Algorithms of intrinsic complexity for point searching in real singular hypersurfaces[1]

B. Bank [2], M. Giusti [3], J. Heintz [4], L. Lehmann [5] L. M. Pardo[6]

June 10, 2010

Dedicated to Tomás Recio
on the occasion of his 60^{th} birthday

Abstract

We treat the general problem of finding real solutions of multivariate polynomial equation systems in the case of a single equation $F = 0$ which is supposed to admit at least one F–regular real solution (where the gradient of F does not vanish) and which has possibly other, F–singular real solutions. We present two families of elimination algorithms of *intrinsic complexity* which solve this problem, one in the case that the real hypersurface defined by F is compact and another without this assumption. In worst case the complexity of our algorithms does not exceed the already known *extrinsic* complexity bound of $(nd)^{O(n)}$ for the elimination problem under consideration, where n is the number of indeterminates of F and d its (positive) degree. In the case

[1]Research partially supported by the following Argentinian, French and Spanish grants: CONICET PIP 2461/01, UBACYT X-098, PICT–2006–02067, BLAN NT05-4-45732 (projet GECKO), MTM 2007-62799.

[2]Humboldt-Universität zu Berlin, Institut für Mathematik, 10099 Berlin, Germany. bank@mathematik.hu-berlin.de

[3]CNRS, Lab. LIX, École Polytechnique, 91228 Palaiseau Cedex, France. Marc.Giusti@Polytechnique.fr

[4]Departamento de Computación, Universidad de Buenos Aires, Ciudad Univ., Pab.I, 1428 Buenos Aires, Argentina, and Departamento de Matemáticas, Estadística y Computación, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain.
joos@dc.uba.ar and heintzj@unican.es

[5]Humboldt-Universität zu Berlin, Institut für Mathematik, 10099 Berlin, Germany. llehmann@mathematik.hu-berlin.de

[6]Departamento de Matemáticas, Estadística y Computación, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain. luis.pardo@unican.es

that F is squarefree and the real variety defined by F is smooth, there exist already algorithms of intrinsic complexity that solve our problem. However these algorithms cannot be used in case that $F = 0$ admits F-singular real solutions.

An elimination algorithm of intrinsic complexity supposes that the polynomial F is encoded by an essentially division-free arithmetic circuit of size L (i.e., F can be evaluated by means of L additions, subtractions and multiplications, using scalars from a previously fixed real ground field, say $\mathbb{Q}$) and that there is given an invariant $\delta(F)$ which (roughly speaking) depends only on the geometry of the complex hypersurface defined by F. The complexity of the algorithm (measured in terms of the number of arithmetic operations in $\mathbb{Q}$) is then *linear* in L and *polynomial* in n, d and $\delta(F)$.

In order to find such a geometric invariant $\delta(F)$ we consider certain deformations of the gradient of F restricted to the complex hypersurface defined by F. These deformations give rise to certain complex varieties which we call the bipolar varieties of the equation $F = 0$. The maximal degree of these bipolar varieties becomes then the essential ingredient of our invariant $\delta(F)$.

By the way, our algorithms find F-regular algebraic sample points for all connected components of the real hypersurface defined by F that are generically smooth (i.e., that contain F-regular points).

Keywords: real polynomial equation solving, intrinsic complexity, singularities, polar and bipolar varieties, degree of varieties

MSC: 68W30, 14P05, 14B05

1 Introduction

Before we start to explain the main results of this paper and their motivations, we introduce some basic notions and notations.

Let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the fields of rational, real and complex numbers, respectively, let $X := (X_1, \ldots, X_n)$ be a vector of indeterminates over $\mathbb{C}$ and let be given a regular sequence $F_1, \ldots, F_p$ of polynomials in $\mathbb{Q}[X]$ defining a closed, $\mathbb{Q}$–definable subvariety S of the n–dimensional complex affine space $\mathbb{A}^n := \mathbb{C}^n$. Thus S is a non–empty equidimensional affine variety of dimension $n - p$, i.e., each irreducible component of S is of dimension $n - p$. Said otherwise, S is a closed subvariety of $\mathbb{A}^n$ of pure codimension p (in $\mathbb{A}^n$).

We call the regular sequence $F_1, \ldots, F_p$ *reduced* if the ideal $(F_1, \ldots, F_p)$ generated in $\mathbb{Q}[X]$ is the ideal of definition of the affine variety S, i.e., if $(F_1, \ldots, F_p)$ is radical. We call $(F_1, \ldots, F_p)$ *strongly reduced* if for any index $1 \leq i \leq p$ the ideal $(F_1, \ldots, F_i)$ is radical. Thus, a strongly reduced regular sequence is always reduced.

A point x of $\mathbb{A}^n$ is called $(F_1, \ldots, F_p)$*–regular* if the Jacobian $J(F_1, \ldots, F_p) := \left[\frac{\partial F_j}{\partial X_k}\right]_{\substack{1 \leq j \leq p \\ 1 \leq k \leq n}}$ has maximal rank p at x. Observe, that for each *reduced* regular

sequence $F_1, \ldots, F_p$ defining the variety S, the locus of $(F_1, \ldots, F_p)$–regular points of S is the same. In this case we call an $(F_1, \ldots, F_p)$–regular point of S simply *regular* (or *smooth*) or we say that S is regular (or smooth) at x. The set S_{reg} of regular points of S is called the *regular locus*, whereas $S_{sing} := S \setminus S_{reg}$ is called the *singular locus* of S. If a point x of S belongs to S_{sing} we say that x is singular or that S is singular at x. Remark that S_{reg} is a non–empty open and S_{sing} a proper closed subvariety of S.

Let $\mathbb{A}^n_{\mathbb{R}} := \mathbb{R}^n$ be the n–dimensional real affine space. We denote by $S_{\mathbb{R}} := S \cap \mathbb{A}^n_{\mathbb{R}}$ the real trace of the complex variety S. Moreover, we denote by $\mathbb{P}^n$ the n–dimensional complex projective space and by $\mathbb{P}^n_{\mathbb{R}}$ its real counterpart. We shall use also the following notations:

$$\{F_1 = 0, \ldots, F_p = 0\} := S \text{ and } \{F_1 = 0, \ldots, F_p = 0\}_{\mathbb{R}} := S_{\mathbb{R}}.$$

We say that a connected component C of $S_{\mathbb{R}}$ is *generically* $(F_1, \ldots, F_p)$ *–regular* if C contains an $(F_1, \ldots, F_p)$–regular point.

We suppose now that there are given natural numbers d, L and ℓ and an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ with p output nodes such that the following conditions are satisfied.

- The degrees $\deg F_1, \ldots, \deg F_p$ of the polynomials $F_1, \ldots, F_p$ are bounded by d.
- The p output nodes of the arithmetic circuit σ represent the polynomials $F_1, \ldots, F_p$ by evaluation.
- The size and the non–scalar depth of the arithmetic circuit σ are bounded by L and ℓ, respectively.

For the terminology and basic facts concerning arithmetic circuits we refer to [27, 15, 13].

The fundamental algorithmic elimination problem which motivates the outcome of the present paper is the search for an invariant and an algorithm Π satisfying the following specification.

(i) The invariant is a function which assigns to $F_1, \ldots, F_p$ a positive integer value $\delta := \delta(F_1, \ldots, F_p)$ of asymptotic order not exceeding $(n\,d)^{O(n)}$, called the degree of the real interpretation of the equation system $F_1 = 0, \ldots, F_p = 0$. *The value $\delta(F_1, \ldots, F_p)$ depends rather on the resulting variety S and its geometry than on the defining polynomials $F_1, \ldots, F_p$ themselves.*

(ii) The algorithm Π decides on input σ whether the variety S contains an $(F_1, \ldots, F_p)$ –regular real *point and, if it is the case, produces for each generically $(F_1, \ldots, F_p)$ –regular connected component of S a suitably encoded real algebraic sample point.*

(iii) In order to achieve this goal, the algorithm Π *performs on input* σ *a computation in* $\mathbb{Q}$ *with* $L(n\,d)^{O(1)}\delta^{O(1)}$ *arithmetic operations (additions, subtractions, multiplications and divisions) which become organized in non–scalar depth* $O(n(\ell + \log n\,d)\log\delta)$ *with respect to the parameters of the arithmetic circuit* σ*.*

The formulation of this problem is somewhat imprecise because of the requirement (i) that the value $\delta(F_1,\ldots,F_p)$ depends "rather on the resulting variety S and its geometry than on the defining polynomials $F_1,\ldots,F_p$ themselves". This is due to the fact that in case that $S_{\mathbb{R}}$ is smooth and $F_1,\ldots,F_p$ is strongly reduced, it is possible to exhibit an algorithm that fulfills conditions (ii) and (iii) and that contains a preprocessing which reduces $F_1,\ldots,F_p$ to a single (elimination) polynomial P such that P depends only on S and has, in particular, the same degree as S. The remaining part of the algorithm is its main subroutine which depends only on S (see [4, 5, 49, 51]).

In view of [27, 15] it seems unlikely that the dependence of the degree of the real interpretation of $F_1 = 0,\ldots,F_p = 0$ on the given equations can be completely reduced to an exclusive dependence on S. However, the quantity $\delta(F_1,\ldots,F_p)$ depends only through $F_1,\ldots,F_p$ on the input circuit σ. We consider therefore $\delta(F_1,\ldots,F_p)$ as an *intrinsic* complexity parameter measuring the size of the input σ. The quantities n, d, L and ℓ are considered as *extrinsic* parameters measuring the size of σ.

In these terms we may say that we search for algorithms Π of *intrinsic* complexity which solve the algorithmic elimination problem expressed by requirement (ii). As already mentioned, in the case that $S_{\mathbb{R}}$ is smooth and $F_1,\ldots,F_p$ is a strongly reduced regular sequence, there exist already algorithms which fit in this pattern, i.e., which have intrinsic complexity.

An important issue is the requirement of (i) that the asymptotic order of $\delta(F_1,\ldots,F_p)$ does not exceed the extrinsic bound $(n\,d)^{O(n)}$. This implies that any algorithm Π that satisfies the specification (i), (ii) and (iii) has a worst case complexity that meets the already known extrinsic bound of $(n\,d)^{O(n)}$ for the elimination problem under consideration (compare the original papers [29, 14, 47, 33, 34, 34, 35, 48, 9] and the forthcoming book [10]). The main asset of such an algorithm Π is its incremental complexity character.

Algorithms of intrinsic complexity for elimination problems over the *complex* numbers (or more generally, over arbitrary algebraically closed fields) were first introduced in [23, 24, 25, 26] (see also [31] and the survey [37]). Decisive progress in direction of computer implementations was made in [28] (see also [32]). This led to the development of the software package "Kronecker" by G. Lecerf [40]. The main procedure of the "Kronecker" software package solves over the complex numbers multivariate circuit represented polynomial equation systems by a reusable and portable algorithm of intrinsic (bit–)complexity character. This algorithm supports

type polymorphism and runs in an exact computer algebra as well as in a numeric environment. In the sequel we shall refer to the underlying theoretical procedure as "Kronecker algorithm" (see Section 4).

The Kronecker software package contains various extensions of its main procedure to other, more ambitious elimination tasks in (complex) algebraic geometry and commutative algebra (see [41, 42, 19] for the theoretical aspects of this extension and [20] for a streamlined presentation of the underlying mathematics).

In the context of wavelet constructions, the Kronecker algorithm and software has become adapted to the real case in [43, 44] for the computation of real solutions of polynomial equation systems by means of polar varieties.

We come now back to the initial real elimination problem. In [2] we solved this problem first for a smooth and compact real hypersurface given by a squarefree equation. For an arbitrary strongly reduced regular sequence $F_1, \ldots, F_p \in \mathbb{Q}[X]$ defining a complex affine variety S with *smooth* and *compact* real trace $S_{\mathbb{R}}$, we solved the problem in [3]. Finally, the problem was tackled in [4, 5, 49, 51] under the single assumption that $S_{\mathbb{R}}$ is smooth.

In all these cases the intrinsic invariant which essentially determines the complexity of the algorithm is a combination of the degree of the original equation system $F_1 = 0, \ldots, F_p = 0$ with the maximal degree of the *generic* polar varieties of suitable type, namely *classic* or *dual*, of the complex variety S (see [26, 25] for the notion of system degree and [4, 5, 6] for motivations, definitions and basic properties of classic and dual polar varieties).

The introduction of the (at this moment) new notion of *dual* polar variety became necessary in order to settle the case when $S_{\mathbb{R}}$ is unbounded. In this situation some of the generic *classic* polar varieties of S may have an empty intersection with $S_{\mathbb{R}}$. This makes classic polar varieties inappropriate for algorithmic applications if $S_{\mathbb{R}}$ is unbounded.

The dual polar varieties are the complex counterpart of Lagrange–multipliers. In [4, 5] we introduced the notion of a *generalized polar variety* of S associated with a given embedding of S into the projective space $\mathbb{P}^n$ and a given non–degenerate hyperquadric of $\mathbb{P}^n$. These generalized polar varieties form an algebraic family which connects the classic with the dual polar varieties of S.

In case that $S_{\mathbb{R}}$ is smooth, but possibly unbounded, the fundamental issue for our algorithmic method is the fact that the dual polar varieties of S cut each connected component of $S_{\mathbb{R}}$ (compare Theorem 1 below for the case that $S_{\mathbb{R}}$ is singular).

The *generic* (classic or dual) polar varieties of S, and therefore also their degrees, depend only on S and not on the particular equations which define S. Thus if the real traces of the generic polar varieties of S are all non–empty, their maximal degree becomes a candidate for an intrinsic invariant which governs over the complexity of an algorithm which satisfies the requirement (ii) above. This was the strategy

followed in [4, 5] which led to a solution of our algorithmic elimination problem in case that $S_{\mathbb{R}}$ is smooth, but possibly unbounded.

In the current paper we present two discrete families of algorithms which solve our problem in the particular case of a complex hypersurface containing smooth real points and possibly also real singularities.

So we start with a polynomial $F \in \mathbb{Q}[X]$ of positive degree d and with an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ, such that σ has a single output node representing F.

We ask for an invariant $\delta := \delta(F)$ of asymptotic order not exceeding $(n\,d)^{O(n)}$, called the *degree of the real interpretation of the equation* $F = 0$, and for an algorithm Π satisfying the following specification.

- *The algorithm* Π *decides on input* σ *whether the variety* $S := \{F = 0\}$ *contains an* F*–regular real point, and, if this is the case produces for each generically* F*–regular connected component of* $S_{\mathbb{R}}$ *a suitably encoded real algebraic sample point.*

- *The algorithm* Π *performs on input* σ *a computation in* $\mathbb{Q}$ *with* $L(n\,d)^{O(1)}\delta^{O(1)}$ *arithmetic operations organized, with respect to the parameters of the arithmetic circuit* σ*, in non–scalar depth* $O(n(\ell + \log nd)\log\delta)$.

Observe that in the case that F is squarefree, the invariant $\delta(F)$ depends only on the complex hypersurface S. In this sense we consider as automatically satisfied the informal requirement above, namely that $\delta(F)$ depends rather on S than on the defining polynomial F itself.

The methods developed in [2, 3, 4, 5, 49, 51] for the case that $S_{\mathbb{R}}$ is smooth, cannot be applied when $S_{\mathbb{R}}$ is singular. This becomes clear observing that in the singular case some of the *generic* classic or dual polar varieties of S may have empty real traces, even if $S_{\mathbb{R}}$ is compact.

Nevertheless, Corollary 2 below asserts the *existence* of generic *dual* polar varieties which cut $S_{\mathbb{R}}$ in smooth points in case that $(S_{reg})_{\mathbb{R}}$ is non–empty.

By suitable deformations of the restriction of the gradient of F to the complex hypersurface S we shall find a way out of this dilemma. We realize these deformations by means of *equidimensional* and *smooth* complex varieties which we call *polar deformations of the equation* $F = 0$.

Polar deformations become realized in two different settings which we call the *classic* and the *dual* model.

It turns out that the degrees of the generic *dual* polar varieties of the polar deformations of the equation $F = 0$, called *bipolar varieties* of the equation $F = 0$, furnish appropriate invariants for the design of two discrete families of procedures (one for

the classic and one for the dual model of polar deformations) which solve on input σ our algorithmic elimination problem for the complex hypersurface S.

The degrees of the bipolar varieties of the different polar deformations of the equation $F = 0$ distinguish themselves by their dependence (or independence) from non–singular linear transformations of the indeterminates $X_1, \ldots, X_n$. Therefore the resulting algorithms have distinct intrinsic character.

In case of the classic model we reach our goal completely (see Theorem 24 and Observation 25 below), whereas in case of the dual model we have to require that $S_{\mathbb{R}}$ is compact and our parallel non–scalar complexity bound is slightly worse than the expected one (see Theorem 18 and Observation 19). Despite of this algorithmic drawback we think that it is worth to expose the subject of polar deformations of $F = 0$ in the dual model, because of the following geometric and algorithmic reasons.

One may ask, in case $(S_{reg})_{\mathbb{R}} \neq \emptyset$, which are the generic polar varieties that contain smooth points of $S_{\mathbb{R}}$. In view of Corollary 2 below, this question makes (only) sense for the *dual* polar varieties. If we would be able to find efficiently equations for such generic dual varieties, we would obtain an algorithm which solves our algorithmic elimination problem and has an intrinsic complexity of the same type as in the case that $S_{\mathbb{R}}$ is smooth.

This leads us to the question how we could find efficiently (rational or algebraic) witness points for strict polynomial inequalities (see end of Section 4 and Section 7 for motivations and a partial answer).

For the search of generic dual polar varieties which cut $S_{\mathbb{R}}$ in smooth points, we have to investigate how dual polar varieties vary with their parameters. This is done in Theorem 12 .

In the present paper the dual model is treated in detail, because it contains additional technical difficulties which easily may become overlooked in the classic model, where the argumentation is much simpler as in the dual case. In this sense, similar or identical arguments will not be repeated in the case of the classic model.

In Section 5 we introduce a unified view of the algorithms developed in Section 4 for the case that $S_{\mathbb{R}}$ is possibly singular and of the algorithms of [2, 4, 5, 49, 51] for the case that $S_{\mathbb{R}}$ is smooth. All these algorithms become interpreted as *walks* in suitable graphs. Theorem 21 reflects Theorem 18 in this context. The complex Kronecker algorithm turns out to be a substantial ingredient of our procedures.

A local version of the complexity statements of Section 6 is contained in [7] and [8], with a substantially different treatment of the corresponding polar deformation of $F = 0$.

For another approach relying on the so–called "critical point method" to find real roots in singular real hypersurfaces we refer to [1] for the general context of this

method and to [50] for the particular case of singular hypersurfaces.

Unfortunately our treatment of possibly singular hypersurfaces has no counterpart in the case of higher codimensional complete intersection varieties. Otherwise the generic polar varieties of a complete intersection varieties S would always be smooth at regular points of S. However this contradicts [6], 3.1.

We shall make an extensive use of different types of polar varieties. The modern concept of (classic) polar varieties was introduced in the 1930's by F. Severi ([54], [53]) and J. A. Todd ([62], [61]), while the intimately related notion of a reciprocal curve goes back to the work of J.-V. Poncelet in the period of 1813–1829.

As pointed out by Severi and Todd, generic polar varieties have to be understood as being organized in certain equivalence classes which embody relevant geometric properties of the underlying algebraic variety S. This view led to the consideration of rational equivalence classes of the generic polar varieties.

Around 1975 a renewal of the theory of polar varieties took place with essential contributions due R. Piene ([46]) (global theory), B. Teissier, D. T. Lê ([39], [58]), J. P. Henry and M. Merle ([36]), A. Dubson ([18], Chapitre IV) (local theory), J. P. Brasselet and others (the list is not exhaustive, see [59],[46] and [12] for a historical account and references). The idea was to use rational equivalence classes of generic polar varieties as a tool which allows to establish numerical formulas in order to classify singular varieties by their intrinsic geometric character ([46]).

On the other hand, first classic and then dual polar varieties became around 12 years ago a fundamental tool for the design of efficient computer procedures of intrinsic complexity which solve suitable instances of our algorithmic elimination problem ([2, 3, 4, 5]). The use of polar varieties made in the present paper is based on certain geometric facts which are developed in [6]. Of particular relevance is a relative degree estimate for polar varieties, namely [6], Theorem 3, which allows us to compare the intrinsic complexities of distinct algorithms.

2 Preliminaries about polar varieties

Let notations be as in the Introduction. When nothing else is said we suppose throughout this section that $F_1, \ldots, F_p \in \mathbb{Q}[X]$ is a *reduced* regular sequence defining a (non–empty) subvariety S of $\mathbb{A}^n$ of pure codimension p.

Let $1 \leq i \leq n-p$ and let $a := [a_{k,l}]_{\substack{1 \leq k \leq n-p-i+1 \\ 0 \leq l \leq n}}$ be a complex $((n-p-i+1)\times(n+1)$–matrix and suppose that $a_* := [a_{k,l}]_{\substack{1 \leq k \leq n-p-i+1 \\ 1 \leq l \leq n}}$ has maximal rank $n-p-i+1$. In case $(a_{1,0}, \ldots, a_{n-p-i+1,0}) = 0$ we denote by $\underline{K}(a) := \underline{K}^{n-p-i}(a)$ and in case $(a_{1,0}, \ldots, a_{n-p-i+1,0}) \neq 0$ by $\overline{K}(a) := \overline{K}^{n-p-i}(a)$ the $(n-p-i)$–dimensional linear subvarieties of the projective space $\mathbb{P}^n$ which for $1 \leq k \leq n-p-i+1$ are spanned by the the points $(a_{k,0} : a_{k,1} : \cdots : a_{k,n})$. In the first case we shall also use the

notations $\underline{K}(a_*)$ and $\underline{K}^{n-p-i}(a_*)$ instead of $\underline{K}(a)$ and $\underline{K}^{n-p-i}(a)$.

The classic and the dual ith polar varieties of S associated with the linear varieties $\underline{K}(a)$ and $\overline{K}(a)$ are defined as the closures of the loci of the $(F_1,\ldots,F_p)$–regular points of S where all $(n-i+1)$–minors of the respective polynomial $((n-i+1)\times n)$–matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1} & \cdots & a_{n-p-i+1,n} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1} & \cdots & \frac{\partial F_p}{\partial X_n} \\ a_{1,1}-a_{1,0}X_1 & \cdots & a_{1,n}-a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-p-i+1,1}-a_{n-p-i+1,0}X_1 & \cdots & a_{n-p-i+1,n}-a_{n-p-i+1,0}X_n \end{bmatrix}$$

vanish. We denote these polar varieties by

$$W_{\underline{K}(a)}(S) := W_{\underline{K}^{n-p-i}(a)}(S) \text{ and } W_{\overline{K}(a)}(S) := W_{\overline{K}^{n-p-i}(a)}(S),$$

respectively. They are of expected pure codimension i in S and do not depend on the particular choice of the reduced regular sequence defining S.

If a is a real $((n-p-i+1)\times(n+1))$–matrix, we denote by

$$W_{\underline{K}(a)}(S_{\mathbb{R}}) := W_{\underline{K}^{n-p-i}(a)}(S_{\mathbb{R}}) := W_{\underline{K}(a)}(S)\cap \mathbb{A}^n_{\mathbb{R}}$$

and

$$W_{\overline{K}(a)}(S_{\mathbb{R}}) := W_{\overline{K}^{n-p-i}(a)}(S_{\mathbb{R}}) := W_{\overline{K}(a)}(S)\cap \mathbb{A}^n_{\mathbb{R}}$$

the real traces of $W_{\underline{K}(a)}(S)$ and $W_{\overline{K}(a)}(S)$.

Observe that this definition of classic and dual polar varieties may be extended to the case that there is given a Zariski open subset O of $\mathbb{A}^n$ such that the equations $F_1=0,\ldots,F_p=0$ intersect transversally at any of their common solutions in O and that S is now the locally closed subvariety of $\mathbb{A}^n$ given by

$$S := \{F_1=0,\ldots,F_p=0\}\cap O,$$

which is supposed to be non-empty.

In Section 4 and 6 we shall need this extended definition of polar varieties in order to establish the notion of a bipolar variety of a given hypersurface. For the moment

let us suppose again that S is the closed subvariety of $\mathbb{A}^n$ defined by the reduced regular sequence $F_1, \ldots, F_p$.

In [4] and [5] we have introduced the notion of dual polar varieties of S (and $S_{\mathbb{R}}$) and motivated by geometric arguments the calculatory definition of these objects. Moreover, we have shown that, for a complex $((n-p-i+1)\times(n+1))$–matrix $a = [a_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ 0\leq l\leq n}}$ with $[a_{k,l}]_{\substack{1\leq k\leq n-p-i+1\\ 1\leq l\leq n}}$ *generic*, the polar varieties $W_{\underline{K}(a)}(S)$ and $W_{\overline{K}(a)}(S)$ are either empty or of pure codimension i in S. Further, we have shown that $W_{\underline{K}(a)}(S)$ and $W_{\overline{K}(a)}(S)$ are normal and Cohen–Macaulay (but non necessarily smooth) at any of their $(F_1, \ldots, F_p)$–regular points (see [6], Corollary 2 and Section 3.1). This motivates the consideration of the so–called *generic* polar varieties $W_{\underline{K}(a)}(S)$ and $W_{\overline{K}(a)}(S)$, associated with complex $((n-p-i+1)\times(n+1))$–matrices a which are generic in the above sense, as invariants of the complex variety S (independently of the given equation system $F_1 = 0, \ldots, F_p = 0$). However, when a generic $((n-p-i+1)\times(n+1))$–matrix a is real, we cannot consider $W_{\underline{K}(a)}(S_{\mathbb{R}})$ and $W_{\overline{K}(a)}(S_{\mathbb{R}})$ as invariants of the real variety $S_{\mathbb{R}}$, since for suitable real generic $((n-p-i+1)\times(n+1))$–matrices these polar varieties may turn out to be empty, whereas for other real generic matrices they may contain points (see Theorem 1, Corollary 2, Theorem 12 and Corollary 13 below).

For our use of the word "generic" we refer to [6], Definition 1.

In case that $S_{\mathbb{R}}$ is smooth and a is real $((n-p-i+1)\times(n+1))$–matrix, the real dual polar variety $W_{\overline{K}(a)}(S_{\mathbb{R}})$ contains at least one point of each connected component of $S_{\mathbb{R}}$, whereas the classic (complex or real) polar varieties $W_{\underline{K}(a)}(S)$ and $W_{\underline{K}(a)}(S_{\mathbb{R}})$ may be empty (see [4] and [5], Proposition 2).

In case of a singular real variety $S_{\mathbb{R}}$ such a strong result cannot be expected. We have the following weaker result.

Theorem 1
Let $1 \leq i \leq n-p$ and let C be a generically $(F_1, \ldots, F_p)$–regular connected component of the real variety $S_{\mathbb{R}}$. Then, with respect to the Euclidean topology, there exists a non–empty, open, semialgebraic subset $O_C^{(i)}$ of $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1)\times(n+1)}$ such that for any $((n-p-i+1)\times(n+1))$–matrix a of $O_C^{(i)}$ the submatrix $a_* \in \mathbb{A}^{(n-p-i+1)\times n}$ has maximal rank $n-p-i+1$, the column vector $a_0 \in \mathbb{A}^{n-p-i+1}$ is non–zero and such that the real dual polar variety $W_{\overline{K}(a)}(S_{\mathbb{R}})$ is generic and contains an $(F_1, \ldots, F_p)$–regular point of C.

Proof
Immediate by [6], Theorem 1. □

As a consequence of Theorem 1 we obtain the following statement (compare [6], Corollary 1).

Corollary 2
Suppose that the real variety $S_{\mathbb{R}}$ *contains an* $(F_1, \ldots, F_p)$*–regular point and let* $1 \leq i \leq n-p$*. Then, with respect to the Euclidean topology, there exists a non-empty, open, semialgebraic subset* $O^{(i)}$ *of* $\mathbb{A}_{\mathbb{R}}^{(n-p-i+1)\times(n+1)}$ *such that for any* $((n-p-i+1)\times(n+1))$*–matrix of* $O^{(i)}$ *the submatrix* $a_* \in \mathbb{A}^{(n-p-i+1)\times n}$ *has maximal rank* $n-p-i+1$*, the column vector* $a_0 \in \mathbb{A}^{n-p-i+1}$ *is non–zero and such that the real dual polar variety* $W_{\overline{K}(a)}(S_{\mathbb{R}})$ *is generic and non–empty.*

We are now going to state and prove a technical result we shall need in Sections 4, 5 and 6.

Let $\overline{X} := (X_1, \ldots, X_{n-1})$ and let be given a Zariski open subset O of $\mathbb{A}^n$ and a complex number $c \in \mathbb{A}^1$ such that the equations $F_1(X) = 0, \ldots, F_p(X) = 0$ and the equations $F_1(\overline{X}, c) = 0, \ldots, F_p((\overline{X}, c) = 0$ intersect transversally at any of their common zeros that belong to O or to $O_c := \{\overline{x} \in \mathbb{A}^{n-1} \mid (\overline{x}, c) \in O\}$, respectively. Denote by $\mu_c : \mathbb{A}^{n-1} \to \mathbb{A}^n$ the embedding of affine spaces defined for $\overline{x} \in \mathbb{A}^{n-1}$ by $\mu_c(\overline{x}) := (\overline{x}, c)$.

We compare now the polar varieties of

$$S := \{F_1(X) = 0, \ldots, F_p(X) = 0\} \cap O$$

and

$$S_c := \{F_1(\overline{X}, c) = 0, \ldots, F_p((\overline{X}, c) = 0\} \cap O_c.$$

Observe that S and S_c are (locally closed) subvarieties of $\mathbb{A}^n$ and $\mathbb{A}^{n-1}$ which we suppose to be non–empty.

Let $1 \leq i < n-p$ and let $a = [a_{k,l}]_{\substack{1\leq k\leq n-p-i \\ 0\leq l\leq n}}$ be a complex $((n-p-i)\times(n+1))$–matrix such that $[a_{k,l}]_{\substack{1\leq k\leq n-p-i \\ 1\leq l\leq n}}$ has maximal rank $n-p-i$.

Lemma 3
Let notations be as above, further let $a' := [a_{k,l}]_{\substack{1\leq k\leq n-p-i \\ 0\leq l\leq n-1}}$ *and* $a'' := \begin{bmatrix} & & \overset{a}{} & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$.

Then, in case $(a_{1,0}, \ldots, a_{n-p-i,0}) \neq 0$*, the affine linear map* $\mu_c : \mathbb{A}^{n-1} \to \mathbb{A}^n$ *induces an isomorphism between the dual polar variety* $W_{\overline{K}^{n-p-i+1}(a')}(S_c)$ *and the closed variety* $W_{\overline{K}^{n-p-i}(a'')}(S) \cap \{X_n - c = 0\}$*. The same is true in case* $(a_{1,0}, \ldots, a_{n-p-i,0}) = 0$ *for the classic polar varieties* $W_{\underline{K}^{n-p-i+1}(a')}(S_c)$ *and* $W_{\underline{K}^{n-p-i}(a'')}(S)$*.*

Proof
Without loss of generality we may assume $(a_{1,0}, \ldots, a_{n-p-i,0}) \neq 0$. Deleting the columns number 0 in the matrices a' and a'' we obtain full rank matrices. Therefore the dual polar varieties $W_{\overline{K}^{n-p-i+1}(a')}(S_c)$ and $W_{\overline{K}^{n-p-i}(a'')}(S)$ are well–defined. It suffices to show that μ_c induces an isomorphism between $W_{\overline{K}^{n-p-i+1}(a')}(S_c) \cap O_c$ and $W_{\underline{K}^{n-p-i}(a'')}(S) \cap \{X_n - c = 0\} \cap O$. From our assumptions we deduce that the mapping μ_c identifies S_c with $S \cap \{X_n - c = 0\}$ and that for each $\overline{x} \in S_c$ the point

$\overline{x}$ is $(F_1(\overline{X},c),\ldots,F_p(\overline{X},c))$–regular and the point $\mu_c(\overline{x})=(\overline{x},c)$ is $(F_1,\ldots,F_p)$–regular. Let $\overline{x}$ be an arbitrary element of S_c. Then all $(n-i)$–minors of the polynomial $((n-i)\times(n-1))$–matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1}(\overline{X},c) & \cdots & \frac{\partial F_1}{\partial X_{n-1}}(\overline{X},c) \\ \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1}(\overline{X},c) & \cdots & \frac{\partial F_p}{\partial X_{n-1}}(\overline{X},c) \\ a_{1,1}-a_{1,0}X_1 & \cdots & a_{1,n-1}-a_{1,0}X_{n-1} \\ \vdots & \vdots & \vdots \\ a_{n-p-i,1}-a_{n-p-i,0}X_1 & \cdots & a_{n-p-i,n-1}-a_{n-p-i,0}X_{n-1} \end{bmatrix}$$

vanish at $\overline{x}$ if and only if all $(n-i+1)$–minors of the polynomial $((n-i+1)\times n)$–matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial X_1}(X) & \cdots & \frac{\partial F_1}{\partial X_{n-1}}(X) & \frac{\partial F_1}{\partial X_n}(X) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_p}{\partial X_1}(X) & \cdots & \frac{\partial F_p}{\partial X_{n-1}}(X) & \frac{\partial F_p}{\partial X_n}(X) \\ a_{1,1}-a_{1,0}X_1 & \cdots & a_{1,n-1}-a_{1,0}X_{n-1} & a_{1,n}-a_{1,0}X_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-p-i,1}-a_{n-p-i,0}X_1 & \cdots & a_{n-p-i,n-1}-a_{n-p-i,0}X_{n-1} & a_{n-p-i,n}-a_{n-p-i,0}X_n \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

vanish at $\mu_c(\overline{x})$. This implies that $\overline{x}$ belongs to $W_{\overline{K}^{n-p-i+1}(a')}(S_c)\cap O_c$ if and only if $\mu(\overline{x})$ belongs to $W_{\overline{K}^{n-p-i}(a'')}(S)\cap\{X_n-c=0\}\cap O$. □

3 The dual model

3.1 Polar deformations in the dual model

Let d, n and i be natural numbers, $1\leq i\leq n-1$, and let $X:=(X_1,\ldots,X_n)$, $\Omega:=(\Omega_1,\ldots,\Omega_{n-i})$ be row vectors and $A:=A_i:=[A_{k,l}]_{\substack{1\leq k\leq n-i\\ 0\leq l\leq n}}$ be an $((n-i)\times(n+1))$–matrix of indeterminates over $\mathbb{C}$. Furthermore, let Λ be a single indeterminate over $\mathbb{C}$ and $F\in\mathbb{R}[X_1,\ldots,X_n]$ an n–variate polynomial over $\mathbb{R}$ of positive degree $\deg F=d$. The polynomial F will be fixed for the rest of this paper.

Let $J(F):=(\frac{\partial F}{\partial X_1},\ldots,\frac{\partial F}{\partial X_n})$ be the gradient (Jacobian) of F. In the sequel we shall generally *not* require that F is reduced (i.e., squarefree). Thus $J(F)$ may vanish identically on some irreducible component of the complex hypersurface $\{F=0\}$.

For a complex $((n-i)\times(n+1))$–matrix $a:=[a_{k,l}]_{\substack{1\leq k\leq n-i\\ 0\leq l\leq n}}$ and a point $x=(x_1,\ldots,x_n)\in\mathbb{A}^n$ we write $A_0:=A_0^{(i)}:=(A_{1,0},\ldots,A_{n-i,0})$, $a_0:=(a_{1,0},\ldots,a_{n-i,0})$, $A_*:=A_*^{(i)}:=[A_{k,l}]_{\substack{1\leq k\leq n-i\\ 1\leq l\leq n}}$ and $a_*:=[a_{k,l}]_{\substack{1\leq k\leq n-i\\ 1\leq l\leq n}}$. Furthermore, we denote by

$A(X) := A_i(X)$ and $a(x)$ the $((n-i)\times n)$–matrices $[A_{k,l} - A_{k,0}X_l]_{\substack{1\le k\le n-i\\ 1\le l\le n}}$ and $[a_{k,l} - a_{k,0}x_l]_{\substack{1\le k\le n-i\\ 1\le l\le n}}$.

Thus, specializing the $((n-i)\times(n+1))$–matrix A to a and the row vector X to x, we obtain a_0, a_*, and $a(x)$ as specializations of A_0, A_* and $A(x)$, respectively. We indicate the rank of a matrix, e.g. of a, by $\mathrm{rk}\,(a)$. As usual we denote by a^T the transposed matrix of a.

For $(\lambda, \omega_1, \ldots, \omega_{n-i}) \in \mathbb{A}^{n-i+1} \setminus \{0\}$ and $\omega := (\omega_1, \ldots, \omega_{n-i})$ we shall write $(\lambda : \omega) := (\lambda : \omega_1 : \cdots : \omega_{n-i})$ for the corresponding point of $\mathbb{P}^{n-i}$.

In the sequel we shall consider the ambient space

$$\mathbb{M}_i := \mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{P}^{n-i}$$

containing the $\mathbb{R}$–definable locally closed variety

$$E_i := \{(x,\ a,\ (\lambda:\omega)) \in \mathbb{M}_i \mid F(x) = 0,\ \mathrm{rk}\ a_* = \mathrm{rk}\ a(x) = n-i,$$
$$a_0\,\omega^T \neq 0,\ J(F)(x)^T\lambda + a(x)^T\omega^T = 0\}.$$

Let $(x,\ a,\ (\lambda:\omega))$ be an arbitrary point of E_i. From $a_0\,\omega^T \neq 0$ and $\mathrm{rk}\ a(x) = n-i$ we deduce first $\omega \neq 0$ and then $J(F)(x) \neq 0$ and $\lambda \neq 0$.

Observation 4
Let x be a point of $\mathbb{A}^n$ satisfying the conditions $F(x) = 0$ and $J(F)(x) \neq 0$. Then there exists a point $(a, (\lambda:\omega))$ of $\mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{P}^{n-i}$ such that $(x, a, (\lambda:\omega))$ belongs to E_i and in particular, E_i is non–empty. If x is a real point, then $(a, (\lambda:\omega))$ may be chosen real.

Proof
Since we have by assumption $J(F)(x) \neq 0$ there exists a complex number $\gamma \in \mathbb{C}\setminus\{0\}$ and a complex $((n-i)\times(n-1))$–matrix b, with columns numbered by $2, \ldots, n$ such that the following conditions are satisfied:

- $-J(F)(x) + \gamma x \neq 0$,
- the complex $((n-i)\times n)$–matrices b_1 and b_2, whose first columns are $-J(F)(x)^T$ and $(-J(F)(x) + \gamma x)^T$ and whose columns number $2, \ldots, n$ are the corresponding columns of b, have maximal rank $n-i$.

Let a be the complex $((n-i)\times(n+1))$–matrix defined by $a_0 = (\gamma, 0, \ldots, 0)$ and $a_* = b_2$, and let $\lambda := 1$ and $\omega = (\omega_1, \ldots, \omega_{n-i}) = (1, 0, \ldots, 0)$. One verifies now easily that the point $(x,\ a,\ (\lambda:\omega))$ belongs to E_i. In particular, if x is a real point, then γ and b, hence also a and $(\lambda:\omega)$ may be chosen real. □

Proposition 5
Let D_i be the closed subvariety of $\mathbb{M}_i$ defined by the condition rk $A_*^{(i)} < n-i$ or rk $A_i(X) < n-i$ or $A_0^{(i)} \cdot \Omega^T = 0$. Then the polynomial equations

$$(1) \qquad F(X) = 0, \qquad \frac{\partial F}{\partial X_l}\Lambda + \sum_{1 \le k \le n-i} (A_{k,l} - A_{k,0}X_l)\Omega_k = 0, \quad 1 \le l \le n,$$

intersect transversally at any of their common solutions in $\mathbb{M}_i \setminus D_i$. Moreover, E_i is exactly the set of solutions of the polynomial equation system (1) outside of the locus D_i.

The set E_i, interpreted as incidence variety between $\mathbb{A}^n$ and $\mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{P}^{n-i}$, dominates the locus of all F–regular points of the complex hypersurface $\{F = 0\}$.

In particular, E_i is an equidimensional algebraic variety which is empty or smooth and of dimension $(n-i)(n+2)-1$. The real variety $E_{\mathbb{R}}^{(i)} := (E_i)_{\mathbb{R}}$ is non–empty if and only if the hypersurface $\{F = 0\}$ contains an F–regular real point.

Proof
Observe that the succinctly written polynomial equation system $J(F)(X)^T\Lambda + A_i(X)^T\Omega^T = 0$ is in fact

$$\frac{\partial F}{\partial X_l}\Lambda + \sum_{1 \le k \le n-i} (A_{k,l} - A_{k,0}X_l)\Omega_k = 0, \quad 1 \le l \le n.$$

Therefore, any point $(x,\ a\ , (\lambda : \omega)) \in \mathbb{M}$ which does not belong to D_i and is a solution of the preceding polynomial equation system satisfies the condition

$$\omega \neq 0, \ \ \lambda \neq 0 \ \text{ and } \ J(F)(x) \neq 0.$$

Hence we may suppose without loss of generality $\lambda := 1$. The polynomial equation system (1) becomes therefore

$$(2) \qquad F(X) = 0, \qquad \frac{\partial F}{\partial X_l}(X) + \sum_{1 \le k \le n-i} (A_{k,l} - A_{k,0}X_l)\Omega_k = 0, \quad 1 \le l \le n.$$

The Jacobian of this system is a polynomial $((n+1) \times ((n-i)(n+2)+n))$–matrix of the following form

$$\mathcal{L}_i :=$$

$$\begin{bmatrix} \frac{\partial F}{\partial X_1} & \cdots & \frac{\partial F}{\partial X_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & & \Omega_1 & \cdots & \Omega_{n-i} & & 0 & \cdots & 0 & -X_1\Omega_1 & \cdots & -X_1\Omega_{n-i} \\ & & & & & & & & & & & & & \cdot & \cdot & \cdot \\ & * & & & A_i(X)^T & & & 0 & & \ddots & & 0 & & \cdot & \cdot & \cdot \\ & & & & & & & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & 0 & \cdots & 0 & \cdots & \Omega_1 & \cdots & \Omega_{n-i} & -X_n\Omega_1 & \cdots & -X_n\Omega_{n-i} \end{bmatrix}.$$

A point $(x,\ a,\ (1 : \omega)) \in \mathbb{M}_i$ which does not belong to D_i satisfies the polynomial equation system (1) if and only if $(x,\ a,\ \omega) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{A}^{n-i}$ is a solution

of (2). Moreover, in this case we have $J(F)(x) \neq 0$ and $\omega \neq 0$. This implies that the $((n+1) \times ((n-i)(n+2)+n))$–matrix $\mathcal{L}_i$ has maximal rank $n+1$ at any solution $(x,\ a,\ \omega)$ of (2) which satisfies the condition $(x,\ a,\ (1:\omega)) \notin D_i$.

Thus the equations of (1) intersect transversally at any of their common solutions in $\mathbb{M}_i \setminus D_i$ and it is also clear from the definitions that these solutions constitute the algebraic variety E_i.

Since the polynomial equation system (2) contains $n+1$ equations in $(n-i)(n+2)+n$ unknowns we conclude now that E_i is empty or equidimensional of dimension $((n-i)(n+2)+n)-(n+1) = (n-i)(n+2)-1$.

If the hypersurface $\{F=0\}$ contains a (real) F–regular point, then Observation 4 implies that E_i (or $E_{\mathbb{R}}^{(i)}$) is not empty. If E_i (or $E_{\mathbb{R}}^{(i)}$) is non–empty it contains a (real) point $(x,\ a,\ (\lambda:\omega))$ with $F(x)=0$, rk $a(x) = n-i$ and $(\lambda:\omega) \in \mathbb{P}^{n-i}$. From rk $a(x) = n-i$ we deduce $J(F)(x) \neq 0$. Therefore, $\{F=0\}$ contains a (real) F–regular point. This implies that E_i dominates the locus of all F–regular points of $\{F=0\}$ and that $E_{\mathbb{R}}^{(i)}$ is non–empty if and only if $\{F=0\}$ contains an F–regular real point. □

The final aim of this paper is the development of geometric tools which allow us to design efficient algorithms that find real F–regular points of the hypersurface $\{F=0\}$. The condition $\Lambda := 1$ in (1) and hence the equation system (2) are not well–suited for this purpose since in this way we obtain rather a description of A_i in terms of X than the opposite. Therefore we prefer to fix one of the entries of Ω and to let move Λ.

On the other side, the algorithmic tools we have at hand require subvarieties of affine spaces with *closed* and *smooth* real traces. In order to satisfy this requirement, we shall replace in Proposition 6 below the polynomial equation system (1) by a more simple one.

For the formulation and the proof of the next result of this section, namely Proposition 6, we introduce the following mathematical objects and notations.

Let $1 \leq h \leq n-i$ and let $B := B_i := [B_{k,l}]_{\substack{1\leq k\leq n-i\\ 1\leq l\leq n}}$ and $\Theta := (\Theta_1, \ldots, \Theta_{n-i})$ be a $((n-i)\times n)$–matrix and a row vector whose entries are new indeterminates $B_{k,l}$ and Θ_k, $1 \leq k \leq n-i$, $1 \leq l \leq n$. We write $B^{(h)}$ for the $((n-i)\times(n+1))$–matrix defined by $(B^{(h)})_0 := (\delta_{k,h})_{1\leq k\leq n-i}$ and $B^{(h)}_* := B$, where $\delta_{k,h}$ denotes the Kronecker symbol given by $\delta_{k,k} = 1$ and $\delta_{k,h} = 0$ for $k \neq h$. Similarly, for $b \in \mathbb{A}^{(n-i)\times n}$ we denote by $b^{(h)}$ the complex $((n-i)\times(n+1))$–matrix defined by $(b^{(h)})_0 := (\delta_{k,h})_{1\leq k\leq n-i}$ and $(b^{(h)})_* := b$.

We introduce now a new ambient space, namely

$$\mathbb{T}_i^{(h)} := \{(x,\ b,\ (\lambda:\vartheta));\ |\ x \in \mathbb{A}^n, b \in \mathbb{A}^{(n-i)\times n}, \lambda \in \mathbb{A}^1$$

and $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i}) \in \mathbb{A}^{n-i}$ with $\vartheta_h \neq 0\}$.

Let

$$H_i^{(h)} := \{(x,\ b,\ (\lambda : \vartheta)) \in \mathbb{T}_i^{(h)} |\ \ F(x) = 0,$$
$$\text{rk } b = \text{rk } b^{(h)}(x) = n - i,\ J(F)(x)^T\lambda + b^{(h)}(x)^T\vartheta^T = 0\}.$$

Observe that $\mathbb{T}_i^{(h)}$ is an algebraic variety which is isomorphic to the affine space $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^{n-i}$ and that $H_i^{(h)}$ is an $\mathbb{R}$–definable locally closed subvariety of $\mathbb{T}_i^{(h)}$. The ambient space $\mathbb{T}_i^{(h)}$ may be linearly embedded in $\mathbb{M}_i$ and this embedding maps $H_i^{(h)}$ into E_i.

Sometimes we shall tacitly identify $\mathbb{T}_i^{(h)}$ with the affine space $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^{n-i}$. This will always be clear by the context.

For $1 \leq k \leq n-i$ and $1 \leq l_1 < \cdots < l_{n-i} \leq n$, let

$$O_{(h;l_1,\ldots,l_{n-i})} := \{a \in \mathbb{A}^{(n-i)\times(n+1)} \mid a = [a_{k,l}]_{\substack{1\leq k\leq n-i \\ 0\leq l\leq n}} \quad \text{with} \quad a_{h,0} \neq 0$$
$$\text{and} \quad \det\left[a_{l_k,l_j}\right]_{1\leq k,j\leq n-i} \neq 0\},$$
$$U_{(l_1,\ldots,l_{n-i})} := \{b \in \mathbb{A}^{(n-i)\times n} \mid b = [b_{k,l}]_{\substack{1\leq k\leq n-i \\ 1\leq l\leq n}} \quad \text{with} \quad \det\left[b_{l_k,l_j}\right]_{1\leq k,j\leq n-i} \neq 0\},$$
$$\mathbb{M}^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}} := \{(x,\ a,\ (\lambda : \omega)) \in \mathbb{M}_i \mid a \in O_{(h;l_1,\ldots,l_{n-i})}\},$$
$$\mathbb{T}^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}} := \{(x,\ b,\ (\lambda : \omega)) \in \mathbb{T}_i \mid b \in U_{(l_1,\ldots,l_{n-i})}\},$$
$$E^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}} := E_i \cap \mathbb{M}^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}}$$

and

$$H^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}} := H_i \cap \mathbb{T}^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}}.$$

Observe that $\left(E^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}}\right)_{\substack{1\leq h\leq n-i \\ 1\leq l_1<,\ldots,l_{n-i}\leq n}}$ and $\left(H^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}}\right)_{\substack{1\leq h\leq n-i \\ 1\leq l_1<,\ldots,l_{n-i}\leq n}}$ are coverings of E_i and $H_i^{(h)}$ by open subvarieties.

We are now able to state and prove the next result.

Proposition 6
Let $1 \leq h \leq n-i$ and $1 \leq l_1 < \cdots < l_{n-i} \leq n$. The $\mathbb{R}$–definable algebraic variety $E^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}}$ is isomorphic to $\mathbb{A}^{n-i} \times H^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}}$. In particular, $H_i^{(h)}$ is an $\mathbb{R}$–definable equidimensional algebraic variety which is empty or smooth and of dimension $(n-i)(n+1)-1$.

Let $D_{(i,h)}$ be the closed subvariety of $\mathbb{T}_i^{(h)}$ defined by the condition rk $B_i < n-i$ or rk $B_i^{(h)}(X) < n-i$.

Then the equations of the system

$$(3)\quad F(X)=0,\qquad \frac{\partial F}{\partial X_l}(X)\,\Lambda+(B_{h,l}-X_l)\Theta_h+\sum_{\substack{1\le k\le n-i\\ k\neq h}} B_{k,l}\,\Theta_k=0,\quad 1\le l\le n,$$

intersect transversally at any of their common solutions in $\mathbb{T}_i^{(h)}\backslash D_{(i,h)}$. *The algebraic variety* $H_i^{(h)}$ *consists exactly of these solutions.*

The set $H_i^{(h)}$, *interpreted as an incidence variety between* $\mathbb{A}^n$ *and* $\mathbb{A}^{(n-i)\times n}\times\mathbb{P}^{n-i}$, *dominates the locus of all* F*–regular points of the complex hypersurface* $\{F=0\}$. *The real variety* $(H_i^{(h)})_{\mathbb{R}}$ *is non–empty if and only if* $\{F=0\}$ *contains an* F*–regular real point.*

Proof

Observe that the succinctly written polynomial equation system

$$J(F)(X)^T\,\Lambda+B^{(h)}(X)^T\Theta^T=0$$

is in fact

$$\frac{\partial F}{\partial X_l}(X)\,\Lambda+(B_{h,l}-X_l)\Theta_h+\sum_{\substack{1\le k\le n-i\\ k\neq h}} B_{k,l}\,\Theta_k=0,\quad 1\le l\le n$$

and that any point $(x,\ b,\ (\lambda:\vartheta))\in\mathbb{T}_i^{(h)}$ with $\vartheta=(\vartheta_1,\ldots,\vartheta_{n-i})$, which does not belong to $D_{(i,h)}$ and is a solution of the polynomial equation system (3), satisfies the condition

$$\vartheta_h\neq 0,\ \ \lambda\neq 0\ \text{ and }\ J(F)(x)\neq 0.$$

Therefore we may suppose without loss of generality $\lambda=1$. The polynomial equation system (3) becomes therefore

$$(4)\quad F(X)=0,\qquad \frac{\partial F}{\partial X_l}(X)+(B_{h,l}-X_l)\Theta_h+\sum_{\substack{1\le k\le n-i\\ k\neq h}} B_{k,l}\,\Theta_k=0,\quad 1\le l\le n.$$

The Jacobian of this system is the polynomial $((n+1)\times(n-i)(n+1)+n)$–matrix

$$J_{i,h}:=\begin{bmatrix}\frac{\partial F}{\partial X_1} & \cdots & \frac{\partial F}{\partial X_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ & & & & & & \Theta_1 & \cdots & \Theta_{n-i} & & 0 & \cdots & 0\\ & * & & & B_h(X)^T & & & 0 & & \ddots & & 0 & \\ & & & & & & 0 & \cdots & 0 & \cdots & \Theta_1 & \cdots & \Theta_{n-i}\end{bmatrix},$$

with

$$B_h(X):=\begin{bmatrix}B_{h,1}-X_1 & \cdots & B_{h,n}-X_n\\ \multicolumn{3}{c}{\left[B_{k,l}\right]_{\substack{1\le k\le n-i,\ k\neq h\\ 1\le l\le n}}}\end{bmatrix}:=\begin{bmatrix}B_{h,1}-X_1 & \cdots & B_{h,n}-X_n\\ B_{1,1} & \cdots & B_{1,n}\\ \vdots & \vdots & \vdots\\ B_{h-1,1} & \cdots & B_{h-1,n}\\ B_{h+1,1} & \cdots & B_{h+1,n}\\ \vdots & \vdots & \vdots\\ B_{n-i,1} & \cdots & B_{n-i,n}\end{bmatrix}.$$

A point $(x,\, b,\, (1 : \vartheta))$ of ${\mathbb{T}_i}^{(h)}$ with $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i})$ which does not belong to $D_{(i,h)}$ satisfies the polynomial equation system (3) if and only if $(x,\, b,\, \vartheta)$ is a solution of (4). Moreover, we have $J(f)(x) \neq 0$ and $\vartheta \neq 0$ in this case. This implies that the $((n+1) \times ((n-i)(n+1)+n))$–matrix $J_{i,h}$ has maximal rank $n+1$ at $(x,\, b,\, \vartheta)$.

Thus the equations of (4) intersect transversally at any of their common solutions in $\mathbb{T}_i^{(h)} \setminus D_{(i,h)}$. It is also clear from the definitions that these solutions form the algebraic variety $H_i^{(h)}$. As in the proof of Proposition 5 one sees that $H_i^{(h)}$ is empty or equidimensional of dimension $(n-i)(n+1)-1$ and dominates the locus of the F–regular points of $\{F = 0\}$.

We are going now to construct for $1 \leq h \leq n-i$ and $1 \leq l_1 < \cdots < l_{n-i} \leq n$ an isomorphism from the algebraic variety $E^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}}$ to $\mathbb{A}^{n-i} \times H^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}}$.

Without loss of generality we may restrict our attention to the case $h := 1$ and $l_1 := 1, \ldots, l_{n-i} := n-i$. We consider therefore

$$U := U_{(1,\ldots,n-i)} = \{b \in \mathbb{A}^{(n-i)\times n} \mid b = [b_{k,l}]_{\substack{1\leq k\leq n-i \\ 1\leq l\leq n}},\ \det [b_{k,l}]_{1\leq k,l\leq n-i} \neq 0\}$$

and

$$O := O_{(1;1,\ldots,n-i)} = \{a \in \mathbb{A}^{(n-i)\times(n+1)} \mid a = [a_{k,l}]_{\substack{1\leq k\leq n-i \\ 1\leq l\leq n}},\, a_{1,0} \neq 0,\ \det [a_{k,l}]_{1\leq k,l\leq n-i} \neq 0\}.$$

Further, we consider the $((n-i)\times(n-i))$–matrix

$$Q := \begin{bmatrix} \frac{1}{A_{1,0}} & \frac{-A_{2,0}}{A_{1,0}} & \cdots & \frac{-A_{n-i,0}}{A_{1,0}} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

whose inverse matrix is

$$Q^{-1} = \begin{bmatrix} A_{1,0} & A_{2,0} & \cdots & A_{n-i,0} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let $A'' = \left[A''_{k,l}\right]_{\substack{1\leq k\leq n-i \\ 1\leq l\leq n}}$ be the $((n-i)\times n)$–matrix $A'' := Q^T A_*$ and let $\Omega'' = (\Omega''_1, \ldots, \Omega''_{n-i})$ be the row vector $\Omega'' := \Omega(Q^T)^{-1}$. Observing the identity $A_0 \cdot Q = (1, 0, \ldots, 0)$ we conclude that $(Q^T A)_0 = (1, 0, \ldots, 0)$ and $(Q^T A)_* = A''$ holds. Moreover we have $\Omega''_1 = A_0 \cdot \Omega^T$.

The entries $A''_{k,l}$ of A'' are rational functions belonging to $\mathbb{Q}(A)$, all well-defined at any point of O and the same is true for the entries of the $((n-i)\times(n-i))$–matrix Q. On the other hand, the entries Ω''_k of Ω'' are polynomials belonging to $\mathbb{Q}[A,\Omega]$.

Let $(x,\, a,\, (\lambda:\omega))$ be a point of $E_O^{(i)}$. Then $q := Q(a)$, and $A''(a)$ and $\widetilde{A}(a) := q^T a$ are well–defined, q is a regular complex $((n-i)\times(n-i))$–matrix and $(x,\, q^T a,\, (\lambda : q^{-1}(\omega)))$ satisfies by the previous commentaries the following conditions:

$$(q^T a)_0 = (1,0,\ldots,0),\ (q^T a)_* = A''(a),\ A''(a)\in U,\ \tilde{A}(a)\in O,$$
$$\Omega''_1(a,\omega)\neq 0,\ rkA''(a) = \text{rk}\,(\widetilde{A}(a))(x) = n-i,$$
$$J(F)(x)^T\lambda + (\widetilde{A}(a))(x)^T\Omega''(a,\omega)^T = 0.$$

Therefore we obtain a morphism of algebraic varieties

$$\varphi_O : E_O^{(i)} \to \mathbb{A}^{n-i}\times H_U^{(i,h)},$$

defined by for $(x,\, a,\, (\lambda:\omega))$ by

$$\varphi_O(x,\, a,\, (\lambda:\omega)) := (a_0,\, x,\, A''(a),\, (\lambda : \Omega''(a,\omega))).$$

Our argumentation implies that φ_O is an isomorphism of algebraic varieties. For any $1\leq h\leq n-i$ and $1\leq l_1<\cdots<l_{n-i}\leq n$ we obtain therefore an isomorphism of algebraic varieties

$$\varphi_{O_{(h;l_1,\ldots,l_{n-i})}} : E^{(i)}_{O_{(h;l_1,\ldots,l_{n-i})}} \to \mathbb{A}^{n-i}\times H^{(i,h)}_{U_{(l_1,\ldots,l_{n-i})}}.$$

Finally, Proposition 5 implies that $(H_i^{(h)})_{\mathbb{R}}$ is non–empty if and only if $\{F=0\}$ contains an F–regular real point. □

For algorithmic applications, Propositions 5 and 6 contain too many open conditions, namely the conditions $\text{rk}\,A_* = \text{rk}\,A(X) = n-i,\ A_0\Omega^T\neq 0$ or $rk\ B = \text{rk}\,B(X) = n-i,\ \Theta_h\neq 0$. Of course, the condition $\text{rk}\,B = \text{rk}\,B(X) = n-i$ may be eliminated by a suitable specialization of the $(n-i)\times n$–matrix B. However, one has to take care that this specialization process does not kill too many F–regular points of the hypersurface $\{F=0\}$. The following result, namely Proposition 7 below, seems to represent a fair compromise. We shall need it later for the task of finding real F–regular points of $\{F=0\}$, in case that $\{F=0\}_{\mathbb{R}}$ is compact.

For the formulation of the following Proposition 7 we need some notation. Let $1\leq h\leq n-i$ and let γ be a non-zero real number. For $b\in\mathbb{A}^i$ with $b=(b_{n-i+1},\ldots,b_n)$ we denote by $b_{(i,h;\gamma)}$ the complex $((n-i)\times n)$–matrix

$$b_{(i,h;\gamma)} := \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \ddots & & & & & \vdots & \\ 0 & \cdots & \gamma & \cdots & 0 & b_{n-i+1} & \cdots & b_n \\ & & & \ddots & & & \vdots & \\ 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

We introduce now the ambient space

$$\mathbb{N}_i{}^{(h)} := \{(x, b, (\lambda : \vartheta)) \mid x \in \mathbb{A}^n, b \in \mathbb{A}^i \text{ and } \vartheta = (\vartheta_1, \ldots, \vartheta_{n-i}) \in \mathbb{A}^{n-i} \text{ with } \vartheta_h \neq 0\}$$

and consider the $\mathbb{R}$–definable subvariety $H_i^{(h,\gamma)}$ of $\mathbb{N}_i{}^{(h)}$ given by

$$H_i^{(h,\gamma)} := \{(x, b, (\lambda : \vartheta)) \in \mathbb{N}_i{}^{(h)} \mid x = (x_1, \ldots, x_n) \in \mathbb{A}^n, F(x) = 0,\ x_h - \gamma \neq 0,\\ J(F)(x)^T \lambda + (b^{(h)}_{(i,h;\gamma)}(x))^T \vartheta^T = 0\}.$$

Observe that $\mathbb{N}_i{}^{(h)}$ is an algebraic variety which is isomorphic to the affine space $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{A}^{n-i}$ and that $H_i^{(h,\gamma)}$ is an $\mathbb{R}$–definable locally closed subvariety of $\mathbb{N}_i{}^{(h)}$. The ambient space $\mathbb{N}_i{}^{(h)}$ may be linearly embedded in $\mathbb{T}_i{}^{(h)}$ and this embedding maps $H_i^{(h,\gamma)}$ into $H_i^{(h)}$. Frequently we shall tacitly identify $\mathbb{N}_i^{(h)}$ with the affine space $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{A}^{n-i}$. This will always be clear by the context.

Let $B^*_{n-i+1}, \ldots, B^*_n$ be new indeterminates.

Proposition 7
Let $1 \leq h \leq n-i$ and let γ be a non–zero real number. Then, outside of the locus given by $\Theta_h\,(X_h - \gamma) = 0$, the polynomial equations of the system

$$\begin{aligned} F(X) &= 0,\\ \frac{\partial F(X)}{\partial X_h}\Lambda + (\gamma - X_h)\Theta_h &= 0,\\ \frac{\partial F(X)}{\partial X_l}\Lambda - X_l\Theta_h + \Theta_l &= 0,\\ 1 \leq l \leq n-i, l \neq h&\\ \frac{\partial F(X)}{\partial X_l}\Lambda + (B^*_l - X_l)\Theta_h &= 0,\\ n-i < l \leq n,& \end{aligned} \tag{5}$$

intersect transversally at each of their common solutions in $\mathbb{N}_i{}^{(h)}$.

Moreover, the polynomial equation system (5) and the open condition $\Theta_h\,(X_h - \gamma) \neq 0$ define the algebraic variety $H_i^{(h,\gamma)}$ which is therefore empty or equidimensional of dimension $n-1$. The varieties $H_i^{(h,\gamma)}$ and $(H_i^{(h,\gamma)})_{\mathbb{R}}$ dominate the locus of all points $x = (x_1, \ldots, x_n)$ of $\{F = 0\}$ and $\{F = 0\}_{\mathbb{R}}$ satisfying the conditions $\frac{\partial F}{\partial X_h}(x) \neq 0$ and $x_h - \gamma \neq 0$. In particular, $(H_i^{(h,\gamma)})_{\mathbb{R}}$ is non–empty and equidimensional of dimension $n-1$ if and only if the hypersurface $\{F = 0\}$ contains a real point $x = (x_1, \ldots, x_n)$ with $\frac{\partial F}{\partial X_h}(x) \neq 0$ and $x_h - \gamma \neq 0$. The polynomials contained in (5) generate in $\mathbb{R}[X, B^*_{n-i+1}, \ldots, B^*_n, \Lambda, \Theta]_{\Theta_h\,(X_h - \gamma)}$ the trivial ideal or form a reduced regular sequence.

Proof
Without loss of generality we may assume $h := 1$. Let $(x, b, (\lambda : \vartheta))$ be a point of $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{P}^{n-i}$ with $x = (x_1, \ldots, x_n), b = (b_{n-i+1}, \ldots, b_n)$, $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i})$ and $\vartheta_1(x_1 - \gamma) \neq 0$ which is a solution of the polynomial equation system (5) in the case $h = 1$.

Without loss of generality we may suppose $\vartheta_1 = 1$. Therefore $(x, b, \lambda, \vartheta)$ represents a solution of the polynomial equation system

$$
\begin{aligned}
F(X) &= 0, \\
\frac{\partial F(X)}{\partial X_1}\Lambda + (\gamma - X_1) &= 0, \\
\frac{\partial F(X)}{\partial X_l}\Lambda - X_l + \Theta_l &= 0, \\
2 \leq l &\leq n - i \\
\frac{\partial F(X)}{\partial X_l}\Lambda + B_l^* - X_l &= 0, \\
n - i < l &\leq n,
\end{aligned}
\tag{6}
$$

and satisfies the condition $x_1 - \gamma \neq 0$. Observe that the conditions (6) and $X_1 - \gamma \neq 0$ imply $\frac{\partial F}{\partial X_1} \neq 0$. Therefore we have $\frac{\partial F}{\partial X_1}(x) \neq 0$. The Jacobian $J_{(x,b,\lambda,\vartheta)}$ of the system (6) at the point $(x, b, \lambda, \vartheta)$ is the complex $((n+1) \times 2n)$–matrix

$$
J_{(x,b,\lambda,\vartheta)} := \begin{bmatrix}
J(F)(x) & 0 & O_{1\times(n-1)} \\
 & \frac{\partial F}{\partial X_1}(x) & O_{1\times(n-1)} \\
* & & \\
 & J(F)_{n-1}(x)^T & I_{n-1}
\end{bmatrix},
$$

with $J(F)_{n-1}(x) := (\frac{\partial F}{\partial X_2}(x), \ldots, \frac{\partial F}{\partial X_n}(x))$. From $\frac{\partial F}{\partial X_1}(x) \neq 0$ we deduce that $J_{(x,b,\lambda,\vartheta)}$ has maximal rank $n + 1$.

Therefore, outside of the locus given by $\Theta_1(X_1 - \gamma) = 0$, the equations of the system (5) intersect transversally at each of their common solutions in $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{P}^{n-i}$.

Let $x = (x_1, \ldots, x_n)$ be an arbitrary complex or real point of the hypersurface $\{F = 0\}$ satisfying the conditions $\frac{\partial F}{\partial X_1}(x) \neq 0$ and $x_1 - \gamma \neq 0$ and let

$$
\vartheta_1 := 1, \quad \lambda := \frac{x_1 - \gamma}{\frac{\partial F}{\partial X_1}(x)}, \quad \vartheta_2 := -\frac{\partial F}{\partial X_2}(x)\lambda + x_2, \quad \ldots \quad , \vartheta_{n-i} := -\frac{\partial F}{\partial X_{n-i}}(x)\lambda + x_{n-i},
$$

$$
b_{n-i+1} := -\frac{\partial F}{\partial X_{n-i+1}}(x)\lambda + x_{n-i+1}, \quad \ldots \quad , b_n := -\frac{\partial F}{\partial X_n}(x)\lambda + x_n,
$$

$$
\vartheta := (\vartheta_1, \ldots, \vartheta_{n-i}) \text{ and } b := (b_{n-i+1}, \ldots, b_n).
$$

Then the point $(x, b, (\lambda : \vartheta)) \in \mathbb{A}^n \times \mathbb{A}^i \times \mathbb{P}^{n-i}$ represents a solution of the polynomial equation system (5) and satisfies the condition $\vartheta_1(x_1 - \gamma) \neq 0$. Therefore the solutions $(x, b, (\lambda : \vartheta)) \in \mathbb{A}^n \times \mathbb{A}^i \times \mathbb{P}^{n-i}$ of (5) with $x = (x_1, \ldots, x_n)$, $\vartheta := (\vartheta_1, \ldots, \vartheta_{n-i})$, $\vartheta_1 = 1$ and $x_1 - \gamma \neq 0$ dominate the locus of all points $x = (x_1, \ldots, x_n)$ of $\{F = 0\}$ with $\frac{\partial F}{\partial X_1}(x) \neq 0$ and $x_1 - \gamma \neq 0$. One sees easily from the definitions that the points of the algebraic variety $H^{(1,\gamma)}$ represents exactly the solutions of (5) which satisfy the condition $\Theta_1(X_1 - \gamma) \neq 0$. Therefore $H^{(1,\gamma)}$ is empty or equidimensional of dimension $n - 1$.

It follows now from our previous argumentation that $H^{(1,\gamma)}$ and $H_{\mathbb{R}}^{(1,\gamma)}$ dominate the locus of all points $x = (x_1, \ldots, x_n)$ of $\{F = 0\}$ and $\{F = 0\}_{\mathbb{R}}$ which satisfy the conditions $\frac{\partial F}{\partial X_1}(x) \neq 0$ and $x_1 - \gamma \neq 0$.

Hence, $H_{\mathbb{R}}^{(1,\gamma)}$ is non–empty (and equidimensional of dimension $n - 1$) if and only if $\{F = 0\}$ contains a real point $x = (x_1, \ldots, x_n)$ with $\frac{\partial F}{\partial X_1}(x) \neq 0$ and $x_1 - \gamma \neq 0$.

The rest of the statement of Proposition 7 follows now by standard arguments of commutative algebra. □

Observation 8
Let notations be as in Proposition 6 and 7. Then the closures of $(H_i^{(h)})_{\mathbb{R}}$ *and* $(H_i^{(h,\gamma)})_{\mathbb{R}}$ *in their respective real ambient spaces need not to be compact, even* $\{F = 0\}_{\mathbb{R}}$ *is so. However, the assumption that* $\{F = 0\}_{\mathbb{R}}$ *is bounded implies that* $(H_i^{(h,\gamma)})_{\mathbb{R}}$ *is compact for sufficiently large* γ.

In the sequel we shall refer for $1 \leq i \leq n - 1$, $1 \leq h \leq n - i$ and $\gamma > 0$ to the equation systems (1),(3) and (5) and the corresponding varieties E_i, $H_i^{(h)}$ and $H_i^{(h,\gamma)}$ as *polar deformations of the equation* $F = 0$ (in the dual model).

The varieties E_i and $H_i^{(h)}$ are inspired in the concept of a *generic* ith dual polar variety of the hypersurface $\{F = 0\}$ whereas the variety $H_i^{(h,\gamma)}$ is inspired in the concept of a *meagerly generic* polar variety of $\{F = 0\}$ (see [6], Section 4, Example 2).

In the next subsection we are going to motivate the notion of a polar deformation of the equation $F = 0$.

3.2 Polar deformations and polar varieties

In this subsection we are going to study, for $1 \leq i \leq n - 1$ and $1 \leq h \leq n - i$ the (existing) link between the varieties E_i and $H_i^{(h)}$ and the ith dual polar varieties of the hypersurface $\{F = 0\}$. We shall now suppose that the polynomial F is reduced, i.e., squarefree.

We consider for $a \in \mathbb{A}^{(n-i)\times(n+1)}$ with $a = [a_{k,l}]_{\substack{1\leq k\leq n-i \\ 1\leq l\leq n}}$, $a_0 \neq 0$ and $\operatorname{rk} a_* = n-i$ the $(n-i+1)$–dimensional linear subvariety $\overline{K}(a) = \overline{K}^{n-i-1}(a)$ of $\mathbb{P}^n$, spanned by the points $(a_{k,0} : \cdots : a_{k,n})$, $1 \leq k \leq n-i$, and the ith dual polar variety of $\{F = 0\}$ associated with $\overline{K}(a)$, namely

$$W_{\overline{K}(a)} := W_{\overline{K}(a)}(\{F = 0\}).$$

Recall that $W_{\overline{K}(a)}$ is the closure of the locus of the F–regular points of $\{F = 0\}$ where all $(n-i+1)$–minors of the polynomial $(n-i+1)\times n$–matrix $T_a = T_a(X)$ defined by

$$T_a := \begin{bmatrix} \frac{\partial F}{\partial X_1} & \cdots & \frac{\partial F}{\partial X_n} \\ a_{1,1} - a_{1,0}X_1 & \cdots & a_{1,n} - a_{1,0}X_n \\ \vdots & \vdots & \vdots \\ a_{n-i,1} - a_{n-i,0}X_1 & \cdots & a_{n-i,n} - a_{n-i,0}X_n \end{bmatrix}$$

vanish.

Let us write $\overline{a} := \overline{a}(X)$ for the polynomial $((n-i+1)\times(n+1))$– matrix

$$\overline{a} := \begin{bmatrix} 0 & J(F)(X) \\ a_0^T & a(X) \end{bmatrix}.$$

We consider now the following set

$$\mathcal{W}_i := \{(x,a) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \mid \operatorname{rk} a_* = \operatorname{rk} a(x) = n-i, \\ \operatorname{rk} \overline{a}(x) = n-i+1,\ x \in W_{\overline{K}(a)}\}.$$

One sees easily that $\mathcal{W}_i$ is a (locally closed) algebraic subvariety of $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)}$ which describes the incidence relation between the ith dual polar varieties of the hypersurface $\{F = 0\}$ and the F–regular points lying on them. With these notations we are able to characterize in terms of ith dual polar varieties the image of the canonical projection of E_i into $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)}$.

Proposition 9
A point $(x,a) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)}$ belongs to $\mathcal{W}_i$ if and only if there exists a point $(\lambda : \omega) \in \mathbb{P}^{n-i}$ such that $(x, a, (\lambda : \omega))$ belongs to E_i.

Proof
Suppose that (x,a) belongs to $\mathcal{W}_i$. Then, in particular, we have $\operatorname{rk} a_* = n-i$, $\operatorname{rk} \overline{a}(x) = n-i+1$ and $x \in W_{\overline{K}(a)}$. From $\operatorname{rk} \overline{a}(x) = n-i+1$ we deduce $a_0 \neq 0$. This implies that $\overline{K}(a)$ is well–defined. From $x \in W_{\overline{K}(a)}$ we conclude that the rows of $T_a(x)$ are linearly dependent. Hence there exists a point $(\lambda : \omega) \in \mathbb{P}^{n-i}$ such that

$$J(F)(x)^T\lambda + a(x)^T\omega^T = 0$$

holds.

From rk $\overline{a}(x) = n - i + 1$ we deduce that we may choose $(\lambda : \omega)$ in such a way that the condition $a_0\,\omega^T \neq 0$ becomes satisfied. Moreover, $x \in W_{\overline{K}(a)}$ implies $F(x) = 0$ and by assumption we have rk $a_* =$ rk $a(x) = n - i$. Therefore the point $(x, a, (\lambda : \omega))$ belongs to E_i.

Suppose now that there is given a point $(x, a, (\lambda : \omega))$ of E_i. Then we have in particular $F(x) = 0$, $a_0\omega^T \neq 0$, rk $a_* =$ rk $a(x) = n-i$ and $J(F)(x)^T\lambda + a(x)^T\omega^T = 0$. From $a_0\omega^T \neq 0$ we deduce $a_0 \neq 0$. This implies again that $\overline{K}(a)$ is well–defined. From $J(F)(x)^T\lambda + a(x)^T\omega^T = 0$ we conclude that the rows of the complex $(n - i + 1) \times n$–matrix $T_a(x)$ are linearly dependent and that therefore all its $(n - i + 1)$–minors vanish. Moreover, rk $a(x) = n - i$ and $(\lambda : \omega) \in \mathbb{P}^{n-i}$ imply $J(F)(x) \neq 0$. Thus, taking into account $F(x) = 0$, we infer that x belongs to $x \in W_{\overline{K}(a)}$. By assumption we have rk $a(x) = n - i$. Therefore, rk $\overline{a}(x) < n - i + 1$ would imply $a_0\omega^T = 0$, a contradiction. Hence we have rk $\overline{a}(x) = n - i + 1$. This implies that the point (x, a) belongs to $\mathcal{W}_i$ □

Let be given an index $1 \leq h \leq n - i$ and a complex $((n - i) \times n)$–matrix b. Recall the notation $b^{(h)}$ for the complex $((n - i) \times (n + 1))$–matrix determined by the conditions $(b^{(h)})_0 := (\delta_{k,h})_{1\leq k\leq n-i}$ and $(b^{(h)})_*$. Let us write $\overline{b^{(h)}} := \overline{b^{(h)}}(X)$ for the polynomial $((n - i + 1) \times (n + 1))$–matrix

$$\overline{b^{(h)}} := \begin{bmatrix} 0 & J(F)(X) \\ (b^{(h)})_0^T & b^{(h)}(X) \end{bmatrix}.$$

We consider now the following set:

$$\mathcal{W}_i^{(h)} := \{(x, b) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \mid \text{rk } b = \text{rk } b^{(h)}(x) = n - i,$$
$$\text{rk } \overline{b^{(h)}}(x) = n - i + 1, \quad x \in W_{\overline{K}(b^{(h)})}\}.$$

Observe that the set $\mathcal{W}_i^{(h)}$ is a (locally closed) algebraic subvariety of $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n}$ which describes the incidence relation between suitable ith dual polar varieties of $\{F = 0\}$ and F–regular points lying on them.

From Proposition 9 we deduce immediately the following result.

Proposition 10
A point $(x, b) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n)}$ belongs to $\mathcal{W}_i^{(h)}$ if and only if there exists a point $(\lambda : \vartheta) \in \mathbb{P}^{n-i}$ with $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i})$ and $\vartheta_h \neq 0$ such that $(x, a, (\lambda : \vartheta))$ belongs to $H_i^{(h)}$.

We are now able to motivate, by the algorithmic problem of solving a single, possibly singular polynomial equation $F = 0$ over the reals, the notion of polar deformation.

In the case that the real variety $\{F = 0\}_{\mathbb{R}}$ contains F–regular points, Corollary 2 guarantees only the *existence* of an non–empty and open, semialgebraic set $O^{(i)}$ of

"directions" of $\mathbb{A}_{\mathbb{R}}^{(n-i)\times(n+1)}$ such that for any $((n-i)\times(n+1))$–matrix $a \in O^{(i)}$ the linear variety $\overline{K}(a)$ is well-defined and the real dual polar variety $W_{\overline{K}(a)}(\{F=0\}_{\mathbb{R}})$ is generic and non–empty.

The problem of finding an explicit semialgebraic description of such a set $O^{(i)}$ and of finding an explicit real $((n-i)\times(n+1))$–matrix belonging to $O^{(i)}$, leads to the consideration of the algebraic varieties $\mathcal{W}_i$ and $\mathcal{W}_i^{(h)}$ as loci, where the smooth points x of $\{F=0\}$ "move" together with suitable directions $a \in \mathbb{A}^{(n-i)\times(n+1)}$ (or $b \in \mathbb{A}^{(n-i)\times n}$) subject to the constraint $x \in W_{\overline{K}(a)}$ (or $x \in W_{\overline{K}(b^{(h)})}$). Unfortunately, the varieties $\mathcal{W}_i$ and $\mathcal{W}_i^{(h)}$ need not to be smooth. In order to repair this defect we consider in this paper the polar deformation varieties E_i and $H_i^{(h)}$ which dominate by Proposition 9 and 10 the varieties $\mathcal{W}_i$ and $\mathcal{W}_i^{(h)}$ and represent natural desingularizations of them.

3.3 A parametric view of the generic dual polar varieties of a real hypersurface

In [6], Section 3.1 we made (without any proof) a comment, saying that generic dual polar varieties of smooth hypersurfaces may become singular. This statement is definitively wrong as the following result shows.

Theorem 11
Let F be reduced (i.e., squarefree), $1 \leq i \leq n-1$ and let a be a generic complex $((n-i)\times(n+1))$–matrix. Then the generic dual polar variety $W_{\overline{K}(a)}$ is smooth at any of its F–regular points.

Proof
Let $\varphi_i : E_i \to \mathbb{A}^{(n-i)\times(n+1)}$ be the morphism of *smooth* algebraic varieties induced by the canonical projection from $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{P}^{n-i}$ onto $\mathbb{A}^{(n-i)\times(n+1)}$ and suppose that the generic polar variety $W_{\overline{K}(a)}$ is not empty.

From [4], [5], Proposition 8 (or alternatively [6], Corollary 2) we deduce that $W_{\overline{K}(a)}$ is equidimensional of dimension $n-i-1$ and contains F–regular points. On the other hand Proposition 5 implies that E_i is equidimensional of dimension $(n-i)(n+2)-1$.

One sees easily that $\varphi_i^{-1}(a)$ is isomorphic to

$$W^* := \{x \in \mathbb{A}^n \mid J(F)(x) \neq 0,\, x \in W_{\overline{K}(a)}\}.$$

Therefore we conclude from the Theorem of Fibers (see e.g. [55]) that the morphism φ_i is dominating (i.e., the constructible set $\varphi_i(E_i)$ is Zariski dense in $\mathbb{A}^{(n-i)\times(n+1)}$). Since by assumption a is a generic element of $\mathbb{A}^{(n-i)\times(n+1)}$, Sard's Theorem (see e.g. [17], [56]) implies that a is a regular value of φ_i. Therefore $\varphi_i^{-1}(a)$, and hence W^*, are smooth. This means that the polar variety $W_{\overline{K}(a)}$ is smooth at any of its

F–regular points. $\square$

For generic classic polar varieties the counterpart of Theorem 11 is a well–known result on generic classic polar varieties of complex hypersurfaces (see the comments in [46], [6] and [2] for an elementary proof).

Theorem 11 and its proof illustrate that there is no hope to generalize the deformation–based methods of this paper to the case of of a general regular sequence of polynomials $F_1, \ldots, F_p$ with $1 < p < n$. Otherwise generic polar varieties of smooth complete intersection varieties of codimension at least two would always be empty or smooth. But this conclusion is wrong in view of [6], Section 3.1.

We are now going to formulate and prove an avatar of Theorem 1 for the most general type of real polar deformation varieties under consideration (see Theorem 12 and Corollary 13 below).

Theorem 12
Suppose that the hypersurface $\{F = 0\}$ contains an F–regular point real point. Let C be a generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$. Then there exists a non–empty, open, semialgebraic subset $O_C^{(i)}$ of $\mathbb{A}^{(n-i)\times(n+1)}$ such that any $a \in O_C^{(i)}$ satisfies the following conditions:

(i) $rk\, a_* = n-i$, $a_0 \neq 0$ and the dual polar variety $W_{\overline{K}(a)}$ is generic and contains an F–regular point of C.

(ii) For any two points $x \in (W_{\overline{K}(a)})_{\mathbb{R}}$ and $(\lambda : \omega) \in \mathbb{P}_{\mathbb{R}}^{n-i}$ with $z := (x, a, (\lambda : \omega)) \in E_{\mathbb{R}}^{(i)}$ there exists a permutation matrix $M \in \mathbb{Z}^{n\times n}$ such that the linear forms $X'_1, \ldots, X'_n, A_{k,l}, 1 \leq k \leq n-i$, $0 \leq l \leq n$ with $(X'_1, \ldots, X'_n) := XM$ form a system of local parameters of $E_{\mathbb{R}}^{(i)}$ at z.

Proof
Let us consider the morphism of smooth real varieties $\psi_i : E_{\mathbb{R}}^{(i)} \to \mathbb{A}_{\mathbb{R}}^{(n-i)\times(n+1)}$ induced by the canonical projection from $\mathbb{A}_{\mathbb{R}}^n \times \mathbb{A}_{\mathbb{R}}^{(n-i)\times(n+1)} \times \mathbb{P}_{\mathbb{R}}^{n-i}$ onto $\mathbb{A}_{\mathbb{R}}^{(n-i)\times(n+1)}$. From Theorem 1 and Sard's Theorem we deduce that there exists a non–empty, open, semialgebraic subset $O_C^{(i)}$ of $\mathbb{A}_{\mathbb{R}}^{(n-i)\times(n+1)}$ such that any $a \in O_C^{(i)}$ is a regular value of the smooth mapping ψ_i and satisfies the condition (i) of the theorem.

Let us consider an arbitrary real $((n-i)\times(n+1))$–matrix a of $O^{(i)}$ and let $x = (x_1, \ldots, x_n) \in (W_{\overline{K}(a)})_{\mathbb{R}}$ and $(\lambda : \omega) \in \mathbb{P}_{\mathbb{R}}^{n-i}$ with $\omega = (\omega_1, \ldots, \omega_{n-i})$ be arbitrary points. Suppose that $z := (x, a, (\lambda : \omega))$ belongs to $E_{\mathbb{R}}^{(i)}$. Without loss of generality we may assume that $\lambda = 1$ holds. Let $\mathcal{L}_i$ be the Jacobian of the polynomial equation system

$$F(X) = 0, \qquad \frac{\partial F}{\partial X_l}(X) + \sum_{1\leq k\leq n-i} (A_{k,l} - A_{k,0}X_l)\Omega_k = 0, \quad 1 \leq l \leq n.$$

An explicit description of the polynomial $(n+1)\times(n+(n+2)(n-i))$–matrix $\mathcal{L}_i$ was given in the proof of Proposition 5.

The matrix $\mathcal{L}_i$ takes at the point z the form

$$\mathcal{L}_i(z) := \begin{bmatrix} \frac{\partial F}{\partial X_1}(x) & \cdots & \frac{\partial F}{\partial X_n}(x) & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & & \omega_1 & \cdots & \omega_{n-i} & & 0 & \cdots & 0 & -x_1\omega_1 & \cdots & -x_1\omega_{n-i} \\ & * & & & a(x)^T & & & 0 & & \ddots & & 0 & & & 0 & \\ & & & & & & 0 & \cdots & 0 & \cdots & \omega_1 & \cdots & \omega_{n-i}-x_n\omega_1 & \cdots & -x_n\omega_{n-i} \end{bmatrix}.$$

Since a is a regular value of the smooth map ψ_i, we conclude that the indeterminates $A_{k,l}$, $1\leq k\leq n-i$, $0\leq l\leq n$ are local parameters of $E^{(i)}_{\mathbb{R}}$ at z. This implies that the $((n+1)\times(2n-i))$–matrix

$$N := \begin{bmatrix} \frac{\partial F}{\partial X_1}(x) & \cdots & \frac{\partial F}{\partial X_n}(x) & 0 & \cdots & 0 \\ & & & & & \\ & * & & & a(x)^T & \\ & & & & & \end{bmatrix}$$

has maximal rank $n+1$. Since z belongs to $E^{(i)}_{\mathbb{R}}$, we have $\operatorname{rk} a(x)^T = \operatorname{rk} a(x) = n-i$. Therefore there are $i+1$ many among the first n columns of N which together with the columns of the $((n+1)\times(n-i))$–matrix.

$$\begin{bmatrix} 0 & \cdots & 0 \\ & a(x)^T & \end{bmatrix}$$

form a non-singular $((n+1)\times(n+1))$–matrix. This implies that there exists $n-i-1$ many, say $X'_1,\ldots,X'_{n-i-1}$, from the indeterminates $X_1,\ldots,X_n$ which together with $A_{k,l}$, $1\leq k\leq n-i$, $0\leq l\leq n$ form a set of local parameters of $E^{(i)}_{\mathbb{R}}$ at z. Since by Proposition 5 we have $\dim E^{(i)}_{\mathbb{R}} = (n-i)(n+2)-1$, we obtain a complete system of local parameters of $E^{(i)}_{\mathbb{R}}$. Observe finally, that there exists a permutation matrix $M\in\mathbb{Z}^{n\times n}$ such that the first $n-i-1$ entries of XM are the indeterminates $X'_1,\ldots,X'_{n-i-1}$. This finishes the proof of the theorem. □

In the case $i=n-1$, Theorem 12 implies the following result.

Corollary 13
Suppose that the hypersurface $\{F=0\}$ contains an F–regular point real point. Then there exists a non–empty, open, semialgebraic subset O of $\mathbb{A}^{n+1}_{\mathbb{R}}$ such that any point $a=(a_0,a_1,\ldots,a_n)$ of O satisfies the following two conditions:

(i) $a_0\neq 0$, $(a_1,\ldots,a_n)\neq 0$ and the (locally closed) subvariety W_a of $\mathbb{A}^n\times\mathbb{A}^1$

defined by the system

$$
\begin{aligned}
F(X) &= 0,\\
\frac{\partial F}{\partial X_l}(X)\Lambda + a_l - a_0 X_l &= 0,\\
1 \leq l &\leq n,\\
\bigvee_{1\leq l\leq n} a_l - a_0 X_l &\neq 0,
\end{aligned}
\tag{7}
$$

is zero-dimensional and of cardinality $\# W_{\overline{K}(a)}$, *the equations of (7) intersect transversally at any point of* W_a *and the real trace* $(W_a)_{\mathbb{R}}$ *of* W_a *is non–empty.*

(ii) For any $(x,\lambda) \in (W_a)_{\mathbb{R}}$ *the point* $z := (x, a, (\lambda : 1))$ *belongs to* $E_{\mathbb{R}}^{(n-1)}$ *and* $A_0, A_1, \ldots, A_n$ *form a system of local parameters of* $E_{\mathbb{R}}^{(n-1)}$ *at* z.

Proof
Since the hypersurface $\{F = 0\}$ contains an F–regular real point, there exists a generically F–regular connected component C of $\{F = 0\}_{\mathbb{R}}$. Apply Theorem 12 for the case $i := n-1$ to C and set $O := O_C^{(n-1)}$. Observing that $a \in O$ implies $W_{\overline{K}(a)}$ generic and $W_a \cong W_{\overline{K}(a)}$, Corollary 13 follows easily from [4, 5], Lemma 7 and Proposition 5. □

We are going now to comment Corollary 13 from an algorithmic point of view.

Let $A = (A_0, \ldots, A_n)$ be a row vector of $n+1$ new indeterminates $A_0, \ldots, A_n$.

Suppose $F \in \mathbb{Q}[X]$ and that the hypersurface $\{F = 0\}$ contains an F–regular real point. Let $1 \leq h \leq n$. From Proposition 5 we conclude that, outside of the locus given by

$$A_0 \cdots A_n \,(A_h - A_0 X_h) = 0,$$

the polynomial equations

$$
\begin{aligned}
F(X) &= 0,\\
\frac{\partial F}{\partial X_l}(X)\,\Lambda + A_l - A_0 X_l &= 0,\\
1 \leq l &\leq n,
\end{aligned}
\tag{8}
$$

intersect transversally at any of their common solutions. This implies that the polynomial equations

$$
\begin{aligned}
F(X) &= 0,\\
-\frac{\partial F}{\partial X_l}(A_h - A_0 X_h) + (A_l - A_0 X_l)\frac{\partial F}{\partial X_h}(X) &= 0,\\
1 \leq l \leq n,\ l &\neq h
\end{aligned}
\tag{9}
$$

intersect transversally in any of their solutions $(x, a) \in \mathbb{A}^n \times \mathbb{A}^{n+1}$ not contained in the locus $A_0 \cdots A_n (A_h - A_0 X_h) = 0$.

Therefore the polynomials which constitute the system (9) generate in $\mathbb{Q}[A, X]_{A_0 \cdots A_n (A_h - A_0 X_h)}$ the trivial ideal or form a reduced regular sequence. Hence the ideal $\mathfrak{a}_h$ generated by these polynomials in $\mathbb{Q}(A)[X]_{(A_h - A_0 X_h)}$ is trivial or a radical complete intersection ideal of dimension zero.

The hypersurface $\{F = 0\}$ contains by assumption a real F–regular point. Thus Corollary 13 implies that there exists $1 \leq h \leq n$ such that $\mathfrak{a}_h$ is a radical complete intersection ideal of dimension zero which vanishes on an F–regular point with coordinates in a suitable real closure K of the field $\mathbb{Q}(A)$. Without loss of generality, we may assume that the variables $X_1, \ldots, X_n$ are in general position with respect to the ideal $\mathfrak{a}_h$ and that in particular the variable X_1 separates the zeros of $\mathfrak{a}_h$ in $K(i)^n$.

For the sake of simplicity we shall suppose $2 \leq h \leq n$. Hence we conclude that there exists polynomials $\varrho_h \in \mathbb{Q}[A]$ and $P_h, G_2^{(h)}, \ldots, G_n^{(h)} \in \mathbb{Q}[A, X_1]$ with $\varrho_h \neq 0$, $\deg_{X_1} P_h \geq 1$ and $\deg_{X_1} G_j^{(h)} < \deg_{X_1} P_h$, $2 \leq j \leq n$, such that P_h is primitive and separable with respect to the variable X_1 and such that

$$P_h,\ \varrho_h X_2 - G_2^{(h)}\ , \ldots,\ \varrho_h X_n - G_n^{(h)}$$

generate the ideal $\mathfrak{a}_h$ in $\mathbb{Q}(A)[X]_{A_h - A_0 X_h}$. We say then that the polynomials $P_h, G_2^{(h)}, \ldots, G_n^{(h)}$ form a *geometric solution* over $\mathbb{Q}(A)$ of the equation system (9) and the open condition $A_h - A_0 X_h \neq 0$ in the variables $X_1, \ldots, X_n$.

The polynomial P_h is uniquely determined by (9) and ϱ_h may be chosen as the numerator of the discriminant of P_h with respect to the indeterminate X_1. This choice determines in turn $G_2^{(h)}, \ldots, G_n^{(h)}$. From Corollary 13 we deduce that $\deg_{X_1} P_h$ is bounded by the degree, say μ, of the $(n-1)$ th generic dual polar variety of $\{F = 0\}$.

Let V_h be the union of all irreducible components of the closed subvariety of $\mathbb{A}^n \times \mathbb{A}^{n+1}$, defined by the polynomial equation system (9) in the unknowns X and A, that are not contained in the locus given by $A_0 A_1 \cdots A_n (A_h - A_0 X_h) = 0$. From [52], Theorem 1, we deduce that the total degree of the polynomials

$$\varrho_h,\ P_h,\ G_2^{(h)}, \ldots, G_n^{(h)} \in \mathbb{Q}[A, X_1]$$

is of order $O(\mu \deg V_h)$.

Suppose that F is given by a division–free arithmetic circuit σ of size L in $\mathbb{Q}[X]$ (thus F has rational coefficients). Let $\delta_1 \leq \mu$ be the degree of the system (9) over $\mathbb{Q}(A)$ outside of the locus given by $A_h - A_0 X_h = 0$ and $\delta := \delta_1 \mu \deg V_h$. Then we have $\delta_1 \leq d^n$ and $\deg V_h \leq (d+1)^n$ and therefore $\delta = d^{O(n)}$.
Then the polynomial $\varrho_h \in \mathbb{Q}[A]$ and the coefficients with respect to X_1 of the polynomials $P_h,\ G_2^{(h)}, \ldots, G_n^{(h)}$ have a representation by a division–free arithmetic

circuit σ^* in $\mathbb{Q}[A]$ of size $L(n\,d)^{O(1)}\delta^2$. The circuit σ^* may be computed from the input circuit σ in time $L(n\,d)^{O(1)}\delta^2$ (see the original contributions [26, 25, 32, 28]and the survey [20] for the notions of geometric solution, system degree and details of the algorithm).

Applying now real quantifier elimination to the formula

$$(\exists\ X_1)\,(P_h(A,X_1)=0\wedge A_0\cdots A_n\,(A_h\varrho_h(A)-A_0G_h^{(h)}(A,X_1))\neq 0\ \wedge\varrho_h(A)\neq 0)$$

we obtain a quantifier free formula $\Psi_h(A)$ in the variables $A_0,\ldots,A_n$ over the elementary language of ordered fields. The formula $\Psi_h(A)$ describes the image of the semialgebraic set

$$\{(a,x_1)\in\mathbb{A}_{\mathbb{R}}^{n+1}\times\mathbb{A}_{\mathbb{R}}^1\mid a=(a_0,\ldots,a_n),\ P_h(a,x_1)=0,$$
$$a_0\cdots a_n(a_h\,\varrho_h(a)-a_0\,G_h^{(h)}(a,x_1))\neq 0,\varrho_h(a)\neq 0\}$$

under the canonical projection $\mathbb{A}_{\mathbb{R}}^{n+1}\times\mathbb{A}_{\mathbb{R}}^1\ \to\ \mathbb{A}_{\mathbb{R}}^{n+1}$. Thus for $a=(a_0,\ldots,a_n)\in\mathbb{A}_{\mathbb{R}}^{n+1}$ the formula $\Psi_h(a)$ is true if and only if there exists a point $x=(x_1,\ldots,x_n)$ of $\mathbb{A}_{\mathbb{R}}^n$ such that (x,a) is a solution of the polynomial equation system (9) with $a_0\cdots a_n(a_h-a_0x_h)\neq 0$. On its turn this implies that $\Psi_h(a)$ is true if and only if there exists a point (x,λ) of $\mathbb{A}_{\mathbb{R}}^n\times\mathbb{A}_{\mathbb{R}}^1$ with $x=(x_1,\ldots,x_n)$ such that (x,a,λ) is a solution of the polynomial equation system (7) with $a_0\cdots a_n(a_h-a_0x_h)\neq 0$, whence $(x,a,(\lambda:1))\in E_{\mathbb{R}}^{(n-1)}$ and $x\in W_{\overline{K}(a)}$. From the choice of h we see that the semialgebraic subset of $\mathbb{A}_{\mathbb{R}}^{n+1}$ defined by the formula $\Psi_h(A)$ has a non–empty interior which contains therefore "generic" rational points. Let $a\in\mathbb{Q}^{n+1}$ be such a point. From the inputs a and σ we are now able to construct in time $L(n\,d)^{O(1)}\,\delta^2$ an F–regular real algebraic point $x\in\mathbb{A}_{\mathbb{R}}^n$ which belongs to the dual polar variety $x\in W_{\overline{K}(a)}$. By [6], Theorem 3 the point x has degree at most μ and belongs to $\{F=0\}_{\mathbb{R}}$.

The crux with this kind of argumentation is the following:
Although we are able to compute in time $L\,(n\,d)^{O(1)}\,\delta^2$ from the arithmetic circuit σ an arithmetic–boolean circuit with $=$ and $>$ decision gates which represents a non–empty open set M_h of points of $\mathbb{A}_{\mathbb{R}}^{n+1}$ that satisfy the formula Ψ_h, we are generally not able to find *efficiently* sample points of M_h, neither rational nor algebraic ones.

An exception is made by certain well-determined singular curves, whose generic dual polar varieties are never empty [45].

By the way, let us mention that the procedures we have in mind for the elimination of just one real existential quantifier are the most classical ones, which may be adapted to the circuit representation of polynomials. There are no precise references to the subject. For technical aspects see [21], Section B.

Fix now an index $1\leq i<n-1$ and suppose that we are able to find a "generic" point $a^*\in\mathbb{Q}^{n+1}$ such that $\Psi_h(a^*)$ holds. Then we may find a $((n-i-1)\times(n+1))$–matrix

$a^{**} \in \mathbb{Q}^{(n-i-1)\times(n+1)}$ such that the rational $((n-i)\times(n+1))$–matrix $a := \begin{bmatrix} a^* \\ a^{**} \end{bmatrix}$ is generic . Hence $W_{\overline{K}(a)}$ is a generic dual polar variety of $\{F = 0\}$. Observe that $W_{\overline{K}(a)}$ contains $W_{\overline{K}(a^*)}$. Since the assertion $\Psi_h(a^*)$ holds we conclude that $W_{\overline{K}(a^*)}$ contains an F–regular real point x. Because x is also contained in $W_{\overline{K}(a)}$, the generic dual polar variety $W_{\overline{K}(a)}$ contains F–regular real points.

This leads us to the problem of finding efficiently for a given consistent system of *strict* inequalities of arithmetic circuit represented polynomials of $\mathbb{Q}[X]$ a rational (or algebraic) point $x \in \mathbb{A}^n_{\mathbb{R}}$ which satisfies all these inequalities. We call such a point a rational (or algebraic) *witness* for the given system.

In the spirit of the dual model, we are going to design in the next section a procedure which decides, under the assumption that $\{F = 0\}_{\mathbb{R}}$ is compact, whether the hypersurface $\{F = 0\}$ contains a real F–regular point, and, if this is the case, returns such a point for each connected component of $\{F = 0\}_{\mathbb{R}}$.

In order to estimate the complexity of this procedure we shall now introduce, with respect to the dual model, different variants of the concept of a *bipolar variety* of the equation $F = 0$. The maximal degree of all bipolar varieties of the equation $F = 0$ will then determine the running time of the procedure.

4 Bipolar varieties in the dual model

Dual polar varieties represent a complex reflection of the Lagrange multipliers. Therefore their geometric meaning concerns more real than complex algebraic varieties. Maybe this is the reason why they, motivated by the aim to find *real* solutions of polynomial equation systems, were only recently introduced in (complex) algebraic geometry.

The definition of the dual polar varieties associated with an equidimensional complex algebraic variety S requires that S is represented as a subvariety of a projective space $\mathbb{P}^n$ which is in turn equipped with a distinguished hyperplane H at infinity and with a non–degenerate hyperquadric Q such that $Q \cap H$ is again non–degenerate.

Of particular interest is the case that S is a smooth subvariety of the affine space $\mathbb{A}^n$, suitably embedded in $\mathbb{P}^n$. This leads to the concepts of an affine and a real dual polar variety (see [4], [5] and [6] for details and motivations.)

In the dual model, the bipolar varieties of the equation $F = 0$ should be introduced as generic dual polar varieties associated with the smooth incidence varieties E_i or $H_i^{(h)}$, $1 \leq i \leq n-1$, $1 \leq h \leq n-i$ (if they are not empty), and should be defined in a "natural" way, only depending on the polynomial F, such that their degree is relevant for the complexity of the problem of finding F–regular real algebraic points belonging to $\{F = 0\}$. We shall see that E_i is not suitable for this task but that

$H_i^{(h)}$ furnishes an appropriate notion of bipolar varieties.

Let us fix $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$ and observe that arbitrary points $(x, a, (\lambda : \omega)) \in E_i$ or $(x, a, (\lambda : \vartheta)) \in H_i^{(h)}$ satisfy the condition $\lambda \neq 0$. Therefore, in principle, we may suppose $\lambda = 1$ and consider E_i and $H_i^{(h)}$ as subvarieties of the respective affine spaces $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{A}^{n-i}$ and $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^{n-i}$.

However, these affine embeddings of E_i and $H_i^{(h)}$ are rather irrelevant for algorithmic considerations, because they require a description of x in terms of a, λ and ω (or alternatively in terms of b, λ and ϑ) and not the opposite.

Consider now an arbitrary point $(x, a, (\lambda : \omega))$ of E_i with $\omega = (\omega_1, \ldots, \omega_{n-i}) \in \mathbb{A}^{n-i}$. Then we have $a_0 \cdot \omega^T \neq 0$ and this implies $\omega \neq 0$. Therefore there exists an index $1 \leq h \leq n-i$ with $\omega_h \neq 0$. For any such h we obtain a different embedding of the affine ambient space $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times(n+1)} \times \mathbb{A}^{n-i}$ in $\mathbb{P}^{(n-i)(n+2)+n}$ and it remains undetermined which embedding we should chose in order to define the bipolar varieties of E_i. Different embeddings lead to completely incompatible generic dual polar varieties that cannot be patched together.

The situation looks different in the case of $H_i^{(h)}$. For any point $(x, b, (\lambda : \vartheta))$ of $H_i^{(h)}$ with $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i})$ we have $\vartheta_h \neq 0$ (this is in fact the deeper meaning of Proposition 6). Therefore by setting $\vartheta_h := 1$ we obtain a canonic embedding of the ambient space $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^{n-i}$ into the projective space $\mathbb{P}^{(n-i)(n+1)+n}$.

Let us be more precise. We associate with $1 \leq h \leq n-i$ the hyperplane at infinity

$$L_h := \{\Theta_h = 0\} := \{(x : b : \lambda : \vartheta) \in \mathbb{P}^{(n-i)(n+1)+n} \mid x \in \mathbb{A}^n, b \in \mathbb{A}^{(n-i)\times n}, \\ \lambda \in \mathbb{A}^1, \vartheta \in \mathbb{A}^{n-i}, \vartheta = (\vartheta_1, \ldots, \vartheta_{n-i}), \vartheta_h = 0\}$$

and the hyperquadric $\mathcal{Q}$ defined by the equation

$$\sum_{1 \leq l \leq n} X_l^2 + \sum_{\substack{1 \leq k \leq n-i \\ 1 \leq l \leq n}} B_{k,l}^2 + \Lambda^2 + \sum_{1 \leq k \leq n-i} \Theta_k^2 = 0.$$

Then $\mathcal{Q}$ and $\mathcal{Q} \cap L_h$ are non–degenerate and

$$(\mathcal{Q} \cap L_h) \cap \mathbb{A}_{\mathbb{R}}^n \times \mathbb{A}_{\mathbb{R}}^{(n-i)\times n} \times \mathbb{A}_{\mathbb{R}}^{n-i}$$

is positive–definite and induces in $\mathbb{A}_{\mathbb{R}}^n \times \mathbb{A}_{\mathbb{R}}^{(n-i)\times n} \times \mathbb{A}_{\mathbb{R}}^{n-i}$ the Euclidean distance.

Similarly, we associate with $1 \leq h \leq n-i$ the hyperplane at infinity

$$\widetilde{L}_h := \{\Theta_h = 0\} := \\ := \{(x : b : \lambda : \vartheta) \in \mathbb{P}^{2n} \mid x \in \mathbb{A}^n, b \in \mathbb{A}^i, \lambda \in \mathbb{A}^1, \vartheta \in \mathbb{A}^{n-i}, \vartheta = (\vartheta_1, \ldots, \vartheta_{n-i}), \vartheta_h = 0\}$$

and the hyperquadric $\mathcal{Q}_h$ defined by the equation

$$\sum_{1 \leq l \leq n} X_l^2 + \sum_{n-i < l \leq n} B_l^{*2} + \sum_{1 \leq k \leq n-i} \Theta_k^2 = 0.$$

Again $\mathcal{Q}_h$ and $\mathcal{Q}_h \cap \widetilde{L}_h$ are non–degenerate and

$$(\mathcal{Q}_h \cap \widetilde{L}_h) \cap \mathbb{A}^n_{\mathbb{R}} \times \mathbb{A}^i_{\mathbb{R}} \times \mathbb{A}^{n-i}_{\mathbb{R}}$$

is positive–definite and induces in $\mathbb{A}^n_{\mathbb{R}} \times \mathbb{A}^i_{\mathbb{R}} \times \mathbb{A}^{n-i}_{\mathbb{R}}$ the Euclidean distance.

This leads us to the following concept.

Definition 14
Let $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$ and let γ be a non-zero real number. In the dual model, the bipolar varieties $\mathfrak{B}_{(i,h,j)}$ and $\mathcal{B}_{(i,h,j;\gamma)}$ are defined as follows:

For $1 \leq j \leq (n-i)(n+1)-1$ let $\mathfrak{B}_{(i,h,j)}$ be the $((n-i)(n+1)-j)$ th generic dual polar variety of $H_i^{(h)}$ and for $1 \leq j \leq n-1$ let $\mathcal{B}_{(i,h,j;\gamma)}$ be the $(n-j)$ th generic dual polar variety of $H_i^{(h,\gamma)}$. We call $\mathfrak{B}_{(i,h,j)}$ the large bipolar variety *of the equation $F=0$ associated with the indices i, h and j.*

For γ generic, we call $\mathcal{B}_{(i,h,j;\gamma)}$ the small bipolar variety *of the equation $F=0$ associated with the indices i, h and j. In this case we shall write*

$$\widetilde{\mathcal{B}}_{(i,h,j)} := \mathcal{B}_{(i,h,j;\gamma)}.$$

The bipolar varieties $\mathfrak{B}_{(i,h,j)}$ and $\mathcal{B}_{(i,h,j;\gamma)}$ are well–defined geometric objects, although the varieties $H_i^{(h)}$ and $H_i^{(h,\gamma)}$ are not closed (compare the definition of the notion of polar variety in Section 2, where we have taken care of this situation).

Let us fix again $1 \leq i \leq n-1$, $1 \leq h \leq n-i$ and a non-zero real number γ. In the sense of [4], [5] we are now going to study different extrinsic descriptions of the bipolar varieties $\mathfrak{B}_{(i,h,j)}$, $1 \leq j \leq (n-i)(n+1)-1$ and $\mathcal{B}_{(i,h,j;\gamma)}$, $1 \leq j \leq n-1$, by means of equations and inequations.

Let

$$\nu = (\nu_1, \ldots, \nu_j),\ \zeta = (\zeta_1, \ldots, \zeta_j),\ [\rho_{r,l}]_{\substack{1 \leq r \leq j \\ 1 \leq l \leq n}},\ [\mu_{r,k}]_{\substack{1 \leq r \leq j \\ 1 \leq k \leq n-i \\ k \neq h}},\ [\beta_{r;k,l}]_{\substack{1 \leq r \leq j \\ 1 \leq k \leq n-i \\ 1 \leq l \leq n}}$$

be row vectors and matrices of generic real (or rational) numbers.

Further, let us write

$$\Theta_1^{(h)} := \Theta_1, \ldots, \Theta_{h-1}^{(h)} := \Theta_{h-1}, \Theta_h^{(h)} := 1, \Theta_{h+1}^{(h)} := \Theta_{h+1}, , \ldots, \Theta_{n-i}^{(h)} := \Theta_{n-i}$$

and

$$\Theta^{(h)} := (\Theta_1^{(h)}, \ldots, \Theta_{n-i}^{(h)}).$$

We consider now two polynomial matrices $T_{(i,h,j)}$ and $T_{(i,h,j;\gamma)}$.

The first one is the $((n+j+1)\times((n-i)(n+1)+n))$–matrix

$$T_{(i,h,j)} := \begin{bmatrix} J(F) & 0 & O_{1\times(n-i-1)} & \\ \frac{\partial(J(F)^T\Lambda+B(X)^T\Theta^{(h)^T})}{\partial(X_1,\ldots,X_n)} & J(F)^T & [B_{l,k}]_{\substack{1\le l\le n\\ 1\le k\le n-i\\ k\neq h}} & \mathcal{I} \\ [\rho_{r,l}-\nu_r X_l]_{\substack{1\le r\le j\\ 1\le l\le n}} & \zeta^T-\Lambda\nu^T & [\mu_{r,k}-\nu_r\Theta_k]_{\substack{1\le r\le j\\ 1\le k\le n-i\\ k\neq h}} & \end{bmatrix},$$

where the index j has the range $1\le j\le (n-i)(n+1)-1$ and $\mathcal{I}$ represents the $((n+j+1)\times(n-i)\,n)$–submatrix

$$\mathcal{I} := \begin{bmatrix} O_{1\times n} & O_{1\times n} & \cdots & O_{1\times n} & O_{1\times n} & \cdots & O_{1\times n} \\ I_n & \Theta_1 I_n & \cdots & \Theta_{h-1}I_n & \Theta_{h+1}I_n & \cdots & \Theta_{n-i}I_n \\ \mathcal{F}_{(h)} & \mathcal{F}_{(1)} & \cdots & \mathcal{F}_{(h-1)} & \mathcal{F}_{(h+1)} & \cdots & \mathcal{F}_{(n-i)} \end{bmatrix}$$

with

$$\mathcal{F}_{(k)} := [\beta_{r;k,l}-\nu_r B_{k,l}]_{\substack{1\le r\le j\\ 1\le l\le n}} \text{ for } 1\le k\le n-i.$$

The second polynomial matrix $T_{(i,h,j;\gamma)}$ is the $((n+j+1)\times 2n)$–matrix

$$T_{(i,h,j;\gamma)} :=$$

$$\begin{bmatrix} J(F) & 0 & & \\ & & \mathcal{C} & \\ \frac{\partial(J(F)^T\Lambda+B^{(h)}_{(h,\gamma)}(X)^T\Theta^{(h)^T})}{\partial(X_1,\ldots,X_n)} & J(F)^T & & \\ [\rho_{r,l}-\nu_r X_l]_{\substack{1\le r\le j\\ 1\le l\le n}} & \zeta^T-\Lambda\nu^T & [\mu_{r,k}-\nu_r\Theta_k]_{\substack{1\le r\le j\\ 1\le k\le n-i\\ k\neq h}} & [\beta_{r;h,l}-\nu_r B^*_l]_{\substack{1\le r\le j\\ n-i<l\le n}} \end{bmatrix},$$

where $1\le j\le n-1$. Here $\mathcal{C}$ denotes the $((n+1)\times n)$–submatrix

$$\mathcal{C} := \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ & \ddots & & & & & \ddots & \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

whose first and $(h+1)$th rows consist only of zeros.

One deduces from Definition 14, Proposition 6 and Proposition 7 that a point $(x,b,(\lambda:\vartheta))$ of $H_i^{(h)}$ or $H_i^{(h,\gamma)}$ with $\vartheta_h=1$ belongs to $\mathfrak{B}_{(i,h,j)}$ or $\mathcal{B}_{(i,h,j;\gamma)}$ if

and only if all $(n+j+1)$–minors of $T_{(i,h,j)}$ or $T_{(i,h,j;\gamma)}$ vanish at $(x, b, \lambda, \vartheta^{(h)})$ in $\mathbb{A}^n \times \mathbb{A}^{(n-i)n} \times \mathbb{A}^{n-i}$, where $\vartheta^{(h)} := (\vartheta_1, \ldots, \vartheta_{h-1}, \vartheta_{h+1}, \ldots, \vartheta_{n-i})$.

Further, from ([4, 5], Proposition 8) we conclude that the bipolar varieties $\mathfrak{B}_{(i,h,j)}$ and $\mathcal{B}_{(i,h,j;\gamma)}$ are of (local) dimension $j-1$ at any point $(x, b, (\lambda : \vartheta))$ of $H_i^{(h)} \cap \mathfrak{B}_{(i,h,j)}$ and $H_i^{(h,\gamma)} \cap \mathcal{B}_{(i,h,j;\gamma)}$. Thus $\mathfrak{B}_{(i,h,j)}$ and $\mathcal{B}_{(i,h,j;\gamma)}$ are empty or equidimensional of dimension $j-1$.

Moreover, from [6] we infer that these bipolar varieties are normal and Cohen–Macaulay at any point of $H_i^{(h)}$ or $H_i^{(h,\gamma)}$ they contain.

In $T_{(i,h,j)}$ we fix any $n+j$ columns which contain the columns corresponding to at least one of the indeteterminates $X_1, \ldots, X_n$ and to $B_{h,1}, \ldots, B_{h,n}$. We characterize this choice by a vector $\underline{t} \in \mathbb{N}^{n+j}$ whose entries are the numbers of the selected columns. We denote by

$$M_{n+j+1}^{(i,h,j,\underline{t})},\ M_{n+j+2}^{(i,h,j,\underline{t})},\ \ldots,\ M_{(n-i)(n+1)+n}^{(i,h,j,\underline{t})}$$

the $(n+j+1)$–minors of $T_{(i,h,j)}$ obtained by adding one by one to the selected columns $\underline{t}$ each other of the columns of $T_{(i,h,j)}$, and, for $1 \leq s \leq j$, we denote by $m_{(i,h,j,\underline{t},s)}$ the $(n+j)$–minor of $T_{(i,h,j)}$ corresponding to the selected columns $\underline{t}$ and all rows excepted the row number $n+s+1$.

Proposition 15
Let $D_{(i,h,j,\underline{t},s)}$ be the closed subvariety of $\mathbb{T}_i^{(h)}$ defined by the condition

$$rk\ B < n-i,\ \ rk\ B^{(h)}(X) < n-i\ \ or\ \ m_{(i,h,j,\underline{t},s)} = 0.$$

Then the polynomial equations of the system

$$F(X) = 0, \quad \frac{\partial F}{\partial X_l}(X)\Lambda + B_{h,l} - X_l + \sum_{\substack{1 \leq k \leq n-i \\ k \neq h}} B_{k,l}\Theta_k = 0,\ 1 \leq l \leq n,$$

$$M_{n+j+1}^{(i,h,j,\underline{t})} = 0\ ,\ \ldots,\ M_{(n-i)(n+1)+n}^{(i,h,j,\underline{t})} = 0$$

intersect transversally at any of their common solutions in $\mathbb{T}_i^{(h)} \setminus D_{(i,h,j,\underline{t},s)}$. They define $\mathfrak{B}_{(i,h,j)} \setminus D_{(i,h,j,\underline{t},s)}$ in $\mathbb{T}_i^{(h)} \setminus D_{(i,h,j,\underline{t},s)}$.

Proof
Obvious by Proposition 6, and the Propositions 6 and 8 of [4, 5]. □

Observe that, for i, h fixed, the bipolar varieties are ordered by inclusion as follows:

$$\overline{H_i^{(h)}} \supsetneq \mathfrak{B}_{(i,h,(n-i)(n+1)-1)} \supset \cdots \supset \mathfrak{B}_{(i,h,1)}$$

(here $\overline{H_i^{(h)}}$ denotes the Zariski closure of $H_i^{(h)}$). The variety $\mathfrak{B}_{(i,h,1)}$ is empty or zero–dimensional. If $\mathfrak{B}_{(i,h,1)}$ is non–empty the chain is strictly decreasing.

Let us fix again $1 \leq i \leq n-1,\ 1 \leq h \leq n-i$ and a non-zero real number γ. From Proposition 7 we deduce that for $\Theta_h = 1$ the equations of the system (5) generate in $\mathbb{R}[X, \Lambda, B^*_{n-i+1}, , \ldots, B^*_n, \Theta^{(h)}]_{X_h-\gamma}$ the vanishing ideal of $H_i^{(h,\gamma)}$ (interpreted as affine subvariety of $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{A}^{(n-i)}$). Therefore, all $(n+j+1)$–minors of $T_{(i,h,j;\gamma)}$ vanish at a point $(x, b, (\lambda : \vartheta))$ of $H_i^{(h,\gamma)}$ with $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-i})$ and $\vartheta_h = 1$ if and only if $(x, b, (\lambda : \vartheta))$ belongs to the affine variety $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma}$, consisting of the elements of $(\mathcal{B}_{(i,h,j;\gamma)})$ which satisfy the condition $X_h - \gamma \neq 0$. In other words, $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma}$ is the locus of $H_i^{(h,\gamma)}$, where all $(n+j+1)$–minors of $T_{(i,h,j;\gamma)}$ vanish.

In $T_{(i,h,j;\gamma)}$ we fix any $n+j$ columns which contain the columns corresponding to at least one of indeterminates $X_1, \ldots, X_n$, to the entries of $\Theta^{(h)}$ and to B^*_l, $n-i < l \leq n$. As before let us characterize this selection by a vector $\underline{t} \in \mathbb{N}^{n+j}$. We denote by

$$M_{n+j+1}^{(i,h,j,\underline{t};\gamma)},\ M_{n+j+2}^{(i,h,j,\underline{t};\gamma)}\ , \ldots,\ M_{2n}^{(i,h,j,\underline{t};\gamma)}$$

the $n+j+1$–minors obtained by adding one by one to the selected columns each other column of $T_{(i,h,j;\gamma)}$, and, for $1 \leq s \leq j$, we denote by $m_{(i,h,j,\underline{t},s;\gamma)}$ the $(n+j)$–minor of $T_{(i,h,j;\gamma)}$ corresponding to selected columns $\underline{t}$ and all rows, excepted the row number $n+s+1$.

Proposition 16
The sequence of polynomials

$$F(X), \quad \frac{\partial F}{\partial X_h}(X)\Lambda + \gamma - X_h,$$

$$\frac{\partial F}{\partial X_l}(X)\Lambda - X_l + \Theta_l,\ 1 \leq l \leq n-i,\ l \neq h, \quad \frac{\partial F}{\partial X_l}(X)\Lambda + B^*_l - X_l,\ n-i < l \leq n,$$

$$M_{n+j+1}^{(i,h,j,\underline{t};\gamma)},\ ,\ldots,\ M_{2n}^{(i,h,j,\underline{t};\gamma)}$$

generates in

$$\mathcal{R} := \mathbb{R}[X, \Lambda, B^*_{n-i+1}, , \ldots, B^*_n, \Theta^{(h)}]_{m_{(i,h,j,\underline{t},s;\gamma)}(X_h-\gamma)}$$

the trivial ideal or forms a reduced regular sequence. The sequence defines in $\mathcal{R}$ the affine variety $(\mathcal{B}_{(i,h,j;\gamma)})_{m_{(i,h,j,\underline{t};\gamma)}(X_h-\gamma)}$ and their entries intersect transversally at any point of $(\mathcal{B}_{(i,h,j;\gamma)})_{m_{(i,h,j,\underline{t};\gamma)}(X_h-\gamma)}$.

Proof
Obvious from Proposition 7, and the Propositions 6 and 8 of [4, 5]. □

Similarly as above, remark that, for i, h, γ fixed, the bipolar varieties $\mathcal{B}_{(i,h,j;\gamma)}$ are ordered by inclusion as follows

$$\overline{H_i^{(h,\gamma)}} \supsetneq \mathcal{B}_{(i,h,n-1;\gamma)} \supset \cdots \supset \mathcal{B}_{(i,h,1;\gamma)}.$$

The variety $\mathcal{B}_{(i,h,1;\gamma)}$ is empty or zero–dimensional. If $\mathcal{B}_{(i,h,1;\gamma)}$ is non–empty then the chain strictly decreases.

Observation 17
Let notations be as in Proposition 15 and Proposition 16 and let $j \geq 2$. The loci of $\mathfrak{B}_{(i,h,j)} \cap H_i^{(h)}$ and $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma} \cap H_i^{(h,\gamma)}$, where, for suitable $\underline{t} \in \mathbb{N}^{n+j}$ and $1 \leq s \leq j$, all minors of the form $m_{(i,h,j,\underline{t},s)}$ and $m_{(i,h,j,\underline{t},s;\gamma)}$ vanish, coincide with $\mathfrak{B}_{(i,h,j-1)} \cap H_i^{(h)}$ and $\mathcal{B}_{i,h,j-1;\gamma} \cap H_i^{(h,\gamma)}$ and are empty or of pure codimension one. Moreover, for each point z of $\mathfrak{B}_{(i,h,1)} \cap H_i^{(h)}$ and $\mathcal{B}_{(i,h,1;\gamma)} \cap H_i^{(h,\gamma)}$ there exist minors of the form $m_{(i,h,1,\underline{t},1)}$ and $m_{(i,h,1,\underline{t},1;\gamma)}$, $\underline{t} \in \mathbb{N}^{n+1}$, respectively, which do not vanish at z.

Proof
Obvious by [6], Lemma 2. □

We denote by $\deg \mathfrak{B}_{(i,h,j)}$, $\deg \mathcal{B}_{(i,h,j;\gamma)}$ and $\deg \widetilde{\mathcal{B}}_{(i,h,j)}$ the geometric degrees of the respective polar varieties in their respective affine ambient spaces (see [30] for definition and properties of the geometric degree of a subvariety of an affine space).

From Lemma 3 and [6], Theorem 13 we deduce that for $1 \leq j \leq n-1$

$$\deg \mathcal{B}_{(i,h,j;\gamma)} \leq \deg \widetilde{\mathcal{B}}_{(i,h,j)} \leq \deg \mathcal{B}_{(i,h,(n-i)n-i+j)}. \tag{10}$$

holds.

Suppose that $\{F=0\}_{\mathbb{R}}$ is compact and contains an F–regular point. Then Observation 8, Proposition 7, Lemma 3 and Corollary 2 imply that $(\widetilde{\mathcal{B}}_{(i,h,j)})_{\mathbb{R}}$ and $(\mathfrak{B}_{(i,h,j)})_{\mathbb{R}}$ are non–empty. This implies $1 \leq \deg \widetilde{\mathcal{B}}_{(i,h,n-1)} \leq \deg \mathcal{B}_{(i,h,(n-i)(n+1)-1)}$.

For $d \geq 2$ and $1 \leq j \leq (n-i)(n+1)-1$ we infer from the Bézout–Inequality [30], [22], [63] the following extrinsic bounds for these degrees (see [5] for details):

$$\deg \mathfrak{B}_{(i,h,j)} \leq d^{n+1} \, (n\,d+j)^{(n-i)(n+1)-j} = (nd)^{\,O((n-i)n)} \tag{11}$$

whence, in particular,

$$\deg \mathfrak{B}_{(n-1,h,j)} \leq \frac{(n\,d^2+d\,j)^{n+1}}{(n\,d+j)^j} \leq (n\,d\,(d+1))^{n+1} = (nd)^{\,O(n)}. \tag{12}$$

Similarly we have for $1 \leq j \leq n-1$

$$\deg \mathcal{B}_{(i,h,j;\gamma)} \leq \deg \widetilde{\mathcal{B}}_{(i,h,j)} \leq d^{n+1}(n\,d+j)^{(n-j)} \leq d\,(n\,d\,(d+1))^n = (nd)^{\,O(n)}. \tag{13}$$

In view of the subsequent algorithmic considerations we notice that the degree estimates (12) and (13) are of order $(nd)^{\,O(n)}$.

We fix now only $1 \leq i \leq n-1$, $1 \leq k \leq n-i$ and a non–zero real number γ.

For $1 \leq l \leq n$ we are going to consider the following closed subvarieties of the affine ambient spaces ${\mathbb{T}_i}^{(h)}$ and ${\mathbb{N}_i}^{(h)}$, which we denote by ${S_l}^{(i,h)}$ and ${S_l}^{(i,h;\gamma)}$:

Let ${S_l}^{(i,h)}$ be the Zariski–closure of the locally closed subset of ${\mathbb{T}_i}^{(h)}$ defined by the conditions

$$F(X) = 0, \qquad \frac{\partial F}{\partial X_{t'}}(X)\,\Lambda + (B_{h,t'} - X_{t'})\Theta_h + \sum_{\substack{1 \leq k \leq n-i \\ k \neq h}} B_{k,t'}\,\Theta_k = 0, \quad 1 \leq t' \leq l,$$

$$rk\ B_i = \mathrm{rk}\ B_i^{(h)}(X) = n - i,$$

and let ${S_l}^{(i,h;\gamma)}$ be the Zariski–closure of the locally closed subset of ${\mathbb{T}_i}^{(h)}$ defined by the conditions

$$\begin{aligned} F(X) &= 0, \\ \frac{\partial F}{\partial X_h}(X)\Lambda + (\gamma - X_h)\Theta_h &= 0, \\ \frac{\partial F}{\partial X_{t'}}(X)\Lambda - X_{t'}\Theta_h + \Theta_{t'} &= 0, \\ 1 \leq t' \leq \min\{l, n-i\},\ t' &\neq h, \\ \frac{\partial F}{\partial X_{t'}}(X)\Lambda + (B^*_{t'} - X_{t'})\Theta_h &= 0, \\ n - i < t' &\leq l, \\ X_h - \gamma &\neq 0. \end{aligned}$$

From the Bézout–Inequality we deduce the estimates

$$\deg {S_l}^{(i,h)} \leq d^{l+1} \tag{14}$$

and

$$\deg {S_l}^{(i,h;\gamma)} \leq d^{l+1}. \tag{15}$$

We associate now with i, h, γ and the real interpretation of the polynomial equation $F = 0$ the following discrete parameters:

$$\delta_{(i,h)} := \max\{\{\deg S_l^{(i,h)} \mid 1 \leq l \leq n\}, \max\{\deg \mathfrak{B}_{(i,h,j)} \mid 1 \leq j \leq (n-i)(n+1) - 1\}\}$$

and

$$\delta_{(i,h;\gamma)} := \max\{\{\deg S_l^{(i,h;\gamma)} \mid 1 \leq l \leq n\}, \max\{\deg \overline{(\mathcal{B}_{(i,h,j;\gamma)})_{X_h - \gamma}} \mid 1 \leq j \leq n-1\}\}$$

For generically chosen γ we write

$$\tilde{\delta}_{(i,h)} := \delta_{(i,h;\gamma)}$$

We observe that the parameter $\delta_{(i,h)}$ remains invariant under linear transformations of the coordinates $X_1, \ldots, X_n$ by unitary complex $(n \times n)$–matrices, whereas the parameters $\delta_{(i,h;\gamma)}$ and $\tilde{\delta}_{(i,h)}$ are coordinate–dependent even for such special coordinate transformations. Therefore we call $\delta_{(i,h)}$ the *unitary–independent degree* of the real interpretation of the equation $F = 0$ associated with i and h. In the same vein we call $\delta_{(i,h;\gamma)}$ and $\tilde{\delta}_{(i,h)}$ the *unitary–dependent degrees* of the real interpretation of $F = 0$ associated with i, h, γ and with i, h, respectively.

In the light of the geometric underpinning of the notion of dual polar varieties exposed in [4], [5], Section 2, the limitation to unitary complex matrices makes sense. The definition of dual polar varieties in intrinsic terms requires as ingredients a non–degenerate hyperquadric and a hyperplane at infinity in the corresponding projective ambient space such that the restriction of the given hyperquadric to the hyperplane at infinity remains non–degenerate. If the chosen hyperquadric represents in the associated real affine space the Euclidean norm, then only unitary matrices leave invariant the given geometric situation. For details we refer to [4], [5], Section 2 and 3.1.

Taking into account the estimate (10) we infer from the Bézout–Inequality that

$$\delta_{(i,h;\gamma)} \leq \tilde{\delta}_{(i,h)} \leq \delta_{(i,h)} \tag{16}$$

holds.

From (11) – (15) we deduce for $d \geq 1$ the *extrinsic* estimates

$$\delta_{(i,h)} = (nd)^{O((n-i)n)}, \tag{17}$$

$$\delta_{(i,h;\gamma)} = (nd)^{O(n)}, \tag{18}$$

$$\tilde{\delta}_{(i,h)} = (nd)^{O(n)}. \tag{19}$$

The estimate (17) is possibly too coarse, whereas the estimates (18) and (19) seem to be tight in worst case. We shall turn back to this subject during our subsequent algorithmic considerations.

We finish this section considering the following algorithmic problem $(\mathbf{P})$:

As input let be given an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ with a single output node, representing a polynomial $F \in \mathbb{Q}[X]$ of (known) degree d and logarithmic height at most η. Accept the input circuit σ if the complex hypersurface contains a real F–regular point. If this is the case, return a finite set of real algebraic sample points for each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$.

We are now going to design for each $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$ a procedure $\Pi_{(i,h)}$ which solves the problem $(\mathbf{P})$ under the assumption that $\{F = 0\}_{\mathbb{R}}$ is compact. Let Z be a new indeterminate.

Procedure $\Pi_{(i,h)}$

Input: An essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ having a single output node.

Input Specification: The circuit σ represents a polynomial $F \in \mathbb{Q}[X]$ of positive degree d and logarithmic height at most η. The semialgebraic set $\{F = 0\}_{\mathbb{R}}$ is compact and the indeterminates $X_1, \ldots, X_n$ are in general position with respect to the complex hypersurface $\{F = 0\}$.

Output: The procedure $\Pi_{(i,h)}$ accepts the input σ if $\{F = 0\}$ contains a real F–regular point. If this is the case, the procdure returns a circuit representation of the coefficients of $n+1$ polynomials $P, G_1, \ldots, G_n \in \mathbb{Q}[Z]$ satisfying for $G := (G_1, \ldots, G_n)$ the following output specification:

- P is monic and separable,
- $\deg G < \deg P \leq \deg \widetilde{\mathcal{B}}_{(i,h,1)} \leq \deg \mathcal{B}_{(i,h,1)}$ with $\deg G := \max\{\deg G_1, \ldots, \deg G_n\}$,
- the zero–dimensional complex affine variety, $\{G(z) \mid z \in \mathbb{A}^1,\ P(z) = 0\}$ contains an F–regular, real algebraic sample point of each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$. In order to represent these sample points, an encoding "à la Thom" of the real zeros of the polynomial P is returned (see e.g. [16] for this kind of encoding).

We fix now $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$.

Design of the procedure $\Pi_{(i,h)}$.

Let be given an essentially division–free circuit σ in $Q[X]$ of size L having a single output node which represents a polynomial $F \in \mathbb{Q}[X]$ satisfying the input specification of the procedure $\Pi_{(i,h)}$. Let d be the (positive) degree of F and η its logarithmic length. We consider the function

$$||\cdot|| : \{F = 0\}_{\mathbb{R}} \to \mathbb{R}$$

induced by the Euclidean norm on $\mathbb{R}^n$. Observe that $||\cdot||$ is continuous and semialgebraic. Since by assumption $\{F = 0\}_{\mathbb{R}}$ is compact, the function $||\cdot||$ is bounded by a positive constant, say K. From the effective Lojasiewicz–Inequality (see [57],Theorem 3) we deduce that there exists an universal constant $c > 0$ (not depending on L, ℓ, d, n or η) which satisfies the condition $\log(\max\{1, K\}) \leq \eta\, d^{c\,n^2}$.

Let us choose a positive integer γ with $\log \gamma > \eta\, d^{c n^2}$ which is representable by a division–free arithmetic circuit in $\mathbb{Z}$ of size and non–scalar depth $O(\log \eta + n^2 \log d)$ and observe that $\gamma > K$ holds.

Therefore any real point $x = (x_1, \ldots, x_n)$ of the hypersurface $\{F = 0\}$ satisfies the condition $x_h - \gamma \neq 0$. Since by assumption the indeterminates $X_1, \ldots, X_n$ are in general position with respect to $\{F = 0\}$, we may suppose without loss of generality

that any generically F–regular connected component of $\{F=0\}_{\mathbb{R}}$ contains also a point x with $\frac{\partial F}{\partial X_h}(x) \neq 0$.

From Proposition 7 and the choice of γ we deduce that the (polynomial) equations of the system

$$
\begin{aligned}
F(X) &= 0,\\
\frac{\partial F}{\partial X_h}(X)\Lambda + \gamma - X_h &= 0,\\
\frac{\partial F}{\partial X_l}(X)\Lambda - X_l + \Theta_l &= 0,\\
1 \leq l \leq n-i,\ l \neq h,&\\
\frac{\partial F}{\partial X_l}(X)\Lambda + B_l^* - X_l &= 0,\\
n-i < l \leq n,&
\end{aligned}
\tag{20}
$$

intersect transversally at each of their real solutions.

Denote by $V := S_n^{(i,h;\gamma)}$ the locally closed algebraic subvariety of $\mathbb{A}^n \times \mathbb{A}^i \times \mathbb{A}^{n-i}$ consisting of the common (complex) solutions of the polynomial equation system (20) which satisfy the condition $X_h - \gamma \neq 0$ and let $V_{\mathbb{R}} := V \cap (\mathbb{A}^n_{\mathbb{R}} \times \mathbb{A}^i_{\mathbb{R}} \times \mathbb{A}^{n-i}_{\mathbb{R}})$ be the real trace of V. Our choice of γ implies that $V_{\mathbb{R}}$ consists of all real solutions of (20) and is therefore closed. Moreover, from our assumptions and Proposition 7 we deduce that V and $V_{\mathbb{R}}$ are empty or smooth of dimension $n-1$ and that the real variety $V_{\mathbb{R}}$ is non–empty if and only if $\{F=0\}_{\mathbb{R}}$ contains an F–regular point. More precisely, for any generically F–regular connected component C of $\{F=0\}_{\mathbb{R}}$ there exists a point (x,b,λ,ϑ) of $V_{\mathbb{R}}$ with $x \in C$, $\frac{\partial F}{\partial X_h}(x) \neq 0$ and $(b,\lambda,\vartheta) \in \mathbb{A}^i_{\mathbb{R}} \times \mathbb{A}^{n-i}_{\mathbb{R}}$. Therefore, a set of algebraic sample points for the connected components of $V_{\mathbb{R}}$ gives rise to a set of algebraic sample points for the generically F–regular connected components of $\{F=0\}_{\mathbb{R}}$.

Suppose now that the hypersurface $\{F=0\}$ contains a real F–regular point. Then the real variety $V_{\mathbb{R}}$ is smooth, of dimension $n-1$ and the polynomials of the system (20) form in $\mathbb{Q}[X, B^*_{n-i+1}, \ldots, B^*_n, \Lambda, \Theta^{(h)}]_{X_h-\gamma}$ a reduced regular sequence. For $1 \leq j \leq n-1$ we deduce from [5], Proposition 2 that the real bipolar variety $(\mathcal{B}_{(i,h,j;\gamma)})_{\mathbb{R}}$ (and hence the complex variety $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma}$ contains at least one point of each connected component of $V_{\mathbb{R}}$. Therefore, $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma}$ and $(\mathcal{B}_{(i,h,j;\gamma)})_{\mathbb{R}}$ are equidimensional of dimension $j-1$. From Proposition 16 and Observation 17 we conclude that for $2 \leq j \leq n-1$ the algebraic variety $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h-\gamma} \setminus (\mathcal{B}_{(i,h,j-1;\gamma)})_{X_h-\gamma}$ is locally definable by reduced regular sequences.

In the same way one sees that the complex variety $(\mathcal{B}_{(i,h,1;\gamma)})_{X_h-\gamma}$ is zero–dimensional and contains for each connected component of $V_{\mathbb{R}}$ a real algebraic sample point.

The algorithm $\Pi_{(i,h)}$ proceeds now by deciding whether $(\mathcal{B}_{(i,h,1;\gamma)})_{X_h-\gamma}$ contains real algebraic points, and, if this is the case, by computing them. The algorithm infers

from these data whether the hypersurface $\{F = 0\}$ contains F–regular real points. If the answer is positive, the data furnish also a finite set of F–regular real algebraic sample points for the generically F–regular connected components of $\{F = 0\}_{\mathbb{R}}$.

At the beginning, the procedure $\Pi_{(i,h)}$ transforms the input circuit σ and the chosen encoding of γ into an essentially division–free arithmetic circuit σ_1 in $\mathbb{Q}[X, B^*_{n-i+1}, \ldots, B^*_n, \Lambda, \Theta^{(h)}]$ of size $O(L + n^2 \log d + \log \eta)$ and non–scalar depth $O(\ell + n^2 \log d + \log \eta)$ such that σ_1 represents the equation system (20) and the polynomial $X_n - \gamma$. Taking the circuit σ_1 as input, the procedure $\Pi_{(i,h)}$ follows now the pattern of the (probabilistic) procedure described in the proofs of [4], Theorem 11 and [4], Theorem 13 in order to decide whether $V_{\mathbb{R}}$ is empty.

If $V_{\mathbb{R}}$ is empty, then the procedure $\Pi_{(i,h)}$ returns the answer that the hypersurface $\{F = 0\}$ does not contain any real F–regular point.

If $V_{\mathbb{R}}$ is non–empty, the procedure $\Pi_{(i,h)}$ produces a circuit representation of the coefficients of $2n + 1$ polynomials $P, G_1, \ldots, G_n, G_{n+1}, \ldots, G_{2n} \in \mathbb{Q}[Z]$ satisfying for $\widetilde{G} := (G_1, \ldots, G_{2n})$ the following output specification:

- P is monic and separable,
- $\deg \widetilde{G} < \deg P \leq \deg \mathcal{B}_{(i,h,1;\gamma)}$,
- $(\mathcal{B}_{(i,h,1;\gamma)})_{X_n - \gamma} = \{\widetilde{G}(z) \mid z \in \mathbb{A}^1, P(z) = 0\}$.

From this representation of the variety $(\mathcal{B}_{(i,h,1;\gamma)})_{X_n-\gamma}$ we deduce now that for $G := (G_1, \ldots, G_n)$ the zero–dimensional variety $\{G(z) \mid z \in \mathbb{A}^1, P(z) = 0\}$ contains an F–regular real algebraic sample point for each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$.

The procedure from [4] and [5], called by $\Pi_{(i,h)}$, is based on the original paradigm [26], [25] of a procedure with intrinsic complexity that solves polynomial equation systems over the *complex* numbers (see also [23, 28, 20]).

We are going now to describe *succinctly* how this procedure is applied to the task of real root finding (Proposition 16 and Observation 17 will play here a key role).

For this purpose we shall freely refer to terminology, mathematical results and subroutines of [28], where the first streamlined version of this procedure was presented and implemented as "Kronecker algorithm" (compare also [32]).

In order to simplify the exposition we shall refrain from the presentation of details which ensure only appropriate genericity conditions for the procedure. The following description requires that the reader is acquainted with the details of the Kronecker algorithm. Although this description may look at first glance intricate, no substantially new idea, which was not explained before, becomes introduced.

Let us consider the polynomial $((n + 2) \times 2n)$–matrix $T_{(i,h,1;\gamma)}$ introduced at the beginning of this section. According to Proposition 16 and the terminology used

in its context, there exist n^2 suitable vectors $\underline{t} \in \mathbb{N}^{n+1}$ which determine each an $(n+1)$–minor $m_{(i,h,1,\underline{t},1;\gamma)}$ and $(n+2)$– minors $M_{n+2}^{(i,h,1,\underline{t};\gamma)}, \ldots, M_{2n}^{(i,h,1,\underline{t};\gamma)}$ satisfying the following condition:

The polynomials

$$F(X), \quad \frac{\partial F}{\partial X_h}(X)\Lambda + \gamma - X_h, \quad \frac{\partial F}{\partial X_l}(X)\Lambda - X_l + \Theta_l,\ 1 \leq l \leq n-i,\ l \neq h,$$

(21)

$$\frac{\partial F}{\partial X_l}(X)\Lambda + B_l - X_l,\ n-i < l \leq n, \quad M_{2n}^{(i,h,1,\underline{t};\gamma)},\ , \ldots,\ M_{n+2}^{(i,h,1,\underline{t};\gamma)}$$

generate in

$$\mathcal{R} := \mathbb{R}[X, \Lambda, B_{n-i+1}^*, , \ldots, B_n^*, \Theta^{(h)}]_{m_{(i,h,1,\underline{t},1\,;\gamma)}(X_h-\gamma)}$$

the trivial ideal or form a reduced regular sequence. Moreover, they define in $\mathcal{R}$ the affine variety $(\mathcal{B}_{(i,h,1;\gamma)})_{m_{(i,h,1,\underline{t},1;\gamma)}(X_h-\gamma)}$ and following Observation 17 and by the choice of γ there exists for each real point z of $\mathcal{B}_{(i,h,1;\gamma)}$ a suitable vector $\underline{t} \in \mathbb{N}^{n+1}$ such that $m_{(i,h,1,\underline{t},1\,;\gamma)}(X_h - \gamma)$ does not vanish at z.

In this situation $\Pi_{(i,h)}$ applies for each of these n^2 vectors $\underline{t} \in \mathbb{N}^{n+1}$ the algorithm "Geometric Solve" of [28] to the polynomial equation system given by (21) and the open condition $m_{(i,h,1,\underline{t},1\,;\gamma)}(X_h - \gamma) \neq 0$.

If the procedure $\Pi_{(i,h)}$ finds no solution for any of these systems, the circuit σ is rejected, because then $\mathcal{B}_{(i,h,1;\gamma)}$, and hence V, do not contain any real point. This in turn implies that $\{F = 0\}$ contains no F–regular real point.
Suppose that this is not the case. Then the procedure $\Pi_{(i,h)}$ plugs together the representations of all solutions found in order to obtain a circuit representation of the coefficients of $2n+1$ polynomials $P, G_1, \ldots, G_{2n} \in \mathbb{Q}[Z]$ satisfying for $\tilde{G} := (G_1, \ldots, G_{2n})$ the following output specification:

- P is monic and separable,
- $\deg \tilde{G} < \deg P \leq \#\,(\mathcal{B}_{(i,h,1;\gamma)})_{(X_h-\gamma)} \leq \deg \mathcal{B}_{(i,h,1;\gamma)}$,
- $(\mathcal{B}_{(i,h,1;\gamma)})_{(X_h-\gamma)} = \{\tilde{G}(z) \mid z \in \mathbb{A}^1, P(z) = 0\}$.

Applying now to the polynomial $P \in \mathbb{Q}[Z]$ any of the known, well–parallelizable computer algebra algorithms for the determination of all real roots of a given univariate polynomial, $\Pi_{(i,h)}$ decides whether $(\mathcal{B}_{(i,h,1;\gamma)})_{X_h-\gamma}$ contains real points. If this is not the case, $\Pi_{(i,h)}$ rejects the input circuit σ and returns the answer that the hypersurface $\{F = 0\}$ does not contain any F–regular real point. Otherwise $\Pi_{(i,h)}$

encodes the real zeros of P "à la Thom" and, together with a circuit representation of the coefficients of the polynomials $G_1, \ldots, G_{2n}$ contained in $\tilde{G}$, the real variety

$$(\mathcal{B}_{(i,h,1;\gamma)})_{\mathbb{R}} = \{\tilde{G}(z) \mid z \in \mathbb{R},\ P(z) = 0\}.$$

The output of $\Pi_{(i,h)}$, consisting of a circuit representation of the coefficients of the univariate polynomials $P, G_1, \ldots, G_n$ and an encoding of the real zeros of P "à la Thom", can be read off from this representation. One verifies easily that P and $G := (G_1, \ldots, G_n)$ satisfy the output specification of $\Pi_{(i,h)}$. In particular:

- P is monic and separable,
- $\deg G < \deg P \leq \#(\mathcal{B}_{(i,h,1;\gamma)})_{(X_h-\gamma)} \leq \deg(\mathcal{B}_{(i,h,1;\gamma)})$,
- the real variety $\{G(z) \mid z \in \mathbb{R}, P(z) = 0\}$ contains an algebraic sample point of each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$.

We are now going to analyze the sequential and the (non–scalar) parallel time complexity of the subroutine of $\Pi_{(i,h)}$ which processes the equation system (20) and the open condition $X_h - \gamma \neq 0$.

Let us first observe that for a given suitable choice of a vector $\underline{t} \in \mathbb{N}^{n+1}$, for a given index $1 \leq j \leq n-1$ and a given vector $\underline{t'} \in \mathbb{N}^{n+j}$ extending $\underline{t}$, the locally closed variety $\mathcal{B}^*_{(i,h,j,\underline{t};\gamma)}$ defined by the polynomial system

$$F(X) = 0, \quad \frac{\partial F}{\partial X_h}(X)\Lambda + \gamma - X_h = 0,$$

$$\frac{\partial F}{\partial X_l}(X)\Lambda - X_l + \Theta_l = 0,\ 1 \leq l \leq n-i,\ l \neq h,$$

$$\frac{\partial F}{\partial X_l}(X)\Lambda + B^*_l - X_l = 0,\ n-i < l \leq n, \tag{22}$$

$$M_{2n}^{(i,h,1,\underline{t};\gamma)} = 0,\ \ldots,\ M_{n+j+1}^{(i,h,1,\underline{t};\gamma)} = 0, \quad m_{(i,h,1,\underline{t},1\,;\gamma)}(X_h - \gamma) \neq 0$$

is a *meagerly generic* polar variety corresponding to the localization of

$$(\mathcal{B}_{(i,h,j;\gamma)})_{m_{(i,h,j,\underline{t'},1\,;\gamma)}(X_h-\gamma)} \text{ of the generic polar variety } \mathcal{B}_{(i,h,j;\gamma)}$$

(see [6], Section 4, and in particular Example 2 for this kind of argumentation and details on meagerly generic polar varieties).

From [6], Theorem 13 we deduce now

$$\deg \mathcal{B}^*_{(i,h,j,\underline{t};\gamma)} \leq \mathcal{B}_{(i,h,j;\gamma)} \leq \delta_{(i,h;\gamma)}.$$

For each suitable chosen vector $\underline{t} \in \mathbb{N}^{n+1}$ we run once the algorithm "Geometric Solve" on the input system (22). This requires each time

$$(L+\log\eta)(n\,d)^{O(1)} \max\{\max\{\deg S_l^{(i,h;\gamma)} | 1 \leq l \leq n\}, \max\{\deg \mathcal{B}^*_{(i,h,j,\underline{t};\gamma)} | 1 \leq j \leq n-1\}$$

$$= (L+\log\eta)(n\,d)^{O(1)}\,(\delta_{(i,h;\gamma)})^2$$

arithmetical operations organized, with respect to the parameters of the arithmetic circuit σ, in non–scalar depth

$$O(n^3(\ell+\log(dn\eta))\log\delta_{(i,h;\gamma)}).$$

Taking into account that we have to run the algorithm "Geometric Solve" in parallel only for n^2 choices of vectors from $\mathbb{N}^{n+1}$, that the univariate polynomial P is of degree at most $\deg \mathcal{B}_{(i,h,1;\gamma)} \leq \delta_{(i,h;\gamma)}$ and that the Thom encoding of the real zeros of P can be found using $O((\deg P)^2) = O(\delta^2_{(i,h;\gamma)})$ arithmetic operations and sign determinations of non–scalar depth $O(\log\delta_{(i,h;\gamma)})$, we see that the sequential and, with respect to the parameters of the arithmetic circuit σ, the non–scalar parallel time of the whole procedure $\Pi_{(i,h)}$ are of order

$$O((L+\log\eta)\,(n\,d)^{O(1)}\,(\delta_{(i,h;\gamma)})^2) = O((L+\log\eta)\,(n\,d)^{O(1)}\,(\tilde{\delta}_{i,h})^2) = (n\,d)^{O(n)}\,\log\eta$$

and

$$O(n^3(\ell+\log(dn\eta))\log\delta_{(i,h;\gamma)}) = O(n^3(\ell+\log(dn\eta))\log\tilde{\delta}_{i,h}) = O(n^4\,\log(dn\eta)\log d),$$

respectively.

We have therefore proven the following complexity statement (compare [4], Theorem 11 and [5], Theorem 13).

Theorem 18
Let n, d, η, i, h, δ, L, ℓ be natural numbers with $d \geq 1$, $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$. Let $X_1, \ldots, X_n$ and Z be indeterminates over $\mathbb{Q}$ and let $X := (X_1, \ldots, X_n)$.

There exists an arithmetic network $\mathcal{N}$ (or arithmetic–boolean circuit) over $\mathbb{Q}$, depending on certain parameters and having size

$$O((L+\log\eta)\,(n\,d)^{O(1)}\,\delta^2) = (n\,d)^{O(n)}\log\eta$$

and non–scalar depth $O(n^3(\ell+\log(nd\eta))\log\delta) = O(n^4\log(dn\eta)\log d)$,

such that $\mathcal{N}$ has for suitable specializations of its parameters the following properties:

Let $F \in \mathbb{Q}[X]$ be a polynomial of degree d and (logarithmic) height η and assume that F is given by an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ. Suppose that $\{F=0\}_{\mathbb{R}}$ is compact, that the variables

$X_1, \ldots, X_n$ are in general position with respect to the complex hypersurface $\{F = 0\}$ and that the unitary–dependent generic degree of the real interpretation of $F = 0$ associated with i and h is bounded by δ (in symbols: $\tilde{\delta}_{(i,h)} \leq \delta$).

Then the algorithm represented by the arithmetic network $\mathcal{N}$ starts from the circuit σ as input and decides whether the hypersurface $\{F = 0\}$ contains a real F–regular point. If this is the case, the algorithm produces a circuit representation of the coefficients of $n+1$ polynomials $P, G_1, \ldots, G_n \in \mathbb{Q}[Z]$ satisfying for $G := (G_1, \ldots, G_n)$ the following output specification:

- *P is monic and separable,*
- *$\deg G < \deg P \leq \delta$,*
- *the complex affine variety $\{G(z) \mid z \in \mathbb{A}^1, P(z) = 0\}$ is zero–dimensional and contains a real F–regular algebraic sample point for each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$.*

In order to represent these sample points the algorithm returns an encoding "à la Thom" of the real zeros of the polynomial P.

For the terminology of arithmetic network and boolean–arithmetic circuit we refer to [64, 65].

Four remarks on the formulation of Theorem 18 are at order.

- If we limit our attention to arithmetic input circuits σ in $\mathbb{Z}[X]$ which depend only on the parameters $0, 1$, then we may replace in the statement of Theorem 18 the quantity $\log \eta$ by ℓ.
- The upper bound $O(n^3(\ell + \log(nd\eta)) \log \delta)$ for the non–scalar depth of the arithmetical network $\mathcal{N}$ is far from being optimal, because it depends on the factor n^3. Only a single factor n is justified by our recursive method, whereas the coarse estimate in the effective Lojasiewicz-Inequality [57] contributes with an extra factor of n^2. In any case, desirable, but maybe difficult to achieve, would be an upper bound of $O(n(\ell + \log(nd\eta)) \log \delta)$ for the non–scalar depth of $\mathcal{N}$.

We state the third remark in the following way:

Observation 19
There exists an universal constant $c > 0$ (independent of the parameters n, d, h, δ, L, ℓ) such that the statement of Theorem 18 remains true if we drop the hypothesis that the indeterminates $X_1, \ldots, X_n$ are in general position with respect to the complex hypersurface $\{F = 0\}$ and if we assume that $\min\{(n\,d)^{cn}, \delta_{(i,h)}\} \leq \delta$ holds.

Proof
In the design of the procedure $\Pi_{(i,h)}$ the genericity assumption on the variables was only used in order to guarantee that the partial derivative $\frac{\partial F}{\partial X_h}$ does not vanish identically on any generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$. It is easy to see that this can be achieved by an orthogonal matrix $M \in \mathbb{A}_{\mathbb{R}}^{n\times n}$ which transforms $X = (X_1, \ldots, X_n)$ into $Y = (Y_1, \ldots, Y_n) := XM$. Let us denote by $\tilde{\delta}_{(i,h)}(Y)$ and $\delta_{(i,h)}(Y)$ the unitary–dependent and unitary–independent degrees, respectively, of the real interpretation of the equation $F(YM^T) = 0$, which are associated with the indices $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$. Then Theorem 18 may be applied to $F(YM^T)$. From (19) and (16) we deduce the estimates

$$\tilde{\delta}_{(i,h)}(Y) = (nd)^{O(n)} \quad \text{and} \quad \tilde{\delta}_{(i,h)}(Y) \leq \delta_{(i,h)}(Y).$$

Since the degree $\delta_{(i,h)}$ is unitary–independent, we have $\delta_{(i,h)}(Y) = \delta_{(i,h)}$, where $\delta_{(i,h)}$ is defined with respect to the original variables $X_1, \ldots, X_n$. This implies the statemet of Observation 19. □

The fourth remark is the following statement.

Observation 20
Theorem 18 asserts only the existence *of a computation that, for a given n–variate input polynomial F of degree d, logarithmic height η and circuit size and non–scalar depth L and ℓ, with variables in general position and $\{F = 0\}_{\mathbb{R}}$ compact, solves the problem* (**P**) *in sequential and non–scalar parallel time $O((L + \log \eta)\,(n\,d)^{O(1)}\,\tilde{\delta}^2_{(i,h)})$ and $O(n^3(\ell + \log(nd\eta))\log \tilde{\delta}_{(i,h)})$, respectively, where $\tilde{\delta}_{(i,h)}$ denotes the unitary–dependent generic degree of the real interpretation of the equation $F = 0$ associated with $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$.*

Theorem 18 refers therefore to the non–uniform *complexity model. In order to realize such a computation in terms of the* uniform *complexity model within the same order of sequential and parallel time, probabilistic methods have to be used (see [32] and [28]). This is achieved by choosing randomly the parameters of the arithmetic network $\mathcal{N}$ of Theorem 18. The same remark applies mutatis mutandis to Observation 19.*

Let us finally comment that the algorithm $\Pi_{(i,h)}$ can be reinterpreted as the following simple minded procedure, inspired by the well-known trick of Rabinowitsch.

Let $F \in \mathbb{Q}[X]$ be a polynomial satisfying the input specification of the procedure $\Pi_{(i,h)}$ and let γ be an integer such that any real point $x = (x_1, \ldots, x_n)$ of the hypersurface $\{F = 0\}$ satisfies the condition $x_h \neq \gamma$. Since $\{F = 0\}_{\mathbb{R}}$ is by assumption compact , such an integer γ exists (recall the beginning of the design of the procedure $\Pi_{(i,h)}$).

Consider now the polynomial equation system

$$\begin{aligned} F(X) &= 0, \\ \frac{\partial F}{\partial X_h}(X)\Lambda + \gamma - X_h &= 0 \end{aligned} \tag{23}$$

and observe that it admits only smooth solutions in $\mathbb{A}^n \times \mathbb{A}^1$ and that its equations generate in $\mathbb{R}[X, \Lambda]$ the trivial or a radical complete intersection ideal. Moreover, observe that the connected components of the real solutions of (23) correspond to the generically F–regular connected components of $\{F = 0\}_{\mathbb{R}}$.

By means of the already mentioned algorithm of [4], Theorem 11 and [5], Theorem 13 we may find for each connected component of

$$\{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid F(x) = 0,\ \frac{\partial F}{\partial X_h}(x)\lambda + \gamma - x_h = 0\}$$

a real algebraic sample point and therefore also for each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$.

For given $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$ the equations (23) form part of the system (20) which is solved by the procedure $\Pi_{(i,h)}$. Without the larger context of the incidence varieties $H_i^{(h,\gamma)}$ and $H_i^{(h)}$ and their real traces, this procedure seems to be arbitrary and depending on the position of the variables $X_1, \ldots, X_n$ and its complexity behavior appears as completely unrelated to the geometry of the complex hypersurface $\{F = 0\}$ and the real variety $\{F = 0\}_{\mathbb{R}}$.

Thanks to the notion of bipolar varieties we become now aware that this is not the case (see Theorem 18 and Observation 19).

5 Walks

We are now going to develop a common view for the procedures $\Pi_{(i,h)}, 1 \leq i \leq n-1,\ 1 \leq h \leq n-i$ described in Section 4 for the task of finding smooth points in possibly singular, real compact hypersurfaces, and the algorithms developed in [2], [3], [4] and [5] for the case of smooth real complete intersection varieties.

Let us fix a polynomial $F \in \mathbb{Q}[X]$ and suppose without loss of generality that the hypersurface $\{F = 0\}$ contains an F–regular real point .

Let $1 \leq i \leq n-1,\ 1 \leq h \leq n-i$ and a suitable integer $\gamma \in \mathbb{N}$ be given. We first analyze the procedure $\Pi_{(i,h)}$ on input σ, where σ is an essentially division–free arithmetic circuit in $\mathbb{Q}[X]$ representing the polynomial F, while $\{F = 0\}_{\mathbb{R}}$ is supposed to be compact with $\{F = 0\}_{\mathbb{R}} = (\{F = 0\}_{X_h - \gamma})_{\mathbb{R}}$.

On input σ we may interpret $\Pi_{(i,h)}$ as a computation which starts with the variety $H_i^{(h,\gamma)}$ defined by the system (5) and "walks down" through the localized bipolar varieties $(\mathcal{B}_{(i,h,j;\gamma)})_{X_h - \gamma}$, beginning with $j := n-1$ and ending with $j := 1$.

In view of Lemma 3 we may interpret the procedure $\Pi_{(i,h)}$ on input σ alternatively as a computation that starts with the variety $H_i^{(h)}$ defined by the system (3) and walks in reverse mode through the *non–generic* dual polar varieties of $H_i^{(h)}$ defined for $1 \leq j \leq n-1$ as follows:

We replace in the $(((n-i)(n+1)+j+1)\times((n-i)(n+1)+n))$–matrix $T_{(i,h,(n-i)n-i+j)}$ introduced at the beginning of Section 4 the rows number $n+j+1,\ldots,(n-i)(n+1)+j$ by unit vectors whose entries are all zero, except one entry of value 1 at the place of the column of $T_{(i,h,(n-i)n-i+j)}$ which corresponds to one of the indeterminates $B_{k,l}, 1 \leq k \leq n-i, 1 \leq l \leq n$ with $(k,l) \notin \{(h,n-i+1),\ldots,(h,n)\}$.

The points of $(H_i^{(h)})$, where the rank of this new matrix is not maximal, form a dual polar variety of $(H_i^{(h)})$ which is non–generic (in fact *meagerly generic* in the sense of [6]). The computation, which represents the alternative interpretation of the procedure $\Pi_{(i,h)}$ on input σ, cuts $(H_i^{(h)})_{X_h-\gamma}$ and the intersections of $(H_i^{(h)})_{X_h-\gamma}$ with the dual polar varieties obtained in this way, by the affine hyperplanes $\{B_{k,l}-\beta_{k,l}=0\}$ with $1 \leq k \leq n-i, 1 \leq l \leq n$, $(k,l) \notin \{(h,n-i+1),\ldots,(h,n)\}$, where $\beta_{k,l}$ is defined as $\beta_{k,l} := 0$ for $k \neq l$, $\beta_{k,k} := 1$ for $k \neq h$ and $\beta_{h,h} := \gamma$.

This construction yields algebraic varieties which are by Lemma 3 isomorphic to

$$(H_i^{(h;\gamma)})_{X_h-\gamma}, (\mathcal{B}_{(i,h,n-1;\gamma)}) \cap (H_i^{(h;\gamma)})_{X_h-\gamma}, \ldots, (\mathcal{B}_{(i,h,1;\gamma)}) \cap (H_i^{(h;\gamma)})_{X_h-\gamma}.$$

We are now going to analyze the main algorithm of [4, 5] in an analoguous way.

First, let us choose for each $1 \leq i \leq n-1$ a generic matrix $b_i = [b_{k,l}]_{\substack{1\leq k\leq n-i\\ 1\leq l\leq n}}$ of $\mathbb{Q}^{(n-i)\times n}$ such that all these matrices become "nested", i.e., for $1 < i \leq n-1$ the matrix b_i forms the first $n-i$ rows of the $((n-i+1)\times n)$–matrix b_{i-1}. The genericity condition for the matrices b_i, $1 \leq i \leq n-1$, will become clear by the context.

Suppose now again that F is given by an essentially division-free arithmetic circuit σ in $\mathbb{Q}[X]$. Under the assumption that the polynomial F is reduced and $\{F=0\}_{\mathbb{R}}$ is non–empty and smooth, this algorithm starts on input σ with the complex hypersurface $\{F=0\}$ and walks down for $h := 1$ trough the generic dual polar varieties

$$W_{\overline{K}(b_1^{(1)})} \supset \cdots \supset W_{\overline{K}(b_{n-1}^{(1)})}$$

associated with the rational matrices $b_1^{(1)},\ldots,b_{n-1}^{(1)}$ (observe that $h := 1$ is the only choice of h which satisfies the condition $1 \leq h \leq n-i$ for any index $1 \leq i \leq n-1$).

Alternatively, we may interprete this algorithm as a procedure that cuts the variety $H_{n-1}^{(1)}$ first with the hyperplanes $\{B_{k,l}-b_{k,l}=0\}, 1 \leq k \leq n-1$, $1 \leq l \leq n$, and then successively with the hyperplanes $\{\Theta_{n-1}=0\},\ldots,\{\Theta_2=0\}$.

Observe that for $1 < i \leq n-1$ the (locally closed) variety

$$\{(x, b', (\lambda : \vartheta), b'') \in (H_i^{(1)} \times \mathbb{A}^{(i-1)\times n}) \,|\, \mathrm{rk} \begin{bmatrix} b' \\ b'' \end{bmatrix} = \mathrm{rk} \begin{bmatrix} b' \\ b'' \end{bmatrix}^{(1)}(x) = n-1\}$$

is isomorphic to

$$H_{n-1}^{(1)} \cap \{\Theta_{n-1} = 0, \ldots, \Theta_{n-i+1} = 0\}$$

and that

$$H_i^{(1)} \cap \{B_{k,l} - b_{k,l} = 0 \,|\, 1 \leq k \leq n-i, 1 \leq l \leq n\}$$

is isomorphic to $W_{\overline{K}(b_i^{(1)})}$.

This shows that we have two different interpretations of essentially the same procedure.

We are now going to generalize this view as walks in the set

$$\Gamma_n := \{(i, h, j) \,|\, 1 \leq i \leq n-1, 1 \leq h \leq n-i, 1 \leq j \leq (n-i)(n+1)\}.$$

Roughly speaking, a walk $\mathcal{W}$ is given by a sequence

$$((i_1, h, j_1), \ldots, (i_m, h, j_m))$$

of "nodes" of Γ_n and a series of affine linear cuts. These cuts become subdivided in m disjoint packets. The first packet of cuts precedes node (i_1, h, j_1). The packet number $2 \leq k \leq m$ becomes inserted between node (i_{k-1}, h, j_{k-1}) and (i_k, h, j_k). The cuts fix arbitrary rational values for some (or none) of the indeterminates $B_{k,l}, 1 \leq k \leq n-i_1, 1 \leq l \leq n$, and value zero for some (or none) of the indeterminates $\Theta_2, \ldots, \Theta_{n-i_1}$.

For $1 \leq l \leq n$, let $S_l^{(i_1,h)}(\mathcal{W})$ be the variety obtained by intersecting $S_l^{(i_1,h)}$ with the cuts of $\mathcal{W}$ preceding the node (i_1, h, j_1). We require that these cuts have to be transversal, that these varieties are non-empty and equidimensional and that for $1 < l \leq n$ the condition

$$\dim S_{l-1}^{(i_1,h)}(\mathcal{W}) = \dim S_l^{(i_1,h)}(\mathcal{W}) + 1$$

is satisfied.

The walk $\mathcal{W}$ becomes now interpreted by the following semantics:

The node (i_1, h, j_1) is interpreted as the variety $S_n^{(i_1,h)}(\mathcal{W})$. For $1 < k \leq m$ the node (i_k, h, j_k) becomes interpreted as the closed variety $\mathcal{W}_{(i_k,h,j_k)}$ which we are going to describe now.

For $j_k = (n-i_k)(n+1)$ let $W_{(i_k,h,j_k)} := H_{i_k}^{(h)}$ and for $1 \leq j_k < (n-i_k)(n+1)$ let $W_{(i_k,h,j_k)}$ be an appropriate, possibly non–generic dual polar variety of $H_{i_k}^{(h)}$, defined by the non–maximality of the rank of the $((n+j_k+1)\times((n-i_k)(n+1)+n))$–matrix

which we obtain similarly as before by replacing in $T_{(i_k,h,j_k)}$ suitable rows by suitable $(n-i_k)(n+1)+n)$–unit–vectors, all compatible according to Lemma 3 with the cuts of $\mathcal{W}$ up to the node (i_k,h,j_k). Then $\mathcal{W}_{(i_k,h,j_k)}$ is obtained by intersecting $W_{(i_k,h,j_k)}$ with the cuts of $\mathcal{W}$ up to the node (i_k,h,j_k) and taking the closure of this intersection in the corresponding ambient space.

We ask the walk $\mathcal{W}$ to fulfill the following requirements:

- $j_1 = (n-i_1)(n+1)$,
- $1 \leq i_1 \leq \cdots \leq i_m \leq n-1$,
- for $1 < k \leq m$ the variety $\mathcal{W}_{(i_k,h,j_k)}$ is non–empty and equidimensional,
- for $1 < k \leq m$ the variety $\mathcal{W}_{(i_{k-1},h,j_{k-1})}$ contains $\mathcal{W}_{(i_k,h,j_k)}$ and satisfies the condition $\dim \mathcal{W}_{(i_{k-1},h,j_{k-1})} = \dim \mathcal{W}_{(i_k,h,j_k)} + 1$,
- $\dim \mathcal{W}_{(i_m,h,j_m)} = 0$
- for $1 < k \leq m$ the cuts $\mathcal{W}$ between the nodes (i_{k-1},h,j_{k-1}) and (i_k,h,j_k) are transversal to $\mathcal{W}_{(i_{k-1},h,j_{k-1})}$ and define $\mathcal{W}_{(i_k,h,j_k)}$ as a subvariety of $\mathcal{W}_{(i_{k-1},h,j_{k-1})}$.

The varieties $\mathcal{W}_{(i_k,h,j_k)}$, $1 \leq k \leq m$, of the walk $\mathcal{W}$ have possibly to be localized by a suitable polynomial in order to satisfy these requirements.

For $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$ the procedure $\Pi_{(i,h)}$ produces on input σ a walk which we denote by $\mathcal{W}_{(i,h)}(F)$ (in fact, there are several, algorithmically equivalent, candidates for $\mathcal{W}_{(i,h)}(F)$).

Similarly, in case that F is reduced and $\{F=0\}_{\mathbb{R}}$ is smooth, the main algorithm of [4, 5] produces on input σ a characteristic walk which we denote by $\mathcal{W}_n(F)$.

Let $\mathcal{W}$ be an arbitrary walk in Γ_n with node sequence $((i_1,h,j_1),\ldots,(i_m,h,j_m))$. The (dual) degree $\delta(\mathcal{W})$ of $\mathcal{W}$ is defined as

$$\delta(\mathcal{W}) := \max\{\max\{\deg S_l^{(i,h)}(\mathcal{W}) \,|\, 1 \leq l \leq n\}, \max\{\deg \mathcal{W}_{(i_k,h,j_k)} \,|\, 1 \leq k \leq m\}\}.$$

Recall the algorithmic problem $(\mathbf{P})$ introduced in Section 4 and suppose that $\{F=0\}$ contains an F–regular real point. We say that the *walk* $\mathcal{W}$ *solves the real sample point problem* $(\mathbf{P})$ *for the equation* $F=0$ if the canonical projection of $(\mathcal{W}_{(i_m,h,j_m)})_{\mathbb{R}}$ into $\mathbb{A}^n_{\mathbb{R}}$ is a (finite) set of real algebraic sample points for the generically F–regular connected components of $\{F=0\}_{\mathbb{R}}$.

Suppose that the polynomial F is represented by an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ.

Applying the Kronecker algorithm to this situation we obtain the following result.

Theorem 21
Let notations and assumptions be as before and suppose that the walk $\mathcal{W}$ *solves the real sample point problem* $(\mathbf{P})$ *for the equation* $\{F = 0\}$. *Then* $\mathcal{W}$ *represents a computation in* $\mathbb{Q}$ *which starts from* σ *and uses* $O(L(nd)^{O(1)}\delta(\mathcal{W})^2)$ *arithmetic operations organized, with respect to the parameters of the arithmetic circuit* σ, *in non–scalar depth* $O(n(\ell + \log(nd)) \log \delta(\mathcal{W}))$ *and whose output encodes a finite set of real algebraic sample points for the generically* F*–regular connected components of* $\{F = 0\}_{\mathbb{R}}$. *The number and degree of these sample points is bounded by* $\delta(\mathcal{W})$.

Proof
The walk $\mathcal{W}$ represents a computation in $\mathbb{Q}$ that calculates from σ first a representation of (complex) algebraic points of $S_n^{(i_1,h)}(\mathcal{W})$ using $O(L(nd)^{O(1)}\delta(\mathcal{W})^2)$ arithmetic operations organized in non-scalar depth $O(n(\ell + \log(nd)) \log \delta(\mathcal{W}))$. The number and degree of these points is bounded by $\delta(\mathcal{W})$. The assumption that $\mathcal{W}$ solves the real sample point problem $(\mathbf{P})$ for the equation $F = 0$ is then used to extend this computation to a representation of a finite set of real algebraic sample points for the generically F–regular connected components of $\{F = 0\}_{\mathbb{R}}$. The number and degree of these sample points is bounded by $\delta(\mathcal{W})$. □

Here, two remarks are at order.

- For $1 \leq i \leq n-1, 1 \leq h \leq n-i$ and $\{F = 0\}_{\mathbb{R}}$ compact containing an F–regular point, Theorem 18, Observation 19 and Theorem 21 applied to $\mathcal{W}_{(i,h)}(F)$ are compatible with Theorem 21 if we consider the constant γ, produced by the procedure $\Pi_{(i,h)}$ on input σ, as precomputed. Observe that we have under this condition $\delta(\mathcal{W}_{(i,h)}(F)) = \delta_{(i,h;\gamma)}$.

  Similarly Theorem 21 is compatible with [4], Theorem 11 and [5], Theorem 13, if we identify $\delta(\mathcal{W}_n(F))$ with the degree of the real interpretation of F in [4] and [5] .

- We have no general criterion at hand to decide which real point finding algorithms for hypersurfaces are of best intrinsic complexity. However, if we limit our attention to the algorithms $\Pi_{(i,h)}, 1 \leq i \leq n-1$, $1 \leq h \leq n-i$, then [6], Theorem 13 implies that $\Pi_{(n-1,1)}$ has the best intrinsic sequential complexity which in worst case is of order $d^{O(n)}$. This means that for $\{F = 0\}_{\mathbb{R}}$ compact, the Rabinowitsch trick inspired algorithm, which consists in solving the polynomial equation system (22) subject to the open condition $X_h - \gamma \neq 0$ for suitable $\gamma \in \mathbb{N}$, has a fairly good intrinsic complexity despite of its coordinate-dependent, extrinsic aspect.

  On the other hand, the algorithm $\Pi_{(n-1,1)}$ comes very close to the "critical point method" applied to point finding in real hypersurfaces (see [1]and [50]).

6 The classic model

Again, as in Section 4, we consider the problem $(\mathbf{P})$ of finding a finite set of real algebraic sample points for the generically F–regular connected components of $\{F = 0\}_{\mathbb{R}}$, where $F \in \mathbb{Q}[X]$ is a circuit represented polynomial. However, in contrast to Theorem 18 and Observation 19, here we do not require that the real hypersurface $\{F = 0\}_{\mathbb{R}}$ is compact.

We are now going to revise the notions and results of Section 3, 4 and 5 from the point of view of *classic polar* varieties. Our fundamental aim is to sketch an algorithm of intrinsic complexity which solves the problem $(\mathbf{P})$ without the requirement of compactness on $\{F = 0\}_{\mathbb{R}}$. Since proofs are almost textually the same as in Sections 3, 4 and 5 we shall omit them here.

Let notations be the same as in Section 3.1 and, for the moment, let us fix indices $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$. We denote by $\widehat{H}_i^{(h)}$ the (locally closed) subvariety of $\mathbb{T}_i^{(h)}$ defined by

$$\widehat{H}_i^{(h)} := \{(x, b, (\lambda : \vartheta)) \in \mathbb{T}_i^{(h)} \mid F(x) = 0,\ \mathrm{rk}\, b = n - i,\ J(F)(x)\lambda + b^T\vartheta^T = 0\}.$$

We have the following avatar of Proposition 6.

Proposition 22
The algebraic subvariety $\widehat{H}_i^{(h)}$ of $\mathbb{T}_i^{(h)}$ is $\mathbb{R}$–definable and empty or equidimensional and smooth of dimension $(n-i)(n+1)-1$.

Let $\widehat{D}_{(i,h)}$ be the closed subvariety of $\mathbb{T}_i^{(h)}$ defined by the condition $\mathrm{rk}\, B_i < n-i$, where B_i is the $((n-i)\times n)$–matrix $B_i := [B_{k,l}]_{\substack{1\leq k\leq n-i\\ 1\leq l\leq n}}$.

Then the equations of the system

$$F(X) = 0, \qquad \frac{\partial F}{\partial X_l}\Lambda + \sum_{1\leq k\leq n-i} B_{k,l}\Theta_k = 0, \quad 1 \leq l \leq n, \tag{24}$$

intersect transversally at any of their common solutions in $\mathbb{T}_i^{(h)} \setminus \widehat{D}_{(i,h)}$. The algebraic variety $\widehat{H}_i^{(h)}$ consists exactly of these solutions.

The set $\widehat{H}_i^{(h)}$, interpreted as incidence variety between $\mathbb{A}^n$ and $\mathbb{A}^{(n-i)\times n} \times \mathbb{P}^{n-i}$, dominates the locus of all F–regular points of the complex hypersurface $\{F = 0\}$. The real variety $(\widehat{H}_i^{(h)})_{\mathbb{R}}$ is non–empty if and only if $\{F = 0\}$ contains an F–regular real point.

For $b \in \mathbb{A}^i$ with $b = (b_{n-i+1}, \ldots, b_n)$ we denote by $b_{(i,h)}$ the complex $((n-i)\times n)$–

matrix

$$b_{(i,h)} := \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & \ddots & & & & & \vdots & \\ 0 & \cdots & 1 & \cdots & 0 & b_{n-i+1} & \cdots & b_n \\ & & & \ddots & & & \vdots & \\ 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

With other words, we write $b_{(i,h)} := b_{(i,h;1)}$.

Let $\widehat{\mathcal{H}}_i^{(h)}$ be the $\mathbb{R}$–definable subvariety of $\mathbb{N}_i^{(h)}$ defined by

$$\widehat{\mathcal{H}}_i^{(h)} := \{(x, b, (\lambda : \vartheta)) \in {\mathbb{N}_i}^{(h)} \mid F(x) = 0,\ J(F)(x)^T\lambda + b_{(i,h)}^T\vartheta^T = 0\}.$$

The counterpart of Proposition 7 is now the following result.

Proposition 23
Outside of the locus given by $\Theta_h = 0$, the polynomial equations of the system

$$F(X) = 0,$$

$$\frac{\partial F(X)}{\partial X_l}\Lambda + \Theta_l = 0,\ 1 \leq l \leq n-i, \tag{25}$$

$$\frac{\partial F(X)}{\partial X_l}\Lambda + B_{h,l}\Theta_h = 0,\ n-i < l \leq n,$$

intersect transversally at each of their common solutions in ${\mathbb{N}_i}^{(h)}$.

Moreover, the polynomial equation system (25) and the open condition $\Theta_h \neq 0$ define the algebraic variety $\widehat{\mathcal{H}}_i^{(h)}$ which is therefore empty or equidimensional of dimension $n-1$. The varieties $\widehat{\mathcal{H}}_i^{(h)}$ and $(\widehat{\mathcal{H}}_i^{(h)})_{\mathbb{R}}$ dominate the locus of all points x of $\{F = 0\}$ and $\{F = 0\}_{\mathbb{R}}$ satisfying the conditions $\frac{\partial F}{\partial X_h}(x) \neq 0$. In particular, $(\widehat{\mathcal{H}}_i^{(h)})_{\mathbb{R}}$ is non–empty and equidimensional of dimension $n-1$ if and only if the hypersurface $\{F = 0\}$ contains a real point x with $\frac{\partial F}{\partial X_h}(x) \neq 0$. The polynomials contained in (25) generate in $\mathbb{R}[X, B_{n-i+1}^*, \ldots, B_n^*, \Lambda, \Theta]_{\Theta_h}$ the trivial ideal or form a reduced regular sequence.

In the classic model, we define for $1 \leq j \leq (n-i)(n+1)-1$ the *large bipolar varieties* $\widehat{\mathfrak{B}}_{(i,h,j)}$ of the equation $F = 0$ associated with the indices i, h and j as the $((n-i)(n+1)-j)$th generic dual polar variety of $\widehat{H}_i^{(h)}$. In the same vein, we define in the classic model for $1 \leq j \leq n-1$ the *small bipolar varieties* $\widehat{\mathcal{B}}_{(i,h,j)}$ of $F = 0$ associated with the indices i, h and j as the $(n-j)$th generic dual polar variety of $\widehat{\mathcal{H}}_i^{(h)}$.

We have the following degree estimates for the bipolar varieties in the classic model:

$$\deg \widehat{\mathcal{B}}_{(i,h,j)} \leq \deg \widehat{\mathfrak{B}}_{(i,h,(n-i)n-i+j)}, \tag{26}$$

$$\deg \widehat{\mathfrak{B}}_{(i,h,j)} = (nd)^{O((n-i)n)}, \tag{27}$$

and

$$\deg \widehat{\mathcal{B}}_{(i,h,j)} = (nd)^{O(n)}, \tag{28}$$

For $1 \leq l \leq n$ let $\widehat{S}_l^{(i,h)}$ be the Zariski–closure of the locally closed subset of ${\mathbb{T}_i}^{(h)}$ defined by the conditions

$$F(X) = 0, \qquad \frac{\partial F}{\partial X_{t'}}(X)\,\Lambda + + \sum_{\substack{1 \leq k \leq n-i \\ k \neq h}} B_{k,t'}\,\Theta_k = 0, \quad 1 \leq t' \leq l,$$

$$rk\ B_i = n - i,$$

and let $\widehat{{S^*}_l^{(i,h)}}$ be the closed subvariety of ${\mathbb{T}_i}^{(h)}$ defined by the conditions

$$\begin{aligned} F(X) &= 0, \\ \frac{\partial F}{\partial X_h}(X)\Lambda + \Theta_h &= 0, \\ \frac{\partial F}{\partial X_{t'}}(X)\Lambda + \Theta_{t'} &= 0, \\ 1 \leq t' \leq \min\{l, n-i\},\ t' &\neq h, \\ \frac{\partial F}{\partial X_{t'}}(X)\Lambda + B^*_{t'}\,\Theta_h &= 0, \\ n - i < t' &\leq l. \end{aligned}$$

The Bézout–Inequality implies the estimates

$$\deg \widehat{S}_l^{(i,h)} \leq d^{l+1} \tag{29}$$

and

$$\deg \widehat{{S^*}_l^{(i,h)}} \leq d^{l+1}. \tag{30}$$

We associate now with i, h and the real interpretation of the polynomial equation $F = 0$ the following discrete parameters:

$$\widehat{\delta}_{(i,h)} := \max\{\{\deg \widehat{S}_l^{(i,h)} \mid 1 \leq l \leq n\}, \max\{\deg \widehat{\mathfrak{B}}_{(i,h,j)} \mid 1 \leq j \leq (n-i)(n+1) - 1\}\}$$

and

$$\widehat{\delta^*}_{(i,h)} := \max\{\{\deg \widehat{{S^*}_l^{(i,h)}} \mid 1 \leq l \leq n\}, \max\{\deg \widehat{\mathfrak{B}}_{(i,h,j)} \mid 1 \leq j \leq n-1\}\}.$$

We observe that the parameter $\widehat{\delta}_{(i,h)}$ remains invariant under arbitrary linear transformations of the coordinates $X_1, \ldots, X_n$ by non-singular complex $(n \times n)$–matrices,

whereas the parameter $\widehat{\delta^*}_{(i,h)}$ is coordinate–dependent. Therefore we call $\widehat{\delta}_{(i,h)}$ the *coordinate–independent degree* of the real interpretation of the equation $F = 0$ associated with i and h. In the same vein we call $\widehat{\delta^*}_{(i,h)}$ the *coordinate–dependent degree* of the real interpretation of $F = 0$ associated with i, h.

Taking into account the estimate (26) we infer from the Bézout–Inequality that

$$\widehat{\delta^*}_{(i,h)} \leq \widehat{\delta}_{(i,h)}$$

holds. From (27) – (30) we deduce for $d \geq 1$ the extrinsic estimates

$$\widehat{\delta}_{(i,h)} = (nd)^{O((n-i)n)}$$

and

$$\widehat{\delta^*}_{(i,h)} = (nd)^{O(n)}.$$

We consider now again the algorithmic problem $(\mathbf{P})$ introduced in Section 4.

Let $1 \leq i \leq n-1$, $1 \leq h \leq n-i$. We are going to sketch a procedure $\widehat{\Pi}_{(i,h)}$ of intrinsic complexity which solves the problem $(\mathbf{P})$ for any circuit represented polynomial $F \in \mathbb{Q}[X]$. Unlike the algorithm $\Pi_{(i,h)}$ of Section 4, the input specification of the procedure $\widehat{\Pi}_{(i,h)}$ does not require that $\{F = 0\}_{\mathbb{R}}$ is compact.

Let Z be a new indeterminate.

Procedure $\widehat{\Pi}_{(i,h)}$

Input: An essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ having a single output node.

Input Specification: The circuit σ represents a polynomial $F \in \mathbb{Q}[X]$ of positive degree d. The indeterminates $X_1, \ldots, X_n$ are in general position with respect to the complex hypersurface $\{F = 0\}$.

Output: The procedure $\widehat{\Pi}_{(i,h)}$ accepts the input σ if $\{F = 0\}$ contains a real F–regular point. If this is the case, the procdure returns a circuit representation of the coefficients of $n+1$ polynomials $\widehat{P}, \widehat{G}_1, \ldots, \widehat{G}_n \in \mathbb{Q}[Z]$ satisfying for $\widehat{G} := (\widehat{G}_1, \ldots, \widehat{G}_n)$ the following output specification:

- $\widehat{P}$ is monic and separable,
- $\deg \widehat{G} < \deg \widehat{P} \leq \deg \widehat{\mathcal{B}}_{(i,h,1)}$.
- The zero–dimensional complex affine variety, $\{\widehat{G}(z) \mid z \in \mathbb{A}^1,\ \widehat{P}(z) = 0\}$ contains an F–regular, real algebraic sample point of each generically F–regular connected component of $\{F = 0\}_{\mathbb{R}}$. In order to represent these sample points, an encoding "à la Thom" of the real zeros of the polynomial $\widehat{P}$ is returned.

The design of the procedure $\widehat{\Pi}_{(i.h)}$ follows the same lines and uses practically the same arguments as the design of the algorithm $\Pi_{(i,h)}$ in Section 4. Unlike the main subroutine of $\Pi_{(i,h)}$ which solves the system (20), the main subroutine of $\widehat{\Pi}_{(i,h)}$ solves the polynomial equation system

$$\begin{aligned} F(X) &= 0, \\ \frac{\partial F}{\partial X_h}(X)\Lambda + 1 &= 0, \\ \frac{\partial F}{\partial X_l}(X)\Lambda + \Theta_l &= 0, \\ 1 \leq l \leq n-i,\ l \neq h, & \\ \frac{\partial F}{\partial X_l}(X)\Lambda + B_l^* &= 0, \\ n-i < l \leq n. & \end{aligned}$$

The procedures $\widehat{\Pi}_{(i,h)}$ $1 \leq i \leq n-1$, $1 \leq h \leq n-i$ give rise to the following complexity result, in the spirit of Theorem 18.

Theorem 24
Let n, d, i, h, $\widehat{\delta}$, L, ℓ be natural numbers with $d \geq 1$, $1 \leq i \leq n-1$ and $1 \leq h \leq n-i$. Let $X_1, \ldots, X_n$ and Z be indeterminates over $\mathbb{Q}$ and let $X := (X_1, \ldots, X_n)$.

There exists an arithmetic network $\widehat{\mathcal{N}}$ over $\mathbb{Q}$, depending on certain parameters and having size

$$L\,(n\,d)^{O(1)}\,\widehat{\delta}^{\,2} = (n\,d)^{\,O(n)}$$

and non–scalar depth

$$O(n(\ell + \log(nd)) \log \widehat{\delta}) = O(n^2 \log(dn) \log d),$$

such that $\widehat{\mathcal{N}}$ has for suitable specializations of its parameters the following properties:

Let $F \in \mathbb{Q}[X]$ be a polynomial of degree d and assume that F is given by an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ. Suppose that the variables $X_1, \ldots, X_n$ are in general position with respect to the complex hypersurface $\{F = 0\}$ and that the coordinate–dependent degree of the real interpretation of $F = 0$ associated with i and h is bounded by $\widehat{\delta}$ (in symbols: $\widehat{\delta^*}_{(i,h)} \leq \widehat{\delta}$).

Then the algorithm represented by the arithmetic network $\widehat{\mathcal{N}}$ starts from the circuit σ as input and decides whether the hypersurface $\{F = 0\}$ contains a real F–regular point. If this is the case, the algorithm produces a circuit representation of the coefficients of $n+1$ polynomials $\widehat{P}, \widehat{G}_1, \ldots, \widehat{G}_n \in \mathbb{Q}[Z]$ satisfying for $\widehat{G} := (\widehat{G}_1, \ldots, \widehat{G}_n)$ the following output specification:

- $\widehat{P}$ *is monic and separable,*

- $\deg \widehat{G} < \deg \widehat{P} \leq \widehat{\delta}$,

- *the complex affine variety* $\{\widehat{G}(z) \mid z \in \mathbb{A}^1, \widehat{P}(z) = 0\}$ *is zero–dimensional and contains a real* F*–regular algebraic sample point for each generically* F*–regular connected component of* $\{F = 0\}_{\mathbb{R}}$.

In order to represent these sample points the algorithm returns an encoding "à la Thom" of the real zeros of the polynomial $\widehat{P}$.

In contrast to Theorem 18, it is not anymore required in Theorem 24 that $\{F = 0\}_{\mathbb{R}}$ is a compact set. In analogy to Observation 19, we have the following result.

Observation 25
There exists an universal constant $c > 0$ *(independent of the parameters* $n, d, h, \widehat{\delta}$, L, ℓ*) such that the statement of Theorem 12 remains true if we drop the hypothesis that the indeterminates* $X_1, \ldots, X_n$ *are in general position with respect to the complex hypersurface* $\{F = 0\}$ *and if we assume that* $\min\{(n\,d)^{cn}, \widehat{\delta}_{(i,h)}\} \leq \widehat{\delta}$ *holds.*

As in Section 4, Theorem 12 and Observation 25 can be transformed to the uniform probabilistic computation model (compare Observation 20).

Furthermore, let us remark that the algorithms $\widehat{\Pi}_{(i,h)}\ 1 \leq i \leq n-1,\ 1 \leq h \leq n-i$, may be reinterpreted as a real solution method for the following, Rabinowitsch trick inspired polynomial equation system:

$$F(X) = 0,$$
$$\frac{\partial F}{\partial X_h}(X)\Lambda + 1 = 0.$$

Our comments on walks in Section 5 can be transfered mutatis mutandis to the context of the classic model in order to prove an analogue statement as Theorem 21.

Summing up, the results and arguments obtained in the dual and the classic model are almost textually the same. The main differences are the following:

- The real point finding procedures $\Pi_{(i,h)},\ 1 \leq i \leq n-1,\ 1 \leq h \leq n-i$, require that the real trace $\{F = 0\}_{\mathbb{R}}$ of the input equation $F = 0$ is compact, whereas the procedures $\widehat{\Pi}_{(i,h)}$ do not require such an assumption.

- The analogues of the statements on the non–emptyness of real dual polar varieties of Section 2, namely Theorem 1 and Corollary 2, are wrong for classic polar varieties. This has to taken into account for a possible reformulation of Theorem 12 and Corollary 13 in the classic model.

7 Witness for real inequalities

At the end of Section 3 we addressed the problem to find efficiently for a given consistent system of strict inequalities of arithmetic circuit represented polynomials of $\mathbb{Q}[X]$ a rational witness, i.e., a point $x \in \mathbb{Q}^n$ which satisfies all these inequalities.

In this section we focus on this problem in case of a single inequality. Moreover, since the problem of finding rational witnesses even for a single inequality involves subtile height estimates from diophantine geometry, we limit our attention to the simpler problem of finding a *real algebraic* witness for a given consistent polynomial inequality.

For this purpose let us consider a squarefree polynomial $F \in \mathbb{Q}[X]$ of positive degree d. We suppose that F is given by an essentially division–free arithmetic circuit σ in $\mathbb{Q}[X]$ of size L and non–scalar depth ℓ.

We shall make use of the following fact.

Proposition 26
The following two conditions for the polynomial F are equivalent.

(i) The polynomial F changes its sign in $\mathbb{A}^n_{\mathbb{R}}$, i.e., there exist points $u, v \in \mathbb{A}^n_{\mathbb{R}}$ such that $F(u)F(v) < 0$ holds.

(ii) The real variety $\{F = 0\}_{\mathbb{R}}$ contains an F–regular point.

Proof
For F irreducible, Proposition 26 is an immediate consequence of [11], Théorème 4.5.1. This implies the equivalence of conditions (i) and (ii) for an arbitrary squarefree polynomial $F \in \mathbb{Q}[X]$ □

For the next result recall from Section 6 that $\widehat{\delta}_{(n-1,1)}$ denotes in the classic model the coordinate–independent degree of the equation $F = 0$ associated with $i := n-1$ and $h := 1$.

Theorem 27
Let notations and assumptions be as before. In the non–uniform deterministic or the uniform probabilistic complexity model there exists an algorithm which on input σ decides whether F changes its sign in $\mathbb{A}^n_{\mathbb{R}}$ and, if this is the case, produces the Thom encoding of two real algebraic witness points $u, v \in \mathbb{A}^n_{\mathbb{R}}$ satisfying the conditions $F(u) > 0$ and $F(v) < 0$.

The algorithm uses $L(n\,d)^{O(1)}(\widehat{\delta}_{(n-1,1)})^2 = (n\,d)^{O(n)}$ arithmetic operations in $\mathbb{Q}$ organized in non–scalar depth $O(n(\ell + \log(n\,d)) \log \widehat{\delta}_{(n-1,1)})$.

Proof
We apply the algorithm $\widehat{\Pi}_{(n-1,1)}$ of Section 6 to the input circuit σ which represents the polynomial F.

The algorithm decides first whether $\{F = 0\}_{\mathbb{R}}$ contains an F–regular point. From Proposition 26 we know that this is the case if and only if the polynomial F changes its sign in $\mathbb{A}^n_{\mathbb{R}}$.

Suppose we get a positive answer. Then the algorithm $\widehat{\Pi}_{(n-1,1)}$ produces the Thom encoding of an F–regular real algebraic point $x = (x_1, \ldots, x_n)$ of $\{F = 0\}_{\mathbb{R}}$ such that the degree of x over $\mathbb{Q}$ is at most $\widehat{\delta}_{(n-1,1)}$. We determine now the signs of $\frac{\partial F}{\partial X_1}(x), \ldots, \frac{\partial F}{\partial X_n}(x)$. Since x is F–regular we may suppose without loss of generality $\frac{\partial F}{\partial X_1}(x) > 0$.

We consider the univariate polynomial $G(X_1) \in \mathbb{R}[X_1]$, $G(X_1) := F(X_1, x_2, \ldots, x_n)$. From $F(x) = 0$ and $\frac{\partial F}{\partial X_1}(x) > 0$ we deduce $G(x_1) = 0$ and $\frac{dG}{dX_1}(x_1) > 0$. With other words, $G(X_1)$ changes its sign at x_1. Applying any of the most classic procedures for the real root finding of univariate polynomials over $\mathbb{R}$ to this situation, we may decide whether $G(X_1)$ has zeros in the intervals $(-\infty, x_1)$ and (x_1, ∞). If for example $G(X_1)$ has no zero in (x_1, ∞) we put $u := (x_1 + 1, x_2, \ldots, x_n)$. Similarly, we put $v := (x_1 - 1, x_2, \ldots, x_n)$ if $G(X_1)$ has no zero in $(-\infty, x_1)$. For the sake of simplicity let us suppose that $G(X_1)$ has zeros in $(-\infty, x_1)$ as well as in (x_1, ∞). Then we compute the roots $a < x_1 < b$ of $G(X_1)$ which come closest to x_1 and put $u := (\frac{x_1+b}{2}, x_2, \ldots, x_n)$ and $v := (\frac{x_1+a}{2}, x_2, \ldots, x_n)$. Observe that we have in any case $F(u) > 0$ and $F(v) < 0$.

For the decision whether the polynomial F changes its sign in $\mathbb{A}^n_{\mathbb{R}}$, the algorithm requires $L(n\,d)^{O(1)}(\widehat{\delta}_{(n-1,1)})^2$ arithmetic operations in $\mathbb{Q}$ organized in non–scalar depth $O(n(\ell + \log(n\,d)) \log \widehat{\delta}_{(n-1,1)})$. If this is the case the procedure $\widehat{\Pi}_{(n-1,1)}$ produces the real algebraic points u and v as witnesses for the strict inequalities $F > 0$ and $F < 0$. The degrees of u and v over $\mathbb{Q}$ are at most $\widehat{\delta}_{(n-1,1)}$. This implies that we can find u and v using at most $L(n\,d)^{O(1)}(\widehat{\delta}_{(n-1,1)})^2$ arithmetic operations in $\mathbb{Q}$ organized, with respect to the parameters of the arithmetic circuit σ, in non–scalar depth $O(n(\ell + \log(n\,d)) \log \widehat{\delta}_{(n-1,1)})$. This yields the complexity bounds of the theorem. □

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