Magnus integrator and successive approximation for solving time-dependent problems

Jürgen Geiser *
Humboldt Univeristät zu Berlin, D- Berlin, Germany

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Abstract

The Magnus expansion has been intensely studied and widely applied for solving explicitly time-dependent problems. Due to its exponential character, it is rather difficult to derive practical algorithms beyond the sixth-order. An alternative method is based on successive approximation methods, that taken into account the temporally inhomogeneous equation (method of Tanabe and Sobolevski). In this work, we show that the recently derived ideas of the successive approximation method in a splitting method. Examples are discussed.

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1 Introduction

In this paper we concentrate on solving linear evolution equations, such as the time-dependent differential equation,

$$\partial_t \, u = A(t)u, \; u(0) = u_0, \tag{1}$$

where A can be an unbounded and time-dependent operator. For solving Hamiltonian problems, it is often the case that $A(t) = T + V(t)$, where only the potential operator $V(t)$ is time-dependent. Our main focus will be to consider and contrast higher order algorithms derived from the Magnus expansion with those from Successive Approximation method. The higher order Magnus algorithms have been well studied by Blanes *et al.*, see their recent comprehensive review[8]. Successive Approximation methods can be applied to a iterative splitting method tested in Refs. [17] and [18].

The Magnus expansion[4, 8] is an attractive and widely applied method of solving explicitly time-dependent problems. However, it requires computing time-integrals and nested commutators to higher orders. Successive approximation is based on recursive integral formulations in which an iterative method is enforce the time dependency.

The paper is outlined as follows: In Section 2, we summarizes the Magnus expansion and its application to Hamiltonian systems. Further, we show how successive approximation method can be applied to any exponential-splitting algorithms. In Section 3 we present the numerical results of the splitting schemes. In Section 4, we present the error analysis In Section 5, we briefly summarize our results.

*email: geiser@mathematik.hu-berlin.de

2 Introduction to splitting methods

In the next subsections, we introduce the underlying splitting methods.

2.1 Splitting method based on the Magnus expansion

The Magnus integrator was introduced as a tool to solve non-autonomous linear differential equations for linear operators of the form

$$\frac{dY}{dt} = A(t)Y(t) \ , \tag{2}$$

with solution

$$Y(t) = \exp(\Omega(t))Y(0) \ . \tag{3}$$

This can be expressed as:

$$Y(t) = \mathcal{T}\left(\exp(\int_0^t A(s)\ ds\right)Y(0) \ , \tag{4}$$

where the time-ordering operator $\mathcal{T}$ is given in [13].

The Magnus expansion is defined as:

$$\Omega(t) = \sum_{n=1}^{\infty} \Omega_n(t) \ , \tag{5}$$

where the first few terms are[8]:

$$\begin{aligned}
\Omega_1(t) &= \int_0^t dt_1 A_1 \\
\Omega_2(t) &= \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [A_1, A_2] \\
\Omega_3(t) &= \frac{1}{6}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([A_1, [A_2, A_3] + [[A_1, A_2], A_3]) \\
&\cdots\cdots \qquad \text{etc.}
\end{aligned} \tag{6}$$

where $A_n = A(t_n)$. In practice, it is more useful to define the nth order Magnus operator

$$\Omega^{[n]}(t) = \Omega(t) + O(t^{n+1}) \tag{7}$$

such that

$$Y(t) = \exp\left[\Omega^{[n]}(t)\right]Y(0) + O(t^{n+1}). \tag{8}$$

Thus the second-order Magnus operator is

$$\begin{aligned}
\Omega^{[2]}(t) &= \int_0^t dt_1 A(t_1) \\
&= tA\left(\frac{1}{2}t\right) + O(t^3)
\end{aligned} \tag{9}$$

and a fourth-order Magnus operator[8] is

$$\Omega^{[4]}(t) = \frac{1}{2}t(A_1 + A_2) - c_3 t^2 [A_1, A_2] \tag{10}$$

where $A_1 = A(c_1 t)$, $A_2 = A(c_2 t)$ and

$$c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \qquad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \qquad c_3 = \frac{\sqrt{3}}{12}. \tag{11}$$

The necessity of doing time integrations and evaluating nested commutators make Magnus integrators beyond the fourth-order rather complex. For the ubiquitous case of

$$A(t) = T + V(t), \tag{12}$$

one has

$$\begin{aligned} e^{\Omega^{[2]}(t)} &= e^{t[T+V(t/2)]} \\ &= e^{\frac{1}{2}tT} e^{tV(t/2)} e^{\frac{1}{2}tT} + O(t^3) \end{aligned} \tag{13}$$

and

$$e^{\Omega^{[4]}(t)} = e^{c_3 t(V_2 - V_1)} e^{t(T + \frac{1}{2}(V_1+V_2))} e^{-c_3 t(V_2 - V_1)} \tag{14}$$

where

$$V_1 = V(c_1 t), \qquad V_2 = V(c_2 t). \tag{15}$$

In the general operator case, because the Magnus expansion generates more terms in the exponential, more complex splittings are necessary. For example, the central exponential in (14) must be further splitted to fourth-order in order to maintain the fourth-order character of the overall algorithm.

Remark 2.1 *The Magnus expansion can be generalized in different ways, e.g., commutator-less expansion, Volsamber iterative method, Floquet-Magnus expansion, etc..[8]. However, none reduces the number of needed operators at high orders.*

2.2 Splitting method based on Successive Approximation

Instead of the Magnus series (5) for solving explicit time-dependent problems, one can also directly implement successive approximation method.

The problem is given as:

$$\frac{\partial Y}{\partial t} = A(t)Y(t), \; a \le t \le b \tag{16}$$

We rewrite:

$$\frac{\partial Y}{\partial t} = A(a)Y(t) + (A(t) - A(a))Y(t) \tag{17}$$

The abstract integral is given as, by the so called Duhamel Principle:

$$Y(t) = \exp((t-a)A(a))Y_0 + \int_a^t \exp((t-s)A(a))(A(s) - A(a))Y(s)\, ds \tag{18}$$

With successive approximation we obtain:

$$Y_1(t) = \exp((t-a)A(a))Y_0, \tag{19}$$

$$\ldots$$

$$Y_{n+1}(t) = \exp((t-a)A(a))Y_0 + \int_a^t \exp((t-s)A(a))(A(s) - A(a))Y_n(s)\, ds \tag{20}$$

and formally we have:

$$Y(t) = \exp((t-a)A(a))Y_0 + \int_a^t \exp((t-s)A(a))R(s,a)Y_0 \, ds \tag{21}$$

$$\tag{22}$$

$$R(t,s) = \sum_{m=1}^{\infty} R_m(t,s), \tag{23}$$

$$R_1(t) = \begin{cases} (A(t)-A(s))\exp((t-s)A(a)) \, ds & , s < t \\ 0 & , s \geq t \end{cases}, \tag{24}$$

$$R_m(t) = \int_s^t R_1(t,\sigma)R_{m-1}(\sigma,t) \, d\sigma. \tag{25}$$

2.3 Algorithm for Successive Approximation

In this section, we will construct a new numerical algorithm in order to use successive approximation as a computational tool. To illustrate how this task can be accomplished, define a solution, for one time step, h, in the interval $[t_n, t_n+h]$, is given by

$$y(t_n+h) = e^{hA_a}y(t_n) + \int_{t_n}^{t_n+h} e^{(t_n+h-s)A_a}(A(s)-A_a)y(s) \, ds \tag{26}$$

where $A_a = A(a)$ is $n \times n$ constant matrix. Successive approximation steps then can be read as

$$y_1(t_n+h) = e^{hA_a}y(t_n), \tag{27}$$

$$y_2(t_n+h) = e^{hA_a}y(t_n) + \int_{t_n}^{t_n+h} e^{(t_n+h-s)A_a}(A(s)-A_a)y_1(s) \, ds \tag{28}$$

$$\ldots.. \tag{29}$$

$$y_k(t_n+h) = e^{hA_a}y(t_n) + \int_{t_n}^{t_n+h} e^{(t_n+h-s)A_a}(A(s)-A_a)y_{k-1}(s) \, ds. \tag{30}$$

After approximating the integrals in each iterations by quadrature formulas, we rewrite the solutions as

$$y_k(t_n+h) = e^{hA_a}y(t_n) + \sum_{j=1}^{s} w_j F(c_j^*), \quad k = 2, ...m \tag{31}$$

where $F(s) = e^{(t_n+h-s)A_a}$, w_j are weights and $c_j^* \in [t_n, t_n+h]$ are nodes.

We simply use the trapezoidal rule for approximating the integrals, we then have following iterative solving scheme,

$$y_k(t_n+h) = e^{hA_a}(I + \frac{h}{2}(A(t_n)-A_a)y(t_n) + \frac{h}{2}(A(t_n+h)-A_a)y_{k-1}(t_n+h) \tag{32}$$

for $k = 2, ...m$. Here $y(t_0) = y_0$ (initial condition), $y(t_n) = y_k(t_{n-1} + h), n = 1, ..N$ and $N = \frac{b-a}{h}$. The algorithm will continue until the following condition is fulfilled,

$$|y_k - y_{k-1}| \leq Tol.$$

It can be easily seen in Equation (32), the scheme involves only one approximation of exponential of a constant matrix. Numerical results related to this algorithm is presented in the next section.

2.4 First Numerical Examples

Consider the following scalar equation,

$$u'(t) = 2u + tu, \quad u(0) = 1, \tag{33}$$

the exact solution is

$$u(t) = e^{-2} e^{\frac{(t+2)^2}{2}}. \tag{34}$$

The comparison of the numerical solution obtained by successive approximation and the exact solution of the scalar equation is shown in Figures (1) and (2) for different time intervals.

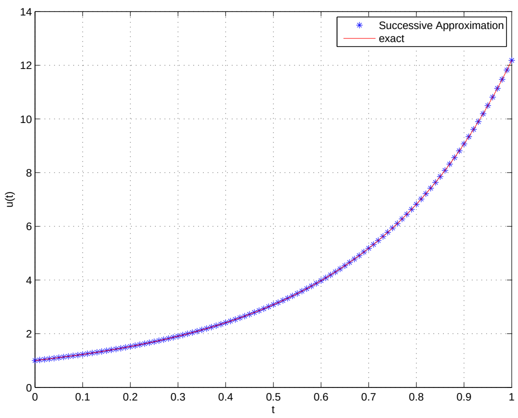

Figure 1: Comparison of approximate solution by successive approximation and exact solution of scalar equation for shorter time scale.

Next example is well known matrix problem [28] of

$$A(t) = \begin{pmatrix} 2 & t \\ 0 & -1 \end{pmatrix}. \tag{35}$$

The exact solution of the problem with $Y(0) = I$ is

$$Y(t) = \begin{pmatrix} e^{2t} & f(t) \\ 0 & e^{-t} \end{pmatrix} \tag{36}$$

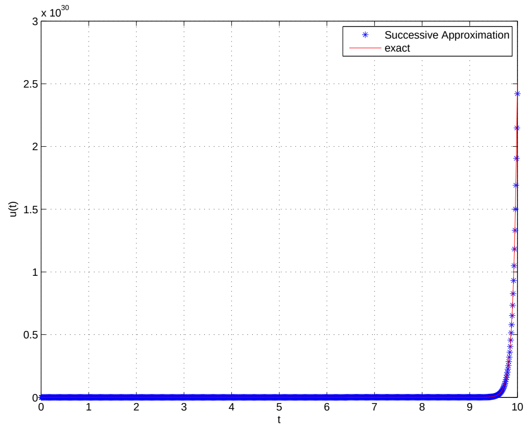

Figure 2: Comparison of approximate solution by successive approximation and exact solution of scalar equation for longer time scale.

with

$$f(t) = \frac{1}{9}e^{-t}(e^{3t} - 1 - 3t) \quad (37)$$

$$= \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^5}{60} + \frac{t^5}{80} + \frac{t^7}{420} + \frac{31t^8}{40320} + \frac{t^9}{6720} + \frac{13t^{10}}{403200} + \frac{13t^{11}}{178200} \quad (38)$$

The comparison of the numerical approximation obtained by successive approximation for $log(f(t))$ and exact $log(f(t))$ is shown in Figure (3).

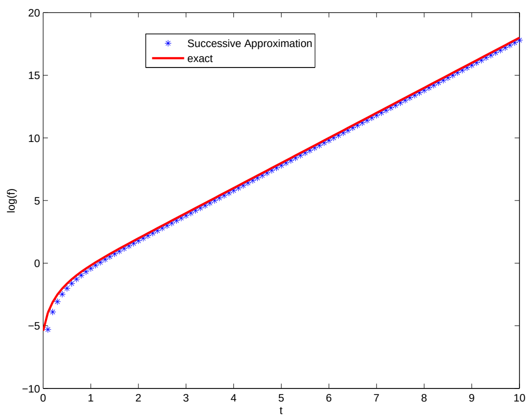

Figure 3: The comparison of the numerical approximation obtained by successive approximation and exact value of $log(f(t))$.

3 Application to Iterative Splitting Methods

In this section, we are proposing the successive approximation scheme embedded into the iterative scheme.

We consider the following time dependent problem,

$$\frac{\partial u}{\partial t} = Au(t) + B(t)u(t), \;\; u(0) = u_0. \tag{39}$$

Our intention is to solve this problem by iterative scheme explained as follows, since this might balance the dominant terms in the equations (39),

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= Au_i + B(t)u_{i-1} \\ \frac{\partial u_{i+1}}{\partial t} &= Au_i + B(t)u_{i+1}, \;\; i = 1, 2, 3...m \end{aligned} \tag{40}$$

The exact solutions of this system of equation then can be written by using the integration constant formula as follows:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \exp(At)Y_0 + \int_0^t \exp((t-s)A)B(s)u_{i-1}(s)\, ds \\ \frac{\partial u_{i+1}}{\partial t} &= \phi(t)Y_0 + \int_0^t \phi(t)\phi^{-1}(s)Au_i(s)\, ds \end{aligned} \tag{41}$$

where $\phi(t)$ is the fundamental set of solution of the second equation in the system (40).

In the light of the previous discussion, $\phi(t)$ can be written in terms of the Magnus series, we then have

$$\phi(t) = \exp(\Omega(t)) \tag{42}$$

$$= \exp(\sum_{j=1}^{\infty} \Omega_j(t)) \tag{43}$$

where

$$\begin{aligned} \Omega_1(t) &= \int_0^t dt_1 B_1 \\ \Omega_2(t) &= \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 [B_1, B_2] \\ \Omega_3(t) &= \frac{1}{6}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 ([B_1, [B_2, B_3] + [[B_1, B_2], B_3]) \\ \cdots\cdots & \quad \text{etc.} \end{aligned} \tag{44}$$

where $B_j = B(t_j)$. In practice, it is more useful to define the jth order Magnus operator

$$\Omega^{[j]}(t) = \Omega(t) + O(t^{j+1}). \tag{45}$$

We first develop a second order scheme by approximating the exact solutions of the system (41) by quadrature formula, since higher order scheme can be easily obtained in the same manner.

The second-order Magnus operator for one time step,

$$\Omega^{[2]}(t) = \int_0^t B(s)\, ds$$

$$= tB\,(t/2) + O(t^3) \tag{46}$$

$$= tB\,(t) + O(t^2), \tag{47}$$

Thus the fundamental set of solution becomes

$$\phi(t) \quad = \quad I + tB\left(t\right) + O(t^2) \tag{48}$$

and then the inverse of the fundamental solution becomes

$$\phi^{-1}(t) \quad = \quad I - tB\left(t\right) + O(t^2), \tag{49}$$

after embedding these approximations into the system (41), the solution for one step, h, in the interval $[t_n, t_n + h]$, is given by

$$\begin{aligned} u_i(t_n + h) &= e^{hA}u_i(t_n) + \int_{t_n}^{t_n+h} e^{(t_n+h-s)A}B(s)u_{i-1}(s)\,ds \qquad (50) \\ u_{i+1}(t_n + h) &= e^{hB(t_n+h/2)}u_{i+1}(t_n) + \int_{t_n}^{t_n+h} e^{hB(t_n+h/2)}(I - hB\left(s\right))Au_i(s)\,ds. \end{aligned}$$

Next, the integrals in the (50) are also approximated by using Gauss-Lobatta points, we then have

$$u_i(t_n + h) \quad = \quad e^{hA}(I + \frac{h}{2}(B(t_n))u_i(t_n) + \frac{h}{2}(B(t_n + h)u_{i-1}(t_n + h), \tag{51}$$

$$u_{i+1}(t_n + h) \quad = \quad e^{hB(t_n+h/2)}(u_{i+1}(t_n) + \frac{h}{2}\left((I - hB(t_n))Au_i(t_n) + (I - hB(t_n + h))Au_i(t_n + h)\right) \tag{52}$$

for $i = 1, 2, ...m$. Here $u(t_0) = u_0 = Y_0, u_i(t_n) = u_{i-1}(t_n) = u(t_n)$.

We summarize our algorithm in the following steps:

- Step 1: Consider the time interval $[t_0, T]$, divide it into N subintervals so that time step is $h = (T - t_0)/N$.
- Step 2: On each subinterval, $[t_n, t_n + h], n = 0, 1..N$, use the algorithm by considering the initial conditions for each step as $u(t_0) = u_0 = Y_0, u_i(t_n) = u_{i-1}(t_n) = u(t_n)$,

$$\begin{aligned} u_i(t_n + h) &= e^{hA}(I + \frac{h}{2}(B(t_n))u_i(t_n) + \frac{h}{2}(B(t_n + h)u_{i-1}(t_n + h), \\ u_{i+1}(t_n + h) &= e^{hB(t_n+h/2)}(u_{i+1}(t_n) + \frac{h}{2}\left((I - hB(t_n))Au_i(t_n) + (I - hB(t_n + h))Au_i(t_n + h)\right) \end{aligned}$$

- Step 3: Check the condition

$$|u_i - u_{i-1}| \leq Tol,$$

  if it is satisfied stop the iteration on this interval,
- Step 4: $u_i(t_n + h) \rightarrow u(t_n + h)$
- Step 5: Repeat this procedure for next interval until the desired time T is reached.

3.1 Numerical examples for time dependent iterative splitting

The comparison of the numerical solution obtained by time dependent iterative splitting and the exact solution of the scalar equation 34 is shown in Figures (4) and (5) for different time intervals, respectively.

The comparison of the numerical approximation obtained by successive approximation for $log(f(t))$ and exact $log(f(t))$ is shown in Figure (6).

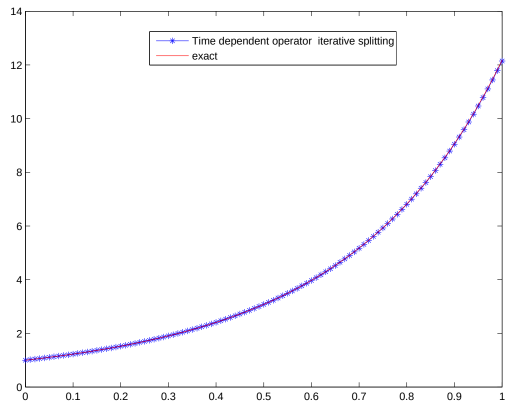

Figure 4: Comparison of approximate solution by time dependent iterative splitting and exact solution of scalar equation for shorter time scale.

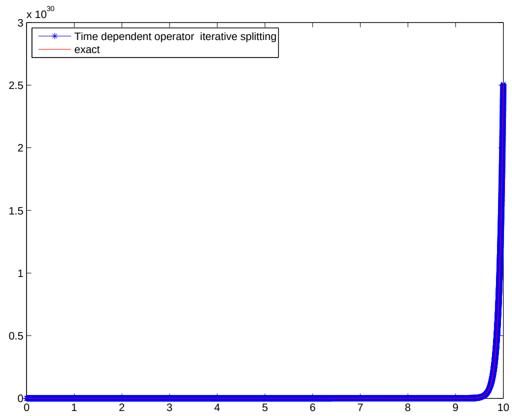

Figure 5: Comparison of approximate solution by time dependent iterative splitting and exact solution of scalar equation for longer time scale and h=0.0001.

3.2 Extension to higher Order Schemes

In this subsection, we present higher order schemes based only time dependent iterative scheme. Since time dependent iterative splitting scheme is generalization of the successive approximation method. One can use the only one iteration (u_i) instead of two iterations (u_i and u_{i+1}), alternately. This choice depends on the operators given in the differential equation. We mean that if A is the dominant term in the expression, it is sufficient to use first algorithm to compute the advanced solution for each time step. Similarly if if $B(t)$ is the dominant term in the expression, it is sufficient to use second algorithm to compute the advanced solution for each time step. Thus our algorithm is flexible with respect to the operators given in partial differential equation. Moreover, in spite of the second order Strang splitting, it needs only one computation of exponential of the matrices, therefore it is efficient algorithm as a computational time. In the following subsections, we consider these cases separately.

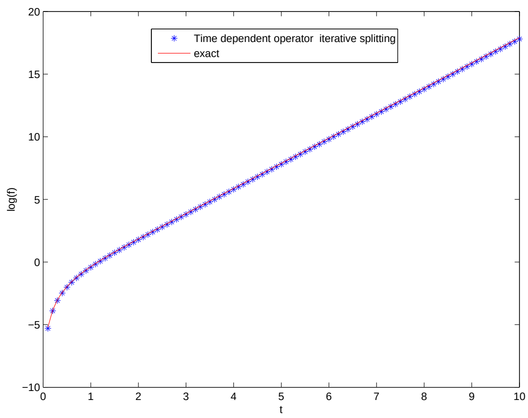

Figure 6: The comparison of the numerical approximation obtained by successive approximation and exact value of $log(f(t))$.

3.2.1 Case 1: The matrix A is dominant term

The higher order scheme may be obtained by using the extrapolation idea as follows:

- Extrapolation idea:

$$u_i(t_n + h) = \frac{2^p u_i(t_n + h/2) - u_i(t_n + h)}{2^p - 1} + O(h^{p+1}) \tag{53}$$

  where p is the order of the method.

- Romberg integration: We use the Romberg Integration to approximate the integral in first equation in the system (41) in higher order as follows,

$$R(0,0) = \frac{1}{2}h(F(t_n) + F(t_n + h) \tag{54}$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\frac{1}{2}F(t_n + h/2) \tag{55}$$

$$R(1,m) = \frac{1}{4^m - 1}(4^m R(1,m-1) - R(0,m-1)) \tag{56}$$

  where $F(x) = \exp((t_n+h-x)A)B(x)u_{i-1}(x)$ The order of the accuracy is then $O((h/2)^{2m+2})$.

- Approximation of exponential function: if A is symmetric matrix, we have $e^{hA} = Pe^{diag[(h\lambda_i)^p]}P^{-1} + O(h^{p+1})$ where λ_i are eigenvalues of A, and the P diagonalizes A.

  Otherwise $e^{hA} = \phi(hA) + O(h^{p+1})$ where $\phi(hA)$ Pade approximation of the exponential.

Therefore the higher order iteration becomes:

$$u_i(t_n + h) = \phi(hA)(I + R(1,(p-1)/2))u_i(t_n) + O(h^{p+1}) \tag{57}$$

where R is the Romberg integration given in Equation (54).

3.2.2 Case 2: The matrix B(t) is dominant term

The higher order treatment can be achieved in the following steps:

- Extrapolation idea explained in the previous section.
- Higher order expansion of the fundamental set of solution:

$$\phi(t) = e^{\int_0^t B(s)\,ds} \quad = \quad e^{\Omega^{[j]}(t)} + O(t^{j+1}). \tag{58}$$

  This can be done by quadrature formula for magnus expansion, commutator free magnus or any other techniques.

- Higher order expansion of the inverse of the fundamental set of solution:

$$\phi(t)^{-1} = e^{-\int_0^t B(s)\,ds} \quad = \quad e^{-\Omega^{[j]}(t)} + O(t^{j+1}). \tag{59}$$

- Higher order approximation of the integral: Romberg integration can be used explained in the previous section.

3.3 Application of the new algorithms to linear time dependent Schrödinger equation

The linear time dependent Schrödinger differential equation in literature is given by

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(x,t) &= \left(-\frac{1}{2\mu}\nabla^2 + V(x,t)\right)\psi(x,t) \qquad (60)\\ &= \mathbf{H}\psi(x,t). \qquad (61)\end{aligned}$$

By defining $\psi = q + ip$, we have following skew symmetric system of equation:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{H} \\ -\mathbf{H} & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{62}$$

The system above is also converted in the following form:

$$u'(t) \quad = \quad \tilde{A}u + \tilde{B}(t)u, \tag{63}$$

where $u = (\mathbf{q}, \mathbf{p})$ is vector valued function, $\tilde{A}$ and $\tilde{B}(t)$ are operators defined as

$$\tilde{\mathrm{A}} \quad = \quad \begin{pmatrix} 0 & \mathbf{A} \\ -\mathbf{A} & 0 \end{pmatrix}, \tag{64}$$

where A is the second order differential operator given by

$$A \quad = \quad -\frac{1}{2\mu}\nabla^2, \tag{65}$$

and $\tilde{B}(t)$ time dependent operator given by

$$\tilde{B}(t) \quad = \quad \begin{pmatrix} 0 & V(x,t)I \\ -V(x,t)I & 0 \end{pmatrix}. \tag{66}$$

Next, for numerical solution of the problem, we define uniform space and time

$$S = \{x_i \in (0,l] : x_i = ih; i = 0,..N, h = l/N\}, \quad (67)$$
$$T = \{t_j \in (0,T] : t_j = jk; j = 0,..M, k = T/M\}, \quad (68)$$

we then have $2N$ dimensional vector to be approximated at each grid points,

$$u(\mathbf{q},\mathbf{p}) = (q_0, q_1.....q_N, p_0, p_1.....p_N) \quad (69)$$

where $p_i = ih, q_i = ihi = 0,..N$. The second order center difference approximation is used for operator A,

$$Au = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}. \quad (70)$$

In this case, the matrix $\tilde{A}, \tilde{B}(t)$ is a $2(N+1) \times 2(N+1)$. The algorithms can be read as:

Case1 : If Ã is dominant term, i.e, $||\tilde{B}(t)|| \to 0 \quad as \quad \mathrm{t} \to \infty$,

$$u_i(t_n + h) = e^{\tilde{A}h}(I + \frac{h}{2}(\tilde{B}(t_n))u_i(t_n) + \frac{h}{2}(\tilde{B}(t_n + h)u_{i-1}(t_n + h),$$

Case2 : If B̃(t) is dominant term, i.e, $||\tilde{A}|| \to 0 \quad as \quad \mathrm{t} \to \infty$,
we then only compute $e^{\tilde{B}(t_n+h/2)}$

$$u_i(t_n + h) = e^{h\tilde{B}(t_n+h/2)}(u_{i+1}(t_n) + \frac{h}{2}\Big((I - h\tilde{B}(t_n))\tilde{A}u_i(t_n) + (I - h\tilde{B}(t_n + h))\tilde{A}u_{i-1}(t_n + h)\Big)$$

Case3 : If there is no dominant terms in the equation, the algorithms written in Case 1 and Case 2 are simultaneously computed.

Remark 3.1 *Since we expand the Laplace operator in the equation by a second order approximation, it is enough to use our second order iterative schemes. Note that the iterations for both cases are solved by any stable second order ode methods.*

3.4 Numerical examples for higher order treatment

We first consider the radial Schrodinger equation

$$\frac{\partial^2 u}{\partial r^2} = f(r,e)u(r) \quad (71)$$

where

$$f(r,E) = 2V(r) - 2E + \frac{l(l+1)}{r^2} \quad (72)$$

The equation (71) can be transformed as a harmonic oscillator with a time dependent spring constant after relabelling $r \to t$ and $u(r) \to q(t)$ and defining

$$k(t,E) = -f(t,E). \quad (73)$$

By redefining the variables as $u(t) = q(t)$ and $\dot{u}(t) = p(t)$, and $Y(t) = (q(t), p(t))$, the Equation (71) can be put into the system of equation as

$$\dot{Y}(t) = A(t)Y(t) \tag{74}$$

and Hamiltonian of the system is written by

$$H = \frac{1}{2}p^2 + \frac{1}{2}k(t,E)q^2. \tag{75}$$

For specific example, the ground state of hydrogen atom can be modeled as Schrodinger equation with the parameters $l = 0, E = -1/2, V(t) = -1/(t-a), a$ is arbitrary constant. Now the time dependent oscillator corresponds to

$$\mathrm{A(t)} = \begin{pmatrix} 0 & 1 \\ f(t) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ f(t) & 0 \end{pmatrix} \equiv T + V(t), \tag{76}$$

with

$$f(t) = (1 - \frac{2}{t-a}). \tag{77}$$

The exact solution for this model with the initial conditions $q(0) = -a, p(0) = 1 + a, a = -0.001$ is

$$q(t) = (t-a)e^{-t}. \tag{78}$$

The comparison of exact and approximation of the hydrogen ground state wave function by various scheme are exhibited in figures (7), (8),(9) and (10) for T=15,20,30 and h=0.0002 and T=15, h=0.0001, respectively.

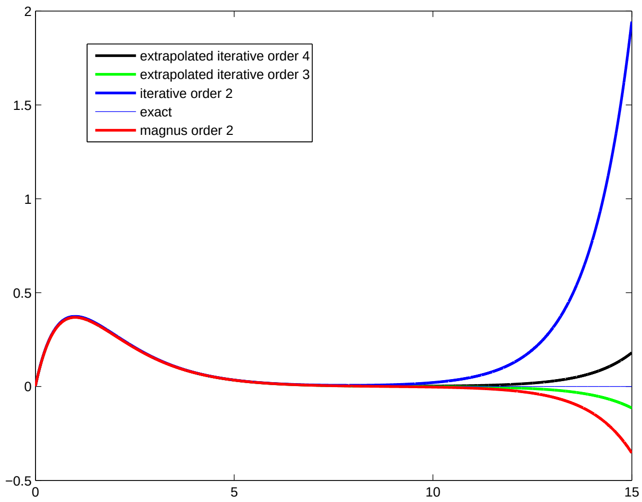

Figure 7: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes in shorter time scale.

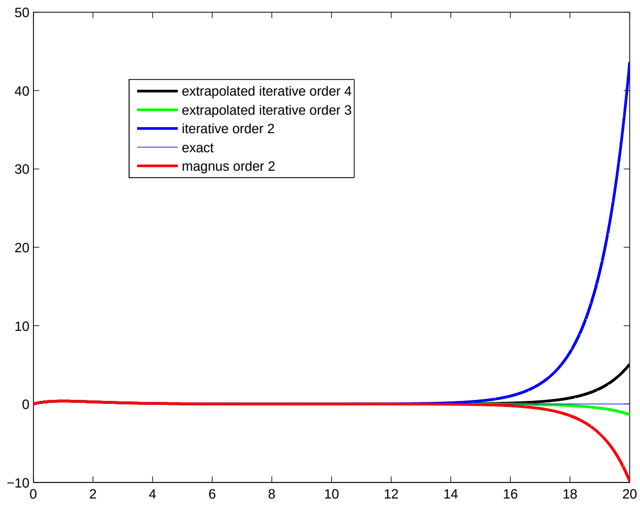

Figure 8: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes in longer time scale.

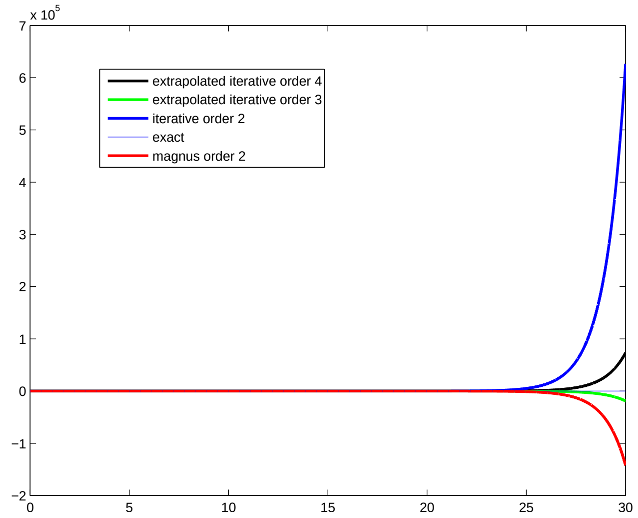

Figure 9: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes in longer time scale.

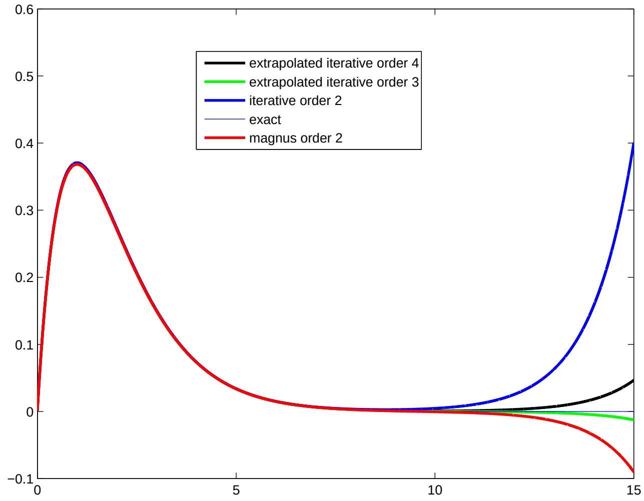

Figure 10: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes for small time step h=0.0001.

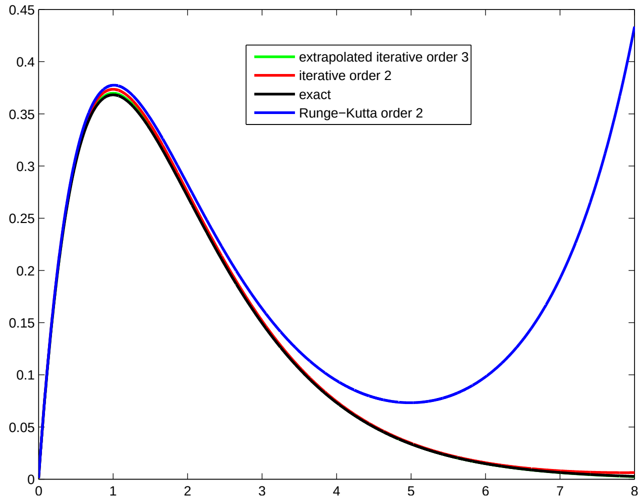

Figure 11: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes for small time step h=0.0002.

In Figure (11), the comparisons of the standard Runge-Kutta order 2, magnus order 3 and iterative method of orders 2,3 with exact solution are exhibited. As can be seen in this figure, standard method does not work for the long time.

In ref, the Shin method of order 4 can be read as

$$\mathrm{T}_4(\Delta t) \quad = \quad -\frac{1}{3}\mathrm{T}_2(\Delta t) + \frac{4}{3}{\mathrm{T}_2}^2(\frac{\Delta t}{2}). \tag{79}$$

where $\mathrm{T}_2(\Delta t) = e^{\Delta t A(t+\frac{1}{2}\Delta t)}$ and ${\mathrm{T}_2}^2(\Delta t) = e^{\Delta t A(t+\frac{3}{2}\Delta t)} e^{\Delta t A(t+\frac{1}{2}\Delta t)}$. We then compare the behaviours of the various fourth order methods with exact solution in Figures (12) and (13) in short time and long time, respectively. Extrapolated fourth order iterative method behaves very well near to internal layer.

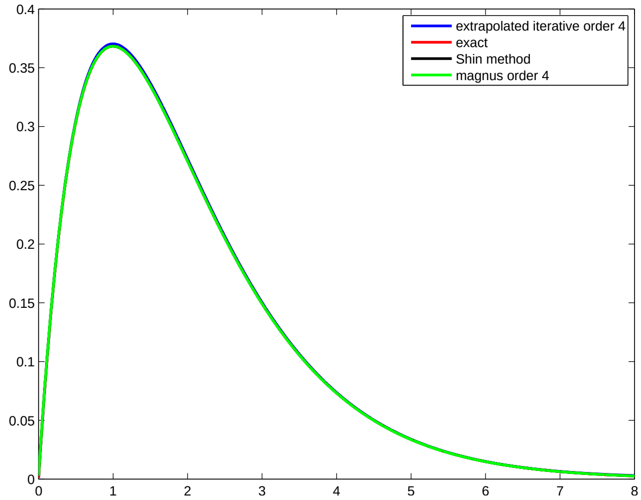

Figure 12: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes.

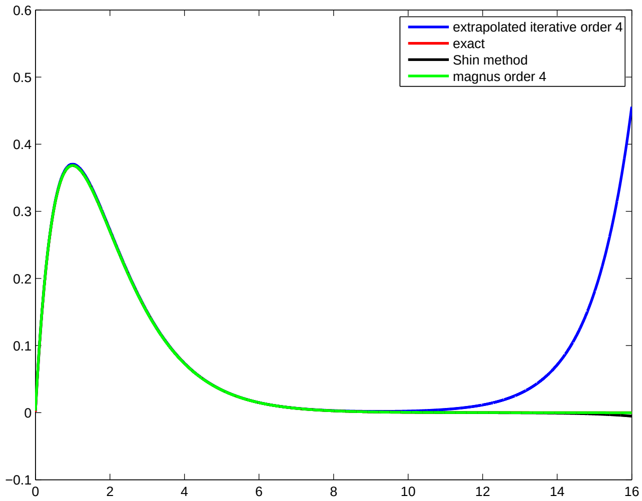

Figure 13: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes for small time step h=0.0002.

Finally, we compare the third order time dependent iterative scheme with third order Crouch-Grossman given as

$$A_1 = A(t_n, y_n); \tag{80}$$

$$A_2 = A(t_n + \frac{3}{4}h, \exp(\frac{3}{4}hA_1)y_n); \tag{81}$$

$$A_3 = A(t_n + \frac{3}{4}h, \exp(\frac{1}{108}hA_2)\exp(\frac{119}{216}hA_1)y_n); \tag{82}$$

$$y_{n+1} = \exp(\frac{13}{51}hA_3)\exp(-\frac{2}{3}hA_2)\exp(\frac{24}{17}hA_1)y_n \tag{83}$$

$$\tag{84}$$

	err_{L_∞}	err_{L_1}
* Iterative order 2	0.0039	0.0106
Runge-Kutta order 2	0.0198	0.0579
* Extrapolated iterative order 3	0.0012	0.0033
Crouch-Grossman method order 3	0.0038	0.0113

Table 1: Comparison of errors for $h = 0.001$ on $[0, 5]$ interval with various methods

	err_{L_∞}	err_{L_1}
* Iterative order 2	0.0086	0.0230
Runge-Kutta order 2	0.0396	0.1141
* Extrapolated iterative order 3	0.0023	0.0064
Crouch-Grossman method order 3	0.0075	0.0223

Table 2: Comparison of errors for $h = 0.002$ on $[0, 5]$ interval with various methods

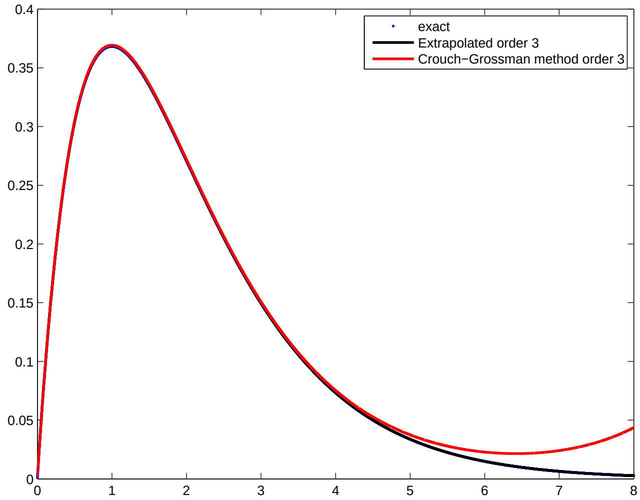

Figure 14: Comparison of exact and approximation of the hydrogen ground state wave function for various schemes for small time step h=0.0002.

	err_{L_∞}	err_{L_1}
* Iterative order 2	0.0039	0.0109
Runge-Kutta order 2	0.0417	0.1090
* Extrapolated iterative order 3	0.0012	0.0034
Crouch-Grossman method order 3	0.0079	0.0210

Table 3: Comparison of errors for $h = 0.001$ on $[0, 6]$ interval with various method

	err_{L_∞}	err_{L_1}
* Iterative order 2	0.0039	0.0122
Runge-Kutta order 2	0.2156	0.4923
* Extrapolated iterative order 3	0.0012	0.0035
Crouch-Grossman method order 3	0.0409	0.0937

Table 4: Comparison of errors for $h = 0.001$ on $[0, 8]$ interval with various method

Remark 3.2 *The numerical results show that our higher order treatment gives a better long time behavior than the standard magnus expansion.*

4 Error analysis

In the following we discuss the error analysis of the timedependent case. We introduce the non-timedependent case and extend to the non-timedependent case.

First we study the consistency and stability of the schemes and then we derive the convergence of the schemes.

4.1 Consistency and Stability analysis

4.1.1 Non-timedependent Case:

We apply the iterative splitting method given as

$$u_i(t) = \exp(At)u_0 + \int_0^t \exp(A(t-s))Bu_{i-1}(s)\, ds, \tag{85}$$

$$u_i(0) = u(0), \tag{86}$$

$$u_{i+1}(t) = \exp(Bt)u_0 + \int_0^t \exp(B(t-s))Au_i(s)\, ds, \tag{87}$$

$$u_{i+1}(0) = u(0), \tag{88}$$

where $i = 1, 3, 5, \dots$ and $u_0(t) = 0$.

We deal with the following assumptions:

Assumption 4.1 *The linear operators $A + B, A, B$ generate C_0 semigroups on $\mathbf{X}$, and the operators A, B satisfy in addition the bounds:*

$$||\exp(At)|| \leq \exp(\omega|t|) \text{ and } ||\exp(Bt)|| \leq \exp(\omega|t|) \tag{89}$$

for some $\omega \geq 0$ and all $t \in \mathbb{R}$.

The following theorem is given the convergence of an iterative operator splitting schemes for one-sided iterations and we assume that exists $B = A^{1-\alpha}$:

Assumption 4.2 *For the consistency proofs we have to assume the following:*

The linear operators $A + B, A, B$ generate analytical semigroups on $\mathbf{X}$, and the operators A, B satisfy in addition the bounds:

$$||B^\alpha \exp(B\tau_n)|| \leq \kappa_1 \tau_n^{-\alpha}. \tag{90}$$

$$||B \exp((A+B)\tau_n)|| \leq \kappa_2 \tau_n^{-1+\alpha}, \tag{91}$$

$$||\exp(A\tau_n)B|| \leq \kappa_3 \tau_n^{-1+\alpha}, \tag{92}$$

$$||A^\beta \exp(A\tau_n)|| \leq \kappa_4 \tau_n^{-\beta}. \tag{93}$$

$$||A^\gamma \exp((A+B)\tau_n)|| \leq \kappa_5 \tau_n^{-\gamma}, \tag{94}$$

where $\alpha, \beta, \gamma \in (0,1)$, $\tau_n = (t^{n+1} - t^n)$ and κ_i for $i = 1, \dots, 5$ are constants, see [30].

Remark 4.3 *For the one stage iterative scheme it is sufficient to have the assumptions: (91) - (94),*

where for the two stage iterative scheme we need all assumptions as for the one-stage schemes and additionally assumption (90).

That means that we assume that operator A and B generates an analytical semigroup with $\exp(At)$ *and* $\exp(Bt)$.

Theorem 4.4 *Let us consider the abstract Cauchy problem in a Banach space* $\mathbf{X}$

$$\begin{aligned} \partial_t c(t) &= Ac(t) + B(t)c(t), \quad 0 < t \leq T \text{ and } x \in \Omega, \\ c(0) &= c_0, \ t \in [0, T], \end{aligned} \tag{95}$$

where $A, B : D(\mathbf{X}) \to \mathbf{X}$ are given linear bounded operators which are generators of the C_0-semigroup and $c_0 \in \mathbf{X}$ is a given element.
We apply the iterative operator splitting scheme given with equations (??) and (??), means we iterate on operator A with $m + 1$ iterative steps and on operator B with m iterative steps.

If the assumptions 4.1 and 4.2 are valid, then

$$||S_i^n - \exp((A + B)n\tau)|| \leq C\tau^{(m+1)\alpha - 1}, \ n\tau \leq T, \tag{96}$$

where $i = 2m+1$ are the iterative steps over A and B, the constant C can be chosen uniformly on bounded time intervals and in particular, independent of n and τ. $\alpha \in (0, 1)$ with the assumption $B = A^{1-\alpha}$.

Proof 4.5 *By applying the telescopic identity we obtain*

$$(S_i^n - \exp((A + B)n\tau)u_0 = \sum_{\nu=0}^{n-1} S_i^{n-\nu-1}(S - \exp((A + B)\tau)) \exp(\nu\tau(A + B))u_0. \tag{97}$$

if we assume the stability bound:

$$||S_i|| \leq \exp(c\omega\tau), \tag{98}$$

with a constant c only depends on the estimation of the method.

Furthermore, if we assume the consistency bound:

$$\begin{aligned} &||S_i^n - \exp((A + B)n\tau)|| \\ &\leq \exp(c\omega T) \sum_{\nu=0}^{n-1} ||(S - \exp(\tau(A + B))) \exp(\nu\tau(A + B))|| \qquad (99) \\ &\leq C\tau^{(m+1)\alpha - 1}, \ n\tau \leq T. \qquad (100) \end{aligned}$$

The desired consistency and stability bound is given in the next subsections.

Consistency analysis

We present the results of the consistency of our iterative method. We assume for the system of operator the generator of an analytical semigroup based on their underlying norms for the Banach space $\mathbf{X}$ and induced operator norm denoted by $|| \cdot ||$.

In the following we discuss the consistency of the 2 stage iterative method, taken into account to iterate over both operators.

Theorem 4.6 *Let us consider the abstract Cauchy problem in a Banach space* $\mathbf{X}$ *given in equation (129). With the operators* $A, B : D(\mathbf{X}) \to \mathbf{X}$ *are linear operators which are generators of the analytical semigroups. We assume* $dom(B) \subset dom(A)$*, so we are restricted to balance the operators. We assume*

$$B = A^{1-\alpha} \tag{101}$$

is the infinitesimal generator of an analytical semigroup for all $\alpha \in (0,1)$*, see [30].*

The consistency error is given as $\mathcal{O}(\tau_n^{(m+1)\alpha})$*, where* $\tau_n = t^{n+1} - t^n$ *and we have equidistant time-steps, with* $n = 1, \ldots, N$*. Further* $m+1$ *are the iterative steps with operator* A*.*

*Then the iteration process (**??**) for* $i = 2m+1$ *for* $m = 0, 1, 2, \ldots$*, where we assume* $m+1$ *iterative steps with operator* A *and* m *iterative steps with operator* B*, is consistent with the order of the consistency* $\mathcal{O}(\tau_n^{\alpha(m+1)})$*, where* $0 \leq \alpha < 1$*.*

Proof 4.7 *Let us consider the iteration (**??**) and (**??**) on the sub-interval* $[t^n, t^{n+1}]$*.*

For the first iterations we have:

$$\partial_t c_1(t) = A c_1(t), \quad t \in (t^n, t^{n+1}], \tag{102}$$

and for the second iteration we have:

$$\partial_t c_2(t) = A c_1(t) + B c_2(t), \quad t \in (t^n, t^{n+1}], \tag{103}$$

In general we have:

For the odd iterations for $i = 1, 3, 5, \ldots, 2m+1$ *for* $m = 0, 1, 2, \ldots$

$$\partial_t c_i(t) = A c_i(t) + B c_{i-1}(t), \quad t \in (t^n, t^{n+1}], \tag{104}$$

where for $c_0(t) \equiv 0$*.*

and for the even iterations for $i = 2, 4, 6, \ldots, 2m$ *for* $m = 1, 2, \ldots$

$$\partial_t c_i(t) = A c_i(t) + B c_{i-1}(t), \quad t \in (t^n, t^{n+1}], \tag{105}$$

where for $c_0(t) \equiv 0$*.*

Means we iterate at least $m+1$ *times over* A *and* m *times over* B*.*

The solutions for the first two iterative steps are given by the variation of constants:

$$c_1(t) = \exp(A(t - t^n)) c(t^n), \quad t \in (t^n, t^{n+1}], \tag{106}$$

$$\begin{aligned} c_2(t) \quad &= \exp(B(t - t^n)) c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(B(t^{n+1} - s)) A c_1(s) ds, \quad t \in (t^n, t^{n+1}]. \end{aligned} \tag{107}$$

For the odd iterations $i = 2m+1$ *for* $m = 0, 1, 2, \ldots$ *we obtain the*

$$c_i(t) = S_i(t) c(t^n) = \exp(A(t - t^n)) c(t^n) + \int_{t^n}^t \exp((t-s)A) B c_{i-1}(s)\, ds, \quad t \in (t^n, t^{n+1}], \tag{108}$$

For the even iterations $i = 2m$ *for* $m = 1, 2, \ldots$ *we obtain the*

$$c_i(t) = S_i(t) c(t^n) = \exp(B(t - t^n)) c(t^n) + \int_{t^n}^t \exp((t-s)B) A c_{i-1}(s)\, ds, \quad t \in (t^n, t^{n+1}], \tag{109}$$

The consistency is given as in the following steps:
For e_1 we have the result of the previous one stage iterative operator method:

$$||e_1|| \leq C\tau^{\alpha}\; ||c(t^n)|| \tag{110}$$

where $\alpha \in (0,1)$, $\tau = (t^{n+1} - t^n)$ and C is a constant depends only on κ_5 and ω.
For e_2 we have to apply the even steps:

$$\begin{aligned} c_2(t^{n+1}) &= \exp(B\tau_n)c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(B(t^{n+1}-s))A\exp((s-t^n)B)c(t^n)\; ds, \end{aligned} \tag{111}$$

$$\begin{aligned} c(t^{n+1}) &= \exp(B\tau_n)c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(B(t^{n+1}-s))A\exp((s-t^n)B)c(t^n)\; ds \\ &+ \int_{t^n}^{t^{n+1}} \exp(B(t^{n+1}-s))A \\ &\int_{t^n}^{s} \exp(B(s-\rho))A\exp((\rho-t^n)(A+B))c(t^n)\; d\rho\; ds. \end{aligned} \tag{112}$$

We obtain:

$$||e_2|| \leq ||\exp((A+B)\tau_n)c(t^n) - c_2|| \tag{113}$$

$$= ||\int_{t^n}^{t^{n+1}} \exp(B(t^{n+1}-s))A \tag{114}$$

$$\int_{t^n}^{s} \exp(B(s-\rho))A\exp((\rho-t^n)(A+B))c(t^n)\; d\rho\; ds||$$

$$= \int_{t^n}^{t^{n+1}} ||\exp(B(t^{n+1}-s))|| \tag{115}$$

$$\int_{t^n}^{s} ||\exp(B(s-\rho))A^{2-\alpha}\exp((\rho-t^n)(A+B))c(t^n)\; d\rho||ds$$

$$= \int_{t^n}^{t^{n+1}} C\int_{t^n}^{s} (s-\rho)^{\alpha-2} d\rho ds||c(t^n)|| \tag{116}$$

$$\leq C\tau^{\alpha}\; ||c(t^n)||$$

where $\alpha \in (0,1)$, $\tau = t^{n+1} - t^n$ and C is a constant only depending on κ_2, κ_5 and ω.

For the general iterative steps, the recursive proof is given in the following. We shift $t^n \to 0$ and $t^{n+1} \to \tau_n$ for simpler calculations, see [24]. The initial conditions are given with $c(0) = c(t^n)$.

For the odd iterative steps $i = 2m+1$ with $m = 0, 1, 2, \ldots$, we have the result of the previous one stage iterative operator method:

$$||e_i|| \leq \tilde{C}\tau_n^{(m+1)\alpha}||c(t^n)||,$$

where $0 \leq \alpha_i < 1$ and $m+1$ are the number of iteration steps over the operator A, $\tilde{C}$ is a constant and depending only on κ_2, κ_5 and ω.

For the even iterative steps $i = 2m+1$ with $m = 1, 2, \ldots$, we have m iterative steps with A and m iterative steps with B. We obtain for c_i and c:

$$
\begin{aligned}
& c_i(\tau_n) = \exp(B\tau_n)c(0) \\
& + \int_0^{\tau_n} \exp(Bs)A \exp((\tau_n - s)A)c(0)\, ds \\
& + \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \exp((\tau_n - s_1 - s_2)B)c(0)\, ds_2\, ds_1 \\
& + \ldots + \\
& + \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \int_0^{\tau_n - s_1 - s_2} \exp(s_3 B)A \ldots \\
& \int_0^{\tau_n - \sum_{j=1}^{i-1} s_j} \exp(Bs_i)A \exp((\tau_n - \sum_{j=1}^{i-1} s_j)A)c(0)\, ds_i \ldots ds_1,
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
& c(\tau_n) = \exp(B\tau_n)c(0) \\
& + \int_0^{\tau_n} \exp(Bs)A \exp((\tau_n - s)A)c(0)\, ds \\
& + \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \exp((\tau_n - s_1 - s_2)B)c(0)\, ds_2\, ds_1 \\
& + \ldots + \\
& + \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \int_0^{\tau_n - s_1 - s_2} \exp(s_3 B)A \ldots \\
& \int_0^{\tau_n - \sum_{j=1}^{i-1} s_j} \exp(Bs_i)A \exp((\tau_n - \sum_{j=1}^{i-1} s_j)A)c(0)\, ds_i \ldots ds_1 \\
& + \int_0^{\tau_n} \exp(As_1)B \int_0^{\tau_n - s_1} \exp(s_2 A)B \int_0^{\tau_n - s_1 - s_2} \exp(s_3 A)B \ldots \\
& \int_0^{\tau_n - \sum_{j=1}^{i} s_j} \exp(As_{i+1})B \exp((\tau_n - \sum_{j=1}^{i} s_j)(A+B))c(0)\, ds_{i+1} \ldots ds_1.
\end{aligned}
\tag{118}
$$

By shifting $0 \to t^n$ and $\tau_n \to t^{n+1}$, we obtain our result:

$$
\begin{aligned}
& ||e_i|| \le || \exp((A+B)\tau_n)c(t^n) - c_i|| \\
& \le \tilde{C}\tau_n^{m\alpha} ||c(t^n)||,
\end{aligned}
\tag{119}
$$

where $0 \le \alpha_i < 1$ and i is the number of iteration steps over the operator A, $\tilde{C}$ is a constant and depending only on κ_2, κ_5 and ω.

In the next section we describe the stability analysis.

Stability Analysis

For stability bound we have the following theorem:

Theorem 4.8 *Let us consider the abstract Cauchy problem in a Banach space* $\mathbf{X}$

$$
||S_i|| \le \exp(c\omega\tau) \tag{120}
$$

where c depends only on the coefficients of the method and ω is a bound for the operators, see assumptions (91) - (94). S_i is given as in equation (108) and τ is the time-step size.

Proof 4.9 *We apply the assumption:*

$B = A^{1-\alpha}$

Based on the definition of S_i we have: For the even iterative steps $i = 2m+1$ with $m = 1, 2, \ldots$, we have m iterative steps with A and m iterative steps with B. We obtain for c_i and c:

$$\begin{aligned} S_i &= \exp(B\tau_n) \\ &+ \int_0^{\tau_n} \exp(Bs)A\exp((\tau_n - s)A)\, ds \\ &+ \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B\exp((\tau_n - s_1 - s_2)B)\, ds_2\, ds_1 \\ &+ \ldots + \\ &+ \int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \int_0^{\tau_n - s_1 - s_2} \exp(s_3 B)A \ldots \\ &\int_0^{\tau_n - \sum_{j=1}^{i-1} s_j} \exp(Bs_i)A\exp((\tau_n - \sum_{j=1}^{i-1} s_j)A)\, ds_i \ldots ds_1, \end{aligned} \tag{121}$$

After application of B we have:

$$\begin{aligned} ||S_i|| &= ||\exp(B\tau_n)|| \\ &+ ||\int_0^{\tau_n} \exp(Bs)A\exp((\tau_n - s)A)\, ds|| \\ &+ ||\int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B\exp((\tau_n - s_1 - s_2)B)\, ds_2\, ds_1|| \\ &+ \ldots + \\ &+ ||\int_0^{\tau_n} \exp(Bs_1)A \int_0^{\tau_n - s_1} \exp(s_2 A)B \int_0^{\tau_n - s_1 - s_2} \exp(s_3 B)A \ldots \\ &\int_0^{\tau_n - \sum_{j=1}^{i-1} s_j} \exp(Bs_i)A\exp((\tau_n - \sum_{j=1}^{i-1} s_j)A)\, ds_i \ldots ds_1||, \end{aligned} \tag{122}$$

$$\leq \exp(\omega\tau) + \sum_{j=1}^{i-1} C_j t^{\alpha j} \leq \exp(\tilde{\omega}\tau) \tag{123}$$

where for all $\omega, C_j \leq 0$ for all $j = 1, \ldots, i-1$ and $\alpha \in (0, 1)$, we find $\tilde{\omega} \leq 0$.

Therefore $||S_i|| \leq \exp(\tilde{\omega}\tau)$ is bounded.

The same can be done for the odd iterations.

Remark 4.10 *Based on the consistency and stability, we have a convergent method of one order less than the consistency order.*

4.2 Timedependent Case

We deal with the perturbation theory [14]. The same proof methods are used for the time-dependent case. The variation of constants can be extended to a time-dependent case, see [30].

We have the following assumptions to our underlying operators:

Assumption 4.11 *We assume that the bounded operators $A + B(t), A, B(t)$ generate C_0 semigroups on $\mathbf{X}$, and the operators A, $B(t)$ satisfy in addition the bounds:*

$$||\exp(At)|| \leq \exp(\omega_1 |t|) \text{ and } ||\exp(\int_0^t B(s)ds)|| \leq \exp(\omega_2 |t|) \tag{124}$$

for some $\omega_1, \omega_2 \geq 0$ and all $t \in \mathrm{I\!R}$.

We apply the iterative splitting method given as

$$u_i(t) = \exp(At)u_0 + \int_0^t \exp(A(t-s))B(s)u_{i-1}(s)\, ds, \tag{125}$$

$$u_i(0) = u(0), \tag{126}$$

$$u_{i+1}(t) = \exp(\int_0^t B(s)ds)u_0 + \int_0^t \exp(\int_0^{t-s} B(s_1)ds_1)Au_i(s)\, ds, \tag{127}$$

$$u_{i+1}(0) = u(0), \tag{128}$$

where $i = 1, 3, 5, \ldots$ and $u_0(t) = 0$.

Theorem 4.12 *Let us consider the abstract Cauchy problem in a Banach space $\mathbf{X}$*

$$\begin{aligned} \partial_t c(t) &= Ac(t) + B(t)c(t), \quad 0 < t \leq T \text{ and } x \in \Omega, \\ c(0) &= c_0, \; t \in [0, T], \end{aligned} \tag{129}$$

where $A, B(t) : D(\mathbf{X}) \to \mathbf{X}$ are given linear bounded operators which are generators of the C_0-semigroup and $c_0 \in \mathbf{X}$ is a given element.

Further, we assume the estimations of the bounded timedependent operator, see [14]:

$$||B(t)\exp(At)x|| \leq \beta ||x||, \tag{130}$$

$\tau_n = (t^{n+1} - t^n)$.

The error of the first time-step is of accuracy $\mathcal{O}(\tau_n^m)$, where $\tau_n = t^{n+1} - t^n$ and we have equidistant time-steps, with $n = 1, \ldots, N$. Then the iteration process (125)–(127) for $i = 1, 3, \ldots, 2m+1$ is consistent with the order of the consistency $\mathcal{O}(\tau_n^{2m+1})$.

Proof 4.13 *For $i = 1$, we have:*

$$c_1(t^{n+1}) = \exp(A\tau_n)c(t^n), \tag{131}$$

and the solution is given as

$$\begin{aligned} &c(t^{n+1}) = \exp(A\tau_n)c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(A(t^{n+1} - s))B(s)\exp(sA + \int_0^s B(\tilde{(s)})d\tilde{s})c(t^n)\, ds, \end{aligned} \tag{132}$$

We obtain:

$$||e_1|| \leq ||\exp((A\tau + \int_{t^n}^{t^{n+1}} B(s)ds))c(t^n) - c_1|| \tag{133}$$

$$= ||\int_{t^n}^{t^{n+1}} \exp(A(t^{n+1} - s))B(s)\exp(sA + \int_0^s B(\tilde{(s)})d\tilde{s})c(t^n)\, ds|| \tag{134}$$

$$\leq C\tau ||B(\xi)|| ||c(t^n)|| \tag{135}$$

where $\xi \in [t^n, t^{n+1}]$ *and the* exp *functions can be estimated by* C.

The same argumentation is used for $i = 2$:

$$
\begin{aligned}
&||e_2|| \leq && (136)\\
&= \int_{t^n}^{t^{n+1}} ||\exp(\int_0^{t^{n+1}-s} B(\tilde{s})d\tilde{s}))A|| && (137)\\
&\int_{t^n}^{s} ||\exp(A(s-\rho))B(s)\exp((\rho - t^n)(A+B))c(t^n)\ d\rho||ds \\
&= \tilde{C}\tau^2 ||A||||B(\xi)||\ ||c(t^n)||
\end{aligned}
$$

where $\xi \in [t^n.t^{n+1}]$

Based on the bounded operators we can apply the recursive argument.

For the odd iterations: $i = 2m+1$, *with* $m = 0, 1, 2, \ldots$, *we obtain for* c_i *and* c:

$$
\begin{aligned}
&||e_i|| \leq \int_0^{\tau_n} ||\exp(As)B(s)|| \qquad (138)\\
&\int_0^{\tau_n - s} ||\exp(\int_0^{\tau_n - s} B(\tilde{s})d\tilde{s})A|| \int_0^{\tau_n - s_1 - s_2} \exp(s_3 A)B(s_3)\ldots \\
&\int_0^{\tau_n - \sum_{j=1}^{i} s_j} \exp(As_{i+1})B(s_{i+1})\exp((\tau_n - \sum_{j=1}^{i} s_j)(A + \int_0^{(\tau_n - \sum_{j=1}^{i} s_j)} B(\tilde{s})d\tilde{s}))c(0)\ ds_{i+1}\ldots ds_1.
\end{aligned}
$$

By shifting $0 \to t^n$ *and* $\tau_n \to t^{n+1}$, *we obtain our result:*

$$||e_i|| \leq \tilde{C}||A||^{m+1}||B(\xi)||^m \tau_n^{2m+1}||c(t^n)||,$$

where $\xi \in [t^n, t^{n+1}]$ *and* $\tilde{C}$ *is a non-timedependent constant.*

The same proof idea can be applied to the even iterative scheme.

Remark 4.14 *Stability is given because of the bounded operators, see assumption 4.11. Because of the boundness of* $B(t)$ *we can also extimate the*

Remark 4.15 *The extension to unbounded operators can also be done with respect to the ideas for the non-timedependent case. Here we have to define one operator to be boundable by the other operators.*

5 Conclusions and Discussions

In this work, we have presented application to successive approximations that are related to iterative splitting schemes. We presented iterative splitting methods of higher accuracy as known standard schemes. In the error analysis we discuss the benefit of the proofs which can be extended to time-dependent cases. By the numerical approximation and choice of the time-dependent operator we have seen benefits in their computational time. Higher order results can also be achieved with at least 2 or 3 iterative steps, which beat standard 3rd and 4th order methods.

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