Error Control in h- and hp-adaptive FEM for Signorini's Problem

Andreas Schröder

Abstract This paper presents a posteriori finite element error estimates for Signorini's problem. The discretization is based on a mixed variational formulation proposed by Haslinger et al. which is extended to higher-order finite elements. The a posteriori error control relies on estimating the discretization error of an auxiliary problem which is given as a variational equation. The estimation consists of error bounds for the discretization error of the auxiliary problem and some further terms which capture the geometrical error and the error in the complementary condition. The derived estimates are applied to h- and hp-adaptive refinement and enrichment strategies. Numerical results confirm the applicability of the theoretical findings. In particular, optimal algebraic and almost exponential convergence rates are obtained.

Keywords hp-FEM · contact problems · error control

1 Introduction

The aim of this paper is to derive error estimates for mixed higher-order finite element discretization schemes for Signorini's problem which plays an import role in mechanical engineering, [11, 12, 18]. A simplified version is considered as a model problem. The discretization is based on a mixed finite element approach introduced by Haslinger et al. in [14, 15, 17]. Although, this approach is originally developed for lower-order finite elements, it can be extended to higher-order finite elements, [25]. The approach relies on a saddle point formulation where the geometrical contact condition given by an obstacle function is captured by a Lagrange multiplier. The restriction for the Lagrange multiplier is just a sign condition and, therefore, more

A. Schröder
Unter den Linden 6
10099 Berlin, Germany
Tel.: +49-2093-2630
Fax: +49-2093-5859
E-mail: andreas.schroeder@mathematik.hu-berlin.de

simple than the original contact condition. However, the multiplier is an additional variable which also has to be discretized. In order to obtain a stable scheme, one has to ensure the discretization spaces to be well balanced with respect to their inf-sup condition. In the used approach, the discretization of the Lagrange multiplier is defined on boundary meshes having a larger mesh size than the mesh size of the primal variable.
In literature, higher-order discretization schemes for contact problems are rarely studied. We refer to [8] for a mixed finite element scheme which avoids different meshes and to [19, 20] for discretization techniques based on a primal, non-mixed formulation.
Modern discretization schemes usually include a posteriori error control and adaptivity. In fact, whenever higher-order finite elements are applied to contact problems, the use of adaptive schemes is inevitable due to the in general limited regularity of the solution. The main idea of the proposed a posteriori error control can be outlined as follows: The mixed formulation consists of a variational equation and a variational inequality. Replacing the Lagrange multiplier by its approximation in the variational equation, we obtain an auxiliary problem whose discretization coincides with the discretization of the mixed formulation. We will show, that the discretization error can be estimated by error bounds related to this auxiliary problem and some additional terms capturing the geometrical error and the error in the complementary condition. The idea to derive error estimations this way was originally proposed by Braess [7] for the obstacle problem. We will extend this idea to Signorini's problem and, in particular, to the discretization schemes given by the mixed variational formulation.
A posteriori error estimates which are based on the primal, non-mixed formulation for lower-order finite elements are proposed in [5, 29] for the obstacle problem and in [16] for Signroni's problem. Estimates for mixed formulations are introduced in [31] for the mortar approach.
In this paper only norm-based estimates are considered. For goal-oriented error estimates, which are necessary in many applications where the quantity of interest is given by a user-defined functional, we refer to [6, 19] for the non-mixed approach. Results for mixed formulations of Signorini's problem are discussed in [26, 28].

This paper is organized as follows: In Section 3, the mixed variational formulation of Signorini's problem is introduced. The higher-order finite element discretization based on the mixed formulation is presented in Section 4. The main part of this work, the derivation of reliable a posteriori estimates, is proposed in Section 5. In Section 6, these estimates are applied to h-adaptive as well as hp-adaptive refinement and enrichment strategies. Numerical results are presented in Section 7, confirming the reliability of the estimates.

2 Notation

Let $\Omega \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a domain with sufficiently smooth boundary $\Gamma := \partial\Omega$. Moreover, let $\Gamma_D \subset \Gamma$ be closed with positive measure and let $\Gamma_C \subset \Gamma \backslash \Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma \backslash \Gamma_D$.

$L^2(\Omega)$, $H^k(\Omega)$ with $k \geq 1$, and $H^{1/2}(\Gamma_C)$ denote the usual Sobolev spaces and

$$H^1(\Omega,\Gamma_D) := \{v \in H^1(\Omega) \mid \gamma(v) = 0 \text{ on } \Gamma_D\}$$

with the trace operator γ. The space $H^{-1/2}(\Gamma_C)$ denotes the topological dual space of $H^{1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$, respectively. Let $(\cdot,\cdot)_{0,\omega}$, $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$, respectively. We define $\|v\|_{0,\omega}^2 := (v,v)_{0,\omega}$ and omit the subscript ω whenever $\omega = \Omega$. Moreover, we state

$$|v|_1^2 := (\nabla v, \nabla v)_0, \qquad \|v\|_1 := \|v\|_0 + |v|_1$$

as the usual equivalent H^1-norms on $H^1(\Omega,\Gamma_D)$ with the gradient operator ∇ in the weak sense. We denote the usual Laplace operator likewise in the weak sense by Δ. Note, the linear and bounded mapping

$$\gamma_C := \gamma_{|\Gamma_C} : H^1(\Omega,\Gamma_D) \to H^{1/2}(\Gamma_C)$$

is surjective due to the assumptions on Γ_C, [18]. As these assumptions are fulfilled in most cases, we can avoid the introduction of complicated $H_{00}^{1/2}(\Gamma_C)$-spaces.
For functions in $L^2(\Omega)$ or $L^2(\Gamma_C)$, the inequality symbols $\geq$ and $\leq$ are defined by means of "almost everywhere". Finally, we define the positive part v_+ of $v \in H^{1/2}(\Gamma_C)$,

$$v_+(x) := \begin{cases} v(x), & \text{if } v(x) \geq 0, \\ 0, & \text{else.} \end{cases}$$

3 Mixed Variational Formulation of Signorini's Problem

Signorini's problem is to find a function $u \in H^1(\Omega,\Gamma_D) \cap H^2(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &\geq g, \quad \partial_n u \geq 0, \quad \partial_n u\,(u-g) = 0 \quad \text{on } \Gamma_C, \end{aligned} \tag{1}$$

where $f \in L^2(\Omega)$. The function $g \in H^{1/2}(\Gamma_C)$ represents an obstacle on the boundary Γ_C. It is well-known, that $u \in H^1(\Omega,\Gamma_D) \cap H^2(\Omega)$ is a solution of Signorini's problem if and only if $u \in K := \{v \in H^1(\Omega,\Gamma_D) \mid \gamma(v) \geq g \text{ on } \Gamma_C\}$ and

$$\forall v \in K: \ (\nabla u, \nabla(v-u))_0 \geq (f,(v-u))_0. \tag{2}$$

Moreover, $u \in H^1(\Omega,\Gamma_D)$ fulfills (2) if and only if u is a minimizer of the functional

$$E(v) := \frac{1}{2}(\nabla v, \nabla v)_0 - (f,v)_0$$

in K. The functional E is strictly convex, continuous and coercive due to Cauchy's and Poincare's inequalities. This implies the existence of a unique minimizer u.

In order to derive a mixed formulation, let

$$\begin{aligned} H_-^{1/2}(\Gamma_C) &:= \{w \in H^{1/2}(\Gamma_C) \mid w \leq 0\}, \\ H_-^{-1/2}(\Gamma_C) &:= \{\mu \in H^{-1/2}(\Gamma_C) \mid \forall w \in H_-^{1/2}(\Gamma_C): \ \langle \mu, w \rangle \geq 0\}. \end{aligned}$$

Using the Hahn-Banach theorem it can be proven that

$$\sup_{\mu \in H_-^{-1/2}(\Gamma_C)} \langle \mu, \gamma_C(v) - g \rangle = \begin{cases} 0, & \text{if } v \in K \\ \infty, & \text{else.} \end{cases}$$

Therefore, we obtain

$$E(u) = \inf_{v \in H^1(\Omega,\Gamma_D)} \sup_{\mu \in H_-^{-1/2}(\Gamma_C)} \mathscr{L}(v,\mu) \tag{3}$$

with the Lagrange functional

$$\mathscr{L}(v,\mu) := E(v) + \langle \mu, \gamma_C(v) - g \rangle$$

on $H^1(\Omega,\Gamma_D) \times H_-^{-1/2}(\Gamma_C)$. This states that, whenever $(u,\lambda) \in H^1(\Omega,\Gamma_D) \times H_-^{1/2}(\Gamma_C)$ is a saddle point of $\mathscr{L}$, that u is a minimizer of E. The existence of a unique saddle point is guaranteed, if there exists a constant $\alpha > 0$ such that

$$\alpha \|\mu\|_{-1/2,\Gamma_C} \leq \sup_{\substack{v \in H^1(\Omega,\Gamma_D) \\ \|v\|_1 = 1}} \langle \mu, \gamma_C(v) \rangle \tag{4}$$

is fulfilled for all $\mu \in H^{-1/2}(\Gamma_C)$, [18]. In fact, it follows from the closed range theorem and the surjectivity of γ_C, that (4) is valid.

Let $\mathscr{L}'_\lambda : H^1(\Omega,\Gamma_D) \to (H^1(\Omega,\Gamma_D))^*$ and $\mathscr{L}'_u : H^{-1/2}(\Gamma_C) \to (H^{-1/2}(\Gamma_C))^* \simeq H^{1/2}(\Gamma_C)$ be the Fréchet derivatives of $\mathscr{L}_\lambda := \mathscr{L}(\cdot,\lambda)$ and $\mathscr{L}_u := \mathscr{L}(u,\cdot)$, respectively. Then, $(u,\lambda) \in H^1(\Omega,\Gamma_D) \times H_-^{-1/2}(\Gamma_C)$ is a saddle point of (3), if and only if the stationary condition

$$\begin{aligned} &\mathscr{L}'_\lambda(u) = 0, \\ &\forall \mu \in H_-^{-1/2}(\Gamma_C) : \ \langle \mu - \lambda, \mathscr{L}'_u(\lambda) \rangle \leq 0 \end{aligned}$$

is fulfilled. Thus, (u,λ) is equivalently characterized by the mixed variational formulation

$$\begin{aligned} &\forall v \in H^1(\Omega,\Gamma_D) : (\nabla u, \nabla v)_0 = (f,v)_0 - \langle \lambda, \gamma_C(v) \rangle, \\ &\forall \mu \in H_-^{-1/2}(\Gamma_C) : \langle \mu - \lambda, \gamma_C(u) - g \rangle \leq 0. \end{aligned} \tag{5}$$

4 Higher-Order Discretization of the Mixed Variational Formulation

We propose a higher-order finite element discretization based on quadrangles or hexahedrons in the following way: Let $\mathscr{T}_h$ and $\mathscr{T}_{C,H}$ be finite element meshes of Ω and Γ_C with mesh sizes h and H, respectively. Let $\Psi_T : [-1,1]^k \to T \in \mathscr{T}_h$, $\Psi_{C,T} : [-1,1]^{k-1} \to T \in \mathscr{T}_{C,H}$ be bijective and sufficiently smooth transformations, and let $p_T, p_{C,T} \in \mathbb{N}$ be degree distributions on $\mathscr{T}_h$ and $\mathscr{T}_{C,H}$. Using the polynomial (Serendipity) tensor product space S_k^q of order q on a reference element $[-1,1]^k$, we set

$$S^p(\mathscr{T}_h) := \left\{ v \in H^1(\Omega,\Gamma_D) \mid \forall T \in \mathscr{T}_h : \ v_{|T} \circ \Psi_T \in S_k^{p_T} \right\}$$

and

$$M^{pc}(\mathscr{T}_{C,H}) := \left\{ \mu \in L^2(\Gamma_C) \mid \forall T \in \mathscr{T}_{C,H} : \ \mu_{|T} \circ \Psi_{C,T} \in S_{k-1}^{p_{C,T}} \right\}.$$

With $M_-^{pc}(\mathscr{T}_{C,H}) := \{\mu_H \in M^{pc}(\mathscr{T}_{C,H}) \mid \mu_H \leq 0\}$ the discrete problem reads: Find $(u_h, \lambda_H) \in S^p(\mathscr{T}_h) \times M_-^{pc}(\mathscr{T}_{C,H})$ such that

$$\begin{aligned} \forall v_h \in S^p(\mathscr{T}_h) &: (\nabla u_h, \nabla v_h)_0 = (f, v_h)_0 - (\lambda_H, \gamma_C(v_h))_{0,\Gamma_C}, \\ \forall \mu_H \in M_-^{pc}(\mathscr{T}_{C,H}) &: (\mu_H - \lambda_H, \gamma_C(u_h) - g)_{0,\Gamma_C} \leq 0. \end{aligned} \tag{6}$$

In order to ensure the existence of a unique solution of (6), we have to verify a discrete version of condition (4),

$$\exists \tilde{\alpha} > 0 : \ \forall \mu_H \in M^{pc}(\mathscr{T}_{C,H}) : \ \tilde{\alpha} \|\mu_H\|_{-1/2,\Gamma_C} \leq \sup_{\substack{v_h \in S^p(\mathscr{T}_h) \\ \|v_h\|_1 = 1}} \langle \mu_H, \gamma_C(v_h) \rangle. \tag{7}$$

To guarantee the discretization scheme to be stable, the constant $\tilde{\alpha}$ has to be independent of h and H. In [17], the discrete inf-sup condition (7) is proven with an h- and H-independent $\tilde{\alpha}$ for uniformly refined meshes and $p_T \equiv 1$, $p_{C,T} \equiv 0$. The essential assumption there is that the quotient h/H is sufficiently small. For this assumption, convergence can also be shown for the proposed scheme. For higher-order approaches, stability and convergence are still open problems.
In our numerical experiments with higher-order finite elements, we obtain stable schemes by using meshes $\mathscr{T}_h$ and $\mathscr{T}_{C,H}$ which imply sufficiently small quotients h/H and $p_{C,T_C}/p_T$ for $T \in \mathscr{T}_h$, $T_C \in \mathscr{T}_{C,H}$ and $T \subset T_C$. In our implementation, we ensure $h/H \leq 0.5$ and $p_{C,T_C} \leq p_T - 1$ using hierarchical meshes with $\mathscr{T}_{C,H}$ being sufficiently coarser than $\mathscr{T}_h$.
From a practical point of view, it is crucial to ensure that the higher-order finite element functions are in $M_-^{pc}(\mathscr{T}_{C,H})$ for $p_{C,T} \geq 1$. It is reasonable to replace $M_-^{pc}(\mathscr{T}_{C,H})$ by

$$\tilde{M}_-^{pc}(\mathscr{T}_{C,H}) := \left\{ \mu_H \in M^{pc}(\mathscr{T}_{C,H}) \mid \forall T \in \mathscr{T}_{C,H} : \ \forall x \in \mathscr{C} : \ \mu_{H|T}(\Psi_{C,T}(x)) \leq 0 \right\}$$

where $\mathscr{C} \subset [-1,1]^{k-1}$ is a sufficiently large set of discrete points. We use Chebycheff points to ensure the additional error to be small.

5 Reliable A Posteriori Error Estimates

In the following, let $(u, \lambda) \in H^1(\Omega, \Gamma_D) \times H_-^{-1/2}(\Gamma_C)$ be the unique solution of (5) and $(u_h, \lambda_H) \in S^p(\mathscr{T}_h) \times M_-^{pc}(\mathscr{T}_{C,H})$ be a solution of (6). The aim of this section is to derive a reliable a posteriori error estimate for $|u - u_h|_1$. The basic idea is to consider the following auxiliary problem: Find $u_0 \in H^1(\Omega, \Gamma_D)$ such that

$$\forall v \in H^1(\Omega, \Gamma_D) : \ (\nabla u_0, \nabla v) = (f, v) - (\lambda_H, \gamma_C(v))_{0,\Gamma_C}. \tag{8}$$

Obviously, the solution u_0 of (8) exists and is unique. Moreover, u_h is a finite element solution of (8). In the sequel, we will show that

$$|u - u_h|_1 \lesssim |u_0 - u_h|_1 + \text{additional terms},$$

where $\lesssim$ abbreviates $\leq$ up to some h- and H-independent constant. Using an arbitrary error estimator η_0 for problem (8), we set

$$\eta := \eta_0 + \text{additional terms}$$

and obtain

$$|u - u_h|_1 \lesssim \eta.$$

Deriving error estimates this way goes back to [7], where this technique was applied to the obstacle problem. Here, we extend this approach for Signorini's problem and, in particular, to discretization schemes given by the mixed variational formulation as introduced in Sections 3 and 4.

In the following, we will make use of Cauchy's inequality

$$(\nabla v, \nabla w)_0 \leq |v|_1 |w|_1$$

for $v, w \in H^1(\Omega, \Gamma_D)$ and of

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \qquad \text{for } a, b \in \mathbb{R},\ \varepsilon > 0 \tag{9}$$

$$(a+b)^2 \leq 2a^2 + 2b^2 \qquad \text{for } a, b \in \mathbb{R}, \tag{10}$$

$$x \leq a + b^{1/2} \qquad \text{for } x, a, b > 0,\ x^2 \leq ax + b. \tag{11}$$

Lemma 1 *There holds*

$$|u - u_h|_1^2 \leq |u_0 - u_h|_1 |u - u_h|_1 + \langle \lambda, \gamma_C(u_h) - g \rangle.$$

Proof. Since $0, 2\lambda \in H_-^{-1/2}(\Gamma_C)$ and $0, 2\lambda_H \in M_-^{pc}(\mathscr{T}_{C,H})$, we have

$$\langle \lambda, \gamma_C(u) - g \rangle = (\lambda_H, \gamma_C(u_h) - g)_{0,\Gamma_C} = 0.$$

Furthermore, there holds $(\lambda_H, \gamma_C(u) - g)_{0,\Gamma_C} \leq 0$. From Cauchy's inequality, we obtain

$$\begin{aligned}
|u - u_h|_1^2 &= (\nabla(u - u_h), \nabla(u - u_h))_0 \\
&= (\nabla(u - u_0), \nabla(u - u_h))_0 + (\nabla(u_0 - u_h), \nabla(u - u_h))_0 \\
&\leq (\lambda_H, \gamma_C(u - u_h))_{0,\Gamma_C} - \langle \lambda, \gamma_C(u - u_h) \rangle + |u_0 - u_h|_1 |u - u_h|_1 \\
&= (\lambda_H, \gamma_C(u) - g)_{0,\Gamma_C} - \langle \lambda, g - \gamma_C(u_h) \rangle + |u_0 - u_h|_1 |u - u_h|_1 \\
&\leq \langle \lambda, \gamma_C(u_h) - g \rangle + |u_0 - u_h|_1 |u - u_h|_1.
\end{aligned}$$

□

Lemma 2 *There holds*

$$\begin{aligned}
\langle \lambda, \gamma_C(u_h) - g \rangle \leq (|u - u_h|_1 + |u_0 - u_h|_1) \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + \\
|(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|.
\end{aligned}$$

Proof. Let $d \in H^1(\Omega, \Gamma_D)$ be the harmonic extension of $(g - \gamma_C(u_h))_+ \in H^{1/2}(\Gamma_C)$, i.e., $d \in W := \{v \in H^1(\Omega, \Gamma_D) \mid \gamma_C(d) = (g - \gamma_C(u_h))_+\}$ with

$$\|d\|_1 = \inf_{v \in W} \|v\|_1.$$

Thus, we have $\|d\|_1 = \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C}$. Moreover, there holds

$$g - \gamma_C(u_h) - \gamma_C(d) = g - \gamma_C(u_h) - (g - \gamma_C(u_h))_+ \leq 0$$

on Γ_C and therefore $g - \gamma_C(u_h) - \gamma_C(d) \in H^{1/2}_-(\Gamma_C)$. Thus, we obtain

$$\begin{aligned}
\langle \lambda, \gamma_C(u_h) - g \rangle &= -\langle \lambda, g - \gamma_C(u_h) - \gamma_C(d) \rangle - \langle \lambda, \gamma_C(d) \rangle \\
&\leq (\nabla u, \nabla d)_0 - (f, d)_0 \\
&= (\nabla(u - u_h), \nabla d) + (\nabla u_h, \nabla d)_0 - (f, d)_0 \\
&\leq |u - u_h|_1 \|d\|_1 + (\nabla u_h, \nabla d)_0 - (f, d)_0 \\
&= |u - u_h|_1 \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + (\nabla u_h, \nabla d)_0 - (f, d)_0.
\end{aligned}$$

Finally, there is

$$\begin{aligned}
(\nabla u_h, \nabla d)_0 - (f, d)_0 &= (\nabla(u_h - u_0), \nabla d)_0 - (\lambda_H, \gamma_C(d))_{0,\Gamma_C} \\
&\leq |u_0 - u_h|_1 \|d\|_1 - (\lambda_H, \gamma_C(d))_{0,\Gamma_C} \\
&\leq \|u_0 - u_h\|_1 \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + |(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|
\end{aligned}$$

which completes the proof. $\square$

Theorem 1 *Let $\varepsilon > 0$, then*

$$\begin{aligned}
|u - u_h|_1 \leq (1 + \varepsilon)|u_0 - u_h|_1 + (1 + \frac{1}{4\varepsilon})\|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + \\
|(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|^{1/2}.
\end{aligned}$$

Proof. From Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
|u - u_h|_1^2 &\leq |u_0 - u_h|_1 |u - u_h|_1 + \langle \lambda, \gamma_C(u_h) - g \rangle \\
&\leq |u - u_h|_1 (|u_0 - u_h|_1 + \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C}) + \\
&\qquad |u_0 - u_h|_1 \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + |(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|).
\end{aligned}$$

Using (9) and (11) proves the theorem,

$$\begin{aligned}
|u - u_h|_1 &\leq |u_0 - u_h|_1 + \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} \\
&\quad + (|u_0 - u_h|_1 \|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} + |(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|)^{1/2} \\
&\leq (1 + \varepsilon)|u_0 - u_h|_1 + (1 + \frac{1}{4\varepsilon})\|(g - \gamma_C(u_h))_+\|_{1/2,\Gamma_C} \\
&\qquad + |(\lambda_H, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|^{1/2}.
\end{aligned}$$

$\square$

Corollary 1 *Let $\eta_0 > 0$ with $|u-u_h|_1 \lesssim \eta_0$ and*

$$\eta^2 := \eta_0^2 + \|(g-\gamma_C(u_h))_+\|_{1/2,\Gamma_C}^2 + |(\lambda_H, (g-\gamma_C(u_h))_+)_{0,\Gamma_C}|. \tag{12}$$

Then,

$$|u-u_h|_1 \lesssim \eta.$$

Proof. Using Theorem 1 and (10), we obtain

$$\begin{aligned} |u-u_h|_1^2 &\lesssim |u_0-u_h|_1^2 + \|(g-\gamma_C(u_h))_+\|_{1/2,\Gamma_C}^2 + |(\lambda_H, (g-\gamma_C(u_h))_+)_{0,\Gamma_C}| \\ &\lesssim \eta_0^2 + \|(g-\gamma_C(u_h))_+\|_{1/2,\Gamma_C}^2 + |(\lambda_H, (g-\gamma_C(u_h))_+)_{0,\Gamma_C}|. \end{aligned}$$

□

Remark 1 The terms in the error estimate of Corollary 1 represent typical error sources in Signorini's problem. The term $(g-\gamma_C(u_h))_+$ measures the error with respect to the geometrical contact condition and the term $|(\lambda_H, (g-\gamma_C(u_h))_+)_{0,\Gamma_C}|$ describes the error in the complementary condition.

Remark 2 The calculation of η in (12) requires the determination or estimation of $\|(g-\gamma(u_h))_+\|_{1/2,\Gamma_C}$. Since $\gamma_C(u_h)$ is piecewise polynomial, [11, Ch. I, Cor.2.1], we have $(g-\gamma(u_h))_+ \in H^1(\Gamma_C)$ provided that $g \in H^1(\Gamma_C)$. Using well-known interpolation results, [21, Thm.7.7.], we get

$$\|(g-\gamma_C(u_h))_+\|_{1/2,\Gamma_C}^2 \lesssim \|(g-\gamma(u_h))_+\|_{0,\Gamma_C} \|(g-\gamma(u_h))_+\|_{1,\Gamma_C}.$$

This leads to the estimate

$$|u-u_h|_1^2 \lesssim \eta_0^2 + \|(g-\gamma(u_h))_+\|_{0,\Gamma_C} \|(g-\gamma(u_h))_+\|_{1,\Gamma_C} + |(\lambda_H, (g-\gamma(u_h))_+)_{0,\Gamma_C}|.$$

The following results bound the discretization error of the Lagrange multiplier.

Lemma 3 *There holds*

$$\|\lambda - \lambda_H\|_{-1/2,\Gamma_C} \lesssim |u-u_0|_1.$$

Proof. The mapping $\hat{\gamma}_C : H^1(\Omega,\Gamma_D)/\ker\gamma_C \to H^{1/2}(\Gamma_C)$ with $\hat{\gamma}_C([v]) := \gamma_C(v)$ and $[v] := v + \ker\gamma_C$ is bijective and continuous. Since $H^1(\Omega,\Gamma_D)$ and $H^{1/2}(\Gamma_C)$ are Banach Spaces, $\hat{\gamma}_C^{-1}$ is also continuous. Let

$$V := \left\{ v \in H^1(\Omega,\Gamma_C) \mid \|v\|_1 \le \|\hat{\gamma}_C^{-1}\| \|\gamma_C(v)\|_{1/2,\Gamma_C} \right\}.$$

In order to show that V is a non-empty set, let $w \in H^{1/2}(\Gamma_C)$ and $v \in H^1(\Omega,\Gamma_D)$ with $\hat{\gamma}_C^{-1}(w) = [v]$. Furthermore, let $\bar{z} \in \ker\gamma_C$ with $\|v-\bar{z}\|_1 = \inf_{z\in\ker\gamma_C} \|v-z\|_1$ and $v^* := v - \bar{z}$. Thus, we obtain

$$\gamma_C(v^*) = \gamma_C(v-\bar{z}) = \gamma_C(v) = \hat{\gamma}_C([v]) = w. \tag{13}$$

Therefore, we have

$$\|v^*\|_1 = \inf_{z\in\ker\gamma_C} \|v-z\|_1 = \|\hat{\gamma}_C^{-1}(w)\| \le \|\hat{\gamma}_C^{-1}\| \, \|w\|_{1/2,\Gamma_C} = \|\hat{\gamma}_C^{-1}\| \, \|\gamma_C(v^*)\|_{1/2,\Gamma_C},$$

which says that $v^* \in V$. Moreover, we can find a $v^* \in V$ for every $w \in H^{1/2}(\Gamma_C)$ such that (13) holds, i.e., $\gamma_C(V) = H^{1/2}(\Gamma_C)$. Using these preparations, we conclude from the definition of the dual norm and Cauchy's inequality, that

$$\begin{aligned}
\|\lambda - \lambda_H\|_{-1/2,\Gamma_C} &= \sup_{w \in H^{1/2}(\Gamma_C)\setminus\{0\}} \frac{\langle \lambda - \lambda_H, w\rangle}{\|w\|_{1/2,\Gamma_C}} \\
&= \sup_{v \in V\setminus\{0\}} \frac{\langle \lambda - \lambda_H, \gamma_C(v)\rangle}{\|\gamma_C(v)\|_{1/2,\Gamma_C}} \\
&= \sup_{v \in V\setminus\{0\}} \frac{(\nabla(u_0 - u), \nabla v)_0}{\|\gamma_C(v)\|_{1/2,\Gamma_C}} \\
&\le \|\hat{\gamma}_C^{-1}\|^{-1} \sup_{v \in V\setminus\{0\}} \frac{(\nabla(u_0 - u), \nabla v)_0}{\|v\|_1} \\
&\le \|\hat{\gamma}_C^{-1}\|^{-1} |u - u_0|_1.
\end{aligned}$$

□

Corollary 2 *Let the assumptions of Corollary 1 be fulfilled. Then, there holds*

$$|u - u_h|_1 + \|\lambda - \lambda_H\|_{-1/2,\Gamma_C} \lesssim \eta.$$

Proof. From Lemma 3 and $\eta_0 \le \eta$, we obtain

$$\begin{aligned}
|u - u_h|_1 + \|\lambda - \lambda_H\|_{-1/2,\Gamma_C} &\lesssim |u - u_h|_1 + |u - u_0|_1 \\
&\lesssim 2|u - u_h|_1 + |u_0 - u_h|_1 \\
&\lesssim \eta + \eta_0 \\
&\lesssim \eta.
\end{aligned}$$

□

In order to apply the error estimates of Corollary 1 and 2, we have to specify an appropriate error estimator η_0 for the variational equation (8). In principle, each error estimator known from the literature of variational equations can be used. We refer to [1,30] for an overview of h-adaptive methods. For hp-adaptivity, we need an error estimator which takes the degree distribution p into account. For the sake of completeness, we state a residual based error estimator proposed by Melenk et al., [22, 23]. Set

$$\eta_0^2 := \sum_{T \in \mathscr{T}_h} \Big((h_T/p_T)^2 R_{0,T}^2 + \sum_{e \in \mathscr{E}_T} (h_e/p_e) R_{0,e}^2 \Big),$$

with

$$\begin{aligned}
p_e &:= \min\left\{ p_T \mid e \in \mathscr{E}_T,\, T \in \mathscr{T}_h \right\},\ e \in \mathscr{E}^\circ \cup \mathscr{E}_\Gamma, \\
R_{0,T} &:= \|f + \Delta u_h\|_{0,T},\ T \in \mathscr{T}_h, \\
R_{0,e} &:= \begin{cases} \frac{1}{2}\|[\partial_n u_h]\|_{0,e}, & e \in \mathscr{E}^\circ, \\ \|\partial_n u_h + \lambda_H\|_{0,e}, & e \in \mathscr{E}_\Gamma, \end{cases}
\end{aligned}$$

where $\mathscr{E}_T$ is the set of edges of $T \in \mathscr{T}_h$, $\mathscr{E}^0$ contains the internal edges and $\mathscr{E}_\Gamma$ the edges on $\Gamma\setminus\Gamma_D$. As usual, $[\cdot]_e$ denotes the jump across an edge $e \in \mathscr{E}^\circ$.

6 h- and hp-Adaptivity

Adaptive strategies are usually based on an error estimate η given by

$$\eta^2 = \sum_{T \in \mathscr{T}_h} \eta_T^2$$

with local error contributions η_T. h-adaptive methods rely on the refinement of mesh elements with large error contributions. Various criterions are proposed in literature, e.g. [4]. A simple and commonly used method is the fixed fraction approach: In each iteration step, a fixed percentage of $\mathscr{T}_h$ is refined which is associated to the largest error contributions. The percentage is described by a parameter $\kappa \in [0,1]$. With $k^\star := \lfloor (1-\kappa)|\mathscr{T}_h| \rfloor + 1$ an iteration step is given as follows:

(1) Determine η_T for all $T \in \mathscr{T}_h$.
(2) Sort $\eta_{T_1} \leq \eta_{T_2} \leq \ldots$.
(3) $\mathscr{K} := \{T \in \mathscr{T}_h \mid \eta_T \geq \eta_{T_{k^\star}}\}$.
(4) Refine all $T \in \mathscr{K}$.

For hp-adaptivity, one has to decide which mesh elements have to be refined and additionally for which mesh elements the polynomial degree has to be increased. Several strategies are discussed in literature, [2,9,23,24]. Many hp-adaptive strategies rely on the estimation of the local regularity of the solution. If the local regularity in a mesh element is sufficient, the polynomial degree should be increased there. Otherwise, the mesh element should be refined. In the following, we propose an hp-strategy which is based on the estimation of the local regularity using two finite element approximations on the same mesh, but with different degree distributions p and $\tilde{p}$. Given those approximations, we can determine the two error estimates $\eta^2 = \sum_{T \in \mathscr{T}_h} \eta_T^2$ and $\tilde{\eta}^2 = \sum_{T \in \mathscr{T}_h} \tilde{\eta}_T^2$ corresponding to p and $\tilde{p}$. This strategy goes back to Süli et al., [27]. The main idea is to assume that the local error contributions η_T and $\tilde{\eta}_T$ for $T \in \mathscr{T}_h$ are approximatively given by

$$\eta_T \approx C_T p_T^{-\rho_T+1} \quad \text{and} \quad \tilde{\eta}_T \approx C_T \tilde{p}_T^{-\rho_T+1}$$

with $\rho_T > 0$. This assumption can be justified by well-known a priori estimates, [3]. Provided that $p_T \neq \tilde{p}_T$, we can approximate ρ_T using

$$\rho_T \approx \frac{\log(\tilde{\eta}_T/\eta_T)}{\log(p_T/\tilde{p}_T)} + 1.$$

The parameter ρ_T can be interpreted as a measure for the local regularity. In this sense, the solution is sufficiently regular, if $\rho_T \geq \max\{p_T, \tilde{p}_T\}$.
Similar to the h-adaptive strategy, the first step of our hp-adaptive strategy is to collect the mesh elements with the largest error contributions in a set $\mathscr{K}$, based on a simple fixed fraction strategy. We set the degree distribution $\tilde{p}_T := p_T + 1$ for all $T \in \mathscr{K}$. In order to ensure that the full local polynomial space is used for elements in $\mathscr{K}$, we additionally set $\tilde{p}_T := p_T + 1$ for all $T \in \mathscr{W}$, where $\mathscr{W}$ contains all elements of $\mathscr{T}_h$ adjacent to elements in $\mathscr{K}$ via an edge. The next step is to calculate a second finite element approximation and to estimate the corresponding discretization error

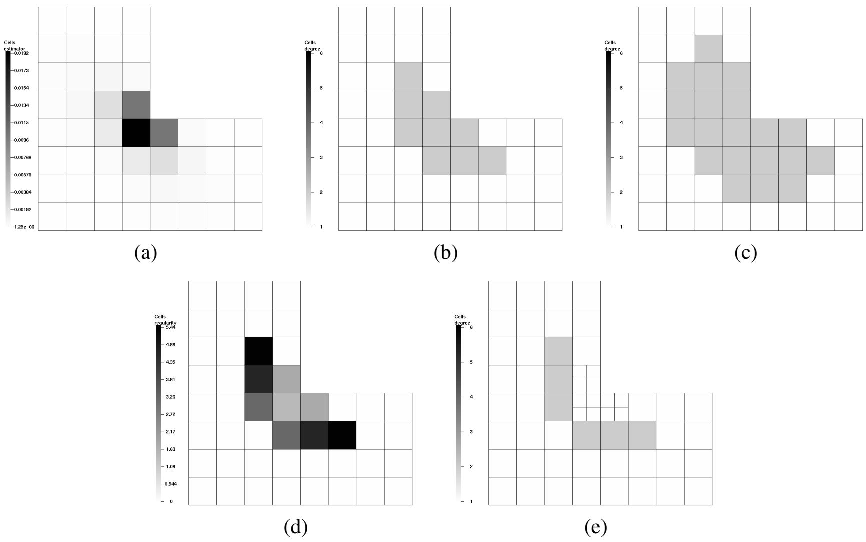


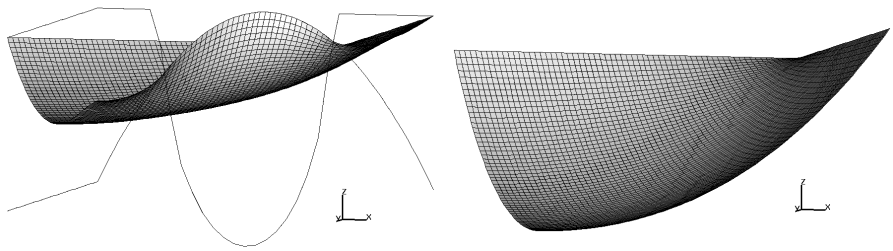
Fig. 1
hp-adaptive refinement strategy: **(a)** η_T for $T \in \mathscr{T}_h$ (step 1), **(b)** $\mathscr{K}$ (step 3), **(c)** $\mathscr{W}$ (step 4), **(d)** ρ_T (step 7), **(e)** *h*-refinement or *p*-enrichment (step 8 and step 9).

by $\tilde{\eta}$. In the last step, we refine all elements with insufficient local regularity, i.e., all $T \in \mathscr{K}$ with $\rho_T < p_T + 1$, and we increase the local polynomial degree for all elements $T \in \mathscr{K}$ with $\rho_T \geq p_T + 1$. The strategy is summarized by the following steps:

(1) Determine η.
(2) Sort $\eta_{T_1} \leq \eta_{T_2} \leq \ldots$.
(3) Set $\mathscr{K} := \{T \in \mathscr{T}_h \mid \eta_T \geq \eta_{T_{k^\star}}\}$.
(4) Set $\mathscr{W} := \{T \in \mathscr{T}_h \backslash \mathscr{K} \mid \exists T_0 \in \mathscr{K} : \emptyset \neq \overline{T} \cap \overline{T_0} \notin \mathscr{V}\}$.
(5) Set $\tilde{p}_T := p_T + 1$ if $T \in \mathscr{K} \cup \mathscr{W}$ and $\tilde{p}_T := p_T$ otherwise.
(6) Determine $\tilde{\eta}$.
(7) Determine ρ_T for all $T \in \mathscr{K}$.
(8) Refine $T \in \mathscr{K}$, if $\rho_T < p_T + 1$.
(9) $p_T := p_T + 1$ for $T \in \mathscr{K}$, if $\rho_T \geq p_T + 1$

In Figures 1(a)-(e), the steps of the *hp*-adaptive strategy are illustrated for the well-known L-shaped domain example with a singularity at the re-entrant corner, [13].

The calculation of the additional error estimate $\tilde{\eta}$ leads to a high effort which is justifiable by the exponential convergence rates of the adaptive scheme, see Section 7. In [23], the intermediate step to determine $\tilde{\eta}$ is omitted. Instead, only the first estimate η is considered in two successive refinement steps. However, error contributions of successive estimates are not necessarily comparable when using adaptive *h*-refinements and p-enrichments. Thus, these one-step strategies have to be applied very carefully. Many *h*- and *hp*-adaptive strategies (including the proposed strategies) rely on the



(a) (b)

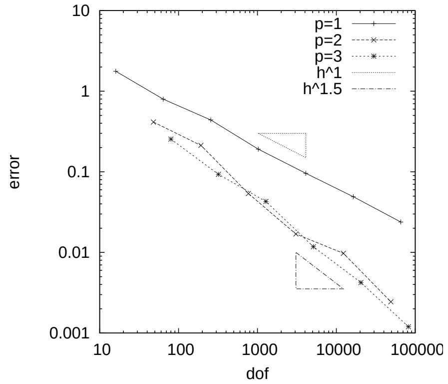
Fig. 2 **(a)** Solution u of Signorini's problem with an obstacle function g on the boundary and the Lagrange multiplier λ. **(b)** Solution of the unrestricted problem.

heuristic assumption, that the error contributions given by η_T reflect the local discretization error. Moreover, the increase of the local accuracy in areas with large error contributions is assumed to significantly reduce the global discretization error. These assumptions are justifiable in most cases and are confirmed by many numerical experiments. Though, convergence and, in particular, optimality are not guaranteed or verified in general. A rigorous verification of convergence and optimality is still an interesting field of research. We refer to Dörfler et al. [10] for results concerning hp-adaptive methods.

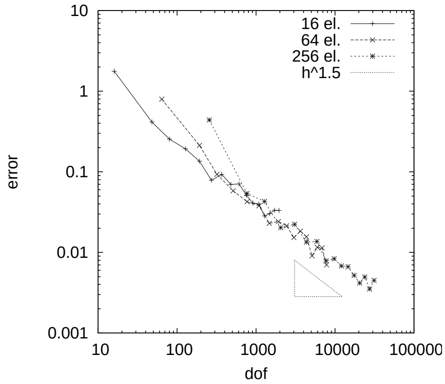
7 Numerical Results

In our numerical experiments, we study Signorini's problem with $\Omega := (-1,1)^2$, $\Gamma_C \subset (-1,1)\times\{-1\}$, $f := -1$, and $g(x_0,x_1) := -x_0^2$. In Figure 2(a), the finite element solution u_h of Signorini's problem is depicted. In addition, the obstacle function g and the discrete Lagrangian multiplier λ_H are sketched in. We observe, that the condition $u_h \geq g$ is approximatively fulfilled. For $u \in H^1(\Omega,\Gamma_D)\cap H^2(\Omega)$, there holds $\lambda \in L^2(\Gamma_C)$ and $\lambda = -\partial_n u$. Thus, we have $\lambda\,(u-g) = 0$ on Γ_C. This condition is also shown in Figure 2(a). In order to give a better visualization, the finite element solution of the unrestricted problem is depicted in Figure 2(b). This problem corresponds to Poisson's problem $-\Delta u = f$ in Ω and $u = 0$ on Γ_D.

In Figure 3, the estimated error is shown which is determined by the error estimate η introduced in Corollaries 1 and 2. As proposed in Remark 2, we replace $\|(g-\gamma_C(u_h))_+\|^2_{1/2,\Gamma_C}$ by $\|(g-\gamma_C(u_h))_+\|_{0,\Gamma_C}\|(g-\gamma_C(u_h))_+\|_{1,\Gamma_C}$. In Figure 3(a), the estimated error obtained by global h-refinements with constant polynomial degree is depicted. As the diagram shows, the estimated convergence rate for $p=1$ is $\mathscr{O}(h)$. This rate is optimal with respect to a priori results, [17]. It is well-known that the solution u of Signorini's problem on convex domains is in $H^2(\Omega)$ and in general



(a)



(b)

Fig. 3 Estimated error: (a) global h-refinement with $p = 1,2,3$. (b) global p-enrichments with 16, 64, and 256 mesh elements.

$u \notin H^k(\Omega)$ for $k \geq 3$. Therefore, we can not expect to achieve the optimal algebraic convergence rate $\mathscr{O}(h^p)$ for $p \geq 2$. In fact, we observe a reduced estimated convergence rate $\mathscr{O}(h^{3/2})$ for $p = 2$ and $p = 3$ in Figure 3(a). The pure p-method with constant meshes exhibits the same limit, cf. Figure 3(b). Note that only the reliability is proven in Section 5.

In Figure 4 the error contributions η_0 and

$$s_0 := \|(g - \gamma_C(u_h))_+\|_{0,\Gamma_C}^{1/2} \|(g - \gamma_C(u_h))_+\|_{1,\Gamma_C}^{1/2},$$
$$s_1 := |(\lambda_{0,H}, (g - \gamma_C(u_h))_+)_{0,\Gamma_C}|^{1/2}$$

are depicted for global h-refinements and global p-enrichments. For $p = 1$, the error contributions s_0 and s_1 seem to be small in comparison to η_0 and may be neglected. For $p \geq 2$, all contributions are of the same order of magnitude.

In Figure 5, the estimated convergence rates are depicted for h-adaptive schemes with polynomial degree $p = 2$ and $p = 3$. As mentioned, we already obtain an optimal convergence rate $\mathscr{O}(h)$ for $p = 1$ by using global refinements. In fact, applying the h-adaptive refinement strategy proposed in Section 6 leads to these global refinements. Thus, adaptive schemes are reasonable for $p \geq 2$ as we can expect a significant improvement of the convergence rates, cf. Figure 5. For $p = 2$ and $p = 3$, the optimal algebraic convergence rate $\mathscr{O}(h^p)$ is achieved. In Figure 6, h-adaptive meshes for $p = 2$ and $p = 3$ are shown. We find local refinements towards both ends of the contact zone and towards the domain's corner which is in Γ_D. Moreover, there are local refinements within the contact zone. In Figure 7(a), the almost exponential estimated convergence rate for hp-adaptive refinements is displayed in comparison to the estimated convergence rate for the h-adaptive refinement with $p = 2$. In Figure 7(b), we can observe the typical geometrical refinement patterns of an hp-adaptive mesh. The corners of the domain and the ends of the contact zone are resolved by h-refinements

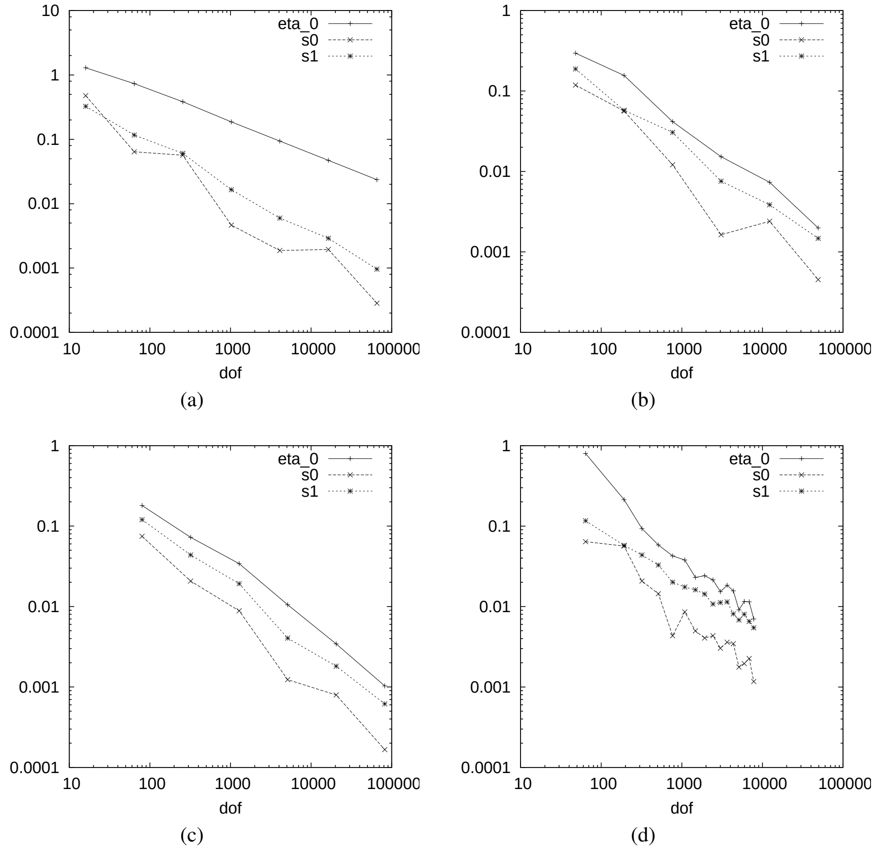


Fig. 4 Error contribution η_0, s_0 and s_1: **(a)** uniform h-refinement with $p = 1$, **(b)** uniform h-refinement with $p = 2$, **(c)** uniform h-refinement with $p = 3$, **(d)** uniform p-enrichment with 64 mesh elements,

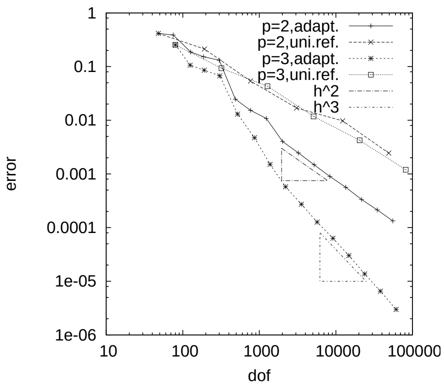
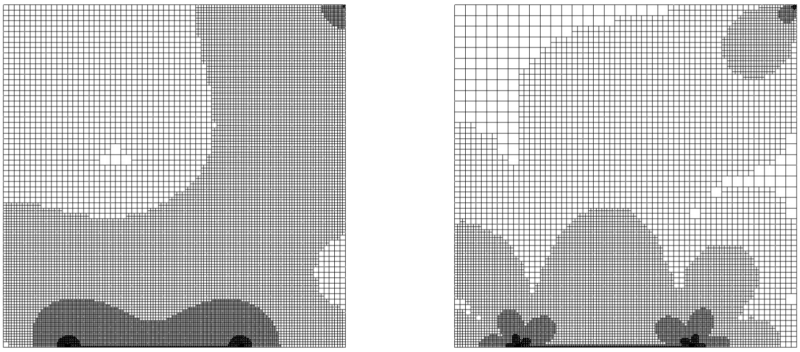
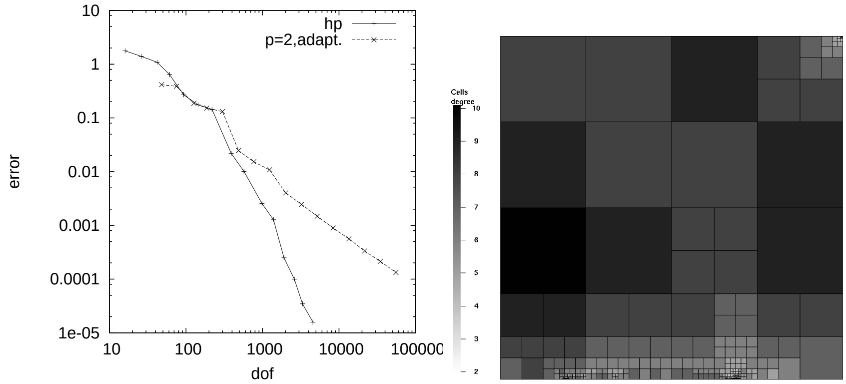


Fig. 5 Estimated convergence rates for h-adaptive refinements.



(a) (b)

Fig. 6 *h*-adaptive meshes: **(a)** $p = 2$, **(b)** $p = 3$.



(a) (b)

Fig. 7 **(a)** estimated convergence rates for *h*- and *hp*-adaptive refinements, **(b)** *hp*-adaptive meshes.

and the polynomial degree is small ($p = 1$ or $p = 2$). Whereas, away from the corners and the contact zone, the polynomial degree is higher.

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