*(y_1) = v_*(y_2) = b_0 - u_0(0)$, $v_*'(y_1) > 0$ and $v_*'(y_2) < 0$. Hence, the functions $v_0(y) := v_*(y + y_j)$, $j = 1, 2$ satisfy (1.3).

The choice with $j = 1$ leads to a non-monotone function v_0 and, hence, to a non-monotone Dirichlet boundary layer at $x = 0$ of the solution $u_{\varepsilon,g}$, produced by Theorem 1.1 (cf. (1.5)). The choice with $j = 1$ leads to a monotone Dirichlet boundary layer.

Similarly one can formulate sufficient conditions for (1.4): If $w'' = f(1, u_0(1) + w)$ has a homoclinic solution w_* with $w_*(\pm\infty) = 0$, then there exists $y_0 \in \mathbb{R}$ such that $w_*'(y_0) = 0$. Hence, the function $w_0(y) := w_*(y + y_0)$ satisfies (1.4). If $w_*(y_0) \ne 0$, then this leads, via Theorem 1.1, to solutions $u_{\varepsilon,g}$ with "large" Neumann boundary layers at $x = 1$. If $w_*(y_0) = 0$ and, hence, $w_* = 0$, this leads to "small" Neumann boundary layers (cf. (1.5)).

Remark 1.4 about the case $\boldsymbol{\varepsilon}$ = const *Suppose that ε is a constant function and that $g = 0$. Then (1.1) reads as*

$$\left.\begin{array}{l} \varepsilon^2 u''(x) = f(x, u(x)), \quad 0 < x < 1, \\ u(0) = b_0, \ \ u'(1) = b_1. \end{array}\right\} \tag{1.12}$$

For those problems J. Hale *and* D. Salazar *showed in [1] existence and asymptotic behavior for $\varepsilon \to 0$ of solutions with monotone or non-monotone Dirichlet boundary layers* *and with "small" or "large" Neumann boundary layers and with internal layers. Their existence proofs were based on a combination of the Liapunov-Schmidt procedure and the implicit function theorem. For that they needed eigenvalue estimates for the differential operator*

$$\varepsilon^2 \frac{d^2}{dx^2} + \partial_2 f(x, \mathcal{U}(x,\varepsilon))$$

with corresponding homogeneous boundary conditions, where $\mathcal{U}(x,\varepsilon)$ is a family of second order approximate solutions to (1.12), i.e. this family satisfies (1.12) with an error of order $O(\varepsilon^2)$.

The proof of our Theorem 1.1 is also based on the implicit function theorem, but we don't need neither the Liapunov-Schmidt procedure nor eigenvalue estimates. Instead we use a lemma of R. Magnus *[2, Lemma 1.3] which helps to verify the assumptions of a quite general implicit function theorem (see our Section 2).*

Existence and asymptotic behavior for $\varepsilon \to 0$ of solutions to (1.12) with monotone Dirichlet boundary layers and with "small" Neumann boundary layers is proved also by upper and lower solution techniques, see, for example, [3, 4].

2 A Generalization of the Implicit Function Theorem

In this section we formulate and prove an implicit function theorem with minimal assumptions concerning continuity with respect to the control parameter. This is just what we need for the proof of our Theorem 1.1.

Our implicit function theorem is very close to that of R. Magnus [2, Theorem 1.2]. The difference is that we work in bundles of Banach spaces and with multi-dimensional control parameters, while Magnus works with a fixed pair of Banach spaces and with scalar control parameters. For other implicit function theorems with weak assumptions concerning continuity with respect to the control parameter see [5, Theorem 7], [6, Theorem 3.4] and [7, Theorem 4.1].

Theorem 2.1 *Let E be a normed vector space and E_0 a subset of E such that zero belongs to the closure of E_0. Further, for any $\varepsilon \in E_0$ let be given normed vector spaces Λ_ε and Banach spaces U_ε and V_ε. Finally, for any $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ let be given maps $F_{\varepsilon,\lambda} \in C^1(U_\varepsilon, V_\varepsilon)$ such that*

$$\|F_{\varepsilon,\lambda}(0)\| \to 0 \quad \textit{for} \quad \|\varepsilon\| + \|\lambda\| \to 0, \tag{2.1}$$

$$\|F'_{\varepsilon,\lambda}(u) - F'_{\varepsilon,\lambda}(0)\| \to 0 \quad \textit{for} \quad \|\varepsilon\| + \|\lambda\| + \|u\| \to 0 \tag{2.2}$$

and

$$\left.\begin{array}{l} \textit{there exist } \varepsilon_0 > 0 \textit{ and } c > 0 \textit{ such that for all } \varepsilon \in E_0 \textit{ and } \lambda \in \Lambda_\varepsilon \textit{ with} \\ \|\varepsilon\| + \|\lambda\| < \varepsilon_0 \textit{ the operators } F'_{\varepsilon,\lambda}(0) \textit{ are invertible and } \|F'_{\varepsilon,\lambda}(0)^{-1}\| \le c. \end{array}\right\} \tag{2.3}$$

Then there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $\delta > 0$ such that for all $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_1$ there exists exactly one $u = u_{\varepsilon,\lambda}$ with $\|u\| < \delta$ and $F_{\varepsilon,\lambda}(u) = 0$. Moreover,

$$\|u_{\varepsilon,\lambda}\| \le 2c\, \|F_{\varepsilon,\lambda}(0)\|. \tag{2.4}$$

Proof For $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_0$ we have $F_{\varepsilon,\lambda}(u) = 0$ if and only if

$$G_{\varepsilon,\lambda}(u) := u - F'_{\varepsilon,\lambda}(0)^{-1} F_{\varepsilon,\lambda}(u) = u. \tag{2.5}$$

Moreover, for such ε and λ and all $u, v \in U_\varepsilon$ we have

$$\begin{aligned} G_{\varepsilon,\lambda}(u) - G_{\varepsilon,\lambda}(v) &= \int_0^1 G'_{\varepsilon,\lambda}(su + (1-s)v)(u-v)ds = \\ &= F'_{\varepsilon,\lambda}(0)^{-1} \int_0^1 \left(F'_{\varepsilon,\lambda}(0) - F'_{\varepsilon,\lambda}(su + (1-s)v)\right)(u-v)ds. \end{aligned}$$

Hence, assumptions (2.2) and (2.3) imply that there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $\delta > 0$ such that for all $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_1$

$$\|G_{\varepsilon,\lambda}(u) - G_{\varepsilon,\lambda}(v)\| \le \frac{1}{2}\|u - v\| \text{ for all } u, v \in K_\varepsilon^\delta := \{w \in U_\varepsilon : \ \|w\| \le \delta\}.$$

Using this and (2.3) again, for all $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_1$ we get

$$\|G_{\varepsilon,\lambda}(u)\| \le \|G_{\varepsilon,\lambda}(u) - G_{\varepsilon,\lambda}(0)\| + \|G_{\varepsilon,\lambda}(0)\| \le \frac{1}{2}\|u\| + c\|F_{\varepsilon,\lambda}(0)\|. \tag{2.6}$$

Hence, assumption (2.1) yields that $G_{\varepsilon,\lambda}$ maps K_ε^δ into K_ε^δ for all $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_1$, if ε_1 is chosen sufficiently small. Now, Banach's fixed point theorem gives a unique in K_ε^δ solution $u = u_{\varepsilon,\lambda}$ to (2.5) for all $\varepsilon \in E_0$ and $\lambda \in \Lambda_\varepsilon$ with $\|\varepsilon\| + \|\lambda\| < \varepsilon_1$. Moreover, (2.6) yields $\|u_{\varepsilon,\lambda}\| \le 1/2\|u_{\varepsilon,\lambda}\| + c\|F_{\varepsilon,\lambda}(0)\|$, i.e. (2.4). ∎

The following lemma is [2, Lemma 1.3], translated to our setting. It gives a criterion how to verify the key assumption (2.3) of Theorem 2.1:

Lemma 2.2 *Let $F'_{\varepsilon,\lambda}(0)$ be Fredholm of index zero for all $\varepsilon \in E_0$ and all $\lambda \in \Lambda_\varepsilon$. Suppose that there do not exist sequences $\varepsilon_1, \varepsilon_2 \ldots \in E_0$, $\lambda_1 \in \Lambda_{\varepsilon_1}, \lambda_2 \in \Lambda_{\varepsilon_2} \ldots$ and $u_1 \in U_{\varepsilon_1}, u_2 \in U_{\varepsilon_2} \ldots$ with $\|u_n\| = 1$ for all $n \in \mathbb{N}$ and $\|\varepsilon_n\| + \|\lambda_n\| + \|F'_{\varepsilon_n,\lambda_n}(0)u_n\| \to 0$ for $n \to 0$. Then (2.3) is satisfied.*

Proof Suppose that (2.3) is not true. Then there exist sequences $\varepsilon_1, \varepsilon_2 \ldots \in E_0$ and $\lambda_1 \in \Lambda_{\varepsilon_1}, \lambda_2 \in \Lambda_{\varepsilon_2} \ldots$ with $\|\varepsilon_n\| + \|\lambda_n\| \to 0$ for $n \to 0$ such that either $F'_{\varepsilon_n,\lambda_n}(0)$ is not invertible or it is but $\|F'_{\varepsilon_n,\lambda_n}(0)^{-1}\| \ge n$ for all $n \in \mathbb{N}$. In the first case there exist $u_n \in U_{\varepsilon_n}$ with $\|u_n\| = 1$ and $F'_{\varepsilon_n,\lambda_n}(0)u_n = 0$ (because $F'_{\varepsilon_n,\lambda_n}(0)$ is Fredholm of index zero). In the second case there exist $v_n \in V_{\varepsilon_n}$ with $\|v_n\| = 1$ and $\||F'_{\varepsilon_n,\lambda_n}(0)^{-1}v_n\| \ge n$, i.e.

$$\||F'_{\varepsilon_n,\lambda_n}(0)u_n\| \le \frac{1}{n} \text{ with } u_n := \frac{F'_{\varepsilon_n,\lambda_n}(0)^{-1}v_n}{\|F'_{\varepsilon_n,\lambda_n}(0)^{-1}v_n\|}.$$

But this contradicts to the assumptions of the lemma. ∎

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Hence, we always suppose the assumptions of Theorem 1.1 to be satisfied. In particular, we use the functions u_0, v_0 and w_0, which are introduced in (1.2), (1.3) and (1.4), and the notation $C^1_+([0,1])$ and $C^{0,1,1}([0,1] \times K^2)$, introduced in (1.6) and (1.9).

3.1 Introduction of Stretched Variables

For $\varepsilon \in C^1_+([0,1])$ we introduce functions $\varphi_\varepsilon, \psi_\varepsilon : [0,1] \to [0,\infty)$ by

$$\varphi_\varepsilon(x) := \int_0^x \frac{dy}{\varepsilon(y)}, \quad \psi_\varepsilon(x) := \int_x^1 \frac{dy}{\varepsilon(y)}.$$

Obviously, φ_ε and ψ_ε are strictly monotone C^2-functions, and

$$\varphi_\varepsilon'(x) = \frac{1}{\varepsilon(x)}, \quad \psi_\varepsilon'(x) = -\frac{1}{\varepsilon(x)}.$$

We look for solutions to (1.1) by means of the ansatz

$$u(x) = u_0(x) + v(\varphi_\varepsilon(x)) + w(\psi_\varepsilon(x)). \tag{3.1}$$

From (3.1) follows

$$\begin{aligned} \varepsilon(x)^2 u'(x) &= \varepsilon(x)^2 u_0'(x) + \varepsilon(x)\left(v'(\varphi_\varepsilon(x)) - w'(\psi_\varepsilon(x))\right), \\ \left(\varepsilon(x)^2 u'(x)\right)' &= \left(\varepsilon(x)^2 u_0'(x)\right)' + v''(\varphi_\varepsilon(x)) + w''(\psi_\varepsilon(x)) + \varepsilon'(x)\left(v'(\varphi_\varepsilon(x)) - w'(\psi_\varepsilon(x))\right). \end{aligned}$$

Therefore, if $v : [0, \varphi_\varepsilon(1)] \to \mathbb{R}$ and $w : [0, \psi_\varepsilon(0)] \to \mathbb{R}$ are solutions to the boundary value problems

$$\left.\begin{array}{l} v'' + \varepsilon'(\varphi_\varepsilon^{-1}(y))v' + 2\varepsilon(\varphi_\varepsilon^{-1}(y))\varepsilon'(\varphi_\varepsilon^{-1}(y))u_0'(\varphi_\varepsilon^{-1}(y)) + \varepsilon(\varphi_\varepsilon^{-1}(y))u_0''(\varphi_\varepsilon^{-1}(y)) = \\ \qquad = f(\varphi_\varepsilon^{-1}(y), u_0(\varphi_\varepsilon^{-1}(y)) + v), \ 0 < y < \varphi_\varepsilon(1), \\ v(0) = b_0 - u_0(0), \ v'(\varphi_\varepsilon(1)) = 0 \end{array}\right\} \tag{3.2}$$

and

$$\left.\begin{array}{l} w'' - \varepsilon'(\psi_\varepsilon^{-1}(y))w' = \\ = f(\psi_\varepsilon^{-1}(y), u_0(\psi_\varepsilon^{-1}(y)) + v(\chi_\varepsilon(y)) + w) - f(\psi_\varepsilon^{-1}(y), u_0(\psi_\varepsilon^{-1}(y)) + v(\chi_\varepsilon(y))) + \\ + g(\psi_\varepsilon^{-1}(y), u_0(\psi_\varepsilon^{-1}(y)) + v(\chi_\varepsilon(y)) + w, \varepsilon(\psi_\varepsilon^{-1}(y))u_0'(\psi_\varepsilon^{-1}(y)) + v'(\chi_\varepsilon(y)) - w'), \\ 0 < y < \psi_\varepsilon(0), \\ w'(0) = \varepsilon(1)(u_0'(1) - b_1), \ w(\psi_\varepsilon(0)) = 0, \end{array}\right\} \tag{3.3}$$

then u, defined by (3.1), is a solution to (1.1). And vice versae: If u is a solution to (1.1) and v is a solution to (3.2), then w, defined by (3.1), is a solution to (3.3). Here we denoted, for the sake of shortness,

$$\chi_\varepsilon(y) := \varphi_\varepsilon(\psi_\varepsilon^{-1}(y)).$$

Remark 3.1 *Obviously, making the ansatz (3.1), one can write down a lot of boundary value problems for v and w, different from (3.2) and (3.3), with the same property that their solutions generate, via (3.1), solutions to (1.1). Our choice of the concrete form of (3.2) and (3.3) is mainly caused by tactical reasons.*

3.2 Solution of the problem for the left boundary layer function

In this subsection we show, by applying Theorem 2.1, that for all small $\varepsilon \in C^1_+([0,1])$ there exists exactly one solution $v \approx v_0$ to (3.2). For that reason we work in the Sobolev space $W^{2,2}(0,\varphi_\varepsilon(1))$ with its usual norm

$$\|v\|_{W^{2,2}(0,\varphi_\varepsilon(1))} := \sqrt{\int_0^{\varphi_\varepsilon(1)} (v(y)^2 + v'(y)^2 + v''(y)^2)\, dy}.$$

Lemma 3.2 *There exist $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in C^1_+([0,1])$ with $\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty < \varepsilon_0$ there exists exactly one solution $v = v_\varepsilon$ to (3.2) with*

$$\|v - v_0\|_{W^{2,2}(0,\varphi_\varepsilon(1))} < \delta.$$

Moreover, there exists $c > 0$ such that

$$\|v_\varepsilon - v_0\|_{W^{2,2}(0,\varphi_\varepsilon(1))} \le c\,(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty)\,. \tag{3.4}$$

Proof We are going to apply Theorem (2.1).

In a first step we introduce the setting of Theorem 2.1:

We set $E := C^1([0,1])$ with its usual norm $\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty$, $E_0 := C^1_+([0,1])$ and $\Lambda_\varepsilon := \{0\}$ for all $\varepsilon \in E_0$. Therefore, in what follows there are no indices λ. Further, for $\varepsilon \in E_0$ we set

$$U_\varepsilon := W^{2,2}(0,\varphi_\varepsilon(1)),\ \ V_\varepsilon := L^2(0,\varphi_\varepsilon(1)) \times \mathbb{R}^2,$$

and $F_\varepsilon = (A_\varepsilon, B_\varepsilon, C_\varepsilon) \in C^1(U_\varepsilon, V_\varepsilon)$ with $A_\varepsilon \in C^1(U_\varepsilon, L^2(0,\varphi_\varepsilon(0)))$ and $B_\varepsilon, C_\varepsilon \in C^1(U_\varepsilon,\mathbb{R})$ is defined by

$$\begin{aligned}A_\varepsilon(v) &:= v'' + v_0'' + \varepsilon'(\varphi_\varepsilon^{-1}(y))(v' + v_0') + \\ &\quad +2\varepsilon(\varphi_\varepsilon^{-1}(y))\varepsilon'(\varphi_\varepsilon^{-1}(y))u_0'(\varphi_\varepsilon^{-1}(y)) + \varepsilon(\varphi_\varepsilon^{-1}(y))u_0''(\varphi_\varepsilon^{-1}(y)) - f(\varphi_\varepsilon^{-1}(y), u_0(\varphi_\varepsilon^{-1}(y)) + v)\end{aligned}$$

and

$$B_\varepsilon(v) := v(0),\ C_\varepsilon(v) := v'(\varphi_\varepsilon(1)) + v_0'(\varphi_\varepsilon(1)).$$

Obviously, we have $F_\varepsilon(v) = 0$ if and only if $v + v_0$ is a solution to (3.2).

In a second step we verify assumption (2.1) of Theorem 2.1:

Because of assumptions (1.2) and (1.3) for any $y \in [0,\varphi_\varepsilon(1)]$ it holds

$$\begin{aligned}&(A_\varepsilon(0))(y) - \varepsilon'(\varphi_\varepsilon^{-1}(y))v_0' - 2\varepsilon(\varphi_\varepsilon^{-1}(y))\varepsilon'(\varphi_\varepsilon^{-1}(y))u_0'(\varphi_\varepsilon^{-1}(y)) - \varepsilon(\varphi_\varepsilon^{-1}(y))u_0''(\varphi_\varepsilon^{-1}(y)) = \\ &\quad = f(0, u_0(0) + v_0(y)) - f(\varphi_\varepsilon^{-1}(y), u_0(\varphi_\varepsilon^{-1}(y)) + v_0(y)) = \\ &\quad = -\int_0^1\int_0^1 \partial_1\partial_2 f(s\varphi_\varepsilon^{-1}(y), u_0(0) + tv_0(y))\varphi_\varepsilon^{-1}(y)v_0(y)dsdt \\ &\quad -\int_0^1\int_0^1 \partial_2^2 f(\varphi_\varepsilon^{-1}(y), u_0(s\varphi_\varepsilon^{-1}(y)) + tv_0(y))u_0'(s\varphi_\varepsilon^{-1}(y))\varphi_\varepsilon^{-1}(y)v_0(y)dsdt.\end{aligned}$$

Further, from the definition of φ_ε follows

$$\varphi_\varepsilon(x) \ge \frac{x}{\|\varepsilon\|_\infty} \text{ for all } x \in [0,1]. \tag{3.5}$$

Hence

$$\varphi_\varepsilon^{-1}(y) \le y\|\varepsilon\|_\infty \text{ for all } y \in [0, \varphi_\varepsilon(1)]. \tag{3.6}$$

Therefore we get

$$\|F_\varepsilon(0)\|_{V_\varepsilon} \le \text{const } (\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty) \tag{3.7}$$

i.e. (2.1) is satisfied. Here we used that v_0 and v_0' decay exponentially. Remark that, if the (2.2) and (2.3) are also satisfied and, hence, Theorem 2.1 works, then its assertion (2.4) together with (3.7) imply the claimed asymptotic estimate (3.4).

In a third step we verify assumption (2.2) of Theorem 2.1: We have $B_\varepsilon'(v) - B_\varepsilon'(0) = C_\varepsilon'(v) - C_\varepsilon'(0) = 0$ and

$$\begin{aligned}
&\|\left(A_\varepsilon'(v) - A_\varepsilon'(0)\right)\overline{v}\|^2_{L^2(0,\varphi_\varepsilon(1))} = \\
&\quad = \int_0^{\varphi_\varepsilon(1)} \left|\int_0^1 \partial_2^2 f(\varphi_\varepsilon^{-1}(y), u_0(\varphi_\varepsilon^{-1}(y)) + sv(y) + v_0(y))ds\right|^2 |\overline{v}(y)v(y)|^2 dy \le \\
&\quad \le \text{const } \max_{0\le y\le\varphi_\varepsilon(1)} |\overline{v}(y)|^2 \int_0^{\varphi_\varepsilon(1)} v(y)^2 dy \le \text{const } \|\overline{v}\|^2_{U_\varepsilon}\|v\|^2_{U_\varepsilon},
\end{aligned}$$

i.e. (2.2) is satisfied.

In the fourth and last step we verify assumption (2.3) of Theorem 2.1. For that we use Lemma 2.2. It is well-known that linear differential operators of the type

$$v \in W^{2,2}(a,b) \mapsto (v'' + p(y)v' + q(y)v, v(a), v'(b)) \in L^2(a,b) \times \mathbb{R}^2$$

with continuous coefficient functions p and q are Fredholm of index zero. Hence, it remains to verify the second assumption of Lemma 2.2.

Let $\varepsilon_n \in C^1_+([0,1])$ and $v_n \in W^{2,2}(0, \varphi_{\varepsilon_n}(1))$ be sequences with

$$\int_0^{\varphi_{\varepsilon_n}(1)} \left(v_n(y)^2 + v_n'(y)^2 + v_n''(y)^2\right) dy = 1 \tag{3.8}$$

and

$$\begin{aligned}
&\|\varepsilon_n\|^2_\infty + \|\varepsilon_n'\|^2_\infty + |v_n(0)|^2 + |v_n'(\varphi_{\varepsilon_n}(1))|^2 + \\
&\quad + \int_0^{\varphi_{\varepsilon_n}(1)} \left(v_n'' + \varepsilon'(\varphi_{\varepsilon_n}^{-1}(y))v_n' - \partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))v_n\right)^2 dy \to 0.
\end{aligned} \tag{3.9}$$

Any of the functions v_n can be extended onto $[0,\infty)$ to a function $\tilde{v}_n \in W^{2,2}(0,\infty)$ in such a way that $\|\tilde{v}_n\|_{W^{2,2}(0,\infty)} \le$ const. In particular, $\tilde{v}_n$ is a bounded sequence in the Hilbert space $W^{2,2}(0,\infty)$. Hence, without loss of generality we can assume that there exists $v_* \in W^{2,2}(0,\infty)$ such that

$$\tilde{v}_n \rightharpoonup v_* \text{ in } W^{2,2}(0,\infty) \text{ for } n \to \infty. \tag{3.10}$$

Moreover, because of the continuous embedding $W^{2,2}(0,\infty) \hookrightarrow W^{1,\infty}(0,\infty)$ it follows that $\tilde{v}_n$ and $\tilde{v}_n'$ are a bounded sequences also in $L^\infty(0,\infty)$, this will be used in the following.

We are going to show that $v_* = 0$. For that reason we derive a variational equation for v_* as follows: Take a smooth test function $\eta : (0,\infty) \to \mathbb{R}$ with compact support. Then we have

$$\int_0^\infty (\tilde{v}_n'(y)\eta'(y) + \partial_2 f(0, u_0(0) + v_0(y))\tilde{v}_n(y)\eta(y))\, dy = v_n'(\varphi_{\varepsilon_n}(1))\eta(\varphi_{\varepsilon_n}(1))$$
$$+ \int_0^{\varphi_{\varepsilon_n}(1)} \left(-v_n''(y) + \partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))v_n(y)\right)\eta(y)dy$$
$$- \int_0^{\varphi_{\varepsilon_n}(1)} \int_0^1 \partial_1\partial_2 f(s\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))\varphi_{\varepsilon_n}^{-1}(y)v_n(y)\eta(y)\, ds\, dy$$
$$- \int_0^{\varphi_{\varepsilon_n}(1)} \int_0^1 \partial_2^2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(s\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))u_0'(s\varphi_{\varepsilon_n}^{-1}(y))\varphi_{\varepsilon_n}^{-1}(y)v_n(y)\eta(y)\, ds\, dy$$
$$+ \int_{\varphi_{\varepsilon_n}(1)}^\infty (\tilde{v}_n'(y)\eta'(y) + \partial_2 f(0, u_0(0) + v_0(y))\tilde{v}_n(y)\eta(y))\, dy. \tag{3.11}$$

The first two terms in the right hand side of (3.11) tend to zero for $n \to \infty$ because of (3.9). The absolute value of third term in the right hand side of (3.11) can be estimated by

$$\int_0^R \int_0^1 \left|\partial_1\partial_2 f(s\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))\varphi_{\varepsilon_n}^{-1}(y)v_n(y)\eta(y)\right| dsdy+$$
$$+ \int_R^{\varphi_{\varepsilon_n}(1)} \int_0^1 \left|\partial_1\partial_2 f(s\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))\varphi_{\varepsilon_n}^{-1}(y)v_n(y)\eta(y)\right| dsdy$$
$$\leq \text{ const } \left(R\varphi_{\varepsilon_n}^{-1}(R) + \int_R^\infty \eta(y)^2 dy\right),$$

where $R \in (0, \varphi_{\varepsilon_n}(1))$ is arbitrary. Remark that (3.5) and (3.6) yield $\varphi_{\varepsilon_n}(1) \to \infty$ for $n \to \infty$ and $\varphi_{\varepsilon_n}^{-1}(R) \to 0$ for $n \to \infty$. Taking first R sufficiently large such that $\int_R^\infty \eta(y)^2 dy$ is small, and then, fixing such R, take n sufficiently large such that $R\varphi_{\varepsilon_n}^{-1}(R)$ is small, we see that the third term in the right hand side of (3.11) tends to zero for $n \to \infty$.

Similarly one shows that the fourth term in the right hand side of (3.11) tends to zero for $n \to \infty$.

Finally the last term in the right hand side of (3.11): Its absolute value can be estimated by a constant times $\int_{\varphi_{\varepsilon_n}(1)}^\infty (\eta(y)^2 + \eta'(y)^2)\, dy$ and, hence, tends to zero for $n \to \infty$.

Using (3.10) and taking the limit $n \to \infty$ in (3.11), we get

$$\int_0^\infty (v_*'(y)\eta'(y) + \partial_2 f(0, u_0(0) + v_0(y))v_*(y)\eta(y))\, dy = 0 \text{ for all } \eta \in C_c^\infty(0,\infty).$$

Therefore v_* is C^2–smooth and satisfies

$$v_*''(y) = \partial_2 f(0, u_0(0) + v_0(y))v_*(y) \text{ for all } y > 0.$$

The function v_0' together with an exponentially growing function constitutes a fundamental system for this linear homogeneous ODE, hence $v_* =$ const v_0'. Moreover, (3.9) and (3.10) and the compact embedding $W^{1,2}(0,1) \hookrightarrow C([0,1])$ yield $v_*(0) = 0$, hence $v_* = 0$.

Now we are going to show that

$$\int_0^{\varphi_{\varepsilon_n}(1)} \left(v_n(y)^2 + v_n'(y)^2 + v_n''(y)^2\right) dy \to 0 \text{ for } n \to \infty, \tag{3.12}$$

which is the needed contradiction to (3.8):

Because of assumption (1.2) there exists a constant $c > 0$ such that

$$\begin{aligned}
c\int_0^{\varphi_{\varepsilon_n}(1)} & \left(v_n(y)^2 + v_n'(y)^2\right) dy \\
&\le \int_0^{\varphi_{\varepsilon_n}(1)} \left(v_n'(y)^2 + \partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)))v_n(y)^2\right) dy \\
&= \int_0^{\varphi_{\varepsilon_n}(1)} \left(-v_n''(y) + \partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + v_0(y))v_n(y)\right) v_n(y)dy \\
&+v_n'(\varphi_{\varepsilon_n}(1))v_n(\varphi_{\varepsilon_n}(1)) - v_n'(0)v_n(0) \\
&- \int_0^{\varphi_{\varepsilon_n}(1)} \int_0^1 \partial_2^2 f(\varphi_{\varepsilon_n}^{-1}(y), u_0(\varphi_{\varepsilon_n}^{-1}(y)) + sv_0(y))v_0(y)v_n(y)^2 ds dy.
\end{aligned} \tag{3.13}$$

The first three terms in the right hand side of (3.13) tend to zero for $n \to \infty$ because of (3.9) and $|v_n(y)| \le$ const. The absolute value of the last term in the right hand side of (3.13) can be estimated by a constant times

$$\int_0^R \tilde{v}_n^2(y)dy + \int_R^\infty |v_0(y)|dy,$$

where $R > 0$ is arbitrary. Now we proceed as above: First take R sufficiently large such that the second term is small. Then fix this R, use the compact embedding $W^{1,2}(0,R) \hookrightarrow C([0,R])$ and take n sufficiently large, such that the first term is small.

For (3.12) it remains to show that $\int_0^{\varphi_{\varepsilon_n}(1)} v_n''(y)^2 dy \to 0$ for $n \to \infty$. But this follows from

$$\|v_n''\|_{L^2(0,\varphi_{\varepsilon_n}(1))} \le \|v_n'' - \partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), v_0)v_n\|_{L^2(0,\varphi_{\varepsilon_n}(1))} + \|\partial_2 f(\varphi_{\varepsilon_n}^{-1}(y), v_0)v_n\|_{L^2(0,\varphi_{\varepsilon_n}(1))}.$$

The first term in the right hand side tends to zero because of (3.9), and the second one because of $\|v_n\|_{L^2(0,\varphi_{\varepsilon_n}(1))} \to 0$ (which was shown above). ■

3.3 Solution of the problem for the right boundary layer function

Let v_ε be the solution to (3.2) for small $\varepsilon \in C_+^1([0,1])$, produced by Lemma 3.2. Inserting $v = v_\varepsilon$ in (3.3) we get

$$\left.\begin{array}{l}
w''(y) - \varepsilon'(\psi_\varepsilon^{-1}(y))w'(y) = f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y)) - f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y)) + \\
\quad +g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y), v_\varepsilon^1(y) - w'(y)),\ 0 < y < \psi_\varepsilon(0), \\
w'(0) = \varepsilon(1)(u_0'(0) - b_1),\ w(\psi_\varepsilon(0)) = 0,
\end{array}\right\} \tag{3.14}$$

where, for the sake of shortness, we denoted

$$v_\varepsilon^0(y) := u_0(\psi_\varepsilon^{-1}(y)) + v_\varepsilon(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y))), \quad v_\varepsilon^1(y) := \varepsilon(\psi_\varepsilon^{-1}(y))u_0'(\psi_\varepsilon^{-1}(y)) + v_\varepsilon'(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y))).$$

A function w is a solution to (3.14) if and only if

$$u(x) = u_0(x) + v_\varepsilon(\varphi_\varepsilon(x)) + w(\psi_\varepsilon(x)) \tag{3.15}$$

is a solution to (1.1). Moreover, using (1.5) and (3.15), we get $u - \mathcal{U}_\varepsilon = (v_\varepsilon - v_0)\circ\varphi_\varepsilon + (w - w_0)\circ\psi_\varepsilon$. Hence, with the notation (1.7) this gives

$$\|(w-w_0)\circ\psi_\varepsilon\|_\varepsilon - \|(v_\varepsilon - v_0)\circ\varphi_\varepsilon\|_\varepsilon \le \|u - \mathcal{U}_\varepsilon\|_\varepsilon \le \|(v_\varepsilon - v_0)\circ\varphi_\varepsilon\|_\varepsilon + \|(w-w_0)\circ\psi_\varepsilon\|_\varepsilon.$$

On the other side, by means of (3.1) one easily calculates that

$$\|(v_\varepsilon - v_0)\circ\varphi_\varepsilon\|_\varepsilon = \|v_\varepsilon - v_0\|_{W^{2,2}(0,\varphi_\varepsilon(1))}, \quad \|(w-w_0)\circ\psi_\varepsilon\|_\varepsilon = \|w - w_0\|_{W^{2,2}(0,\psi_\varepsilon(0))}.$$

Hence, from (3.4) follows

$$\|u - \mathcal{U}_\varepsilon\|_\varepsilon \le c(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty) + \|w - w_0\|_{W^{2,2}(0,\psi_\varepsilon(0))} \le 2c(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty) + \|u - \mathcal{U}_\varepsilon\|_\varepsilon.$$

Finally, the continuous embedding $W^{1,2}(0,\infty) \hookrightarrow L^\infty(0,\infty)$ yields that there exists a positive constant such that for all $\varepsilon \in C_+^1([0,1])$ with $\|\varepsilon\|_\infty \le 1$ and all $u \in C^2([0,1])$ it holds

$$\begin{aligned} \|(u-\mathcal{U}_\varepsilon)\|_\infty + \|\varepsilon(u-\mathcal{U}_\varepsilon)'\|_\infty &= \|(u-\mathcal{U}_\varepsilon)\circ\varphi_\varepsilon^{-1}\|_{C^1([0,\varphi_\varepsilon(1)])} \le \\ &\le \text{const } \|(u-\mathcal{U}_\varepsilon)\circ\varphi_\varepsilon^{-1}\|_{W^{2,2}(0,\varphi_\varepsilon(1))} = \text{const } \|u-\mathcal{U}_\varepsilon\|_\varepsilon. \end{aligned} \tag{3.16}$$

Therefore, Theorem 1.1 is proved if the following Lemma is proved:

Lemma 3.3 *There exist $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in C_+^1([0,1])$ and all $g \in C^{0,1,1}([0,1]\times K^2)$ with*

$$\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty + \sqrt{\int_0^1 \frac{dx}{\varepsilon(x)}}\,(\|g\|_\infty + \|\partial_2 g\|_\infty + \|\partial_3 g\|_\infty) < \varepsilon_0$$

there exists exactly one solution $w = w_{\varepsilon,g}$ to (3.14) with

$$\|w - w_0\|_{W^{2,2}(0,\psi_\varepsilon(0))} < \delta.$$

Moreover, there exists $c > 0$ such that

$$\|w_{\varepsilon,g} - w_0\|_{W^{2,2}(0,\psi_\varepsilon(0))} \le c\left(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty + \sqrt{\int_0^1 \frac{dx}{\varepsilon(x)}}\,\|g\|_\infty\right). \tag{3.17}$$

Proof We proceed as in the proof of Lemma 3.2, i.e. we apply Theorem 2.1 again.

In a first step we introduce the setting of Theorem 2.1. We set $E := C^1([0,1])$ with its usual norm $\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty$ and $E_0 := C^1_+([0,1])$ as in Lemma 3.2. Further, for $\varepsilon \in C^1_+([0,1])$ we set (cf. (1.9))

$$\Lambda_\varepsilon := C^{0,1,1}([0,1]\times K^2) \text{ with the norm } \sqrt{\int_0^1 \frac{dx}{\varepsilon(x)}}\,(\|g\|_\infty + \|\partial_2 g\|_\infty + \|\partial_3 g\|_\infty)$$

and

$$U_\varepsilon := W^{2,2}(0,\psi_\varepsilon(0)),\;\; V_\varepsilon := L^2(0,\psi_\varepsilon(0))\times\mathbb{R}^2 \text{ with their usual norms.}$$

Finally, for $\varepsilon \in C^1_+([0,1])$ and $g \in C^1([0,1]\times K^2)$ we define $F_{\varepsilon,g} = (A_{\varepsilon,g}, B_{\varepsilon,g}, C_{\varepsilon,g}) \in C^1(U_\varepsilon, V_\varepsilon)$ with $A \in C^1(U_\varepsilon, L^2(0,\psi_\varepsilon(0)))$ and $B, C \in C^1(U_\varepsilon,\mathbb{R})$ by

$$\begin{aligned}(A_{\varepsilon,g}(w))\,(y) := {}& w'' + w_0'' - \varepsilon'(\psi_\varepsilon^{-1}(y))(w'(y) + w_0'(y))\\ &+ f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y)) - f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y) + w_0(y))\\ &- g(\psi_\varepsilon^{-1}(y), \kappa(v_\varepsilon^0(y) + w(y) + w_0(y)), \kappa(v_\varepsilon^1(y) - w'(y) - w_0'(y)))\end{aligned}$$

and

$$B_{\varepsilon,g}(w) := w'(0) - \varepsilon(1)(u_0'(0) - b_1),\;\; C_{\varepsilon,g}(w) := w(\psi_\varepsilon(0)).$$

Here $\kappa : \mathbb{R} \to [0,1]$ is a C^∞ cut off function with $\kappa(z) = 1$ for $|z| \le 4k$ and $\kappa(z) = 0$ for $|z| \ge 5k$. Obviously, for sufficiently small $\|\varepsilon\|_\infty$ and $\|w - w_0\|_{W^{2,2}(0,\psi_\varepsilon(0))}$ we have $\kappa(v_\varepsilon^0(y) + w(y) + w_0(y)) = v_\varepsilon^0(y) + w(y) + w_0(y)$ and $\kappa(v_\varepsilon^1(y) - w'(y) - w_0'(y)) = v_\varepsilon^1(y) - w'(y) - w_0'(y)$ for all $y \ge 0$ and, hence, $F_{\varepsilon,g}(w) = 0$ if and only if $w + w_0$ is a solution to (3.14). Moreover, for such ε and w it holds

$$\begin{aligned}(A'_{\varepsilon,g}(w)\bar w)(y) = {}& \bar w'' - \varepsilon'(\psi_\varepsilon^{-1}(y))w'(y) - \partial_2 f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y) + w_0(y))\bar w(y)\\ &- \partial_2 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y) + w_0(y), v_\varepsilon^1(y) - w'(y) - w_0'(y))\bar w(y)\\ &- \partial_3 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w(y) + w_0(y), v_\varepsilon^1(y) - w'(y) - w_0'(y))\bar w'(y)\end{aligned}$$

and $B'_{\varepsilon,g}(w)\bar w = \bar w'(0)$, $C'_{\varepsilon,g}(w)\bar w = \bar w(\psi_\varepsilon(0))$.

In a second step we verify assumption (2.1) of Theorem 2.1: We have

$$\begin{aligned}&(A_{\varepsilon,g}(0))(y) + \varepsilon'(\psi_\varepsilon^{-1}(y))w_0'(y))\\ &\quad + g(\psi_\varepsilon^{-1}(y), \kappa(v_\varepsilon^0(y) + w_0(y)), \kappa(v_\varepsilon^1(y) - w'(y) - w_0'(y)))\\ &\quad = f(1, u_0(1) + w_0(y)) + f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0) - f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y))\\ &\quad = w_0(y)\int_0^1 \big(\partial_2 f(1, u_0(1) + tw_0(y)) - \partial_2 f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + tw_0(y))\big)\,dt\\ &\quad = w_0(y)\int_0^1\int_0^1 (1 - \psi_\varepsilon^{-1}(y))\Big(\partial_1\partial_2 f(s + (1-s)\psi_\varepsilon^{-1}(y), u_0(1) + tw_0(y))\\ &\quad + \partial_2^2 f(\psi_\varepsilon^{-1}(y), u_0(s + (1-s)\psi_\varepsilon^{-1}(y)) + tw_0(y))u_0'(s + (1-s)\psi_\varepsilon^{-1}(y))\Big)\\ &\quad + v_\varepsilon(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y)))\partial_2^2 f(\psi_\varepsilon^{-1}(y), u_0(\psi_\varepsilon^{-1}(y)) + sv_\varepsilon(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y))) + tw_0(y))dsdt.\end{aligned}$$

Therefore

$$\|A_{\varepsilon,g}(0)\|^2_{L^2(0,\psi_\varepsilon(0))} \leq$$
$$\leq \text{const} \int_0^{\psi_\varepsilon(0)} \left(\|g\|^2_\infty + w_0(y)^2 \left(\|\varepsilon'\|_\infty + (1-\psi_\varepsilon^{-1}(y))^2 + v_\varepsilon(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y)))^2\right)\right) dy. \tag{3.18}$$

Moreover, it holds

$$\psi_\varepsilon(x) \geq \frac{1-x}{\|\varepsilon\|_\infty} \text{ and } 1-\psi_\varepsilon^{-1}(y) \leq y\|\varepsilon\|_\infty \tag{3.19}$$

and, hence,

$$\int_0^{\psi_\varepsilon(0)} w_0(y)^2(1-\psi_\varepsilon^{-1}(y))^2 dy \leq \text{const}\|\varepsilon\|^2_\infty.$$

Finally, we have

$$\int_0^{\psi_\varepsilon(0)} v_\varepsilon(\varphi_\varepsilon(\psi_\varepsilon^{-1}(y)))^2 w_0(y)^2 dy = \int_0^1 v_\varepsilon(\varphi_\varepsilon(x))^2 w_0(\psi_\varepsilon(x))^2 \frac{dx}{\varepsilon(x)} \leq$$
$$\leq \text{const} \left(\int_0^{1/2} w_0(\psi_\varepsilon(x))^2 \frac{dx}{\varepsilon(x)} + \int_{1/2}^1 v_\varepsilon(\varphi_\varepsilon(x))^2 \frac{dx}{\varepsilon(x)}\right) \leq$$
$$\leq \text{const} \left(\int_0^{1/2} w_0(\psi_\varepsilon(x))^2 \frac{dx}{\varepsilon(x)} + \int_{1/2}^1 v_0(\varphi_\varepsilon(x))^2 \frac{dx}{\varepsilon(x)} + \|\varepsilon\|_\infty + \|\varepsilon'\|_\infty\right) =$$
$$= \text{const} \left(\int_0^{\psi_\varepsilon(1/2)} w_0(y)^2 dy + \int_{\varphi_\varepsilon(1/2)}^{\varphi_\varepsilon(1)} v_0(y)^2 dy + \|\varepsilon\|_\infty + \|\varepsilon'\|_\infty\right) \leq$$
$$\leq \text{const} \left(\int_{\psi_\varepsilon(0)}^{\psi_\varepsilon(1/2)} e^{-2\alpha y} dy + \int_{\varphi_\varepsilon(1/2)}^{\varphi_\varepsilon(1)} e^{-2\alpha y} dy + \|\varepsilon\|_\infty + \|\varepsilon'\|_\infty\right) \leq$$
$$\leq \text{const} \left(\|\varepsilon\|_\infty + \|\varepsilon'\|_\infty\right).$$

Here we used (3.4), (3.5) and (3.19). Inserting this into (3.18) we get

$$\|A_{\varepsilon,g}(0)\|^2_{L^2(0,\psi_\varepsilon(0))} + |B_{\varepsilon,g}(0)|^2 + |C_{\varepsilon,g}(0)|^2 \leq \text{ const } \left(\|\varepsilon\|^2_\infty + \|\varepsilon'\|^2_\infty + \psi_\varepsilon(0)\|g\|^2_\infty\right),$$

i.e. (2.1) is satisfied. Remark that, if the (2.2) and (2.3) are also satisfied and, hence, Theorem 2.1 works, then its assertion (2.4) implies the claimed asymptotic estimate (3.17).

In a third step we verify assumption (2.2) of Theorem 2.1: We have

$$B'_{\varepsilon,g}(w) - B'_{\varepsilon,g}(0) = C'_{\varepsilon,g}(w) - C'_{\varepsilon,g}(0) = 0$$

and

$$\|\left(A'_{\varepsilon,g}(w) - A'_{\varepsilon,g}(0)\right)\overline{w}\|^2_{L^2(0,\psi_\varepsilon(0))} =$$
$$= \int_0^{\psi_\varepsilon(0)} \Big(\overline{w}(y)\Big(\int_0^1 \partial_2^2 f(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y) + sw(y))ds\, w(y)$$
$$-\partial_2 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y) + w(y), v_\varepsilon^1(y) - w_0'(y) - w'(y))$$
$$+\partial_2 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y), v_\varepsilon^1(y) - w_0'(y))\Big)$$
$$-\overline{w}'(y)\left(-\partial_3 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y) + w(y), v_\varepsilon^1(y) - w_0'(y) - w'(y))\right.$$
$$\left.+\partial_3 g(\psi_\varepsilon^{-1}(y), v_\varepsilon^0(y) + w_0(y), v_\varepsilon^1(y) - w_0'(y))\right)\Big)^2 dy.$$

Therefore

$$\| \left(F'_{\varepsilon,g}(w) - F'_{\varepsilon,g}(0)\right) \overline{w}\|^2_{V_\varepsilon} \le$$
$$\le \text{const} \left(\|w\|^2_{U_\varepsilon} + \int_0^1 \frac{dx}{\varepsilon(x)} \left(\|\partial_2 g\|^2_\infty + \|\partial_3 g\|^2_\infty\right) \right) \|\overline{w}\|^2_{U_\varepsilon},$$

i.e. (2.2) is satisfied.

In the fourth and last step we verify assumption (2.3) of Theorem 2.1. For that we use Lemma 2.2 again.

Let $\varepsilon_n \in C^1_+([0,1])$, $g_n \in C^{0,1,1}([0,1] \times K^2)$ and $w_n \in W^{2,2}(0, \psi_{\varepsilon_n}(0))$ be sequences with

$$\int_0^{\psi_{\varepsilon_n}(0)} \left(w_n(y)^2 + w_n'(y)^2 + w_n''(y)^2\right) dy = 1 \tag{3.20}$$

and

$$\begin{aligned} &\|\varepsilon_n\|^2_\infty + \|\varepsilon_n'\|^2_\infty + \int_0^1 \frac{dx}{\varepsilon(x)} \left(\|\partial_2 g_n\|^2_\infty + \|\partial_3 g_n\|^2_\infty\right) + |w_n'(0)|^2 + |w_n(\varphi_{\varepsilon_n}(1)|^2 \\ &+ \int_0^{\psi_{\varepsilon_n}(0)} (w_n'' - \varepsilon'(\psi^{-1}_{\varepsilon_n}(y))w_n'(y) - \partial_2 f(\psi^{-1}_{\varepsilon_n}(y), v^0_\varepsilon(y) + w_0(y))v_n(y) \\ &-\partial_2 g_n(\psi^{-1}_\varepsilon(y), v^0_\varepsilon(y) + w_0(y), v^1_\varepsilon(y) - w_0'(y)) \\ &+\partial_3 g_n(\psi^{-1}_\varepsilon(y), v^0_\varepsilon(y) + w_0(y), v^1_\varepsilon(y) - w_0'(y)))^2 dy \to 0. \end{aligned} \tag{3.21}$$

As in the proof of Lemma 3.2 we can assume that w_n is the restriction on $[0, \psi_\varepsilon(0)]$ of a function $\tilde{w} \in W^{2,2}(0, \psi_\varepsilon(0))$, that $\tilde{w}_n$ and $\tilde{w}_n'$ are bounded sequences also in $L^\infty(0,\infty)$ and that there exists $w_* \in W^{2,2}(0,\infty)$ such that

$$\tilde{w}_n \rightharpoonup w_* \text{ in } W^{2,2}(0,\infty) \text{ for } n \to \infty. \tag{3.22}$$

Moreover, as in the proof of 3.2 one can show that

$$\int_0^\infty (w_*'(y)\eta'(y) + \partial_2 f(0, u_0(1) + w_0(y))w_*(y)\eta(y))\, dy = 0 \text{ for all } \eta \in C_c^\infty(0,\infty),$$

i.e. $w_* =$ const w_0'. Therefore, assumption $w_0'(0) = 0$ from (1.4) yields $w_*(0) = 0$. Moreover, (3.21), (3.10) and the compact embedding $W^{2,2}(0,1) \hookrightarrow C^1([0,1])$ imply $w_*'(0) = 0$, i.e. $w_* = 0$.

Using this, as in the proof of Lemma 3.2 we can construct a contradiction to the assumption (3.20). ■

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