A COMBINED BDF-SEMISMOOTH NEWTON APPROACH FOR TIME-DEPENDENT BINGHAM FLOW*

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Abstract. This paper is devoted to the numerical simulation of time-dependent convective Bingham flow in cavities. Motivated by a primal-dual regularization of the stationary model, a family of regularized time-dependent problems is introduced. Well posedness of the regularized problems is proved and convergence of the regularized solutions to a solution of the original multiplier system is verified. For the numerical solution of each regularized multiplier system, a fully-discrete approach is studied. A stable finite element approximation in space, together with a second order backward differentiation formula for the time discretization are proposed. The discretization scheme yields a system of Newton differentiable nonlinear equations in each time step, for which a semismooth Newton algorithm is utilized. We present two numerical experiments to verify the main properties of the proposed approach.

1. Introduction

The numerical simulation of viscoplastic materials has recently received an increasing amount of attention due to its importance in several industrial processes and natural phenomena. The Bingham model, in particular, is widely used in food industry (production of sauces and pastes) and in other important fields of application (see e.g. [4, 17, 20]).

The main characteristic of a Bingham flow is given by the presence of a yield stress: the material behaves like solid in regions where the stresses are small and like an incompressible fluid in regions where the stresses go beyond a given threshold. Since no exact information of the solid-liquid zones is known beforehand, the numerical simulation of such materials becomes challenging.

Based on a variational inequality formulation of the problem, several methods have been proposed for its numerical solution. The methodologies can be broadly classified into two families: direct *global regularization* and *multiplier approaches*.

Direct global regularization techniques replace the nondifferentiable term (related to the presence of the plasticity threshold) by a C^∞-approximation. In that manner, the problem changes of nature to a partial differential equation and many known numerical techniques may be applied (see [12] and the references therein). As the regularization parameter vanishes, however, important analytical and numerical drawbacks may come into play (see [9]).

Multiplier approaches, on the other hand, are based on a reformulation of the modelling variational inequality as a system involving a Lagrange multiplier. The Uzawa method [16], the augmented Lagrangian method and, more recently, a penalty-Newton-Uzawa-CG method [9] have been proposed to cope with such systems, mainly in the stationary case. For the time dependent problem, a suitable time discretization is usually proposed, which makes it possible to exploit the efficiency of the methods developed for the stationary problem. Particular members of

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the latter are operator splitting methods (see [9, 24]), which in addition look after an uncoupled system in order to avoid problems related to the incompressibility condition and the convective term.

In this paper, utilizing a multiplier approach and based on the efficiency of the semismooth Newton method (**SSN**) for the numerical solution of the stationary Bingham model (see [8, 7]), we propose a solution algorithm for the time-dependent flow.

To exploit the efficiency of the semismooth Newton method applied to the stationary problem, a suitable time discretization scheme has to be considered in order to deal with the nonlinear convective term. By means of a second order backward differentiation formula (**BDF**) we are able to obtain an appropriate nonlinear system in each time step, for which the superlinear convergent algorithm developed in [7] can be applied.

The propsed algorithm is based on a family of regularized problems, which are motivated by a Tikhonov regularization of the stationary problem presented in [7]. Differently from [7], however, the well-posedess cannot be justified by means of an auxiliar optimization problem. Instead, monotonicity and compactness techniques are considered. Also the convergence of the regularized solutions and multipliers is proved.

The outline of the paper is a follows. In Section 2 the Bingham model is presented and a characterization of it in form of a variational inequality of the second kind and of a multiplier system are given. In Section 3 a family of regularized problems for the approximation of the original model is proposed. The well-posedness of the regularized problems and its convergence properties are investigated. A full discretization scheme for the numerical approximation of each regularized problem is given in Section 4. A finite element space discretization, together with a backward differentiation formula for the time discretization and a semismooth Newton method for the solution of the nonlinear systems arising in each time step, are proposed. Finally, in Section 5 two detailed numerical experiments are presented. In particular, the property of steady state in finite time, which typically fails when C^∞-regularization approaches are used, is verified experimentally.

2. Time-dependent Bingham model

Let Ω be an open bounded domain of $\mathbb{R}^d$, $d = 2$, with Lipschitz boundary Γ. Let $T > 0$, $Q := \Omega\times]0,T[$ and $\Sigma := \Gamma\times]0,T[$. Throughout this paper we will use the notation $L^p(W)$, $1 \le p \le \infty$, for the spaces

$$L^p(0,T;W) := \left\{ f : [0,T] \to\ W\ :\ \int_0^T \|f(t)\|_W^p\, dt < \infty \right\},$$

where W is a Banach space. These spaces are endowed with the norm

$$\|f\|_{L^p(W)} := \left(\int_0^T \|f(t)\|_W^p\, dt \right)^{\frac{1}{p}}.$$

Throughout, $\mathbf{L}^2(\Omega)$ stands for the product space $(L^2(\Omega))^d$ and $\mathbf{H}^1(\Omega)$ for $(H^1(\Omega))^d$, where $H^1(\Omega)$ corresponds to the usual Sobolev space.

The norm in a Banach space X is denoted by $\|\cdot\|_X$ and the duality product between X' and X by $\langle\cdot,\cdot\rangle_{X',X}$. If X is a Hilbert space, we denote by $(\cdot,\cdot)_X$ its scalar product. Further, $\langle\cdot\,,\cdot\rangle$ and $|\cdot|$ stand for the Euclidian inner product and its associated norm, respectively.

The Frobenius scalar product in $\mathbb{R}^{d\times d}$ and its associated norm are defined by

$$(A : B) := \mathrm{tr}(AB^\top) \text{ and } \|A\| := \sqrt{(A : A)}, \quad \text{for } A, B \in \mathbb{R}^{d\times d},$$

where tr stands for the trace of the matrix. Further, we define $\mathbf{L}^{2\times 2}(\Omega)$ as the space of (2×2)-matrices of $L^2(\Omega)$-functions. We endow this space with the norm $\|\mathbf{p}\|_{\mathbf{L}^{2\times 2}}$, which is induced by the following scalar product

$$(\mathbf{p}\,,\,\mathbf{q})_{\mathbf{L}^{2\times 2}} := \int_\Omega (\mathbf{p}(x) : \mathbf{q}(x))\,dx. \tag{2.1}$$

Since $\mathbf{L}^{2\times 2}(\Omega)$ endowed with the scalar product (2.1) is isomorph to the space $(L^2(\Omega))^{d\times d}$ endowed with the usual $\mathbf{L}^2(\Omega)$-scalar product (see [7, Sec. 5.1]), it constitutes a Hilbert space.

Let $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support. The space $\mathcal{V}$ is defined by

$$\mathcal{V} := \left\{\mathbf{v} \in (\mathcal{D}(\Omega))^d \,:\, \operatorname{div}\mathbf{v} = 0\right\},$$

and we define the spaces

$$V := \text{closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\Omega) \quad \text{and} \quad H := \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega).$$

Under the requirements on Ω, we can represent V and H also in the following form (see [26, pg. 13])

$$\begin{aligned} V &= \left\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \,:\, \operatorname{div}\mathbf{v} = 0\right\}, \\ H &= \left\{\mathbf{w} \in \mathbf{L}^2(\Omega) \,:\, \operatorname{div}\mathbf{w} = 0\ ,\ \mathbf{w}\cdot\vec{n}\,|_\Gamma = 0\right\}, \end{aligned}$$

where $\vec{n}$ denotes the unit outward normal along the boundary. Throughout the paper we additionally use the notation $\mathbf{u}(t) := \mathbf{u}(x,t)$.

The time-dependent Bingham flow is modeled by the following boundary value-problem: find a velocity field $\mathbf{y} : Q \to \mathbb{R}^d$ and a scalar pressure field $p : Q \to \mathbb{R}$ such that

$$\begin{cases} \partial_t \mathbf{y}(t) + (\mathbf{y}(t)\cdot\nabla)\mathbf{y}(t) = \operatorname{Div}\boldsymbol{\sigma}(t) - \nabla p(t) + \mathbf{f}(t) \text{ in } Q, \\ \operatorname{div}\mathbf{y}(t) = 0, \\ \mathbf{y} = \mathbf{0} \ \text{ on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0, \\ \boldsymbol{\sigma}(t) = 2\mu\mathcal{E}\mathbf{y}(t) + \sqrt{2}g\frac{\mathcal{E}\mathbf{y}(t)}{\|\mathcal{E}\mathbf{y}(t)\|}, \quad \text{if } \ \mathcal{E}\mathbf{y}(t) \neq 0, \\ \|\boldsymbol{\sigma}(t)\| \leq g, \ \text{ if } \ \ \mathcal{E}\mathbf{y}(t) = 0, \end{cases} \tag{2.2}$$

where $\mu > 0$ stands for the viscosity coefficient, $g > 0$ for the plasticity threshold (yield stress), $\mathbf{f}$ is a body force and Div is the row-wise divergence operator. The deviatoric part of the Cauchy stress tensor is denoted by $\boldsymbol{\sigma}$ and $\mathcal{E}$ stands for the deformation or rate of strain tensor, whose components are given by

$$\mathcal{E}_{ij}(\mathbf{y}) := \frac{1}{2}\left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i}\right), \ \text{ for } \mathbf{y} = (y_1, \cdots, y_d)^\top.$$

In order to obtain a solution to system (2.2) a-priori information about the regions where the material behaves as a rigid body or as a viscoplastic flow has to be at hand. Since in most of the cases this is not possible, an equivalent formulation of the problem in form of a parabolic variational inequality of the second kind (see [10, 9]) is usually studied.

2.1. **Variational inequality.** A weak formulation of problem (2.2) is given in the following way: find $\mathbf{y}(t) \in V$ a.e. on $(0,T)$ such that

$$\begin{aligned} &(\partial_t \mathbf{y}(t)\,,\,\mathbf{v} - \mathbf{y}(t))_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}(t), \mathbf{v} - \mathbf{y}(t)) + \mathbf{c}(\mathbf{y}(t), \mathbf{y}(t), \mathbf{v} - \mathbf{y}(t)) \\ &\qquad + \widetilde{g}\mathbf{j}(\mathbf{v}) - \widetilde{g}\mathbf{j}(\mathbf{y}(t)) \geq (\mathbf{f}\,,\,\mathbf{v} - \mathbf{y}(t))_{\mathbf{L}^2}\,, \text{ for all } \mathbf{v} \in V, \\ &\qquad \mathbf{y}(0) = \mathbf{y}_0, \end{aligned} \tag{$\boldsymbol{\mathcal{P}}$}$$

where $\widetilde{g} := \sqrt{2}g$, $\partial_t \mathbf{y}(t) := \frac{\partial \mathbf{y}(t)}{\partial t}$ and

$$\mathbf{a}(\mathbf{y},\mathbf{v}) := 2\mu \int_\Omega \mathcal{E}\mathbf{y} : \mathcal{E}\mathbf{v}\, dx, \tag{2.3}$$

$$\mathbf{j}(\mathbf{v}) := \int_\Omega \|\mathcal{E}\mathbf{v}\|\, dx \tag{2.4}$$

$$\mathbf{c}(\mathbf{y},\mathbf{v},\mathbf{u}) := \int_\Omega \langle (\mathbf{y}\cdot\nabla)\,\mathbf{v}\ ,\ \mathbf{u}\rangle\, dx. \tag{2.5}$$

Problem $(\boldsymbol{\mathcal{P}})$ corresponds to the variational formulation of system (2.2), and it was first presented in [10, Sec. 5].

Remark 2.1. Korn's inequality (see e.g. [5, Cor. 11.2.22]) implies the existence of a positive constant α_0 such that

$$\mathbf{a}(\mathbf{u},\mathbf{u}) \geq \alpha_0 \|\mathbf{u}\|^2_{\mathbf{H}^1_0},\ \text{for all } \mathbf{u}\in\mathbf{H}^1_0(\Omega).$$

Thus, we conclude that $\mathbf{a}(\cdot,\cdot)$ is a bilinear and coercive form in $\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)$.

Furthermore, it is known that $\mathbf{c}(\cdot,\cdot,\cdot)$ is a trilinear, continuous form in $\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)$ (see [26, pg. 109]). This fact implies the existence of a positive constant κ such that

$$\mathbf{c}(\mathbf{u},\mathbf{v},\mathbf{w}) \leq \kappa \|\mathbf{u}\|_{\mathbf{H}^1_0} \|\mathbf{v}\|_{\mathbf{H}^1_0} \|\mathbf{w}\|_{\mathbf{H}^1_0},\ \text{for all } \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}^1_0(\Omega).$$

Next, an existence theorem for problem $(\boldsymbol{\mathcal{P}})$ is stated.

Theorem 2.1. *Let $\mathbf{f}\in L^2(V')$ and $\mathbf{y}_0\in H$. There exists a unique solution $\overline{\mathbf{y}}\in L^2(V)$ to $(\boldsymbol{\mathcal{P}})$, such that $\partial_t\overline{\mathbf{y}}\in L^2(V')$. Moreover, if $\Omega\subset\mathbb{R}^2$ is of class $\mathcal{C}^2$ and $\mathbf{f}$, $\mathbf{y}_0$ satisfy*

$$\mathbf{f},\ \partial_t\mathbf{f}\in L^2(V'),\ \mathbf{f}(0)\in H \ \textit{and}\ \mathbf{y}_0\in V\cap\mathbf{H}^2(\Omega), \tag{2.6}$$

then, problem $(\boldsymbol{\mathcal{P}})$ has a unique solution $\overline{\mathbf{y}}\in C(0,T;V)$, with $\partial_t\overline{\mathbf{y}}\in L^2(V)\cap L^\infty(H)$.

Proof. See [10, Chap. VI, Thm. 3.1 and Thm 4.1]. □

2.2. **Multiplier approach.** From the variational inequality $(\boldsymbol{\mathcal{P}})$ an alternative characterization of the solution using a multiplier approach can be obtained. With such a characterization, a partial differential equation involving a multiplier, together with additional complementarity relations, is obtained.

In the following theorem the existence of such a multiplier is stated and a nonlinear system of equations, which characterize both the velocity field and the multiplier, is presented.

Theorem 2.2. *There exists a function $\mathbf{q}(t)\in\mathbf{L}^{2\times2}(\Omega)$ a.e. in $(0,T)$ which satisfies, together with $\overline{\mathbf{y}}\in L^2(V)$, the following system of equations*

$$\begin{aligned}(\partial_t\overline{\mathbf{y}}(t)\ \mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\overline{\mathbf{y}}(t),\mathbf{v}) + (\mathbf{q}(t)\ \mathcal{E}\mathbf{v})_{\mathbf{L}^{2\times2}}&\\ +\mathbf{c}(\overline{\mathbf{y}}(t),\overline{\mathbf{y}}(t),\mathbf{v}) &= (\mathbf{f}(t)\ ,\ \mathbf{v})_{\mathbf{L}^2}\ , \textit{ for all } \mathbf{v}\in V\end{aligned} \tag{2.7a}$$

$$\|\mathbf{q}(x,t)\| \leq \widetilde{g} \textit{ a.e. in } Q \tag{2.7b}$$

$$(\mathbf{q}(x,t):\mathcal{E}\overline{\mathbf{y}}(x,t)) = \widetilde{g}\|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \textit{ a.e. in } Q \tag{2.7c}$$

$$\overline{\mathbf{y}}(0) = \mathbf{y}_0. \tag{2.7d}$$

Proof. See [10, Chap. VI, Thm. 9.1] and [9, Thm. 2, p. 38]. □

The active and inactive sets for (2.7) are respectively defined by

$$\boldsymbol{\mathcal{A}} := \{(x,t) \in Q \,:\, \|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \neq 0\} \qquad \text{and} \qquad \boldsymbol{\mathcal{I}} := \Omega \setminus \boldsymbol{\mathcal{A}},$$

and they correspond to the regions where the material behaves like incompressible fluid and like solid, respectively.

System (2.7) represents an equivalent formulation of problem ($\boldsymbol{\mathcal{P}}$). Since the multiplier $\mathbf{q}$ is not necessarily unique (see [10, 9]), numerical instabilities in the approximation of system (2.7) may occur.

3. Regularized problems

To overcome the ill-posedness of system (2.7), we approximate its solution by solving a sequence of regularized systems. The proposed family of regularized problems is motivated by a Tikhonov regularization of the stationary Bingham model studied in [7, 8]. In those references the numerical behavior of such a regularization together with the application of semismooth Newton methods was investigated, and a comparison with other up-to-date methods was carried out.

For a parameter $\gamma > 0$, the regularized problem consists in: find $\mathbf{y}_\gamma(t) \in V$ and $\mathbf{q}_\gamma(t) \in \mathbf{L}^{2\times 2}(\Omega)$ a.e. in $(0,T)$ such that

$$\begin{aligned}(\partial_t \mathbf{y}_\gamma(t)\ \mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}_\gamma(t), \mathbf{v}) + (\mathbf{q}_\gamma(t)\,,\, \mathcal{E}\mathbf{v})_{\mathbf{L}^{2\times 2}} \\ +\mathbf{c}(\mathbf{y}_\gamma(t), \mathbf{y}_\gamma(t), \mathbf{v}) = (\mathbf{f}(t)\,,\, \mathbf{v})_{\mathbf{L}^2}\,, \text{ for all } \mathbf{v} \in V,\end{aligned} \tag{3.1a}$$

$$\mathbf{q}_\gamma(x,t) = \begin{cases} \widetilde{g}\frac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|} \text{ a.e. in } \boldsymbol{\mathcal{A}}_\gamma \\ \gamma \mathcal{E}\mathbf{y}_\gamma(x,t) \text{ a.e. in } \boldsymbol{\mathcal{I}}_\gamma, \end{cases} \tag{3.1b}$$

$$\mathbf{y}_\gamma(x,0) = \mathbf{y}_0. \tag{3.1c}$$

Here, the active and inactive sets for system (3.1) are respectively defined by

$$\boldsymbol{\mathcal{A}}_\gamma := \{(x,t) \in Q \,:\, \gamma\|\mathcal{E}\mathbf{y}_\gamma(x,t)\| \geq \widetilde{g}\} \qquad \text{and} \qquad \boldsymbol{\mathcal{I}}_\gamma := Q \setminus \boldsymbol{\mathcal{A}}_\gamma.$$

Note that, by definition, $\|\mathbf{q}_\gamma(x,t)\| \leq \widetilde{g}$ a.e. in Q.

It is easy to see that we can rewrite equation (3.1b) as follows:

$$\mathbf{q}_\gamma(x,t) = \widetilde{g}\gamma \frac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\max\left(\widetilde{g}, \gamma\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|\right)}, \text{ a.e. in } Q. \tag{3.2}$$

Next, we define the operator $\mathbf{b} : V \to\ \mathbf{L}^{2\times 2}(\Omega)$ by

$$\mathbf{b}(\mathbf{y}_\gamma) := \frac{\widetilde{g}\gamma\, \mathcal{E}\mathbf{y}_\gamma}{\max\left(\widetilde{g}, \gamma\|\mathcal{E}\mathbf{y}_\gamma\|\right)}.$$

Since $1/\max\left(\widetilde{g}, \gamma\|\mathcal{E}\mathbf{y}_\gamma\|\right) \in L^\infty(\Omega)$, the operator is well-defined. Consequently, we may rewrite system (3.1) as follows:

$$\begin{cases} (\partial_t \mathbf{y}_\gamma(t)\,,\, \mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}_\gamma(t), \mathbf{v}) + (\mathbf{b}(\mathbf{y}(t))\ \mathcal{E}\mathbf{v})_{\mathbf{L}^{2\times 2}} \\ \qquad +\mathbf{c}(\mathbf{y}_\gamma(t), \mathbf{y}_\gamma(t), \mathbf{v}) = (\mathbf{f}(t)\,,\, \mathbf{v})_{\mathbf{L}^2}\,, \text{ for all } \mathbf{v} \in V. \\ \qquad \mathbf{y}(0) = \mathbf{y}_0. \end{cases} \tag{3.3}$$

Differently from the stationary case, system (3.3) does not correspond to the necessary condition of an optimization problem. Therefore, in order to justify existence and uniqueness of solutions to (3.3), techniques from the theory of partial differential equations have to be used.

In the following theorem such an existence result is obtained by combining a monotonicity argument, to cope with the regularized multiplier representation, and a compactness technique, in order to pass to the limit in the trilinear form.

Theorem 3.1. *Let $f \in L^2(V')$ and $\mathbf{y}_0 \in H$. There exists a unique solution $\mathbf{y}_\gamma \in L^2(V)$ for problem* (3.3).

Proof. See the Appendix. $\square$

In the following theorem we analyze the convergence of the sequence of regularized solutions $(\mathbf{y}_\gamma, \mathbf{q}_\gamma)$ of (3.1) towards a solution $(\overline{\mathbf{y}}, \overline{\mathbf{q}})$ of (2.7). For that purpose, an assumption with respect to the problem data is needed.

Assumption 3.2. *There exists a constant $\alpha > 0$, such that*

$$(\alpha_0 - \kappa \|\overline{\mathbf{y}}(t)\|_{\mathbf{H}^1}) \geq \alpha, \ \ a.e \text{ in } [0,T],$$

where α_0 and κ are the same constants as in Remark 2.1.

Theorem 3.3. *If Assumption 3.2 hold, then the solutions $(\mathbf{y}_\gamma, \mathbf{q}_\gamma)$ of the regularized system* (3.1) *converge, as $\gamma \to \infty$, to a solution $(\overline{\mathbf{y}}, \overline{\mathbf{q}})$ of the original problem* (2.7) *in the following way: $\mathbf{y}_\gamma \to \overline{\mathbf{y}}$, strongly in $L^2(V)$, and $\mathbf{q}_\gamma \rightharpoonup \overline{\mathbf{q}}$, weakly in $L^2(\mathbf{L}^{2\times 2}(\Omega))$.*

Proof. Let us start by recalling that $(\overline{\mathbf{y}}(t), \overline{\mathbf{q}}(t))$ and $(\mathbf{y}_\gamma(t), \mathbf{q}_\gamma(t))$ satisfy equations (2.7a) and (3.1a), respectively. Thus, by subtracting (2.7a) from (3.1a), we have, a.e. in $(0,T)$, that

$$\begin{aligned} &\int_\Omega \langle \partial_t(\mathbf{y}_\gamma(t) - \overline{\mathbf{y}}(t)) \, , \, \mathbf{v} \rangle \, dx + \mu \int_\Omega (\mathcal{E}(\mathbf{y}_\gamma(t) - \overline{\mathbf{y}}(t)) : \mathcal{E}\mathbf{v}) \, dx \\ &\qquad + \mathbf{c}(\mathbf{y}_\gamma(t), \mathbf{y}_\gamma(t), \mathbf{v}) - \mathbf{c}(\overline{\mathbf{y}}(t), \overline{\mathbf{y}}(t), \mathbf{v}) = \int_\Omega (\overline{\mathbf{q}}(t) - \mathbf{q}_\gamma(t) : \mathcal{E}\mathbf{v} \, dx), \end{aligned} \tag{3.4}$$

for all $\mathbf{v} \in V$. Next, since Ω is a bounded set in $\mathbb{R}^d$, we know that $\mathbf{c}(\cdot,\cdot,\cdot)$ is a trilinear continuous form in $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$. Therefore, by an easy computation we get that

$$\begin{aligned} \mathbf{c}(\mathbf{y}_\gamma(t), \mathbf{y}_\gamma(t), \mathbf{v}) - \mathbf{c}(\overline{\mathbf{y}}(t), \overline{\mathbf{y}}(t), \mathbf{v}) &= \mathbf{c}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), \overline{\mathbf{y}}, \mathbf{v}) \\ &+ \mathbf{c}(\overline{\mathbf{y}}(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), \mathbf{v}) + \mathbf{c}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), \mathbf{v}), \ \text{ for all } \mathbf{v} \in V, \end{aligned}$$

which implies, choosing $v = (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t) \in V$, a.e. in $[0,T]$ and due to the properties of the form $\mathbf{c}(\cdot,\cdot,\cdot)$ (see [26, Lem. 1.3, p. 109]), that

$$\begin{aligned} &\mathbf{c}(\mathbf{y}_\gamma(t), \mathbf{y}_\gamma(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)) - \mathbf{c}(\overline{\mathbf{y}}(t), \overline{\mathbf{y}}(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)) \\ &\qquad = \mathbf{c}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), \overline{\mathbf{y}}(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)). \end{aligned} \tag{3.5}$$

Thus, by taking $\mathbf{v} = (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)$, a.e. in $(0,T)$, in (3.4) and using (3.5), we obtain that

$$\begin{aligned} &\int_\Omega \langle \partial_t(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t) \, , \, (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t) \rangle \, dx + \mu \int_\Omega \|\mathcal{E}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t))\|^2 \, dx \\ &\qquad + \mathbf{c}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t), \overline{\mathbf{y}}(t), (\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)) = \int_\Omega (\overline{\mathbf{q}}(t) - \mathbf{q}_\gamma(t) : \mathcal{E}((\mathbf{y}_\gamma - \overline{\mathbf{y}})(t))) \, dx. \end{aligned} \tag{3.6}$$

Next, we establish pointwise bounds for $((\overline{\mathbf{q}} - \mathbf{q}_\gamma) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}}))$ in the following disjoint sets $\boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{A}}_\gamma$, $\boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{I}}_\gamma$, $\boldsymbol{\mathcal{A}}_\gamma \cap \boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{I}}_\gamma \cap \boldsymbol{\mathcal{I}}$.

On $\boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{A}}_\gamma$: Here, we use the facts that $\|\overline{\mathbf{q}}(x,t)\| = \|\mathbf{q}_\gamma(x,t)\| = \widetilde{g}$ and $\mathbf{q}_\gamma(x,t) = \widetilde{g} \frac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|}$. Thus, due to Cauchy-Schwarz inequality and (2.7c), we have the following pointwise estimate

$$\begin{aligned} ((\overline{\mathbf{q}} - \mathbf{q}_\gamma)(x,t) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}})(x,t)) &\leq \|\overline{\mathbf{q}}(x,t)\| \|\mathcal{E}\mathbf{y}_\gamma(x,t)\| - \widetilde{g} \|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \\ &\quad - \left(\widetilde{g} \frac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|} : \mathcal{E}\mathbf{y}_\gamma(x,t) \right) + \|\mathbf{q}_\gamma(x,t)\| \|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \\ &= \widetilde{g} \|\mathcal{E}\mathbf{y}_\gamma(x,t)\| - \widetilde{g} \|\mathcal{E}\overline{\mathbf{y}}(x,t)\| - \widetilde{g} \|\mathcal{E}\mathbf{y}_\gamma(x,t)\| + \widetilde{g} \|\mathcal{E}\overline{\mathbf{y}}(x,t)\| = 0. \end{aligned} \tag{3.7}$$

On $\boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{I}}_\gamma$: Here, we know that $\mathcal{E}\mathbf{y}_\gamma(x,t) = \gamma^{-1}\mathbf{q}_\gamma(x,t)$, $\|\mathbf{q}_\gamma(x,t)\| < \widetilde{g}$ and $\|\overline{\mathbf{q}}(x,t)\| = \widetilde{g}$. Hence, from the Cauchy-Schwarz inequality and (2.7c), we get

$$
\begin{aligned}
((\overline{\mathbf{q}} - \mathbf{q}_\gamma)(x,t) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}})(x,t)) &\le \gamma^{-1}\|\overline{\mathbf{q}}(x,t)\|\|\mathbf{q}_\gamma(x,t)\| - \widetilde{g}\|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \\
&\quad -\gamma^{-1}\|\mathbf{q}_\gamma(x,t)\|^2 + \|\mathbf{q}_\gamma(x,t)\|\|\mathcal{E}\overline{\mathbf{y}}(x,t)\| \\
&\quad < \gamma^{-1}(\widetilde{g}^2 - \|\mathbf{q}_\gamma(x,t)\|^2) < \tfrac{\widetilde{g}^2}{\gamma}.
\end{aligned}
\tag{3.8}
$$

On $\boldsymbol{\mathcal{A}}_\gamma \cap \boldsymbol{\mathcal{I}}$: In this set it holds that $\mathcal{E}\overline{\mathbf{y}}(x,t) = 0$ and $\mathbf{q}_\gamma(x,t) = \widetilde{g}\frac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|}$. Then, due to the Cauchy-Schwarz inequality and (2.7b), we have that

$$
\begin{aligned}
((\overline{\mathbf{q}} - \mathbf{q}_\gamma)(x,t) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}})(x,t)) &= \|\overline{\mathbf{q}}(x,t)\|\|\mathcal{E}\mathbf{y}_\gamma(x,t)\| - \widetilde{g}\tfrac{\mathcal{E}\mathbf{y}_\gamma(x,t)}{\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|} : \mathcal{E}\mathbf{y}_\gamma(x,t) \\
&\le \widetilde{g}\|\mathcal{E}\mathbf{y}_\gamma(x,t)\| - \widetilde{g}\|\mathcal{E}\mathbf{y}_\gamma(x,t)\| = 0.
\end{aligned}
\tag{3.9}
$$

On $\boldsymbol{\mathcal{I}}_\gamma \cap \boldsymbol{\mathcal{I}}$: Here, we have that $\mathcal{E}\overline{\mathbf{y}}(x,t) = 0$ and $\mathcal{E}\mathbf{y}_\gamma(x,t) = \gamma^{-1}\mathbf{q}_\gamma(x,t)$. Thus, the Cauchy-Schwarz inequality and (2.7b) imply that

$$
\begin{aligned}
((\overline{\mathbf{q}} - \mathbf{q}_\gamma)(x,t) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}})(x,t)) &= \gamma^{-1}\|\overline{\mathbf{q}}(x,t)\|\|\mathbf{q}_\gamma(x,t)\| - \gamma^{-1}\|\mathbf{q}_\gamma(x,t)\|^2 \\
&\le \gamma^{-1}\left(\widetilde{g}^2 - \|\mathbf{q}_\gamma(x,t)\|^2\right) < \tfrac{\widetilde{g}^2}{\gamma}.
\end{aligned}
\tag{3.10}
$$

Since $\boldsymbol{\mathcal{A}}_\gamma \cap \boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{I}}_\gamma$, $\boldsymbol{\mathcal{A}}_\gamma \cap \boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{I}}_\gamma \cap \boldsymbol{\mathcal{I}}$ provide a disjoint partitioning of Q, estimates (3.7), (3.8), (3.9) and (3.10) imply that

$$
\textstyle\int_\Omega ((\overline{\mathbf{q}} - \mathbf{q}_\gamma)(x,t) : \mathcal{E}(\mathbf{y}_\gamma - \overline{\mathbf{y}})(x,t))\,dx < \int_\Omega \frac{\widetilde{g}^2}{\gamma}\,dx, \text{ a.e. in } [0,T]. \tag{3.11}
$$

Due to the coercivity of the form $\mathbf{a}(\cdot,\cdot)$ and the continuity of $\mathbf{c}(\cdot,\cdot,\cdot)$ (see Remark 2.1), we conclude, from (3.6), that there exist two positive constants α_0 and κ, such that

$$
\frac{1}{2}\frac{d}{dt}\|(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)\|^2_{\mathbf{L}^2} + (\alpha_0 - \kappa\|\overline{\mathbf{y}}(t)\|_{\mathbf{H}^1_0})\|(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)\|^2_{\mathbf{H}^1_0} < \frac{\widetilde{g}^2}{\gamma} meas(\Omega), \text{ a.e. in } [0,T],
$$

which, thanks to the Assumption 3.2, yields that

$$
\frac{1}{2}\frac{d}{dt}\|(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)\|^2_{\mathbf{L}^2} + \alpha\|(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)\|^2_{\mathbf{H}^1_0} < \frac{\widetilde{g}^2}{\gamma} meas(\Omega), \text{ a.e. in } [0,T]. \tag{3.12}
$$

Now, by integrating (3.12) in $(0,T)$, we obtain that

$$
\frac{1}{2}\|\mathbf{y}_\gamma(T) - \overline{\mathbf{y}}(T)\|^2_{\mathbf{L}^2} + \alpha\int_0^T \|(\mathbf{y}_\gamma - \overline{\mathbf{y}})(t)\|^2_{\mathbf{H}^1_0}\,dt < T\,meas(\Omega)\frac{\widetilde{g}^2}{\gamma}. \tag{3.13}
$$

Thus, from (3.13), we conclude that $\mathbf{y}_\gamma \to \overline{\mathbf{y}}$ strongly in $L^2(V)$ and $\mathbf{y}_\gamma(T) \to \overline{\mathbf{y}}(T)$ strongly in $\mathbf{L}^2(\Omega)$, as $\gamma \to \infty$.

In order to prove the convergence of the multipliers $\mathbf{q}_\gamma$, first note that $\|\mathbf{q}_\gamma(x,t)\| \le \widetilde{g}$, a.e. in Q and, therefore, $\{\mathbf{q}_\gamma\}$ is bounded in $L^2(\mathbf{L}^{2\times 2}(\Omega))$. Consequently, it is possible to extract a subsequence (denoted in the same way) such that

$$
\mathbf{q}_\gamma \rightharpoonup \mathbf{q}^* \in L^2(\mathbf{L}^{2\times 2}(\Omega)), \text{ as } \gamma \to \infty. \tag{3.14}
$$

Moreover, since the set $\left\{\mathbf{q} \in L^2(\mathbf{L}^{2\times 2}(\Omega)) : \|\mathbf{q}(x,t)\| \le \widetilde{g}, \text{ a.e in } Q\right\}$ is convex and closed, we conclude that

$$
\|\mathbf{q}^*(x,t)\| \le \widetilde{g}, \text{ a.e in } Q. \tag{3.15}
$$

Next, we know that $\mathbf{y}_\gamma$ converges to $\overline{\mathbf{y}}$ strongly in $L^2(\mathbf{H}^1(\Omega))$. Moreover, we have that $\partial_t\mathbf{y}_\gamma \in H^{-1}(\mathbf{H}^{-1})$ and that $\partial_t \in \mathcal{L}(L^2(\mathbf{H}^1(\Omega)), H^{-1}(\mathbf{H}^{-1}(\Omega)))$. Thus, we can conclude that

$$
\partial_t\mathbf{y}_\gamma \rightharpoonup \partial_t\overline{\mathbf{y}} \in L^2(\mathbf{H}^1), \text{ as } \gamma \to \infty. \tag{3.16}
$$

Furthermore, thanks to (3.14), (3.16) and to the weakly sequentially continuity of $\mathbf{c}(\cdot,\cdot,\cdot)$, we may pass to the limit in (3.1a) and obtain that

$$\begin{aligned}(\partial_t\overline{\mathbf{y}}(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\overline{\mathbf{y}}(t),\mathbf{v}) + \mathbf{c}(\overline{\mathbf{y}}(t),\overline{\mathbf{y}}(t),\mathbf{v}) + (\mathbf{q}^*(t)\,,\,\mathcal{E}(\mathbf{v}))_{\mathbf{L}^{2\times 2}(\Omega)}&\\ = (\mathbf{f}(t)\,,\,\mathbf{v})_{\mathbf{L}^2}\,,\ \text{a.e. in } (0,T) \text{ and for all } \mathbf{v}\in V.&\end{aligned}$$

By taking $\mathbf{v}=\overline{\mathbf{y}}(t)$, we get that

$$\begin{aligned}(\partial_t\overline{\mathbf{y}}(t)\,,\,\overline{\mathbf{y}}(t))_{\mathbf{L}^2} + \mathbf{a}(\overline{\mathbf{y}}(t),\overline{\mathbf{y}}(t)) + (\mathbf{q}^*(t)\,,\,\mathcal{E}\overline{\mathbf{y}}(t))_{\mathbf{L}^{2\times 2}(\Omega)}&\\ = (\mathbf{f}(t)\,,\,\overline{\mathbf{y}}(t))_{\mathbf{L}^2}\,,\ \text{a.e. in } (0,T).&\end{aligned}\tag{3.17}$$

On the other hand, by taking $\mathbf{v}=0$ and $\mathbf{v}=2\overline{\mathbf{y}}(t)$ in $(\boldsymbol{\mathcal{P}})$, we obtain that

$$(\partial_t\overline{\mathbf{y}}(t)\,,\,\overline{\mathbf{y}}(t))_{\mathbf{L}^2} + \mathbf{a}(\overline{\mathbf{y}}(t),\overline{\mathbf{y}}(t)) + \widetilde{g}\,\mathbf{j}(\overline{\mathbf{y}}(t)) = (\mathbf{f}(t)\,,\,\overline{\mathbf{y}}(t))_{\mathbf{L}^2}\,,\ \text{a.e. in } (0,T).\tag{3.18}$$

Thus, by subtracting (3.17) from (3.18), we get

$$\int_\Omega \left[\widetilde{g}\,\|\mathcal{E}\overline{\mathbf{y}}(t)\| - (\mathbf{q}^*(t) : \mathcal{E}\overline{\mathbf{y}}(t))\right] dx = 0,\ \text{a.e. in } (0,T),$$

which implies, due to the Cauchy-Schwarz inequality, that

$$\widetilde{g}\|\mathcal{E}\overline{\mathbf{y}}(x,t)\| = (\mathbf{q}^*(x,t) : \mathcal{E}\overline{\mathbf{y}}(x,t)),\ \text{a.e. in } Q\tag{3.19}$$

□

In the next theorem, we obtain an equivalent system to (3.1), which considers the pressure in an explicit way.

Proposition 3.4. *There exists a unique $p_\gamma(t)\in L^2_0(\Omega)$, a.e. in $[0,T]$, such that*

$$\begin{aligned}(\partial_t\mathbf{y}_\gamma(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}_\gamma(t),\mathbf{v}) + (\mathbf{q}_\gamma(t)\,,\,\mathcal{E}\mathbf{v})_{\mathbf{L}^{2\times 2}} + \mathbf{c}(\mathbf{y}_\gamma(t),\mathbf{y}_\gamma(t),\mathbf{v})&\\ - (p_\gamma(t)\,,\,\operatorname{div}\mathbf{v})_{L^2} = (\mathbf{f}(t)\,,\,\mathbf{v})_{\mathbf{L}^2}\,,\ \textit{for all } \mathbf{v}\in\mathbf{H}^1_0(\Omega),&\end{aligned}$$

$$(r\,,\,\operatorname{div}\mathbf{y}_\gamma(t))_{L^2} = 0,\quad \textit{for all } r\in L^2(\Omega).$$

Proof. First, we recall that $(\mathbf{y}_\gamma(t),\,\mathbf{q}_\gamma(t))\in V\times\mathbf{L}^{2\times 2}(\Omega)$ satisfy equation (3.1a), *i.e.*,

$$\begin{aligned}(\partial_t\mathbf{y}(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + a(\mathbf{y}_\gamma(t),\mathbf{v}) + (\mathbf{q}_\gamma(t),\mathbf{v})_{\mathbf{L}^{2\times 2}}&\\ +\mathbf{c}(\mathbf{y}_\gamma(t),\mathbf{y}_\gamma(t),\mathbf{v}) = (\mathbf{f}\,,\,\mathbf{v})_{\mathbf{L}^2}\,,\ \text{a.e. in } [0,T] \text{ and for all } \mathbf{v}\in V.&\end{aligned}\tag{3.20}$$

Since $\operatorname{div}\mathbf{y}_\gamma(t)=0$ and $\operatorname{div}\mathbf{v}=0$, the result is a direct consequence of de Rham's Theorem (see [26, Rem. 1.9] and [13, Th. 3.6, p. 34]). □

Thanks to Proposition 3.4, we obtain the following equivalent variational formulation of system (3.1): find $(\mathbf{y}_\gamma(t),\mathbf{q}_\gamma(t),p_\gamma(t))\in\mathbf{H}^1_0(\Omega)\times\mathbf{L}^{2\times 2}(\Omega)\times L^2_0(\Omega)$, a.e. in $[0,T]$, such that

$$\begin{aligned}(\partial_t\mathbf{y}_\gamma(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}_\gamma(t),\mathbf{v}) + (\mathbf{q}_\gamma(t)\,,\,\mathcal{E}\mathbf{v})_{\mathbf{L}^{2\times 2}} + \mathbf{c}(\mathbf{y}_\gamma(t),\mathbf{y}_\gamma(t),\mathbf{v})&\\ - (p_\gamma(t)\,,\,\operatorname{div}\mathbf{v})_{L^2} = (\mathbf{f}(t)\,,\,\mathbf{v})_{\mathbf{L}^2}\,,\ \text{for all } \mathbf{v}\in\mathbf{H}^1_0(\Omega),&\end{aligned}\tag{3.21a}$$

$$(r\,,\,\operatorname{div}\mathbf{y}_\gamma(t))_{L^2} = 0,\quad \text{for all } r\in L^2(\Omega),\tag{3.21b}$$

$$\max(\widetilde{g},\gamma\|\mathcal{E}\mathbf{y}_\gamma(x,t)\|)\mathbf{q}_\gamma(x,t) = g\gamma\mathcal{E}\mathbf{y}_\gamma(x,t),\quad \text{a.e. in } Q,\tag{3.21c}$$

$$\mathbf{y}(x,0) = \mathbf{y}_0.\tag{3.21d}$$

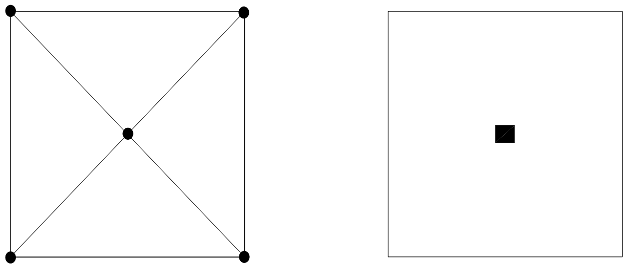

FIGURE 1. (cross-grid $\mathbb{P}_1$)-$\mathbb{Q}_0$ elements. $\bullet$ are the velocity nodes and $\blacksquare$ the pressure nodes.

Remark 3.1. Hereafter we identify the space $\mathbf{L}^{2\times 2}(\Omega)$ with the space $(L^2(\Omega))^4$. The space $(L^2(\Omega))^4$ is endowed with the usual L^2-scalar product (see [7, Sec. 5.1]), *i.e.*,

$$(\mathbf{q},\, \mathbf{p})_{(L^2)^4} := \int_\Omega \langle \mathbf{q}(x)\,,\, \mathbf{p}(x)\rangle\, dx.$$

4. Numerical approach

In this section, we propose a discretization scheme for system (3.21) which allows to exploit the efficiency of the semismooth Newton method developed for the stationary Bingham model. Similar to [7], a first order finite element approximation for the space variable with (cross-grid $\mathbb{P}_1$)-$\mathbb{Q}_0$ elements is considered. With this choice, the same test functions for the velocity gradient and the dual variable are utilized and a direct relation between them is obtained. This is of importance for the accurate determination of active and inactive sets.

For the time variable discretization, the aim is to use an advancing scheme that leads us to convection-independent systems in each time step. In that way, the semismooth Newton algorithm developed in [7] to solve such nonlinear systems of equations can be used.

4.1. **Space Discretization.** We start by constructing a semi-discrete version of system (3.21) with a first order finite element approximation. In our particular case, we choose the so called (cross-grid $\mathbb{P}_1$)-$\mathbb{Q}_0$ elements (see Figure 1) and define the finite dimensional Hilbert spaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega)$, $\mathbf{W}^h \subset (L^2(\Omega))^4$ and $\mathbf{U}^h \subset L^2(\Omega)$ by

$$\begin{aligned}
\mathbf{V}^h &:= (V_h \cap H^1(\Omega))^2, \text{ where } V^h := \left\{ v^h \in C(\Omega) \,:\, v^h|_T \in \Pi_1, \text{ for all } T \in \mathcal{T}^h \right\},\\
\mathbf{W}^h &:= \left\{ (q_1^h, q_2^h, q_3^h, q_4^h) \in (L^2(\Omega))^4 \,:\, q_j^h|_T \in \Pi_0, \text{ for } j = 1, \dots, 4 \text{ and } T \in \mathcal{T}^h \right\},\\
\mathbf{U}^h &:= O_h \cap L^2(\Omega), \text{ where } O_h := \left\{ r^h \in C(\Omega) \,:\, r^h|_Q \in \Pi_0, \text{ for all } Q \in \mathcal{Q}^h \right\}.
\end{aligned}$$

with $\dim \mathbf{V}^h = 2n$, $\dim \mathbf{W}^h = 4m$ and $\dim \mathbf{U}^h = l$, $n, m, l \in \mathbb{N}$, respectively. Here, $\mathcal{Q}^h$ is a regular quadrangulation of Ω, and $\mathcal{T}^h$ is the regular triangulation obtained by dividing any square in $\mathcal{Q}^h$ by using its two main diagonals ([23, Sec. 6]). In order to simplify the analysis, we assume that Ω has a polygonal boundary.

Remark 4.1. Let us mention that the (cross-grid $\mathbb{P}_1$)-$\mathbb{Q}_0$ elements satisfy the Ladyzhenskaya-Babuska-Brezzi (LBB) condition and, therefore, lead to stable approximation of Stokes like systems (see [23, p. 435]).

As a consequence of the chosen space discretization, we obtain the following semi-discrete approximation of system (3.21) (see [23, Sec. 13.3] and [7, Sec. 5] for further details):

$$
\begin{cases}
\mathbf{M}^h \frac{\partial}{\partial t}\vec{\mathbf{y}}(t) + \mathbf{A}^h_\mu \vec{\mathbf{y}}(t) + \mathbf{Q}^h \vec{\mathbf{q}}(t) + \mathbf{C}^h(\vec{\mathbf{y}}(t))\,\vec{\mathbf{y}}(t) + B^h \vec{p}(t) = \vec{\mathbf{f}}(t) \\
-(B^h)^\top \vec{\mathbf{y}}(t) = 0 \\
\max\left(\widetilde{g}\vec{e}, \gamma N(\mathcal{E}^h \vec{\mathbf{y}(t)})\right) \star \vec{\mathbf{q}}(t) = \gamma \widetilde{g} \mathcal{E}^h \vec{\mathbf{y}}(t), \\
\vec{\mathbf{y}}(0) = \vec{\mathbf{y}}_0.
\end{cases}
\tag{4.1}
$$

where $\vec{e} \in \mathbb{R}^{4m}$ denotes the vector of all ones and $\vec{\mathbf{y}}(t) \in \mathbb{R}^{2n}$, $\vec{\mathbf{q}}(t) \in \mathbb{R}^{4m}$, $\vec{p}(t) \in \mathbb{R}^l$ and $\vec{\mathbf{y}}_0(t) \in \mathbb{R}^{2n}$ are the time dependent vectors of coefficients in the finite element representation of the triplet $(\mathbf{y}^h(t), \mathbf{q}^h(t), p^h(t))$ and the initial condition $\mathbf{y}_0(t)$, respectively. $\mathbf{A}^h_\mu \in \mathbb{R}^{2n\times 2n}$ and $\mathbf{M}^h \in \mathbb{R}^{2n\times 2n}$ are the stiffness and mass matrices, respectively, and the matrices $\mathbf{Q}^h \in \mathbb{R}^{2n\times 4m}$ and $B^h \in \mathbb{R}^{2n\times l}$ are obtained in the usual way, from the bilinear forms $(\cdot, \mathcal{E}(\cdot))_{(L^2)^4}$ and $-(\cdot, \operatorname{div}\cdot)_{L^2}$, using the basis for $\mathbf{V}^h$, $\mathbf{W}^h$ and $\mathbf{U}^h$. The matrix $\mathbf{C}^h(\vec{\mathbf{y}}(t))$ is defined by

$$
\mathbf{C}^h(\vec{\mathbf{y}}(t))_{ij} := \sum_{\ell=1}^{2n} \mathbf{y}_\ell \, \mathbf{c}(\boldsymbol{\varphi}_\ell, \boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i),
$$

where $\boldsymbol{\varphi}_j$, $j = 1, \dots, 2n$ are the basis functions of $\mathbf{V}^h$. Further, the discrete approximation $\mathcal{E}^h$ of the deformation tensor and the right hand side $\vec{\mathbf{f}}$ are constructed by using these basis functions (see [1, Sec. 6] and [7, Sec. 5]). Finally, the function $N : \mathbb{R}^{4m} \to \mathbb{R}^{4m}$ is defined by

$$
N(\vec{\mathbf{q}})_i = N(\vec{\mathbf{q}})_{i+m} = \cdots = N(\vec{\mathbf{q}})_{i+4m} := |(q_i, q_{i+m}, \dots, q_{i+4m})^\top|
$$

for $\vec{\mathbf{q}} \in \mathbb{R}^{4m}$ and $i = 1, \dots, m$.

4.2. **Time Discretization by a BDF.** When a fully implicit method (*e.g.* the one step θ-method) is used to solve a Navier-Stokes type equation, a nonlinear and convective system of equations, which changes in every time step, must be solved in each time iteration. This usually implies the increase of the computational cost.

In order to avoid this issue, semi-implicit methods can be taken into account. The characteristic of some of these methods is that they allow to obtain systems whose associated matrix is a Stokes type one and does not change in every time step (see [23, Sec. 13.4]). These type of semi-implicit methods can be constructed in many ways. For instance, it is possible to approximate the linear terms in the equation by a Crank-Nicholson method and the nonlinear convective term $C^h(\vec{\mathbf{y}})$ by the explicit Adams-Bashforth method (see [23, p. 440]).

In this paper, we focus on a semi-implicit method constructed using the second order backward differentiation formula (BDF2), which also presents the same kind of property: a system whose associated matrix is a Stokes type one and does not change in every time step.

When applied to an equation $u' = \Upsilon(u)$, the BDF2 scheme reads as follows

$$
u^{k+2} - \frac{4}{3}u^{k+1} + \frac{1}{3}u^k = \frac{2}{3}k\Upsilon^{k+2}, \text{ for } 0 \le k \le \mathcal{N} - 2, \text{ and } u^0, u^1 \text{ given},
$$

where u^k stands for the approximation of the solution u at each time step.

By following a similar argumentation as in [2, p. 370] and [23, pp. 440-441], we formulate a second order backward differentiation-Galerkin method as follows: at each time level $t_{k+1} =$

$(k+1)\delta t$, for $k = 0, \dots, \mathcal{N}-1$, solve the system

$$(4.2) \qquad \begin{cases} \left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}_\mu^h\right) \vec{\mathbf{y}}^{\,k+2} + \mathbf{Q}^h \vec{\mathbf{q}}^{\,k+2} + B^h \vec{p}^{\,k+2} = \widetilde{\mathbf{F}}^{\,k+2} \\ -(B^h)^\top \vec{\mathbf{y}}^{\,k+2} = 0 \\ \max\left(\widetilde{g}\vec{e}, \gamma N(\mathcal{E}^h \vec{\mathbf{y}}^{\,k+2})\right) \star \vec{\mathbf{q}}^{\,k+2} = \gamma \widetilde{g} \mathcal{E}^h \vec{\mathbf{y}}^{\,k+2}, \\ \vec{\mathbf{y}}^{\,0} = \vec{\mathbf{y}}_0, \end{cases}$$

where $\vec{\mathbf{y}}^{\,k}$ represents the approximation of $\vec{\mathbf{y}}(t_k)$ and the right hand side $\widetilde{\mathbf{F}}^{\,k+2}$ is given by

$$\widetilde{\mathbf{F}}^{\,k+2} := \vec{\mathbf{f}}^{\,k+2} - \mathbf{C}^h(\vec{\bar{\mathbf{y}}}^{\,k})\vec{\bar{\mathbf{y}}}^{\,k} + \frac{2}{\delta t}\mathbf{M}^h \vec{\mathbf{y}}^{\,k+1} - \frac{1}{2\delta t}\mathbf{M}^h \vec{\mathbf{y}}^{\,k},$$

with $\vec{\bar{\mathbf{y}}} := 2\vec{\mathbf{y}}^{\,k+1} - \vec{\mathbf{y}}^k$. Further, in order to calculate the initialization steps $\vec{\mathbf{y}}^{\,0}$ and $\vec{\mathbf{y}}^{\,1}$, we use the one-step backward Euler scheme. This initialization is suggested in [2, p. 370], and can be developed in the following way: first, we choose $\vec{\mathbf{y}}^{\,0} = \vec{\mathbf{y}}_0$. Next, we calculate two intermediate values $(\vec{\mathbf{y}}_0^{\,2/3}, \vec{\mathbf{q}}^{\,2/3}, \vec{p}^{\,2/3})$ and $(\vec{\mathbf{y}}_0^{\,4/3}, \vec{\mathbf{q}}^{\,4/3}, \vec{p}^{\,4/3})$ by solving the next two systems

$$(4.3) \qquad \begin{cases} \left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}_\mu^h\right) \vec{\mathbf{y}}_0^{\,2/3} + \mathbf{Q}^h \vec{\mathbf{q}}^{\,2/3} + B^h \vec{p}^{\,2/3} = \vec{\mathbf{f}}^{\,2/3} - \mathbf{C}^h(\vec{\mathbf{y}}^{\,0})\,\vec{\mathbf{y}}^{\,0} + \mathbf{M}^h \vec{\mathbf{y}}^{\,0} \\ -(B^h)^\top \vec{\mathbf{y}}_0^{\,2/3} = 0 \\ \max\left(\widetilde{g}\vec{e}, \gamma N(\mathcal{E}^h \vec{\mathbf{y}}_0^{\,2/3})\right) \star \vec{\mathbf{q}}^{\,2/3} = \gamma \widetilde{g} \mathcal{E}^h \vec{\mathbf{y}}_0^{\,2/3}, \end{cases}$$

$$(4.4) \qquad \begin{cases} \left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}_\mu^h\right) \vec{\mathbf{y}}_0^{\,4/3} + \mathbf{Q}^h \vec{\mathbf{q}}^{\,4/3} + B^h \vec{p}^{\,4/3} = \vec{\mathbf{f}}^{\,4/3} - \mathbf{C}^h(\vec{\mathbf{y}}^{\,0})\,\vec{\mathbf{y}}^{\,0} + \mathbf{M}^h \vec{\mathbf{y}}_0^{\,2/3} \\ -(B^h)^\top \vec{\mathbf{y}}_0^{\,4/3} = 0 \\ \max\left(\widetilde{g}\vec{e}, \gamma N(\mathcal{E}^h \vec{\mathbf{y}}_0^{\,2/3})\right) \star \vec{\mathbf{q}}^{\,4/3} = \gamma \widetilde{g} \mathcal{E}^h \vec{\mathbf{y}}_0^{\,4/3}, \end{cases}$$

and finally, we define

$$\vec{\mathbf{y}}^{\,1} := \frac{1}{2}(\vec{\mathbf{y}}_0^{\,2/3} + \vec{\mathbf{y}}_0^{\,4/3}).$$

In [2, p. 371] is proved that if we use the initialization defined above, $\delta t \leq C\, h^{4/5}$, with h the size of the spatial mesh and $C > 0$, and if Assumption 3.2 holds, the scheme is second order accurate with respect to the time variable.

4.3. **Semismooth Newton algorithm.** For the sake of readability let us firts recall some basic notions about semismooth Newton methods (see e.g. [19]).

Definition 4.2. Let X and Z be Banach spaces and $D \subset X$ an open subset. The mapping $F : D \to Z$ is called Newton differentiable on the open subset $V \subset D$ if there exists a generalized derivative $G : V \to L(X, Z)$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|_X} \|F(x+h) - F(x) - G(x+h)h\|_Z = 0,$$

for every $x \in V$.

Proposition 4.1. *If x^* is a solution of $F(x) = 0$, F is Newton differentiable in an open neighborhood V containing x^* with generalized derivative G. If $\{\|G(y)^{-1}\|_{\mathcal{L}(Z,X)} : y \in V\}$ is bounded, then the Newton iterations*

$$x_{k+1} = x_k - G(x_k)^{-1}F(x_k)$$

converge superlinearly to x^*, provided that $\|x_0 - x^*\|_X$ is sufficiently small.

For our specific problem, the nonlinear system (4.2) to be solved in each time step, as well as the nonlinear systems (4.3) and (4.4), can be written in the following general form

$$
\begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{Q} \\ -\mathcal{B}^\top & \mathbf{0} & \mathbf{0} \\ -\gamma\tilde{g}\mathcal{K} & \mathbf{0} & \mathcal{D}(\vec{\mathbf{y}}) \end{pmatrix} \begin{pmatrix} \vec{\mathbf{y}} \\ \vec{p} \\ \vec{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} \mathcal{H} \\ 0 \\ 0 \end{pmatrix}, \tag{4.5}
$$

where the matrices $\mathcal{A} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{B} \in \mathbb{R}^{2n\times l}$ and $\mathcal{Q} \in \mathbb{R}^{2n\times 4m}$ are sparse and constant with respect to the variables $\vec{\mathbf{y}} \in \mathbb{R}^{2n}$, $\vec{\mathbf{q}} \in \mathbb{R}^{4m}$ and $\vec{p} \in \mathbb{R}^{l}$, $\mathcal{K} \in \mathbb{R}^{4m\times 2n}$ stands for a linear operator and the right hand side $\mathcal{H}$ depends on known values of previous time steps.

The matrix $\mathcal{D}(\vec{\mathbf{y}})$ is given by $\mathcal{D}(\vec{\mathbf{y}}) := diag(m(\vec{\mathbf{y}}))$, where the vector function $m(\vec{\mathbf{y}})$ is defined by

$$
m(\vec{\mathbf{y}}) := \max(\tilde{g}\vec{e}, \gamma\mathcal{N}(\mathcal{K}\vec{\mathbf{y}})).
$$

Here, $\mathcal{N} : \mathbb{R}^{4m} \to \mathbb{R}^{4m}$ represents a nonlinear and Newton differentiable function. Therefore, since the max function is Newton differentiable when defined from $\mathbb{R}^s$ to $\mathbb{R}^t$, for all $s, t \in \mathbb{N}$ (see [19]), we can conclude that $\mathcal{D}(\vec{\mathbf{y}})$ is also Newton differentiable with respect to the variable $\vec{\mathbf{y}}$ (see [22, 7, 25]). Furthermore, the Newton derivative is given by

$$
\mathcal{D}'(\vec{\mathbf{y}}) := \gamma\chi_{\mathcal{A}}\mathcal{N}'(\mathcal{K}(\vec{\mathbf{y}}))\mathcal{K}, \tag{4.6}
$$

where $\mathcal{N}'$ is the Newton derivative of $\mathcal{N}$ and $\chi_{\mathcal{A}}$ is defined by $\chi_{\mathcal{A}} := diag(\vec{\ell})$, where

$$
\vec{\ell}_i := \begin{cases} 1 & \text{if } \mathcal{N}(\mathcal{K}(\vec{\mathbf{y}}))_i \geq \frac{\tilde{g}}{\gamma} \\ 0 & \text{else} \end{cases} \tag{4.7}
$$

Remark 4.3. The function $\chi_{\mathcal{A}}$ is the indicator function of the active set associated to the equation $-\gamma\tilde{g}\mathcal{K}\,\vec{\mathbf{y}} + \mathcal{D}(\vec{\mathbf{y}})\,\vec{\mathbf{q}} = 0$. In the particular case of system (4.2), this function allows us to characterize the active set $\boldsymbol{\mathcal{A}}_\gamma$ (see system (3.1)), which represents an approximation of the regions in which the material behaves as a visco-plastic fluid.

The structure of (4.5) is therefore similar as the one obtained for the stationary model (see [7, p. 15]) and the semismooth Newton algorithm developed in [7] may be used to solve the nonlinear systems given by (4.5).

Indeed, system (4.5) can be reformulated as the following operator equation

$$
F(\vec{\mathbf{y}}, \vec{p}, \vec{\mathbf{q}}) = \begin{pmatrix} \mathcal{A}\vec{\mathbf{y}} + \mathcal{B}\vec{p} + \mathcal{Q}\vec{\mathbf{q}} - \mathcal{H} \\ -\mathcal{B}^\top\vec{\mathbf{y}} \\ -\gamma\tilde{g}\mathcal{K}\vec{\mathbf{y}} + \mathcal{D}(\vec{\mathbf{q}}) \end{pmatrix} = 0
$$

and its Newton step can be written as:

$$
\begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{Q} \\ -\mathcal{B}^\top & \mathbf{0} & \mathbf{0} \\ -\gamma\tilde{g}\mathcal{K} + \mathcal{D}'(\vec{\mathbf{y}})\,diag(\vec{\mathbf{q}}) & \mathbf{0} & \mathcal{D}(\vec{\mathbf{y}}) \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{y}} \\ \delta_p \\ \delta_{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} -\mathcal{A}\vec{\mathbf{y}} - \mathcal{B}\vec{p} - \mathcal{Q}\vec{\mathbf{q}} + \mathcal{H} \\ \mathcal{B}^\top\vec{\mathbf{y}} \\ \gamma\tilde{g}\mathcal{K}\vec{\mathbf{y}} - \mathcal{D}(\vec{\mathbf{y}})\vec{\mathbf{q}} \end{pmatrix}, \tag{4.8}
$$

where $\mathcal{D}'(\vec{\mathbf{y}})$ is given by (4.6).

Note that the matrix in the left hand side of (4.8) is a Stokes-type matrix. There are several approaches for the efficient solution of this kind of systems. In this paper we use a penalization of the equation $-\mathcal{B}^\top\vec{\mathbf{y}}$, representing the incompressibility constraint. The idea is to consider the following equation

$$
-\mathcal{B}^\top\vec{\mathbf{y}} + \varsigma\vec{p} = 0,
$$

with, $\varsigma > 0$ sufficiently small. This procedure is motivated by continuous stabilization procedures developed for the finite element approximation of Stokes and Navier-Stokes equations (see [3, 18]).

The procedure described above leads us to the following system of equations

$$
\text{(4.8')} \qquad \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{Q} \\ -\mathcal{B}^\top & \varsigma I & \mathbf{0} \\ -\gamma\tilde{g}\mathcal{K} + \mathcal{D}'(\vec{\mathbf{y}})\,diag(\vec{\mathbf{q}}) & \mathbf{0} & \mathcal{D}(\vec{\mathbf{y}}) \end{pmatrix} \begin{pmatrix} \delta_{\mathbf{y}} \\ \delta_p \\ \delta_{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} -\mathcal{A}\vec{\mathbf{y}} - \mathcal{B}\vec{p} - \mathcal{Q}\vec{\mathbf{q}} + \mathcal{H} \\ \mathcal{B}^\top\vec{\mathbf{y}} - \varsigma\vec{p} \\ \gamma\tilde{g}\mathcal{K}\vec{\mathbf{y}} - \mathcal{D}(\vec{\mathbf{y}})\vec{\mathbf{q}} \end{pmatrix},
$$

where I represents the identity matrix.

Note that, since $\mathcal{D}(\vec{\mathbf{y}})$ is invertible, system (4.8') is an uncoupled nonlinear system of equations for the Newton residuum $\delta_{\mathbf{y}}, \delta_p, \delta_{\mathbf{q}}$. In fact, $\delta_{\mathbf{q}}$ and δ_p can be calculated by

$$
\text{(4.9)} \qquad \delta_{\mathbf{q}} \;=\; -\vec{\mathbf{q}} + \mathcal{D}(\vec{\mathbf{y}})^{-1}\left[\gamma\tilde{g}\mathcal{K}\vec{\mathbf{y}} + (\gamma\tilde{g}\mathcal{K} - \mathcal{D}'(\vec{\mathbf{y}})diag(\vec{\mathbf{q}}))\delta_{\mathbf{y}}\right]
$$

$$
\text{(4.10)} \qquad \delta_p \;=\; \frac{1}{\varsigma}\left(\mathcal{B}^\top\vec{\mathbf{y}} + \mathcal{B}^\top\delta_{\mathbf{y}}\right) - \vec{p},
$$

and for the calculation of $\delta_{\mathbf{y}}$, we need to solve the following system of equations

$$
\mathcal{M}\delta_{\mathbf{y}} = \mathcal{F},
$$

where $\mathcal{M} \in \mathbb{R}^{2m\times 2m}$ and $\mathcal{F} \in \mathbb{R}^{2m}$ are given by

$$
\begin{aligned}
\mathcal{M} \;&:=\; \left[\mathcal{A} + \mathcal{Q}\mathcal{D}(\vec{\mathbf{y}})^{-1}(\gamma\tilde{g}\mathcal{K} - \mathcal{D}'(\vec{\mathbf{y}})diag(\vec{\mathbf{q}})) + \tfrac{1}{\varsigma}\mathcal{B}\mathcal{B}^\top\right] \\
\mathcal{F} \;&:=\; -\mathcal{A}\vec{\mathbf{y}} + \mathcal{H} + \mathcal{Q}\mathcal{D}(\vec{\mathbf{y}})^{-1}\gamma\tilde{g}\mathcal{K}\vec{\mathbf{y}} - \tfrac{1}{\varsigma}\mathcal{B}\mathcal{B}^\top\vec{\mathbf{y}}
\end{aligned}
$$

In order to guarantee the existence of solution for the above system of linear equations, positive definiteness of the system matrix $\mathcal{M}$ must be proved.

In the particular case of the stationary Bingham flow, which in [7] is analyzed as a minimization problem, it is proved that positive definiteness of matrix $\mathcal{M}$ also implies that $\delta_{\mathbf{y}}$ is a descent direction of the associated energy functional. This fact guarantees global convergence of the proposed semismooth-Newton algorithm (see [7, Lem. 6.3]). Further, it is proved that the matrix $\mathcal{M}$ is uniformly positive definite if a condition like $\mathcal{N}(\vec{\mathbf{q}})_i \le g$ is fulfilled for all $i = 1,\dots,4m$. Thus, in order to guarantee the positive definiteness of the matrix $\mathcal{M}$, a projection procedure of the multiplier $\vec{\mathbf{q}}$ on the feasible set $\{\vec{\phi} \in \mathbb{R}^{4m} : \mathcal{N}(\vec{\phi})_i \le g, \text{ for all } i = 1,\dots,4m\}$ is performed when constructing $\mathcal{M}$. Due to this globalization procedure, we obtain a modified matrix $\widehat{\mathcal{M}}$, which is always positive definite. Moreover, in spite of the modifications of the system matrix in the Newton step, a superlinear rate of convergence is guaranteed under the assumption that the matrix $\mathcal{A}$ is positive definite (see [7, Sec. 6] and [25, Th. 4.1, Th. 4.2]).

In the case of time-dependent Bingham flow, the proposed approach is equivalent to analyze the problem, in each time step k, as a minimization problem of a given energy functional. Thus, the analysis developed in [7, Sec. 6] can be extended to the solution of system (4.8').

We then propose to use the following algorithm to solve this problem.

Algorithm (SSN)

1. Initalize $(\vec{\mathbf{y}}_0, \vec{\mathbf{q}}_0, \vec{p}_0) \in \mathbb{R}^{2n} \times \mathbb{R}^{4m} \times \mathbb{R}^l$ and set $k = 0$.
2. Estimate the active sets, *i.e.*, determine $\chi_{\mathcal{A}}$.
3. Project $\vec{\mathbf{q}}$ in the feasible set and construct matrix $\widehat{\mathcal{M}}$. Solve
$$\widehat{\mathcal{M}}\delta_{\mathbf{y}} = \mathcal{F}.$$
4. Compute $\delta_{\mathbf{q}}$ and δ_p according to (4.9) and (4.10) respectively.
5. Update $\vec{\mathbf{y}}_{k+1} := \vec{\mathbf{y}}_k + \delta_{\mathbf{y}}$, $\vec{\mathbf{q}}_{k+1} := \vec{\mathbf{q}}_k + \delta_{\mathbf{q}}$ and $\vec{p}_{k+1} := \vec{p}_k + \delta_p$.
6. Stop, or set $k := k+1$ and go to step 2.

Next, we propose a combined algorithm for the numerical solution of the time-dependent Bingham flow problem.

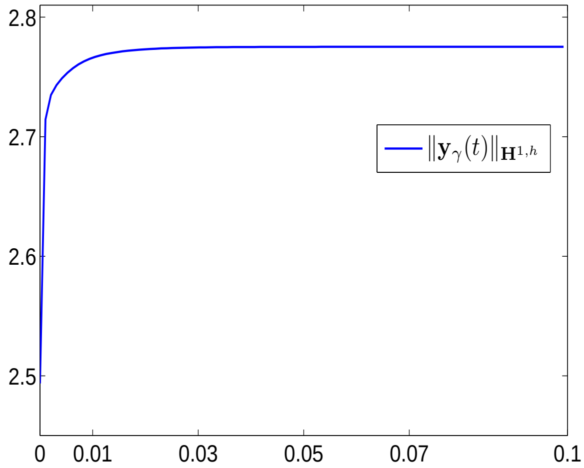

FIGURE 2. Experiment 1. $\mathbf{H}^1$-norm of the velocity field $\mathbf{y}(t)$. Variation of the kinetic energy of the flow.

Algorithm (BDF2-SSN)

(1) Set $\vec{\mathbf{y}}^{\,0} := \vec{\mathbf{y}}_0$. By using Algorithm **SSN**, solve systems (4.3) and (4.4) to obtain $\vec{\mathbf{y}}_0^{\,2/3}$ and $\vec{\mathbf{y}}_0^{\,4/3}$, respectively. Set $\vec{\mathbf{y}}^{\,1} := \frac{1}{2}(\vec{\mathbf{y}}_0^{\,2/3} + \vec{\mathbf{y}}_0^{\,4/3})$ and set $k = 0$.

(2) For $k = 0, \ldots, \mathcal{N} - 2$, define

$$\mathcal{F}(\vec{\mathbf{y}}^{\,k+2}, \vec{\mathbf{q}}^{\,k+2}, \vec{p}^{k+2}) := \begin{pmatrix} \left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}_\mu^h\right) \vec{\mathbf{y}}^{\,k+2} + \mathbf{Q}^h \vec{\mathbf{q}}^{\,k+2} + B^h \vec{p}^{k+2} - \widetilde{\mathbf{F}}^{\,k+2} \\ -(B^h)^\top \vec{\mathbf{y}}^{\,k+2} + \varsigma \vec{p}^{k+2} \\ -\gamma \widetilde{g} \mathcal{E}^h \vec{\mathbf{y}}^{\,k+2} + \max\left(\widetilde{g}\vec{e}, \gamma N(\mathcal{E}^h \vec{\mathbf{y}}^{\,k+2})\right) \star \vec{\mathbf{q}}^{\,k+2} \end{pmatrix},$$

and solve the nonlinear algebraic equation $\mathcal{F}(\vec{\mathbf{y}}^{\,k+2}, \vec{\mathbf{q}}^{\,k+2}, \vec{p}^{k+2}) = 0$, by using Algorithm **SSN**.

Remark 4.4. Note that, since the mass matrix $\mathbf{M}^h$ is positive definite, the matrix $\left(\frac{3}{2\delta t}\mathbf{M}^h + \mathbf{A}_\mu^h\right)$ is also positive definite (see [23, p. 148]), which implies that the approximated solution $\mathbf{y}^k$ is obtained, in any time iteration k, at a local superlinear rate.

5. COMPUTATIONAL RESULTS

In this section, we present two experiments which show the behavior of the Algorithm **BDF2-SSN**. First, we analyze the wall driven cavity flow and then the flow in a reservoir, *i.e.*, considering homogeneous Dirichlet boundary conditions.

Let us discuss the parameters of the algorithm. Let $\|\delta_h^k\| := \|\delta_\mathbf{y}\|_{\mathbf{H}^{1,h}} + \|\delta_\mathbf{q}\|_{(L^{2,h})^4} + \|\delta_p\|_{L^{2,h}}$, where the upper index k represents each time step and $\|\cdot\|_{\mathbf{H}^{1,h}}$, $\|\cdot\|_{(L^{2,h})^4}$ and $\|\cdot\|_{L^{2,h}}$ stand for the discrete versions of $\|\cdot\|_{\mathbf{H}^1}$, $\|\cdot\|_{(L^2)^4}$ and $\|\cdot\|_{L^2}$, respectively. We stop the inner algorithm **SSN**, in each time step k, as soon as $\|\delta_h^k\|$ is lower than $\sqrt{\epsilon}$, where ϵ denotes the machine accuracy ($\approx$ 2.2204e-016). Additionally, we choose $\varsigma := \sqrt{\epsilon}$, and fix the regularization parameter $\gamma = 10^3$. We consider uniform space meshes whose components have all the same area and measure the

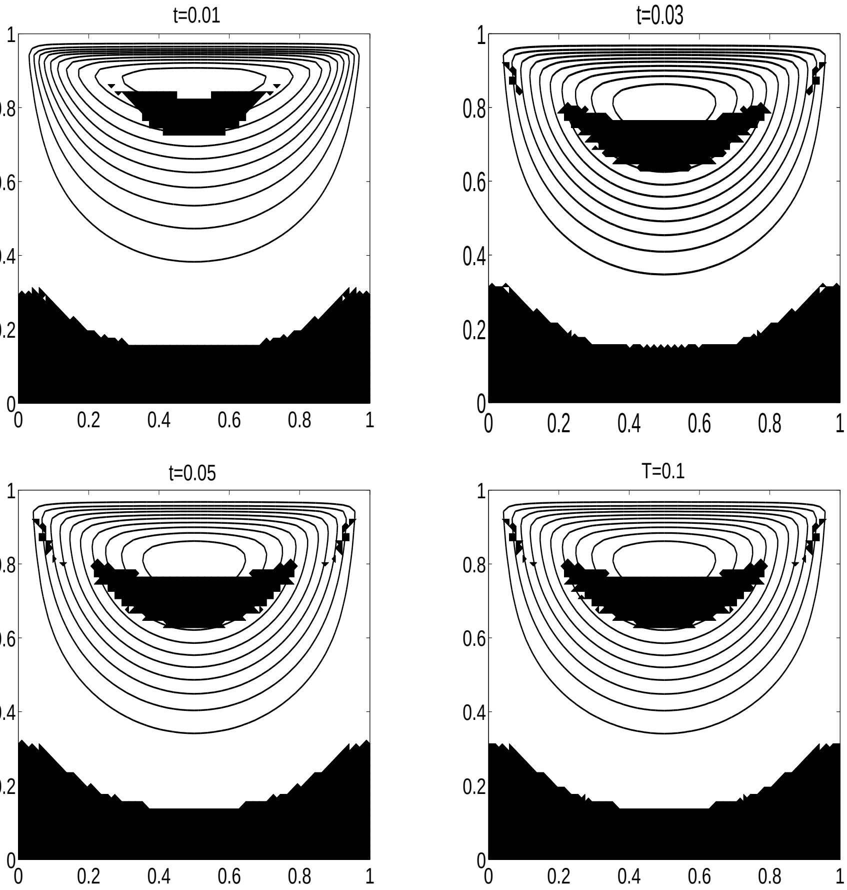

FIGURE 3. Experiment 1. Streamlines and rigid (black) and plastic (white) zones in the flow for: $t = 0.01$, $t = 0.03$, $t = 0.05$ and $T = 0.1$.

size of these meshes by the constant radius of the inscribed circumferences of the triangles in the mesh, represented by h. Further, we define the time step size as $\delta t := 0.1 * (h^{4/5})$.

Let us recall that the following experiments are based on the tests and examples presented in [9, 14, 15, 16, 24]. Furthermore, the obtained results with the proposed methodology are in good agreement with the results presented in these contributions.

5.1. **Experiment 1.** Here we analyze the wall driven cavity flow in the time interval $[0, 0.1]$. We consider that $\Omega :=]0,1[\times]0,1[$, $\Gamma_D := \{(x_1,x_2) \,:\, 0 < x_1 < 1 \; x_2 = 1\}$ and the following boundary condition

$$\mathbf{y}_D(x_1,x_2) := \left\{ \begin{array}{lll} \mathbf{0} & \text{if} & (x_1,x_2) \in \Gamma \setminus \Gamma_D, \\ \{1,0\} & \text{if} & (x_1,x_2) \in \Gamma_D. \end{array} \right.$$

We study a flow given by $\mathbf{f} = 0$, $g = 2.5$ and $\mu = 1$. We choose $\vec{\mathbf{y}}^{\,0} = \mathbf{0}$ and $\vec{\mathbf{y}}^{\,1} = \frac{1}{2}(\vec{\mathbf{y}}_0^{\,2/3} + \vec{\mathbf{y}}_0^{\,4/3})$, where the intermediate values $\vec{\mathbf{y}}_0^{\,2/3}$ and $\vec{\mathbf{y}}_0^{\,4/3}$ are calculated according to (4.3) and (4.4), respectively. Further, we consider a space mesh given by $h = 0.0033$ ($\approx 1/300$) and a time mesh given by δt= 0.001 ($= 0.1 * (h^{4/5})$)

t	0.01	0.03	0.05	0.1
	0.0187	0.1136	0.0905	–
$\|\delta_h^k\|$	4.63e-5	4.27e-4	8.48e-5	2.19e-4
	2.75e-9	1.48e-8	3.99e-9	4.05e-9
# it	8	4	3	2

TABLE 1. Values of $\|\delta_h^k\|$ in the last three inner iterations of the algorithm **SSN** and total number of inner iterations, for $t = 0.01$, $t = 0.03$, $t = 0.05$ and $T = 0.1$.

In Figure 2, we show the evolution of $\|\mathbf{y}(t)\|_{\mathbf{H}^{1,h}}$ in the time interval $[\delta t, 0.1]$. This figure gives insight into the variation of the kinetic energy of the flow. Since the norm of the velocity profile $\|\mathbf{y}(t)\|_{\mathbf{H}^{1,h}}$ tends to a constant limit as $t \to \infty$, we can conclude that the kinetic energy of the flow also tends to be constant as $t \to \infty$. This fact implies that the flow will reaches a steady state for a finite t (see [14, p. 954]).

In Figure 3, the calculated rigid and plastic regions, as well as the streamlines of the flow, for several instants, are presented. We can observe the evolution of the flow until $T = 0.1$.

With respect to the behavior of Algorithm **SSN**, the average number of iterations is 4.27. That means that the inner **SSN** algorithm only needs to solve, in average, approximately four $2n \times 2n$ systems of equations per time iteration. Further, in Table 1, we show the values of $\|\delta_h^k\|$ in the last three inner iterations of the algorithm **SSN**, for several time steps, as well as the total number of inner iterations in each of these time steps. Here, it is possible to appreciate the fast decay of the residuum in the last iterations. This fact helps us to show the superlinear convergence rate of the inner algorithm in each time step.

5.2. **Experiment 2.** In this experiment, we consider the flow in a reservoir, *i.e.,* we consider homogeneous Dirichlet boundary conditions and a forcing term given by

$$\mathbf{f}(x_1, x_2) := 300(x_2 - 0.5, 0.5 - x_1)$$

Further, we take $g = 10$ and $\mu = 1$, and we consider a space mesh given by $h = 0.0033$ ($\approx 1/300$) and a time mesh given by δt= 0.001 ($= 0.1 * (h^{4/5})$).

In Figure 4, the calculated rigid and plastic regions, as well as the streamlines in the flow, for several instants, are presented. We can observe the evolution of the flow until $T = 0.1$.

In this experiment, we investigate the behavior of the flow when the sources of energy, which provoke the flow, are cut off. Thus, we analyze the behavior of the fluid in two time intervals $[0, 0.1]$ and $[0.1, \infty)$. For the first interval we assume that the given forcing term rules the flow. We take the initial values $\vec{\mathbf{y}}^{\,0} = \mathbf{0}$ and $\vec{\mathbf{y}}^1 = \frac{1}{2}(\vec{\mathbf{y}}_0^{2/3} + \vec{\mathbf{y}}_0^{4/3})$, where the intermediate values $\vec{\mathbf{y}}_0^{2/3}$ and $\vec{\mathbf{y}}_0^{4/3}$ are calculated according to (4.3) and (4.4), respectively.

Then, at $t = 0.1$, we stop the flow by eliminating the only source of energy for the flow, given by the forcing term. Therefore, in the time interval $[0.1, \infty)$ we consider that $\mathbf{f} = \mathbf{0}$. In this interval, we consider the initial values $\vec{\mathbf{y}}^{\,0} = \vec{\mathbf{y}}^{0.1}$ and $\vec{\mathbf{y}}^1 = \frac{1}{2}(\vec{\mathbf{y}}_{0.1}^{2/3} + \vec{\mathbf{y}}_{0.1}^{4/3})$, where the intermediate values $\vec{\mathbf{y}}_{0.1}^{2/3}$ and $\vec{\mathbf{y}}_{0.1}^{4/3}$ are also calculated according to (4.3) and (4.4), respectively.

In Figure 5, we present the evolution of $\|\mathbf{y}(t)\|_{\mathbf{H}^{1,h}}$ in the time interval $[0, \infty)$. As in the previous experiment, here we observe that the kinetic energy of the flow tends to a constant value implying that the fluid will reach a steady state in finite time. Moreover, as expected, the kinetic energy goes fast to zero, since there is no source of energy to provoke movement. This fact shows that our approach fulfills the propertie that $\mathbf{y}(t) \to 0$ in finite time if $\mathbf{f} = 0$ and $\mathbf{y} = \mathbf{0}$ on Σ.

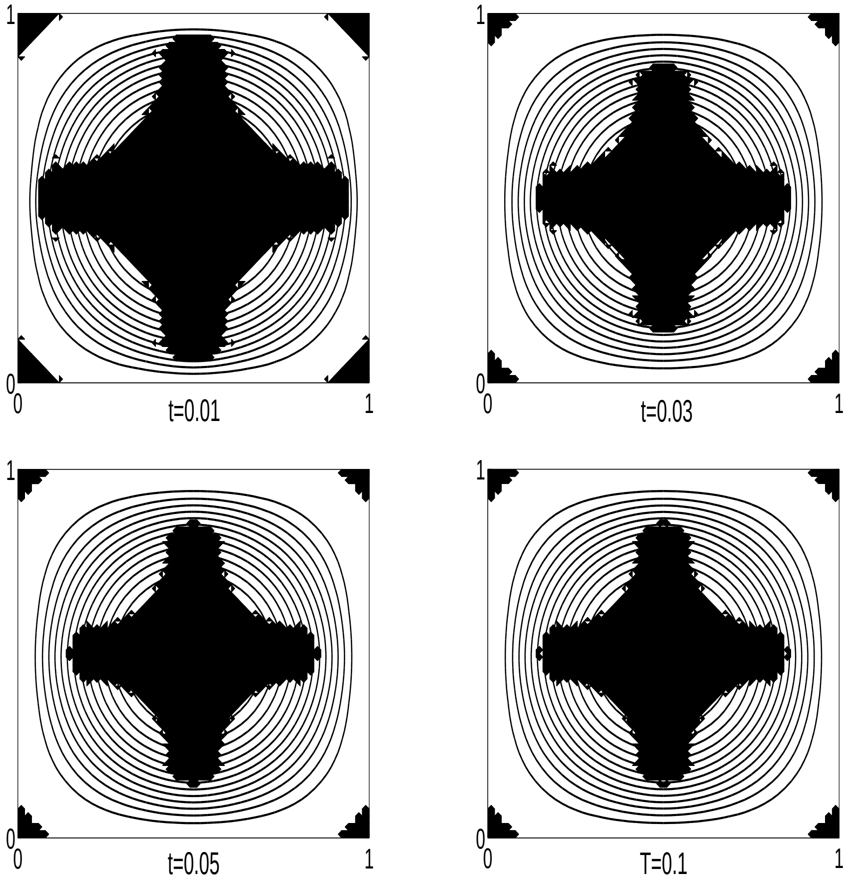

FIGURE 4. Experiment 2. Streamlines and rigid (black) and plastic (white) zones in the flow for: $t = 0.01$, $t = 0.03$, $t = 0.05$ and $T = 0.1$.

APPENDIX

Proof of Theorem 3.1. Let us consider a base $\{\mathbf{w}_1, \ldots, \mathbf{w}_m, \ldots\} \subset V$, free and total in V, and let us introduce the following problem, for $m \in \mathbb{N}$: find $\mathbf{y}_m(t) = \sum_{j=1}^{m} g_{jm}(t)\,\mathbf{w}_j$ such that

$$(5.1)\qquad \begin{cases} (\partial_t \mathbf{y}_m(t)\,,\, \mathbf{w}_i)_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y}_m(t), \mathbf{w}_i) + \mathbf{b}(\mathbf{y}_m(t), \mathcal{E}\mathbf{w}_i) \\ \qquad\qquad + \mathbf{c}(\mathbf{y}_m(t), \mathbf{y}_m(t), \mathbf{w}_i) = (\mathbf{f}(t)\,,\, \mathbf{w}_i)_{\mathbf{L}^2}\,, \text{ for } i = 1, \ldots, m, \\ \qquad\qquad \mathbf{y}_m(0) = \mathbf{y}_{0m} \in V_m := span\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}, \end{cases}$$

with $\lim_{m\to\infty} \mathbf{y}_{0m} = \mathbf{y}_0 \in H$.

We divide the proof in four steps.

i)Existence and uniqueness of solutions for (5.1). First, note that system (5.1) constitutes a

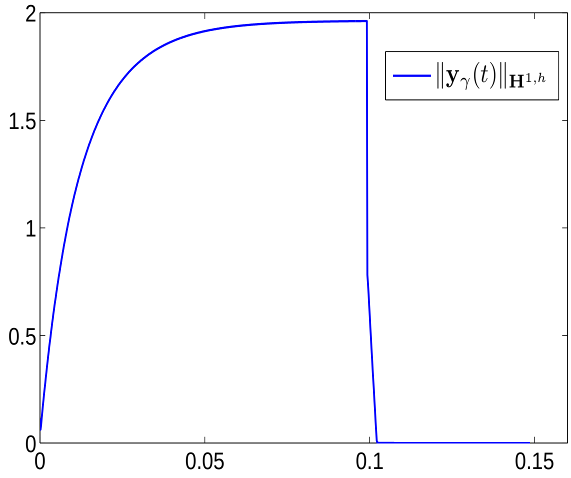

FIGURE 5. Experiment 2. $\mathbf{H}^1$-norm of the velocity field $\mathbf{y}(t)$. Variation of the kinetic energy of the flow

nonlinear differential system with respect to the variables $g_{1m}, \dots, g_{mm}$, namely

$$\begin{aligned}\textstyle\sum_{j=1}^m \partial_t g_{jm}(t)\,(\mathbf{w}_j\,,\,\mathbf{w}_i)_{\mathbf{L}^2} + \gamma \sum_{j=1}^m g_{jm}(t)\,(\mathcal{E}\mathbf{w}_j\,,\,\mathcal{E}\mathbf{w}_i)_{\mathbf{L}^{2\times 2}} \\ +\widetilde{g}\gamma\left(\frac{\sum_{j=1}^m g_{jm}(t)\mathcal{E}\mathbf{w}_j}{\max(\widetilde{g},\gamma\|\sum_{j=1}^m g_{jm}(t)\mathcal{E}\mathbf{w}_j\|)}\,,\,\mathcal{E}\mathbf{w}_i\right)_{\mathbf{L}^{2\times 2}} \\ +\textstyle\sum_{j,k=1}^m \mathbf{c}(\mathbf{w}_j,\mathbf{w}_k,\mathbf{w}_i)g_{im}(t)g_{km}(t) = (\mathbf{f}(t)\,,\,\mathbf{w}_i)_{\mathbf{L}^2}\,.\end{aligned}$$

By inverting the matrix $(\mathbf{w}_j,\,\mathbf{w}_i)_{\mathbf{L}^2}$, the system can be reformulated as a standard ODE system. Moreover, we have that

$$\left\|\frac{\widetilde{g}\gamma\sum_{j=1}^m g_{jm}(t)\mathcal{E}\mathbf{w}_j}{\max(\widetilde{g},\gamma\|\sum_{j=1}^m g_{jm}(t)\mathcal{E}\mathbf{w}_j\|)}\right\|_{\mathbf{L}^{2\times 2}} \le \widetilde{g},$$

which implies that the Carathéodory hypotheses are verified (see [6, p. 43] or [11, p. 45]) and, therefore, there exists a maximal solution for (3.2), on some interval $[0, t_m]$. The choice $t_m = T$ can be made from the a-priori estimates derived next.

ii)A-priori estimate. Multiplying the equations in (5.1) by $g_{im}(t)$ and adding them, we obtain that

$$\begin{aligned}(\mathbf{y}_m'(t)\,,\,\mathbf{y}_m(t))_{\mathbf{L}^2} + \mu\|\mathbf{y}_m(t)\|_V^2 \quad &+ \quad (\mathbf{b}(\mathbf{y}_m(t))\,,\,\mathcal{E}\mathbf{y}_m(t))_{\mathbf{L}^2} \\ &+ \quad \mathbf{c}(\mathbf{y}_m(t),\mathbf{y}_m(t),\mathbf{y}_m(t)) = (\mathbf{f}(t)\,,\,\mathbf{y}_m(t))_{\mathbf{L}^2}\,,\end{aligned}$$

which, from the properties of $\mathbf{c}(\cdot,\cdot,\cdot)$ and since $(\mathbf{b}(\mathbf{v}),\mathcal{E}\mathbf{v})_{\mathbf{L}^2} \ge 0$, for all $\mathbf{v} \in V$, implies that

$$\begin{aligned}\tfrac{d}{dt}\|\mathbf{y}_m(t)\|_H^2 + 2\mu\|\mathbf{y}_m(t)\|_V^2 \quad &\le \quad 2\|\mathbf{f}(t)\|_{V'}\|\mathbf{y}_m(t)\|_V \\ &\le \quad \mu\|\mathbf{y}_m(t)\|_V^2 + \tfrac{1}{\mu}\|\mathbf{f}(t)\|_{V'}^2.\end{aligned}$$

Consequently, proceeding as in the proof of [26, Th. 3.1, p. 193], it follows that $\mathbf{y}_m$ is uniformly bounded in $L^\infty(H) \cap L^2(V)$. Moreover, $\|\mathbf{y}_m(T)\|_H$ is bounded and, therefore, $t_m = T$ in step $i)$.

iii)Boundedness in $\mathcal{H}^\theta(\mathbb{R},V,H)$. First, let $\tilde{\mathbf{u}}$ represent the extension of $\mathbf{u}(t)$ from $[0,T]$ to $\mathbb{R}$, for $x\in\Omega$, *i.e.*, $\tilde{\mathbf{u}}(x,t):\Omega\times\mathbb{R}\to V$ and

$$\tilde{\mathbf{u}}(x,t) := \begin{cases} \mathbf{u}(x,t) & \text{if } (x,t)\in Q \\ 0 & \text{elsewhere.}\end{cases}$$

Furthermore, let $\widehat{\mathbf{u}}(\tau)$ represent the Fourier transform of the function $\tilde{\mathbf{u}}$. Next, by following [26, p. 185], we define the space $\mathcal{H}^\theta(\mathbb{R},V,H)$ by

$$\mathcal{H}^\theta(\mathbb{R},V,H) := \left\{\mathbf{u}\in L^2(\mathbb{R};V)\,:\,\tau\to|\tau|^\theta\widehat{\mathbf{u}}(\tau)\in L^2(\mathbb{R},H)\right\},\ \theta>0,$$

Proceeding as in [26] we define the operators $\mathbf{A}:V\to V'$, $\mathbf{B}:V\to V'$ and $\mathbf{C}:V\to V'$ by

$$\begin{aligned} \langle\mathbf{A}\mathbf{y},\mathbf{v}\rangle_{V',V} &= \mathbf{a}(\mathbf{y},\mathbf{v}),\ \text{for all } \mathbf{v}\in V,\\ \langle\mathbf{B}\mathbf{y},\mathbf{v}\rangle_{V',V} &= (\mathbf{b}(\mathbf{y}),\mathcal{E}\mathbf{v})_{\mathbf{L}^2}\,,\ \text{for all } \mathbf{v}\in V\\ \langle\mathbf{C}\mathbf{y},\mathbf{v}\rangle_{V',V} &= \mathbf{c}(\mathbf{y},\mathbf{y},\mathbf{v}),\ \text{for all } \mathbf{v}\in V,\end{aligned} \tag{5.2}$$

In order to proof that $\tilde{\mathbf{y}}_m$ is bounded in $\mathcal{H}^\theta(\mathbb{R},V,H)$, it suffices to verify that

$$\int_0^T\|\mathbf{f}_m(t)\|_{V'}\,dt\le \mathit{const.},\ \text{for all } m, \tag{5.3}$$

where $\mathbf{f}_m(t):=\mathbf{f}(t)-\mu\mathbf{A}\mathbf{y}_m(t)-\mathbf{B}\mathbf{y}_m(t)-\mathbf{C}\mathbf{y}_m(t)$ (see [26, pp. 193-194]).

Since $\|\mathbf{b}(\mathbf{w})\|_{\mathbf{L}^2}\le\widetilde{g}$, for all $\mathbf{w}\in V$, we have that

$$\begin{aligned}\|\mathbf{B}\mathbf{y}_m\|_{V'} &= \sup_{\|\mathbf{v}\|_V=1}|\langle\mathbf{B}\mathbf{y}_m(t)\,,\,\mathbf{v}\rangle_{V',V}|\\ &= \sup_{\|\mathbf{v}\|_V=1}|(\mathbf{b}(\mathbf{y}_m(t)),\mathcal{E}\mathbf{v})_{\mathbf{L}^2}|\\ &\le \sup_{\|\mathbf{v}\|_V=1}\|\mathbf{b}(\mathbf{y}_m(t))\|_{\mathbf{L}^2}\|\mathcal{E}\mathbf{v}\|_{\mathbf{L}^2}\le\widetilde{g}.\end{aligned} \tag{5.4}$$

Furthermore, we know that $\|\mathbf{C}\mathbf{w}\|_{V'}\le c\|\mathbf{w}\|_V^2$, for all $\mathbf{w}\in V$ (see [26, p. 191]), which implies, together with (5.4), that

$$\int_0^T\|\mathbf{f}_m(t)\|_{V'}\,dt\le\int_0^T\big(\|\mathbf{f}(t)\|_{V'}+\mu\|\mathbf{y}_m(t)\|_V+\widetilde{g}+c\|\mathbf{y}_m(t)\|_V^2\big)\,dt.$$

Due to the fact that $\mathbf{y}_m(t)$ is uniformly bounded in $L^\infty(H)\cap L^2(V)$, we verify (5.3). Consequently, we get that $\tilde{\mathbf{y}}$ is bounded in $\mathcal{H}^\theta(\mathbb{R},V,H)$ for some $\theta>0$ (for more details see [26, p. 194]).

iv)Convergence along a subsequence. Since by step *ii*) the sequence $\{\mathbf{y}_m\}$ is uniformly bounded in $L^\infty(H)\cap L^2(V)$, there exists a subsequence $\{\mathbf{y}_\ell\}\subset\{\mathbf{y}_m\}$ such that

$$\begin{aligned}&\mathbf{y}_\ell\overset{*}{\rightharpoonup}\mathbf{y}\ \text{in } L^\infty(H)\ \text{weakly star,}\\ &\mathbf{y}_\ell\rightharpoonup\mathbf{y}\ \text{in } L^2(V)\ \text{weakly,}\\ &\mathbf{y}_\ell(T)\rightharpoonup\xi\ \text{in } L^2(\Omega)\ \text{weakly.}\end{aligned}$$

Additionally, since $\|\mathbf{B}\mathbf{w}\|_{V'}\le\widetilde{g}$, for all $\mathbf{w}\in V$, it follows that

$$\mathbf{B}\mathbf{y}_\ell\rightharpoonup\chi\,\text{weakly in } L^2(V').$$

Since $\tilde{\mathbf{y}}_m$ is bounded in $\mathcal{H}^\theta(\mathbb{R},V,H)$, for some $\theta>0$, it follows (see [26, Th. 2.2, p. 186]) that

$$\mathbf{y}_\ell\to\mathbf{y}\,\text{strongly in } L^2(H)$$

and, consequently, (see [26, Lem. 3.2, p. 196]), we have that

$$\int_0^T\mathbf{c}(\mathbf{y}_\ell,\mathbf{y}_\ell,\mathbf{v})\,dt\to\int_0^T\mathbf{c}(\mathbf{y},\mathbf{y},\mathbf{v})\,dt,\ \text{for all } \mathbf{v}\in C^1(\overline{Q}).$$

Multiplying the equations in (5.1) by a continuous differentiable function $\psi : [0,T] \to \mathbb{R}$, such that $\psi(T) = 0$, and integrating by parts yields

$$-\int_0^T (\mathbf{y}_\ell(t)\,,\,\psi'(t)\,\mathbf{w}_i)_{\mathbf{L}^2} + \int_0^T \mathbf{a}(\mathbf{y}_\ell(t),\psi(t)\,\mathbf{w}_i) + \int_0^T \langle \mathbf{B}\mathbf{y}_\ell(t)\,,\,\psi(t)\,\mathbf{w}_i\rangle_{V',V}$$
$$+\int_0^T \mathbf{c}(\mathbf{y}_\ell(t),\mathbf{y}_\ell(t),\psi(t)\mathbf{w}_i) = \psi(0)\,(\mathbf{y}_{0\ell}\,,\,\mathbf{w}_i)_{\mathbf{L}^2} + \int_0^T (\mathbf{f}(t)\,,\,\psi(t)\mathbf{w}_i)_{\mathbf{L}^2}\,.$$

By passing to the limit in the previous equation, we obtain that

$$\begin{aligned}&-\int_0^T (\mathbf{y}(t)\,,\,\psi'(t)\,\mathbf{v})_{\mathbf{L}^2} + \int_0^T \mathbf{a}(\mathbf{y}(t),\psi(t)\,\mathbf{v}) + \int_0^T \langle \chi\,,\,\psi(t)\,\mathbf{v}\rangle_{V',V}\\ &\quad+\int_0^T \mathbf{c}(\mathbf{y}(t),\mathbf{y}(t),\psi(t)\mathbf{v}) = \psi(0)\,(\mathbf{y}_0\,,\,\mathbf{v})_{\mathbf{L}^2} + \int_0^T (\mathbf{f}(t)\,,\,\psi(t)\mathbf{v})_{\mathbf{L}^2}\,,\end{aligned}\tag{5.5}$$

for all $\mathbf{v}$ which is a linear combination of $\mathbf{w}_i$. Since, $\overline{\text{span}\{\mathbf{w}_1,\ldots,\mathbf{w}_m,\ldots\}} = V$, equation (5.5) holds for all $\mathbf{v} \in V$. Furthermore, taking $\psi \in \mathcal{D}(0,T)$, we get that $\mathbf{y}$ satisfies

$$(\partial_t \mathbf{y}(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + \mathbf{a}(\mathbf{y},\mathbf{v}) + \langle \chi\,,\,\mathbf{v}\rangle_{V',V} + \mathbf{c}(\mathbf{y},\mathbf{y},\mathbf{v}) = (\mathbf{f}(t)\,,\,\mathbf{v})_{\mathbf{L}^2}\,,\ \text{for all } \mathbf{v}\in V,\tag{5.6}$$

in distributional sense.

Next, in order to verify the initial and final condition, we consider the system

$$(\tilde{\mathbf{y}}_\ell(t)\,,\,\mathbf{w}_j)_{\mathbf{L}^2} + \left\langle \widetilde{\mathbf{A}\mathbf{y}_\ell}(t)\,,\,\mathbf{w}_j\right\rangle_{V',V} + \left\langle \widetilde{\mathbf{B}\mathbf{y}_\ell}(t)\,,\,\mathbf{w}_j\right\rangle_{V',V} + \left\langle \widetilde{\mathbf{C}\mathbf{y}_\ell}(t)\,,\,\mathbf{w}_j\right\rangle_{V',V} =$$
$$\left(\widetilde{\mathbf{f}}(t)\,,\,\mathbf{w}_j\right)_{\mathbf{L}^2} + (\mathbf{y}_{0\ell}\,,\,\mathbf{w}_j)_{\mathbf{L}^2}\,\delta(t-0) - (\mathbf{y}_\ell(T)\,,\,\mathbf{w}_j)_{\mathbf{L}^2}\,\delta(t-T),$$

where $\tilde{\mathbf{y}}_\ell$, $\widetilde{\mathbf{A}\mathbf{y}_\ell}$, $\widetilde{\mathbf{B}\mathbf{y}_\ell}$, $\widetilde{\mathbf{C}\mathbf{y}_\ell}$ and $\widetilde{\mathbf{f}}$ stand for the extension to $\mathbb{R}$ of $\mathbf{y}_\ell$, $\mathbf{A}\mathbf{y}_\ell$, $\mathbf{B}\mathbf{y}_\ell$, $\mathbf{C}\mathbf{y}_\ell$ and $\mathbf{f}$, respectively. Passing to the limit as $\ell \to \infty$ yields

$$(\tilde{\mathbf{y}}(t)\,,\,\mathbf{w}_j)_{\mathbf{L}^2} + \left\langle \widetilde{\mathbf{A}\mathbf{y}}(t)\,,\,\mathbf{w}_j\right\rangle_{V',V} + \langle \widetilde{\chi}(t)\,,\,\mathbf{w}_j\rangle_{V',V} + \left\langle \widetilde{\mathbf{C}\mathbf{y}}(t)\,,\,\mathbf{w}_j\right\rangle_{V',V} =$$
$$\left(\widetilde{\mathbf{f}}(t)\,,\,\mathbf{w}_j\right)_{\mathbf{L}^2} + (\mathbf{y}_0\,,\,\mathbf{w}_j)_{\mathbf{L}^2}\,\delta(t-0) - (\xi\,,\,\mathbf{w}_j)_{\mathbf{L}^2}\,\delta(t-T),\ \text{for all } j,$$

which by density implies that

$$\left\{\begin{aligned}&(\tilde{\mathbf{y}}(t)\,,\,\mathbf{v})_{\mathbf{L}^2} + \left\langle \widetilde{\mathbf{A}\mathbf{y}}(t)\,,\,\mathbf{v}\right\rangle_{V',V} + \langle \widetilde{\chi}(t)\,,\,\mathbf{v}\rangle_{V',V} + \left\langle \widetilde{\mathbf{C}\mathbf{y}}(t)\,,\,\mathbf{v}\right\rangle_{V',V} =\\ &\qquad\left(\widetilde{\mathbf{f}}(t)\,,\,\mathbf{v}\right)_{\mathbf{L}^2} + (\mathbf{y}_0\,,\,\mathbf{v})_{\mathbf{L}^2}\,\delta(t-0) - (\xi\,,\,\mathbf{v})_{\mathbf{L}^2}\,\delta(t-T).\end{aligned}\right.\tag{5.7}$$

Now, since (5.6) holds in distributional sense, it follows, by restricting (5.7) to $[0,T]$ and comparing with (5.6), that

$$\mathbf{y}(0) = \mathbf{y}_0 \quad \text{and} \quad \xi = \mathbf{y}(T).$$

Finally, it has to be verified that $\chi = \mathbf{B}\mathbf{y}$. Since, $\mathbf{B}\mathbf{y}$ corresponds to the derivative of the convex functional

$$\int_\Omega \boldsymbol{\Psi}(\mathcal{E}\mathbf{y}),$$

with $\boldsymbol{\Psi} : \mathbb{R}^{d\times d} \to \mathbb{R}$ defined by

$$\boldsymbol{\Psi} := \begin{cases} \widetilde{g}\|A\| - \frac{\widetilde{g}^2}{2\gamma} & \text{if } \|A\| \geq \frac{\widetilde{g}}{\gamma}\\ \frac{\gamma}{2}\|A\|^2 & \text{if } \|A\| < \frac{\widetilde{g}}{\gamma},\end{cases}$$

it constitutes a monotone and hemicontinuous operator (see [21, p. 158]). Therefore,

$$\chi_\ell := \int_0^T \langle \mathbf{B}\mathbf{y}_\ell(t) - \mathbf{B}\mathbf{v}(t)\,,\,\mathbf{y}_\ell(t) - \mathbf{v}(t)\rangle_{V',V}\,dt \geq 0,\ \text{for all } \mathbf{v}\in L^2(V).\tag{5.8}$$

From (5.1) and the properties of $\mathbf{c}(\cdot,\cdot,\cdot)$, it follows that

$$\int_0^T \langle \mathbf{B}\mathbf{y}_\ell(t)\,,\,\mathbf{y}_\ell(t)\rangle_{V',V}\;dt = \int_0^T (\mathbf{f}(t)\,,\,\mathbf{y}_\ell(t))_{\mathbf{L}^2}\;dt$$
$$- \int_0^T \langle \mathbf{A}\mathbf{y}_\ell(t)\,,\,\mathbf{y}_\ell(t)\rangle_{V',V}\;dt + \tfrac{1}{2}\|\mathbf{y}_{0\ell}\|_H^2 - \tfrac{1}{2}\|\mathbf{y}_\ell(T)\|_H^2,$$

and, therefore,

$$\chi_\ell = \int_0^T (\mathbf{f}(t)\,,\,\mathbf{y}_\ell(t))_{\mathbf{L}^2}\,dt - \int_0^T \langle \mathbf{A}\mathbf{y}_\ell(t)\,,\,\mathbf{y}_\ell(t)\rangle_{V',V}\;dt + \tfrac{1}{2}\|\mathbf{y}_{0\ell}\|_H^2$$
$$-\tfrac{1}{2}\|\mathbf{y}_\ell(T)\|_H^2 - \int_0^T \langle \mathbf{B}\mathbf{y}_\ell(t)\,,\,\mathbf{v}\rangle_{V',V}\;dt - \int_0^T \langle \mathbf{B}\mathbf{v}(t)\,,\,\mathbf{y}_\ell - \mathbf{v}\rangle_{V',V}\;dt,$$

which, since $\liminf \|\mathbf{y}_\ell(T)\|_H^2 \geq \|\mathbf{y}(T)\|_H^2$, implies that

$$\begin{aligned}\limsup_{\ell\to\infty} \chi_\ell \leq \int_0^T (\mathbf{f}(t)\,,\,\mathbf{y})_{\mathbf{L}^2}\,dt - \int_0^T \langle \mathbf{A}\mathbf{y}(t)\,,\,\mathbf{y}(t)\rangle_{V',V}\;dt + \tfrac{1}{2}\|\mathbf{y}_0\|_H^2 \\ -\tfrac{1}{2}\|\mathbf{y}(T)\|_H^2 - \int_0^T \langle \chi(t)\,,\,\mathbf{v}\rangle_{V',V}\,dt - \int_0^T \langle \mathbf{B}\mathbf{v}\,,\,\mathbf{y}(t)-\mathbf{v}\rangle_{V',V}\,.\end{aligned} \tag{5.9}$$

Multiplying (3.3) by $\mathbf{y}$ and integrating by parts with respect to t, we obtain that

$$\begin{aligned}\int_0^T \Big((\mathbf{f}(t)\,,\,\mathbf{y}(\mathbf{t}))_{\mathbf{L}^2} - \langle \mathbf{A}\mathbf{y}(t)\,,\,\mathbf{y}(t)\rangle_{V',V}\Big)\,dt \\ +\tfrac{1}{2}\|\mathbf{y}_0\|_H^2 - \tfrac{1}{2}\|\mathbf{y}(T)\|_H^2 = \int_0^T \langle \chi(t)\,,\,\mathbf{y}(t)\rangle_{V',V}\,dt,\end{aligned} \tag{5.10}$$

which, together with (5.8) and (5.9), implies that

$$\int_0^T \langle \chi(t) - \mathbf{B}\mathbf{v}\,,\,\mathbf{y}(t)-\mathbf{v}\rangle_{V',V}\,dt \geq 0, \text{ for all } \mathbf{v}\in V. \tag{5.11}$$

Using the hemicontinuity of $\mathbf{B}$ and taking $\mathbf{v} = \mathbf{y} - \lambda\mathbf{w}$, $\lambda > 0$ with $\mathbf{w} \in L^2(V)$, we obtain, from (5.11), that

$$\lambda \int_0^T \langle \chi(t) - \mathbf{B}(\mathbf{y}(t) - \lambda\mathbf{w})\,,\,\mathbf{w}\rangle_{V',V}\,dt \geq 0,$$

which implies that

$$\int_0^T \langle \chi(t) - \mathbf{B}(\mathbf{y}(t) - \lambda\mathbf{w})\,,\,\mathbf{w}\rangle_{V',V}\,dt \geq 0. \tag{5.12}$$

Taking the limit as $\lambda \to 0$ in (5.12), we get, by Lebesgue's Theorem, that

$$\int_0^T \langle \chi(t) - \mathbf{B}\mathbf{y}(t)\,,\,\mathbf{w}\rangle_{V',V}\,dt \geq 0, \text{ for all } \mathbf{w}\in V,$$

and, therefore, $\chi = \mathbf{B}\mathbf{y}$.

For the uniqueness, the proof is similar to the one for the unregularized problem given in [10]. $\square$

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