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Preprint Nr. 2010 - 12

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On the existence of weak solutions to a coupled system of two turbulent flows

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Abstract In this paper, we study a model problem for the stationary turbulent motion of two fluids in disjoint bounded domains Ω_1 and Ω_2 such that $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \varnothing$. The specific difficulty of this problem arises from the boundary condition which characterizes the interaction of the fluid motions along Γ.

We prove the existence of a weak solution to the problem under consideration which is more regular than the solution obtained in [3]. Moreover, we establish some regularity results for any weak solution. Our discussion is heavily based on the results in appendices 1 and 2 which seem to be of independent interest.

1. Introduction

Let Ω_1 and Ω_2 be bounded domains in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) such that

$$\Omega_1 \cap \Omega_2 = \varnothing, \qquad \Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \varnothing,$$

$$\partial\Omega_i \text{ Lipschitz}, \qquad \Gamma \subset \partial\Omega_i \text{ relatively open } (i = 1, 2).$$

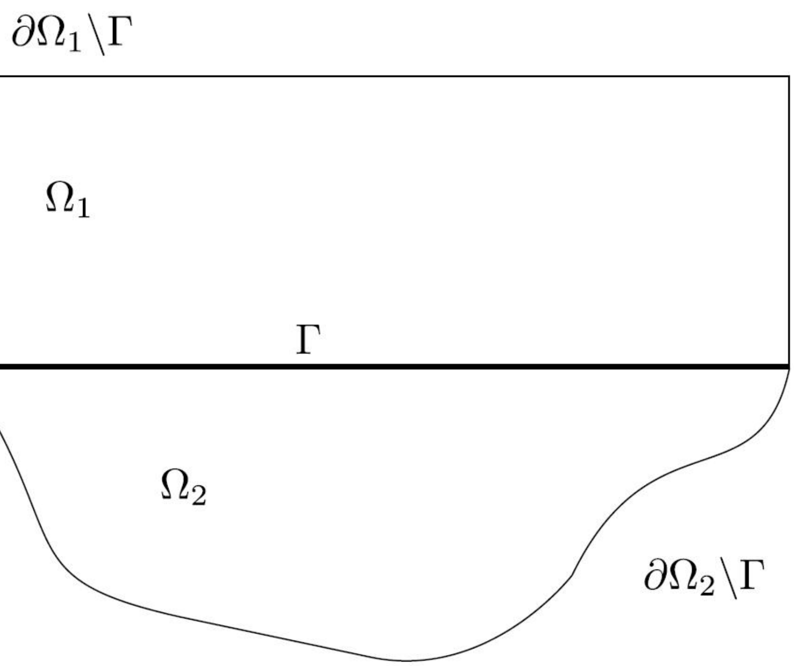

Figure 1

We consider the following system of PDEs in Ω_i $(i = 1, 2)$

$$-\mathrm{div}(\nu_i(k_i)\boldsymbol{D}(\boldsymbol{u}_i)) + \nabla p_i = \boldsymbol{f}_i \quad \text{in } \Omega_i, \tag{1.1}$$

$$\mathrm{div}\, \boldsymbol{u}_i = 0 \quad \text{in } \Omega_i, \tag{1.2}$$

$$-\Delta k_i = \mu_i(k_i)|\boldsymbol{D}(\boldsymbol{u}_i)|^2 \quad \text{in } \Omega_i \tag{1.3}$$

where

$\boldsymbol{u}_i = (u_{i1}, \dots, u_{id})$ = mean velocity, p_i = mean pressure,
k_i = mean turbulent kinetic energy

are the unknown functions. For a vector field $\boldsymbol{u} = (u_1, \dots, u_d)$ we use the notations

$$\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top), \qquad |\boldsymbol{D}(\boldsymbol{u})|^2 = \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u}).$$

The coefficients ν_i and μ_i are assumed to be uniformly bounded. We notice that the special case $\nu_i(k_i) = \nu_{i0} + \nu_{iT}(k_i)$ where

$\nu_{i0} = \text{const} > 0$ — dynamic viscosity of the fluid,

$0 \le \nu_{iT}(k_i) \le \text{const}$ — eddy viscosity,

as well as the two cases

$$\mu_i(k_i) = \nu_i(k_i) \quad \text{or} \quad \mu_i(k_i) = \nu_{iT}(k_i)\ ^{1)}$$

are included in our discussion.

Finally, $\boldsymbol{f}_i$ represents an external force in Ω_i.

The system (1.1) - (1.3) belongs to the class of one-equation RANS (Reynolds Averaged Navier-Stokes) models. The triple $(\boldsymbol{u}_i, k_i, p_i)$ $(i = 1, 2)$ characterizes the stationary turbulent motion of a viscous fluid in Ω_i, where the convection term in the fluid equations as well as in the turbulent kinetic energy equations is neglected.

A discussion of RANS models can be found in [2; pp. 304-316], [12; pp. 182-196, 216-252], [18; 319-337] (with $\mu(k) = \nu_T(k)$), and in [14] within the context of oceanography. Related problems (but without turbulence effects) are studied in [17]. The stationary turbulent motion of a fluid with unbounded eddy viscosities of the type $\nu_T(k) = c_0\sqrt{k}$ (Kolmogorov 1942, Prandtl 1945) has been studied in [7] and [13]. ■

We complete (1.1) - (1.3) by the following boundary conditions which link both systems of PDEs in Ω_1 and Ω_2 through the interface Γ:

$$\text{(1.4)} \qquad \begin{cases} \boldsymbol{u}_i = \boldsymbol{0} \quad \text{on} \quad \partial\Omega_i \setminus \Gamma, \\ \boldsymbol{u}_i \cdot \boldsymbol{n}_i = 0 \quad \text{on} \quad \Gamma, \\ \nu_i(k_i)(\boldsymbol{D}(\boldsymbol{u}_i)\boldsymbol{n}_i)_{\boldsymbol{\tau}} + |\boldsymbol{u}_i - \boldsymbol{u}_j|(\boldsymbol{u}_i - \boldsymbol{u}_j)_{\boldsymbol{\tau}} = 0 \quad \text{on} \quad \Gamma \ \ (i \neq j), \end{cases}$$

1) If $\mu_i = \nu_i$, system (1.1), (1.3) has some common features with the thermistor equations (see, e. g., Howison, S. D.; Rodrigues, J. F.; Shillor, M., *Stationary solutions to the thermistor problem.* J. Math. Analysis Appl. 174 (1993), 573-588; Cimatti, G., *The stationary thermistor problem with a current limiting device.* Proc. Royal Soc. Edinb. 116A (1990), 79-84). We notice that the assumption $\mu_i = \nu_i$ significantly simplifies the arguments of the passage to the limit in (1.3) with approximate solutions (cf. [7] and Gallouët, T.; Lederer, J.; Lewandowski, R.; Murat, F.; Tartar, L., *On a turbulent system with unbounded eddy viscosities.* Nonlin. Analysis 52 (2003), 1051-1068).

$$(1.5) \qquad k_i = 0 \quad \text{on} \quad \partial\Omega_i \setminus \Gamma, \qquad\qquad k_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \quad \text{on} \quad \Gamma$$

where

$$\boldsymbol{n}_i \;=\; (n_{i1}, \ldots, n_{id}) \;=\; \text{unit outward normal on } \partial\Omega_i,$$

$$\boldsymbol{\xi}_{\boldsymbol{\tau}} \;=\; \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{n}_i)\boldsymbol{n}_i \qquad (\boldsymbol{\xi} \in \mathbb{R}^d),$$

$$(1.6) \qquad 0 \le G_i(t) \le c_0 t, \quad |G_i(t) - G_i(\bar{t})| \le c_0|t - \bar{t}| \quad \forall\, t, \bar{t} \in [0, +\infty) \quad (c_0 = \text{const} > 0)$$

$(i = 1, 2)$. In (1.4), the boundary conditions on the (fixed) interface Γ model the situation when the interface is nonpermeable for both fluids which, however, do not completely adhere to the interface. Along this interface the fluids exhibit a partial slip which produces kinetic energy (cf. [3; pp. 69-73] for more details).

The boundary value problem (1.1) - (1.5) (with $\nabla\boldsymbol{u}_i$ in place of $\boldsymbol{D}(\boldsymbol{u}_i)$ in (1.1), (1.3) and (1.4)) has been investigated in [3]. In this paper, the authors prove the existence of a solution $\{\boldsymbol{u}_1, k_1, p_1; \boldsymbol{u}_2, k_2, p_2\}$ to (1.1)-(1.5) where (1.1) is satisfied in the usual weak sense (cf. our definition in Section 2), while (1.4) is satisfied in the sense of transposition of the Laplacean $-\Delta$ under zero boundary conditions. The aim of the present paper is to give an existence proof for a weak solution to (1.1)-(1.5) (in the sense of the definition of Section 2). Our proof is shorter and more transparent than the one in [3]. Moreover, we establish some regularity results on $(\boldsymbol{u}_i, k_i)$.

Our paper is organized as follows. In Section 2, we introduce the notion of weak solution $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ to (1.1)-(1.5). By appealing to standard references, we show the existence of a pressure p_i associated with the pair $(\boldsymbol{u}_i, k_i)$ $(i = 1, 2)$. Section 3 contains our main existence result. It's proof is based on a straightforward application of the Schauder [2)] fixed point theorem. A higher integrability result on $\nabla\boldsymbol{u}_i$ is established in Section 4. From this result we deduce the local existence of the second order derivatives of k_i. In Appendix 1 we study in great detail the problem of whether a function which belongs to a Sobolev-Slobodeckij space over Γ and equals zero on $\partial\Omega \setminus \Gamma$, is a trace of a Sobolev function defined in Ω. The solution of this problem is fundamental to the homogenization of the boundary condition (1.5). Finally, Appendix 2 is concerned with the inhomogeneous Dirichlet problem for the Poisson equation with right hand side in L^1.

2. Weak formulation of (1.1)-(1.5)

Let $W^{1,q}(\Omega)$ $(1 \le q < +\infty)$ denote the usual Sobolev space. We define

$$W_0^{1,q}(\Omega) := \{\varphi \in W^{1,q}(\Omega) : \varphi = 0 \quad \text{a. e. on} \quad \partial\Omega\}.$$

[2)] We notice that the Schauder fixed point theorem has been also used in: Bernardi, C.; Chacon, T.; Lewandowski, R.; Murat, F., *Existence d'une solution pour un modèle de deux fluides turbulentes couplés.* C. R. Acad. Sci. Paris, Ser. I, 328 (1999), 993-998. In comparison with this paper, our existence theorem for a weak solution $\{\boldsymbol{u}_1, k_1, p_1; \boldsymbol{u}_2, k_2, p_2\}$ to (1.1)-(1.5) (see Section 3) involves more regularity of k_1, k_2 (see Remark 2.2 for details).

Spaces of vector-valued function will be denoted by bold letters, e. g., $\boldsymbol{L}^q(\Omega) := [L^q(\Omega)]^d$, $\boldsymbol{W}^{1,q}(\Omega) := [W^{1,q}(\Omega)]^d$ etc. Next, define

$$\begin{aligned}\boldsymbol{V}_i \;:=\; &\{\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega_i) : \operatorname{div} \boldsymbol{v} = 0 \quad a.\ e.\ in \quad \Omega_i,\\ &\boldsymbol{v} = \boldsymbol{0} \quad a.\ e.\ on \quad \partial\Omega_i \setminus \Gamma, \quad \boldsymbol{v}\cdot\boldsymbol{n}_i = 0 \quad a.\ e.\ on \quad \Gamma\}\end{aligned}$$

$(i = 1, 2)$.

Without any further reference, throughout the paper we suppose

$$\left[\begin{array}{l} \textit{there exist constants } \nu_*, \nu^* \textit{ and } \mu^* \textit{ such that} \\ 0 < \nu_* \le \nu_i(t) \le \nu^* < +\infty, \quad 0 \le \mu_i(t) \le \mu^* < +\infty \qquad \forall\, t \in \mathbb{R} \qquad (i = 1, 2). \end{array}\right.$$

Definition *Let* $\boldsymbol{f}_i \in \boldsymbol{L}^{2^*}(\Omega_i)$ [3)] $(i = 1, 2)$. *The functions* $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ *are called weak solution to* (1.1)-(1.5) *if*

$$(2.1)\quad (\boldsymbol{u}_i, k_i) \in \boldsymbol{V}_i \times \bigcap_{1 \le q < \frac{d}{d-1}} W^{1,q}(\Omega_i) \quad (i = 1, 2),$$

$$(2.2)\quad \begin{cases} \displaystyle\int_{\Omega_1} \nu_1(k_1)\boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \int_{\Omega_2} \nu_2(k_2)\boldsymbol{D}(\boldsymbol{u}_2) : \boldsymbol{D}(\boldsymbol{v}_2) + \\ \displaystyle\quad + \int_\Gamma |\boldsymbol{u}_1 - \boldsymbol{u}_2|(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)\,dS = \\ \displaystyle = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 + \int_{\Omega_2} \boldsymbol{f}_2 \cdot \boldsymbol{v}_2 \quad \forall\ (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2, \end{cases}$$

$$(2.3)\quad \begin{cases} \textit{for some } \ r > d, \\ \displaystyle\int_{\Omega_i} \nabla k_i \cdot \nabla\varphi = \int_{\Omega_i} \mu_i(k_i)|\boldsymbol{D}(\boldsymbol{u}_i)|^2\varphi \quad \forall\ \varphi \in W_0^{1,r'}(\Omega_i)\ ^{4)}, \end{cases}$$

$$(2.4)\qquad k_i = 0 \quad a.\ e.\ on \quad \partial\Omega_i \setminus \Gamma, \quad k_i = G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \quad a.\ e.\ on \quad \Gamma.$$

Remark 2.1 (**existence of a pressure**) Define

[3)] By q^* we denote Sobolev embedding exponent for $W^{1,q}(\Omega)$ ($\Omega \subset \mathbb{R}^N$ bounded, Lipschitzian; $N \ge 2$), i. e. $q^* = \frac{Nq}{N-q}$ if $1 \le q < N$, and $1 \le q^* < +\infty$ if $q = N$. If $q > N$, then $W^{1,q}(\Omega) \subset C(\bar\Omega)$ continuously.

[4)] Notice that $r > N$ iff $1 < r' < \frac{N}{N-1}$.

$$\boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i) := \{\boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega_i) : \begin{array}{lll} \boldsymbol{w} = \boldsymbol{0} & a.\ e.\ on & \partial\Omega_i \setminus \Gamma, \\ \boldsymbol{w}\cdot\boldsymbol{n}_i = 0 & a.\ e.\ on & \Gamma\} \end{array}$$

$(i = 1, 2)$. Clearly, $\boldsymbol{V}_i$ is a closed subspace of $\boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i)$. We have:

Let $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ *be a weak solution to* (1.1)-(1.5). *Then there exists* $p_i \in L^2(\Omega_i)$ *with* $\int\limits_{\Omega_i} p_i = 0$ *such that*

$$\text{(2.2')} \quad \begin{cases} \displaystyle\int\limits_{\Omega_i} \nu_i(k_i)\boldsymbol{D}(\boldsymbol{u}_i) : \boldsymbol{D}(\boldsymbol{w}) + (-1)^{i+1}\int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot\boldsymbol{w}dS = \\ \displaystyle = \int\limits_{\Omega_i} \boldsymbol{f}_i\cdot\boldsymbol{w} + \int\limits_{\Omega_i} p_i \operatorname{div}\boldsymbol{w} \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i). \end{cases}$$

In addition, there holds

$$\text{(2.2")} \quad \|p_i\|_{L^2} \le c\Big(\|\nabla\boldsymbol{u}_i\|_{\boldsymbol{L}^2} + \|\boldsymbol{f}_i\|_{\boldsymbol{L}^{2^*}}\Big).$$

To prove this, we first note the following

Proposition *Let* $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ *be a bounded Lipschitz domain and let* $1 < r < +\infty$. *Then, for every* $f \in L^r(\Omega)$ *with* $\int\limits_{\Omega} f = 0$, *there exists* $\boldsymbol{v} \in \boldsymbol{W}^{1,r}_0(\Omega)$ *such that*

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= f \quad a.\ e.\ in \quad \Omega, \\ \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^r} &\le c\|f\|_{L^r}. \end{aligned}$$

For a proof, see, e. g. [9; Chap. III, Thm. 3.2], [22; Chap. II, Lemma 2.1.1, a)].

We now proceed as follows. For $\boldsymbol{w} \in \boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i)$, define

$$\mathcal{F}_i(\boldsymbol{w}) := \int\limits_{\Omega_i} \nu_i(k_i)\boldsymbol{D}(\boldsymbol{u}_i) : \boldsymbol{D}(\boldsymbol{w}) + (-1)^{i+1}\int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot\boldsymbol{w}dS - \int\limits_{\Omega_i} \boldsymbol{f}_i\cdot\boldsymbol{w}$$

$(i = 1, 2)$. It is easy to check that $\mathcal{F}_i$ is a linear continuous functional on $\boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i)$. By (2.2), $\mathcal{F}_i(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}_i$.

Next, the above Proposition implies that the mapping

$$A : \boldsymbol{v} \mapsto A\boldsymbol{v} = \text{div } \boldsymbol{v}$$

is surjective from $\boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i)$ onto the space

$$\Big\{ f \in L^2(\Omega_i) : \int\limits_{\Omega_i} f = 0 \Big\}.$$

Now, following word by word the arguments of the proof in [9; Chap. III, Thm. 5.2] or [22; Chap. II, Lemma 2.11, b)] we obtain the existence of a $p_i \in L^2(\Omega_i)$ with $\int\limits_{\Omega_i} p_i = 0$ such that

$$\mathcal{F}_i(\boldsymbol{w}) = \int\limits_{\Omega_i} p_i \text{div } \boldsymbol{w} \quad \forall \ \boldsymbol{w} \in \boldsymbol{W}^{1,2}_{0,\Gamma}(\Omega_i),$$

i. e., (2.2') holds.

Estimate (2.2") is readily seen.

Remark 2.2 In [3; Thm. 5.2, pp. 88-89] the notion of (weak) solution to (1.1)-(1.5) means that k_i belongs to the Sobolev-Slobodeckij space $W^{s,2}(\Omega_i)$ $(0 < s < \frac{1}{2})$, and that (1.3) is satisfied in the sense of transposition of $-\Delta$ (cf. [3; p. 78]). In contrast to that paper, our definition of weak solution to (1.1)-(1.5) involves more regularity of k_i [5)].

Indeed, for any $0 < s < \dfrac{1}{2}$ we have $\dfrac{2d}{2+d-2s} < \dfrac{d}{d-1}$. Thus, if

$$\frac{2d}{2d-2s} < q < \frac{d}{d-1},$$

then

$$1 - \frac{d}{q} > s - \frac{d}{2},$$

and therefore

$$W^{1,q}(\Omega_i) \subset W^{s,2}(\Omega_i)$$

(see, e. g., [24; p. 328]). Hence, $k_i \in \bigcap\limits_{1 \le q < \frac{d}{d-1}} W^{1,q}(\Omega_i)$ implies $k_i \in W^{s,2}(\Omega_i)$ for all $0 < s < \dfrac{1}{2}$.

5) See also Appendix 2.

Finally, let $k_i \in W^{1,q}(\Omega_i)$ $(1 \le q < \frac{d}{d-1})$ satisfy (2.3) and (2.4). Integration by parts on the left hand side of (2.3) gives, for any $\varphi \in W^{2,2}(\Omega_i) \cap W_0^{1,2}(\Omega_i)$,

$$-\int\limits_{\Omega_i} k_i \Delta\varphi + \int\limits_{\Gamma} G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2)\boldsymbol{n}_i \cdot \nabla\varphi dS = \int\limits_{\Omega_i} \mu_i(k_i)|\boldsymbol{D}(\boldsymbol{u}_i)|^2\varphi,$$

i. e., k_i satisfies (1.3) in the sense of transposition of $-\Delta$ under zero boundary conditions on φ (cf. [3; p. 78]). ■

3. Existence of a weak solution

The following theorem is the main result of our paper.

Theorem *Let $\Omega_i \subset \mathbb{R}^d$ $(i = 1, 2;\ d = 2$ or $d = 3)$ be bounded domains of class $\mathcal{C}^1$* [6]*. Suppose that assumption (A)* [7] *is satisfied.*

Then, for every $\boldsymbol{f}_i \in \boldsymbol{L}^{2^*}(\Omega_i)$ $(i = 1, 2)$ there exists a weak solution $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ to (1.1)-(1.5). In addition,

$$k_i \ge 0 \quad a.\ e.\ in \quad \Omega_i, \tag{3.1}$$

$$\sum_{i=1}^{2} \|\boldsymbol{u}_i\|^2_{\boldsymbol{W}^{1,2}(\Omega_i)} + \int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^3 dS \ \le\ c\sum_{j=1}^{2} \|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_j)}, \tag{3.2}$$

$$\left\{\begin{array}{l} \textit{for every} \quad 1 \le q < \dfrac{d}{d-1} \quad \textit{there exists} \quad c = \text{const} \quad \textit{such that} \\[2ex] \|k_i\|_{\boldsymbol{W}^{1,q}(\Omega_i)} \le c\sum\limits_{j=1}^{2} \|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_j)} \\ \textit{where} \quad c = c(q) \to +\infty \quad \textit{as} \quad q \to \dfrac{d}{d-1}, \end{array}\right. \tag{3.3}$$

$$\left\{\begin{array}{l} \textit{for every} \quad \Omega_i' \Subset \Omega_i \quad \textit{and every} \quad \delta > 0, \\[2ex] \displaystyle\int\limits_{\Omega_i'} \frac{|\nabla k_i|^2}{(1+k_i)^{1+\delta}} \le \frac{c}{\delta}\sum_{j=1}^{2} \|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_j)}, \\[2ex] \textit{where} \quad c \to +\infty \quad \textit{as} \quad \text{dist}\,(\Omega_i', \partial\Omega_i) \to 0. \end{array}\right. \tag{3.4}$$

[6] The condition $\Omega_i \in \mathcal{C}^1$ we need in order to apply Theorem A2.1.

[7] See p. 22

Proof We consider the space $L^1(\Omega_1)\times L^1(\Omega_2)$ equipped with the norm

$$\|(k_1,k_2)\| := \sum_{i=1}^{2} \|k_i\|_{L^1(\Omega_i)}.$$

For appropriate $R>0$ which will be fixed below, we set

$$\mathcal{K}_R := \{(k_1,k_2)\in L^1(\Omega_1)\times L^1(\Omega_2) : \|(k_1,k_2)\|\le R\}.$$

Then, for any $(k_1,k_2)\in\mathcal{K}_R$ we show that there exists exactly one $(\boldsymbol{u}_1,\boldsymbol{u}_2)\in \boldsymbol{V}_1\times \boldsymbol{V}_2$ which satisfies (2.2). With $(\boldsymbol{u}_1,k_1;\boldsymbol{u}_2,k_2)$ at hand, we deduce from Theorem A2.1 the existence and uniqueness of a pair $(\hat{k}_1,\hat{k}_2)\in W^{1,q}(\Omega_1)\times W^{1,q}(\Omega_2)$ $(1<q<\dfrac{d}{d-1}$ arbitrary) which solves (2.3) with the given L^1-function $\mu_i(k_i)|D(\boldsymbol{u}_i)|^2$ on the right hand side, and with given $G_i((|\boldsymbol{u}_1-\boldsymbol{u}_2|^2)$ on Γ $(i=1,2)$. This gives rise to introduce a mapping $\mathcal{T}:\mathcal{K}_R\to\mathcal{K}_R$ by

$$\mathcal{T}(k_1,k_2) := (\hat{k}_1,\hat{k}_2).$$

We then prove:

(i) $\mathcal{T}$ *is continuous*;

(ii) $\mathcal{T}(\mathcal{K}_R)$ *is precompact.*

From Schauder's fixed it follows that there exists $(k_1^*,k_2^*)\in\mathcal{K}_R$ such that $\mathcal{T}(k_1^*,k_2^*) = (k_1^*,k_2^*)$.

Now, with the fixed point $(\boldsymbol{k}_1^*,\boldsymbol{k}_2^*)$ at hand, we obtain the existence and uniqueness of a pair $(\boldsymbol{u}_1^*,\boldsymbol{u}_2^*)\in \boldsymbol{V}_1\times\boldsymbol{V}_2$ which satisfies (2.2) (with $(\boldsymbol{k}_1^*,\boldsymbol{k}_2^*)$ in place of (k_1,k_2) therein). By the definition of $\mathcal{T}$, the functions $\{\boldsymbol{u}_1^*,k_1^*;\boldsymbol{u}_2^*,k_2^*\}$ are a weak solution to (1.1)-(1.5). ■

We turn to the details of the proof.

Definition of $\mathcal{T}:\mathcal{K}_R\to\mathcal{K}_R$. The space $\boldsymbol{V}_1\times\boldsymbol{V}_2$ is a Hilbert space with respect to the scalar product

$$\langle(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{v}_1,\boldsymbol{v}_2)\rangle := \sum_{i=1}^{2}\int_{\Omega}\nabla\boldsymbol{u}_i\cdot\nabla\boldsymbol{v}_i.$$

By $\|\!|\cdot|\!\| := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ we denote the associated norm.

1) *The mapping* $(k_1,k_2)\mapsto(\boldsymbol{u}_1,\boldsymbol{u}_2)$. Given any $(k_1,k_2)\in L^1(\Omega_1)\times L^1(\Omega_2)$, we prove the existence and uniqueness of a pair $(\boldsymbol{u}_1,\boldsymbol{u}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2$ which satisfies (2.2). To do this, we replace (2.2) by an operator equation in $\boldsymbol{V}_1\times\boldsymbol{V}_2$ to which an abstract existence and uniqueness theorem applies.

Firstly, for any (fixed) $(k_1,k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$ we introduce a linear bounded mapping $\mathcal{A}_{(k_1,k_2)} : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \to \boldsymbol{V}_1 \times \boldsymbol{V}_2$ by

$$\langle \mathcal{A}_{(k_1,k_2)}(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{v}_1,\boldsymbol{v}_2)\rangle := \sum_{i=1}^{2} \int\limits_{\Omega_i} \nu_i(k_i)\boldsymbol{D}(\boldsymbol{u}_i) : \boldsymbol{D}(\boldsymbol{v}_i).$$

By Korn's equality,

$$\langle \mathcal{A}_{(k_1,k_2)}(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{u}_1,\boldsymbol{u}_2)\rangle \geq c_0 |\!|\!|(\boldsymbol{u}_1,\boldsymbol{u}_2)|\!|\!|^2 \quad (c_0 = \text{const} > 0)$$

for all $(\boldsymbol{u}_1,\boldsymbol{u}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$ (c_0 independent of (k_1,k_2)).

Secondly, observing the continuity of the trace mapping $\gamma : W^{1,2}(\Omega) \to L^4(\partial\Omega)$ ($d = 2$ and $d = 3$; see, e. g., [8], [11], [24; pp. 281-282, 329-330]) we obtain, for every $(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{v}_1,\boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$ [8)],

$$\begin{aligned}
&\left| \int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)dS \right| \leq \\
&\quad \leq \left(\int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^{\frac{8}{3}} dS \right)^{\frac{3}{4}} \left(\int\limits_{\Gamma} |\boldsymbol{v}_1 - \boldsymbol{v}_2|^4 dS \right)^{\frac{1}{4}} \\
&\quad \leq c \left(\sum_{i=1}^{2} \|\boldsymbol{u}_i\|^2_{\boldsymbol{L}^{\frac{8}{3}}(\partial\Omega_i)} \right) \sum_{j=1}^{2} \|\boldsymbol{v}_j\|_{\boldsymbol{L}^4(\partial\Omega_j)}{}^{9)} \\
&\quad \leq c |\!|\!|(\boldsymbol{u}_1,\boldsymbol{u}_2)|\!|\!|^2 |\!|\!|(\boldsymbol{v}_1,\boldsymbol{v}_2)|\!|\!|.
\end{aligned}$$

We now introduce a (nonlinear) mapping $\mathcal{B} : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \to \boldsymbol{V}_1 \times \boldsymbol{V}_2$ by

$$\langle \mathcal{B}(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{v}_1,\boldsymbol{v}_2)\rangle := \int\limits_{\Gamma} |\boldsymbol{u}_1 - \boldsymbol{u}_2|(\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)dS.$$

By elementary calculus,

$$\begin{aligned}
&\langle \mathcal{B}(\boldsymbol{u}_1,\boldsymbol{u}_2) - \mathcal{B}(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2),(\boldsymbol{u}_1,\boldsymbol{u}_2) - (\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\rangle \geq \\
&\quad \geq \int\limits_{\Gamma} (|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 - |\bar{\boldsymbol{u}}_1 - \bar{\boldsymbol{u}}_2|^2)(|\boldsymbol{u}_1 - \boldsymbol{u}_2| - |\bar{\boldsymbol{u}}_1 - \bar{\boldsymbol{u}}_2|)dS \geq 0
\end{aligned}$$

---

8) For notational simplicity, in this section we use the same notation for a function in $W^{1,q}(\Omega)$ and its trace.

9) Throughout the paper, we denote by c positive constants which may change their numerical value but do not depend on the functions under consideration.

and

$$\|\!|\mathcal{B}(\boldsymbol{u}_1,\boldsymbol{u}_2)-\mathcal{B}(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\|\!| \ \le\ c(\|\!|(\boldsymbol{u}_1,\boldsymbol{u}_2)\|\!|+\|\!|(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\|\!|)\sum_{i=1}^{2}\|\boldsymbol{u}_i-\bar{\boldsymbol{u}}_i\|_{\boldsymbol{W}^{1,2}(\Omega_i)}$$

for all $(\boldsymbol{u}_1,\boldsymbol{u}_2)$, $(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2$

Thus,

$$\begin{cases} \mathcal{A}_{(k_1,k_2)}+\mathcal{B} \quad \textit{is continuous on the whole of} \quad \boldsymbol{V}_1\times\boldsymbol{V}_2 \\ \textit{and maps bounded sets into bounded sets,} \end{cases}$$

$$\begin{cases} \langle(\mathcal{A}_{(k_1,k_2)}+\mathcal{B})(\boldsymbol{u}_1,\boldsymbol{u}_2)-(\mathcal{A}_{(k_1,k_2)}+\mathcal{B})(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2),(\boldsymbol{u}_1,\boldsymbol{u}_2)-(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\rangle\ge \\ \ge c_0\|\!|(\boldsymbol{u}_1,\boldsymbol{u}_2)-(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\|\!|^2 \qquad \forall\,(\boldsymbol{u}_1,\boldsymbol{u}_2),(\bar{\boldsymbol{u}}_1,\bar{\boldsymbol{u}}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2. \end{cases}$$

From [27; Thm. 26.A, p. 557] it follows that for every $\boldsymbol{f}_i\in\boldsymbol{L}^{2^*}(\Omega_i)$ (i=1,2) there exists exactly one $(\boldsymbol{u}_1,\boldsymbol{u}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2$ such that

$$\langle(\mathcal{A}_{(k_1,k_2)}+\mathcal{B})(\boldsymbol{u}_1,\boldsymbol{u}_2),(\boldsymbol{v}_1,\boldsymbol{v}_2)\rangle=\sum_{i=1}^{2}\int_{\Omega_i}\boldsymbol{f}_i\cdot\boldsymbol{v}_i \quad \forall\,(\boldsymbol{v}_1,\boldsymbol{v}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2, \tag{3.5}$$

i. e., (2.2) holds with the given $(k_1,k_2)\in L^1(\Omega_1)\times L^1(\Omega_2)$. In addition, we have

$$\sum_{i=1}^{2}\|\boldsymbol{u}_i\|^2_{\boldsymbol{W}^{1,2}(\Omega_i)}+\int_\Gamma|\boldsymbol{u}_1-\boldsymbol{u}_2|^3dS\le c\sum_{j=1}^{2}\|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_i)}, \tag{3.6}$$

where the constant c does not depend on (k_1,k_2).

2) *The mapping* $(\boldsymbol{u}_1,\boldsymbol{u}_2)\mapsto(\hat{k}_1,\hat{k}_2)$. Let $1<q<\dfrac{d}{d-1}$. Let $(\boldsymbol{u}_1,\boldsymbol{u}_2)\in\boldsymbol{V}_1\times\boldsymbol{V}_2$ denote the solution to (3.5) (uniquely determined by $(k_1,k_2)\in L^1(\Omega_1)\times L^1(\Omega_2)$) which has been obtained by the preceding step 1).

Define

$$\tilde{h}_i:=\begin{cases} G_i(|\boldsymbol{u}_1-\boldsymbol{u}_2|^2) & \textit{a. e. on} \quad \Gamma, \\ 0 & \textit{a. e. on} \quad \partial\Omega_i\setminus\Gamma \end{cases}$$

(G_i as in (1.6); $i=1,2$). By Corollary A1.1,

$$\tilde{h}_i\in W^{1-\frac{1}{q},q}(\partial\Omega_i),\quad \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)}\le c\sum_{j=1}^{2}\|\boldsymbol{u}_j\|^2_{\boldsymbol{W}^{1,2}(\Omega_j)}. \tag{3.7}$$

Now, from Theorem A2.1 and Theorem A2.2, 1° we obtain the existence and uniqueness of a pair $(\hat{k}_1, \hat{k}_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ such that

$$\hat{k}_i \geq 0 \quad a.\ e.\ in \quad \Omega_i, \tag{3.8}$$

$$\int\limits_{\Omega_i} \nabla \hat{k}_i \cdot \nabla \varphi_i = \int\limits_{\Omega_i} \mu_i(k_i) |\boldsymbol{D}(\boldsymbol{u}_i)|^2 \varphi_i \quad \forall\, \varphi_i \in W^{1,q'}(\Omega_i), \tag{3.9}$$

$$\hat{k}_i = \tilde{h}_i \quad a.\ e.\ on \quad \partial\Omega_i, \tag{3.10}$$

$$\|\hat{k}_i\|_{W^{1,q}(\Omega_i)} \leq c\Big(\|\, |\boldsymbol{D}(\boldsymbol{u}_i)|^2\|_{L^1(\Omega_i)} + \ \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)}\Big), \tag{3.11}$$

$$\begin{cases} for\ every \quad \Omega_i' \Subset \Omega_i \quad and\ every \quad \delta > 0, \\ \\ \dfrac{\nabla \hat{k}_i}{(1+\hat{k}_i)^{\frac{1+\delta}{2}}} \in \boldsymbol{L}^2(\Omega_i'), \\ \\ \displaystyle\int\limits_{\Omega_i} \frac{\nabla \hat{k}_i}{(1+\hat{k}_i)^{1+\delta}} \leq \frac{c}{\delta}\Big(\|\, |\boldsymbol{D}(\boldsymbol{u}_i)|^2\|_{L^1(\Omega_i)} + \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)}\Big), \\ where \quad c \to +\infty \quad as \quad \mathrm{dist}(\Omega_i', \partial\Omega_i) \to 0. \end{cases} \tag{3.12}$$

We notice that the constants c in (3.7), (3.11) and (3.12) do not depend on (k_1, k_2). By combining (3.7) and (3.11) we find

$$\|(\hat{k}_1, \hat{k}_2)\| \leq c \sum_{i=1}^{2} \|\boldsymbol{f}_i\|^2_{\boldsymbol{L}^{2^*}(\Omega_i)} \ =: R. \tag{3.13}$$

3) Let us consider $\mathcal{K}_R$ [10] with R as in (3.13). For $(k_1, k_2) \in \mathcal{K}_R$, define

$$\mathcal{T} : (k_1, k_2) \mapsto (\boldsymbol{u}_1, \boldsymbol{u}_2) \mapsto \mathcal{T}(k_1, k_2) := (\hat{k}_1, \hat{k}_2),$$

where $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is as in step 1), $(\hat{k}_1, \hat{k}_2)$ as in step 2). Then $\mathcal{T}$ is a well-defined (single valued) mapping of $\mathcal{K}_R$ into itself. [11]

(i) $\mathcal{T}$ **is continuous**. Let be $(k_{1m}, k_{2m}) \in \mathcal{K}_R$ $(m \in \mathbb{N})$ such that

$$k_{im} \to k_i \quad \text{strongly in} \quad L^1(\Omega_i) \quad \text{as} \quad m \to \infty \quad (i = 1, 2).$$

[10] Recall $\mathcal{K}_R := \{(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2) : \|(k_1, k_2)\| \leq R\}$.
[11] In fact, $\mathcal{T}$ maps the whole of $L^1(\Omega_1) \times L^1(\Omega_2)$ into $\mathcal{K}_R$.

Clearly, $(k_1, k_2) \in \mathcal{K}_R$. Without loss of generality, we may assume that

$$k_{im} \to k_i \quad \text{a. e. in} \quad \Omega_i \quad \text{as} \quad m \to \infty \quad (i = 1, 2). \tag{3.14}$$

We prove that

$$\mathcal{T}(k_{1m}, k_{2m}) \to \mathcal{T}(k_1, k_2) \quad \text{strongly in} \quad L^1(\Omega_1) \times L^1(\Omega_2) \quad \text{as} \quad m \to \infty.$$

To begin with, we introduce the following notation. For (k_{1m}, k_{2m}), let $(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m}) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$ denote the uniquely determined solution of

$$\langle (\mathcal{A}_{(k_{1m}, k_{2m})} + \mathcal{B})(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m}), (\boldsymbol{v}_1, \boldsymbol{v}_2) \rangle = \sum_{i=1}^{2} \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i \quad \forall\ (\boldsymbol{v}_i, \boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2. \tag{3.5$_m$}$$

Clearly,

$$\sum_{i=1}^{2} \|\boldsymbol{u}_{im}\|^2_{\boldsymbol{W}^{1,2}(\Omega_i)} + \int_{\Gamma} |\boldsymbol{u}_{1m} - \boldsymbol{u}_{2m}|^3 dS \le c \sum_{i=1}^{2} \|\boldsymbol{f}_i\|^2_{\boldsymbol{L}^{2^*}(\Omega_i)}. \tag{3.6$_m$}$$

Analogously, for the limit element (k_1, k_2), let $(\boldsymbol{u}_1, \boldsymbol{u}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$ denote the uniquely determined solution to (3.5). This solution satisfies (3.6).

We claim

$$(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m}) \to (\boldsymbol{u}_1, \boldsymbol{u}_2) \quad \text{strongly in} \quad \boldsymbol{W}^{1,2}(\Omega_1) \times \boldsymbol{W}^{1,2}(\Omega_2) \quad \text{as} \quad m \to \infty. \tag{3.15}$$

To prove this, we first note that from (3.6$_m$) it follows that there exists a subsequence $\{(\boldsymbol{u}_{1m_s}, \boldsymbol{u}_{2m_s})\}$ $(s \in \mathbb{N})$ such that

$$(\boldsymbol{u}_{1m_s}, \boldsymbol{u}_{2m_s}) \to (\bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2) \quad \text{weakly in} \quad \boldsymbol{W}^{1,2}(\Omega_1) \times \boldsymbol{W}^{1,2}(\Omega_2) \quad \text{as} \quad s \to \infty.$$

Using the compactness of the embedding $W^{1,2}(\Omega) \subset L^r(\partial\Omega)$ $(1 \le r < 4;\ d = 2$ resp. $d = 3)$, we obtain

$$\langle \mathcal{B}(\boldsymbol{u}_{1m_s}, \boldsymbol{u}_{2m_s}), (\boldsymbol{v}_1, \boldsymbol{v}_2) \rangle \to \mathcal{B}(\bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2), (\boldsymbol{v}_1, \boldsymbol{v}_2) \rangle \quad \forall\ (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$$

as $m \to \infty$. With the help of (3.14) the passage to the limit $s \to \infty$ in (3.5$_m$) gives

$$\langle (\mathcal{A}_{(k_1, k_2)} + \mathcal{B})(\bar{\boldsymbol{u}}_1, \bar{\boldsymbol{u}}_2 \rangle = \sum_{i=1}^{2} \int_{\Omega} \boldsymbol{f}_i \cdot \boldsymbol{v}_i \quad \forall\ (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2.$$

Comparing this and (3.5) we find $\bar{\boldsymbol{u}}_i = \boldsymbol{u}_i$ $(i = 1, 2)$. Therefore the whole sequence $\{(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m})\}$ converges weakly in $\boldsymbol{W}^{1,2}(\Omega_1) \times \boldsymbol{W}^{1,2}(\Omega_2)$ to $(\boldsymbol{u}_1, \boldsymbol{u}_2)$.

We now form the difference between (3.5_m) and (3.5), and use the test function $\boldsymbol{v}_i = \boldsymbol{u}_{im} - \boldsymbol{u}_i$ $(i = 1, 2)$. Observing the monotonicity of $\mathcal{B}$, we find

$$\begin{aligned}
\nu_* \sum_{i=1}^{2} \int_{\Omega_i} |\boldsymbol{D}(\boldsymbol{u}_{im} - \boldsymbol{u}_i)|^2 \ &\leq \ \sum_{i=1}^{2} \int_{\Omega_i} \nu_i(k_{im})(\boldsymbol{D}(\boldsymbol{u}_{im}) - D(\boldsymbol{u}_i)) : \boldsymbol{D}(\boldsymbol{u}_{im} - \boldsymbol{u}_i) \\
&\leq \ \sum_{i=1}^{2} \int_{\Omega_i} (-\nu_i(k_{im}) + \nu_i(k_i))\boldsymbol{D}(\boldsymbol{u}_i) : \boldsymbol{D}(\boldsymbol{u}_{im} - \boldsymbol{u}_i) \\
&\quad \to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Whence (3.15).

Next, set $(\hat{k}_{1m}, \hat{k}_{2m}) := \mathcal{T}(k_{1m}, k_{2m})$ $(m \in \mathbb{N})$ and $(\hat{k}_1, \hat{k}_2) := \mathcal{T}(k_1, k_2)$. Let $1 < q < \dfrac{d}{d-1}$. By the definition of $\mathcal{T}$, the pair $(\hat{k}_{1m}, \hat{k}_{2m}) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ is uniquely determined by (k_{1m}, k_{2m}) and $(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m})$ through

$$(3.9_m) \qquad \int_{\Omega_i} \nabla \hat{k}_{im} \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_{im}) |\boldsymbol{D}(\boldsymbol{u}_{im})|^2 \varphi_i \quad \forall\ \varphi_i \in W_0^{1,q'}(\Omega_i),$$

$$(3.10_m) \qquad \hat{k}_{im} = \tilde{h}_{im} \quad a.\ e.\ on \quad \partial\Omega_i,$$

where $\tilde{h}_{im} \in W^{1-\frac{1}{q},q}(\partial\Omega_i)$ is defined by

$$\tilde{h}_{im} := \begin{cases} G_i(|\boldsymbol{u}_{1m} - \boldsymbol{u}_{2m}|^2) & a.\ e.\ on \ \ \Gamma, \\ 0 & a.\ e.\ on \ \ \partial\Omega_i \setminus \Gamma \end{cases}$$

(see Theorem A2.1). From (3.7) (with $\boldsymbol{u}_{im}$ in place of $\boldsymbol{u}_i$) it follows that

$$\|\tilde{h}_{im}\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \leq c \sum_{j=1}^{2} \|\boldsymbol{u}_{jm}\|^2_{\boldsymbol{W}^{1,2}(\Omega_j)} \leq \text{const.}$$

We obtain

$$(3.16) \qquad \tilde{h}_{im} \to \tilde{h}_i \quad \text{weakly in} \quad W^{1-\frac{1}{q},q}(\partial\Omega_i) \quad \text{as} \quad m \to \infty,$$

where $\tilde{h}_i$ is defined as above, i. e.

$$\tilde{h}_i := \begin{cases} G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) & a.\ e.\ on \ \ \Gamma, \\ 0 & a.\ e.\ on \ \ \partial\Omega_i \setminus \Gamma \end{cases}$$

$(i = 1, 2)$. To see (3.16), we first note that (3.15) implies $\boldsymbol{u}_{im} \to \boldsymbol{u}_i$ strongly in $\boldsymbol{L}^4(\partial\Omega_i)$ as $m \to \infty$ ($d = 2$ resp. $d = 3$). Therefore

$$G_i(|\boldsymbol{u}_{1m} - \boldsymbol{u}_{2m}|^2) \to G_i(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2) \quad \text{strongly in} \quad L^2(\Gamma) \quad \text{as} \quad m \to \infty.$$

Since $W^{1-\frac{1}{q},q}(\partial\Omega_i)$ is reflexive, (3.16) is now readily seen by routine arguments.

To proceed, we note that $\hat{k}_{im}$ satisfies the estimate

$$\begin{aligned}\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} &\leq c\Big(\||\boldsymbol{D}(\boldsymbol{u}_{im})|^2\|_{L^1(\Omega_i)} + \|\tilde{h}_{im}\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)}\Big) \quad \text{[cf. (3.11)]}\\ &\leq c\sum_{j=1}^{2} \|\boldsymbol{u}_{jm}\|^2_{\boldsymbol{W}^{1,2}(\Omega_j)}\\ &\leq c\sum_{j=1}^{2} \|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_j)} \quad \text{[by (3.6}_m\text{)]}\end{aligned}$$

$(i = 1, 2;\ m \in \mathbb{N})$. Hence there exists a subsequence $\{\hat{k}_{im_t}\}$ $(t \in \mathbb{N})$ such that

$$\hat{k}_{im_t} \to \bar{k}_i \quad \text{weakly in} \quad W^{1,q}(\Omega_i) \quad \text{as} \quad t \to \infty.$$

Using (3.14), (3.15) and (3.16) the passage to the limit $t \to \infty$ in (3.9_{m_t}) and (3.10_{m_t}) gives

$$\begin{aligned}\int_{\Omega_i} \nabla\bar{k}_i \cdot \nabla\varphi_i &= \int_{\Omega_i} \mu_i(k_i)|\boldsymbol{D}(\boldsymbol{u}_i)|^2\varphi_i \quad \forall\ \varphi_i \in W_0^{1,q'}(\Omega_i),\\ \bar{k}_i &= \tilde{h}_i \quad a.\ e.\ on \quad \partial\Omega_i.\end{aligned}$$

Combining this and (3.9), (3.10) we get

$$\begin{aligned}\int_{\Omega_i} \nabla(\bar{k}_i - \hat{k}_i) \cdot \nabla\varphi_i &= 0 \quad \forall\ \varphi_i \in W_0^{1,q'}(\Omega_i),\\ \bar{k}_i - \hat{k}_i &= 0 \quad a.\ e.\ on \quad \partial\Omega_i.\end{aligned}$$

By theorem A2.1, $\bar{k}_i = \hat{k}_i$ a. e. in Ω_i $(i = 1, 2)$. It follows that the whole sequence $\{\hat{k}_{im}\}$ converges weakly in $W^{1,q}(\Omega_i)$ to $\hat{k}_i$ as $m \to \infty$. Therefore, by the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$,

$$\hat{k}_{im} \to \hat{k}_i \quad \text{strongly in} \quad L^1(\Omega_i) \quad \text{as} \quad m \to \infty,$$

i. e., $\mathcal{T}$ is continuous.

(ii) $\mathcal{T}(\mathcal{K}_R)$ **is precompact.** Let $(\hat{k}_{1m}, \hat{k}_{2m}) \in \mathcal{T}(\mathcal{K}_R)$ $(m \in \mathbb{N})$. Then $(\hat{k}_{1m}, \hat{k}_{2m}) = \mathcal{T}(k_{1m}, k_{2m})$, where $(k_{1m}, k_{2m}) \in \mathcal{K}_R$. As above, let $(\boldsymbol{u}_{1m}, \boldsymbol{u}_{2m}) \in \boldsymbol{V}_1 \times \boldsymbol{V}_2$ denote the uniquely determined solutions to (3.5_m). The existence and uniqueness argument used at the end of the proof of the continuity of $\mathcal{T}$ (cf. Theorem A2.1), implies that $(\hat{k}_{1m}, \hat{k}_{2m}) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ and $1 < q < \dfrac{d}{d-1}$) and (3.9_m) and (3.10_m) hold. It follows that

$$\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} \leq c\sum_{j=1}^{2} \|\boldsymbol{f}_j\|^2_{\boldsymbol{L}^{2^*}(\Omega_j)} \quad (i = 1, 2; m \in \mathbb{N})$$

(cf. above). By the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$, there exists a subsequence $\{\hat{k}_{im_s}\}$ $(s \in \mathbb{N})$ and an element $(l_1, l_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$ such that

$$\hat{k}_{im_s} \to l_i \quad \text{strongly in} \quad L^1(\Omega_i) \quad \text{as} \quad s \to \infty,$$

i. e. $\mathcal{T}(\mathcal{K}_R)$ is precompact.

By Schauder's fixed point theorem, there exists $(k_1^*, k_2^*) \in \mathcal{K}_R$ such that $\mathcal{T}(k_1^*, k_2^*) = (k_1^*, k_2^*)$. The proof of the theorem is complete.

4. Regularity properties of weak solutions

In this section, we establish regularity properties for any weak solution $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ to (1.1)–(1.5) (see Sect. 2 for the definition).

Theorem 4.1 (**Local regularity**) *Let $\boldsymbol{f}_i \in \boldsymbol{L}^2(\Omega_i)\,(i = 1, 2)$. Then there exists $\sigma > 2$ such that for every weak solution $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ to* (1.1)–(1.5) *there holds*

$$\nabla \boldsymbol{u}_i \in \boldsymbol{L}^\sigma_{\mathrm{loc}}(\Omega_i), \quad k_i \in W^{2,\frac{\sigma}{2}}_{\mathrm{loc}}(\Omega_i).$$

Indeed, the local higher integrability of $\nabla \boldsymbol{u}_i$ follows from [6; Prop. 4.1]. It follows $|\boldsymbol{D}(\boldsymbol{u}_i)|^2 \in \boldsymbol{L}^{\frac{\sigma}{2}}_{\mathrm{loc}}(\Omega_i)$. Then $k_i \in W^{2,\frac{\sigma}{2}}_{\mathrm{loc}}(\Omega_i)$ is a consequence of Theorem A 2.1, (A2.7).

Theorem 4.2 (**global higher integrability of $\nabla \boldsymbol{u}_i$**) *Assume that*

$$\overline{\Gamma} \cap (\partial\Omega_i \setminus \Gamma) \quad \textit{is Lipschitz} \quad (i = 1, 2)\ ^{12)}$$

Let $\boldsymbol{f}_i \in \boldsymbol{L}^2(\Omega_i)$. Then there exists $\rho > 2$ such that for every weak solution $\{\boldsymbol{u}_1, k_1; \boldsymbol{u}_2, k_2\}$ to (1.1)–(1.5) *there holds*

$$\nabla \boldsymbol{u}_i \in \boldsymbol{L}^\rho(\Omega_i).$$

This result is a special case of [26; Thm. 2.1]. ■

We notice that the higher integrability of the gradient has been used in [3] for the uniqueness of the weak solution to (1.1)–(1.5) in the case $d = 2$. It has been also used in [4].

Appendix 1. Extension of a function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial\Omega \setminus \Gamma$

.

[12)] See [26; (1.24a), (1.24b)] for details.

$\boxed{1}$ Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For $0 < s < 1$ and $1 < q < +\infty$ we consider the Sobolev-Slobodeckij space

$$W^{s,q}(\partial\Omega) := \left\{ w \in L^q(\partial\Omega) : \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|w(x)-w(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y < +\infty \right\}$$

with the norm

$$\|w\|_{W^{s,q}(\partial\Omega)} := \left(\|w\|^q_{L^q(\partial\Omega)} + \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|w(x)-w(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y \right)^{\frac{1}{q}}$$

(see, e. g., [8], [19] for details).

Let $\Gamma \subset \partial\Omega$ be **relatively open**. We have

$\boxed{1.1}$ Let $w \in W^{s,q}(\partial\Omega)$. If $w = 0$ a. e. on $\partial\Omega \setminus \Gamma$, then

$$\int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|w(x)-w(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y =$$

$$\text{(A1.1)} \qquad = \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{|w(x)-w(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y +$$

$$+ \int\limits_{\Gamma} |w(y)|^q \left(\int\limits_{\partial\Omega\setminus\Gamma} \frac{1}{|x-y|^{N-1+sq}} dS_x \right) dS_y$$

$$+ \int\limits_{\partial\Omega\setminus\Gamma} \left(\int\limits_{\Gamma} \frac{|w(x)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y$$

This follows from the additivity of the integral.

We notice that the second and third integral on the right hand side of (A1.1) are equal. Indeed, we have

$$\int_{\Gamma} \left(\int_{\partial\Omega \smallsetminus \Gamma} \frac{|w(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y =$$

$$= \int_{\partial\Omega \smallsetminus \Gamma} \left(\int_{\Gamma} \frac{|w(y)|^q}{|x-y|^{N-1+sq}} dS_y \right) dS_x \quad \text{[by Fubini-Tonelli]}$$

$$\text{(A1.2)} \qquad = \int_{\partial\Omega \smallsetminus \Gamma} \left(\int_{\Gamma} \frac{|w(x)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y$$

[change of notation of the variables x and y].

1.2 Let $g \in L^q(\Gamma)$ $(1 < q < +\infty)$, let $0 < s < 1$ and assume that

$$\text{(A1.3)} \qquad \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x-y|^{N-1+sq}} dS_x dSy < +\infty,$$

$$\text{(A1.4)} \qquad \int_{\Gamma} |g(y)|^q \Big(\int_{\partial\Omega \smallsetminus \Gamma} \frac{1}{|x-y|^{N-1+sq}} dS_x \Big) dS_y < +\infty.$$

Define

$$\tilde{g} := \begin{cases} g & \text{a. e.} \quad \text{on } \Gamma, \\ 0 & \text{a. e.} \quad \text{on } \partial\Omega \smallsetminus \Gamma. \end{cases}$$

Then $\tilde{g} \in W^{s,q}(\partial\Omega)$.

Indeed, firstly $\tilde{g} \in L^q(\partial\Omega)$. Secondly, from (A1.3) and (A1.4) it follows

$$+\infty > \int\limits_{\Gamma} \left(\int\limits_{\Gamma} \frac{|g(x)-g(y)|^q}{|x-y|^{N-1+sq}} dS_x + \int\limits_{\partial\Omega\setminus\Gamma} \underbrace{\frac{|\tilde{g}(x)-g(y)|^q}{|x-y|^{N-1+sq}}}_{\tilde{g}(x)=0} dS_x \right) dS_y$$

$$+ \int\limits_{\partial\Omega\setminus\Gamma} \left(\int\limits_{\Gamma} \underbrace{\frac{|g(x)-\tilde{g}(y)|^q}{|x-y|^{N-1+sq}}}_{\tilde{g}(y)=0} dS_x + \underbrace{0}_{\tilde{g(x)}=\tilde{g}(y)=0} \right) dS_y \text{ [observe(A1.2) with } g \text{ in place of } w]$$

$$= \int\limits_{\partial\Omega} \left(\int\limits_{\partial\Omega} \frac{|\tilde{g}(x)-\tilde{g}(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y.$$

■

Remark A1.1 Under the above assumptions, for $y \in \Gamma$ define

$$\omega(y) = \omega_{s,q}(y) := \int\limits_{\partial\Omega\setminus\Gamma} \frac{1}{|x-y|^{N-1+sq}} dS_x.$$

We have

1) *ω is continuous on Γ,*

2) $\omega(y) \leq \dfrac{\text{mes}(\partial\Omega \setminus \Gamma)}{(\text{dist}(y, \partial\Omega\setminus\Gamma))^{N-1+sq}} < +\infty,$

3) *let $x_0 \in \partial\Omega \setminus \Gamma$, $\text{dist}(x_0, \Gamma) = 0$; if there exists $a_0 > 0$, $\rho_0 > 0$ such that $\text{mes}((\partial\Omega\setminus\Gamma) \cap B_\rho(x_0)) \geq a_0\rho^{N-1}$ for all $0 < \rho \leq \rho_0$ [13] then*

$$\lim_{y\in\Gamma, y\to x_0} \omega(y) = +\infty.$$

Condition (A1.4) reads

(A1.4') $$\int\limits_{\Gamma} \omega(y)|g(y)|^q dS_y < +\infty.$$

Thus, condition (A1.4) (resp. (A1.4')) expresses a decay property of g near the boundary $\partial\Gamma$. ■

[13] $B_\rho(x_0) = \{\xi \in \mathbb{R}^N : |\xi - x_0| < \rho\}$ We notice that the condition on mes $((\partial\Omega\setminus\Gamma)\cap B_\rho(x_0))$ occurs in the discussion of Campanato spaces; (see [8; pp. 209-245], [10; p. 32]) for more details.

The above discussion gives rise to introduce the following

Definition *Let* $0 < s < 1$*, let* $1 < q < +\infty$ *and let be* ω *as in Remark* A1.1. *Then*

$$W_{00}^{s,q}(\Gamma) := \left\{ g \in W^{s,q}(\Gamma) : \int\limits_{\Gamma} \omega(y)|g(y)|^q dS_y < +\infty \right\}$$

(cf. the definition of $H_{00}^{\frac{1}{2}}(\Omega)$ in [16; Chap. 1, Thm. 11.7 (with $\mu = 0$ therein)] and the notation $H_{00}^{\frac{1}{2}}(\Gamma)$ in [3; pp. 73, 80 etc.]).

Let $\gamma : W^{1,q}(\Omega) \to W^{1-\frac{1}{q},q}(\partial\Omega)$ $(1 < q < +\infty)$ denote the trace mapping (see, e. g., [8], [11], [19], [24; pp. 281-282, 329-330]). To make things clearer, we also write γ_Ω in place of γ.

Summarizing our preceding discussion, we have:

1° *Let* $h \in W^{1,q}(\Omega)$ *satisfy* $\gamma(h) = 0$ *a. e. on* $\partial\Omega \setminus \Gamma$. *Then*

$$\gamma(h)|_\Gamma \in W_{00}^{1-\frac{1}{q},q}(\Gamma).$$

2° *Let* $g \in W_{00}^{1-\frac{1}{q},q}(\Gamma)$. *Define*

$$\tilde{g} := \begin{cases} g & a.\ e.\ on \quad \Gamma, \\ 0 & a.\ e.\ on \quad \partial\Omega \setminus \Gamma. \end{cases}$$

Then there exists $h \in W^{1,q}(\Omega)$ *such that*

$$\gamma(h) = \tilde{g} \quad a.\ e.\ on \quad \Gamma.$$

Indeed, 1° follows immediately from 1.1. To verify 2°, we notice that our above discussion gives $\tilde{g} \in W^{1-\frac{1}{q},q}(\partial\Omega)$. The claim then follows from the inverse trace theorem (see [8], [19], [24; p. 332]). ■

1.3 We now study the extension of **any** function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial\Omega \setminus \Gamma$ (i. e. without the decay property (A1.4)).

Let $\{e_1, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^N$. We introduce

Assumption (A) *For every* $x \in \bar{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ *there exists*

(i) *a Euclidean basis* $\{f_1, \ldots, f_N\}$ in $\mathbb{R}^N$ [14],

(ii) *an open cube* $\Delta = \{\tau \in \mathbb{R}^{N-1} : \max\{|\tau_1|, \ldots, |\tau_{N-1}| < \delta\}$,

[14] $\{f_1, \ldots, f_N\}$ originates from $\{e_1, \ldots, e_N\}$ by shift and rotation.

(iii) *a Lipschitz function* $a : \Delta \to \mathbb{R}$

such that in terms of local coordinates $\xi \in \mathrm{span}\{f_1, \ldots, f_N\}$ [15] *there holds*

1) $x = (0, \ldots, 0, a(0))$,

2.1) $\{\xi \in \mathbb{R}^d : \xi' \in \Delta,\ a(\xi') < \xi_N < a(\xi') + \delta\} \subset \Omega$,

2.2) $\{\xi \in \mathbb{R}^d : \xi' \in \Delta,\ \xi_N = a(\xi')\} \subset \partial\Omega$,

2.3) $\{\xi \in \mathbb{R}^d : \xi' \in \Delta, -\delta < \xi_{N-1} < 0,\ \xi_N = a(\xi')\} \subset \Gamma$

(cf. figure 2).

For what follows we need some more notations.

$$\begin{aligned}
\Delta^- &:= \{\xi' \in \Delta : -\delta < \xi_{N-1} < 0\}, \\
\Delta^+ &:= \{\xi' \in \Delta : 0 < \xi_{N-1} < \delta\}
\end{aligned}$$

and

$$\phi(\xi) := \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N-1} \\ a(\xi') + \xi_N \end{pmatrix}, \quad \xi = (\xi', \xi_N) \in \Delta \times (-\delta, \delta),$$

$$U := \phi(\Delta \times (-\delta, \delta)).$$

[15] For $\xi = \mathrm{span}\{f_1, \ldots, f_N\}$ we write $\xi = (\xi', \xi_N)$, $\quad \xi' = (\xi_1, \ldots, \xi_{N-1})$.

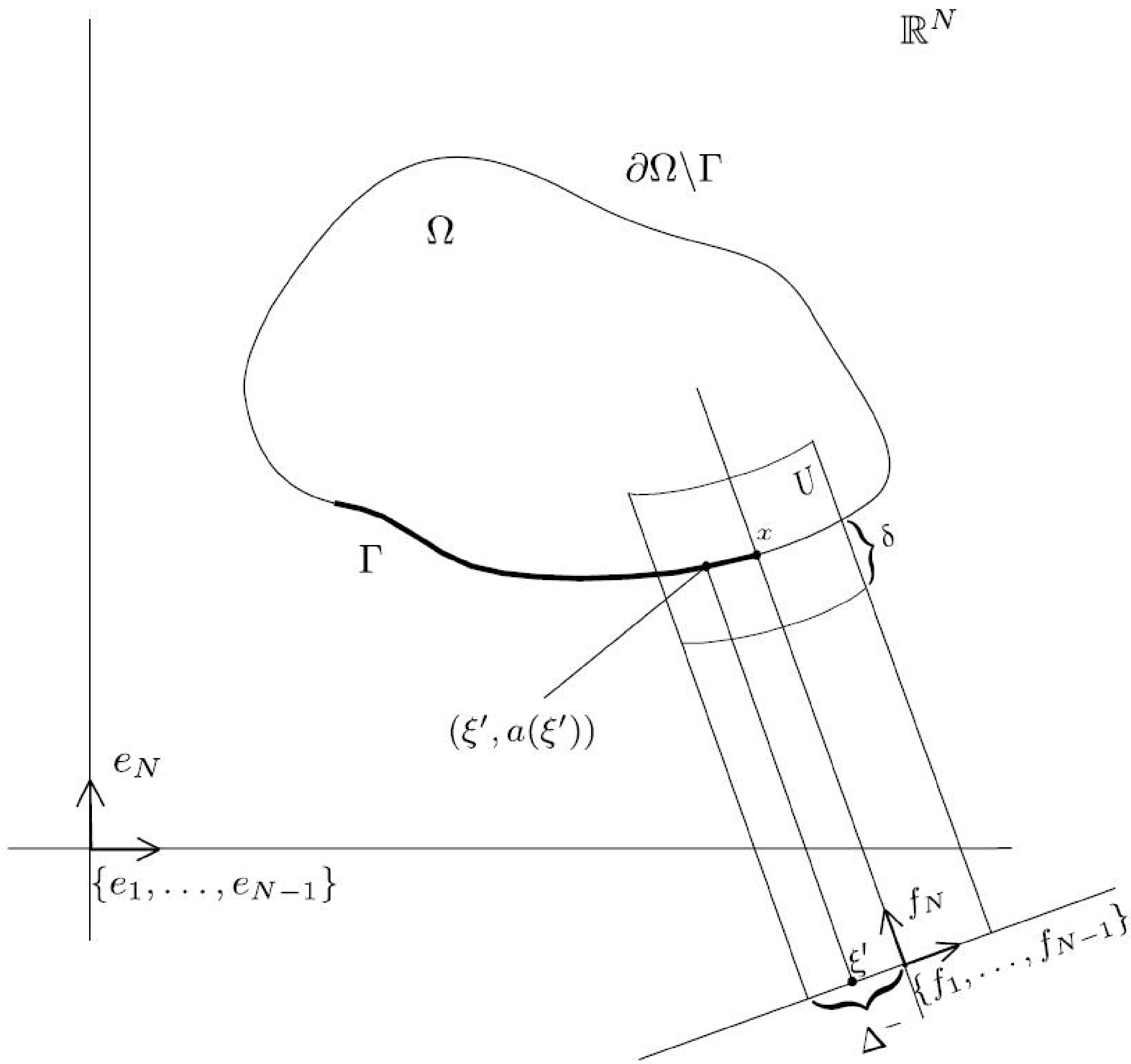
$\mathbb{R}^N$
$\partial\Omega\backslash\Gamma$
Ω
U
x
δ
Γ
$(\xi', a(\xi'))$
e_N
$\{e_1, \ldots, e_{N-1}\}$
f_N
$\{f_1, \ldots, f_{N-1}\}$
ξ'
Δ^-

Figure 2

We obtain

$$\Delta = \Delta^- \cup \{\xi' \in \Delta : \xi_N = 0\} \cup \Delta^+,$$

$$|\xi' - \hat{\xi}'|_{\mathbb{R}^{N-1}} \le |\phi(\xi) - \phi(\hat{\xi}')|_{\mathbb{R}^N} \le c_0|\xi' - \hat{\xi}')|_{\mathbb{R}^{N-1}} \quad \forall\, \xi, \hat{\xi} \in \Delta \times (-\delta, \delta),$$

$$\phi^{-1}(\eta) := \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{N-1} \\ \eta_N - a(\eta') \end{pmatrix}, \quad \eta = (\eta', \eta_N) \in U.$$

Then conditions 1) and 2.1)- 2.3) can be equivalently stated as follows:

1') $\phi(0) = (0, \dots, 0, a(0))^\top$,

2.1') $\phi(\Delta \times (0, \delta)) = \Omega \cap U$,

2.2') $\phi(\Delta \times \{0\}) = \partial\Omega \cap U$,

2.3') $\phi(\Delta^- \times \{0\}) = \Gamma \cap U$.

Theorem A1.1 *Let assumption* (A) *be satisfied and let* $1 < q < +\infty$. *For* $g \in W^{s,q}(\Gamma)$, *define*

$$\tilde{g} := \begin{cases} g & a.\ e.\ on \quad \Gamma, \\ 0 & a.\ e.\ on \quad \partial\Omega \setminus \Gamma. \end{cases}$$

If $s < \dfrac{1}{q}$, *then* $\tilde{g} \in W^{s,q}(\partial\Omega)$ *and*

(A1.5) $$\|\tilde{g}\|_{W^{s,q}(\partial\Omega)} \le c\|g\|_{W^{s,q}(\Gamma)}.$$

Proof The definition of the Lipschitz continuity of $\partial\Omega$ implies the existence of Euclidean coordinate systems $\{f_{\alpha 1}, \dots, f_{\alpha N}\}$ in $\mathbb{R}^N$, open cubes $\Delta_\alpha \subset \mathbb{R}^{N-1}$ and Lipschitz functions $a_\alpha : \Delta_\alpha \to \mathbb{R}$ ($\alpha = 1, \dots, m$) such that 2.1) and 2.2) hold with Δ_α and a_α in place of Δ and a, respectively (see, e. g. [8; pp. 304-306], [10; pp. 21-25], [11; pp. 5-7]). It follows $\partial\Omega \subset \bigcup_{\alpha=1}^{m} U_\alpha$, where

$$U_\alpha := \phi_\alpha(\Delta_\alpha \times (-\delta_\alpha, \delta_\alpha))$$

(recall $\phi_\alpha(\xi) = (\xi', a_\alpha(\xi')+\xi_N)^\top, \quad \xi = (\xi', \xi_N) \in \Delta\times(-\delta, \delta))$. By 2.2), $x_\alpha = (0, \ldots, 0, a_\alpha(0)) \in \partial\Omega$.

If $\Gamma \cap U_\alpha \subset \Gamma$ or $(\partial\Omega \setminus \Gamma) \cap U_\alpha \subset \partial\Omega \setminus \Gamma$ there is nothing to prove. Therefore, it suffices to consider a local representation $\{\{f_{\alpha 1}, \ldots, f_{\alpha N}\}, \Delta_\alpha, a_\alpha\}$ of $\partial\Omega$ such that $x_\alpha \in \bar{\Gamma} \cap (\partial\Omega \setminus \Gamma)$. Then 2.3) of assumption (A) implies

$$\{\xi \in \mathbb{R}^d : \xi' \in \Delta, -\delta_\alpha < \xi_{N-1} < 0, \quad \xi_N = a_\alpha(\xi')\} \;=\; \Gamma \cap U_\alpha.$$

For notational simplicity, in what follows we omit the index α.

Let $g \in W^{s,q}(\Gamma)$. By 2.3),

$$\int\limits_{\Gamma\cap U}\int\limits_{\Gamma\cap U} \frac{|g(x)-g(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y =$$

$$= \int\limits_{\Delta^-}\int\limits_{\Delta^-} \frac{|g\circ\phi(\xi', a(\xi')) - g\circ\phi(\eta', a(\eta'))|^q}{|\phi(\xi', a(\xi')) - \phi(\eta', a(\eta'))|^{N-1+sq}} \sqrt{1+|\nabla a(\xi')|^2}\sqrt{1+|\nabla a(\eta')|^2} d\xi' d\eta'$$

$$\geq c \int\limits_{\Delta^-}\int\limits_{\Delta^-} \frac{|g\circ\phi(\xi', a(\xi')) - g\circ\phi(\eta', a(\eta'))|^q}{|\xi'-\eta'|^{N-1+sq}} d\xi' d\eta'.$$

Next, define $z(\xi') := g\circ\phi(\xi', a(\xi'))$ for a. e. $\xi' \in \Delta^-$, and

$$\tilde{z} := \begin{cases} z & \text{a. e. in} \quad \Delta^-, \\ 0 & \text{a. e. in} \quad \Delta^+. \end{cases}$$

Then $z \in W^{s,q}(\Delta^-)$, and

$$\tilde{z} = \tilde{g}\circ\phi \quad \text{a. e. in} \quad \Delta, \qquad \tilde{g} = \tilde{z}\circ\phi^{-1} \quad \text{a. e. in} \quad \partial\Omega\cap U.$$

Now from [25; Thm. 3.5] (see also [16; Chap. 1, Thm. 11.4] for $q = 2$) it follows that

$$\text{(A1.6)} \qquad \tilde{z} \in W^{s,q}(\Delta), \qquad \|\tilde{z}\|_{W^{s,q}(\Delta)} \leq c\|z\|_{W^{s,q}(\Delta^-)}.$$

We obtain

$$\int\limits_{\partial\Omega\cap U}\int\limits_{\partial\Omega\cap U}\frac{|\tilde g(x)-\tilde g(y)|^q}{|x-y|^{N-1+sq}}dS_xdS_y =$$

$$= \int\limits_{\Delta}\int\limits_{\Delta}\frac{|\tilde g\circ\phi(\xi',a(\xi'))-\tilde g\circ\phi(\eta',a(\eta'))|^q}{|\phi(\xi',a(\xi'))-\phi(\eta',a(\eta'))|^{N-1+sq}}\times$$

$$\times\sqrt{1+|\nabla a(\xi')|^2}\sqrt{1+|\nabla a(\eta')|^2}d\xi'd\eta' \quad \text{[by (2.1)]}$$

$$\leq c_1\int\limits_{\Delta}\int\limits_{\Delta}\frac{|\tilde g\circ\phi(\xi',a(\xi'))-\tilde g\circ\phi(\eta',a(\eta'))|^q}{|\xi'-\eta'|^{N-1+sq}}d\xi'd\eta' \quad \text{[by (iii)]}$$

$$\leq c_2\int\limits_{\Delta^-}\int\limits_{\Delta^-}\frac{|g\circ\phi(\xi',a(\xi'))-g\circ\phi(\eta',a(\eta'))|^q}{|\xi'-\eta'|^{N-1+sq}}d\xi'd\eta' \quad \text{[by (A1.6)]}$$

$$\leq c_3\int\limits_{\Gamma\cap U}\int\limits_{\Gamma\cap U}\frac{|w(x)-w(y)|^q}{|x-y|^{N-1+sq}}dS_xdS_y \quad \text{[by (ii) and 2.3)]}$$

The proof of the theorem is now easily completed by standard arguments. ■

Remark A1.2 If $s=\dfrac{1}{q}$, then the statement of Theorem A1.1 fails.

2 Let $\Omega_1,\Omega_2\subset\mathbb{R}^d$ ($d=2$ or $d=3$) be bounded domains such that

$$\Omega_1\cap\Omega_2=\varnothing,\qquad \Gamma:=\bar\Omega_1\cap\bar\Omega_2\neq\varnothing,$$

$$\partial\Omega_i \text{ Lipschitz, } \Gamma \text{ relatively open in } \partial\Omega_i\ (i=1,2)$$

(cf. Section 1). Let $\gamma_{\Omega_i}: W^{1,q}(\Omega_i)\to W^{1-\frac{1}{q},q}(\partial\Omega_i)$ $(1<q<+\infty)$ denote the trace mapping (cf. above). In what follows, we write $\gamma_i=\gamma_{\Omega_i}$. For $\boldsymbol{u}_i\in\boldsymbol{W}^{1,2}(\Omega_i)$ the trace $\gamma_i(\boldsymbol{u}_i)$ is understood componentwise. By Sobolev's embedding theorem,

$$\text{(A1.7)}\qquad \begin{cases} |\boldsymbol{u}_i|^2\in W^{1,q}(\Omega_i) \quad \text{where}\\ 1\leq q<2 \quad \text{arbitrary if} \quad d=2,\qquad q=\dfrac{3}{2} \quad \text{if} \quad d=3.\end{cases}$$

Then $\gamma_i(|\boldsymbol{u}_i|^2)\in W^{1-\frac{1}{q},q}(\partial\Omega_i)$. ■

Let us consider

$$\boldsymbol{u}_i\in\boldsymbol{W}^{1,2}(\Omega_i),\quad \gamma_i(\boldsymbol{u}_i)=\boldsymbol{0}\quad \text{a.e. on}\quad \partial\Omega_i\setminus\Gamma.$$

For notational simplicity, set $\boldsymbol{v}_i := \gamma_i(\boldsymbol{u}_i)$ a. e. on Γ. Then $\boldsymbol{v}_i \in \boldsymbol{W}^{\frac{1}{2},2}(\Gamma)$, $|\boldsymbol{v}_i|^2 \in W^{1-\frac{1}{q},q}(\Gamma)$ [16)] and

$$\text{(A1.8)} \qquad \int_\Gamma |\boldsymbol{v}_i(y)|^{2q} \Bigg(\int_{\partial\Omega_i \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \Bigg) dS_y < +\infty$$

(cf. (A1.1). To homogenize boundary condition (2.3), we have to consider the following

Problem $(\mathcal{P})$ *Define* $g := |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$ *a. e. on* Γ*, and*

$$\tilde{g}_i := \begin{cases} g & a.\ e.\ on \quad \Gamma, \\ 0 & a.\ e.\ on \quad \partial\Omega_i \setminus \Gamma. \end{cases}$$

Does there exist $\tilde{h}_i \in W^{1,q}(\Omega_i)$ *such that* $\gamma_i(\tilde{h}_i) = \tilde{g}_i$ *a. e. on* $\partial\Omega_i$*?*

An answer to this problem can be given by imposing the following condition on the geometry of Ω_1 and Ω_2 "near to the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$":

Assumption (B) *For every* $y \in \Gamma$*, there holds*

$$\int_{\partial\Omega_1 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x = \int_{\partial\Omega_2 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \quad \text{for all} \quad y \in \Gamma$$

(q as in (A1.7))

We obtain the following result.

Let assumption (B) be satisfied. Let be $\boldsymbol{u}_i \in \boldsymbol{W}^{1,2}(\Omega_i), \gamma_i(\boldsymbol{u}_i) = \boldsymbol{0}$ *a. e. on* $\Omega_i \setminus \Gamma\ (i = 1, 2)$*. Set* $\boldsymbol{v}_i := \gamma_i(\boldsymbol{u}_i)$ *a. e. on* Γ*. If*

$$\text{(A1.9)} \qquad |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 \in W^{1-\frac{1}{q},q}(\Gamma) \qquad (q \text{ as in (A1.7)}),$$

then there exists $\widetilde{h}_i \in W^{1,q}(\Omega_i)$ *such that*

$$\gamma_i(\widetilde{h}_i) = \widetilde{g}_i \quad a.e.\ on \quad \partial\Omega_i.$$

16) The definition of the trace mapping implies

$$(\gamma(\varphi))^2 = (\varphi|_\Gamma)^2 = \varphi^2|_\Gamma = \gamma(\varphi^2)$$

for every $\varphi \in C^1(\bar{\Omega})$. Thus, by approximation

$$|\boldsymbol{v}_i|^2 = \sum_{l=1}^d (\gamma_i(u_{il}))^2 = \sum_{l=1}^d \gamma_i(u_{il}^2) = \gamma_i\Bigg(\sum_{l=1}^d u_{il}^2\Bigg) = \gamma_i(|\boldsymbol{u}_i|^2).$$

Indeed, combining (A1.8) and assumption (B) we find

$$\int_\Gamma |(\boldsymbol{v}_1 - \boldsymbol{v}_2)(y)|^{2q} \left(\int_{\partial\Omega_i \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \right) dS_y < +\infty \quad (i = 1, 2).$$

Observing (A1.9) we see that (A1.3) and (A1.4) are satisfied with $g = |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$, $N = d$, $s = 1 - \dfrac{1}{q}$ and $\Omega = \Omega_i$. The claim follows from $\boxed{1.2}$ above. ■

It is easily verified that this result continues to hold for $G_i(|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2)$ in place of $|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$.

We notice that assumption (B) is satisfied if Ω_1 and Ω_2 obey an appropriate symmetry property with respect to Γ.

Remark A1.2 Assumption (A1.9) is equivalent to

(A1.9') $$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \in W^{1-\frac{1}{q},q}(\Gamma).$$

This is readily seen when observing the elementary identity

$$|\boldsymbol{a} - \boldsymbol{b}|^2 - |\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}}|^2 = |\boldsymbol{a}|^2 - |\hat{\boldsymbol{a}}|^2 + (|\boldsymbol{b}|^2 - |\hat{\boldsymbol{b}}|^2) - 2(\boldsymbol{a} \cdot \boldsymbol{b} - \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}})$$

$(\boldsymbol{a}, \hat{\boldsymbol{a}}, \boldsymbol{b}, \hat{\boldsymbol{b}} \in \mathbb{R}^d)$.

Remark A1.3 We notice that (A1.9') is true in case $d = 2$. To see this, first observe that $W^{\frac{1}{2},2}(\partial\Omega_i) \subset L^r(\partial\Omega_i)$ $(1 \le r < +\infty$ arbitrary). We obtain, for every $1 \le q < 2$,

$$\int_\Gamma \int_\Gamma \frac{|\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^q}{|x-y|^q} |\boldsymbol{v}_j(x)|^q dS_x dS_y \le$$

$$\le \int_\Gamma \left(\int_\Gamma \frac{|\boldsymbol{v}_i(x) - \boldsymbol{v}_i(y)|^2}{|x-y|^2} dS_x \right)^{\frac{q}{2}} \left(\int_\Gamma |\boldsymbol{v}_j(x)|^{\frac{2q}{2-q}} dS_x \right)^{\frac{2-q}{2}} dS_y$$

$$\le (\text{mes}\,\Gamma)^{\frac{2-q}{2}} \|\boldsymbol{v}_i\|^q_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_j\|^q_{\boldsymbol{L}^{\frac{2q}{2-q}}(\Gamma)}$$

$(i, j = 1, 2; i \ne j)$. Whence (A1.9').

We obtain: if $d = 2$ and assumption (B) holds, then problem $(\mathcal{P})$ has a solution. ■

Theorem A1.2 *Suppose that* $\Gamma \cap (\partial\Omega_i \setminus \Gamma)$ $(i = 1, 2)$ *satisfies assumption* (A). *Let* $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \boldsymbol{W}^{\frac{1}{2},2}(\Gamma)$. *Define* $g := |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$ *a. e. on* Γ, *and*

$$\tilde{g}_i := \begin{cases} g & a.\ e.\ on \quad \Gamma, \\ 0 & a.\ e.\ on \quad \partial\Omega_i \setminus \Gamma. \end{cases}$$

Then

$$\tilde{g}_i \in \bigcap_{1 \le q < \frac{d}{d-1}} W^{1-\frac{1}{q},q}(\partial\Omega_i),$$

$$\|\tilde{g}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \le c\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{L}^r(\Gamma)},$$

where

$$\begin{aligned} c &= c(q) \to +\infty \quad as \quad q \to \frac{d}{d-1}, \quad \Big(1 \le q < \frac{d}{d-1}\Big), \\ r &= \frac{2q}{2-q} \quad if \quad d = 2, \qquad r = 4 \quad if \quad d = 3. \end{aligned}$$

Proof $\boxed{d = 2}$ First, notice $W^{\frac{1}{2},2}(\partial\Omega) \subset L^r(\partial\Omega)$ $(1 \le r < +\infty)$ continuously.
Observing that

$$\Big| |\boldsymbol{a} - \boldsymbol{b}|^2 - |\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}}|^2 \Big| \le |\boldsymbol{a} - \boldsymbol{b} - (\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}})|\ |\boldsymbol{a} - \boldsymbol{b} + (\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}})|, \qquad \boldsymbol{a}, \hat{\boldsymbol{a}}, \boldsymbol{b}, \hat{\boldsymbol{b}} \in \mathbb{R}^N,$$

we obtain by the aid of Hölder's inequality, for every $1 \le q < 2$,

$$\begin{aligned} &\int_\Gamma \int_\Gamma \frac{|g(x) - g(y)|^q}{|x - y|^q}\ dS_x dS_y \le \\ &\le \left(\int_\Gamma \int_\Gamma \frac{|\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x) - (\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y))|^2}{|x - y|^2}\ dS_x dS_y \right)^{\frac{q}{2}} \times \\ &\quad \times \left(\int_\Gamma \int_\Gamma |\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x) + (\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y))|^{\frac{2q}{2-q}} dS_x dS_y \right)^{\frac{2-q}{q}} \\ &\le c\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^q_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^q_{\boldsymbol{L}^{\frac{2q}{2-q}}(\Gamma)}. \end{aligned}$$

Thus, $g \in W^{1-\frac{1}{q},q}(\Gamma)$ and

$$
\begin{aligned}
\|g\|^q_{W^{1-\frac{1}{q}}(\Gamma)} &\leq \left(\int\limits_\Gamma |\boldsymbol{v}_1 - \boldsymbol{v}_2|^{\frac{2q}{2-q}} \right)^{2-q} + \int\limits_\Gamma \int\limits_\Gamma \frac{|g(x)-g(y)|^q}{|x-y|^q} \, dS_x dS_y \\
&\leq c\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^q_{\boldsymbol{W}^{\frac{1}{2}}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^q_{\boldsymbol{L}^{\frac{2q}{2-q}}(\Gamma)}.
\end{aligned}
$$

On the other hand, Theorem A1.1 (with $\Omega = \Omega_i$, $s = 1 - \dfrac{1}{q}, s < \dfrac{1}{q}$) gives

$$
\tilde{g}_i \in W^{1-\frac{1}{q}}(\partial\Omega_i), \quad \|\tilde{g}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \leq c\|g\|_{W^{1-\frac{1}{q},q}(\Gamma)}.
$$

Whence the claim.

$\boxed{d=3}$ Then $W^{\frac{1}{2},2}(\partial\Omega) \subset L^4(\partial\Omega)$ continuously. Hence $g \in L^2(\Gamma)$. We divide the proof into two steps.

Step 1 For every $0 < \delta < 1$, there holds

$$
\text{(A1.10)} \qquad \begin{cases} g \in W^{\frac{1-\delta}{2},\frac{4}{3}}(\Gamma), \\ \displaystyle\int\limits_\Gamma \int\limits_\Gamma \frac{|g(x)-g(y)|^{\frac{4}{3}}}{|x-y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \leq c\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^{\frac{4}{3}}_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^{\frac{4}{3}}_{\boldsymbol{L}^4(\Gamma)}. \end{cases}
$$

Indeed, with the help of the above inequality for $\boldsymbol{a}, \hat{\boldsymbol{a}}, \boldsymbol{b}, \hat{\boldsymbol{b}} \in \mathbb{R}^N$ and Hölder's inequality we find

$$
\begin{aligned}
&\int\limits_\Gamma \int\limits_\Gamma \frac{|g(x)-g(y)|^{\frac{4}{3}}}{|x-y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \leq \\
&\leq c \int\limits_\Gamma \int\limits_\Gamma \frac{\Big| |\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x)|^2 - |\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y)|^2 \Big|^{\frac{4}{3}}}{|x-y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \\
&\leq c \int\limits_\Gamma \int\limits_\Gamma \frac{\Big|\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x) - (\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y))|^{\frac{4}{3}}}{|x-y|^2} \times \\
&\quad \times \frac{|\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x) + (\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y))|^{\frac{4}{3}}}{|x-y|^{\frac{2(1-\delta)}{3}}} dS_x dS_y
\end{aligned}
$$

$$\text{(A1.11)} \qquad \leq c\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^{\frac{4}{3}}_{W^{\frac{1}{2},2}(\Gamma)} \times$$
$$\times \left(\int_\Gamma \int_\Gamma \frac{(|\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x)| + |\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y)|)^4}{|x-y|^{2(1-\delta)}} dS_x dS_y \right)^{\frac{1}{3}}$$

(notice that $W^{\frac{1}{2},2}(\Gamma) \subset L^4(\Gamma)$).

Next, by elementary integral calculus it is easily seen that there exists a positive constant K_0 such that

$$\int_{\partial\Omega_i} \frac{1}{|x-y|^{2(1-\delta)}} dS_y \leq K_0 \qquad \forall\, x \in \partial\Omega_i \quad (i = 1, 2)$$

($K_0 = K_0(\delta) \to +\infty$ as $\delta \to 0$). Then the second double integral on the right hand side of (A1.11) can be estimated as follows

$$\int_\Gamma \int_\Gamma \frac{(|\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x)| + |\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y)|)^4}{|x-y|^{2(1-\delta)}} dS_x dS_y \leq$$

$$\leq 16 \int_\Gamma |\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x)|^4 \left(\int_\Gamma \frac{1}{|x-y|^{2(1-\delta)}} dS_y \right) dS_x$$
$$+16 \int_\Gamma |\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y)|^4 \left(\int_\Gamma \frac{1}{|x-y|^{2(1-\delta)}} dS_x \right) dS_y$$

$$\leq 32 K_0 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^4_{\boldsymbol{L}^4(\Gamma)}.$$

Inserting this estimate into (A1.11) we find (A1.10) ($c = c(\delta) \to +\infty$ as $\delta \to 0$).

Step 2 From Theorem A1.1 (with $\Omega = \Omega_i$, $s = \dfrac{1-\delta}{2}, q = \dfrac{4}{3}$) and (A1.10) it follows that

$$\tilde{g} \in W^{\frac{1-\delta}{2},\frac{4}{3}}(\partial\Omega_i),$$
$$\begin{aligned}
\|\tilde{g}\|_{W^{\frac{1-\delta}{2},\frac{4}{3}}(\partial\Omega_i)} &\leq c\|g\|_{W^{\frac{1-\delta}{2},\frac{4}{3}}(\Gamma)} = \\
&= c\left(\|g\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|\tilde{g}(x) - \tilde{g}(y)|^{\frac{4}{3}}}{|x-y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \right)^{\frac{3}{4}} \\
&\leq (\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{\boldsymbol{L}^{\frac{8}{3}}(\Gamma)} + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{L}^4(\Gamma)})
\end{aligned}$$

(A1.12) $$\le (\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_{\boldsymbol{W}^{\frac{1}{2},2}(\Gamma)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{W}^4(\Gamma)}.$$

To proceed, we notice the continuous embedding

(A1.13) $$W^{\frac{1-\delta}{2},\frac{4}{3}}(\partial\Omega_i) \subset W^{\frac{1-\delta}{2}-\alpha,\frac{4}{3-2\alpha}}(\partial\Omega_i) \qquad \left(0 < \alpha < \frac{1-\delta}{2}\right)$$

(see, e. g., [1], [25; p. 328, $n = d - 1 = 2$ in (8)]).

Now, consider q such that $\frac{4}{3} < q < \frac{3}{2}$. Define

$$\delta := \frac{2(3-2q)}{q}, \qquad \alpha := \frac{1-2\delta}{6}.$$

It follows

$$\frac{1-\delta}{2} - \alpha = 1 - \frac{1}{q}, \qquad \frac{4}{3-2\alpha} = q.$$

By combining (A1.12) and (A1.13) we obtain the statement of Theorem A1.2 when $d = 3$.

■

Corollary A1.1 *Suppose that $\Gamma \cap (\partial\Omega_i \setminus \Gamma)$ $(i = 1, 2)$ satisfies assumption (A). Let be $\boldsymbol{u}_i \in \boldsymbol{W}^{1,2}(\Omega_i)$ such that*

$$\gamma_i(\boldsymbol{u}_i) = \boldsymbol{0} \quad \text{a. e. on} \quad \partial\Omega_i \setminus \Gamma.$$

Define

$$\tilde{h}_i := \begin{cases} G_i(|\gamma_1(\boldsymbol{u}_1) - \gamma_2(\boldsymbol{u}_2)|^2) & \text{a. e. on} \quad \Gamma, \\ 0 & \text{a. e. on} \quad \partial\Omega_i \setminus \Gamma \end{cases}$$

(G_i as in (1.6); $i = 1, 2$).

Then, for every $1 \le q < \frac{d}{d-1}$,

$$\tilde{h}_i \in W^{1-\frac{1}{q},q}(\partial\Omega_i), \ \ \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \le c \sum_{j=1}^{2} \|\boldsymbol{u}_j\|^2_{\boldsymbol{W}^{1,2}(\Omega_j)},$$

where $c = c(q) \to +\infty$ *as* $q \to \dfrac{d}{d-1}$.

Proof As above, for notational simplicity, set $\boldsymbol{v}_i := \gamma_i(\boldsymbol{u}_i)$ and $h_i := G_i(|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2)$ a. e. on Γ $(i = 1, 2)$. Then

$$\tilde{h}_i := \begin{cases} h_i & a.\ e.\ on \quad \Gamma, \\ 0 & a.\ e.\ on \quad \partial\Omega_i \setminus \Gamma \end{cases}$$

and

$$|h_i(x) - h_i(y)| \le c_0 \Big| |\boldsymbol{v}_1(x) - \boldsymbol{v}_2(x)|^2 - |\boldsymbol{v}_1(y) - \boldsymbol{v}_2(y)| \Big| \quad \text{for a. e.} \quad x, y \in \Gamma.$$

It is readily seen that the proof of Theorem A1.2 can be repeated word by word with h_i and $\tilde{h}_i$ in place of g and $\tilde{g}_i$, respectively. We obtain

$$\tilde{h}_i \in \bigcap_{1 \le q < \frac{d}{d-1}} W^{1-\frac{1}{q},q}(\partial\Omega_i),$$

$$\|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \le c \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{W}^{\frac{1}{2}(\Gamma)}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\boldsymbol{L}^r(\Gamma)},$$

where r is as in Theorem A1.2.

Combining this and the continuity of the trace mapping $\gamma_i : W^{1,2}(\Omega_i) \to W^{\frac{1}{2},2}(\partial\Omega_i)$ we get the assertion of the corollary.

Appendix 2. The inhomogeneous Dirichlet problem for the Poisson equation with right hand side in L^1

Let $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded domain with boundary $\partial\Omega \in \mathcal{C}^1$. We consider the following boundary value problem:

(A2.1) $\qquad -\Delta u = f \quad \text{in} \quad \Omega,$

(A2.2) $\qquad u = g \quad \text{on} \quad \partial\Omega.$

Our basic existence result concerning weak solutions to this problem is

Theorem A2.1 *Assume*

$$f \in L^1(\Omega), \quad g \in W^{1-\frac{1}{q},q}(\partial\Omega) \quad \left(1 < q < \tfrac{N}{N-1}\right)$$

Then, there exists exactly one $u \in W^{1,q}(\Omega)$ *such that*

(A2.3) $$\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega f\varphi \quad \forall\, \varphi \in W_0^{1,q'}(\Omega),$$

(A2.4) $$u = g \quad \text{on} \quad \partial\Omega,$$

(A2.5) $$\|u\|_{W^{1,q}} \le c(\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q},q}})$$

Moreover, for every $\Omega' \subset\subset \Omega$ and every $\delta > 0$ there holds

(A2.6) $$\begin{cases} \dfrac{\nabla u}{(1+|u|)^{\frac{1+\delta}{2}}} \in \boldsymbol{L}^2(\Omega'), \\ \displaystyle\int_{\Omega'} \frac{|\nabla u|^2}{(1+|u|)^{1+\delta}} \le \frac{c}{\delta}(\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q},q}}) \\ \textit{where} \quad c \to +\infty \quad \textit{as} \quad \operatorname{dist}(\Omega', \partial\Omega) \to 0. \end{cases}$$

If, in addition, $f \in L^r_{\text{loc}}(\Omega)$ $(r > 1)$ then

(A2.7) $$u \in W^{2,r}_{\text{loc}}(\Omega).$$

Proof We begin by noting the following result. For every $1 < q < +\infty$ there exists a positive constant C_q such that, for any $v \in W_0^{1,q}(\Omega)$,

(A2.8) $$\|\nabla v\|_{\boldsymbol{L}^q} \le C_q \sup \left\{ \frac{\displaystyle\int_\Omega \nabla v \cdot \nabla\varphi}{\|\nabla\varphi\|_{\boldsymbol{L}^{q'}}}; \quad \varphi \in W_0^{1,q'}(\Omega), \;\; \varphi \ne 0 \right\}$$

(see. [21; Thm. 4.2, p. 191]).

Next, by the inverse trace theorem, there exists $h \in W^{1,q}(\Omega)$ such that

$$\gamma(h) = g \quad \text{a. e. on} \quad \partial\Omega, \quad \|h\|_{W^{1,q}} \le c\|g\|_{W^{1-\frac{1}{q},q}}.$$

Then we can find functions $f_m, h_m \in C^\infty(\bar\Omega)$ $(m \in \mathbb{N})$ such that

$$f_m \to f \quad \text{strongly in} \quad L^1(\Omega), \quad h_m \to h \quad \text{strongly in} \quad W^{1,q}(\Omega)$$

as $m \to \infty$. The Riesz representation theorem for linear continuous functionals on the Hilbert space $W_0^{1,2}(\Omega)$ provides the existence and uniqueness of a $v_m \in W_0^{1,2}(\Omega)$ satisfying

(A2.9) $$\int_\Omega \nabla v_m \cdot \nabla\varphi = \int_\Omega (f_m\varphi + (\partial_i h_m)\partial_i\varphi) \quad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

Now, let $1 < q < \frac{N}{N-1}$. Observing that $W^{1,q'}(\Omega) \subset C(\bar{\Omega})$ we obtain

$$\Big| \int\limits_{\Omega} (f_m \varphi + (\partial_i h_m) \partial_i \varphi) \Big| \le c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}) \|\varphi\|_{W^{1,q'}} \quad \forall\, \varphi \in W_0^{1,q'}(\Omega).$$

Combining this estimate and (A2.8), (A2.9) gives

$$\|\nabla v_m\|_{\boldsymbol{L}^q} \le c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}).$$

Define $u_m := v_m + h_m$ $(m \in \mathbb{N})$. Then $u_m \in W^{1,2}(\Omega)$ and

$$\text{(A2.10)} \qquad \int\limits_{\Omega} \nabla u_m \cdot \nabla \varphi = \int\limits_{\Omega} f_m \varphi \qquad \forall\, \varphi \in W_0^{1,2}(\Omega),$$

$$\text{(A2.11)} \qquad u_m = h_m \quad \text{a. e. on} \quad \partial\Omega \quad \text{[in the sense of traces]},$$

$$\text{(A2.12)} \qquad \|\nabla u_m\|_{\boldsymbol{L}^q} \le c(\|f_m\|_{L^q} + \|h_m\|_{W^{1,q}}).$$

From (A2.12) we conclude (by passing to a subsequence if necessary) that $u_m \to u$ weakly in $W^{1,q}(\Omega)$ as $m \to \infty$. By a routine argument, $u = g$ a. e. on $\partial\Omega$ (in the sense of traces). The passage to the limit $m \to \infty$ in (A2.10), (A2.11) gives (A2.3), (A2.4), respectively. Finally, taking the $\liminf\limits_{m\to\infty}$ on both sides of (A2.12) provides (A2.5).

The uniqueness of u follows from (A2.5).

To prove the interior estimate (A2.6), let $\delta > 0$. We consider the function

$$\phi(t) = \phi_\delta(t) := \left(1 - \frac{1}{(1+|t|)^\delta}\right) \operatorname{sign} t, \quad t \in \mathbb{R}.$$

Clearly,

$$|\phi(t)| \le 1, \quad \phi'(t) = \frac{\delta}{(1+|t|)^{1+\delta}} \quad \forall\, t \in \mathbb{R}.$$

Let $\zeta \in C_c^1(\Omega)$ be a cut-off function for Ω', i. e. $\zeta \equiv 1$ on Ω' and $0 \le \zeta \le 1$ in Ω. Then the function $\varphi = \phi(u_m)\zeta^2$ is admissible in (A2.10). By (A2.12),

$$\begin{aligned} \delta \int\limits_{\Omega'} \frac{|\nabla u_m|^2}{(1+|u_m|)^{1+\delta}} &\le \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| \int\limits_{\Omega} |\nabla u_m| \\ &\le \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| (\text{mes } \Omega)^{\frac{1}{q'}} \cdot c\Big(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}\Big). \end{aligned}$$

Thus,

(A2.13) $$\int\limits_{\Omega'} \frac{|\nabla u_m|^2}{(1+|u_m|)^{1+\delta}} \le \frac{C}{\delta} \quad \forall\, m \in \mathbb{N} \;\; (C = \text{ const}).$$

As above, we may assume that $u_m \to u$ weakly in $W^{1,q}(\Omega)$ and, in addition, $u_m \to u$ a. e. in Ω. These convergence properties together with (A2.13) imply

$$\frac{\nabla u_m}{(1+|u_m|)^{\frac{1+\delta}{2}}} \to \frac{\nabla u}{(1+|u|)^{\frac{1+\delta}{2}}} \quad \text{weakly in} \quad \boldsymbol{L}^2(\Omega') \quad as \quad m \to \infty.$$

Whence (2.6).

To prove (A2.7), we first note that $W^{1,q}(\Omega) \subset L^{\frac{Nq}{N-q}}(\Omega)$. Now, let B_R be a ball such that $B_{2R} \subset \Omega$. Let $1 < r \le \frac{Nq}{N-q}$. Then $u \in L^r(B_{2R})$ and

$$\left| \int\limits_{B_{2R}} u \Delta\varphi \right| \le \|f\|_{L^r(B_{2R})} \|\varphi\|_{L^{r'}(B_{2R})} \quad \forall\, \varphi \in C_c^\infty(B_{2R}) \quad \text{[by (A2.3)]}.$$

From [20; Thm. 9.5 (3), p. 144] it follows

$$u \in W^{2,r}(B_R),\; \|u\|_{W^{2,r}(B_R)} \le c(\|f\|_{L^r(B_{2R})} + \|u\|_{L^r(B_{2R})}).$$

Hence, (A2.7) holds for all values of r satisfying $1 < r \le \frac{Nq}{N-q}$. By a bootstrapping argument, (A2.7) can be proved for any $r > \frac{Nq}{N-q}$. ■

Remark A2.1 We notice that the existence and uniqueness result stated in Theorem A2.1, follows from the L^p-theory of linear elliptic boundary value problems developed in [15], provided the boundary $\partial\Omega$ is sufficiently smooth. Theorem A2.1 is also an immediate consequence of [20; Thm. 10.7, pp. 181-182; $\partial\Omega \in \mathcal{C}^1$].

On the other hand, the existence of a weak solution $u \in \bigcap\limits_{1<q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ to linear elliptic equations in divergence form with bounded measurable coefficients, right hand sides in L^1 and zero boundary condition has been proved in [23] by a duality argument.

Remark A2.2 Our approximation procedure for solving boundary value problem (A2.1), (A2.2) permits to prove additional properties of the weak solution $u \in W^{1,q}(\Omega)$ $(1 < q < \frac{N}{N-1})$ (for instance, the interior estimate (A2.6)). Moreover, we have

Theorem A2.2 *Let the assumptions of Theorem A2.1 hold. Let* $u \in W^{1,q}(\Omega)$ *satisfy (A2.3)-(A2.5). Then*

1° *if* $f \ge 0$ *a. e. in* Ω *and* $g \ge 0$ *a. e. on* $\partial\Omega$, *then*

$$u \ge 0 \quad a.\ e.\ in \quad \Omega;$$

2° *if* $f \in L^r_{\mathrm{loc}}(\Omega)$ $(r > \frac{N}{2})$, *then*

$$\operatorname*{ess\,sup}_{\Omega'} |u| < +\infty \quad \forall\, \Omega' \subset\subset \Omega;$$

3° *if* $f \in L^r(\Omega)$ $(r > \frac{N}{2})$, $\operatorname*{ess\,sup}_{\partial\Omega} |g| < +\infty$, *then*

$$\operatorname*{ess\,sup}_{\Omega} |u| < +\infty.$$

This theorem can be proved by the methods developed in [5] and [23].

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