Time-dependent fluid transport: Analytical Framework

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Abstract. We introduce a solver method for time-dependent mobile and immobile transport regions. The motivation is driven by transport processes in porous medias (e.g. waste disposal, chemical deposition processes).
We analyze the coupled transport-reaction equation with mobile and immobile areas.
We apply analytical methods, such as Laplace-transformation, and extend the linear case to the time-dependent case.
The analytical methods are based on variation of constants and Laplace-transformation to ordinary differential equations.
In such analytical solutions can be embedded into a finite volume methods which are only spatial dependent and improve their application to time-dependent convection and reaction problems. At the end of the article we illustrate the analytical solutions for different benchmark problems.

Key words: advection-reaction equation, mobile-immobile transport, Laplace transformation, analytical solutions, finite volume methods.
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1 Introduction

We study real-life problems in the direction of deposition processes given by transport and reaction models.

The modeling is based an a homogenization of the underlying media, see [3] and [4].

The equations are coupled with the reaction terms and are presented as follows.

$$\partial_t R_i u_i + e_i(t) \nabla \cdot \mathbf{v}\, u_i = -\lambda_i\ f_i(t)\ R_i\ u_i + \lambda_{i-1}\ R_{i-1}\ f_{i-1}(t)\ u_{i-1} \tag{1}$$
$$+\beta(-u_i + g_i) \ \text{ in } \Omega \times (0,T)\ ,$$
$$u_{i,0}(x) = u_i(x,0) \ \text{ on } \Omega\ , \tag{2}$$

$$\partial_t R_i g_i = -\lambda_i \; R_i \; f_i(t) \; g_i + \lambda_{i-1} \; R_{i-1} \; f_{i-1}(t) \; g_{i-1} \tag{3}$$
$$+\beta(-g_i + u_i) \;\text{ in } \Omega \times (0,T) \;,$$
$$g_{i,0}(x) = g_i(x,0) \;\text{ on } \Omega \;, \tag{4}$$
$$i = 1, \ldots, m \;,$$

where m is the number of equations and i is the index of each component. The unknown mobile concentrations $u_i = u_i(x,t)$ are considered in $\Omega \times (0,T) \subset I\!R^n \times I\!R^+$, where n is the spatial dimension. The unknown immobile concentrations $g_i = g_i(x,t)$ are considered in $\Omega \times (0,T) \subset I\!R^n \times I\!R^+$, where n is the spatial dimension. The retardation factors R_i are constant and $R_i \geq 0$. The kinetic part is given by the factors λ_i. They are constant and $\lambda_i \geq 0$.

Further $e_i(t), f_i(t) : I\!R^+ \to I\!R^+, \; i = 1, \ldots, m$ are the polynomial functions of the time-dependent convection and reaction term.

For the initialization of the kinetic part, we set $\lambda_0 = 0$. The kinetic part is linear and irreversible, so the successors have only one predecessor. The initial conditions are given for each component i as constants or linear impulses. For the boundary conditions we have trivial inflow and outflow conditions with $u_i = 0$ at the inflow boundary. The transport part is given by the velocity $\mathbf{v} \in I\!R^n$ and is piecewise constant, see [13] and [14]. The exchange between the mobile and immobile part is given by β.

The paper is organized as follows. One of the main contributions are the one-dimensional analytical solutions. The application for reaction equations are described in section 2. In section 3, the construction of the analytical solutions for the convection reaction equations in different situations is presented. The verification of the new discretization method in various numerical examples is performed in section 4. At the end of this paper we introduce future works.

2 Analytical solutions for time-dependent reaction equation

In the next section we deal with the following system with piecewise constant velocities for the coupled transport in one dimension. The equation is given as

$$R_i \partial_t u_i = -R_i \; f_i(t) \lambda_i u_i + R_{i-1} f_{i-1}(t) \lambda_{i-1} u_{i-1} \;, \; t \in [0,T], \tag{5}$$

for $i = 1, \ldots, m$, whereas m denotes the number of equations. The unknowns $u_i = u_i(x,t)$ denote the contaminant concentrations. They are reacting with constant rates λ_i and R_i are the retardation factors. Further $T \in I\!R^+$ is the final time.

We assume a irreversible form of a decay chain, e.g. $\lambda_0 = 0$ and $f_0(t) = 0, \; \forall t \in [0,T]$, and for each contaminant given single source terms $\lambda_{i-1} u_{i-1}$.

Lemma 1. *We deal with the linear differential equation (24) where $f_j : I\!R \to I\!R, \; j = 1, \ldots, i$ are a Riemann integrable functions.*

Then the analytical solution is given as:

$$c_1(t) = c_{01} \exp(-\lambda_1 \int_0^t f(s)\, ds), \tag{6}$$

$$\begin{aligned} c_i(t) &= c_{0i} \exp(-\lambda_i \int_0^t f(s)\, ds) \\ &+ \frac{R_{i-1}}{R_i} \lambda_{i-1} \exp(-\lambda_i \int_0^t f(s)\, ds) \int_0^t \exp(\int_0^s \lambda_i f_i(s_1) ds_1) f_{i-1}(s) c_{i-1}(s)\, ds \end{aligned} \tag{7}$$

where $c(0) = (c_{01}, \ldots, c_{0i})^t$ *are the initial condition. We assume* i *components.*

Proof. The idea is based on the integration rule of a linear differential equation and we apply the recursive arguments.

Example 1. We obtain the analytical solution for $i = 3$ with $f_1, f_2, f_3 : I\!R \to I\!R$:

$$c_1(t) = c_{01} \exp(-\lambda_1 \int_0^t f_1(s)\, ds), \tag{8}$$

$$\begin{aligned} c_2(t) &= c_{02} \exp(-\lambda_2 \int_0^t f_2(s)\, ds) \\ &+ c_{01} \frac{R_1}{R_2} \lambda_1 \exp(-\lambda_2 \int_0^t f_2(s_1)) ds_1 \\ &\cdot \int_0^t \exp(-\lambda_1 \int_0^s f_1(s_1) ds_1 + \int_0^s \lambda_2 f_2(s_1) ds_1) f_1(s)\, ds \end{aligned} \tag{9}$$

$$\begin{aligned} c_3(t) &= c_{03} \exp(-\lambda_3 \int_0^t f_3(s)\, ds) \\ &+ c_{02} \frac{R_2}{R_3} \lambda_2 \exp(-\lambda_3 \int_0^t f_3(s_1) ds_1 \\ &\cdot \int_0^t \exp(- \int_0^s \lambda_2 f_2(s_1) ds_1 + \int_0^s \lambda_3 f_3(s_1) ds_1) f_2(s)\, ds \\ &+ c_{01} \frac{R_1}{R_3} \lambda_1 \lambda_2 \exp(-\lambda_3 \int_0^t f_3(s_1)) ds_1 \\ &\cdot \int_0^t \exp(- \int_0^s \lambda_2 f_2(s_1) ds_1 + \int_0^s \lambda_3 f_3(s_1) ds_1) \\ &\cdot \left(\int_0^s \exp(- \int_0^{s_2} \lambda_1 f_1(s_1) ds_1 + \int_0^{s_2} \lambda_2 f_2(s_1) ds_1) f_1(s_2)\, ds_2 \right) f_2(s)\, ds \end{aligned} \tag{10}$$

Remark 1. The closed forms are only possible for a helpful simplification with $f_i(t) = f(t), \forall i = 1, \ldots, m$, here we obtain an analytical integrable for of the integral in equation (32). We can use the primitive
$\int_0^t \exp(\int_0^s \lambda f(s_1) ds_1) f(s)\, ds = \frac{1}{\lambda} \exp(\int_0^s \lambda f(s_1) ds_1$ while $\lambda \in I\!R$ is a constant.
In general, the integral in equation (32) can be solved with numerical integration.

2.1 Special case

In the following we derive an analytical solution of for a special case of the time-dependent functions.

Lemma 2. *We deal with the linear differential equation (24) and assume:* $f_1(t) = \ldots = f_i(t) = f(t)$, *where* $f : I\!R \to I\!R$ *is a Riemann integrable function.*

Then the analytical solution is given as:

$$c_1(t) \ = c_{01} \exp(-\lambda_1 \int_0^t f(s) \ ds), \tag{11}$$

$$c_i(t) \ = c_{0i} \exp(-\lambda_i \int_0^t f(s) \ ds) \tag{12}$$

$$+ \sum_{m=1}^{i-1} c_{0m} \frac{R_m}{R_i} (\prod_{j=m}^{i-1} \lambda_j) \sum_{j=m}^{i} \frac{\exp(-\lambda_j \int_0^t f(s) \ ds)}{\prod_{k=m,k\neq j}^{i} (\lambda_k - \lambda_j)}$$

where $c(0) = (c_{01}, \ldots, c_{0i})^t$ *are the initial condition. We assume* i *components.*

Proof. We apply the integration rule of a linear differential equation and the complete induction:

$i = 1$: We solve:

$$\partial_t R_1 c_1(t) \ = -\lambda_1 R_1 f(t) c_1(t), \tag{13}$$

$$c_1(0) = c_{01} \ , \tag{14}$$

and we obtain

$$c_1(t) \ = c_{01} \exp(-\lambda_1 \int_0^t f(s) \ ds). \tag{15}$$

In the induction step, we assume the solution of c_i is correct and given we go to the step:
$i \to i + 1$

We solve:

$$\partial_t R_{i+1} c_{i+1}(t) \ = -\lambda_{i+1} R_{i+1} f(t) c_{i+1}(t) + \lambda_i R_i f(t) c_i(t), \tag{16}$$

$$c_{i+1}(0) = c_{0,i+1} \ , \tag{17}$$

and we obtain

$$c_{i+1}(t) \ = \exp(-\lambda_{i+1} \int_0^t f(s) \ ds) \tag{18}$$

$$\left(c_{0,i+1} + \int_0^t \lambda_i \frac{R_i}{R_{i+1}} f(s) c_i(s) \ \exp(\lambda_{i+1} \int_0^s f(\tilde{s}) \ d\tilde{s}) \ ds \right). \tag{19}$$

We reset $c_i(s)$ by the assumed analytical solution and obtain:

$$c_{i+1}(t) = \exp(-\lambda_{i+1} \int_0^t f(s)\, ds) \tag{20}$$
$$\Big(c_{0,i+1} + \int_0^t \lambda_i \frac{R_i}{R_{i+1}} f(s) c_{0i} \exp(-\lambda_i \int_0^t f(s)\, ds) \ \exp(\lambda_{i+1} \int_0^s f(\tilde{s})\, d\tilde{s})\, ds$$
$$+ \int_0^t \lambda_i \frac{R_i}{R_{i+1}} f(s) \sum_{m=1}^{i-1} c_{0m} \frac{R_m}{R_i} (\prod_{j=m}^{i-1} \lambda_j) \sum_{j=m}^{i} \frac{\exp(-\lambda_j \int_0^t f(s)\, ds)}{\prod_{k=m,k\neq j}^{i} (\lambda_k - \lambda_j)} \ \exp(\lambda_{i+1} \int_0^s f(\tilde{s})\, d\tilde{s})\, ds \Big).$$

we obtain

$$c_{i+1}(t) = \exp(-\lambda_{i+1} \int_0^t f(s)\, ds) \tag{21}$$
$$\Big(c_{0,i+1} + \lambda_i \frac{R_i}{R_{i+1}} \frac{1}{\lambda_{i+1} - \lambda_i} c_{0i} \Big(\exp((\lambda_{i+1} - \lambda_i) \int_0^t f(s)\, ds) - 1 \Big)$$
$$+ \int_0^t \sum_{m=1}^{i-1} c_{0m} \frac{R_m}{R_{i+1}} (\prod_{j=m}^{i} \lambda_j) \sum_{j=m}^{i+1} \frac{\exp((\lambda_{i+1} - \lambda_j) \int_0^t f(s)\, ds)}{\prod_{k=m,k\neq j}^{i+1} (\lambda_k - \lambda_j)} \Big).$$

where we have derived the formula (32) for $i+1$ and we are done.

In the following we apply an example for a given $f(t)$.

Example 2. We obtain the analytical solution for the special case $f_1(t) = \ldots = f_m(t) = t$:

$$c_1(t) = c_{01} \exp(-\lambda_1 \frac{t^2}{2}), \tag{22}$$
$$c_i(t) = c_{0i} \exp(-\lambda_i \frac{t^2}{2}) + \sum_{m=1}^{i-1} c_{0m} \frac{R_m}{R_i} (\prod_{j=m}^{i-1} \lambda_j) \sum_{j=m}^{i} \frac{\exp(-\lambda_j \frac{t^2}{2})}{\prod_{k=m,k\neq j}^{i} (\lambda_k - \lambda_j)} \tag{23}$$

where $c(0) = (c_{01}, \ldots, c_{0i})^t$ are the initial condition. We assume i components.

Definition 1. *We define for the products and summations:*
1.) $\prod_{j=m}^{i} a_j = 1$ *for* $m = 1, i = 0$,
otherwise we use the notation.
2.) $\sum_{j=m}^{i} a_j = 0$ *for* $m = 1, i = 0$,
otherwise we use the notation.

Remark 2. For reversible reaction processes, there exists also analytical solutions, see [7]. For such solutions, we have to solve a coupled linear equation system.

3 Analytical solutions for convection-reaction equation with time-dependent convection and reaction

In the following we construct the solutions for the convection-reaction equations in different initial conditions and coupled situations.

3.1 Piecewise-constant and piecewise-linear initial conditions

In the next section we deal with the following system with piecewise constant velocities for coupled transport in one dimension. The equation is given as

$$\partial_t u_i + v_i \; e_i(t) \; \partial_x u_i = -\lambda_i \; f_i(t) \; u_i + \lambda_{i-1} \; f_{i-1}(t) \; u_{i-1} \; , \tag{24}$$

for $i = 1, \ldots, m$, whereas m denotes the number of equations. The unknowns $u_i = u_i(x,t)$ denote the contaminant concentrations. They are transported with constant (and in general different) velocities v_i and decay with constant reaction rates λ_i. The spatiotemporal domain is given by $(0, \infty) \times (0, T)$. Further $e_i, f_i : \mathbb{R}^+ \to \mathbb{R}^+$ are Riemann-integrable functions.

We assume a simple (irreversible) form of a decay chain, e.g. $\lambda_0 = 0$, and for each contaminant given single source terms $\lambda_{i-1} u_{i-1}$. For simplification, we assume that $v_i > 0$ for $i = 1, \ldots, m$. The case $v_i < 0$ can be treated analogously. Owing to (24), all velocities v_i must have the same sign and must be piecewise constant for the cell i. Furthermore we do not allow piecewise equal parameters for the case $v_i = v_{i-1}$ and $\lambda_i = \lambda_{i-1}$, for $i = 2, \ldots, m$. In special solutions we will allow these cases.

The analytical solutions for equal retardation factors can be found in [32]. We enlarge the solutions for different retardation factors and special initial conditions.

We will derive the analytical solutions with piecewise linear initial conditions, but all other piecewise polynomial functions could be derived as shown in the following.

For the boundary conditions we use zero concentrations at the inflow boundary $x = 0$. The initial conditions are defined for $x \in (0, 1)$,

$$\begin{aligned} u_1(x,0) &= \begin{cases} ax + b \; , & x \in (0,1) \\ 0 & , \text{otherwise} \end{cases} \; , \\ u_i(x,0) &= 0 \; , \quad i = 2, \ldots, m \; , \end{aligned} \tag{25}$$

where a and b are arbitrary constants.

We use the Laplace transformation for the translation of the partial differential equation to the ordinary differential equation. The transformations for this case are given in [6], [17] and [20].

In equation (24) we apply the Laplace transformation given in [1] and [5]. For that we need to define the transformed function $\hat{u} = \hat{u}(s,t)$:

$$\hat{u}_i(s,t) := \int_0^\infty u_i(x,t) \, e^{-sx} \, dx \, . \tag{26}$$

From (24), the functions $\hat{u}_i$ satisfy the transformed equations

$$\partial_t \hat{u}_1 = -\left(\lambda_1 \; f_1(t) \; + s v_1 \; e_1(t) \;\right) \hat{u}_1 \;, \tag{27}$$

$$\partial_t \hat{u}_i = -\left(\lambda_i \; f_i(t) \; + s v_i \; e_i(t) \;\right) \hat{u}_i + \lambda_{i-1} \; f_{i-1}(t) \hat{u}_{i-1} \;, \tag{28}$$

and the transformed initial conditions for $s \in (0, \infty)$,

$$\hat{u}_1(s, 0) = \left(\frac{a}{s^2} + \frac{b}{s} \right) (1 - e^{-s}) + \frac{a}{s} e^{-s} \,, \tag{29}$$

$$\hat{u}_i(s, 0) = 0 \,, \quad i = 2, \ldots, m \,. \tag{30}$$

Lemma 3. *We deal with the linear differential equations (43) and (28) where $e_i, f_i : I\!\!R \rightarrow I\!\!R$, $m = 1, \ldots, m$ are a Riemann integrable functions.*

Then the analytical solution of equations (43) and (28) are given as:

$$\hat{u}_1(t) \; = \hat{u}_{01} \exp(-\lambda_1 \int_0^t f_1(\tilde{t}) d\tilde{t} - s v_1 \int_0^t e_1(\tilde{t}) d\tilde{t}), \tag{31}$$

$$\begin{aligned} \hat{u}_i(t) \; &= \hat{u}_{0i} \exp(-\lambda_i \int_0^t f_i(\tilde{t}) d\tilde{t} - s v_i \int_0^t e_i(\tilde{t}) d\tilde{t}) && (32) \\ &+ \hat{u}_{0i-1} \lambda_{i-1} \; \exp(-\lambda_i \int_0^t f_i(\tilde{t}) d\tilde{t} - s v_i \int_0^t e_i(\tilde{t}) d\tilde{t}) && (33) \\ &\cdot \int_0^t \exp(\lambda_i \int_0^{s_1} f_i(\tilde{t}) d\tilde{t} + s v_i \int_0^{s_1} e_i(\tilde{t}) d\tilde{t}) \; f_{i-1}(s_1) \hat{u}_{i-1}(s_1) \; ds_1 \end{aligned}$$

where $\hat{u}(0) = (\hat{u}_{01}, \ldots, \hat{u}_{0i})^t$ are the initial condition. We assume i components.

Proof. The idea is based on the integration rule of a linear differential equation and we apply the recursive arguments.

Example 3. We obtain the analytical solution for $i = 1, 2, 3$ with $e_i, f_i : I\!\!R \rightarrow I\!\!R$:

$$\hat{u}_1(t) \; = \hat{u}_{01} \exp(-\lambda_1 \int_0^t f_1(\tilde{t}) d\tilde{t} - s v_1 \int_0^t e_1(\tilde{t}) d\tilde{t}), \tag{34}$$

$$\begin{aligned} \hat{u}_2(t) \; &= \hat{u}_{02} \exp(-\lambda_2 \int_0^t f_2(\tilde{t}) d\tilde{t} - s v_2 \int_0^t e_2(\tilde{t}) d\tilde{t}) && (35) \\ &+ \hat{u}_{01} \lambda_1 \; \exp(-\lambda_2 \int_0^t f_2(\tilde{t}) d\tilde{t} - s v_2 \int_0^t e_2(\tilde{t}) d\tilde{t}) \\ &\cdot \int_0^t \exp\Bigg(- \lambda_1 \int_0^{s_1} f_1(\tilde{t}) d\tilde{t} - s v_1 \int_0^{s_1} e_1(\tilde{t}) d\tilde{t} \\ &+ \lambda_2 \int_0^{s_1} f_2(\tilde{t}) d\tilde{t} + s v_2 \int_0^{s_1} e_2(\tilde{t}) d\tilde{t} \Bigg) f_1(s_1) \; ds_1 \end{aligned}$$

$$\hat{u}_3(t) \; = \hat{u}_{03} \exp(-\lambda_3 \int_0^t f_3(\tilde{t}) d\tilde{t} - s v_3 \int_0^t e_3(\tilde{t}) d\tilde{t}) \tag{36}$$

$$+\hat{u}_{02}\lambda_2 \ \exp(-\lambda_3 \int_0^t f_3(s_1) ds_1 - s v_3 \int_0^t e_3(s_1) ds_1)$$
$$\cdot \int_0^t \exp\Big(- \lambda_2 \int_0^{s_1} f_2(\tilde{t}) d\tilde{t} - s v_2 \int_0^{s_1} e_2(\tilde{t}) d\tilde{t}$$
$$+\lambda_3 \int_0^{s_1} f_3(\tilde{t}) d\tilde{t} + s v_3 \int_0^{s_1} e_3(\tilde{t}) d\tilde{t} \Big) f_2(s_1) \ ds_1$$
$$+\hat{u}_{01}\lambda_1\lambda_2 \ \exp(-\lambda_3 \int_0^t f_3(\tilde{t}) d\tilde{t} - s v_3 \int_0^t e_3(\tilde{t}) d\tilde{t})$$
$$\cdot \int_0^t \exp\Big(- \lambda_2 \int_0^{s_1} f_2(\tilde{t}) d\tilde{t} - s v_2 \int_0^{s_1} e_2(\tilde{t}) d\tilde{t}$$
$$+\lambda_3 \int_0^{s_1} f_3(\tilde{t}) d\tilde{t} + s v_3 \int_0^{s_1} e_3(\tilde{t}) d\tilde{t} \Big)$$
$$\cdot \Bigg(\int_0^{s_1} \exp\Big(- \lambda_1 \int_0^{s_2} f_1(\tilde{t}) d\tilde{t} - s v_1 \int_0^{s_2} e_1(\tilde{t}) d\tilde{t}$$
$$+\lambda_2 \int_0^{s_2} f_2(\tilde{t}) d\tilde{t} + s v_2 \int_0^{s_2} e_2(\tilde{t}) d\tilde{t} \Big)$$
$$\cdot f_1(s_2) \ ds_2 \Bigg) f_2(s_1) \ ds_1$$

Remark 3. The closed forms are only possible for a helpful simplification with $f_i(t) = f(t), \forall i = 1, \ldots, m$, here we obtain an analytical integrable for of the integral in equation (32). We can use the primitive
$\int_0^t \exp(\int_0^s \lambda f(s_1) ds_1) f(s) \ ds = \frac{1}{\lambda} \exp(\int_0^s \lambda f(s_1) ds_1$ while $\lambda \in I\!R$ is a constant.
In general, the integral in equation (32) can be solved with numerical integration.

3.2 Special case

We assume $f(t) = e_i(t) = f_i(t), \forall \ i = 1, \ldots, m$.

Based on this assumption, we can follow the linear case, see also below the proof.

We denote for further solutions:

$$\Lambda_i = \prod_{j=1}^{i-1} \lambda_j \ . \tag{37}$$

The equation (28) is solved with the solution methods for the ordinary differential equation, described in [17], and the more general case is presented in [6].

Lemma 4. *The exact solution of (43) and (28) are given as:*

$$\hat{u}_1 = \hat{u}_1(s,0) \, e^{-(\lambda_1 + s v_1) \ \int_0^t f(\tilde{t}) \ d\tilde{t}} \ , \tag{38}$$

for $i = 2, \ldots, m$,

$$\hat{u}_i = \hat{u}_1(s,0) \, \Lambda_i \sum_{j=1}^{i} e^{-(\lambda_j + s v_j) \int_0^t f(\tilde{t}) \, d\tilde{t}} \prod_{\substack{k=1 \\ k \neq j}}^{i} (s(v_k - v_j) + \lambda_k - \lambda_j)^{-1} \ . \quad (39)$$

Proof. The case $i = 1$ is trivial:

$$\partial_t \hat{u}_1 = - \left(\lambda_1 \ f(t) \ + s v_1 \ f(t) \ \right) \hat{u}_1 \ , \quad (40)$$

$$(41)$$

we have to inset the solution in equation (40):

$$\hat{u}_1 = \hat{u}_1(s,0) \, e^{-(\lambda_1 + s v_1) \ \int_0^t f(\tilde{t}) \, d\tilde{t}} \ , \quad (42)$$

and we obtain:

$$\partial_t \hat{u}_1 = - \left(\lambda_1 \ f(t) \ + s v_1 \ f(t) \ \right) \hat{u}_1 \ , \quad (43)$$

$$(44)$$

which is the result.

The proof idea for $i > 1$ is given to the transformation to the linear case. see [16].

The derivation of $\hat{u}_i$ is given as:

$$\partial_t \hat{u}_i = f(t) \hat{\partial}_t u_{i,linear}, \quad (45)$$

where the linear solutions $u_{i,linear}$ can also be written with kernel $e^{-(\lambda_j + s v_j) \int_0^t f(\tilde{t}) \, d\tilde{t}}$ instead of kernel $e^{-(\lambda_j + s v_j) t}$. The linear solutions are given and proofed in [16].

One can easily re-substitute the linear to the time-dependent kernel with out loosing any generality.

So we have to fulfill:

$$\partial_t \hat{u}_i = f(t) \left(- \left(\lambda_i \ + s v_i \ \right) \hat{u}_i + f(t) \lambda_{i-1} \hat{u}_{i-1} \right) \ , \quad (46)$$

and we have

$$f(t) \partial_t \hat{u}_{i,linear} = f(t) \left(- \left(\lambda_i \ + s v_i \ \right) \hat{u}_{i,linear} + f(t) \lambda_{i-1} \hat{u}_{i-1,linear} \right) \ , \quad (47)$$

$$\partial_t \hat{u}_{i,linear} = - \left(\lambda_i \ + s v_i \ \right) \hat{u}_{i,linear} + \lambda_{i-1} \hat{u}_{i-1,linear} \ , \quad (48)$$

while the linear solutions with the time-dependent kernel fulfilled the equation (48) and we are done.

The analytical solution in (39) can have a singular point for a single value of s. Nevertheless, this causes no difficulties when we apply the inverse Laplace transformation and thus we do not need to discuss this issue any further.

To obtain the exact solution of (24), we must apply the inverse Laplace transformation on (43). For that we have to apply some algebraic manipulations.

For the first case, let us assume that $v_j \neq v_k$ and $\lambda_j \neq \lambda_k$ for $j \neq k$ and $\forall j, k = 1, \ldots, m$. Then we can denote

$$\lambda_{kj} = \lambda_{jk} := \frac{\lambda_j - \lambda_k}{v_j - v_k} . \tag{49}$$

Furthermore, for the next transformation, we require that the values λ_{jk} are different for each pair of indices j and k.

The factors $\Lambda_{j,i}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$ and the factor $\Lambda_{jk,i}$ with $\lambda_{jk} \neq \lambda_{jl}$ for $k \neq l$ are given by

$$\Lambda_{j,i} = \left(\prod_{\substack{k=1 \\ k \neq j}}^{i} \frac{1}{\lambda_k - \lambda_j} \right) , \quad \Lambda_{jk,i} = \left(\prod_{\substack{l=1 \\ l \neq j \\ l \neq k}}^{i} \frac{\lambda_{jl}}{\lambda_{jl} - \lambda_{jk}} \right) , \tag{50}$$

where we have the following assumptions:

$$1.\ v_j \neq v_k \ \ \forall j, k = 1, \ldots, m, \text{ for } j \neq k , \tag{51}$$
$$2.\ \lambda_j \neq \lambda_k \ \ \forall j, k = 1, \ldots, m, \text{ for } j \neq k , \tag{52}$$
$$3.\ \lambda_{jk} \neq \lambda_{jl} \ \ \forall j, k, l = 1, \ldots, m, \text{ for } j \neq k \ \wedge j \neq l \wedge k \neq l, \tag{53}$$
$$4.\ v_j \neq v_k \text{ and } \lambda_j \neq \lambda_k \ \ \forall j, k = 1, \ldots, m, \text{ for } j \neq k . \tag{54}$$

From (50), the last term in (39) for a given index j can be rewritten in the following form,

$$\prod_{\substack{k=1 \\ k \neq j}}^{i} (s(v_k - v_j) + \lambda_k - \lambda_j)^{-1} = \Lambda_{j,i} \sum_{\substack{k=1 \\ k \neq j}}^{i} \frac{\lambda_{jk}}{s + \lambda_{jk}} \Lambda_{jk,i} . \tag{55}$$

From (29), adopted in (45) and (39), the standard inverse Laplace transformation can be used and the solution u_i for (24) is given by

$$\begin{aligned} & u_1(x,t) \\ & = \exp(-\lambda_1 \int_0^t f(\tilde{t}) \, d\tilde{t}) \\ & \cdot \begin{cases} 0 & , 0 \leq x < v_1 \int_0^t f(\tilde{t}) \, d\tilde{t} \\ a(x - v_1 \int_0^t f(\tilde{t}) \, d\tilde{t}) + b & , v_1 \int_0^t f(\tilde{t}) \, d\tilde{t} \leq x < v_1 \int_0^t f(\tilde{t}) \, d\tilde{t} + 1 \\ 0 & , v_1 \int_0^t f(\tilde{t}) \, d\tilde{t} + 1 \leq x \end{cases} , \end{aligned} \tag{56}$$

$$u_i(x,t) = \Lambda_i \left(\sum_{j=1}^{i} \exp(-\lambda_j \int_0^t f(\tilde{t}) \, d\tilde{t}) \Lambda_{j,i} \sum_{\substack{k=1 \\ k \neq j}}^{i} \Lambda_{jk,i} A_{jk} \right) , \tag{57}$$

$$A_{jk} = \begin{cases} 0 & , 0 \le x < v_j \int_0^t f(\tilde{t})\, d\tilde{t} \\ a(x - v_j \int_0^t f(\tilde{t})\, d\tilde{t}) & \\ +(b - \frac{a}{\lambda_{jk}})(1 - \exp(-\lambda_{jk}(x - v_j \int_0^t f(\tilde{t})\, d\tilde{t}))) & , v_j t \le x < v_j \int_0^t f(\tilde{t})\, d\tilde{t} + 1 \\ (b - \frac{a}{\lambda_{jk}} + a) \exp(-\lambda_{jk}(x - v_j \int_0^t f(\tilde{t})\, d\tilde{t} - 1)) & \\ -(b - \frac{a}{\lambda_{jk}}) \exp(-\lambda_{jk}(x - v_j \int_0^t f(\tilde{t})\, d\tilde{t})) & , v_j \int_0^t f(\tilde{t})\, d\tilde{t} + 1 \le x \end{cases} .$$

3.3 General initial conditions for piecewise linear convection-reaction equations with time-dependent reaction

To generalize our initial conditions we combine the subsection 3.1 and **??**.

The equation is given as

$$R_i \partial_t u_i + v_i f(t) \partial_x u_i = -R_i f(t) \lambda_i u_i + R_{i-1} f(t) \lambda_{i-1} u_{i-1} , \tag{58}$$

for $i = 1, \ldots, m$, whereas m denotes the number of equations. The unknowns $u_i = u_i(x,t)$ denote the contaminant concentrations. They are transported with constant (and, in general, different) velocities v_i and decay with constant reaction rates λ_i. The spatiotemporal domain is given by $(0, \infty) \times (0, T)$. Further, R_i is the retardation factor that respects the acceleration or restriction of the time scales.

We have the same assumptions as in the previous sections.

For the boundary conditions we use zero concentrations at the inflow boundary $x = 0$. The initial conditions are defined for $x \in (0, 1)$,

$$u_p(x, 0) = \begin{cases} \sum_{q=1}^{Q} b_{p,q} x + c_{p,q} & , x \in [x_q, x_{q+1}] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{59}$$

$$p = 1, \ldots, m , \tag{60}$$

where $b_{p,q}$ and $c_{p,q}$ are arbitrary constants of the piecewise quadratic function and $[x_q, x_{q+1}]$ is the interval of the function, Q is the number of intervals.

We use the Laplace transformation for the translation of the partial differential equation to the ordinary differential equation. The transformations for this case are given in [6], [17] and [20].

In equation (58) we apply the Laplace transformation given in [1] and [5]. For that we need to define the transformed function $\hat{u} = \hat{u}(s,t)$:

$$\hat{u}_i(s,t) := \int_0^\infty u_i(x,t)\, e^{-sx}\, dx . \tag{61}$$

From (58), the functions $\hat{u}_i$ satisfy the transformed equations

$$\partial_t \hat{u}_1 = -\left(\lambda_1 f(t) + s v_1 f(t)\right) \hat{u}_1 , \tag{62}$$

$$\partial_t \hat{u}_i = -\left(\lambda_i f(t) + s v_i f(t)\right) \hat{u}_i + \lambda_{i-1} f(t) \hat{u}_{i-1} , \tag{63}$$

and the transformed initial conditions for $s \in (0, \infty)$,

$$\hat{u}_p(s,0) = \sum_{q=1}^{Q} (\left(\frac{b_{p,q}}{s^2} + \frac{c_{p,q}}{s} \right) (1 - e^{-s})) \tag{64}$$

$$+ (\frac{b_{p,q}}{s}) \, e^{-s}) \, ,$$

$$p = 1, \ldots, m \, . \tag{65}$$

We denote for further solutions:

$$\Lambda_{i,p} = \prod_{j=p}^{i-1} \lambda_j \, . \tag{66}$$

Equation (62)-(63) is solved with the solution methods for the ordinary differential equation, described in [17], and the more general case is presented in [6].

Thus we find the exact solution of (62)-(63):

$$\hat{u}_1 = \hat{u}_1(s,0) \exp(-(\lambda_1 \int_0^t f(\tilde{t}) d\tilde{t} + s v_1 \int_0^t f(\tilde{t}) d\tilde{t})), \tag{67}$$

$$\hat{u}_i = \hat{u}_i(s,0) \exp(-(\lambda_i \int_0^t f(\tilde{t}) d\tilde{t} + s v_i \int_0^t f(\tilde{t}) d\tilde{t}))$$

$$+ \sum_{p=1}^{i-1} \hat{u}_p(s,0) \; \Lambda_{i,p} \; \sum_{j=p}^{i} e^{-(\lambda_j + s v_j) \int_0^t f(\tilde{t}) d\tilde{t}} \prod_{\substack{k=p \\ k \neq j}}^{i} \left(s(v_k - v_j) + \lambda_k - \lambda_j \right)^{-1} \, , \tag{68}$$

for $i = 1, \ldots, m$.

The analytical solution in (67)-(68) can have a singular point for a single value of s. Nevertheless, this causes no difficulties when we apply the inverse Laplace transformation and thus we do not need to discuss this issue any further.

To obtain the exact solution of (62)-(63), we must apply the inverse Laplace transformation on (67)-(68). For that we have to apply some algebraic manipulations.

For the first case, let us assume that $v_j \neq v_k$ and $\lambda_j \neq \lambda_k$ for $j \neq k$ and $\forall j, k = 1, \ldots, m$. Then we can denote

$$\lambda_{kj} = \lambda_{jk} := \frac{\lambda_j - \lambda_k}{v_j - v_k} \, . \tag{69}$$

Furthermore, for the next transformation, we require that the values λ_{jk} are different for each pair of indices j and k.

The factors $\Lambda_{j,i}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$ and the factor $\Lambda_{jk,i}$ with $\lambda_{jk} \neq \lambda_{jl}$ for $k \neq l$ are given by

$$\Lambda_{j,i,p} = \left(\prod_{\substack{k=p \\ k \neq j}}^{i} \frac{1}{\lambda_k - \lambda_j} \right) \; , \; \Lambda_{jk,i,p} = \left(\prod_{\substack{l=p \\ l \neq j \\ l \neq k}}^{i} \frac{\lambda_{jl}}{\lambda_{jl} - \lambda_{jk}} \right) \; , \tag{70}$$

where we have the following assumptions:

$$1.\ v_j \neq v_k \ \ \forall j,k = 1,\dots,m,\ \text{for } j \neq k\ , \tag{71}$$
$$2.\ \lambda_j \neq \lambda_k \ \ \forall j,k = 1,\dots,m,\ \text{for } j \neq k\ , \tag{72}$$
$$3.\ \lambda_{jk} \neq \lambda_{jl} \ \ \forall j,k,l = 1,\dots,m,\ \text{for } j \neq k\ \wedge j \neq l \wedge k \neq l, \tag{73}$$
$$4.\ v_j \neq v_k \text{ and } \lambda_j \neq \lambda_k \ \ \forall j,k = 1,\dots,m,\ \text{for } j \neq k\ . \tag{74}$$

From (70), the last term in (67)-(68) for a given index j can be rewritten in the following form,

$$\prod_{\substack{k=p\\k\neq j}}^{i} \left(s(v_k - v_j) + \lambda_k - \lambda_j\right)^{-1} = \mathit{\Lambda}_{j,i,p} \sum_{\substack{k=p\\k\neq j}}^{i} \frac{\lambda_{jk}}{s + \lambda_{jk}} \mathit{\Lambda}_{jk,i,p}\ . \tag{75}$$

From (64) adopted in (62) and (63), the standard inverse Laplace transformation can be used and the solution u_i for (61) is given by

$$\begin{aligned} &u_1(x,t) \\ &= \exp(-\lambda_1 \int_0^t f(\tilde{t})d\tilde{t}) \\ &\cdot \sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_1 \int_0^t f(\tilde{t})d\tilde{t} + x_q \\ b_{1,q}(x - v_1 \int_0^t f(\tilde{t})d\tilde{t}) + c_{1,q} & , v_1 \int_0^t f(\tilde{t})d\tilde{t} + x_q \leq x < v_1 \int_0^t f(\tilde{t})d\tilde{t} + x_{q+1} \\ 0 & , v_1 \int_0^t f(\tilde{t})d\tilde{t} + x_{q+1} \leq x \end{cases} , \end{aligned} \tag{76}$$

$$\begin{aligned} &u_i(x,t) \\ &= \exp(-\lambda_i \int_0^t f(\tilde{t})d\tilde{t}) \\ &\cdot \sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_i \int_0^t f(\tilde{t})d\tilde{t} + x_q \\ b_{i,q}(x - v_1 \int_0^t f(\tilde{t})d\tilde{t}) + c_{i,q} & , v_i \int_0^t f(\tilde{t})d\tilde{t} + x_q \leq x < v_i \int_0^t f(\tilde{t})d\tilde{t} + x_{q+1} \\ 0 & , v_i \int_0^t f(\tilde{t})d\tilde{t} + x_{q+1} \leq x \end{cases} , \\ &+ \sum_{p=1}^{i-1} \mathit{\Lambda}_{i,p} \left(\sum_{j=p}^{i} \exp(-\lambda_j \int_0^t f(\tilde{t})d\tilde{t}) \mathit{\Lambda}_{j,i,p} \sum_{\substack{k=p\\k\neq j}}^{i} \mathit{\Lambda}_{jk,i,p} A_{jk,p} \right) , \end{aligned} \tag{77}$$

$$A_{jk,p} = \sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_q \\ b_{i,q}(x - (v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_i)) \\ +(c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}}) \\ \cdot(1 - \exp(-\lambda_{jk}(x - (v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_q)))) & , v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_q \leq x \\ & , < v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_{q+1} \\ (c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}} + b_{i,q}) \\ \cdot \exp(-\lambda_{jk}(x - (v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_{q+1}))) \\ -(c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}}) \\ \cdot \exp(-\lambda_{jk}(x - (v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_q))) & , v_j \int_0^t f(\tilde{t}) d\tilde{t} + x_{q+1} \leq x \end{cases}$$

with $i = 2, \ldots, m$.

where Q is the number of piecewise linear intervals.

Remark 4. Here, we have derived analytical solutions of (58) with general initial conditions (59) that are used to verify numerical results or design discretization methods with embedded analytical support functions, see [18] and [33]. Modifications of the solutions can be done respecting to dominant components and improved notations to stabilize the numerical computations.

3.4 Solution for the mobile and immobile parts

Here we construct semi-analytical solutions for the mobile and immobile parts. We deal with the equations:

$$\partial_t u_i + v_i f(t) \partial_x u_i = -\lambda_i f(t) u_i + \lambda_{i-1} f(t) u_{i-1} + \beta(-u_i + g_i) \ , \qquad (78)$$

$$\partial_t g_i = -\lambda_i f(t) g_i + \lambda_{i-1} f(t) g_{i-1} + \beta(-g_i + u_i) \ , \qquad (79)$$

where the function $f : I\!R \to I\!R$ is integrable.

We propose a splitting method to decouple the mobile and the immobile parts of the equations.

We set A as the operator for the mobile part and B as the operator for the immobile part.

The following iteration scheme solves the problem.

The iterative time-splitting method

The following algorithm is based on the iteration with fixed splitting discretization step-size τ. On the time interval $[t^n, t^{n+1}]$ we solve the following sub-problems consecutively for $j = 0, 2, \ldots 2m$.

$$\frac{\partial U_j(x,t)}{\partial t} = AU_j(x,t) \ + \ BU_{j-1}(x,t), \text{ with } U_j(t^n) = U^n \qquad (80)$$

$$U_0(x, t^n) = U^n \ , \ U_{-1} = 0,$$

$$\begin{aligned}
&\text{and } U_j(x,t) = U_{j-1}(x,t) = u_1 \ , \ \text{on } \partial\Omega \times (0,T) \ , \\
&\frac{\partial U_{j+1}(x,t)}{\partial t} = \ AU_j(x,t) \ + \ BU_{j+1}(x,t), \\
&\text{with } \ U_{j+1}(x,t^n) = U^n \ , \\
&\text{and } U_j(x,t) = U_{j-1}(x,t) = U_1 \ , \ \text{on } \partial\Omega \times (0,T) \ ,
\end{aligned} \tag{81}$$

where $U^n = (u,g)^t$ is the vector of the mobile and immobile solutions and is the known split approximation at the time level $t = t^n$ (see [8]).

Now we apply to our concrete solutions:
First the mobile solution equation, given in (78):

$$\begin{aligned}
&u_{1,uncoupled}(x,t) \\
&= \exp(-\lambda_1 \int_0^t f(s)ds) \\
&\sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_1 \int_0^t f(s)ds + x_q \\ b_{1,q}(x - v_1 \int_0^t f(s)ds) + c_{1,q} & , v_1 \int_0^t f(s)ds + x_q \leq x < v_1 \int_0^t f(s)ds + x_{q+1} \\ 0 & , v_1 \int_0^t f(s)ds + x_{q+1} \leq x \end{cases} ,
\end{aligned} \tag{82}$$

$$\begin{aligned}
&u_{i,uncoupled}(x,t) \\
&= \exp(-\lambda_i \int_0^t f(s)ds) \\
&\sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_i \int_0^t f(s)ds + x_q \\ b_{i,q}(x - v_1 \int_0^t f(s)ds) + c_{i,q} & , v_i \int_0^t f(s)ds + x_q \leq x < v_i \int_0^t f(s)ds + x_{q+1} \\ 0 & , v_i \int_0^t f(s)ds + x_{q+1} \leq x \end{cases} , \\
&+ \sum_{p=1}^{i-1} \Lambda_{i,p} \left(\sum_{j=p}^{i} \exp(-\lambda_j \int_0^t f(s)ds) \Lambda_{j,i,p} \sum_{\substack{k=p \\ k \neq j}}^{i} \Lambda_{jk,i,p} A_{jk,p} \right) ,
\end{aligned} \tag{83}$$

$$A_{jk,p} = \sum_{q=1}^{Q} \begin{cases} 0 & , 0 \leq x < v_j \int_0^t f(s)ds + x_q \\ \\ b_{i,q}(x - (v_j \int_0^t f(s)ds + x_i)) & \\ +(c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}}) & \\ \cdot(1 - \exp(-\lambda_{jk}(x - (v_j \int_0^t f(s)ds + x_q)))) & , v_j \int_0^t f(s)ds + x_q \leq x \\ & , < v_j \int_0^t f(s)ds + x_{q+1} \\ \\ (c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}} + b_{i,q}) & \\ \cdot \exp(-\lambda_{jk}(x - (v_j \int_0^t f(s)ds + x_{q+1}))) & \\ -(c_{i,q} - \frac{b_{i,q}}{\lambda_{jk}}) & \\ \cdot \exp(-\lambda_{jk}(x - (v_j \int_0^t f(s)ds + x_q))) & , v_j \int_0^t f(s)ds + x_{q+1} \leq x \end{cases}$$

with $i = 2, \ldots, m$.

where Q is the number of piecewise linear intervals.

Second the immobile solution, given in (79):

$$g_{1,uncoupled}(t) = g_{01} \exp(-\lambda_1 \int_0^t f(s)ds), \tag{84}$$

$$g_{i,uncoupled}(t) = g_{0i} \exp(-\lambda_i \int_0^t f(s)ds) \tag{85}$$
$$+ \sum_{m=1}^{i-1} g_{0m} (\Pi_{j=m}^{i-1} \lambda_j) \sum_{j=m}^{i} \frac{\exp(-\lambda_j \int_0^t f(s)ds)}{\prod_{k=m,k\neq j}^{i} (\lambda_k - \lambda_j)}$$

where $g(0) = (g_{01}, \ldots, g_{0i})^t$ are the initial condition. We assume i components.

The general iterative scheme is given as:

Algorithm 31 *On the time interval* $[0, t]$ *we solve the following sub-problems consecutively, for* $j = 1, 2, 3, \ldots M$ *and for the components* $i = 1, \ldots, m$.

$$u_{i,coupled,j}(x,t) = u_{i,uncoupled}(x,t) \tag{86}$$
$$+ \int_0^t u_{i,uncoupled}(x, t-s) \beta g_{i,coupled,j-1}(x,s) \, ds$$
$$\textit{with } \; u_{i,coupled,j}(x,0) = u_{i,uncoupled}(x,0),$$
$$g_{i,coupled,j}(x,t) = g_{i,uncoupled}(t) \tag{87}$$
$$+ \int_0^t g_{i,uncoupled}(t-s) \beta u_{i,coupled,j-1}(x,s) \, ds$$
$$\textit{with } \; g_{i,coupled,j}(x,0) = g_{i,uncoupled}(0),$$

where the initialization $u_{i,coupled,0}(x,t)$, $g_{i,coupled,0}(x,t)$ *is an approximation of* $u_i(x,t), g_i(x,t)$ *and can be chosen in our linear case to be* 0, *for example the uncoupled solutions* $u_{i,coupled,0}(x,t) = u_{i,uncoupled}(x,t)$ *and* $g_{i,coupled,0}(x,t) = g_{i,uncoupled}(x,t)$. *For the integrals we have to apply quadrature rules, which have to be of order* $O(t^j)$, *where* $j = 1, 2, 3, \ldots$ *are the iterative steps. We finish our algorithm after* $2 - 5$ *iteration steps, or after a sufficient small error :* $\max_{i=1}^m (|u_{i,coupled,j+1}(x,t) - u_{i,coupled,j}(x,t)|, |g_{i,coupled,j+1}(x,t) - g_{i,coupled,j}(x,t)|) \leq err$.

Example 4. For a special function $f(t) = t$, we obtain the integration :

$\int_0^t f(s)ds = t^2/2$,
which is used in the analytical formulas in Algorithm 31.

Remark 5. Here we present an algorithm based on the Jacobian formulation by using step $j-1$-iterative solution of u_i in the equation 87. We can improve the algorithm to a Gauss-Seidel formulation, by using j iterative solution of u_i.

Remark 6. For recursively iterations one can apply quadrature rules as Gauss-Legendre or Romberg schemes. Such schemes can be programmed more efficient.

4 Numerical experiments

4.1 First experiment: Decay of ten species (time-dependent reaction)

We use ascending parameters for the retardation factors. The retardation factors are given as $R_1 = 10, R_2 = 9, R_3 = 8, R_4 = 7, R_5 = 6 \ldots R_{10} = 1$. The reaction factors are given as $\lambda_1 = 2.0, \lambda_2 = 1.8, \lambda_3 = 1.6, \lambda_4 = 1.4, \ldots, \lambda_{10} = 0.0$. Further $f_i(t) = t$, $i = 1, \ldots, m$.

We apply equation (22) and (23).

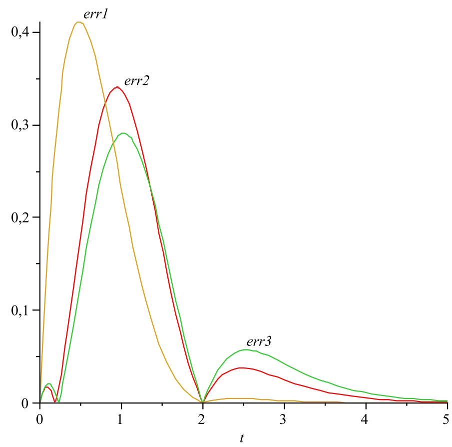

Fig. 1. Experiment with ten descending species with error between the quadratic and linear solutions (experiment for $t \in [0, 5]$).

Remark 7. The analytical solution show that the quadratic time-dependency is important at the beginning of the initialization process. After $t \geq 5$, we have nearly the same results as for the linear case.

4.2 Second experiment: descending retardation factor with ten species (Transport and Decay)

We use ascending parameters for the retardation factors. The retardation factors are given as $R_1 = 10, R_2 = 9, R_3 = 8, R_4 = 7, R_5 = 6 \ldots R_{10} = 1$. The reaction factors are given as $\lambda_1 = 2.0, \lambda_2 = 1.8, \lambda_3 = 1.6, \lambda_4 = 1.4, \ldots, \lambda_{10} = 0.0$. Further $f_i(t) = t$, $i = 1, \ldots, m$.

With the assumptions:

- $\lambda_j \neq \lambda_k$

– $v_j \neq v_k$

– $\lambda_{jk} \neq \lambda_{jl}$

The initial conditions are given as

$$u_1(x,0) = \begin{cases} ax+b \ , \ x \in (0,1) \\ 0 \qquad\quad , \text{otherwise} \end{cases} ,$$
$$u_i(x,0) = 0 \ , \quad i = 2, \ldots, m \ , \tag{88}$$

where $a = 1$ and $b = 1$.

The velocity is given as $v = 1$, where each species velocity is given as $v_i = \frac{v}{R_i}, \ i = 1, \ldots, 10$.

The end time is given as $t \in [0,T], \ T = 10$ and the spacial domain $x \in [0,10]$.

Here we apply the reduced equation (56) and (57).

The idea is to select the dominated decay chains and to apply them in the scheme.

The linear cases is given in Figure 2.

The quadratic cases is given in Figure 3.

We have the following color series: $u(1)$=red, $u(2)$=blue, $u(3)$=green, $u(4)$=yellow, $u(5)$=black, $u(6)$=cyan, $u(7)$=violet, $u(8)$=coral, $u(9)$=brown, $u(10)$=orange.

The error between the linear and quadratic cases is given in Figure 4.

Remark 8. Here we can see the influence of at least 10 species to our analytical equations. We can concentrate on the dominant species and save so computational time without loosing accuracy. Such analytical results helps to generalize analytical solutions with efficient computations.

4.3 3th experiment: descending retardation factor with general initial conditions

We use ascending parameters for the retardation factors. The retardation factors are given as $R_1 = 16, R_2 = 8, R_3 = 4, R_4 = 2, R_5 = 1$. The reaction factors are given as $\lambda_1 = 0.4, \lambda_2 = 0.3, \lambda_3 = 0.2, \lambda_4 = 0.1, \lambda_5 = 0.0$.

The initial conditions are given with the parameters :

1.) Initial condition 1 : $x_1 = 0.0, x_2 = 1.0$

$b_{1,1} = 1.0, c_{1,1} = 1.0$
$b_{2,1} = -1.0, c_{2,1} = 1.0$
$b_{3,1} = 1.0, c_{3,1} = 0.0$
$b_{4,1} = 2.0, c_{4,1} = 1.0$
$b_{5,1} = -1.0, c_{5,1} = 2.0$

2.) Initial condition 2 : $x_2 = 1.0, x_3 = 2.0$

$b_{1,2} = 3.0, c_{1,2} = 1.0$
$b_{2,2} = -1.0, c_{2,2} = 2.0$
$b_{3,2} = 2.0, c_{3,2} = 0.0$
$b_{4,2} = 1.0, c_{4,2} = 1.0$

$b_{5,2} = -1.0, c_{5,2} = 1.0$

3.) Initial condition 3 : $x_3 = 2.0, x_4 = 3.0$
$b_{1,3} = 0.0, c_{1,3} = 1.0$
$b_{2,3} = -1.0, c_{2,3} = 2.0$
$b_{3,3} = 1.0, c_{3,3} = 1.0$
$b_{4,3} = -1.0, c_{4,3} = 1.0$
$b_{5,3} = 1.0, c_{5,3} = 2.0$

For the time-dependent function $f(t)$, we apply:

$$f(t) = t \rightarrow \int_0^t f(s) ds = t^2/2 \quad (89)$$

$$f(t) = t^2 \rightarrow \int_0^t f(s) ds = t^3/3 \quad (90)$$

Here we use our equation (76)-(78) and $q = 3$.

The idea is to select the dominated decay chains and to apply them in the scheme, see the results for the linear case is given in Figure 5. Here we have the following color bars: $u(1)$=red, $u(2)$=yellow ocher, $u(3)$=red.

The results for the linear case is given in Figure 6.

The results for the linear case is given in Figure 7

The errors between the linear and quadratic case in Figure 8.

The errors between the linear and cubic case in Figure 9.

The errors between the quadratic and cubic case in Figure 10.

We have the following color series: $err(1)$=red, $err(2)$=yellow ocher, $err(3)$=red.

Remark 9. The general initial conditions and the coupling to all successive species is done in the last experiment. Here we can see the influence of general initial conditions for all species to their successors. The time-dependent cases are different in the initialization process, so for $t \geq 6$ we have a lower influence. By the way the important resolution for the mixed initialization is also important and influenced the time-dependent case.

Here it is important to deal with the time-dependent quadratic or cubic case to skip the error, which will occur in the linear case.

5 Conclusions

We derived analytical solutions of time-dependent convection dominant equations with general initial conditions. The analytical solutions are given for general time-dependent functions and we analyze a closed formulation to a special time-dependent case.

Such test functions can be embedded to discretization methods for the convection diffusion reaction equation.

We couple mobile and immobile problems with an iterative method and apply to each problem part the analytical solutions.

For complex computations of large systems of time-dependent convection-reaction problems, we can use such methods in the initialization process for a more detailed computation.

In future the decomposition methods and analytically-improved methods can be generalized for non-linear problems in time and space.

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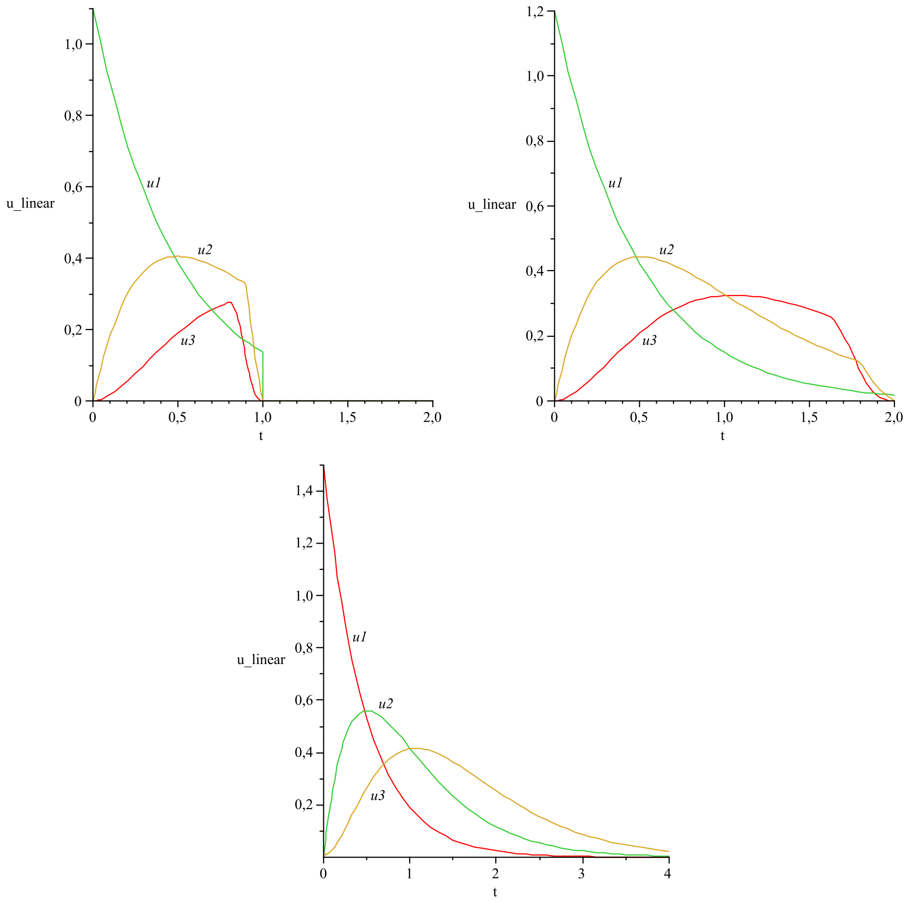

Fig. 2. Experiment with ten descending parameters for the linear time-dependent case (Figure left: $x = 0.1, t \in [0, 1]$, Figure right: $x = 0.2, t \in [0, 2]$), lower Figure: $x = 0.5, t \in [0, 4]$.

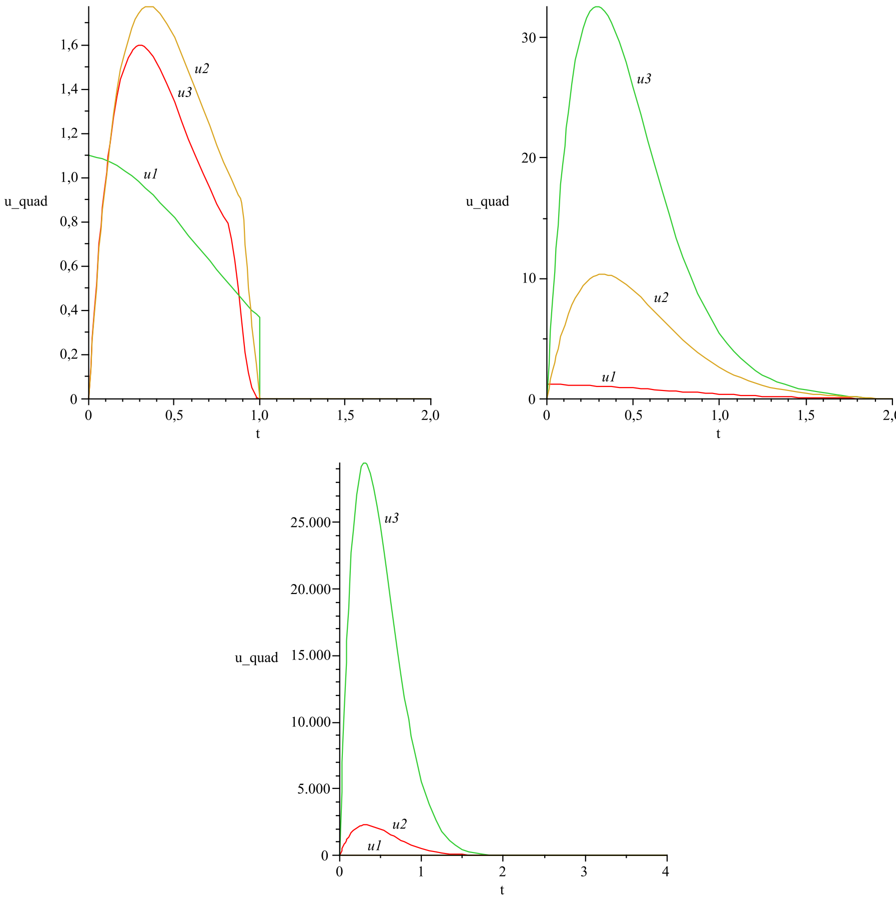

Fig. 3. Experiment with ten descending parameters for the quadratic time-dependent case(Figure left: $x = 0.1, t \in [0, 1]$, Figure right: $x = 0.2, t \in [0, 2]$), lower Figure: $x = 0.5, t \in [0, 4]$.

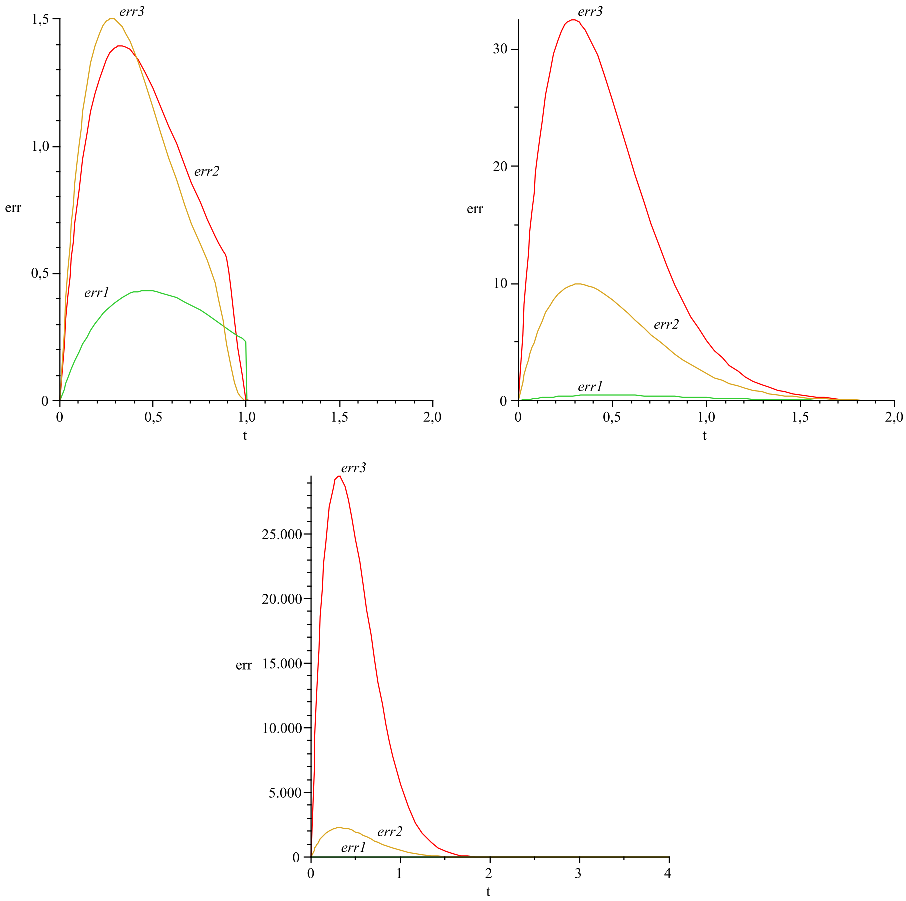

Fig. 4. Experiment with ten descending parameters and the error between the linear and the quadratic time-dependent case(Figure left: $x = 0.1, t \in [0,1]$, Figure right: $x = 0.2, t \in [0,2]$), lower Figure: $x = 0.5, t \in [0,4]$.

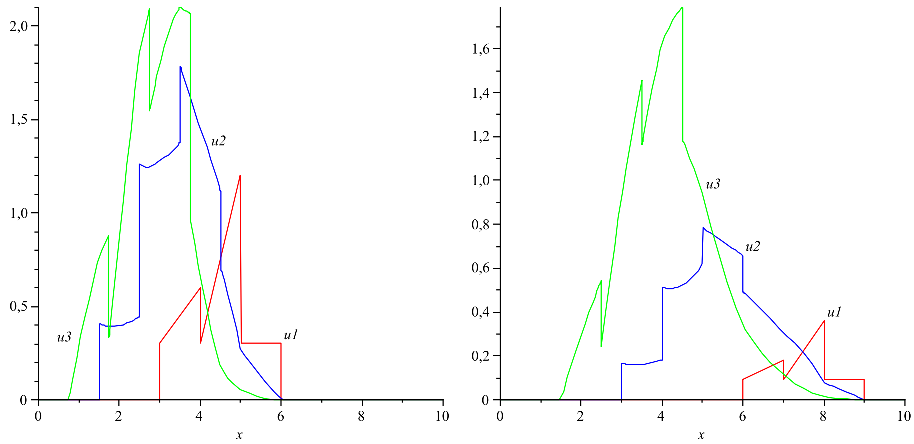

Fig. 5. Experiment with general initial conditions and linear time-dependent case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).

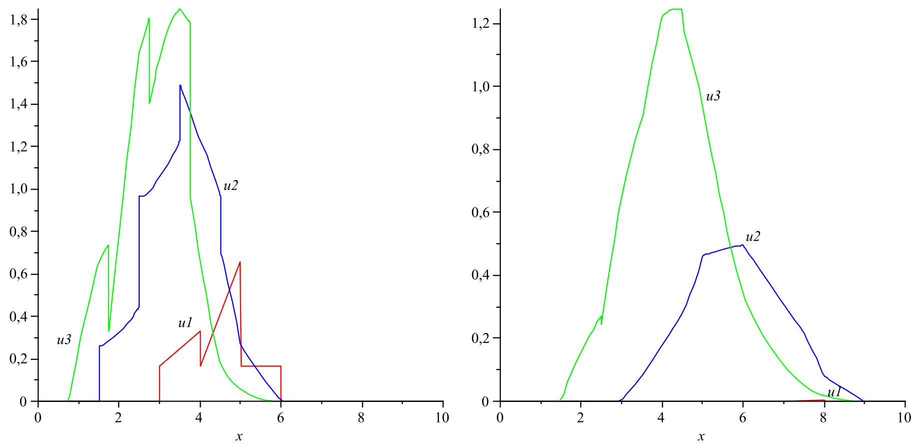

Fig. 6. Experiment with general initial conditions and quadratic time-dependent case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).

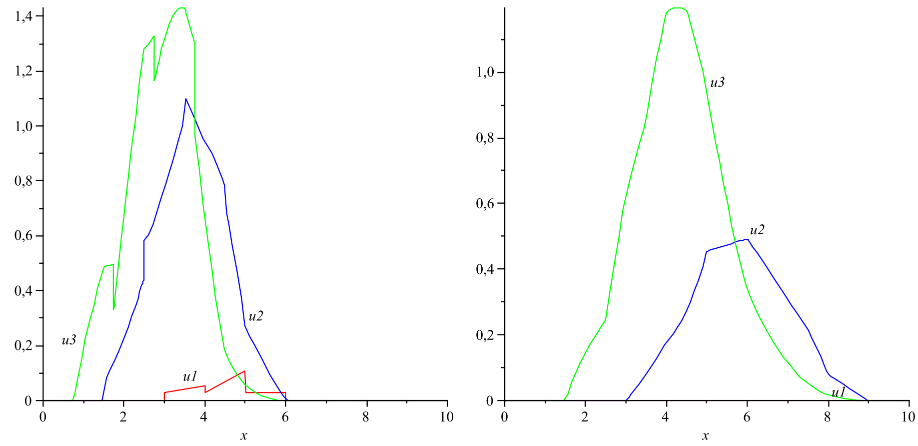

Fig. 7. Experiment with general initial conditions and cubic time-dependent case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).

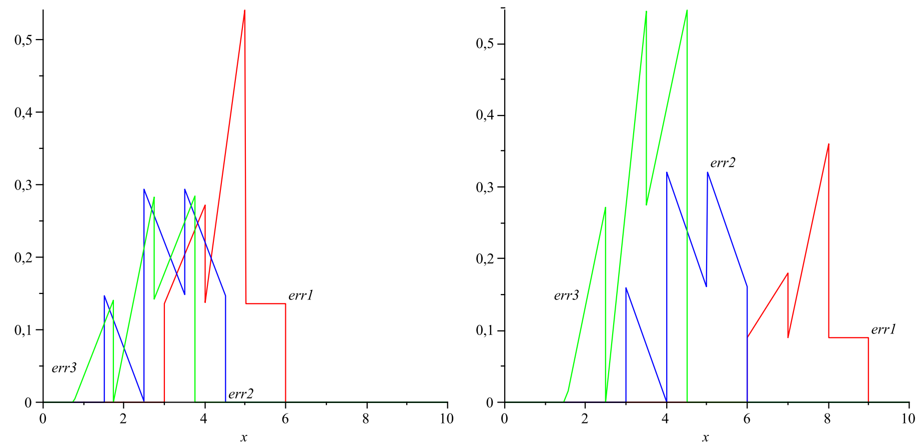

Fig. 8. Experiment with general initial conditions and error between the linear and quadratic case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).

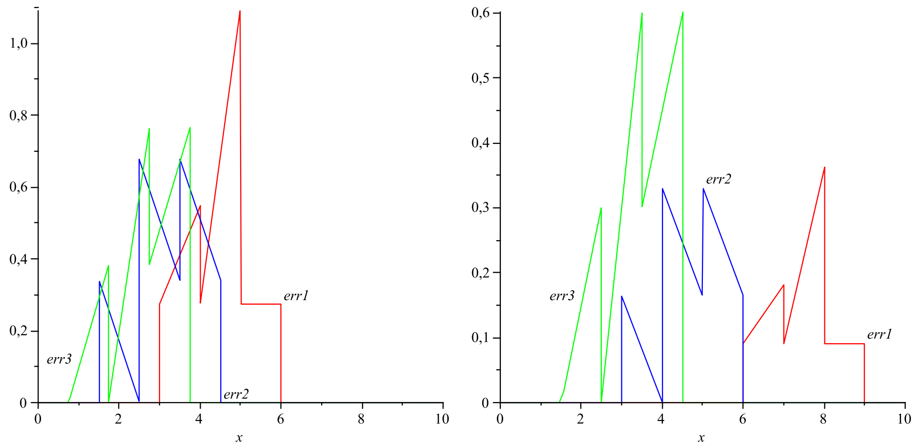

Fig. 9. Experiment with general initial conditions and error between the linear and cubic case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).

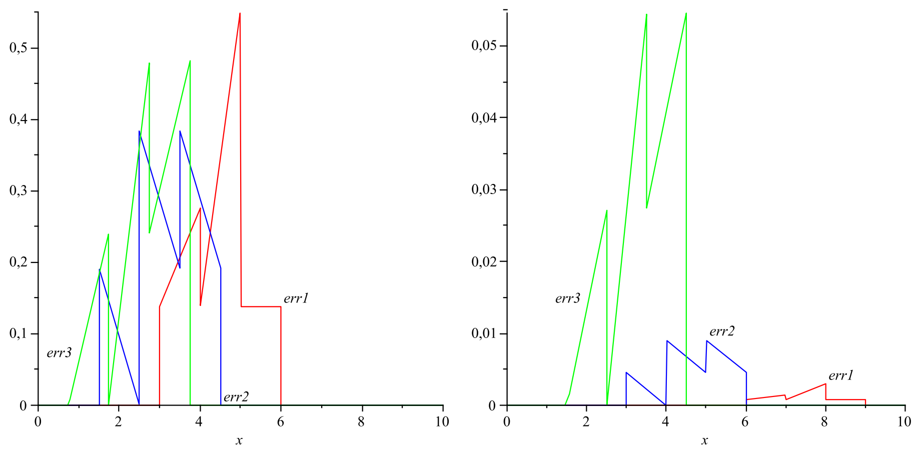

Fig. 10. Experiment with general initial conditions and error between the quadratic and cubic case (left figure $x \in [0, 10], t = 3.0$, right figure $x \in [0, 10], t = 6.0$).