

Mixed finite element methods for two-body contact problems

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Abstract

This paper presents mixed finite element methods of higher-order for two-body contact problems of linear elasticity. The discretization is based on a mixed variational formulation proposed by Haslinger et al. which is extended to higher-order finite elements. The main focus is on the convergence of the scheme and on a priori estimates for the h - and p -method. For this purpose, a discrete inf-sup condition is proven which guarantees the stability of the mixed method. Numerical results confirm the theoretical findings.

Keywords:

higher-order FEM, contact problems, mixed methods

1. Introduction

The aim of this paper is to derive mixed finite element methods of higher-order for two-body contact problems in linear elasticity. The discretization approach is based on mixed finite elements for contact problems introduced by Haslinger et al. in [14, 16, 18]. This approach was originally developed for low-order finite elements. In this paper, we extend it to higher-order discretizations and to two-body contact problems. The approach relies on a saddle point formulation. The introduced Lagrange multiplier is defined on the surface of one of the bodies in contact and enforces the geometrical contact condition via a sign condition.

To guarantee the uniqueness of the solution of the mixed scheme and to show its convergence one has to provide a uniform discrete inf-sup condition which balances the discretization spaces of the primal variable and of the Lagrange multiplier. It is an essential assumption to show the convergence of the mixed scheme without regularity assumptions, to derive a priori estimates and to determine convergence rates based on these estimates.

In this work, the higher-order discretization of the primal variable is given via a conforming ansatz using tensor product polynomials. The discretization space of the discrete Lagrange multiplier is also based on such tensor products. To include the sign condition, we enforce the discrete Lagrange multiplier to be positive only in Gauss quadrature points leading to a non-conforming discretization. This approach was already suggested in [7] for frictional contact problems. We show the convergence of the mixed scheme and discuss some arguments as proposed by Haslinger et al. and Lhalouani et al., cf. [5, 14, 15, 16, 22] to determine convergence rates for low-order discretizations of the Lagrange multiplier. The main result is the derivation of convergence rates with respect to higher-order discretizations in both variables. The essential ingredient is to intensively utilize the discretization of the Lagrange multiplier via its definition in Gauss points. This enables to apply higher-order interpolations as introduced in [3] as well as quadrature rules for the exact integration of polynomials.

This work also deals with the verification of a uniform discrete inf-sup condition. For low-order finite elements and one-body contact problems, the discrete inf-sup condition is proven in [14, 16]. An essential assumption of the proof is that the discretization of the Lagrange multiplier is defined on boundary meshes with a different mesh size than that of the primal variable. We show that in the higher-order approach, this assumption can, in principle, be

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avoided using different polynomial degrees. In the proof of the discrete inf-sup condition, we use approximation results of the p -method of finite elements and some inverse estimates for higher-order polynomials, cf. [2, 11]. In particular, we adapt the proof of the discrete inf-sup condition for frictional one-body contact problems as described in [25].

Higher-order discretization schemes for contact problems are rarely studied in literature, especially for mixed variational formulations. We refer to [21] for finite element discretizations based on primal, non-mixed formulations, to [7] for mixed methods using a mortar approach and to [24] for boundary element methods. Mixed methods with quadratic finite elements are described in [17, 19].

The paper is organized as follows: In Section 2, the two-body contact problem and its mixed variational formulation are introduced. The convergence of the mixed scheme and general a priori estimates are discussed in Section 3. In Section 4 the discretization of higher-order is presented, its convergence is proven and convergence rates are determined. A uniform discrete inf-sup condition is proven in Section 5. Finally, numerical results confirming the theoretical findings are discussed in Section 6.

2. Two-body contact problem and its mixed variational formulation

We consider the deformation of two bodies being in contact. They are represented by the domains $\Omega^l \subset \mathbb{R}^k$, $k \in \{2, 3\}$, $l \in \{1, 2\}$, with sufficiently smooth boundaries $\Gamma^l := \partial\Omega^l$ and are clamped at some boundary parts which are represented by the closed sets $\Gamma_D^l \subset \Gamma^l$ with positive measure. The boundary parts of the bodies where the bodies possibly get in contact are described by open sets Γ_C^l where we assume $\bar{\Gamma}_C^l \subsetneq \Gamma^l \setminus \Gamma_D^l$ and $\Gamma_N^l := \Gamma^l \setminus (\Gamma_D^l \cup \bar{\Gamma}_C^l)$. Volume and surface forces act on the bodies. They are described by functions $f^l \in L^2(\Omega^l; \mathbb{R}^k)$ and $q^l \in L^2(\Gamma_N^l; \mathbb{R}^k)$. The resulting deformation is described by displacement fields $v^l \in H^1(\Omega^l; \mathbb{R}^k)$ with the linearized strain tensor $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^\top)$. The stress tensor describing the linear-elastic material law is defined as $\sigma^l(v)_{ij} := C_{ijkl}^l \varepsilon(v)_{kl}$, where $C_{ijkl}^l \in L^\infty(\Omega)$ with $C_{ijkl}^l = C_{jilk}^l = C_{klij}^l$ and $C_{ijkl}^l \tau_{ij} \tau_{kl} \geq \kappa \tau_{ij}^2$ for all $\tau \in L^2(\Omega; \mathbb{R}^{k \times k})$ with $\tau_{ij} = \tau_{ji}$ and a constant $\kappa > 0$. We set $H_D^1(\Omega^l) := \{v \in H^1(\Omega^l; \mathbb{R}^k) \mid \gamma_{\Gamma_D^l}^l(v) = 0, i = 1, \dots, k\}$ for the trace operator $\gamma^l \in L(H^1(\Omega^l), L^2(\Gamma^l))$ and define $(\sigma_n^l(v))_i := \sigma_{ij}^l(v) n_j^l$, $v_n^l := v_i^l n_i^l$, $\sigma_{nn}^l(v) := \sigma_{ij}^l(v) n_i^l n_j^l$, $\sigma_{nr}^l(v) := \sigma_n^l(v) - \sigma_{nn}^l(v) n$ with outer normal n^l of Γ^l . For a bijective, sufficiently smooth mapping $\Phi : \Gamma_C^1 \rightarrow \Gamma_C^2$ and $x \in \Gamma_C^1$, we define

$$\tilde{n}(x) := \begin{cases} \frac{\Phi(x)-x}{|\Phi(x)-x|}, & x \neq \Phi(x), \\ n^1(x) = -n^2(x), & x = \Phi(x) \end{cases}$$

and the gap function $g(x) := |x - \Phi(x)|$. Furthermore, we set $[v^1, v^2]_{\tilde{n}}(x) := v_i^1(x) \tilde{n}_i - v_i^2(\Phi(x)) \tilde{n}_i$ for functions v^1 and v^2 on Γ_C^1 and Γ_C^2 , respectively. The two-body contact problem is thus to find displacement fields u^1 and u^2 such that

$$\begin{aligned} -\operatorname{div} \sigma^l(u^l) &= f^l \text{ in } \Omega^l, \\ u^l &= 0 \text{ on } \Gamma_D^l, \\ \sigma_n^l(u^l) &= q^l \text{ on } \Gamma_N^l, \\ \sigma_{nr}^l(u^l) &= 0 \text{ on } \Gamma_C^l, \\ [u^1, u^2]_{\tilde{n}} &\leq g, \quad \sigma_{\tilde{n}\tilde{n}}^1(u^1) \leq 0, \quad \sigma_{\tilde{n}\tilde{n}}^1(u^1)([u^1, u^2]_{\tilde{n}} - g) = 0 \text{ on } \Gamma_C^1. \end{aligned}$$

In this paper, the following notational conventions are used. The space $H^{-1/2}(\Gamma_C^1)$ denotes the topological dual space of $H^{1/2}(\Gamma_C^1)$ with norms $\|\cdot\|_{-1/2, \Gamma_C^1}$ and $\|\cdot\|_{1/2, \Gamma_C^1}$, respectively. Let $(\cdot, \cdot)_{0, \omega}$ and $(\cdot, \cdot)_{0, \Gamma'}$ be the usual L^2 -scalar products on $\omega \subset \mathbb{R}^k$ and $\Gamma' \subset \Gamma_C^1$, respectively. For $v \in H_D^1(\Omega^l)$ and $w \in L^2(\Gamma')$, we define $\|v\|_{0, \Omega^l}^2 := (v_i, v_i)_{0, \Omega^l}$ and $\|w\|_{0, \Gamma'}^2 := (w, w)_{0, \Gamma'}$. Furthermore, the usual H^1 -norm on $H_D^1(\Omega^l)$ is denoted by $\|\cdot\|_{1, \Omega^l}$. We define $\gamma_N^l \in L(H_D^1(\Omega), L^2(\Gamma_N^l, \mathbb{R}^k))$ as $\gamma_N^l(v)_i = \gamma_{\Gamma_N^l}^l(v)_i$ and $\mathcal{H}_D := H_D^1(\Omega^1) \times H_D^1(\Omega^2)$, which is a Hilbert space with the norm $\|v\|_1^2 := \sum_{l=1,2} \|v^l\|_{1, \Omega^l}^2$ for $v \in \mathcal{H}_D$. We set $\gamma_{C\tilde{n}} \in L(\mathcal{H}_D, H^{1/2}(\Gamma_C^1))$ as $\gamma_{C\tilde{n}}(v) := [\gamma_C^1(v^1), \gamma_C^2(v^2)]_{\tilde{n}}$ which is surjective due to the assumptions on Γ_C^1 , cf. [8]. Finally, we introduce some interpolation spaces $H^{1+\theta}(\Omega^l)$ and $H^{-1/2+\theta}(\Gamma_C^1)$ for $\theta > 0$ which are defined via $H^{1+\theta}(\Omega^l) := [H^1(\Omega^l), H^2(\Omega^l)]_{\theta, 2}$ and $H^{-1/2+\theta}(\Gamma_C^1) := [H^{-1/2}(\Gamma_C^1), H^{1/2}(\Gamma_C^1)]_{\theta, 2}$ with norms $\|\cdot\|_{1+\theta, \Omega^l}$ and $\|\cdot\|_{-1/2+\theta, \Gamma_C^1}$, respectively, cf. [23, 26].

It is well-known, that the solution of the two-body contact problem $u \in \mathcal{H}_D$ is also a solution $u \in K := \{v \in \mathcal{H}_D \mid \gamma_{C\bar{n}}(v) \leq g\}$ of the variational inequality

$$a(u, v - u) \geq \ell(v - u)$$

for all $v \in K$, where $a(u, v) := \sum_{l=1,2} (\sigma_{ij}(u^l), \varepsilon_{ij}(v^l))_0$ and $\ell(v) := \sum_{l=1,2} \left((f_i^l, v_i^l)_0 + (q_i^l, \gamma_N^l(v^l))_{0, \Gamma_N^l} \right)$. The inequality above is fulfilled if and only if u is a minimizer of the functional $E(v) := \frac{1}{2}a(v, v) - \ell(v)$ in K . Due to Cauchy's and Korn's inequalities a is continuous and \mathcal{H}_D -elliptic, i.e., there exist constants ν_0 and ν_1 , so that

$$a(u, v) \leq \nu_0 \|u\|_1 \|v\|_1, \quad \nu_0 \|v\|_1^2 \leq a(v, v) \quad (2.1)$$

for all $u, v \in \mathcal{H}_D$. Therefore, the functional E is strictly convex, continuous and coercive. This implies the existence of a unique minimizer u due to the convexity and closedness of K . Given the Lagrange functional $\mathcal{L}(v, \mu) := E(v) + \langle \mu, \gamma_{C\bar{n}}(v) - g \rangle$ on $\mathcal{H}_D \times H_+^{-1/2}(\Gamma_C^1)$, the Hahn-Banach theorem yields

$$E(u) = \inf_{v \in \mathcal{H}_D} \sup_{\mu \in H_+^{-1/2}(\Gamma_C^1)} \mathcal{L}(v, \mu) \quad (2.2)$$

for $H_+^{1/2}(\Gamma_C^1) := \{w \in H^{1/2}(\Gamma_C^1) \mid w \geq 0\}$ and $H_+^{-1/2}(\Gamma_C^1) := \{\mu \in H^{-1/2}(\Gamma_C^1) \mid \forall w \in H_+^{1/2}(\Gamma_C^1) : \langle \mu, w \rangle \geq 0\}$. Note that we assume $g \in H^{1/2}(\Gamma_C^1)$. Thus, u is a minimizer of E , whenever $(u, \lambda) \in \mathcal{H}_D \times H_+^{-1/2}(\Gamma_C^1)$ is a saddle point of \mathcal{L} . The existence of a unique saddle point is guaranteed, if there exists a constant $\alpha > 0$ such that

$$\alpha \|\mu\|_{-1/2, \Gamma_C^1} \leq \sup_{v \in \mathcal{H}_D, \|v\|_1=1} \langle \mu, \gamma_{C\bar{n}}(v) \rangle \quad (2.3)$$

for all $\mu \in H^{-1/2}(\Gamma_C^1)$, cf. [10]. In fact, it follows from the closed range theorem and the surjectivity of $\gamma_{C\bar{n}}$, that (2.3) is valid. Due to the stationarity condition, $(u, \lambda) \in \mathcal{H}_D \times H_+^{-1/2}(\Gamma_C^1)$ is a saddle point of \mathcal{L} if and only if it fulfills the mixed variational formulation

$$\begin{aligned} a(u, v) &= \ell(v) - \langle \lambda, \gamma_{C\bar{n}}(v) \rangle, \\ \langle \mu - \lambda, \gamma_{C\bar{n}}(u) - g \rangle &\leq 0 \end{aligned} \quad (2.4)$$

for all $v \in \mathcal{H}_D$ and $\mu \in H_+^{-1/2}(\Gamma_C^1)$.

3. Discretization of the mixed variational formulation

Let \mathcal{S}_h^p and \mathcal{M}_H^q be finite dimensional subspaces of \mathcal{H}_D and $H^{-1/2}(\Gamma_C^1)$, respectively. Furthermore, let $\mathcal{M}_{H,+}^q$ be a convex and closed subset of \mathcal{M}_H^q . Here, h, H, p , and q denote some parameters specifying the finite element discretizations as introduced in Section 5. The discrete saddle point problem of the two-body contact problem is to find a discrete saddle point $(u_h^p, \lambda_H^q) \in \mathcal{S}_h^p \times \mathcal{M}_{H,+}^q$, such that

$$\mathcal{L}(u_h^p, \lambda_H^q) = \inf_{v_h^p \in \mathcal{S}_h^p} \sup_{\mu_H^q \in \mathcal{M}_{H,+}^q} \mathcal{L}(v_h^p, \mu_H^q). \quad (3.1)$$

Again, by the stationarity condition, we conclude that the discrete saddle point is equivalently characterized by

$$\begin{aligned} a(u_h^p, v_h^p) &= \ell(v_h^p) - (\lambda_H^q, \gamma_{C\bar{n}}(v_h^p))_{0, \Gamma_C^1}, \\ (\mu_H^q - \lambda_H^q, \gamma_{C\bar{n}}(u_h^p) - g)_{0, \Gamma_C^1} &\leq 0 \end{aligned} \quad (3.2)$$

for all $v_h^p \in \mathcal{S}_h^p$ and all $\mu_H^q \in \mathcal{M}_{H,+}^q$. It is easy to see that the first component of the discrete saddle point is the unique minimizer of the minimization problem

$$E(u_h^p) = \min_{v_h^p \in K_{hh}^{pq}} E(v_h^p)$$

with $K_{hh}^{pq} := \{v_h^p \in \mathcal{S}_h^p \mid \forall \mu_H^q \in \mathcal{M}_{H,+}^q : (\mu_H^q, \gamma_{C\bar{n}}(v_h^p) - g)_{0, \Gamma_C^1} \leq 0\}$. Moreover, u_h^p fulfills

$$a(u_h^p, v_h^p - u_h^p) \geq \ell(v_h^p - u_h^p) \quad (3.3)$$

for all $v_h^p \in K_{hh}^{pq}$. To guarantee the existence of a saddle point, a discrete inf-sup condition on $\mathcal{S}_h^p \times \mathcal{M}_H^q$ has to be satisfied.

Theorem 3.1. *If there is a constant $\alpha > 0$ such that*

$$\alpha \|\mu_H^q\|_{-1/2, \Gamma_C^1} \leq \sup_{v_h^p \in \mathcal{S}_h^p, \|v_h^p\|_1=1} (\mu_H^q, \gamma_{C\bar{n}}(v_h^p))_{0, \Gamma_C^1} \quad (3.4)$$

for all $\mu_H^q \in \mathcal{M}_H^q$, then there exists a unique discrete saddle point of the two-body contact problem (3.1).

Proof. Using (3.4), we conclude by standard arguments (e.g. [20, Lem. 3.2]), that

$$\mathcal{M}_H^q \ni \mu_H^q \mapsto \sup_{v_h^p \in \mathcal{S}_h^p} -\mathcal{L}(v_h^p, \mu_H^q)$$

is coercive. The assertion follows from the closedness and convexity of $\mathcal{M}_{H,+}^q$ and [10, Prop IV.2.3 and Remark IV.2.1]. The uniqueness is a direct consequence of (3.4). \square

Remark 3.2. We call the discretization scheme (3.2) stable, if there exists a unique discrete saddle point independently of the discretization level. In other words, to guarantee the discretization schemes (3.2) to be stable, (3.4) has to be uniformly fulfilled, i.e. the constant α in (3.4) has to be independent of h , H , p and q .

The convergence of the mixed method can be stated without any regularity assumptions using standard techniques of convex analysis. Only the coercivity of a and the approximation properties of \mathcal{S}_h^p and \mathcal{M}_H^q are used. Here, we present a modification of Theorem 1.1.5.3 in [18]. In the following, a sequence $\{v_h^p\}$ with $v_h^p \in \mathcal{S}_h^p$ converges to $v \in \mathcal{H}_D$ if $v_h^p \rightarrow v$ as $h \rightarrow 0$ for a fixed p or as $p \rightarrow \infty$ for a fixed h . Similarly, the convergence of a sequence $\{\mu_H^q\}$ with $\mu_H^q \in \mathcal{M}_{H,+}^q$ is defined. Moreover, we omit $h, H \rightarrow 0$ and $p, q \rightarrow \infty$ using the usual lim-notation.

Theorem 3.3. *Let condition (3.4) be fulfilled. Moreover, assume that*

- (i) for all $v \in \mathcal{H}_D$, there exists a sequence $\{v_h^p\}$ with $v_h^p \in \mathcal{S}_h^p$ which strongly converges to v ,
- (ii) for all $\mu \in H_+^{-1/2}(\Gamma_C^1)$ there exists a sequence $\{\mu_H^q\}$ with $\mu_H^q \in \mathcal{M}_{H,+}^q$ which strongly converges to μ ,
- (iii) for all sequences $\{\mu_H^q\}$ with $\mu_H^q \in \mathcal{M}_{H,+}^q$ weakly converging to $\mu \in H_+^{-1/2}(\Gamma_C^1)$, there holds $\mu \in H_+^{-1/2}(\Gamma_C^1)$,
- (iv) there exists a bounded sequence $\{\bar{v}_h^p\}$ with $\bar{v}_h^p \in \mathcal{S}_h^p$ and $(\mu_H^q, \gamma_{C\bar{n}}(\bar{v}_h^p) - g)_{0, \Gamma_C^1} \leq 0$ for all $\mu_H^q \in \mathcal{M}_{H,+}^q$.

Then, the sequence $\{u_h^p\}$ strongly converges to u and the sequence of discrete Lagrange multipliers $\{\lambda_H^q\}$ weakly converges to λ .

Proof. Since $\bar{v}_h^p \in K_{hH}^{pq}$, we obtain from (3.3) that

$$\nu_1 \|u_h^p\|_1^2 \leq a(u_h^p, u_h^p) \leq a(u_h^p, \bar{v}_h^p) - \ell(\bar{v}_h^p - u_h^p) \leq (\nu_0 \|\bar{v}_h^p\|_1 + \|\ell\|) \|u_h^p\|_1 + \|\ell\| \|\bar{v}_h^p\|_1.$$

Thus, we have $\|u_h^p\|_1 \leq \nu_1^{-1}(\nu_0 \|\bar{v}_h^p\|_1 + \|\ell\|) + (\nu_1^{-1} \|\ell\| \|\bar{v}_h^p\|_1)^{1/2}$ and, therefore, the sequence $\{u_h^p\}$ is bounded. From (3.4) and (3.2), we obtain

$$\alpha \|\lambda_H^q\|_{-1/2, \Gamma_C^1} \leq \sup_{v_h^p \in \mathcal{S}_h^p, \|v_h^p\|_1=1} (\lambda_H^q, \gamma_{C\bar{n}}(v_h^p))_{0, \Gamma_C^1} \leq \nu_1 \|u_h^p\|_1 + \|\ell\|,$$

which implies that $\{\lambda_H^q\}$ is also bounded. Due to the reflexivity of \mathcal{H}_D and $H^{-1/2}(\Gamma_C^1)$, there exist subsequences $\{u_h^{\bar{p}}\} \subset \{u_h^p\}$ and $\{\lambda_H^{\bar{q}}\} \subset \{\lambda_H^q\}$ which weakly converge to $u^* \in \mathcal{H}_D$ and $\lambda^* \in H^{-1/2}(\Gamma_C^1)$, respectively. From (iii), we have $\lambda^* \in H_+^{-1/2}(\Gamma_C^1)$. Let $\{v_h^{\bar{p}}\}$ and $\{\mu_H^{\bar{q}}\}$ strongly converge to $v \in \mathcal{H}_D$ and $\mu \in H_+^{-1/2}(\Gamma_C^1)$, respectively, as assumed in (i) and (ii). It is easy to see that $a(u_h^{\bar{p}}, v_h^{\bar{p}})$, $(\mu_H^{\bar{q}}, \gamma_{C\bar{n}}(u_h^{\bar{p}}))_{0, \Gamma_C^1}$ and $(\lambda_H^{\bar{q}}, \gamma_{C\bar{n}}(v_h^{\bar{p}}))_{0, \Gamma_C^1}$ converge to $a(u^*, v)$, $(\mu, \gamma_{C\bar{n}}(u^*))_{0, \Gamma_C^1}$ and $(\lambda^*, \gamma_{C\bar{n}}(v))_{0, \Gamma_C^1}$, respectively. Passing to the limit in (3.2) yields

$$a(u^*, v) = \ell(v) - (\lambda^*, \gamma_{C\bar{n}}(v))_{0, \Gamma_C^1}, \quad (3.5)$$

$$(\mu, \gamma_{C\bar{n}}(u^*) - g)_{0, \Gamma_C^1} \leq \liminf (\lambda_H^{\bar{q}}, \gamma_{C\bar{n}}(u_h^{\bar{p}}) - g)_{0, \Gamma_C^1}. \quad (3.6)$$

Since $v \mapsto a(v, v)$ is convex and continuous and, therefore, weakly lower semi-continuous, we obtain from (3.2), that

$$a(u^*, u^*) + \liminf (\lambda_H^{\bar{q}}, \gamma_{C\bar{n}}(u_h^{\bar{p}}))_{0, \Gamma_C^1} \leq \liminf (a(u_h^{\bar{p}}, u_h^{\bar{p}}) + (\lambda_H^{\bar{q}}, \gamma_{C\bar{n}}(u_h^{\bar{p}}))_{0, \Gamma_C^1}) = \liminf \ell(u_h^{\bar{p}}) = \ell(u^*).$$

Hence, using (3.5) with $v := u^*$ and (3.6), we find

$$(\mu, \gamma_{C\bar{n}}(u^*) - g)_{0, \Gamma_C^1} \leq \liminf (\lambda_H^{\bar{q}}, \gamma_{C\bar{n}}(u_h^{\bar{p}}) - g)_{0, \Gamma_C^1} = \ell(u^*) - a(u^*, u^*) - (\lambda^*, g)_{0, \Gamma_C^1} = (\lambda^*, \gamma_{C\bar{n}}(u^*) - g)_{0, \Gamma_C^1}. \quad (3.7)$$

Since (v, μ) is arbitrarily chosen, (3.5) and (3.7) imply that (u^*, λ^*) is a saddle point. Due to the uniqueness, we conclude $(u^*, \lambda^*) = (u, \lambda)$ and, additionally, that the entire sequence $\{(u_h^p, \lambda_H^q)\}$ converges to (u, λ) weakly. To show that $\{u_h^p\}$ converges to u strongly, we consider

$$\begin{aligned} 0 &\leq a(u - u_h^p, u - u_h^p) = a(u, u) - 2a(u, u_h^p) + \ell(u_h^p) - (\lambda_H^q, \gamma_{C\bar{n}}(u_h^p))_{0, \Gamma_C^1} \\ &= a(u, u) - 2a(u, u_h^p) + \ell(u_h^p) - (\lambda_H^q, g)_{0, \Gamma_C^1} \rightarrow -a(u, u) + \ell(u) - \langle \lambda, \gamma_{C\bar{n}}(u) \rangle + \langle \lambda, \gamma_{C\bar{n}}(u) - g \rangle = 0. \end{aligned}$$

□

Remark 3.4. Obviously, condition (iv) in Theorem 3.3 is fulfilled if $g \in \gamma_{C\bar{n}}(\mathcal{S}_h^p)$, and, in particular, if $g = 0$. We refer to Section 5 for the verification of the conditions (i)-(iii) with respect to a given discretization.

In the following, we discuss some general a priori estimates similar to those introduced in [5, 14, 15, 16, 22]. The important assumption is given by the discrete inf-sup condition (3.4). For notational simplicity, \lesssim abbreviates \leq up to a positive constant which is independent of \mathcal{S}_h^p and \mathcal{M}_H^q .

Lemma 3.5. *There holds*

$$\begin{aligned} \|u - u_h^p\|_1^2 &\lesssim \|u - u_h^p\|_1 (\|u - v_h^p\|_1 + \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1}) + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \|u - v_h^p\|_1 + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle \\ &\quad + \langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u) - g \rangle \end{aligned}$$

for all $v_h^p \in \mathcal{S}_h^p$, $\mu_H^q \in \mathcal{M}_{H,+}^q$ and $\mu \in H_+^{-1/2}(\Gamma_C^1)$.

Proof. We find

$$\langle \lambda_H^q - \lambda, g - \gamma_{C\bar{n}}(u_h^p) \rangle \leq \langle \mu_H^q, g - \gamma_{C\bar{n}}(u_h^p) \rangle - \langle \lambda, g - \gamma_{C\bar{n}}(u_h^p) \rangle = \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u_h^p) - u \rangle.$$

Thus, we obtain

$$\begin{aligned} \langle \lambda_H^q - \lambda, \gamma_{C\bar{n}}(u - u_h^p) \rangle &\leq \langle \mu - \lambda, g - \gamma_{C\bar{n}}(u_h^p) \rangle + \langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u - u_h^p) \rangle \\ &= \langle \lambda_H^q - \lambda, g - \gamma_{C\bar{n}}(u_h^p) \rangle + \langle \mu - \lambda_H^q, g - \gamma_{C\bar{n}}(u) \rangle \\ &\leq \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u_h^p) - u \rangle + \langle \mu - \lambda_H^q, g - \gamma_{C\bar{n}}(u) \rangle. \end{aligned}$$

Due to (2.1), there holds

$$\begin{aligned} \|u - u_h^p\|_1^2 &\lesssim a(u - u_h^p, u - v_h^p) + a(u - u_h^p, v_h^p - u_h^p) \\ &= a(u - u_h^p, u - v_h^p) + \langle \lambda_H^q - \lambda, \gamma_{C\bar{n}}(v_h^p) - u \rangle + \langle \lambda_H^q - \lambda, \gamma_{C\bar{n}}(u - u_h^p) \rangle \\ &\lesssim \|u - u_h^p\|_1 \|u - v_h^p\|_1 + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \|u - v_h^p\|_1 + \|u - u_h^p\|_1 \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1} + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle \\ &\quad + \langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u) - g \rangle. \end{aligned}$$

□

Similar to [18, Theorem 1.1.5.1], we conclude the following a priori estimation.

Theorem 3.6. *Assume condition (3.4) to be fulfilled. Then,*

$$\|u - u_h^p\|_1^2 + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1}^2 \lesssim \|u - v_h^p\|_1^2 + \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1}^2 + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle + \langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u) - g \rangle$$

for all $v_h^p \in \mathcal{S}_h^p$ and $\mu_H^q \in \mathcal{M}_{H,+}^q$ as well as all $\mu \in H_+^{-1/2}(\Gamma_C^1)$.

Proof. From condition (3.4), we obtain

$$\begin{aligned} \|\mu_H^q - \lambda_H^q\|_{-1/2, \Gamma_C^1} &\lesssim \sup_{v_h^p \in \mathcal{S}_h^p, \|v_h^p\|_1=1} \langle \mu_H^q - \lambda_H^q, \gamma_{C\bar{n}}(v_h^p) \rangle = \sup_{v_h^p \in \mathcal{S}_h^p, \|v_h^p\|_1=1} \langle \mu_H^q, \gamma_{C\bar{n}}(v_h^p) \rangle + a(u_h^p, v_h^p) - \ell(v_h^p) \\ &= \sup_{v_h^p \in \mathcal{S}_h^p, \|v_h^p\|_1=1} \langle \mu_H^q - \lambda, \gamma_{C\bar{n}}(v_h^p) \rangle + a(u_h^p - u, v_h^p) \lesssim \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1} + \|u - u_h^p\|_1. \end{aligned}$$

Thus,

$$\|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \leq \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1} + \|\mu_H^q - \lambda_H^q\|_{-1/2, \Gamma_C^1} \lesssim \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1} + \|u - u_h^p\|_1. \quad (3.8)$$

From Lemma 3.5 and (3.8) as well as Young's inequality $2ab \leq \epsilon a^2 + \epsilon^{-1} b^2$ for $a, b, \epsilon > 0$, we obtain

$$\|u - u_h^p\|_1^2 \lesssim \epsilon \|u - u_h^p\|_1^2 + (\epsilon^{-1} + 1/2) (\|u - v_h^p\|_1^2 + \|\lambda - \mu_H^q\|_{-1/2, \Gamma_C^1}^2) + \langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle + \langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u) - g \rangle.$$

Choosing a suitable $\epsilon > 0$ together with (3.8) yields the assertion. \square

4. Higher-order discretizations

In this section, we propose a higher-order finite element discretization and verify the assumptions of Theorem 3.3 to show its convergence. Moreover, we derive convergence rates of the mixed scheme. Here and in the following sections, we assume that the domains Ω^l are Lipschitz and polygonal. The discretization is based on quasi-uniform finite element meshes \mathcal{T}^l and \mathcal{E} of parallelograms or parallelepipeds which decompose Ω^l and Γ_C^1 with mesh sizes h_l and H , respectively. Let $\Psi_T^l : [-1, 1]^k \rightarrow T \in \mathcal{T}^l$, $\Phi_E : [-1, 1]^{k-1} \rightarrow E \in \mathcal{E}$ be affine transformations and let $p_l, q \in \mathbb{N}$. Using the polynomial tensor product space S_k^r of order r on the reference element $[-1, 1]^k$, we define

$$\mathcal{S}_h^p := \{v = (v^1, v^2) \in \mathcal{H}_D \mid \forall T_l \in \mathcal{T}^l : v|_{T_l} \circ \Psi_T^l \in S_k^{p_l}, l = 1, 2\}$$

with $h := (h_1, h_2)$ and $p := (p_1, p_2)$. Furthermore, we define

$$\mathcal{M}_H^q := \{\mu \in L^2(\Gamma_C^1) \mid \forall E \in \mathcal{E} : \mu|_E \circ \Phi_E \in S_{k-1}^q\}.$$

To complete the discretization of the mixed formulation, we have to specify the subset $\mathcal{M}_{H,+}^q$. This is done in the following way,

$$\mathcal{M}_{H,+}^q := \{\mu_H^q \in \mathcal{M}_H^q \mid \forall E \in \mathcal{E} : \forall x \in C_q : \mu_{HE}^q(\Phi_E(x)) \geq 0\} \quad (4.1)$$

with the finite set $C_q \subset [-1, 1]^{k-1}$ of the $(q+1)^{k-1}$ Gauss-quadrature points. We note that polynomials \hat{P} of order $2q+1$ on $[-1, 1]^{k-1}$ are exactly integrated by the resulting quadrature rule, i.e., with some weights $\alpha_{\hat{x}} \geq 0$, there holds $\int_{[-1, 1]^{k-1}} \hat{P}(\hat{x}) d\hat{x} = \sum_{\hat{x} \in C_q} \alpha_{\hat{x}} \hat{P}(\hat{x})$. Furthermore, for polynomials P on $E \in \mathcal{E}$ we have

$$\int_E P(x) dx = \sum_{\hat{x} \in C_q} \beta_{\hat{x}} P(\Phi_E(\hat{x})) \quad (4.2)$$

with $\beta_{\hat{x}} := \alpha_{\hat{x}} |\det \nabla \Phi_E(\hat{x})^\top \nabla \Phi_E(\hat{x})| \geq 0$, $\hat{x} \in C_q$.

To show Conditions (i)-(iv) of Theorem 3.3, we introduce the interpolation operators I_H^q and J_p^h which map continuous functions into \mathcal{M}_H^q and \mathcal{S}_h^p , respectively. The operator I_H^q is defined via the transformed Gauss quadrature points $\Phi_E(\hat{x})$, $\hat{x} \in C_q$, on each $E \in \mathcal{E}$ whereas J_p^h uses transformed Gauss-Lobatto-Points. There holds

$$\|v - I_H^q(v)\|_{0, \Gamma_C^1} \lesssim H^{\min(q+1, \theta)} / (q+1)^\theta \|v\|_{\theta, \Gamma_C^1}, \quad (4.3)$$

$$\|w - J_p^h(w)\|_1 \lesssim \sum_{l=1,2} h_l^{\min(p_l, \theta_l)} / p_l^{\theta_l} \|w^l\|_{1+\theta_l} \quad (4.4)$$

for all $v \in H^\theta(\Gamma_C^1)$ with $\theta > (k-1)/2$, cf. [3, Thm 3.4, Thm. 5.2], and all $w = (w^1, w^2) \in \mathcal{H}_D$ with $w^l \in H^{1+\theta_l}(\Omega^l)$, $\theta_l > 1/2$, cf. [3, Thm. 5.9]. Moreover, we make use of the inverse estimate

$$\|\mu_H^q\|_{-1/2+\theta, \Gamma_C^1} \lesssim \frac{\max\{1, q\}^{2\theta}}{H^\theta} \|\mu_H^q\|_{-1/2, \Gamma_C^1} \quad (4.5)$$

for all $\mu_H^q \in \mathcal{M}_H^q$, cf. [11, Thm. 3.5, Thm 3.9].

Lemma 4.1. Let $\{\mu_H^q\}$ with $\mu_H^q \in \mathcal{M}_H^q$ be a bounded sequence in $H^{-1/2}(\Gamma_C^1)$ and $v \in H^\theta(\Gamma_C^1)$ with $\theta > (k-1)/2$. For $q \geq 1$, it holds

$$|(\mu_H^q, v - I_H^q(v))_0| \lesssim H^{\min(q+1, \theta)-1/2} / (q+1)^{\theta-1} \|v\|_{\theta, \Gamma_C^1}.$$

Proof. From Cauchy's inequality and the inverse estimate (4.5) as well as the interpolation estimate (4.3), we have

$$|(\mu_H^q, v - I_H^q(v))_{0, \Gamma_C^1}| \leq \|\mu_H^q\|_{0, \Gamma_C^1} \|v - I_H^q(v)\|_{0, \Gamma_C^1} \lesssim H^{\min(q+1, \theta)-1/2} / (q+1)^{\theta-1} \|v\|_{\theta, \Gamma_C^1} \|\mu_H^q\|_{-1/2, \Gamma_C^1}.$$

Since the sequence $\{\mu_H^q\}$ is assumed to be bounded in $H^{-1/2}(\Gamma_C^1)$, we obtain the assertion. \square

Theorem 4.2. Let the discrete inf-sup condition (3.4) be valid. Moreover, let $g \in \gamma_{C\bar{n}}(\mathcal{S}_h^p)$. Then, $\{u_h^p\}$ strongly converges to u and $\{\lambda_H^q\}$ weakly converges to λ .

Proof. Obviously, Condition (i) is fulfilled due to (4.4) and the density of $H^{1+\theta_l}(\Omega')$ in $H^1(\Omega')$, $\theta_l > 1/2$. Also Condition (iv) is valid due to the assumption on g , cf. Remark 3.4. To show Condition (ii), let $\mu \in H_+^{-1/2}(\Gamma_C^1)$ and $\epsilon > 0$. Due to the density of $H^\theta(\Gamma_C^1)$ in $H^{-1/2}(\Gamma_C^1)$ with $\theta > (k-1)/2$, there exists a function $\mu_\epsilon \in H^\theta(\Gamma_C^1) \cap H_+^{-1/2}(\Gamma_C^1)$ with $\|\mu - \mu_\epsilon\|_{-1/2, \Gamma_C^1} \leq \epsilon$. For a fixed q there exists an H so that $\|\mu_\epsilon - I_H^q(\mu_\epsilon)\|_{0, \Gamma_C^1} \leq \epsilon$. Define $\mu_H^q := I_H^q(\mu_\epsilon)$, then $\mu_H^q \in \mathcal{M}_{H,+}^q$ and

$$\|\mu - \mu_H^q\|_{-1/2, \Gamma_C^1} \leq \|\mu - \mu_\epsilon\|_{-1/2, \Gamma_C^1} + \|\mu_\epsilon - \mu_H^q\|_{-1/2, \Gamma_C^1} \lesssim \epsilon.$$

The same holds for a fixed H , so that Condition (ii) is proven.

To show Condition (iii), let the sequence $\{\mu_H^q\}$ weakly converge to $\mu \in H^{-1/2}(\Gamma_C^1)$ and $v \in H_+^{-1/2}(\Gamma_C^1) \cap H^{1+\theta}(\Gamma_C^1)$ with $\theta > 0$. There holds

$$|(\mu_H^q, I_H^q(v))_{0, \Gamma_C^1} - \langle \mu, v \rangle| \leq |(\mu_H^q, v - I_H^q(v))_{0, \Gamma_C^1}| + |\langle \mu, v \rangle - (\mu_H^q, v)_{0, \Gamma_C^1}|. \quad (4.6)$$

Due to the weak convergence of $\{\mu_H^q\}$ the last summand in (4.6) tends to zero. The sequence $\{\mu_H^q\}$ is bounded in $H^{-1/2}(\Gamma_C^1)$. Due to Lemma 4.1 also the first summand tends to zero. Altogether, $(\mu_H^q, I_H^q(v))_{0, \Gamma_C^1}$ converges to $\langle \mu, v \rangle$ and we obtain from (4.2)

$$\langle \mu, v \rangle = \lim (\mu_H^q, I_H^q(v))_{0, \Gamma_C^1} = \lim \sum_{E \in \mathcal{E}} \sum_{\hat{x} \in C_q} \beta_{\hat{x}} \mu_H^q(\Phi_E(\hat{x})) v(\Phi_E(\hat{x})) \geq 0.$$

Since $H^{1+\theta}(\Gamma_C^1)$ is dense in $H^{1/2}(\Gamma_C^1)$, there holds $\mu \in H_+^{-1/2}(\Gamma_C^1)$. \square

Remark 4.3. In principle, we can also apply other finite sets C to approximatively ensure the sign condition in (4.1). In Section 6 we discuss the use of Chebychev points by some numerical experiments. For the justification of this approach, we refer to [9], where some bounds of polynomials are considered fulfilling pointwise restrictions in Chebychev points.

Remark 4.4. Note that the discretization given by (4.1) is non-conforming for $q \geq 2$, i.e., $\mathcal{M}_{H,+}^q \not\subset H_+^{-1/2}(\Gamma_C^1)$. The conforming definition of $\mathcal{M}_{H,+}^q$ by $\{\mu_H^q \in \mathcal{M}_H^q \mid \forall E \in \mathcal{E} : \mu_{H|E}^q \geq 0\} \subset H_+^{-1/2}(\Gamma_C^1)$ may seem to be more natural. However, ensuring higher-order finite element functions to be in this set is not obvious for $q \geq 2$.

To obtain convergence rates using Theorem 3.6, we have to discuss the expressions $\langle \lambda - \mu_H^q, \gamma_{C\bar{n}}(u) - g \rangle$ and $\langle \lambda_H^q - \mu, \gamma_{C\bar{n}}(u) - g \rangle$, which dominate the overall error estimations. First, we consider the case $q = 0$ and discuss some arguments similar to those introduced in [22]. In this case, there holds $\mathcal{M}_{H,+}^0 \subset H_+^{-1/2}(\Gamma_C^1)$ and, in particular, $\langle \lambda_H^0 - \mu, \gamma_{C\bar{n}}(u) - g \rangle = 0$ setting $\mu := \lambda_H^0 \in H_+^{-1/2}(\Gamma_C^1)$. In the following, we assume $u_i^l \in H^{1+\theta_l}(\Omega)$, $\lambda \in H^\theta(\Gamma_C^1)$ and $\gamma_{C\bar{n}}(u) - g \in H^{\tilde{\theta}}(\Gamma_C^1)$ with $0 \leq \theta, \tilde{\theta} \leq 1$ and $\theta_l > 1/2$. For $q = 0$, we have $\Pi_H(\lambda) \in \mathcal{M}_{H,+}^0$ where Π_H is piecewisely given by the integral mean value (i.e., the L^2 -projection onto piecewise constant functions). Due to $\|\lambda - \Pi_H(\lambda)\|_{0, \omega} \lesssim H^\theta \|\lambda\|_{\theta, \Gamma_C^1}$ for $\lambda \in H^\theta(\Gamma_C^1)$, cf. [4], as well as Cauchy's inequality we obtain

$$\langle \lambda - \Pi_H(\lambda), \gamma_{C\bar{n}}(u) - g \rangle = \int_{\Gamma_C^1} (\lambda - \Pi_H(\lambda)) (\gamma_{C\bar{n}}(u) - g - \Pi_H(\gamma_{C\bar{n}}(u) - g)) ds \lesssim H^{\theta+\tilde{\theta}}.$$

Furthermore, the definition of the dual norm yields

$$\|\lambda - \Pi_H(\lambda)\|_{-1/2, \Gamma_C^1} = \sup_{w \in H^{1/2}(\Gamma_C^1), \|w\|_{1/2, \Gamma_C^1} = 1} \int_{\Gamma_C^1} (\lambda - \Pi_H(\lambda))(w - \Pi_H(w)) ds \lesssim H^{\theta+1/2}.$$

Assuming condition (3.4), we obtain from (4.4) and Theorem 3.6,

$$\|u - u_h^p\|_1 + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \lesssim H^{(\theta+\tilde{\theta})/2} + \sum_{l=1}^2 h_l^{\min(p_l, \theta_l)} / p_l^{\theta_l}. \quad (4.7)$$

An alternative approach is proposed in [14, 15, 16] for $k = 2$, where the set of points of Γ_C^1 in which $\gamma_{C\bar{n}}(u) - g$ changes from negative to zero is assumed to be finite. The number of segments $\tilde{\mathcal{E}} \subset \mathcal{E}$ which contains such points is, therefore, bounded independently of H . Assuming $\theta_l \geq 1$, we have either $0 = -\sigma_{\bar{n}\bar{n}}(u) = \lambda = \Pi_H(\lambda)$ or $\gamma_{C\bar{n}}(u) - g = 0$ on $E \in \mathcal{E} \setminus \tilde{\mathcal{E}}$. Provided that $\gamma_{C\bar{n}}(u) - g \in W^{1, \infty}(E)$, $E \in \tilde{\mathcal{E}}$, we have $\|\gamma_{C\bar{n}}(u) - g\|_{\infty, E} \lesssim H \|\nabla(\gamma_{C\bar{n}}(u) - g)\|_{\infty, E}$ and obtain by Cauchy's inequality

$$\begin{aligned} \langle \lambda - \Pi_H(\lambda), \gamma_{C\bar{n}}(u) - g \rangle &\leq \sum_{E \in \tilde{\mathcal{E}}} \|\lambda - \Pi_H(\lambda)\|_{0, E} \|\gamma_{C\bar{n}}(u) - g\|_{\infty, E} H^{1/2} \\ &\leq H^{1/2} \|\lambda - \Pi_H(\lambda)\|_{0, \Gamma_C^1} \left(\sum_{E \in \tilde{\mathcal{E}}} \|\gamma_{C\bar{n}}(u) - g\|_{\infty, E}^2 \right)^{1/2} \lesssim H^{\theta+3/2} \end{aligned}$$

and, therefore,

$$\|u - u_h^p\|_1 + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \lesssim H^{\theta+1/2} + H^{\theta/2+3/4} + \sum_{l=1}^2 h_l^{\min(p_l, \theta_l)} / p_l^{\theta_l}. \quad (4.8)$$

Remark 4.5. For $p_l = 1$ and $\theta = 1/2$, (4.8) corresponds to the result $\mathcal{O}(h + H)$ as shown in [16] for the Signorini problem. In [22], the order of convergence is stated by $\mathcal{O}(h^{3/4})$ where $h = H$ and $p_l = 1$ is assumed. We obtain the same result for $\theta = 1/2$ and $\tilde{\theta} = \theta_l = 1$ with respect to (4.7). These regularity assumptions are implicitly assumed in [22].

For $q \geq 1$, we may proceed as follows.

Theorem 4.6. *Let the discrete inf-sup condition (3.4) be uniformly fulfilled. Furthermore, let $\theta, \tilde{\theta} > (k-1)/2$, $\theta_l > 1/2$ and $q \geq 1$. Then,*

$$\|u - u_h^p\|_1 + \|\lambda - \lambda_H^q\|_{-1/2, \Gamma_C^1} \lesssim H^{\min(q+1, \theta)/2} / (q+1)^{\theta/2} + H^{\min(q+1, \tilde{\theta})/2-1/4} / (q+1)^{\tilde{\theta}/2-1/2} + \sum_{l=1}^2 h_l^{\min(p_l, \theta_l)} / p_l^{\theta_l}.$$

Proof. Obviously, there holds $I_H^q(\lambda) \in \mathcal{M}_{H,+}^q$ and, therefore,

$$\langle \lambda - I_H^q(\lambda), \gamma_{C\bar{n}}(u) - g \rangle \lesssim \|\lambda - I_H^q(\lambda)\|_0 \lesssim H^{\min(q+1, \theta)} / (q+1)^\theta.$$

We conclude from Theorem 4.2 that the sequence $\{\lambda_H^q\}$ weakly converges and is, therefore, bounded in $H^{-1/2}(\Gamma_C^1)$. Thus, we obtain from Lemma 4.1 and (4.2)

$$\begin{aligned} \langle \lambda_H^q, \gamma_{C\bar{n}}(u) - g \rangle &\leq |(\lambda_H^q, \gamma_{C\bar{n}}(u) - g - I_H^q(\gamma_{C\bar{n}}(u) - g))_{0, \Gamma_C^1}| + (\lambda_H^q, I_H^q(\gamma_{C\bar{n}}(u) - g))_{0, \Gamma_C^1} \\ &\lesssim H^{\min(q+1, \tilde{\theta})-1/2} / (q+1)^{\tilde{\theta}-1} + \sum_{E \in \tilde{\mathcal{E}}} \sum_{\hat{x} \in C} \beta_{\hat{x}} \lambda_H^q(\Phi_E(\hat{x})) (\gamma_{C\bar{n}}(u) - g)(\Phi_E(\hat{x})). \end{aligned} \quad (4.9)$$

Since $\lambda_H^q(\Psi_C(\hat{x})) \geq 0$ and $\beta_{\hat{x}} \geq 0$ for $\hat{x} \in C$ as well as $\gamma_{C\bar{n}}(u) - g \leq 0$ on Γ_C^1 , the sum in (4.9) is non-positive and can, therefore, be omitted. Theorem 3.6 together with (4.4) yields the assertion. \square

Remark 4.7. The convergence rates stated in Theorem 4.6 may seem to be suboptimal due to the use of the inverse estimate in Lemma 4.1 and due to the fact that I_H^q is not an orthogonal projection (in contrast to Π_H). We refer to the numerical experiments in Section 6, where considerably higher convergence rates can be observed. Furthermore, positive rates with respect to q require high regularity assumptions on $\gamma_{C\bar{n}}(u) - g$ with $\tilde{\theta} > 1$.

5. The discrete inf-sup condition

To show the discrete inf-sup condition (3.4) to be uniformly fulfilled, we follow the proof of Lemma 3.1 in [16], where this condition is derived for low-order discretization schemes for a simplified Signorini problem. Moreover, we adapt the proof of the discrete inf-sup condition for a frictional contact problem given in [25].

Lemma 5.1. *For $\mu \in H^{-1/2}(\Gamma_C^1)$, there exists a function $u^\mu = (u^{\mu,1}, u^{\mu,2}) \in \mathcal{H}_D$, such that*

$$\sum_{l=1,2} (\varepsilon_{ij}(u^{\mu,l}), \varepsilon_{ij}(v^l))_0 + (u^{\mu,l}, v^l)_0 = \langle \mu, \gamma_{C\bar{n}}(v) \rangle \quad (5.1)$$

for all $v = (v^1, v^2) \in \mathcal{H}_D$. Additionally, there holds $C_1 \|\mu\|_{-1/2, \Gamma_C} \leq \|u^\mu\|_1$ for a constant $C_1 > 0$.

Proof. See [25, Lemma 2]. □

Obviously, the variational problem (5.1) is equivalent to

$$\begin{aligned} (\varepsilon_{ij}(u^{\mu,1}), \varepsilon_{ij}(v^1))_0 + (u^{\mu,1}, v^1)_0 &= \langle \mu, \gamma_{C\bar{n}}(v^1, 0) \rangle, \\ (\varepsilon_{ij}(u^{\mu,2}), \varepsilon_{ij}(v^2))_0 + (u^{\mu,2}, v^2)_0 &= \langle \mu, \gamma_{C\bar{n}}(0, v^2) \rangle \end{aligned}$$

for all $v^l \in H_D^1(\Omega^l)$. We call the variational problem (5.1) regular, if $u^{\mu,l} \in H^{1+\theta_l}(\Omega^l)$, $0 < \theta_l \leq 1$, and

$$\|u^{\mu,l}\|_{1+\theta_l, \Omega^l} \leq C_2 \|\mu\|_{-1/2+\theta_l, \Gamma_C^1} \quad (5.2)$$

for all $\mu \in H^{-1/2+\theta_l}(\Gamma_C^1)$ with a constant $C_2 > 0$.

Theorem 5.2. *Assume the variational problem (5.1) to be regular for $\theta_l \leq 1/2$ and let h, H, p, q be chosen such that*

$$\Pi(h, H, p, q) := \sum_{l=1}^2 (h_l H^{-1} \max\{1, q\}^2 p_l^{-1})^{\theta_l}$$

is sufficiently small, then (3.4) holds for a constant $\alpha > 0$ independent of h, H, p and q .

Proof. For $\mu_H^q \in \mathcal{M}_H^q$, let $u_h^{\mu_H^q, p} = (u_h^{\mu_H^q, p, 1}, u_h^{\mu_H^q, p, 2}) \in \mathcal{S}_h^p$ be uniquely determined by

$$\sum_{l=1,2} (\varepsilon_{ij}(u_h^{\mu_H^q, p, l}), \varepsilon_{ij}(v_h^{p, l}))_0 + (u_h^{\mu_H^q, p, l}, v_h^{p, l})_0 = (\mu_H^q, \gamma_{C\bar{n}}(v_h^p))_{0, \Gamma_C^1}$$

for all $v_h^p = (v_h^{p,1}, v_h^{p,2}) \in \mathcal{S}_h^p$. For $v \in \mathcal{H}_D$, define the norm $\|v\|_1^2 := \sum_{l=1,2} (\varepsilon_{ij}(v^l), \varepsilon_{ij}(v^l))_{0, \Omega^l} + \|v^l\|_{0, \Omega^l}^2$ which is equivalent to the H^1 -norm $\|\cdot\|_1$. In particular, there exists a constant $\beta > 0$ such that $\|v\|_1 \leq \beta \|v\|_1$.

Using the same arguments as in [2, Section 4.2], (4.4) holds true for the solution $u_h^{\mu_H^q} \in H^{1+\theta_l}(\Omega^l)$ of (5.1) with $0 < \theta_l \leq 1/2$. Thus, applying the Galerkin orthogonality and the regularity assumption as well as the inverse estimate (4.5), we obtain

$$\|u_h^{\mu_H^q} - u_h^{\mu_H^q, p}\|_1 \lesssim \|u_h^{\mu_H^q} - J_h^p(u_h^{\mu_H^q})\|_1 \lesssim \sum_{l=1,2} (h_l/p_l)^{\theta_l} \|u_h^{\mu_H^q, l}\|_{1+\theta_l} \lesssim \sum_{l=1,2} (h_l/p_l)^{\theta_l} \|\mu_H^q\|_{-1/2+\theta_l, \Gamma_C^1} \lesssim \Pi(h, H, p, q) \|\mu_H^q\|_{-1/2, \Gamma_C^1}.$$

Therefore, for a sufficiently small value of $\Pi(h, H, p, q)$, we obtain

$$\|u_h^{\mu_H^q} - u_h^{\mu_H^q, p}\|_1 \leq (C_1 - \beta^{-1}\alpha) \|\mu_H^q\|_{-1/2, \Gamma_C^1} \quad (5.3)$$

with $0 < \alpha < \beta C_1$. From Lemma 5.1, we obtain

$$\begin{aligned} \sup_{v_h \in \mathcal{S}_h^p \setminus \{0\}} \frac{(\mu_H^q, \gamma_{C\bar{n}}(v_h))_{0, \Gamma_C^1}}{\|v_h^p\|_1} &\geq \frac{(\mu_H^q, \gamma_{C\bar{n}}(u_h^{\mu_H^q, p}))_{0, \Gamma_C^1}}{\|u_h^{\mu_H^q, p}\|_1} \geq \beta \|u_h^{\mu_H^q, p}\|_1 \\ &\geq \beta \|u_h^{\mu_H^q}\|_1 - \beta \|u_h^{\mu_H^q} - u_h^{\mu_H^q, p}\|_1 \geq \beta C_1 \|\mu_H^q\|_{-1/2, \Gamma_C} - \beta \|u_h^{\mu_H^q} - u_h^{\mu_H^q, p}\|_1 \geq \alpha \|\mu_H^q\|_{-1/2, \Gamma_C}. \end{aligned}$$

□

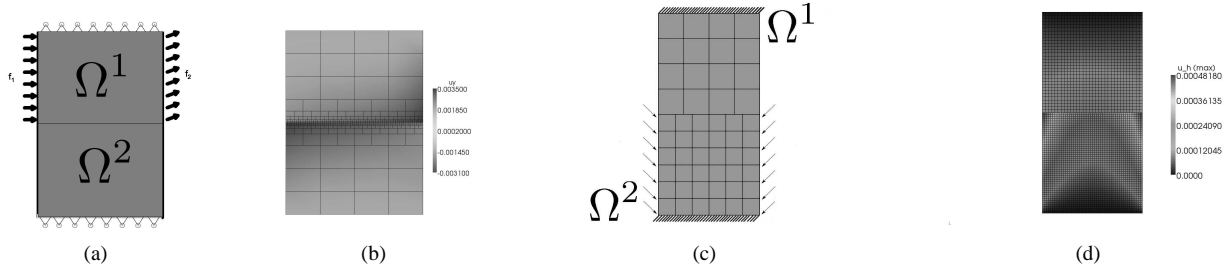


Figure 1: (a) Configuration of the first example, (b) resulting displacements in y-direction of the first example, (c) configuration of the second example, (d) maximum displacement u_h .

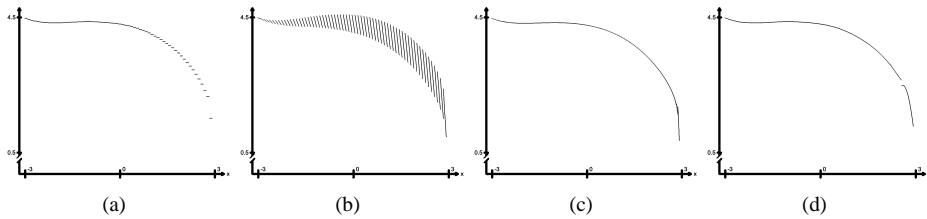


Figure 2: Lagrange multiplier: (a) $p = 2, q = 0, H = h$, (b) $p = 2, q = 1, H = h$, (c) $p = 2, q = 1, H = 2h$, (d) $p = 2, q = 2, H = 4h$.

Remark 5.3. From the practical point of view, the result of Theorem 5.2 does not seem to be satisfactory as it is not clear when $\Pi(h, H, p, q)$ is small enough such that (5.3) holds. Nevertheless, Theorem 5.2 justifies the modification of the discretization scheme by coarsening the mesh \mathcal{E} or by decreasing the polynomial degree q to obtain a stable scheme. In Section 6, numerical results confirm this theoretical observation.

Remark 5.4. To verify the variational problem (5.1) to be regular, we may apply some regularity results on elliptic boundary value problems with natural boundary conditions. We refer to [6, 12] and, in particular, to [13, Ch. 9] for more details.

6. Numerical results

In this section, we consider some numerical experiments and discuss the theoretical findings. In the first experiment, we study the stability properties of the mixed scheme. As stated in Theorem 5.2, the discrete inf-sup condition is uniformly fulfilled if the number $\Pi(h, H, p, q)$ is sufficiently small. To reduce $\Pi(h, H, p, q)$, we can vary h and H or p and q or both. It is noted that varying h and H implies that the Lagrange multiplier is possibly defined on a coarser mesh which may lead to a high implementational complexity. Using a surface mesh \mathcal{E} , which is inherited from the interior mesh \mathcal{T}^I , the implementational effort is essentially smaller. However, in this case we have $h/H = 1$ and can only vary p and q to keep $\Pi(h, H, p, q)$ small.

To illustrate the relations between h, H, p and q , we consider the contact of two linear elastic bodies which are represented by $\Omega_1 = [-3, 3] \times [4, 8]$ and $\Omega_2 = [-3, 3] \times [0, 4.005]$, cf. Figure 1(a). The body Ω_1 is subjected to Neumann boundary conditions given by $f_1 = (1, 0)$ on its left side and by $f_1 = (1, 0.5)$ on its right side. Young's moduli are set to $E_1 = E_2 = 500$ and Poisson's ratios to $\nu_1 = \nu_2 = 0.4$. Figure 1(b) shows the resulting displacements in y-direction. Note that the domains Ω^1 and Ω^2 overlap which leads to the contact of the domains.

In Figure 2, Lagrange multipliers are depicted for $p = 2, q = 0, 1, 2$ and different quotients of the mesh sizes $h = h^1 = h^2$ and H . We observe that the Lagrange multiplier in Figure 2(b) with $p = 2, q = 1$ and $H = h$ seems to oscillate when using this configuration. This oscillation phenomenon may be interpreted as a one-dimensional

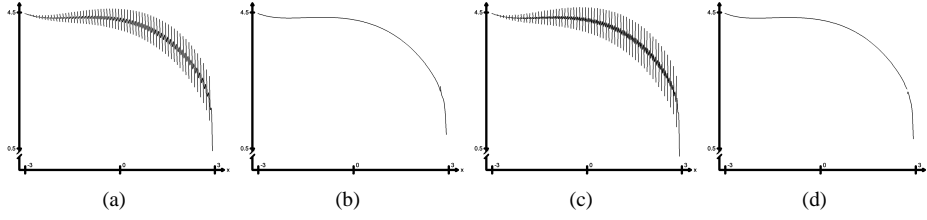


Figure 3: Lagrange multiplier: **(a)** $p = 4, q = 3, H = h$, **(b)** $p = 4, q = 3, H = 2h$, **(c)** $p = 5, q = 4, H = h$, **(d)** $p = 5, q = 4, H = 2h$.

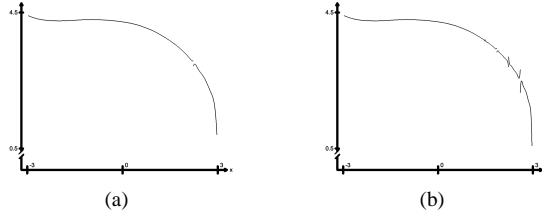


Figure 4: Lagrange multiplier: **(a)** $p = 2, q = 6, H = 8h$, **(b)** $p = 2, q = 6, H = 4h$.

checkerboard instability, which suggests that the Lagrange multiplier is not unique. In this case, it is not a reasonable approximation of contact forces and is, therefore, physically meaningless. For other configurations, the value of $\Pi(h, H, p, q)$ is reduced and the Lagrange multiplier does not oscillate. The mixed scheme seems to be stable. In this example we use meshes which match for both domains.

For higher-order spaces with $q \geq 3$, we obtain similar results. In particular, we observe that $q = p - 1$ and $H = 2h$ lead to a stable discretization. We refer to Figure 3(a),(b) for $p = 4, q = 3, H = h$ and $H = 2h$ as well as to Figure 3(c),(d) for $p = 5, q = 4, H = h$ and $H = 2h$, respectively.

We also test configurations with $q > p$ and observe that such configurations result in a stable discretization if H is chosen sufficiently coarse. This is illustrated in Figure 4, where p and q are set to $p = 2$ and $q = 6$. Choosing $H = 8h$ we obtain a stable discretization without oscillations, cf. Figure 4(a), whereas for $H = 4h$ the Lagrange multiplier oscillates, cf. Figure 4(b). In the experiments we use Gauss points as well as Chebychev points to define the set C . Both approaches yield the same stability results.

In the second experiment, we use non-matching meshes and discuss the stability properties. Moreover, we study the convergence rates. Here, the two linear elastic bodies are represented by $\Omega_1 = [0, 10] \times [9.9995, 19.9995]$ and $\Omega_2 = [0, 10] \times [0, 10]$. The configuration is shown in Figure 1(c). Young's moduli are set to $E_1 = 1.5 \cdot 10^5$ and $E_2 = 2 \cdot 10^5$, Poisson's ratios to $\nu_1 = 0.2$ and $\nu_2 = 0.4$, respectively. On the left and right side of Ω_2 Neumann boundary conditions with $f = (\pm 0.5, -5)$ are prescribed. In Figure 1(d) the resulting maximum displacement is shown.

Table 1: Error of the displacement variable u_h in the energy norm and convergence rates

DoF	p=1,q=0	rates	p=1,q=1	rates
124	2.17E-04	-	-	-
456	1.01E-04	1.09	1.11E-04	-
1744	5.65E-05	0.84	5.73E-05	0.95
6816	2.99E-05	0.92	3.06E-05	0.90
26944	1.58E-05	0.91	1.58E-05	0.95
107136	8.13E-06	0.95	8.12E-06	0.95

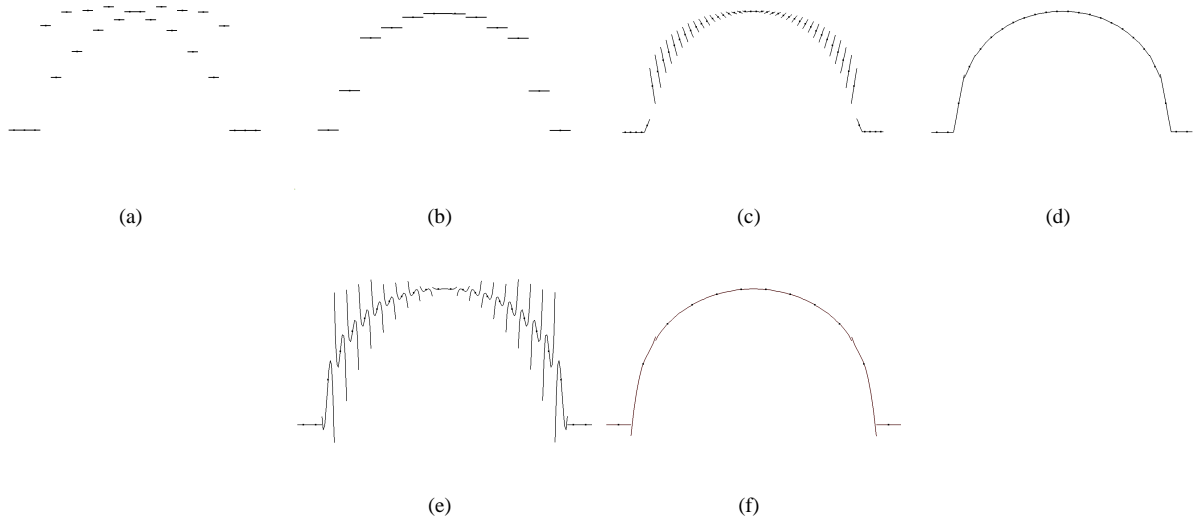


Figure 5: Lagrange multiplier: **(a)** $p = 1, q = 0, H = \min\{h^1, h^2\}$, **(b)** $p = 1, q = 0, H = 2 \max\{h^1, h^2\}$, **(c)** $p = 2, q = 1, H = \min\{h^1, h^2\}$, **(d)** $p = 2, q = 1, H = 2 \max\{h^1, h^2\}$, **(e)** $p = 4, q = 3, H = \min\{h^1, h^2\}$, **(f)** $p = 4, q = 3, H = 2 \max\{h^1, h^2\}$.

Table 2: L^2 -error of Lagrange multiplier λ_H and convergence rates

DoF	p=1,q=0	rates	p=1,q=1	rates
2	5.60	-	-	-
4	2.40	1.22	2.28	-
8	2.13	0.16	2.36	-0.04
16	0.90	1.24	1.21	0.95
32	0.53	0.75	0.26	2.20
64	0.30	0.83	0.14	0.85
128	0.11	1.33	0.11	0.37

As in the first experiment, we study different discretizations of the primal variable and the Lagrange multiplier. In Figure 5, the discrete Lagrange multiplier is depicted for various polynomial degrees and mesh sizes. Again, we observe that the choice $p = q - 1$ and $H = \min\{h^1, h^2\}$ lead to an unstable discretization, where the coarsening of the surface mesh \mathcal{E} via $H = 2 \min\{h^1, h^2\}$ stabilizes the scheme. Note that the non-conforming property of the discretization can be seen in Figure 5(e) and (f) where the discrete Lagrange multiplier is partly negative.

To study the convergence of the mixed scheme, we calculate reference solutions on meshes which are given by at least one additional uniform refinement of the meshes at the finest level. For instance, in the case $p = 1$, the reference solution is determined with more than $5.25 \cdot 10^6$ degrees of freedom (DoF) on Ω_1 and with more than $1.18 \cdot 10^7$ DoF on Ω_2 . In the case $p = 2$, we use more than $3.94 \cdot 10^6$ DoF on Ω_1 and more than $8.86 \cdot 10^6$ DoF on Ω_2 . We measure the error of the displacement variable in the energy norm and the error of the Lagrange multiplier in the L^2 -Norm.

In Tables 1, 2, 3 and 4, the discretization errors of the displacement variable and the Lagrange multiplier are shown. Moreover, the convergence rates are depicted, which is the number κ with $\|u - u_h^p\|_1 = \mathcal{O}(h^\kappa)$ and $\|\lambda - \lambda_H^q\|_0 = \mathcal{O}(h^\kappa)$, respectively. We observe that the convergence of the mixed scheme with $p = 1$ and $q = 0$ is nearly 1 which corresponds to the results in (4.7) and (4.8). The convergence rates of the Lagrange multiplier vary more or less near to 1. Note that the discretization error of the Lagrange multiplier is determined in the L^2 -norm and not in the $H^{-1/2}$ -norm. For $p = 2$ and $q = 1$ we obtain almost the same convergence rates. The reason may be the lack of regularity of u and

Table 3: Error of the displacement variable u_h in the energy norm and convergence rates

DoF	p=2,q=1	rates	p=2,q=0	rates
352	7.81E-05	0	5.96E-05	-
1328	3.67E-05	1.09	3.39E-05	0.81
5152	1.86E-05	0.97	1.72E-05	0.97
20288	9.04E-06	1.04	9.13E-06	0.91
80512	4.75E-06	0.92	4.77E-06	0.93

Table 4: L^2 -error of Lagrange multiplier λ_H and convergence rates

DoF	p=2,q=1	rates	p=2,q=0	rates
4	2.28	0	2.37	-
8	2.32	-0.02	1.97	0.26
16	1.13	1.03	0.88	1.15
32	0.25	2.16	0.52	0.75
64	0.13	0.86	0.29	0.84
128	0.10	0.43	0.12	1.23

λ . However, the constant in the convergence study seems to be smaller which may suggest the use of higher-order schemes in the sense of the h -method, cf. Figure 6(a) and (b). It is noted that these results are comparable with the results in [19] where similar experiments are studied in the context of a mixed scheme with lowest and second order Lagrange multipliers. In Table 5, we consider the p -method of the mixed scheme, i.e., the mesh sizes h and H are fixed and the polynomial degrees p and q are increased. The polynomial degree for the Lagrange multiplier is set to $q = p - 1$ and the mesh size to $H = 2h$ to ensure the stability of the scheme. Again, we use Gauss points and Chebychev points to ensure the sign condition of the Lagrange multipliers in the definition of $\mathcal{M}_{H,+}^q$. The convergence rates are determined by $\kappa = \ln(\|e_i\|_1/\|e_j\|_1)/\ln(N_j/N_i)$ where e_i is the error and N_i the number of unknowns in the i -th row in Table 5. Note that the numerical findings indicate that the a priori results in Theorem 4.6 may be suboptimal. Indeed, we do not expect $\gamma_{C\bar{n}}(u) - g$ to be sufficiently regular in realistic problems so that the determined convergence rates will match with the computational results. Figure 6(c) shows the convergence of the h - and the p -method with respect to the number of degrees of freedom. We observe that the convergence rate of the p -method seems to be slightly better than that of the h -method.

To illustrate the mixed scheme to be applicable to real world applications, we finally consider a 3D contact problem which is taken from a grinding simulation, cf. [1]. In Figure 7(a), a workpiece with a mounted point in front is depicted. In this simulation, we discretize the displacement variable with trilinear finite elements. The Lagrange multiplier is discretized with piecewise constant functions and is defined on the surface mesh of the mounted point. In the simulation, $2h \leq H$ is assumed. To achieve a suitable resolution, the mesh of the workpiece and the mesh of the mounted point are adaptively refined in their contact zones. The deformation of the workpiece and the mounted point

Table 5: Error of the displacement variable u_h in the energy norm and convergence rates using Gauss- and Chebychev points

DoF	Gauss	rates	Chebychev	rates
1744	5.18E-5	-	5.18E-5	-
5152	1.74E-5	1.00	1.74E-5	1.00
8560	1.61E-5	0.14	1.61E-5	0.14
13632	1.03E-5	0.96	1.06E-5	0.90
20368	7.56E-6	0.78	7.71E-6	0.80
28768	6.02E-6	0.65	5.98E-6	0.73
38832	4.81E-6	0.75	-	-

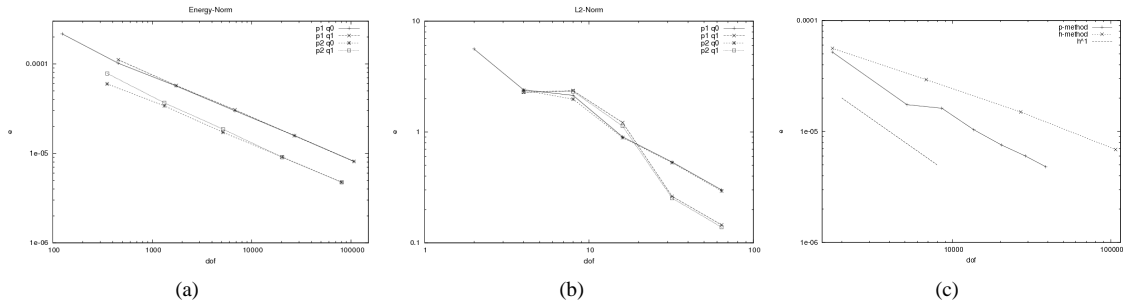


Figure 6: (a) Error of the displacements u_h , (b) error of λ_H , (c) error of u_h using the p -method.

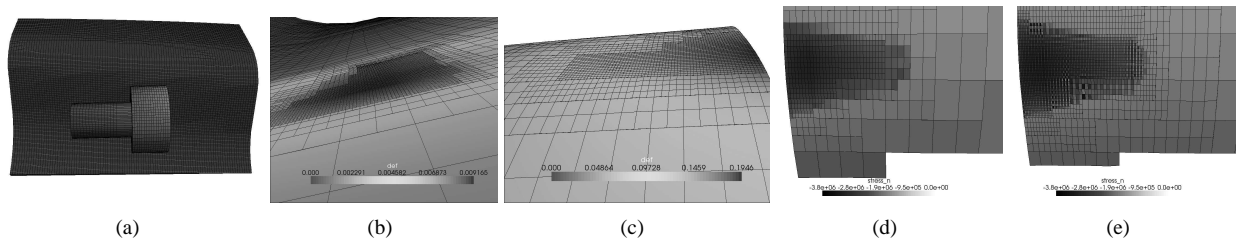


Figure 7: (a) workpiece with mounted point in front, (b) deformation of the workpiece, (c) deformation of the mounted point, (d) Lagrange multiplier without checkerboard patterns, (e) with checkerboard patterns.

is depicted in Figure 7(b)-(c). The adaptively resolved discrete Lagrange multiplier in the contact zone is shown in Figure 7(d). It does not have oscillations or checkerboard patterns due to $2h \leq H$. In this case, the discrete Lagrange multiplier can be interpreted as a contact force which is of particular interest in this simulation to control the grinding process. In Figure 7(e), the Lagrange multiplier with $h = H$ is depicted. We observe undesired checkerboard patterns which indicate that the discrete Lagrange multiplier is not unique.

7. Conclusion

In this paper, a mixed method of higher-order for two-body contact problems is proposed. It relies on a saddle point formulation where a Lagrange multiplier, defined on the surface of one of the bodies, captures the geometrical contact condition. The main results are given by the convergence of the scheme and a uniform discrete inf-sup condition which is an essential assumption to prove the stability and convergence of the mixed method. It is shown that the discrete inf-sup condition is uniformly fulfilled if the quotients of the mesh sizes and the polynomial degrees are sufficiently small. It can be observed in numerical experiments that the variation of these quotients may avoid instability effects. Another aspect of this paper is the provision of convergence rates with respect to a higher-order discretization. The essential ingredient is to enforce the sign condition in Gauss points enabling to use higher-order interpolation and to exploit the weak convergence of the discrete Lagrange multiplier.

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