Minimal Entropy Martingale Measure for Lévy Processes

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May 14, 2009

Abstract

Let X be a real-valued Lévy process under $\mathbb{P}$ in its natural filtration. The minimal entropy martingale measure is defined as an absolutely continuous martingale measure that minimizes the relative entropy with respect to $\mathbb{P}$. We show in this paper that the sufficient conditions for its existence, known in literature, are also necessary and give an explicit formula for the infimum of the relative entropy.

Key words: Lévy processes, martingale measures, relative entropy, f-divergences, minimal entropy martingale measure, Eshscer martingale transform, mathematical finance, incomplete markets

AMS subject classification: primary 60G51, secondary 91B28

1 Introduction

In the last years, financial models based on Lévy processes have become very popular as possible alternatives to the traditional Black-Scholes model. The exponential Lévy model, i.e. a model in which the asset price process has the form $S_t = S_0 e^{X_t},\ t \in [0,T]$, with a real valued Lévy process X is very flexible and analytically tractable and is able to reproduce many stylized facts of the financial time series such as volatility clustering, jumps, heavy tails and skewness.

The main problem with the exponential Lévy models is that they in general generate an incomplete market: there are infinitely many equivalent martingale (or risk neutral) measures (EMMs) and hence the option prices are not unique. A popular approach is to fix a suitable

*This work was supported by the Deutsche Telekom Stiftung and the International Research Training Group 1339 SMCP.

equivalent martingale measure, which determines the option prices by an expectation of the discounted payoff taken with respect to this measure. Probably the easiest approach is taking the Esscher transform (EEMM) studied by Gerber and Shiu, [GS94], i.e. the unique (if it exists) martingale measure from the exponential family of the process X. Another classical method is minimization of a functional over equivalent martingale measures. Well studied examples are the minimal martingale measure by Föllmer and Schweizer [FS91], the minimal entropy martingale measure (MEMM) studied by Chan, [Cha99], Miyahara, [Miy99], Frittelli, [Fri00], Fujiwara and Miyahara, [FM03], Esche and Schweizer, [ES05] and the f^q-minimal martingale measure (qMMM) by Jeanblanc, Klöppel and Miyahara, [JKM07].

In this paper we choose the relative entropy as a functional to be minimized. This method is mathematically elegant and has a financial interpretation within the context of a utility maximization, [Fri00], [GR01]. The dual representation of the asset price process through a stochastic exponential of other Lévy process provides a close link between MEMM and the Escher martingale measure. We provide a complete characterization of the MEMM: we show that the well-known sufficient conditions for the existence of MEMM (i.e., [FM03], Theorem 3.1 or [ES05], Theorem B) are also necessary. Finally, we present a representation of the infimum of the entropy process through the cumulant-generating function of the Lévy process no matter whether the MEMM exists or not. In the latter case we also construct a sequence of absolutely continuous martingale measures $(\mathbb{P}^*_n)_{n\in\mathbb{N}}$ whose entropy processes approach the infimum. Our results are valid for arbitrary Lévy processes. Applying the duality representation these results can be translated easily for the exponential Lévy model.

This paper is structured as follows. In the second section we formulate the basic problem in more detail and state sufficient conditions for the existence of the MEMM, proved by Esche and Schweizer, [ES05]. In the third section we present our main results, which allow a complete characterisation of the MEMM for both Lévy and exponential Lévy processes. The translation of our results for the exponential Lévy models is presented in Section 4. The last section contains a discussion of the results of Hubalek and Sgarra and the comparison to our results as well as some steps in their proofs that are unclear to us. Appendix A summarizes required results on exponential transforms and measure changes for Lévy processes. For a better readability, some omitted proofs are presented in Appendix B.

2 Setup and results of Esche and Schweizer

Let $(\Omega, \mathscr{F}, \mathbb{F}\, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} = \mathbb{F}^X = (\mathscr{F}^X_t)_{t\in[0,T]}$ is the $\mathbb{P}$-augmentation of the natural filtration of a real-valued Lévy process X. Let (σ^2, ν, γ) denote the characteristic triplet of X with respect to the truncation function $x\mathbf{1}_{\{|x|\leq 1\}}$. The entropy process for a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ with $\mathbb{Q} \ll \mathbb{P}$ is defined by

$$I_t(\mathbb{Q} \mid \mathbb{P}) := \mathbb{E}^{\mathbb{Q}}\left(\log\left(\frac{d\mathbb{Q}\mid_{\mathscr{F}_t}}{d\,\mathbb{P}\mid_{\mathscr{F}_t}}\right)\right) \in [0,\infty], \quad t \in [0,T].$$

Furthermore, we introduce the following sets of probability measures on $(\Omega, \mathscr{F})$:

$$\begin{aligned}
\mathscr{Q}_a(X) &:= \{\mathbb{Q} \ll \mathbb{P} \mid X \text{ is a local } \mathbb{Q}\text{-martingale}\},\\
\mathscr{Q}_e(X) &:= \{\mathbb{Q} \sim \mathbb{P} \mid X \text{ is a local } \mathbb{Q}\text{-martingale}\} \subseteq \mathscr{Q}_a(X),\\
\mathscr{Q}_f(X) &:= \{\mathbb{Q} \in \mathscr{Q}_a(X) \mid I_t(\mathbb{Q} \mid \mathbb{P}) < \infty \; \forall\, t \in [0,T]\},\\
\mathscr{Q}_l(X) &:= \{\mathbb{Q} \in \mathscr{Q}_a(X) \mid X \text{ is a Lévy process under } \mathbb{Q}\}.
\end{aligned}$$

The *minimal entropy equivalent martingale measure* (MEMM or $\mathbb{P}^{\text{MEMM}}$) is defined by the property that it minimizes the relative entropy among all measures $\mathbb{Q} \in \mathscr{Q}_a(X)$. The Esscher martingale measure (ESMM) for a process X, if it exists, is a martingale measure with density given by $\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}} = \text{const}\, e^{\vartheta^* X_T}$.
We assume that the paths of the process X are neither increasing nor decreasing.

Assumption A: $X \not\equiv 0$, the paths of X are $\mathbb{P}$-a.s. not monotone.

Remark 2.1. *The first fundamental theorem of asset pricing ([CS], Theorem 4.6) yields that the exponential Lévy model $S = e^X$ satisfies the no-arbitrage condition (NFLVR), if and only if the paths of S are not monotone (and hence, the paths of the exponential transform of X). Therefore, this assumption is not restrictive.*

Remark 2.2. *We shall see later that the Assumption A implies the existence of a probability measure $\mathbb{Q} \in \mathscr{Q}_e(R) \cap \mathscr{Q}_f(R) \cap \mathscr{Q}_l(R)$. Frittelli shows in [Fri00], Theorem 2.2, that in this case the MEMM is equivalent to $\mathbb{P}$ if it exists. Moreover, it follows from [ES05], Theorem A, that X is a Lévy process under $\mathbb{P}^{MEMM}$, hence X is a proper $\mathbb{P}^{MEMM}$-martingale. So, the notion "minimal entropy equivalent martingale measure" is justified.*

A Lévy process X is in general a semimartingale under $\mathbb{Q} \ll P$. However, as Esche and Schweizer show in [ES05], X is again a Lévy process under the MEMM, if it exists. Furthermore, they state sufficient conditions for the existence of the MEMM. Moreover, they show that in this case the MEMM is equal to the *Esscher martingale measure* (ESMM or $\mathbb{P}^{\text{ESMM}}$) for the exponential transform R. More precisely, we have

Theorem 2.1 ([ES05], Theorem A)**.** *Let R be a real-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with characteristic triplet (σ^2, ν, γ). Suppose that $\mathscr{Q}_e(R) \cap \mathscr{Q}_f(R) \cap \mathscr{Q}_l(R) \neq \emptyset$. If there exists a probability measure $\mathbb{P}^* \in \mathscr{Q}_a(R)$ with*

$$I_t(\mathbb{P}^* \mid \mathbb{P}) \leq I_t(\mathbb{Q} \mid \mathbb{P}), \quad \forall\, \mathbb{Q} \in \mathscr{Q}_a(R),\, \forall\, t \in [0,T], \tag{2.1}$$

then R is a Lévy process under $\mathbb{P}^*$.

Theorem 2.2 ([ES05], Theorem B)**.** *Let R be a real-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with characteristic triplet (σ^2, ν, γ). Suppose that there exists a $\vartheta^* \in \mathbb{R}$ satisfying*

$$\int_{\mathbb{R}\setminus\{0\}} |xe^{\vartheta^* x} - h(x)|\nu(\,dx) < \infty, \tag{ES1}$$

$$\gamma + \sigma^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} \left(xe^{\vartheta^* x} - h(x)\right)\nu(\,dx) = 0. \tag{ES2}$$

Then, the MEMM exists and coincides with the ESMM.

3 Main Results

In this section we present the reverse results to [ES05] and obtain the complete characterisation of the MEMM for a Lévy process R that satisfies the Assumption A. We show that the assumptions (ES1) and (ES2) are not only sufficient but also necessary for the existence of the MEMM. We also give a representation of the infimum of the relative entropy process, regardless of whether or not the MEMM exists, by means of the cumulant generating function and find a sequence of absolutely continuous martingale measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ that converges to the infimum of the entropy process if the MEMM does not exist.
In [HS06] Hubalek and Sgarra investigate the existence question of the MEMM for the exponential Lévy model. The authors use arguments that base on particular properties of an exponential Lévy process. Also, some of the steps of their main results are not clear to us. In section 5 we discuss these results and present a comparison of both approaches.

Our first result states, that the infimum of the relative entropy over $\mathscr{Q}_e(R)$ equals the negative value of the cumulant generating function of the process R, given by

$$\varphi(\vartheta) = \ln \mathbb{E}(e^{\vartheta R_1}) = \gamma\vartheta + \frac{1}{2}\sigma^2\vartheta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{\vartheta x} - 1 - \vartheta h(x))\nu(\,dx) < \infty,$$

for $\vartheta \in \Theta_{\text{EXP}}(R, \mathbb{P})$, where $\Theta_{\text{EXP}}(R, \mathbb{P}) := \{\vartheta \in \mathbb{R} \,:\, \int_{\{|x|>1\}} e^{\vartheta x}\nu(\,dx) < \infty\}$.

Theorem 3.1. *It holds that*

$$\inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{Q}_a(R)\} = -t\inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}, \quad \forall\, t \in [0, T].$$

Proof. The proof is given in Appendix B. □

The second main result provides the complete characterization of the MEMM. For a Lévy process R with characteristic triplet (σ^2, ν, γ) we define the function κ by

$$\kappa(\vartheta) := \frac{\mathbb{E}(R_1 e^{\vartheta R_1})}{\mathbb{E}(e^{\vartheta R_1})},$$

for a $\varphi \in \Theta^1_{\text{EXP}}(R, \mathbb{P})$, where $\Theta^1_{\text{EXP}}(R, \mathbb{P}) := \{\vartheta \in \Theta_{\text{EXP}}(R, \mathbb{P}) \ : \ \int_{\{|x|>1\}} |x| e^{\vartheta x} \nu(\,dx) < \infty\}$. If the minimal point ϑ^* satisfies $\kappa(\vartheta^*) = 0$, then the MEMM exists and coincides with the ESMM. Otherwise, the infimum of the entropy is not achieved over $\mathscr{Q}_a(R)$. Finally, we will show that the condition $\kappa(\vartheta^*) = 0$ is equivalent to the conditions (ES1) and (ES2). Esche and Schweizer show in [ES05], that (ES1) and (ES2) are sufficient for the existence of the MEMM. Similar results for an exponential Lévy process can be found in [Miy99], [FM03]. We complete these results by showing that for a Lévy process R that satisfies the Assumption A, the conditions (ES1) and (ES2) are also necessary for the existence of the MEMM.

Theorem 3.2. *Suppose $\vartheta^* \in \mathbb{R}$ satisfies $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) \ : \ \vartheta \in \mathbb{R}\}$. If $\kappa(\vartheta^*) = 0$, then there exists a unique probability measure $\mathbb{P}^* \in \mathscr{Q}_e(R) \cap \mathscr{Q}_f(R) \cap \mathscr{Q}_l(R)$ satisfying*

$$I_t(\mathbb{P}^* \,|\, \mathbb{P}) \leq I_t(\mathbb{Q}|\, \mathbb{P}), \quad \forall\, t \in [0, T],\, \forall\, \mathbb{Q} \in \mathscr{Q}_a(R).$$

$\mathbb{P}^*$ is the ESMM and its Radon-Nikodým density with respect to $\mathbb{P}$ is given by

$$\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}} = e^{\vartheta^* R_T - T\varphi(\vartheta^*)}.$$

If κ is not defined in ϑ^* or if it holds that $\kappa(\vartheta^*) \neq 0$, then the infimum of the relative entropy is not attained over $\mathscr{Q}_a(R)$.

Proof. The proof is given in Appendix B. □

Remark 3.1. *If we consider a Lévy process on the infinite time horizon, the statements of Theorem 3.1 and 3.2 can be adopted for this case if we replace the properties "equivalent" and "absolutely continuous" by "locally equivalent" and "locally absolutely continuous".*

If ϑ^* is an inner point of $\Theta_{\text{EXP}}(R, \mathbb{P})$, then $\varphi'(\vartheta^*) = \kappa(\vartheta^*) = 0$ and the MEMM exists. If ϑ^* is a boundary point of $\Theta_{\text{EXP}}(R, \mathbb{P})$, then the one-sided derivative $\lim_{\vartheta \to \vartheta^*} \varphi'(\vartheta) = \kappa(\vartheta^*)$, $\vartheta \in \Theta_{\text{EXP}}(R, \mathbb{P})$ is not necessary equal to zero. We give an example of a Lévy process R with a triplet (σ^2, ν, γ), for which both cases $\kappa(\vartheta^*) = \lim_{\vartheta \to \vartheta^*} \varphi'(\vartheta) < 0$ and $\kappa(\vartheta^*) = \lim_{\vartheta \to \vartheta^*} \varphi'(\vartheta) = 0$ occur depending on the value of the parameter γ.

Example 3.1. *Let $a > 0$ and R be a Lévy process with a characteristic triplet (σ^2, ν, γ) given by*

$$\begin{aligned}
\sigma^2 &> 0 \quad \textit{arbitrary}, \\
\gamma &= -2\left(\int_0^1 \frac{e^{ax} - 1}{x}\, dx + \int_1^\infty \frac{1}{x^2}\, dx\right) - 2\sigma^2 a < 0, \\
\nu(\,dx) &= \begin{cases} x^{-2}, & 0 < x \leq 1, \\ e^{-ax} x^{-3}, & x > 1, \\ 0, & \textit{otherwise,} \end{cases}
\end{aligned}$$

with respect to the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$. Since $\sigma^2 > 0$, the process R satisfies the Assumption A. It holds for an arbitrary $\vartheta \in \mathbb{R}$ that

$$\varphi(\vartheta) < \infty \Longleftrightarrow \int_1^\infty \frac{e^{(\vartheta-a)x}}{x^3}\,dx < \infty \Longleftrightarrow \vartheta \leq a.$$

Hence, $\Theta_{EXP}(R, \mathbb{P}) = (-\infty, a]$. For an arbitrary $\vartheta \in (0, a)$ we have

$$\varphi'(\vartheta) \leq \gamma + \sigma^2 a + \int_0^1 \frac{e^{ax}-1}{x}\,dx + \int_1^\infty \frac{1}{x^2}\,dx = \frac{\gamma}{2} < 0.$$

The function φ is strictly convex and hence, decreasing on $(-\infty, a]$. Monotone convergence theorem yields $\lim_{\vartheta\uparrow a}\varphi'(\vartheta) = \frac{\gamma}{2} < 0$. Applying the dominated convergence theorem we obtain

$$\frac{\mathbb{E}(R_1 e^{aR_1})}{\mathbb{E}(e^{aR_1})} = \frac{\gamma}{2} < 0.$$

Hence, according to Theorem 3.2, the MEMM does not exist. Theorem 3.1 provides a representation of the infimum of the relative entropy:

$$\inf_{\mathbb{Q}\in\mathscr{Q}_a(R)} I_t(\mathbb{Q}|\,\mathbb{P}) = -t\varphi(a) = -t\left(\gamma a + \frac{1}{2}\sigma^2 a^2 + \int_0^1 \frac{e^{ax}-1-ax}{x^2}\,dx + \int_1^\infty \frac{1-e^{-ax}}{x^3}\,dx\right).$$

Now let Y be a Lévy process with the characteristic triplet $(\sigma^2, \nu, \gamma/2)$. a is again the infimum point of the cumulant generating function of Y_1 and it holds that $\kappa(a) = 0$. The MEMM does exist.

The proof of Theorem 3.1 offers an approximating sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}} \subseteq \mathscr{Q}_a(R)\cap\mathscr{Q}_l(R)$ if the infimum of the relative entropy is not attained. We formulate this result in a lemma.

Lemma 3.1. *There exists a sequence of probability measures $(\mathbb{P}^*_n)_{n\in\mathbb{N}} \subseteq \mathscr{Q}_a(R)\cap\mathscr{Q}_f(R)\cap\mathscr{Q}_l(R)$, such that*

$$\lim_{n\to\infty} I_t(\mathbb{P}^*_n\,|\,\mathbb{P}) = \inf_{\mathbb{Q}\in\mathscr{Q}_a(R)} I_t(\mathbb{Q}|\,\mathbb{P}), \quad \forall t \in [0, T].$$

Example 3.2 (α-stable Lévy processes)**.** *Let us consider an arbitrary α-stable symmetric Lévy process. Its characteristic triplet is given by $(0, \nu, 0)$, where $\nu(\,dx) = \frac{1}{|x|^{\alpha+1}}\,dx$, with $\alpha \in (0, 2)$ and truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$. It holds that $\Theta_{EXP}(R, \mathbb{P}) = \{0\}$, hence,*

$$\inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q}\in\mathscr{Q}_a(R)\} = -t\varphi(0) = 0, \quad \forall\, t \in [0, T].$$

We consider the following sequence of measure changes, given by Girsanov parameters $(0, \mathbf{1}_{\{|x|\leq n\}})_R$. The corresponding probability measures $\mathbb{P}^*_n$ lie in $\mathscr{Q}_a(R)\cap\mathscr{Q}_l(R)$. The relative entropy process is given by

$$I_t(\mathbb{P}_n\,|\,\mathbb{P}) = t\frac{2\alpha}{n^\alpha} \longrightarrow 0 \quad \textit{for } n \to \infty.$$

According to Theorem 3.2, the MEMM does not exist, since the expectation of R_1 under $\mathbb{P}$ does not exist. Yet, the sequence $(\mathbb{P}_n)_{n\in\mathbb{N}} \subseteq \mathcal{Q}_a(R) \cap \mathcal{Q}_f(R) \cap \mathcal{Q}_l(R)$ approximates the infimum.

It is not hard to show that the sufficient conditions for the existence of MEMM from Theorem 3.2 are equivalent to (ES1) and (ES2).

Lemma 3.2. *For an arbitrary $\vartheta^* \in \Theta_{EXP}(R,\mathbb{P})$ the following conditions are equivalent.*

1. $\kappa(\vartheta^*) = 0$.
2. *It holds that* $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) : \vartheta \in \mathbb{R}\}$ *and* $\kappa(\vartheta^*) = 0$.
3. ϑ^* *satisfies (ES1) and (ES2) from [ES05], Theorem B.*

Proof. 2. $\to$ 3. Let $\vartheta_0 \in \Theta_{\mathrm{EXP}}(R,\mathbb{P})$ with $\varphi(\vartheta_0) = \inf\{\varphi(\vartheta) : \vartheta \in \mathbb{R}\}$. Suppose that $\vartheta_0 \neq \vartheta^*$. Without loss of generality, assume $\vartheta_0 < \vartheta^*$. The function φ is differentiable on $I = (\vartheta_0, \vartheta^*)$ and $\varphi'(\vartheta) = \kappa(\vartheta)$, $\vartheta \in I$. Due to minimality of φ in ϑ_0 it holds that $\kappa(\vartheta) > 0$ on I. Then again κ is an increasing function and $\kappa(\vartheta^*) = 0$, hence, $\kappa(\vartheta) < 0$ for $\vartheta \in I$. We obtain a contradiction.
2. $\to$ 3. It holds that $\vartheta^* \in \Theta^1_{\mathrm{EXP}}(R,\mathbb{P}) \subseteq \Theta_{\mathrm{EXP}}(R,\mathbb{P})$. Let us consider a measure change with Girsanov parameters $(\vartheta^*, e^{\vartheta^* x})_R$. We have

$$0 = \kappa(\vartheta^*) = \frac{\mathbb{E}(R_1 e^{\vartheta^* R_1})}{\mathbb{E}(e^{\vartheta^* R_1})} = \mathbb{E}^*(R_1),$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^{\vartheta^*} \in \mathcal{Q}_l(R)$. On the other hand,

$$\begin{aligned}
\mathbb{E}^*(R_1) &= \gamma^* + \int_{\mathbb{R}\setminus\{0\}} (x - h(x))\nu^*(\,dx) \\
&= \gamma + \sigma^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} (e^{\vartheta^* x} - 1)h(x)\nu(\,dx) + \int_{\mathbb{R}\setminus\{0\}} (x - h(x))e^{\vartheta^* x}\nu(\,dx) \\
&= \gamma + \sigma^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} (xe^{\vartheta^* x} - h(x))\nu(\,dx).
\end{aligned}$$

We obtain (ES2). Furthermore,

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}} |xe^{\vartheta^* x} - h(x)|\nu(\,dx) \leq \int_{\{|x|\geq 1\}} |x|e^{\vartheta^* x}\nu(\,dx) + \int_{\{|x|\geq 1\}} |h(x)|\nu(\,dx) \\
&+ \int_{\{|x|<1\}} |xe^{\vartheta^* x} - h(x)|\nu(\,dx).
\end{aligned}$$

The first term is by assumption finite, also the second, since h is a bounded function. $xe^{\vartheta^* x} - h(x)$ behaves like x^2 for $x \to 0$ and is therefore integrable. We obtain (ES1).

3. → 1. We have $\vartheta^* \in \Theta^1_{\mathrm{EXP}}(R, \mathbb{P})$, the expectation under $\mathbb{P}^{\vartheta^*}$ exists and it is

$$\kappa(\vartheta^*) = \mathbb{E}^*(R_1) = \gamma + \sigma^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} (xe^{\vartheta^* x} - h(x))\nu(\,dx) = 0.$$

□

4 MEMM for exponential Lévy processes

Suppose that the asset price $(S_t)_{t\in[0,T]}$ follows an exponential Lévy model $S_t = e^{X_t}, \quad t \in [0,T]$, for a real-valued Lévy process X. The next result gives justification for dealing with an arbitrary Lévy process, even if we would like to characterize the MEMM for the exponential Lévy process:
S can be represented as a stochastic exponential of another Lévy process R, $S = \mathscr{E}(R)$. Appendix A presents the notion of stochastic exponential and the required results. Theorem 4.1 yields that S is a martingale if and only if R is a martingale. Since the entropy process depends on the measure and not the process we consider, the MEMM is the same for both processes S and R.

Theorem 4.1. *Suppose S follows an exponential Lévy process $S = e^X$ for a Lévy process X on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}^X, \mathbb{P})$. It holds that*

1. *S has a representation as a stochastic exponential of a Lévy process R with $\Delta R_t > -1$ for all $t \in [0,T]$.*
2. *S is a (local) martingale if and only if R is a local martingale.*
3. *If S (resp. R) is a local martingale, then it is a martingale.*

Proof. 1. is discussed in Section A.1.
2. Suppose S is a local martingale. S is also a càd process, and it follows from Proposition 1.28, [JS87] that $\tau_n := \inf\{t \in [0,T] \,:\, S_t < \frac{1}{n}\}$, $n \in \mathbb{N}$, is a stopping time. We have $\tau_n \uparrow T$, $n \to \infty$, and $\frac{1}{S_{t\wedge\tau_n}}$ is bounded for all $n \in \mathbb{N}$. Hence, $\frac{1}{S_-}$ is a locally bounded , predictable process and due to 4.34 (b), [JS87], the stochastic integral is also a local martingale. The claim follows from the representation (A.3) of R. Suppose now that R is a local martingale. We apply the same arguments to the representation (A.1) of S.
3. It follows from [Sid79] that if the Lévy process R or the exponential Lévy process $S = e^X$ are local martingales then they are proper martingales. □

Using the dual representation $S = e^X = \mathscr{E}(R)$ and the correspondence between the characteristic triplets of X and R, which can be found in Lemma A.2, we translate our main results for the special case of exponential Lévy model. From Theorem 3.2 we know, that, if

MEMM exists, then it is the Esscher transform for the process R. We express the necessary and sufficient conditions for the existence of MEMM in terms of the characteristics of X.

Lemma 4.1. *Let X be a Lévy process, $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space and $S = e^X$. Suppose that the trajectories of S satisfy the Assumption A. There exist the MEMM for S if and only if there exists a $\vartheta^* \in \mathbb{R}$ satisfying*

$$\int_{\{x>1\}} e^x e^{\vartheta^*(e^x-1)} \nu(\,dx) < \infty \quad \textit{and} \tag{MEMM1}$$

$$0 = \gamma + \sigma^2 \left(\frac{1}{2} + \vartheta^* \right) + \int_{\mathbb{R}\setminus\{0\}} ((e^x - 1)e^{\vartheta^*(e^x-1)} - h(x))\nu(\,dx). \tag{MEMM2}$$

In this case, X is a Lévy process under $\mathbb{P}^{MEMM}$ with characteristic triplet $(\sigma^2, \nu^*, \gamma^*)$, given by

$$\nu^*(\,dx) = e^{\vartheta^*(e^x-1)}\nu(\,dx),$$
$$\gamma^* = \gamma + \sigma^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} h(x)(e^{\vartheta^*(e^x-1)} - 1)\nu(\,dx).$$

The infimum of the relative entropy is given by

$$I_t(\mathbb{P}^{MEMM} \,|\, \mathbb{P}) = -t \left(\gamma\vartheta^* + \frac{1}{2}\sigma^2\vartheta^*(\vartheta^* + 1) + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\vartheta^*(e^x-1)} - 1 - \vartheta^* h(x) \right) \nu(\,dx) \right).$$

Proof. We write S as a stochastic exponential of a Lévy process R. From the characteristic triplet of R it is easy to see, that R also satisfies the Assumption A (i.e., neither R nor $-R$ are subordinators). Theorem 3.2 and Lemma 3.2 provide necessary and sufficient conditions for the existence of the MEMM:

$$\kappa_R(\vartheta^*) = 0 \quad \text{for} \quad \vartheta^* \in \mathbb{R} \text{ with } \varphi_R(\vartheta^*) = \inf\{\varphi_R(\vartheta) \,:\, \vartheta \in \mathbb{R}\}$$

or, equivalently, the existence of a $\vartheta^* \in \Theta_{\mathrm{EXP}}(R, \mathbb{P})$ satisfying

$$\gamma_R + \sigma_R^2\vartheta^* + \int_{\mathbb{R}\setminus\{0\}} (xe^{\vartheta x} - h(x))\nu_R(\,dx) = 0,$$

where φ_R denotes the cumulant generating function of the Lévy process R under $\mathbb{P}$. If the MEMM for S exists, then it is the ESMM for the process R and its density is given by

$$\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}} = e^{\vartheta^* R_T - T\varphi_R(\vartheta^*)}.$$

Using the relation between the characteristic triplets of X and R, stated in Lemma A.2, we obtain (MEMM1) and (MEMM2). $\square$

5 Discussion

In conclusion we would like to compare our results with the results of Hubalek and Sgarra, [HS06] and discuss unclear steps in the proof of Theorem 8, [HS06]. For a better readability we summarize the notational difference in a table.

The statements of Theorems 8 and 9, [HS06], are comparable to our results in the special case of an exponential Lévy process. Proposition 4 in [HS06], used to prove Theorem 8, provides from our point of view an incomplete proof for the existence of the ESMM for the process R. Hubalek and Sgarra use the result from Theorem A, [ES05]: X is a Lévy process under $\mathbb{P}^{\mathrm{MEMM}}$ and the Girsanov parameters are given by $(\eta_0, y_0(x))_X$. They obtain $y_0(x) = e^{\vartheta^*(e^x-1)}$ and

$$\vartheta^* := \frac{1}{\beta}\int_B y_0(x)\ln y_0(x)\nu(\,dx), \tag{5.1}$$

The density of $\mathbb{P}^{\mathrm{MEMM}}$ with respect to $\mathbb{P}$ is given by

$$\begin{aligned}\frac{d\,\mathbb{P}^{\mathrm{MEMM}}}{d\,\mathbb{P}} &= \exp\Big\{\eta_0 W_T - \frac{\eta_0^2\sigma^2 T}{2} + \lim_{\epsilon\downarrow 0}\Big(\sum_{0<s\le T}\vartheta^*\big(e^{\Delta X_s}-1\big) - T\int_{\{|x|>\epsilon\}}\big(e^{\vartheta^*(e^x-1)}-1\big)\nu(\,dx)\Big)\Big\}\\ &= \exp\Big\{\vartheta^* R_T - \varphi_R(\vartheta^*)T + (\eta_0-\vartheta^*)W_T - \frac{1}{2}\sigma^2 T(\eta_0^2 - (\vartheta^*)^2)\Big\}\end{aligned}$$

and $(\eta_0, y_0(x))_X$ would correspond to the ESMM for R , if one can show that $\vartheta^* = \eta_0$. This is however not obvious. The assignment (5.1) does not provide a concrete value for ϑ^*:

$$\vartheta^* = \frac{\int_B \vartheta^* e^{\vartheta^*(e^x-1)}(e^x-1)\nu(\,dx)}{\int_B e^{\vartheta^*(e^x-1)}(e^x-1)\nu(\,dx)} \equiv \vartheta^*.$$

In [HS06] only exponential Lévy processes, $S = e^X$, are treated. In this case the exponential transform R of X has certain properties (e.g., the jumps are bounded from below, the cumulant generating function exists for $\vartheta < 0$) that simplify some of the steps of their proofs. It is also not obvious how these methods can be generalised for an arbitrary Lévy process X, therefore, our proofs do not base on the statements and steps from [HS06].

Notational differences:

Hubalek and Sgarra	Krol and Küchler	meaning
$\tilde{X}$	R	exponential transform of X
(c, U, b)	(σ^2, ν, γ)	the characteristic triplet of X
$(\psi_0, y(x))$	$(\eta_0, y(x))$	Girsanov parameters
$\tilde{\kappa}$	φ_R	cumulant generating function of $\tilde{X}$ and R respectively

A Exponential transforms and measure changes for Lévy processes

In this section we gather the required results on exponential transforms and measure changes for Lévy processes. For more details the interested reader is referred to the books by Jacod and Shiryaev, [JS87] and Cont and Tankov, [CT03]. Thoughout this section we consider a real-valued Lévy process X on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is the augmented natural filtration of X. (σ^2, ν, γ) denotes the characteristic triplet of X with respect to the truncation function $h(x) = x\mathbf{1}_{\{|x|\le 1\}}$.

A.1 Stochastic exponentials

Definition A.1 ([CT03], Definition 2.1)**.** *Let X be a real-valued semimartingale. The stochastic exponential $\mathscr{E}(X)$ is defined as the unique solution Z to the stochastic differential equation*

$$Z_t = 1 + \int_0^t Z_{u-}\, dX_u. \tag{A.1}$$

Z is given by

$$Z_t = \exp\left\{X_t - \frac{\sigma^2}{2}t\right\} \prod_{0<u\le t} (1 + \Delta X_u) e^{-\Delta X_u}, \quad t \in [0, T]. \tag{A.2}$$

The function $X \mapsto \mathscr{E}(X)$ can be inverted:

Lemma A.1 ([KS02], Lemma 2.2)**.** *Let Z be a semimartingale such that Z, Z_- are $\mathbb{R}\setminus\{0\}$-valued. Then there exists a unique semimartingale X such that $X_0 = 0$ and $Z = Z_0\mathscr{E}(X)$. It is given by*

$$X_t = \int_0^t \frac{1}{Z_{s-}}\, dZ_s, \quad t \in [0, T]. \tag{A.3}$$

X is called the stochastic logarithm of Z *and is denoted by $\mathscr{L}(Z) := X$.*

Definition A.2 ([KS02], Definition 2.5)**.** *For any real-valued semimartingale X with $X_0 = 0$, we call $R := \mathscr{L}(e^X)$ the* exponential transform of X. *Conversely, we call $X := \log(\mathscr{E}(R)$ the* logarithmic transform *of any real-valued semimartingale R with $R_0 = 0$ and $\Delta R_t > -1$.*

A very useful property is that X is a Lévy process if and only if R is a Lévy process (with $\Delta R_t > -1$). The next result gives the correspondence of the respective characteristic triplets.

Lemma A.2 ([KS02], Lemma 2.7). *The exponential transform R of a Lévy process X is again a Lévy process. Its characteristic triplet $(\sigma_R^2, \nu_R, \gamma_R)$ is given by*

$$\gamma_R = \gamma_X + \frac{\sigma_X^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (h(e^x - 1) - h(x))\nu_X(dx), \tag{A.4}$$

$$\sigma_R = \sigma_X, \tag{A.5}$$

$$\nu_R(dx) = (\nu_X \circ g^{-1})(dx). \tag{A.6}$$

Conversely, the logarithmic transform X of a Lévy process R with $\Delta R > 1$ and $(\sigma_R^2, \nu_R, \gamma_R)$ is again a Lévy process, with triplet (σ^2, ν, γ) as follows:

$$\gamma = \gamma_R - \frac{\sigma_R^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (h(\log(1+x)) - h(x))\nu_R(dx), \tag{A.7}$$

$$\sigma = \sigma_R, \tag{A.8}$$

$$\nu(dx) = (\nu_R \circ g)(dx), \quad \text{with } g(x) = e^x - 1. \tag{A.9}$$

A.2 Measure changes for Lévy processes

We now turn to the description of equivalent probability measures, such that X remains a Lévy process under the new measure.

Theorem A.1 ([CT03], Proposition 9.8). *Suppose $\mathbb{P}$ and $\mathbb{P}^*$ are two probability measures on $(\Omega, \mathscr{F})$ and X is a Lévy process under $\mathbb{P}$ and $\mathbb{P}^*$ with characteristic triplets (σ^2, ν, γ) and $((\sigma^*)^2, \nu^*, \gamma^*)$ respectively. The following statements are equivalent.*

1. *$\mathbb{P}^*|_{\mathscr{F}_t}$ and $\mathbb{P}|_{\mathscr{F}_t}$ are equivalent for at least one $t \in (0,T]$ (and hence for all $t \in [0,T]$).*
2. *The characteristics satisfy $\sigma = \sigma^*$, $\nu \sim \nu^*$, where $y(x)$, given by $\frac{d\nu^*}{d\nu} = y(x)$, satisfies*

$$\int_{\mathbb{R}\setminus\{0\}} (\sqrt{y(x)} - 1)^2 \nu(dx) < \infty. \tag{A.10}$$

If $\sigma = 0$ it must additionally hold that $\gamma^* - \gamma = \int_{\mathbb{R}\setminus\{0\}} h(x)(\nu^* - \nu)(dx)$.

Suppose (1) and (2) are satisfied. It holds that $\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{U_T}$ $\mathbb{P}$-a.s. for a Lévy process $U = (U_t)_{t\in[0,T]}$ with

$$U_t = \eta W_t - \frac{1}{2}\eta^2\sigma^2 t + \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} (\log y(x) - y(x) + 1) J_X(du, dx)$$
$$+ \int_{(0,t]}\int_{\mathbb{R}\setminus\{0\}} (y(x) - 1)\tilde{J}_X(du, dx),$$

where η satisfies

$$\gamma^* - \gamma - \int_{\mathbb{R}\setminus\{0\}} h(x)(\nu^* - \nu)(dx) = \sigma^2 \eta$$

and is equal to zero if $\sigma^2 = 0$. J_X and $\tilde{J}_X$ denote the jump and compensated jump measures of X respectively.

We also formulate a modification of the Girsanov theorem.

Theorem A.2. *Suppose $T > 0$, $\eta \in \mathrm{I\!R}$ and y a function $y : \mathrm{I\!R} \to [0, \infty)$ satisfying A.10 Then the process $N = (N_t)_{t\in[0,T]}$, given by*

$$N_t = \eta\sigma W_t + \int_0^t \int_{\mathrm{I\!R}\setminus\{0\}} (y(x) - 1)\tilde{J}_X(\,du,\,dx), \quad t \in [0, T],$$

is well-defined and $\frac{d\,\mathrm{I\!P}^*}{dP} = \mathscr{E}(N)_T$ defines a probability measure P^* on $(\Omega, \mathscr{F})$ with $\mathrm{I\!P}^* \ll \mathrm{I\!P}$. X is again a Lévy process under $\mathrm{I\!P}^*$, its characteristic triplet $((\sigma^*)^2, \nu^*, \gamma^*)$ is given by

$$\begin{aligned}
\sigma^* &= \sigma, \\
\nu^*(\,dx) &= y(x)\nu(\,dx), \\
\gamma^* &= \gamma + \int_{\mathrm{I\!R}\setminus\{0\}} h(x)(\nu^* - \nu)(\,dx) + \sigma^2\eta.
\end{aligned}$$

B Omitted proofs

The following lemma summarizes some of the properties of the cumulant generating function φ and presents a link between φ and κ.

Lemma B.1. *Let R be a Lévy process on a filtered probability space $(\Omega, \mathscr{F}, \mathrm{I\!F}, \mathrm{I\!P})$.*

1. *$\Theta_{EXP}(R, \mathrm{I\!P})$ is a non-empty interval.*
2. *For an arbitrary $\vartheta_0 \in \mathring{\Theta}_{EXP}(R, \mathrm{I\!P})$ it holds that $\vartheta_0 \in \Theta^1_{EXP}(R, \mathrm{I\!P})$ and $\varphi'(\vartheta_0) = \kappa(\vartheta_0)$.*

Suppose that the Lévy process R satisfies additionally the Assumption A. Then it holds that

3. *φ is a strictly convex function on $\Theta_{EXP}(R, \mathrm{I\!P})$ and $\varphi(\vartheta) \to \infty$ for $|\vartheta| \to \infty$.*
4. *There exists a unique $\vartheta^* \in \mathrm{I\!R}$ satisfying $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) : \vartheta \in \mathrm{I\!R}\}$.*

Proof.
1. The claim follows from $\varphi(0) = 0$ and the fact that $\Theta_{\mathrm{EXP}}(R, \mathrm{I\!P})$ is a convex set (see e.g. [DCD86], pages 126-128).
2. For an arbitrary $\vartheta_0 \in \mathring{\Theta}_{\mathrm{EXP}}(R, \mathrm{I\!P})$ there exists an $\epsilon > 0$ satisfying $(\vartheta_0 - 2\epsilon, \vartheta_0 + 2\epsilon) \subseteq$

$\mathring{\Theta}_{\text{EXP}}(R, \mathbb{P})$. We can choose $n_1, n_2 \in \mathbb{N}$ sufficiently large so that $x < e^{\epsilon x}$ for all $x > n_1$ and $|x| < e^{-\epsilon x}$ for all $x < -n_2$. We obtain

$$\int_{\{|x|>1\}} |x| e^{\vartheta_0 x} \nu(dx) \leq \int_{[-n_2, n_1]} |x| e^{\vartheta_0 x} \nu(dx) + \int_{\{x>n_1\}} e^{(\epsilon+\vartheta_0)x} \nu(dx) + \int_{\{x<-n_2\}} e^{(-\epsilon+\vartheta_0)x} \nu(dx).$$

The first integral is bounded from above by $(n_1 \vee n_2) \int_{\mathbb{R}\setminus\{0\}} e^{\vartheta_0 x} \nu(dx)$ and is finite. The latter integrals are also finite, since both $\vartheta_0 - \epsilon$ and $\vartheta_0 + \epsilon$ lie in the interval $\Theta_{\text{EXP}}(R, \mathbb{P})$. Hence, $\vartheta_0 \in \Theta^1_{\text{EXP}}(R, \mathbb{P})$. Moreover, φ is infinitely differentiable on $\mathring{\Theta}_{\text{EXP}}(R, \mathbb{P})$ and from $\mathbb{E}(R_1 e^{\vartheta R_1}) = \left(e^{\varphi(\vartheta)}\right)' = e^{\varphi(\vartheta)} \varphi'(\vartheta), \quad \vartheta \in \mathring{\Theta}_{\text{EXP}}(R, \mathbb{P})$, we obtain

$$\varphi'(\vartheta) = \frac{\mathbb{E}(R_1 e^{\vartheta R_1})}{\mathbb{E}(e^{\vartheta R_1})} = \kappa(\vartheta) \quad \text{on } \mathring{\Theta}_{\text{EXP}}(R, \mathbb{P}).$$

3. The process R satisfies the Assumption A, if and only if neither R nor $-R$ are subordinators ([CT03], Proposition 3.10). It holds that

$$\varphi''(\vartheta) = \sigma^2 + \int_{\mathbb{R}\setminus\{0\}} x^2 e^{\vartheta x} \nu(dx) > 0, \quad \vartheta \in \mathring{\Theta}_{\text{EXP}}(R, \mathbb{P}).$$

Hence, φ is a strictly convex function on the interior of $\Theta_{\text{EXP}}(R, \mathbb{P})$. If the boundary point of $\mathring{\Theta}_{\text{EXP}}(R, \mathbb{P})$ belongs to $\Theta_{\text{EXP}}(R, \mathbb{P})$ and φ is one-sided continuous in this point then φ is strictly convex also on $\Theta_{\text{EXP}}(R, \mathbb{P})$. Let ϑ_0 be without loss of generality the right boundary point of $\Theta_{\text{EXP}}(R, \mathbb{P})$. We consider the case $\vartheta_0 \geq 0$. The case $\vartheta_0 < 0$ is treated analogously. Define the function $v : \Theta_{\text{EXP}}(R, \mathbb{P}) \to \mathbb{R}$, by

$$v(\vartheta) = e^{\vartheta x} - 1 - \vartheta x \mathbf{1}_{\{|x|\leq 1\}}, \quad x \in \mathbb{R}\setminus\{0\}.$$

and let $(\vartheta_n)_{n\in\mathbb{N}} \subseteq \Theta_{\text{EXP}}(R, \mathbb{P})$ be an arbitrary sequence that converges to ϑ_0. The sequence $(v(\vartheta_n))_{n\in\mathbb{N}}$ is monotone increasing for $x > 0$ and monotone decreasing for $x < 0$. It holds that

$$v(\vartheta_0) \leq \left(e^{\vartheta_0 x} - 1 - \vartheta_0 x \mathbf{1}_{\{0<x\leq 1\}}\right) \mathbf{1}_{\{x>0\}} + \left(e^{\vartheta_1 x} - 1 - \vartheta_1 x \mathbf{1}_{\{-1<x\leq 0\}}\right) \mathbf{1}_{\{x<0\}} \in L^1(\nu).$$

The dominated convergence theorem yields that $\varphi(\vartheta_n) \to \varphi(\vartheta)$, hence the function φ is continuous in ϑ_0.

Let $I = \operatorname{supp}(\mathbb{P} \circ R_1^{-1})$. Due to Assumption A, It holds that $0 \in \mathring{I}$. Therefore, $\varphi(\vartheta) \to \infty$ for $|\vartheta| \to \infty$.

4. The statement is obvious for $\Theta_{\text{EXP}}(R, \mathbb{P}) = \{0\}$. Let $\Theta_{\text{EXP}}(R, \mathbb{P})$ be an interval. The uniqueness of the minimum point ϑ^* follows directly from the strict convexity of φ. The set $I = \{\vartheta \in \mathbb{R} : \varphi(\vartheta) \leq 1\} \supseteq \{0\}$ is bounded, due to $\varphi(\vartheta) \to \infty$ for $|\vartheta| \to \infty$. Let $a := \inf\{\varphi(\vartheta) : \vartheta \in \mathbb{R}\}$. Since φ is strictly convex, it holds that $a > -\infty$. We consider a sequence $(\vartheta_n)_{n\in\mathbb{N}} \subseteq I$ with $\varphi(\vartheta_n) \to a$. This sequence is bounded, so it has a subsequence (ϑ_{n_j}), that converges to a $\vartheta^* \in \mathbb{R}$. Fatou lemma yields

$$\mathbb{E}(e^{\vartheta^* R_1}) = \mathbb{E}(\lim_{n\to\infty} e^{\vartheta_{n_j} R_1}) \leq \liminf_{n\to\infty} \mathbb{E}(e^{\vartheta_{n_j} R_1}) = e^a < \infty.$$

Hence, $\vartheta^* \in \Theta_{\text{EXP}}(R, \mathbb{P})$. The function φ is continuous on $\Theta_{\text{EXP}}(R, \mathbb{P})$, so $\varphi(\vartheta_{n_j}) \to \varphi(\vartheta^*)$. The claim follows from the strict convexity of the cumulant generating function. □

B.1 Proof of Theorem 3.1

The lemma B.1 yields a unique $\vartheta^* \in \mathbb{R}$ with $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) : \vartheta \in \mathbb{R}\}$. First we investigate the case when ϑ^* lies in the interior of $\Theta_{\text{EXP}}(R, \mathbb{P})$ and hence, in $\Theta^1_{\text{EXP}}(R, \mathbb{P})$ It holds that $\varphi'(\vartheta^*) = \kappa(\vartheta^*) = 0$. The process $Z = (Z_t)_{t \in [0,T]}$, given by $Z_t = e^{\vartheta^* R_t - t\varphi(\vartheta^*)}$, $t \in [0, T]$, is a $\mathbb{P}$-martingale with $\mathbb{E}(Z_t) = 1$, hence, Z defines a density process of an equivalent measure $\mathbb{P}^*$. The Girsanov parameters are given by $(\vartheta^*, e^{\vartheta^* x})$, hence $\mathbb{P}^* \in \mathscr{Q}_e(R) \cap \mathscr{Q}_l(R)$.
The process of relative entropy of $\mathbb{P}^*$ with respect to $\mathbb{P}$ is given by

$$I_t(\mathbb{P}^* \,|\, \mathbb{P}) = \mathbb{E}^*(\vartheta^* R_t - t\varphi(\vartheta^*)) = \vartheta^* \,\mathbb{E}^*(R_t) - t\varphi(\vartheta^*) = -t\varphi(\vartheta^*), \quad t \in [0, T],$$

Hence,

$$\inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{Q}_a(R)\} \leq -t \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}, \quad \forall t \in [0, T].$$

We show that also the reverse inequality holds. Let $\mathbb{Q}$ be an arbitrary measure in $\mathscr{Q}_a(R)$. R is a local $\mathbb{Q}$-martingale, hence, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \uparrow T$, $n \to \infty$, such that $(R_{t\wedge\tau_n})_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale for all $n \in \mathbb{N}$. From [JS87], Theorem III.3.4(2) we obtain $\mathbb{P}^*_{\mathscr{F}_{t\wedge\tau_n}} \ll \mathbb{P}_{\mathscr{F}_{t\wedge\tau_n}}$ and it holds that

$$\frac{d\,\mathbb{P}^*\,|_{\mathscr{F}_{t\wedge\tau_n}}}{d\,\mathbb{P}\,|_{\mathscr{F}_{t\wedge\tau_n}}} = e^{\vartheta^* R_{t\wedge\tau_n} - (t\wedge\tau_n)\varphi(\vartheta^*)}, \quad t \in [0, T].$$

$\log\left(\frac{d\,\mathbb{P}^*\,|_{\mathscr{F}_{t\wedge\tau_n}}}{d\,\mathbb{P}\,|_{\mathscr{F}_{t\wedge\tau_n}}}\right)$ is $\mathbb{Q}$ integrable and from Lemma 2.1. in [FM03] we obtain:

$$\begin{aligned} I_t(\mathbb{Q}|\,\mathbb{P}) &\geq I_{t\wedge\tau_n}(\mathbb{Q}|\,\mathbb{P}) \geq \int \log\left(\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}}\Big|_{\mathscr{F}_{t\wedge\tau_n}}\right) d\mathbb{Q} \\ &= \vartheta^*\,\mathbb{E}^{\mathbb{Q}}(R_{t\wedge\tau_n}) - \mathbb{E}^{\mathbb{Q}}(t\wedge\tau_n)\varphi(\vartheta^*) = -\,\mathbb{E}^{\mathbb{Q}}(t\wedge\tau_n)\varphi(\vartheta^*) \\ &\to -t\inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\} \quad \text{as } n \to \infty, \end{aligned}$$

and the claim is proved.
Let us consider the more interesting case when ϑ^* is the boundary point of $\Theta_{\text{EXP}}(R, \mathbb{P})$. We shall construct a sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}} \subseteq \mathscr{Q}_a(R)$ those entropy processes approximate the infimum of relative entropy over $\mathscr{Q}_a(R)$. The jump measure ν of the Lévy process R has an unbounded support. Otherwise it would mean that $\Theta_{\text{EXP}}(R, \mathbb{P}) = \mathbb{R}$ and the infimum point ϑ^* would be an interior point of $\Theta_{\text{EXP}}(R, \mathbb{P})$. We assume without loss of generality that for all $n \in \mathbb{N}$, $\text{supp}(\nu) \cap [-n-1, n) \neq \emptyset$, as well as $\text{supp}(\nu) \cap (n, n+1] \neq \emptyset$ holds.
For all $n \in \mathbb{N}$ let us consider the following absolutely continuous measure changes given by the Girsanov parameters $(\eta_n, y_n(x))_R = (0, \mathbf{1}_{[-n,n]})$. It holds that

$$\frac{d\,\mathbb{P}_n}{d\,\mathbb{P}} = \mathscr{E}(N)_T \quad \text{with}$$

$$N_t = \int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} (y(x) - 1)\tilde{J}_R(\,du,\,dx) = -\int_{(0,t]} \int_{\mathbb{R}\setminus\{0\}} \mathbf{1}_{[-n,n]^c} J_R(\,du,\,dx) + t\nu([-n,n]^c)$$

Under the new measure R is again a Lévy process with the characteristic triplet $(\sigma^2, \nu|_{[-n,n]}, \gamma)$. We have

$$\begin{aligned}
\mathscr{E}(N)_t &= \exp\{N_t - \frac{1}{2}\langle N^c, N^c\rangle_t\} \prod_{0\le s\le t}(1+\Delta N_s)e^{-\Delta N_s} \\
&= e^{t\nu([-n,n]^c)} \prod_{0\le s\le t}(1+\Delta N_s), \quad t\in[0,T] \\
&= \begin{cases} 0 & \text{if } \Delta R_s(\omega) > n \text{ for at least one } 0 < s \le t, \\ e^{t\nu([-n,n]^c)} & \text{if } \Delta R_s(\omega) \le n \text{ for all } 0 < s \le t \end{cases} \\
&= \frac{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}} = \mathbf{1}_{B_t^n} e^{t\nu([-n,n]^c)}, \quad B_t^n := \{\omega\in\Omega \,:\, |\Delta R_s(\omega)| \le n,\ \forall\, 0 < s \le t\}.
\end{aligned}$$

It holds that $\Theta_{\text{EXP}}(R, \mathbb{P}_n) = \mathbb{R}$, and

$$\varphi_n(\vartheta) = \log \mathbb{E}_n(e^{\vartheta R_1}) = \log \int_{B_1^n} e^{\vartheta R_1}\, d\,\mathbb{P} - \nu([-n,n]^c).$$

The function φ is a strictly convex finite function, satisfying $\varphi_n(\vartheta)\to\infty$ for $|\vartheta|\to\infty$.
Let us show the existence of a subsequence (φ_{n_k}) with $\inf_{\vartheta\in\mathbb{R}}\varphi_n(\vartheta) \to \inf_{\vartheta\in\mathbb{R}}\varphi(\vartheta)$ for $n\to\infty$. The measure $\nu([-n,n]^c)$ is independent of ϑ and converges to zero as $n\to\infty$, hence, we can consider without loss of generality the following sequence

$$\xi_n(\vartheta) = \log \int_{B_1^n} e^{\vartheta R_1}\, d\,\mathbb{P}$$

and assume, that φ_n is strictly convex for all $n\in\mathbb{N}$ and $\varphi_n(\vartheta)\to\infty$ for $|\vartheta|\to\infty$. Then, ξ_n, $n\in\mathbb{N}$, is again a strictly convex, $\mathbb{R}$-valued function with $\xi_n(\vartheta)\to\infty$ for $|\vartheta|\to\infty$. The sequence $(\xi_n(\vartheta))_{n\in\mathbb{N}}$ is increasing for all $\vartheta\in\mathbb{R}$ and converges to $\varphi(\vartheta)$, in the following sense

$$\xi_n(\vartheta)\to\infty \text{ for } n\to\infty,\ \text{if } \varphi(\vartheta)=\infty, \qquad \xi_n(\vartheta)\to\varphi(\vartheta) \text{ for } n\to\infty,\ \text{if } \varphi(\vartheta)<\infty.$$

Let $\vartheta_n\in\mathbb{R}$ be such that $\xi_n(\vartheta_n) = \inf\{\xi_n(\vartheta) \,:\, \vartheta\in\mathbb{R}\}$. We show that there exists a subsequence $\xi_{n_k}(\vartheta_{n_k})$ with $\xi_{n_k}(\vartheta_{n_k}) \to \varphi(\vartheta^*) = \inf\{\varphi(\vartheta) \,:\, \vartheta\in\mathbb{R}\}$. We distinguish three cases: $\vartheta_n = \vartheta^*$ for infinitely many $n\in\mathbb{N}$ (in this case we are done), $\vartheta_n > \vartheta^*$ for infinitely many $n\in\mathbb{N}$ and $\vartheta_n < \vartheta^*$ for infinitely many $n\in\mathbb{N}$.
We consider the second case. Let $(\vartheta_n)_{n\in\mathbb{N}}$ be this subsequence. We first show that $(\vartheta_n)_{n\in\mathbb{N}}$ is bounded from above.
It holds that $\xi_1(\vartheta)\to\infty$ for $\vartheta\to\infty$, so there exists a $\bar\vartheta > \vartheta_1$ such that $\xi_1(\bar\vartheta) > \varphi(\vartheta^*)$. Hence, ξ_1 is monotone increasing for all $\vartheta > \vartheta_1$. It holds that $\vartheta_n \le \bar\vartheta$ for all $n\in\mathbb{N}$. Otherwise, there would exist an $n_0\in\mathbb{N}$ with $\vartheta_{n_0} > \bar\vartheta$ and

$$\xi_{n_0}(\vartheta^*) \ge \xi_{n_0}(\vartheta_{n_0}) \ge \xi_1(\vartheta_{n_0}) \overset{\vartheta_{n_0}>\bar\vartheta>\vartheta_1}{>} \xi_1(\bar\vartheta) > \varphi(\vartheta^*).$$

On the other hand, $\xi_{n_0}(\vartheta^*) \le \varphi(\vartheta^*)$, since $(\xi_n(\vartheta))_{n\in\mathbb{N}}$ is increasing and converges to $\varphi(\vartheta)$. Therefore, $(\vartheta_n)_{n\in\mathbb{N}} \subseteq [\vartheta^*, \bar\vartheta]$. Thus, the sequence $(\vartheta_n)_{n\in\mathbb{N}}$ has a convergent subsequence with a limit $\vartheta' \geq \vartheta^*$. For better notation we again denote this subsequence by $(\vartheta_n)_{n\in\mathbb{N}}$. Assume, it holds that $\vartheta' > \vartheta^*$. Then there exists a ϑ_0 with $\vartheta^* < \vartheta_0 < \vartheta_n < \vartheta'$ for all $n \in \mathbb{N}$ sufficiently large. Hence, ξ_n is strictly decreasing on $[\vartheta^*, \vartheta']$ and

$$\xi_n(\vartheta_0) < \xi_n(\vartheta^*), \quad \forall n \geq n_0.$$

Passing to the limit we obtain a contradiction to the minimality of φ in ϑ^*. Hence, ϑ' coincides with ϑ^*.

We now show that $(\xi_n(\vartheta_n))_{n\in\mathbb{N}}$ converges to $\varphi(\vartheta^*)$. For an arbitrary $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ it holds that $\varphi(\vartheta^*) - \xi_n(\vartheta^*) < \epsilon$ and specially $\xi_{n_0}(\vartheta^*) > \varphi(\vartheta^*) - \epsilon$. Moreover, we have for all $n \geq n_0$: $\xi_{n_0}(\vartheta_n) \leq \xi_n(\vartheta_n)$. There exists an $m_0 \in \mathbb{N}$ (w.l.o.g. $m_0 \geq n_0$), so that for all $m \geq m_0$ it holds that $\xi_{n_0}(\vartheta^*) - \epsilon < \xi_{n_0}(\vartheta_m) < \xi_{n_0}(\vartheta^*) + \epsilon$ Hence, for all $m \geq m_0$

$$\varphi(\vartheta^*) - 2\epsilon < \xi_{n_0}(\vartheta^*) - \epsilon < \xi_{n_0}(\vartheta_n) < \xi_n(\vartheta_n) < \xi_n(\vartheta^*) \leq \varphi(\vartheta^*).$$

Letting ϵ to zero proves the claim.

The density

$$\frac{d\,\mathbb{P}_n^*}{d\,\mathbb{P}_n} = e^{\vartheta_n R_T - T\varphi_n(\vartheta)}, \quad \mathbb{P}_n^* \sim \mathbb{P}_n \ll \mathbb{P},$$

defines a probability measure $\mathbb{P}_n^* \in \mathscr{Q}_a(R) \cap \mathscr{Q}_l(R)$. The relative entropy process is given by

$$I_t(\mathbb{P}_n^* \,|\, \mathbb{P}) = \int \log\left(\frac{d\,\mathbb{P}_n^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}}\right) d\,\mathbb{P}_n^* = \underbrace{\int \log\left(\frac{d\,\mathbb{P}_n^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}\right) d\,\mathbb{P}_n^*}_{-\varphi_n(\vartheta_n)t} + \int \log\left(\frac{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}}\right) d\,\mathbb{P}_n^*.$$

We show that the latter integral converges to zero as $n \to \infty$. It holds that

$$\begin{aligned}
\int_\Omega \log\left(\frac{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}}\right) d\,\mathbb{P}_n^* &= \int_\Omega \log\left(\frac{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}}\right) \frac{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}} \frac{d\,\mathbb{P}_n^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}\, d\,\mathbb{P} \\
&= \int_\Omega \log\left(\mathbf{1}_{B_t^n} e^{t\nu([-n,n]^c)}\right) \mathbf{1}_{B_t^n} e^{t\nu([-n,n]^c)} \frac{d\,\mathbb{P}_n^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}\, d\,\mathbb{P} \\
&= \int_\Omega t\nu([-n,n]^c) \mathbf{1}_{B_t^n} e^{t\nu([-n,n]^c)} \frac{d\,\mathbb{P}_n^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}_n\,|_{\mathscr{F}_t}}\, d\,\mathbb{P} \qquad (\log 0 \cdot 0 := 0) \\
&= t\nu([-n,n]^c)\, \mathbb{P}_n^*(B_t^n) \to 0 \quad \text{for } n \to \infty.
\end{aligned}$$

We obtain $\lim_{n\to\infty} I_t(\mathbb{P}_n^* \,|\, \mathbb{P}) = -t \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}$, $t \in [0,T]$, and hence,

$$\inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{Q}_a\} \leq -t \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}, \quad \forall t \in [0,T].$$

Now we show the inverse inequality. The steps of this proof are similar to those in Theorem 3.1, [FM03]. Let us fix a $t \in [0,T]$, and $\mathbb{Q}_0 \in \mathscr{Q}_a(R)$ with $I_t(\mathbb{Q}_0|\,\mathbb{P}) < \infty$. For every $n \in \mathbb{N}$ we define again $\mathbb{P}_n$, $\mathbb{P}_n^*$, ϑ_n, ϑ^* as before, and denote for the simplicity of notation as $\mathbb{P}, \mathbb{P}_n, \mathbb{Q}_0$

the restrictions of these measures on $\mathscr{F}_t$. There exits an $n_0 \in \mathbb{N}$ such that $\mathbb{Q}_0(B_t^n) > 0$ for all $n \geq n_0$. Let us define another sequence of measures $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathscr{F}_t)$ by

$$\mathbb{Q}_n(A) := \mathbb{Q}_0(A \mid B_t^n), \quad A \in \mathscr{F}_t.$$

Let $A \in \mathscr{F}_t$ be an arbitrary set with $\mathbb{P}_n(A) = 0$ and hence, $\mathbb{P}_n(B_t^n \cap A) = 0$. Since $\mathbb{Q}_0 \ll \mathbb{P}$, it also holds that $\mathbb{Q}_0(B_t^n \cap A) = 0$, i.e., $\mathbb{Q}_n(A) = 0$. We have $\mathbb{Q}_n \ll \mathbb{P}_n$. The dominated convergence theorem yields

$$\frac{d\mathbb{Q}_n}{d\,\mathbb{P}_n} = \frac{e^{-t\nu([-n,n]^c)}}{\mathbb{Q}_0(B_t^n)} \frac{d\mathbb{Q}_0}{d\,\mathbb{P}}.$$

$$\begin{aligned}
I_t(\mathbb{Q}_n |\, \mathbb{P}_n) &= \frac{1}{\mathbb{Q}_0(B_t^n)} \int_{B_t^n} \log\left(\frac{d\mathbb{Q}_n}{d\,\mathbb{P}_n}\right) d\mathbb{Q}_0 \\
&= \frac{1}{\mathbb{Q}_0(B_t^n)} \int_{B_t^n} \log\left(\frac{e^{-t\nu([-n,n]^c)}}{\mathbb{Q}_0(B_t^n)} \frac{d\mathbb{Q}_0}{d\,\mathbb{P}}\right) d\mathbb{Q}_0 \\
&= \frac{1}{\mathbb{Q}_0(B_t^n)} \int_{B_t^n} \log\left(\frac{d\mathbb{Q}_0}{d\,\mathbb{P}}\right) d\mathbb{Q}_0 - [\underbrace{\log(\mathbb{Q}_0(B_t^n))}_{\to 0} + \underbrace{t\nu([-n,n]^c)}_{\to 0}] \to I_t(\mathbb{Q}_0 |\, \mathbb{P}).
\end{aligned}$$

With the same arguments we have for all $m \in \mathbb{N}$, $t \in [0,T]$

$$\int_\Omega R_{t\wedge\tau_m}\, d\mathbb{Q}_n = \frac{1}{\mathbb{Q}_0(B_t^n)} \int_{B_t^n} R_{t\wedge\tau_m}\, d\mathbb{Q}_0 \overset{n\to\infty}{\longrightarrow} \int_\Omega R_{t\wedge\tau_m}\, d\mathbb{Q}_0 = 0.$$

The random variable

$$\log\left(\frac{d\,\mathbb{P}_n^*}{d\,\mathbb{P}_n}\Big|_{\mathscr{F}_{t\wedge\tau_m}}\right) = \vartheta_n R_{t\wedge\tau_m} - (t\wedge\tau_m)\varphi_n(\vartheta_n)$$

is $\mathbb{Q}_n$-integrable for an arbitrary but fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}$ since it holds that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_n}(|\vartheta_n R_{t\wedge\tau_m} - (t\wedge\tau_m)\varphi_n(\vartheta_n)|) &\leq |\vartheta_n|\, \mathbb{E}^{\mathbb{Q}_n}(|R_{t\wedge\tau_m}|) + |\varphi_n(\vartheta_n)|\, \mathbb{E}^{\mathbb{Q}_n}(|t\wedge\tau_m|) \\
&\leq |\vartheta_n| \frac{1}{\mathbb{Q}_0(B_t^n)}\, \mathbb{E}^{\mathbb{Q}_0}(|R_{t\wedge\tau_m}|) + t|\varphi_n(\vartheta_n)| < \infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\underbrace{I_t(\mathbb{Q}_n |\, \mathbb{P}_n)}_{\to I_t(\mathbb{Q}_0 |\, \mathbb{P})} &\geq I_{t\wedge\tau_m}(\mathbb{Q}_n |\, \mathbb{P}_n) \\
&\geq \int \log\left(\frac{d\,\mathbb{P}_n^*}{d\,\mathbb{P}_n}\Big|_{\mathscr{F}_{t\wedge\tau_m}}\right) d\mathbb{Q}_n \\
&= \underbrace{\vartheta_n}_{\to\vartheta^*} \underbrace{\mathbb{E}^{\mathbb{Q}_n}(R_{t\wedge\tau_m})}_{\to 0} - \underbrace{\mathbb{E}^{\mathbb{Q}_n}(t\wedge\tau_m)}_{\to \mathbb{E}^{\mathbb{Q}_0}(t\wedge\tau_m)} \underbrace{\varphi_n(\vartheta_n)}_{\to\varphi(\vartheta^*)}.
\end{aligned}$$

We obtain

$$I_t(\mathbb{Q}_0 |\, \mathbb{P}) \geq -\varphi(\vartheta^*)\, \mathbb{E}^{\mathbb{Q}_0}(t\wedge\tau_m) \to -\varphi(\vartheta^*)t = -t\inf\{\varphi(\vartheta) \;:\; \vartheta \in \mathbb{R}\} \quad \text{for } m \to \infty.$$

The reverse inequality follows and the theorem is proved. □

C Proof of Theorem 3.2

Before proving this result we need to show that a Lévy process satisfying Assumption A, also satisfies the assumptions of Theorem A, [ES05].

Theorem C.1. *Let R be a real-valued Lévy process on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with a characteristic triplet (σ^2, ν, γ) satisfying Assumption A. Then there exists a probability measure $\mathbb{Q}$ with $\mathbb{Q} \in \mathscr{Q}_e(R) \cap \mathscr{Q}_f(R) \cap \mathscr{Q}_l(R)$.*

Proof. The proof can be carried out similarly to the proof of Theorem 7 in [HS06]. Hubalek and Sgarra show the corresponding result for the exponential Lévy process. □

We will also make use of the following result.

Theorem C.2 ([CM03], Theorem 2.2.)**.** *Let Y be a random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying $0 \in \mathit{supp}\,(\mathbb{P} \circ Y^{-1})$ and*

$$\mathscr{E} = \{\mathbb{Q} \in \mathscr{M}_1(\Omega, \mathscr{F}) \,:\, \mathbb{Q} \ll \mathbb{P},\, \mathbb{E}^{\mathbb{Q}}(|Y|) < \infty,\, \mathbb{E}^{\mathbb{Q}}(Y) = 0\}.$$

1. *It holds that $\inf\{I(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{E}\} = -\inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}$.*
2. *If there exists a $\vartheta^* \in \mathbb{R}$ with $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}$ and $\kappa(\vartheta^*) = 0$, then the infimum of $I(\mathbb{Q}|\,\mathbb{P})$ over $\mathscr{E}$ is attained at the unique probability measure $\mathbb{P}^* \in \mathscr{E}$ and its density is given by*

$$\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}} = \mathit{const}\, e^{\vartheta^* Y}.$$

Otherwise, the infimum of $I(\mathbb{Q}|\,\mathbb{P})$ over $\mathscr{E}$ is not attained.

C.1 Proof of Theorem 3.2

Existence: Suppose $\vartheta^* \in \mathbb{R}$ satisfies $\varphi(\vartheta^*) = \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}$ and $\kappa(\vartheta^*) = 0$.

$$\frac{d\,\mathbb{P}^*}{d\,\mathbb{P}} = e^{\vartheta^* R_T - T\varphi(\vartheta^*)}$$

defines an equivalent martingale measure $\mathbb{P}^*$. R is a Lévy process under this measure and it holds that

$$I_t(\mathbb{P}^* \,|\, \mathbb{P}) = \mathbb{E}^* \left(\log \left(\frac{d\,\mathbb{P}^*\,|_{\mathscr{F}_t}}{d\,\mathbb{P}\,|_{\mathscr{F}_t}} \right) \right) = \mathbb{E}^*(\vartheta^* R_t - t\varphi(\vartheta^*)) = -t\varphi(\vartheta^*).$$

Theorem 3.1 yields that $\mathbb{P}^*$ is the MEMM.

Non-existence: Suppose $\kappa(\vartheta^*) \neq 0$ or $\kappa(\vartheta^*)$ does not exist. Assume, that the MEMM $\mathbb{P}^{\text{MEMM}}$ exists. Theorem C.1 yields that $\mathscr{Q}_e(R) \cap \mathscr{Q}_f(R) \cap \mathscr{Q}_l(R) \neq \emptyset$, hence, R is a Lévy process under $\mathbb{P}^{\text{MEMM}}$ (Theorem A, [ES05]) and a proper martingale. For an arbitrary $t \in (0, T]$ define the following set

$$\mathscr{E}_t = \{\mathbb{Q} \in \mathscr{M}_1(\Omega, \mathscr{F}_t) \,:\, \mathbb{Q} \ll \mathbb{P}\,|_{\mathscr{F}_t},\, R_t \in L^1(\mathbb{Q}),\, \mathbb{E}^{\mathbb{Q}}(R_t) = 0\}.$$

It holds that $\mathbb{P}^{\text{MEMM}}\,|_{\mathscr{F}_t} \in \mathscr{E}_t$ and it follows from Theorem C.2

$$\inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{E}_t\} = -t \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}. \tag{C.1}$$

Theorem 3.1 provides the following representation of the relative entropy:

$$I_t(\mathbb{P}^{\text{MEMM}}\,|\,\mathbb{P}) = -t \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\} \overset{\text{(C.1)}}{=} \inf\{I_t(\mathbb{Q}|\,\mathbb{P}) \,:\, \mathbb{Q} \in \mathscr{E}_t\}.$$

Hence, $\mathbb{P}^{\text{MEMM}}\,|_{\mathscr{F}_t}$ minimizes the relative entropy over $\mathscr{E}_t$. Theorem C.2 yields

$$\frac{\mathbb{E}(R_t e^{\vartheta^* R_t})}{\mathbb{E}(e^{\vartheta^* R_t})} = 0 \quad \text{for} \quad \vartheta^* \in \mathbb{R} \text{ with } \varphi(\vartheta^*) = \inf\{\varphi(\vartheta) \,:\, \vartheta \in \mathbb{R}\}.$$

Hence, $\kappa(\vartheta^*) = 0$, we obtain a contradiction. □

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