Projector based treatment of linear constant coefficient DAEs

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Abstract

Linear DAEs with constant coefficients have been well understood by way of the theory of matrix pencils for quite a long time, and this is the reason why they are only briefly discussed in monographs. We want to consider them in detail here, not because we believe that the related linear algebra has to be invented anew, but as we intend to give a sort of guide for the extensive discussion on linear DAEs with time-varying coefficients and on nonlinear DAEs.

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Linear constant coefficient DAEs

Linear DAEs with constant coefficients have been well understood by way of the theory of matrix pencils for quite a long time, and this is the reason why they are only briefly discussed in monographs. We want to consider them in detail here, not because we believe that the related linear algebra has to be invented anew, but as we intend to give a sort of guide for the extensive discussion on linear DAEs with time-varying coefficients and on nonlinear DAEs. Later on, in particular, when investigating time-dependent linear DAEs, we will repeat many arguments given here by proceeding pointwise.
This paper is organized as follows. Section 1 records well known facts on regular matrix pairs and describes the structure of the related DAEs. The other sections serve as an introduction to the projector based analysis. Section 2 and 3 provide the basic tool of this analysis, the sequence of matrices G_i and the accompanying admissible projectors and characteristic values. Section 4 provides a new view of regular Kronecker index μ matrix pairs. They yield singular matrices $G_{\mu-1}$ but nonsingular G_μ. Conversely, in Section 5, we show that any matrix pair corresponding to a singular $G_{\mu-1}$ and a nonsingular G_μ must be a regular pair with Kronecker index μ. Applying the matrix sequence one can determine the complete structure of a regular matrix pair as well as its finite spectrum (Section 6). Section 7 touches some questions concerning singular matrix pairs and the related DAEs.
Let us emphasize that, for constant coefficient linear DAEs, we are given a famous tool for understanding the DAE structure by the Weierstraß-Kronecker canonical form. The DAE inherits regularity and index from the matrix pair. However, for time-varying linear DAEs and for general nonlinear DAEs there are no such tools, but the characterization by means of a corresponding matrix (function) sequence works well. In particular, a regularity notion is primarily bound to nonsingular G_μ.

1 Regular matrix pairs and the Weierstraß-Kronecker canonical form

In this section we deal with the equation

$$Ex'(t) + Fx(t) = q(t), \ t \in \mathcal{I}, \tag{1}$$

formed by the ordered pair $\{E, F\}$ of real valued $m \times m$ matrices E, F. For given functions $q : \mathcal{I} \to \mathbb{R}^m$ being at least continuous on the interval $\mathcal{I} \subseteq \mathbb{R}$, we are looking for continuous solutions $x : \mathcal{I} \to \mathbb{R}^m$ having a continuously differentiable component Ex. We use the notation $Ex'(t)$ for $(Ex)'(t)$. Special interest is directed to homogeneous equations

$$Ex'(t) + Fx(t) = 0, \quad t \in \mathbb{R}. \tag{2}$$

For $E = I$, the special case of explicit ODEs is covered. Now, in the more general setting, the ansatz $x_*(t) = e^{\lambda_* t} z_*$ well-known for explicit ODEs, yields

$$Ex_*'(t) + Fx_*(t) = e^{\lambda_* t}(\lambda_* E + F)z_*,$$

hence, x_* is a nontrivial particular solution of the DAE (2) if λ_* is a zero of the polynomial $p(\lambda) := \det(\lambda E + F)$, $\lambda \in \mathbb{C}$, and $z_* \neq 0$ satisfies $(\lambda_* E + F)z_* = 0$. λ_* and z_* are called

generalized eigenvalue and eigenvector, respectively.
This shows the meaning of the polynomial $p(\lambda)$ and the related family of matrices $\lambda E+F$ named *matrix pencil* formed by $\{E,F\}$.

Example 1.1 *The matrices*

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\quad \text{and } F=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

lead to the DAE system

$$\begin{aligned} x_1'-x_1 &= 0,\\ x_2'+x_3 &= 0,\\ x_2 &= 0. \end{aligned}$$

The polynomial $p(\lambda)$ is given by

$$p(\lambda)=\det(\lambda E+F)=\det\begin{bmatrix}\lambda-1&0&0\\0&\lambda&1\\0&1&0\end{bmatrix}=1-\lambda$$

implying $\lambda_*=1$ and $z_*=(1\,0\,0)^T$ to be a generalized eigenvalue and eigenvector. Obviously, $x_*(t)=e^{\lambda_* t}z_*=(e^t\,0\,0)^T$ is a non-trivial solution of the differential-algebraic equation.

If E is nonsingular, the homogeneous equation (2) represents an implicit regular ODE. Its fundamental solution system forms an m-dimensional subspace in C^1. What happens if E is singular? Is there a class of equations, i.e., pairs $\{E,F\}$, such that equation (2) has a finite-dimensional solution space? The answer is closely related to the notion of regular pairs.

Definition 1.2 *The ordered pair $\{E,F\}$, and also the matrix pencil formed by $\{E,F\}$, are called regular if the polynomial $p(\lambda):=\det(\lambda E+F)$, $\lambda\in\mathbb{C}$, does not vanish identically. Otherwise $\{E,F\}$ is said to be singular.*

A pair $\{E,F\}$ with nonsingular E is always regular, and its polynomial p is of degree m. In case of singular matrices E, the polynomial degree is lower.

Proposition 1.3 *For any regular pair $\{E,F\}$ with singular E there exist nonsingular real valued $m\times m$ matrices L and K, and integers $1\le l\le m$, $\mu\le l$, such that*

$$LEK=\begin{bmatrix}I&\\&N\end{bmatrix}\begin{matrix}\}m-l\\ \}l\end{matrix}\quad,\quad LFK=\begin{bmatrix}W&\\&I\end{bmatrix}\begin{matrix}\}m-l\\ \}l\end{matrix}\quad,\tag{3}$$

where N is nilpotent of order μ, i.e., $N^\mu=0$, $N^{\mu-1}\neq 0$. The integers l and μ as well as the eigenstructure of the blocks N and W are uniquely determined by the pair $\{E,F\}$.

Proof: Since $\{E, F\}$ is a regular pair, there is a number $c \in \mathbb{R}$ such that $cE + F$ is nonsingular. Put $\tilde{E} := (cE + F)^{-1}E$, $\tilde{F} := (cE + F)^{-1}F = I - c\tilde{E}$, $\mu = \operatorname{ind}\tilde{E}$, $r = \operatorname{rank}\tilde{E}^{\mu}$, $S = [s_1 ... s_m]$, where $s_1, \ldots, s_r$ and $s_{r+1}, \ldots, s_m$ are basises of $\operatorname{im}\tilde{E}^{\mu}$ and $\ker\tilde{E}^{\mu}$, respectively. Lemma A.10 provides the special structure of the product $S^{-1}\tilde{E}S$, namely,

$$S^{-1}\tilde{E}S = \begin{bmatrix} \tilde{M} & 0 \\ 0 & \tilde{N} \end{bmatrix},$$

with a nonsingular $r \times r$ block $\tilde{M}$ and a nilpotent $(m - r) \times (m - r)$ block $\tilde{N}$. $\tilde{N}$ has nilpotency index μ. Compute

$$S^{-1}\tilde{F}S = I - cS^{-1}\tilde{E}S = \begin{bmatrix} I - c\tilde{M} & 0 \\ 0 & I - c\tilde{N} \end{bmatrix}.$$

The block $I - c\tilde{N}$ is nonsingular due to the nilpotency of $\tilde{N}$. Denote

$$L := \begin{bmatrix} \tilde{M}^{-1} & 0 \\ 0 & (I - c\tilde{N})^{-1} \end{bmatrix} S^{-1}(cE+F)^{-1}, \quad K := S, \quad N := (I - c\tilde{N})^{-1}\tilde{N}, \quad W := \tilde{M}^{-1} - cI,$$

so that we arrive at the representation

$$LEK = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad LFK = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}.$$

Since $\tilde{N}$ and $(I - c\tilde{N})^{-1}$ commute, it holds that $N^l = ((I - c\tilde{N})^{-1}\tilde{N})^l = ((I - c\tilde{N})^{-1})^l\tilde{N}^l$, and N inherits the nilpotency of $\tilde{N}$, hence $N^{\mu} = 0$, $N^{\mu-1} \neq 0$. Put $l := m - r$. It remains to verify that the integers l and μ as well as the eigenstructure of N and W are independent of the transformations L and K. Assume that there is a further collection $\tilde{l}, \tilde{\mu}, \tilde{L}, \tilde{K}, \tilde{r} = m - \tilde{l}$ such that

$$\tilde{L}E\tilde{K} = \begin{bmatrix} I_{\tilde{r}} & 0 \\ 0 & \tilde{N} \end{bmatrix}, \quad \tilde{L}F\tilde{K} = \begin{bmatrix} \tilde{W} & 0 \\ 0 & I_{\tilde{l}} \end{bmatrix}.$$

Considering the polynomial

$$\begin{aligned} p(\lambda) = \det(\lambda E + F) &= \det(L^{-1})\det(\lambda I_r + W)\det(K^{-1}) \\ &= \det(\tilde{L}^{-1})\det(\lambda I_{\tilde{r}} + \tilde{W})\det(\tilde{K}^{-1}) \end{aligned}$$

we realize that the values r and $\tilde{r}$ must coincide, hence $l = \tilde{l}$. Derive further, with $U := \tilde{L}L^{-1}$, $V := \tilde{K}^{-1}K$,

$$U\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = \tilde{L}EK = \begin{bmatrix} I & 0 \\ 0 & \tilde{N} \end{bmatrix}V, \quad U\begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} = \tilde{L}FK = \begin{bmatrix} \tilde{W} & 0 \\ 0 & I \end{bmatrix}V,$$

that is in detail

$$\begin{bmatrix} U_{11} & U_{12}N \\ U_{21} & U_{22}N \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ \tilde{N}V_{21} & \tilde{N}V_{22} \end{bmatrix}, \quad \begin{bmatrix} U_{11}W & U_{12} \\ U_{21}W & U_{22} \end{bmatrix} = \begin{bmatrix} \tilde{W}V_{11} & \tilde{W}V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

Comparing the entries of these matrices we find the relations $U_{12}N = V_{12}$ and $U_{12} = \tilde{W}V_{12}$, which lead to $U_{12} = \tilde{W}U_{12}N = ... = \tilde{W}^{\mu}U_{12}N^{\mu} = 0$. Analogously we derive $U_{21} = 0$. Then, the blocks $U_{11} = V_{11}$, $U_{22} = V_{22}$ must be nonsingular. It results that

$$V_{11}W = \tilde{W}V_{11}, \quad V_{22}N = \tilde{N}V_{22}$$

holds true, that is, the matrices N and $\tilde{N}$ as well as W and $\tilde{W}$ are similar, and in particular, $\mu = \tilde{\mu}$ is valid. □

The real valued matrix N has the eigenvalue zero only, and can be transformed into its Jordan canonical form by means of a real valued similarity transformation. Therefore, in Proposition 1.3, the transformation matrices L and K can be chosen such that N is in Jordan canonical form.
Proposition 1.3, as well as the given proof also hold true for complex valued matrices. It is a well known result of Weierstraß and Kronecker (cf. [Gan70]). The pair given in (3) is called *Weierstraß-Kronecker canonical form* of the pair $\{E, F\}$.

Definition 1.4 *The Kronecker index μ of a regular pair $\{E, F\}$ with singular E is defined to be the nilpotency order μ in the Weierstraß-Kronecker canonical form (3). If E is nonsingular, put $\mu = 0$. We write* $\operatorname{ind}\{E, F\} = \mu$.

Via the Weierstraß-Kronecker canonical form of a regular pair $\{E, F\}$, the structure of the corresponding DAE (1), (2) is easily discovered. Scaling of (1) by L and transforming $x = K \begin{bmatrix} y \\ z \end{bmatrix}$ leads to the equivalent decoupled system

$$\begin{aligned} y'(t) + Wy(t) &= p(t), \quad &(4)\\ Nz'(t) + z(t) &= r(t), \quad t \in \mathcal{I}, \quad &(5) \end{aligned}$$

with $Lq =: \begin{bmatrix} p \\ r \end{bmatrix}$. The first equation (4) represents a standard explicit ODE. The second one has the only solution

$$z(t) = \sum_{j=0}^{\mu-1} (-1)^j N^j r^{(j)}(t), \tag{6}$$

provided that r is smooth enough. This becomes clear after recursive use of (5) since

$$z = r - Nz' = r - N(r - Nz')' = r - Nr' + N^2 z'' = r - Nr' + N^2(r - Nz')'' = ...$$

Expression (6) shows the dependence of the solution x on derivatives of the source or perturbation term q. The higher the index μ, the more differentiations are involved. Only in the index-one case we have $N = 0$, hence $z(t) = r(t)$, and no derivatives are involved. Since numerical differentiations in these circumstances may cause considerably trouble, it is very important to know the index μ as well as details on the structure responsible for a higher index when modeling and simulating with DAEs in practice.
The general solution of the homogeneous DAE (2), if the pair $\{E, F\}$ is regular, is of the form

$$x(t) = K \begin{bmatrix} e^{-tW} \\ 0 \end{bmatrix} y_0, \quad y_0 \in \mathbb{R}^{m-l},$$

that means, the solution space has dimension $m - l$.

Theorem 1.5 *The homogeneous DAE (2) has a finite-dimensional solution space if and only if the pair $\{E, F\}$ is regular.*

Proof: As we have seen before, if the pair $\{E, F\}$ is regular, then the solutions of (2) form an $(m - l)$-dimensional space.
Conversely, let $\{E, F\}$ be a singular pair, i.e., $\det(\lambda E + F) \equiv 0$. For any set of $m + 1$ different real values $\lambda_1, \dots, \lambda_{m+1}$ we find nontrivial vectors $\eta_1, \dots, \eta_{m+1} \in \mathbb{R}^m$ such that $(\lambda_i E + F)\eta_i = 0,\ i = 1, \dots, m + 1$, and a nontrivial linear combination $\sum_{i=1}^{m+1} \alpha_i \eta_i = 0$. The function $x(t) = \sum_{i=1}^{m+1} \alpha_i e^{\lambda_i t} \eta_i$ does not vanish identically, and it satisfies the DAE (2) as well as the initial condition $x(0) = 0$. For disjoint $(m + 1)$-element sets there always arise different solutions, and, consequently, there are more than countably many different solutions of a homogeneous IVP of (2). □

Example 1.6 *(cf. [GM89]) The pair $\{E, F\}$, with $m = 4$,*

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is singular. In detail, equation (1) reads

$$\begin{aligned} (x_1 + x_2)' \ + \ x_2 &= q_1, \\ x_4' \qquad\quad &= q_2, \\ x_3 &= q_3, \\ x_3' \qquad\quad &= q_4. \end{aligned}$$

What does the solution space of the corresponding homogeneous DAE (2) look like? Obviously, the component x_3 vanishes identically and x_4 is an arbitrary constant function. The remaining equation $(x_1 + x_2)' + x_2 = 0$ is satisfied by any arbitrary continuous x_2, and the resulting expression for x_1 is:

$$x_1(t) = c - x_2(t) - \int_0^t x_2(s) ds,$$

c being a further arbitrary constant. Clearly, this solution space does not have finite dimension, which confirms the assertion of Theorem 1.5. Indeed, the regularity assumption is violated since

$$p(\lambda) = \det(\lambda E + F) = \det \begin{bmatrix} \lambda & \lambda + 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix} = 0.$$

Notice that, in case of nontrivial perturbations q, the consistency condition $q_3' = q_4$ must be valid for solvability. In practice, such unbalanced models should be avoided. However, in large dimensions m, this might not be a trivial task.

Definition 1.7 *A DAE (1) with the constant coefficient pair $\{E, F\}$ is said to be* <u>*regular*</u> *or* <u>*regular with Kronecker index*</u> $\mu = \operatorname{ind}\{E, F\}$ *if this pair $\{E, F\}$ is regular with Kronecker index μ.*

Let us take a closer look at the subsystem (5), which is specified by the nilpotent matrix N. We may choose the transformation matrices L and K in such a way that N has Jordan canonical form, say

$$N = \operatorname{diag}[J_1, \dots, J_s], \tag{7}$$

with s nilpotent Jordan blocks

$$J_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in L(\mathbb{R}^{k_i}),\ i = 1, \dots, s,$$

where $k_1 + \cdots + k_s = l, \quad \mu = \max\{k_i : i = 1, \dots, s\}$. The Kronecker index μ is the maximal order of a Jordan block of N.
The Jordan canonical form (7) shows the further decoupling of the subsystem (5) in accordance with the Jordan structure into s lower-dimensional equations

$$J_i \zeta_i'(t) + \zeta_i(t) = r_i(t),\ \ i = 1, \dots, s.$$

Now we observe that $\zeta_{i,2}$,...,ζ_{i,k_i} are components involved with derivatives whereas the derivative of the first component $\zeta_{i,1}$ is not involved. Notice that the value of $\zeta_{i,1}(t)$ depends on the $(k_i - 1)$-th derivative of $r_{i,k_i}(t)$ for all $i = 1, ..., s$ since

$$\zeta_{i,1}(t) = r_{i,1}(t) - \zeta_{i,2}'(t) = r_{i,1}(t) - r_{i,2}'(t) + \zeta_{i,3}'(t) = ... = \sum_{j=1}^{k_i} (-1)^{j-1} r_{i,j}^{(j-1)}(t).$$

Example 1.8 *Choosing $m = 5$ and the nilpotent matrix*

$$N = \begin{bmatrix} 0 & & & & \\ & 0 & 1 & & \\ & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix}$$

we have

$$s = 3, \quad J_1 = [0], \quad J_2 = J_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the nilpotency index $\mu = 2$. The detailed system (5) reads as

$$\begin{array}{rcl} z_1 & = & r_1, \\ z_3' + z_2 & = & r_2, \\ z_3 & = & r_3, \\ z_5' + z_4 & = & r_4, \\ z_5 & = & r_5. \end{array} \tag{8}$$

Its solution is given by

$$\begin{aligned} z_1 &= r_1, \\ z_2 &= r_2 - r_3', \\ z_3 &= r_3, \\ z_4 &= r_4 - r_5', \\ z_5 &= r_5. \end{aligned}$$

Here $r = 0$ implies $z = 0$, i.e., the homogeneous equation has the trivial solution only.

2 Basic sequences of matrices and admissible projectors

Our aim is now a suitable rearrangement of terms within the equation

$$Ex'(t) + Fx(t) = q(t), \tag{9}$$

which allows for a similar insight into the structure of the DAE to that given by the Weierstraß-Kronecker canonical form. However, we do not use transformations but apply a projector based decoupling concept, and we work in terms of the original equation setting.

The basic construction is very simple. Put $G_0 := E$, $B_0 := F$, $N_0 := \ker G_0$. Let $Q_0 \in L(\mathbb{R}^m)$ be a projector onto N_0, and $P_0 := I - Q_0$ the complementary one. Using the projector properties (see Appendix 8) $Q_0^2 = Q_0$, $Q_0P_0 = 0$, $P_0 + Q_0 = I$ and $G_0 = G_0P_0$ we may rewrite (9) as

$$\begin{aligned} G_0x' + B_0x &= q, \\ G_0P_0x' + B_0(Q_0 + P_0)x &= q, \\ (\underbrace{G_0 + B_0Q_0}_{=:G_1})(P_0x' + Q_0x) + \underbrace{B_0P_0}_{=:B_1}x &= q, \\ G_1(P_0x' + Q_0x) + B_1x &= q. \end{aligned}$$

Next, let Q_1 be a projector onto $N_1 := \ker G_1$ and $P_1 := I - Q_1$ the complementary one. We rearrange the last equation to

$$\begin{aligned} G_1P_1(P_0x' + Q_0x) + B_1(Q_1 + P_1)x &= q, \\ \underbrace{(G_1 + B_1Q_1)}_{G_2}\ [P_1(P_0x' + Q_0x) + Q_1x] + \underbrace{B_1P_1}_{B_2}x &= q, \end{aligned} \tag{10}$$

and so on. The goal is a matrix G_κ with maximal possible rank m in front of the term containing the derivative x'. This will allow us to multiply the last equation obtained this way by G_κ^{-1}. Multiplying further by suitable projectors we find a decoupled DAE system that can be solved as simply as a DAE in Weierstraß-Kronecker canonical form.

The projector based decoupling procedure to be described now is not at all restricted to pairs of square matrices. Although our interest mainly concerns square DAEs, to be able to consider several aspects of over- and underdetermined DAEs we construct the basic sequences of matrices and accompanying projectors for ordered pairs $\{E, F\}$ of general rectangular matrices $E, F \in L(\mathbb{R}^m, \mathbb{R}^k)$.

Start with $G_0 := E,\ B_0 := F,\ N_0 := \ker G_0$, and let $Q_0 \in L(\mathbb{R}^m)$ denote a projector onto N_0, and $P_0 := I - Q_0$ the complementary one.
Then, for $i \geq 0$, put

$$\begin{aligned} G_{i+1} &:= G_i + B_i Q_i, \quad N_{i+1} := \ker G_{i+1}, \\ B_{i+1} &:= B_i P_i, \end{aligned} \tag{11}$$

and introduce $Q_{i+1} \in L(\mathbb{R}^m)$ being a projector onto N_{i+1}, $P_{i+1} := I - Q_{i+1}$. Denote $r_i := \operatorname{rank} G_i$ and introduce the product of projectors $\Pi_i := P_0 \cdots P_i$. These ranks and products of projectors will play a special role later on. From $B_{i+1} = B_i P_i = B_0 \Pi_i$ we derive the inclusion $\ker \Pi_i \subseteq \ker B_{i+1}$ as an inherent property of our construction.

Let us stress again that we are aiming at a matrix G_κ the rank of which is as high as possible. However, how can one know whether the maximal rank has been reached? Appropriate criteria would be very helpful. In important cases, in particular for regular DAEs, one meets full rank matrices G_κ, that is, $r_\kappa = \min\{m, k\}$.

In general, since $G_i = G_{i+1} P_i$, the images of the G_i satisfy the inclusion relations

$$\operatorname{im} G_0 \subseteq \operatorname{im} G_1 \subseteq \cdots \subseteq \operatorname{im} G_i \subseteq \operatorname{im} G_{i+1},$$

and hence

$$r_0 \leq r_1 \leq \cdots \leq r_i \leq r_{i+1}.$$

A further basic property of the sequence (11) is the inclusion

$$N_{i-1} \cap N_i \subseteq N_i \cap N_{i+1}, \quad i \geq 1. \tag{12}$$

Namely, if $G_{i-1} z = 0$ and $G_i z = 0$ are valid for a vector $z \in \mathbb{R}^m$, which corresponds to $P_{i-1} z = 0$ and $z = Q_i z$, then we can conclude that

$$G_{i+1} z = G_i z + B_i Q_i z = B_i z = B_{i-1} P_{i-1} z = 0.$$

From (12) we learn that a nontrivial intersection $N_{i_*-1} \cap N_{i_*}$ leads to matrices G_i being not injective for all $i > i_*$. Consequently, we will not find an injective matrix G_κ. As we will realize in Section 5 (see also Proposition 6.5), such a nontrivial intersection indicates immediately a singular pair $\{E, F\}$.

Example 2.1 *For the DAE*

$$\begin{array}{cccccc} x_1' & & + \ x_1 & +x_2 & +x_3 & = q_1, \\ & x_3' & & +x_2 & & = q_2, \\ & & x_1 & & +x_3 & = q_3, \end{array}$$

the first matrices of our sequence are

$$G_0 = E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ B_0 = F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

As a nullspace projector onto $\ker G_0$ *we choose*

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and obtain } G_1 = G_0 + B_0 Q_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = B_0 P_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since G_1 *is singular, we turn to the next level. We choose as a projector onto* $\ker G_1$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and arrive at } G_2 = G_1 + B_1 Q_1 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

The matrix G_2 *is nonsingular, hence the maximal rank is reached and we stop constructing the sequence. Looking at the polynomial* $p(\lambda) = \det(\lambda E + F) = 2\lambda$ *we know this DAE to be regular. Later on we will see that a nonsingular matrix* G_2 *is typical for regularity with Kronecker index two. Observe further that the nullspaces* N_0 *and* N_1 *intersect trivially, and that the projector* Q_1 *is chosen such that* $\Pi_0 Q_1 Q_0 = 0$ *is valid, or equivalently,* $N_0 \subseteq \ker \Pi_0 Q_1$.

Example 2.2 *Here we deal with the singular matrix pair from Example 1.6, that is with*

$$G_0 = E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_0 = F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing

$$Q_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{yields} \quad G_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix G_1 *is singular. We turn to the next level. We pick*

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{which implies} \quad G_2 = G_0.$$

We continue constructing

$$Q_{2j} = Q_0, \;\; G_{2j+1} = G_1, \;\; Q_{2j+1} = Q_1, \;\; G_{2j+2} = G_0, \;\; j \geq 1.$$

Here we have $r_i = 3$ *for all* $i \geq 0$. *The maximal rank three is met already by* G_0, *but there is no criterion in sight which would indicate this in time. Furthermore,* $N_i \cap N_{i+1} = \{0\}$ *holds true for all* $i \geq 0$, *which means that there is no step indicating a singular pencil via property 12. Notice that the product* $\Pi_0 Q_1 Q_0 = P_0 Q_1 Q_0$ *does not vanish as it does in the previous example.*

The rather bad experience with Example 2.2 leads us to the idea to refine the choice of the projectors by incorporating more information from the previous steps, in particular that from the previous nullspaces. So far, just the image spaces of the projectors Q_i are prescribed. We refine the construction by prescribing certain appropriate parts of their nullspaces, too. More precisely, we put parts of the previous nullspaces into the current one.

In general, when constructing the sequence (11), we proceed as follows. At any level we decompose

$$N_0 + \cdots + N_{i-1} = \widehat{N_i} \oplus X_i, \quad \widehat{N_i} := (N_0 + \cdots + N_{i-1}) \cap N_i, \tag{13}$$

where X_i is any complement to $\widehat{N_i}$ in $N_0 + \cdots + N_{i-1}$. Then we choose Q_i in such a way that the condition

$$X_i \subseteq \ker Q_i \tag{14}$$

is met. This is always possible since the subspaces $\widehat{N_i}$ and X_i intersect trivially (see Appendix, Lemma A.6). It restricts to some extend the choice of the projectors. However, a great variety of possible projectors is left.

If the intersection $\widehat{N_i} = (N_0 + \cdots + N_{i-1}) \cap N_i$ is trivial, then we have

$$X_i = N_0 + \cdots + N_{i-1} \subseteq \ker Q_i.$$

This is the case in Example 2.1, and it is typical for regular DAEs.

Definition 2.3 *For $\kappa \in \mathbb{N}$, the projectors $Q_0, \ldots, Q_\kappa$ in the matrix sequence (11) are said to be* <u>*admissible*</u> *for $\{E, F\}$ if condition (14) is valid for $i = 1, \ldots, \kappa$. Q_0 is always admissible. Q_0, ..., Q_κ are called* <u>*regular admissible*</u> *if they are admissible with trivial intersections $\widehat{N_1}, \ldots, \widehat{N_\kappa}$.*

The projectors in Example 2.1 are admissible but the projectors in Example 2.2 are not. We revisit Example 2.2 and provide admissible projectors now.

Example 2.4 *Consider once again the singular pair from Examples 1.6 and 2.2. We start the sequence with the same matrices G_0, B_0, Q_0, G_1 as described in Example 2.2 but now we use an admissible projector Q_1. The nullspaces of G_0 and G_1 are given by*

$$N_0 = \operatorname{span} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \textit{and } N_1 = \operatorname{span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This allows us to choose

$$Q_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which satisfies the condition $X_1 \subseteq \ker Q_1$*, where* $X_1 = N_0$ *and* $\widehat{N}_1 = N_0 \cap N_1 = \{0\}$*. It yields*

$$G_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we find $N_2 = \operatorname{span} \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ *and with*

$$N_0 + N_1 = N_0 \oplus N_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right),$$

we have $N_2 \subseteq N_0 + N_1$*,* $N_0 + N_1 + N_2 = N_0 + N_1$ *as well as* $\widehat{N}_2 = (N_0 + N_1) \cap N_2 = N_2$*. A possible complement* X_2 *to* $\widehat{N}_2$ *in* $N_0 + N_1$ *and an appropriate projector* Q_2 *are*

$$X_2 = \operatorname{span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This leads to $G_3 = G_2$*, and the nontrivial intersection* $N_2 \cap N_3$ *indicates (cf.* (12)*) that all further matrices* G_i *are singular. Proposition 6.5 below says that this indicates at the same time a singular matrix pencil. In the next steps, for* $i \geq 3$*, it results that* $N_i = N_2$ *and* $G_i = G_2$*.*

We stress once more that for all our projectors Q_i their image is fixed to be N_i. For admissible projectors Q_i, also a part of $\ker Q_i$ is fixed. However, there remains a great variety of possible projectors, since the subspaces X_i are not uniquely determined and further represent just a part of $\ker Q_i$. Of course, we could restrict the variety of projectors by prescribing special subspaces. This might be useful with respect to computational aspects.

Definition 2.5 *The admissible projectors* $Q_0, \ldots, Q_\kappa$ *are called widely orthogonal ones if* $Q_0 = Q_0^*$*, and*

$$X_i = \widehat{N}_i^{\perp} \cap (N_0 + \cdots + N_{i-1}), \tag{15}$$

and

$$\ker Q_i = [N_0 + \cdots + N_i]^{\perp} \oplus X_i, \quad i = 1, \ldots, \kappa, \tag{16}$$

hold true.

The widely orthogonal projectors are completely fixed and they surely have some advantages.

The next assertions collect useful properties of admissible projectors and the corresponding matrix sequences (11) for a given pair $\{E, F\}$. In particular, the special role of the products $\Pi_i = P_0 \cdots P_i$ is revealed. We emphasize this by using mainly the short notation Π_i.

Proposition 2.6 *Let $Q_0, \dots, Q_\kappa$ be admissible for the pair $\{E, F\}$. Then the following holds true for $i = 1, \dots, \kappa$:*

(1) $\ker \Pi_i = N_0 + \cdots + N_i$,

(2) *The products $\Pi_i = P_0 \cdots P_i$ and $\Pi_{i-1}Q_i = P_0 \cdots P_{i-1}Q_i$, are projectors again.*

(3) $N_0 + \cdots + N_{i-1} \subseteq \ker \Pi_{i-1}Q_i$,

(4) $B_i = B_i \Pi_{i-1}$,

(5) $\widehat{N_i} \subseteq N_i \cap \ker B_i = N_i \cap N_{i+1} \subseteq \widehat{N}_{i+1}$.

(6) *If $Q_0, \dots, Q_\kappa$ are widely orthogonal, then* $\operatorname{im} \Pi_i = [N_0 + \cdots + N_i]^\perp$, $\Pi_i = \Pi_i^*$ *and* $\Pi_{i-1}Q_i = (\Pi_{i-1}Q_i)^*$.

(7) *If $Q_0, \dots, Q_\kappa$ are regular admissible, then* $\ker \Pi_{i-1}Q_i = \ker Q_i$ *and* $Q_iQ_j = 0$ *for* $j = 0, \dots, i-1$.

Proof: (1) ($\Rightarrow$) To show $\ker \Pi_i \subseteq N_0 + \cdots + N_i$ for $i = 1, \dots, \kappa$, we consider

$$0 = \Pi_i z = P_0 \cdots P_i z = \prod_{k=0}^{i} (I - Q_k) z.$$

Expanding the right hand expression, we obtain

$$z = \sum_{k=0}^{i} Q_k H_k z \ \in N_0 + \cdots + N_i$$

with suitable matrices H_k.
($\Leftarrow$) The other direction will be proven by induction. Starting the induction with $i = 0$, we observe that $\ker \Pi_0 = \ker P_0 = N_0$. We suppose that $\ker \Pi_{i-1} = N_0 + \cdots + N_{i-1}$ is valid. Because of

$$N_0 + \cdots + N_i = X_i + \widehat{N_i} + N_i$$

each $z \in N_0 + \cdots + N_i$ can be written as $z = x_i + \bar{z}_i + z_i$ with

$$x_i \in X_i \subseteq N_0 + \cdots + N_{i-1} = \ker \Pi_{i-1}, \quad \bar{z}_i \in \widehat{N_i} \subseteq N_i, \quad z_i \in N_i.$$

Since Q_i is admissible, we have $X_i \subseteq \ker Q_i$ and $N_i = \operatorname{im} Q_i$. Consequently,

$$\Pi_i z = \Pi_{i-1}(I - Q_i)z = \Pi_{i-1}(I - Q_i)x_i = \Pi_{i-1}x_i = 0$$

which implies $N_0 + \cdots + N_i \subseteq \ker \Pi_i$ to be true.

(2) From (1) we know that $\operatorname{im} Q_j = N_j \subseteq \ker \Pi_i$ for $j \le i$. It follows that

$$\Pi_i P_j = \Pi_i (I - Q_j) = \Pi_i.$$

Consequently, $\Pi_i^2 = \Pi_i$ and $\Pi_i \Pi_{i-1} = \Pi_i$ implying

$$(\Pi_{i-1}Q_i)^2 = \Pi_{i-1}(I - P_i)\Pi_{i-1}Q_i = \Pi_{i-1}Q_i - \Pi_i\Pi_{i-1}Q_i = \Pi_{i-1}Q_i - \Pi_iQ_i = \Pi_{i-1}Q_i.$$

(3) For any $z \in N_0 + \cdots N_{i-1}$, we know from (1) that $\Pi_{i-1}z = 0$ and $\Pi_i z = 0$. Thus

$$\Pi_{i-1}Q_i z = \Pi_{i-1}z - \Pi_i z = 0.$$

(4) By construction of B_i (see (11)), we find $B_i = B_0\Pi_{i-1}$. Using (2), we get that

$$B_i = B_0\Pi_{i-1} = B_0\Pi_{i-1}\Pi_{i-1} = B_i\Pi_{i-1}.$$

(5) First, we show that $\widehat{N_i} \subseteq N_i \cap \ker B_i$. For $z \in \widehat{N_i} = (N_0 + \cdots + N_{i-1}) \cap N_i$ we find $\Pi_{i-1}z = 0$ from (1) and, hence, $B_i z = B_0\Pi_{i-1}z = 0$ using (4). Next,

$$N_i \cap \ker B_i = N_i \cap N_{i+1}$$

since $G_{i+1}z = (G_i + B_iQ_i)z = B_i z$ for any $z \in N_i = \operatorname{im} Q_i = \ker G_i$. Finally,

$$\widehat{N}_{i+1} = (N_0 + \cdots + N_i) \cap N_{i+1} \text{ implies immediately that } N_i \cap N_{i+1} \subseteq \widehat{N}_{i+1}.$$

(6) We use induction for showing that $\operatorname{im} \Pi_i = [N_0 + \cdots + N_i]^\perp$. Starting with $i = 0$, we know that $\operatorname{im} \Pi_0 = N_0^\perp$ since $Q_0 = Q_0^*$.
Since $X_i \subseteq N_0 + \cdots + N_{i-1}$ (see (15)) we derive from (1) that $\Pi_{i-1}X_i = 0$. Regarding (16), we find

$$\operatorname{im} \Pi_i = \Pi_{i-1}\operatorname{im} P_i = \Pi_{i-1}([N_0 + \cdots + N_i]^\perp + X_i) = \Pi_{i-1}([N_0 + \cdots + N_i]^\perp).$$

Using $[N_0 + \cdots + N_i]^\perp \subseteq [N_0 + \cdots + N_{i-1}]^\perp = \operatorname{im} \Pi_{i-1}$ we conclude

$$\operatorname{im} \Pi_i = \Pi_{i-1}([N_0 + \cdots + N_i])^\perp = [N_0 + \cdots + N_i]^\perp.$$

In consequence, Π_i is the orthoprojector onto $[N_0 + \cdots + N_i]^\perp$ along $N_0 + \cdots + N_i$, i.e., $\Pi_i = \Pi_i^*$. It follows that

$$\Pi_{i-1}Q_i = \Pi_{i-1} - \Pi_i = \Pi_{i-1}^* - \Pi_i^* = (\Pi_{i-1} - \Pi_i)^* = (\Pi_{i-1}Q_i)^*.$$

(7) Let $\widehat{N_i} = 0$ be valid. Then, $X_i = N_0 + \cdots + N_{i-1} = N_0 \oplus \cdots \oplus N_{i-1}$ and, therefore,

$$\ker \Pi_{i-1} \overset{(1)}{=} N_0 \oplus \cdots \oplus N_{i-1} = X_i \subseteq \ker Q_i.$$

It implies $Q_iQ_j = 0$ for $j = 0, \ldots, i-1$. Furthermore, for any $z \in \ker \Pi_{i-1}Q_i$, we have $Q_i z \in \ker \Pi_{i-1} \subseteq \ker Q_i$, which means that $z \in \ker Q_i$. □

Remark 2.7 *If $Q_0, \ldots, Q_\kappa$ are regular admissible projectors, and $\Pi_0, \ldots, \Pi_\kappa$ are symmetric, then $Q_0, \ldots, Q_\kappa$ are widely orthogonal. This is a consequence of the properties*

$$\operatorname{im} \Pi_i = (\ker \Pi_i)^\perp = (N_0 \oplus \cdots \oplus N_i)^\perp, \quad \ker Q_i = \operatorname{im} \Pi_i \oplus X_i \quad \textit{for } i = 1, \ldots, \kappa.$$

In general, if there are nontrivial intersections $\widehat{N}_i$, widely orthogonal projectors are given, if the Π_i are symmetric and, additionally $Q_i\Pi_i = 0$, $P_i(I - \Pi_{i-1}) = (P_i(I - \Pi_{i-1}))^*$ hold. In the regular case these properties are always given.

Once more we emphasize that the matrix sequence depends on the choice of the admissible projectors. However, the properties that are important for us later on are independent of the choice of the projectors, as the following theorem shows.

Theorem 2.8 *For any pair $\{E, F\}$, the subspaces $N_0 + \cdots + N_i$, $\widehat{N_i}$ and $\operatorname{im} G_i$, as well as the integers $r_0 \le r_1 \le \cdots \le r_i$, are independent of the special choice of admissible projectors for $i \ge 0$.*

Proof: All claimed properties are direct and obvious consequences of Lemma 2.10 below. □

Definition 2.9 *For a given pair $\{E, F\}$, if the sequence (11) is built with admissible projectors, the integers $r_i := \operatorname{rank} G_i$, $i \ge 0$, $u_i := \dim \widehat{N_i}$ $i \ge 1$, are called characteristic values.*

Lemma 2.10 *Let $Q_0, \ldots, Q_\kappa$ and $\bar{Q}_0, \ldots, \bar{Q}_\kappa$ be any two admissible projector sequences for the pair $\{E, F\}$, and N_j, $\bar{N}_j$, G_j, $\bar{G}_j$ etc. the corresponding subspaces and matrices. Then it holds that*

(1) $\bar{N}_0 + \cdots + \bar{N}_j = N_0 + \cdots + N_j$ *for* $j = 0, \ldots, \kappa$,

(2) *and*

$$\bar{G}_j = G_j Z_j, \quad \bar{B}_j = B_j + G_j \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl}, \quad \textit{for } j = 0, \ldots, \kappa,$$

with nonsingular matrices $Z_0, \ldots, Z_{\kappa+1}$ given by $Z_0 := I$, $Z_{j+1} := Y_{j+1} Z_j$,

$$\begin{aligned} Y_1 &:= I + Q_0(\bar{Q}_0 - Q_0) = I + Q_0 \bar{Q}_0 P_0, \\ Y_{j+1} &:= I + Q_j(\bar{\Pi}_{j-1} \bar{Q}_j - \Pi_{j-1} Q_j) + \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl} \bar{Q}_j, \end{aligned}$$

where $\mathfrak{A}_{jl} = \bar{\Pi}_{j-1}$ for $l = 0, \ldots, j-1$.

(3) $\bar{G}_{\kappa+1} = G_{\kappa+1} Z_{\kappa+1}$ *and* $\bar{N}_0 + \cdots + \bar{N}_{\kappa+1} = N_0 + \cdots + N_{\kappa+1}$,

(4) $(\bar{N}_0 + \cdots + \bar{N}_{j-1}) \cap \bar{N}_j = (N_0 + \cdots + N_{j-1}) \cap N_j$ *for* $j = 1, \ldots, \kappa + 1$.

Remark 2.11 *The introduction of $\mathfrak{A}_{il}$ seems to be unnecessary at this point. However, in the case of DAEs with time-dependent coefficients, the corresponding terms for $\mathfrak{A}_{il}$ are not as easy as here.*

Proof: We prove (1) and (2) together by induction. For $i = 0$ we have

$$\bar{G}_0 = E = G_0, \quad \bar{B}_0 = F = B_0, \quad \bar{N}_0 = \ker \bar{G}_0 = \ker G_0 = N_0, \quad Z_0 = I.$$

To apply induction we suppose the relations

$$\bar{N}_0 + \cdots + \bar{N}_j = N_0 + \cdots + N_j, \tag{17}$$

$$\bar{G}_j = G_j Z_j, \quad \bar{B}_j = B_j + G_j \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl} \tag{18}$$

to be valid for $j \leq i$ with nonsingular Z_j as described above, and

$$Z_j^{-1} = I + \sum_{l=0}^{j-1} Q_l \mathfrak{C}_{jl}$$

with certain $\mathfrak{C}_{jl}$. Comparing $\bar{G}_{i+1}$ and G_{i+1} we write

$$\bar{G}_{i+1} = \bar{G}_i + \bar{B}_i \bar{Q}_i = G_i Z_i + \bar{B}_i \bar{Q}_i Z_i + \bar{B}_i \bar{Q}_i (I - Z_i) \tag{19}$$

and consider the last term in more detail. We have, due to the form of Y_l, induction assumption (17) and $\operatorname{im}(Y_j - I) \subseteq N_0 + \cdots + N_{j-1} = \ker \Pi_{j-1}$ given for all $j \geq 0$ (see Proposition 2.6), that

$$N_0 + \cdots + N_{j-1} \subseteq \ker \Pi_{j-1} Q_j, \quad \bar{N}_0 + \cdots + \bar{N}_{j-1} \subseteq \ker \bar{\Pi}_{j-1} \bar{Q}_j, \quad j \leq i, \tag{20}$$

and therefore,

$$Y_{j+1} - I = (Y_{j+1} - I) \Pi_{j-1}, \quad j = 1, \ldots, i. \tag{21}$$

This implies

$$\operatorname{im}(Y_j - I) \subseteq \ker (Y_{j+1} - I), \quad j = 1, \ldots, i. \tag{22}$$

Concerning $Z_j = Y_j Z_{j-1}$ and using (22), a simple induction proof shows

$$Z_j - I = \sum_{l=1}^{j} (Y_l - I), \quad j = 1, \ldots, i,$$

to be satisfied. Consequently,

$$\operatorname{im}(I - Z_i) \subseteq N_0 + \cdots + N_{i-1} = \bar{N}_0 + \cdots + \bar{N}_{i-1} \subseteq \ker \bar{Q}_i.$$

Using (19), we get

$$\bar{G}_{i+1} = G_i Z_i + \bar{B}_i \bar{Q}_i Z_i.$$

which leads to

$$\bar{G}_{i+1} Z_i^{-1} = G_i + \bar{B}_i \bar{Q}_i = G_i + B_i Q_i + (\bar{B}_i \bar{Q}_i - B_i Q_i).$$

We apply the induction assumption (18) to find

$$\bar{G}_{i+1} Z_i^{-1} = G_{i+1} + B_i (\bar{Q}_i - Q_i) + G_i \sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il} \bar{Q}_i.$$

Induction assumption (17) and Proposition 2.6 imply $\ker \bar{\Pi}_{i-1} = \ker \Pi_{i-1}$ and hence

$$B_i = B_0 \Pi_{i-1} = B_0 \Pi_{i-1} \bar{\Pi}_{i-1} = B_i \bar{\Pi}_{i-1}.$$

Finally,

$$\begin{aligned}\bar{G}_{i+1}Z_i^{-1} &= G_{i+1}+B_i(\bar{\Pi}_{i-1}\bar{Q}_i-\Pi_{i-1}Q_i)+G_{i+1}\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{Q}_i\\ &= G_{i+1}+B_iQ_i(\bar{\Pi}_{i-1}\bar{Q}_i-\Pi_{i-1}Q_i)+G_{i+1}\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{Q}_i \;=\; G_{i+1}Y_{i+1},\end{aligned}$$

which means that

$$\bar{G}_{i+1}=G_{i+1}Y_{i+1}Z_i=G_{i+1}Z_{i+1}. \tag{23}$$

Next, we will show Z_{i+1} to be nonsingular. Owing to the induction assumption, we know that Z_i is nonsingular. Considering the definition of Z_{i+1} we have to show Y_{i+1} to be nonsingular. Firstly,

$$\Pi_iY_{i+1}=\Pi_i \tag{24}$$

since $\operatorname{im}Q_j\subseteq\ker\Pi_i$ for $j\le i$. This follows immediately from the definition of Y_{i+1} and Proposition 2.6 (1). Using the induction assumption (17), Proposition 2.6 and Lemma A.2, we find

$$\Pi_j\bar{\Pi}_j=\Pi_j,\quad \bar{\Pi}_j\Pi_j=\bar{\Pi}_j\quad\text{and}\quad \Pi_j\Pi_j=\Pi_j\qquad\text{for } j=0,...,i.$$

This implies that

$$\Pi_{i-1}(Y_{i+1}-I)=\Pi_{i-1}(Y_{i+1}-I)\Pi_i \tag{25}$$

because

$$\begin{aligned}\Pi_{i-1}(Y_{i+1}-I) &\overset{\text{Prop.2.6(1)}}{=} \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i-\Pi_{i-1}Q_i)\\ &= (\Pi_{i-1}-\Pi_i)(\bar{\Pi}_{i-1}\bar{Q}_i-\Pi_{i-1}Q_i)\\ &= \Pi_{i-1}(\bar{Q}_i-Q_i) \;=\; \Pi_{i-1}(P_i-\bar{P}_i)\\ &= \Pi_i-\Pi_{i-1}\bar{\Pi}_{i-1}\bar{P}_i \;=\; \Pi_i-\Pi_{i-1}\bar{\Pi}_i\\ &= \Pi_i-\Pi_{i-1}\bar{\Pi}_i\Pi_i \;=\; (I-\Pi_{i-1}\bar{\Pi}_i)\Pi_i.\end{aligned}$$

The equations (24) and (25) imply

$$\Pi_{i-1}(Y_{i+1}-I)=\Pi_{i-1}(Y_{i+1}-I)\Pi_i=\Pi_{i-1}(Y_{i+1}-I)\Pi_iY_{i+1}$$

and, consequently,

$$\begin{aligned}I &= Y_{i+1}-(Y_{i+1}-I)\overset{(21)}{=}Y_{i+1}-(Y_{i+1}-I)\Pi_{i-1}\\ &= Y_{i+1}-(Y_{i+1}-I)\Pi_{i-1}\{(I-\Pi_{i-1})Y_{i+1}+\Pi_{i-1}\}\\ &= Y_{i+1}-(Y_{i+1}-I)\Pi_{i-1}\{Y_{i+1}-\Pi_{i-1}(Y_{i+1}-I)\}\\ &= Y_{i+1}-(Y_{i+1}-I)\Pi_{i-1}\{Y_{i+1}-\Pi_{i-1}(Y_{i+1}-I)\Pi_iY_{i+1}\}\\ &= (I-(Y_{i+1}-I)\{I-\Pi_{i-1}(Y_{i+1}-I)\Pi_i\})\,Y_{i+1}.\end{aligned}$$

This means that Y_{i+1} is nonsingular and

$$Y_{i+1}^{-1}=I-(Y_{i+1}-I)\{I-\Pi_{i-1}(Y_{i+1}-I)\Pi_i\}.$$

Then $Z_{i+1} = Y_{i+1}Z_i$ is also nonsingular, and

$$Z_{i+1}^{-1} = Z_i^{-1}Y_{i+1}^{-1} = (I + \sum_{l=0}^{i-1} Q_l \mathfrak{C}_{il})Y_{i+1}^{-1} = I + \sum_{l=0}^{i} Q_l \mathfrak{C}_{i+1\,l}$$

with certain coefficients $\mathfrak{C}_{i+1\,l}$. From (23) we conclude $\bar{N}_{i+1} = Z_{i+1}^{-1}N_{i+1}$, and, due to the special form of Z_{i+1}^{-1},

$$\bar{N}_{i+1} \subseteq N_0 + \cdots + N_{i+1},\ \bar{N}_0 + \cdots + \bar{N}_{i+1} \subseteq N_0 + \cdots + N_{i+1}.$$

Owing to the property $\operatorname{im}(Z_{i+1} - I) \subseteq N_0 + \cdots + N_i = \bar{N}_0 + \cdots + \bar{N}_i$, it holds that

$$N_{i+1} = Z_{i+1}\bar{N}_{i+1} = (I + (Z_{i+1} - I))\bar{N}_{i+1} \subseteq \bar{N}_0 + \cdots + \bar{N}_{i+1}.$$

Thus, $N_0 + \cdots + N_{i+1} \subseteq \bar{N}_0 + \cdots + \bar{N}_{i+1}$ is valid. For symmetry reasons we have

$$N_0 + \cdots + N_{i+1} = \bar{N}_0 + \cdots + \bar{N}_{i+1}.$$

Finally, we derive from the induction assumption that

$$\begin{aligned}
\bar{B}_{i+1} &= \bar{B}_i\bar{P}_i = \big(B_i + G_i \sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il}\big)\bar{P}_i \\
&= B_iP_i\bar{P}_i + B_iQ_i\bar{P}_i + G_{i+1}\sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il}\bar{P}_i \\
&= B_iP_i + B_iQ_i\bar{\Pi}_i + G_{i+1}\sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il}\bar{P}_i \ = \ B_{i+1} + G_{i+1}\sum_{l=0}^{i} Q_l \mathfrak{A}_{i+1,l}
\end{aligned}$$

with $\mathfrak{A}_{i+1,l} = \mathfrak{A}_{il}\bar{P}_i$, $l = 0, \dots, i-1$, $\mathfrak{A}_{i+1,l} = \bar{\Pi}_i$, and therefore, for $l \leq i-1$,

$$\mathfrak{A}_{i+1,l} = \mathfrak{A}_{il}\bar{P}_i = \mathfrak{A}_{i-1,l}\bar{P}_{i-1}\bar{P}_i = \mathfrak{A}_{l+1,l}\bar{P}_{l+1}\cdots\bar{P}_i = \bar{\Pi}_l\bar{P}_{l+1}\cdots\bar{P}_i = \bar{\Pi}_i.$$

We have proved assertions (1) and (2), and (3) is a simple consequence. Next we prove assertion (4). By assertion (1) from Lemma 2.6, we have $N_0 + \cdots + N_i = \ker \Pi_i$ and

$$\begin{aligned}
G_{i+1} &= G_0 + B_0Q_0 + \cdots + B_iQ_i = G_0 + B_0Q_0 + B_1P_0Q_1 + \cdots + B_iP_0\cdots P_{i-1}Q_i \\
&= G_0 + B_0(Q_0 + P_0Q_1 + \cdots + P_0\cdots P_{i-1}Q_i) \\
&= G_0 + B_0(I - P_0\cdots P_i) = G_0 + B_0(I - \Pi_i).
\end{aligned}$$

This leads to the description

$$\begin{aligned}
\widehat{N}_{i+1} = (N_0 + \cdots + N_i) \cap N_{i+1} &= \{z \in \mathbb{R}^m : \Pi_i z = 0,\ G_0z + B_0(I - \Pi_i)z = 0\} \\
&= \{z \in \mathbb{R}^m : z \in N_0 + \cdots + N_i,\ G_0z + B_0z = 0\} \\
&= \{z \in \mathbb{R}^m : z \in \bar{N}_0 + \cdots + \bar{N}_i,\ \bar{G}_0z + \bar{B}_0z = 0\} \\
&= (\bar{N}_0 + \cdots + \bar{N}_i) \cap \bar{N}_{i+1}.
\end{aligned}$$

□

3 Equivalence transformations

Given a matrix pair $\{E, F\}$, $E, F \in L(\mathbb{R}^m, \mathbb{R}^k)$, and nonsingular matrices $L \in L(\mathbb{R}^k)$, $K \in L(\mathbb{R}^m)$, we form

$$\bar{E} = LEK, \quad \bar{F} = LFK. \tag{26}$$

The DAEs corresponding to the pairs $\{E, F\}$ and $\{\bar{E}, \bar{F}\}$ are

$$\begin{aligned} Ex'(t) + Fx(t) &= q(t), \\ \bar{E}\bar{x}'(t) + \bar{F}\bar{x}'(t) &= \bar{q}(t), \end{aligned}$$

which are related to each other by the transformation $x = K\bar{x}$ and premultiplication by L resp. L^{-1}, $\bar{q} = Lq$. In this sense, these DAEs are solution equivalent. How are the matrix functions and admissible projectors for $\{E, F\}$ and $\{\bar{E}, \bar{F}\}$ related? The answer is simple.

Theorem 3.1 *If two matrix pairs $\{E, F\}$ and $\{\bar{E}, \bar{F}\}$ are related via (26), then they have common characteristic values. In detail,*

$$r_i = \bar{r}_i, \quad i \geq 0, \qquad u_i = \bar{u}_i, \quad i \geq 1.$$

If $\{E, F\}$ has the admissible projectors $Q_0, \ldots, Q_\kappa$, then $\{\bar{E}, \bar{F}\}$ has the admissible projectors $\bar{Q}_0, \ldots, \bar{Q}_\kappa$ with $\bar{Q}_i := K^{-1}Q_iK$ for $i = 0, \ldots, \kappa$.

Proof: The transformations $\bar{G}_0 = LG_0K$, $\bar{B}_0 = LB_0K$, $\bar{N}_0 = K^{-1}N_0$ are given at the beginning, and $\bar{Q}_0 := K^{-1}Q_0K$ is admissible. Compute $\bar{G}_1 = \bar{G}_0 + \bar{B}_0\bar{Q}_0 = LG_1K$, $\bar{r}_1 = r_1$, then

$$\bar{N}_1 = K^{-1}N_1, \; \bar{N}_0 \cap \bar{N}_1 = K^{-1}(N_0 \cap N_1).$$

Put $\bar{X}_1 := K^{-1}X_1$ such that $\bar{N}_0 = (\bar{N}_0 \cap \bar{N}_1) \oplus \bar{X}_1$ and noting that $\bar{Q}_1 := K^{-1}Q_1K$ has the property $\ker \bar{Q}_1 \supseteq \bar{X}_1$, this means that $\bar{Q}_0, \bar{Q}_1$ are admissible. At the level i, we have

$$\bar{G}_i = LG_iK, \; \bar{N}_0 + \cdots + \bar{N}_{i-1} = K^{-1}(N_0 + \cdots + N_{i-1}), \; \bar{N}_i = K^{-1}N_i, \; \bar{r}_i = r_i,$$

and $\bar{Q}_i := K^{-1}Q_iK$ satisfies condition $\ker \bar{Q}_i \supseteq \bar{X}_i$ with

$$\bar{X}_i := K^{-1}X, \; \bar{N}_0 + \cdots + \bar{N}_{i-1} = [(\bar{N}_0 + \cdots + \bar{N}_{i-1}) \cap \bar{N}_i] \oplus \bar{X}_i.$$

The conclusion is, in particular, that the characteristic values of a matrix pair are invariant with respect to equivalence transformations. □

4 Admissible projectors for a DAE in Weierstraß-Kronecker canonical form

Here we deal with the matrix pair $\{E, F\}$ given by the $m \times m$ structured matrices

$$E = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & N \end{array}\right] \begin{array}{l} \}m-l \\ \}l \end{array}, \quad F = \left[\begin{array}{c|c} W & 0 \\ \hline H & I \end{array}\right] \begin{array}{l} \}m-l \\ \}l \end{array}, \tag{27}$$

where N is a nilpotent, upper triangular $l \times l$ matrix, $l > 0$,

$$N = \begin{array}{c} \begin{bmatrix} 0 & N_{1,2} & \cdots & N_{1,\mu} \\ & \ddots & & \vdots \\ & & \ddots & N_{\mu-1,\mu} \\ & & & 0 \end{bmatrix} \end{array} \begin{array}{l} \}l_1 \\ \\ \}l_{\mu-1} \\ \}l_\mu \end{array}, \tag{28}$$

$l_1 \geq \cdots \geq l_\mu \geq 1$, $l_1 + \cdots + l_\mu = l$, and the blocks $N_{i,i+1}$ with l_i rows and l_{i+1} columns have full column rank, that means, $\ker N_{i,i+1} = \{0\},\ i = 1, \ldots, \mu - 1$. Then N has nilpotency index μ; that is $N^\mu = 0,\ N^{\mu-1} \neq 0$, and l_i equals the number of its Jordan blocks of order $\geq i,\ i = 1, \ldots, \mu$.
This special form of the nilpotent block is closely related with the tractability index concept, in particular with the decouplings provided by admissible projectors (see Section 5).
The Jordan canonical form of such a nilpotent matrix N consists of $l_1 - l_2$ (nilpotent) Jordan chains of order one, $l_2 - l_3$ chains of order two, and so on up to $l_{\mu-1} - l_\mu$ chains of order $\mu - 1$, and l_μ chains of order μ. Any nilpotent matrix can be put into the structural form (28) by means of a similarity transformation. Thus, without loss of generality we may suppose this special form.

The polynomial $p(\lambda) := \det(\lambda E + F) = \det(\lambda I + W)$ has degree $m - l$. If $l = m$ then $p(\lambda) \equiv 1$. This pair $\{E, F\}$ is regular and represents a slight generalization of the classical Weierstraß-Kronecker canonical form discussed in Section 1 (cf. (3)), where the block H is absent.

In accordance with the structure of E and F in (27) we write $z \in \mathbb{R}^m$ as

$$z = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_\mu \end{bmatrix}, \quad z_0 \in \mathbb{R}^{m-l}, \quad z_i \in \mathbb{R}^{l_i}, \quad i = 1, \ldots, \mu.$$

Now we construct a matrix sequence (11) by admissible projectors. Thereby, on the next three pages in the present section, the letter N is used twofold: N_i, with a single subscript, indicates one of the subspaces, and $N_{j,k}$, with double subscript, means an entry of a matrix.
Put $G_0 = E,\ \ B_0 = F$.
Since $N_0 = \ker G_0 = \{z \in \mathbb{R}^m : z_0 = 0,\ z_\mu = 0,\ \ldots, z_2 = 0\}$ we choose

$$Q_0 = \left[\begin{array}{c|cccc} 0 & & & & \\ \hline & I & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \\ \end{array}, \quad \Pi_0 = P_0 = \left[\begin{array}{c|cccc} I & & & & \\ \hline & 0 & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \\ \end{array},$$

which leads to

$$G_1 = \left[\begin{array}{c|ccccc} I & & & & & \\ \hline & I & N_{1,2} & \cdots & \cdots & N_{1,\mu} \\ & & 0 & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & N_{\mu-1,\mu} \\ & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \\ \\ \end{array}, \quad B_1 = \left[\begin{array}{c|cccc} W & & & & \\ \hline & 0 & & & \\ & & I & & \\ H & & & \ddots & \\ & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \\ \end{array},$$

$N_1 = \{z \in \mathbb{R}^m : z_0 = 0,\ z_\mu = 0,\ \ldots,\ z_3 = 0,\ z_1 + N_{1,2}z_2 = 0\}$, $N_1 \cap N_0 = 0$. Choosing

$$Q_1 = \left[\begin{array}{c|ccccc} 0 & & & & & \\ \hline & 0 & I - N_{1,2} & & & \\ & & I & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \}l_2 \\ \\ \\ \\ \end{array}, \quad P_1 = \left[\begin{array}{c|ccccc} I & & & & & \\ \hline & I & N_{1,2} & & & \\ & & 0 & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \}l_2 \\ \\ \\ \\ \end{array},$$

$$\Pi_1 = \left[\begin{array}{c|ccccc} I & & & & & \\ \hline & 0 & & & & \\ & & 0 & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \}l_2 \\ \\ \\ \\ \end{array},$$

we meet the condition $N_0 \subseteq \ker Q_1$, which means that $Q_1 Q_0 = 0$, and find

$$G_2 = \left[\begin{array}{c|ccccc} I & & & & & \\ \hline & I & N_{1,2} & \cdots & \cdots & N_{1,\mu} \\ & & I & N_{2,3} & & \vdots \\ & & & 0 & \ddots & \vdots \\ & & & & \ddots & N_{\mu-1,\mu} \\ & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \}l_2 \\ \\ \\ \\ \end{array}, \quad B_2 = \left[\begin{array}{c|ccccc} W & & & & & \\ \hline & 0 & & & & \\ & & 0 & & & \\ H & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \}l_2 \\ \\ \\ \\ \end{array},$$

$N_2 = \{z \in \mathbb{R}^m : z_0 = 0,\ z_\mu = 0,\ \ldots,\ z_4 = 0,\ z_2 + N_{23}z_3 = 0,\ z_1 + N_{12}z_2 + N_{13}z_3 = 0\}$, $(N_0 + N_1) \cap N_2 = (\ker \Pi_1) \cap N_2 = \{0\}$. Suppose that we are on level i and that we have $Q_0, \ldots, Q_{i-1}$ being admissible,

$$Q_{i-1} = \left[\begin{array}{c|cccccc} 0 & & & & & & \\ \hline & 0 & & & * & & \\ & & \ddots & & \vdots & & \\ & & & 0 & * & & \\ & & & & I & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \}l_i \\ \\ \\ \\ \end{array}, \quad \Pi_{i-1} = \left[\begin{array}{c|ccccc} I & & & & & \\ \hline & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & I & \\ & & & & & \ddots \\ & & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \}l_i \\ \\ \\ \end{array},$$

$Q_{i-1}(N_0 + \cdots + N_{i-2}) = Q_{i-1}\mathrm{im}\,(I - \varPi_{i-2}) = \mathrm{im}\,Q_{i-1}(I - \varPi_{i-2}) = \{0\}$,

$$G_i = \left[\begin{array}{c|cccccc} I & & & & & & \\ \hline & I & N_{1,2} & \cdots & \cdots & \cdots & N_{1,\mu} \\ & & \ddots & \ddots & & & \vdots \\ & & & I & N_{i,i+1} & & \vdots \\ & & & & 0 & \ddots & \vdots \\ & & & & & \ddots & N_{\mu-1,\mu} \\ & & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \}l_i \\ \\ \\ \\ \end{array}, \quad B_i = \left[\begin{array}{c|cccccc} W & & & & & & \\ \hline & 0 & & & & & \\ & & \ddots & & & & \\ H & & & 0 & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \}l_i \\ \\ \\ \\ \end{array}. \tag{29}$$

It follows that

$$\begin{aligned} N_i = \{z \in \mathbb{R}^m : z_0 = 0,\, z_\mu = 0,\, \ldots,\, z_{i+2} = 0, \\ z_i + N_{i,i+1}z_{i+1} = 0,\, \ldots,\, z_1 + N_{12}z_2 + \cdots + N_{1,i+1}z_{i+1} = 0\}, \\ (N_0 + \cdots + N_{i-1}) \cap N_i = (\ker \varPi_{i-1}) \cap N_i = \{0\}. \end{aligned}$$

Choosing

$$Q_i = \left[\begin{array}{c|ccccccc} 0 & & & & & & & \\ \hline & 0 & & & * & & & \\ & & \ddots & & \vdots & & & \\ & & & 0 & * & & & \\ & & & & I & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \}l_{i+1} \\ \\ \\ \\ \end{array}, \quad P_i = \left[\begin{array}{c|ccccccc} I & & & & & & & \\ \hline & I & & & * & & & \\ & & \ddots & & \vdots & & & \\ & & & I & * & & & \\ & & & & 0 & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \}l_{i+1} \\ \\ \\ \\ \end{array},$$

$$\varPi_i = \left[\begin{array}{c|cccccc} I & & & & & & \\ \hline & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \\ \}l_{i+1} \\ \\ \\ \end{array},$$

we meet the admissibility condition (14), as $Q_i(I - \varPi_{i-1}) = 0$, and arrive at

$$G_{i+1} = \left[\begin{array}{c|cccccc} I & & & & & & \\ \hline & I & N_{1,2} & \cdots & \cdots & \cdots & N_{1,\mu} \\ & & \ddots & \ddots & & & \vdots \\ & & & I & N_{i+1,i+2} & & \vdots \\ & & & & 0 & \ddots & \vdots \\ & & & & & \ddots & N_{\mu-1,\mu} \\ & & & & & & 0 \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \}l_{i+1} \\ \\ \\ \\ \end{array}, B_{i+1} = \left[\begin{array}{c|cccccc} W & & & & & & \\ \hline & 0 & & & & & \\ & & \ddots & & & & \\ H & & & 0 & & & \\ & & & & I & & \\ & & & & & \ddots & \\ & & & & & & I \end{array}\right] \begin{array}{l} \\ \}l_1 \\ \\ \}l_{i+1} \\ \\ \\ \\ \end{array}.$$

This verifies formulas (29) to provide the right pair G_i, B_i at level i, $i \geq 1$, for $\{E, F\}$ as in (27). Obviously, we obtain precisely a nonsingular matrix G_μ, but $G_{\mu-1}$ is singular. The characteristic values of our pair $\{E, F\}$ are $u_i = 0$, $i \geq 1$, and

$$r_i = m - \dim N_i = m - l_{i+1} < m,\ i = 0, \ldots, \mu - 1,\ r_i = m,\ i \geq \mu.$$

The next proposition records this result.

Proposition 4.1 *The matrix sequence (11) built with admissible projectors for the special pair $\{E, F\}$ given by (27), (28) consists of singular matrices $G_0, \ldots, G_{\mu-1}$ and a nonsingular G_μ. The characteristic values are: $u_i = 0$ for $i = 1, \ldots, \mu$, and*
$r_i = m - \dim N_i = m - l_{i+1}$ for $i = 0, \ldots, \mu - 1$, $r_\mu = m$.

For a DAE in Weierstraß-Kronecker canonical form (27) with its structured part N (28) the decoupling into the basic components is given a priori (cf. (4), (5)). The so-called “slow” subsystem

$$y'(t) + Wy(t) = p(t)$$

is a standard explicit ODE, hence an integration problem, whereas the so-called ”fast” subsystem

$$Nz'(t) + z(t) = r(t) - Hy(t)$$

contains exclusively algebraic relations and differentiation problems.
The admissible projectors expose these two basic structures as well as a further subdivision of the differentiation problems, too: The proper state variable is comprised by $\Pi_{\mu-1}$ while $I - \Pi_{\mu-1}$ collects all other variables:

$$\Pi_{\mu-1} = \left[\begin{array}{c|ccc} I & & & \\ \hline & 0 & & \\ & & \ddots & \\ & & & 0 \end{array}\right],\ I - \Pi_{\mu-1} = \left[\begin{array}{c|ccc} 0 & & & \\ \hline & I & & \\ & & \ddots & \\ & & & I \end{array}\right].$$

Those variables that are not differentiated at all and those variables that have to be differentiated i-times are comprised by

$$Q_0 = \left[\begin{array}{c|ccc} 0 & & & \\ \hline & I & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{array}\right] \}l_1 \quad \text{and } \Pi_{i-1}Q_i = \left[\begin{array}{c|ccccc} 0 & & & & & \\ \hline & 0 & & & & \\ & & \ddots & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & 0 \end{array}\right] \}l_{i+1},$$

respectively.
These decoupling properties of the projectors will also be valid for more general DAEs.

Example 4.2 *Reconsideration of the DAE from Example 2.1 that is not in Weierstraß-Kronecker canonical form, with the projectors*

$$\Pi_1 = P_0P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix},\ Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ P_0Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The DAE itself can be rewritten without any differentiations of equations as

$$(-x_1 + x_3)' = q_2 + q_3 - q_1, \tag{30}$$

$$x_1 = \frac{1}{2}(q_3 - (-x_1 + x_3)), \tag{31}$$

$$x_2 = (q_1 - q_3) - x_1'. \tag{32}$$

Obviously, $\Pi_1 x$ reflects the proper state variable $-x_1+x_3$, for which an explicit ODE (30) *is given. $P_0 Q_1 x$ refers to the variable x_1 that is described by the algebraic equation* (31) *when the solution $-x_1 + x_3$ is already given by* (30)*. Finally, $Q_0 x$ reflects the variable x_2 which can be determined by* (32)*. Note, that the variable x_1 has to be differentiated here.*

5 Decoupling of regular DAEs by admissible projectors

In this section we deal with $m \times m$ matrices E, F. If they form a regular matrix pair $\{E, F\}$ with Kronecker index $\mu \in \mathbb{N}$, then the matrices $G_0, \dots, G_{\mu-1}$ generated by means of admissible projectors according to (11) are singular and G_μ is the first nonsingular matrix occurring in the sequence (see Proposion 4.1 and Theorem 3.1).
What do we know about the reverse implication? If a nonsingular matrix G_μ is met in the sequence (11), then do we have a regular pair $\{E, F\}$? Does its Kronecker index equal μ ? We will have positive answers to both questions at the end of this section.
The nonsingular matrix G_μ allows for a projector based decoupling so that the decoupled version of the given DAE looks quite similar to the Weierstraß-Kronecker canonical form. We stress that, at the same time, our discussion should serve as a model for a corresponding decoupling of time-dependent linear DAEs for which we do not have a Weierstraß-Kronecker canonical form. As already mentioned, when constructing the matrices G_j we have in mind a rearrangement of terms within the original DAE

$$Ex'(t) + Fx(t) = q(t) \tag{33}$$

such that the solution components $\Pi_{\mu-1}x(t)$ and $(I - \Pi_{\mu-1})x(t)$ are separated as far as possible and the nonsingular matrix G_μ occurs in front of the derivative $(\Pi_{\mu-1}x(t))'$. Let the matrix sequence (11) starting from $G_0 = E$, $B_0 = F$ be realized up to G_μ which is nonsingular. Let $\mu \in \mathbb{N}$ be the smallest such index.
Consider the accompanying admissible projectors $Q_0, \dots, Q_\mu$.
We have $Q_\mu = 0,\ P_\mu = I,\ \Pi_\mu = \Pi_{\mu-1}$ for trivial reasons. Due to Proposition 2.6, the intersections $\widehat{N_i}$ are trivial,

$$\widehat{N_i} = N_i \cap (N_0 + \cdots + N_{i-1}) = \{0\},\ \ i = 1, \dots, \mu - 1,$$

and therefore

$$N_0 + \cdots + N_{i-1} = N_0 \oplus \cdots \oplus N_{i-1}, \quad X_i = N_0 \oplus \cdots \oplus N_{i-1}, \quad i = 1, \dots, \mu - 1. \tag{34}$$

From (34) we derive the relations

$$Q_iQ_j = 0, \quad j = 0, \dots, i-1, \quad i = 1, \dots, \mu-1, \tag{35}$$

which are very helpful in computations. Recall from Section 2 the properties

$$\begin{array}{rcl} G_iP_{i-1} & = & G_{i-1}, \quad B_i = B_i\varPi_{i-1}, \ \ i = 1, \dots, \mu, \\ G_iQ_j & = & B_jQ_j, \ \ j = 0, \dots, i-1, \quad i = 0, \dots, \mu-1, \end{array}$$

which will be used frequently.

Applying $G_0 = G_0P_0 = G_0\varPi_0$ we rewrite the DAE (33) as

$$G_0(\varPi_0x(t))' + B_0x(t) = q(t), \tag{36}$$

and then, with $B_0 = B_0P_0 + B_0Q_0 = B_0\varPi_0 + G_1Q_0$, as

$$G_1P_1P_0(\varPi_0x(t))' + B_0\varPi_0x(t) + G_1Q_0x(t) = q(t).$$

Now we use the relation

$$\begin{array}{rcl} G_1P_1P_0 & = & G_1\varPi_0P_1P_0 + G_1(I-\varPi_0)P_1P_0 \\ & = & G_1\varPi_1 - G_1(I-\varPi_0)Q_1 \\ & = & G_1\varPi_1 - G_1(I-\varPi_0)Q_1\varPi_0Q_1 \end{array}$$

to replace the first term. This yields

$$G_1(\varPi_1x(t))' + B_1x(t) + G_1\{Q_0x(t) - (I-\varPi_0)Q_1(\varPi_0Q_1x(t))'\} = q(t).$$

Proceeding further by induction we suppose

$$\begin{array}{rcl} G_i(\varPi_ix(t))' & + & B_ix(t) \\ & + & G_i\displaystyle\sum_{l=0}^{i-1}\{Q_lx(t) - (I-\varPi_l)Q_{l+1}(\varPi_lQ_{l+1}x(t))'\} = q(t) \end{array} \tag{37}$$

and, in the next step, using the properties $G_{i+1}P_{i+1}P_i = G_i$, $B_iQ_i = G_{i+1}Q_i$, $G_iQ_l = G_{i+1}Q_l$, $l = 0, \dots, i-1$, and

$$\begin{aligned} P_{i+1}P_i\varPi_i &= \varPi_iP_{i+1}P_i\varPi_i + (I-\varPi_i)P_{i+1}P_i\varPi_i \\ &= \varPi_{i+1} - (I-\varPi_i)Q_{i+1} \\ &= \varPi_{i+1} - (I-\varPi_i)Q_{i+1}\varPi_iQ_{i+1}, \end{aligned}$$

we reach

$$\begin{array}{rcl} G_{i+1}(\varPi_{i+1}x(t))' & + & B_{i+1}x(t) \\ & + & G_{i+1}\displaystyle\sum_{l=0}^{i}\{Q_lx(t) - (I-\varPi_l)Q_{l+1}(\varPi_lQ_{l+1}x(t))'\} = q(t), \end{array}$$

so that expression (37) can be used for all $i = 1, \ldots, \mu$.
In particular, we obtain

$$\begin{aligned} G_\mu(\Pi_\mu x(t))' &+ B_\mu x(t) \\ &+ G_\mu \sum_{l=0}^{\mu-1} \{Q_l x(t) - (I - \Pi_l) Q_{l+1} (\Pi_l Q_{l+1} x(t))'\} = q(t). \end{aligned} \tag{38}$$

Taking into account that $Q_\mu = 0,\ P_\mu = I,\ \Pi_\mu = \Pi_{\mu-1}$, and scaling with G_μ^{-1} we derive the equation

$$(\Pi_{\mu-1}x(t))' + G_\mu^{-1} B_\mu x(t) + \sum_{l=0}^{\mu-1} Q_l x(t) - \sum_{l=0}^{\mu-2} (I - \Pi_l) Q_{l+1} (\Pi_l Q_{l+1} x(t))' = G_\mu^{-1} q(t). \tag{39}$$

In turn, equation (39) can be decoupled into two parts, the explicit ODE with respect to $\Pi_{\mu-1}x(t)$,

$$(\Pi_{\mu-1}x(t))' + \Pi_{\mu-1} G_\mu^{-1} B_\mu x(t) = \Pi_{\mu-1} G_\mu^{-1} q(t), \tag{40}$$

and the remaining equation

$$\begin{aligned} (I - \Pi_{\mu-1}) G_\mu^{-1} B_\mu x(t) &+ \sum_{l=0}^{\mu-1} Q_l x(t) \\ &- \sum_{l=0}^{\mu-2} (I - \Pi_l) Q_{l+1} (\Pi_l Q_{l+1} x(t))' = (I - \Pi_{\mu-1}) G_\mu^{-1} q(t). \end{aligned} \tag{41}$$

Next, we show that equation (41) uniquely defines the component $(I - \Pi_{\mu-1})x(t)$ in terms of $\Pi_{\mu-1}x(t)$.
We decouple equation (41) once again into μ further parts according to the decomposition

$$I - \Pi_{\mu-1} = Q_0 P_1 \cdots P_{\mu-1} + Q_1 P_2 \cdots P_{\mu-1} + \cdots + Q_{\mu-2} P_{\mu-1} + Q_{\mu-1}. \tag{42}$$

Notice that $Q_i P_{i+1} \cdots P_{\mu-1},\ i = 0, \ldots, \mu-2$ are projectors, too, and

$$\begin{aligned} Q_i P_{i+1} \cdots P_{\mu-1} Q_i &= Q_i, \\ Q_i P_{i+1} \cdots P_{\mu-1} Q_j &= 0, \quad \text{if } i \neq j, \\ Q_i P_{i+1} \cdots P_{\mu-1} (I - \Pi_l) Q_{l+1} &= Q_i (I - \Pi_l) Q_{l+1} = 0, \quad \text{for } l = 0, \ldots, i-1, \\ Q_i P_{i+1} \cdots P_{\mu-1} (I - \Pi_i) Q_{i+1} &= Q_i Q_{i+1}. \end{aligned}$$

Hence, multiplying (41) by $Q_i P_{i+1} \cdots P_{\mu-1},\ i = 0, \ldots, \mu-2$, and $Q_{\mu-1}$ yields

$$\begin{aligned} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} B_\mu x(t) &+ Q_i x(t) - Q_i Q_{i+1} (\Pi_i Q_{i+1} x(t))' \\ &- \sum_{l=i+1}^{\mu-2} Q_i P_{i+1} \cdots P_l Q_{l+1} (\Pi_l Q_{l+1} x(t))' = Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} q(t), \\ &\qquad i = 0, \ldots, \mu-2, \end{aligned} \tag{43}$$

$$Q_{\mu-1} G_\mu^{-1} B_\mu x(t) + Q_{\mu-1} x(t) = Q_{\mu-1} G_\mu^{-1} q(t). \tag{44}$$

Equation (44) uniquely determines the component $Q_{\mu-1}x(t)$ as

$$Q_{\mu-1} x(t) = Q_{\mu-1} G_\mu^{-1} q(t) - Q_{\mu-1} G_\mu^{-1} B_\mu x(t),$$

and the formula contained in (43) for $i = \mu - 2$ gives

$$Q_{\mu-2}x(t) = Q_{\mu-2}P_{\mu-1}G_\mu^{-1}q(t) - Q_{\mu-2}P_{\mu-1}G_\mu^{-1}B_\mu x(t) - Q_{\mu-2}Q_{\mu-1}(\mathit{\Pi}_{\mu-2}Q_{\mu-1}x(t))',$$

and so on, i.e., in a consecutive manner we obtain expressions determining the components $Q_i x(t)$ in dependence on $\mathit{\Pi}_{\mu-1}x(t)$ and $Q_{i+j}x(t)$, $j = 1, \dots, \mu - 1 - i$.
To compose an expression for the whole solution $x(t)$ there is no need for the components $Q_i x(t)$ themselves, $i = 0, \dots, \mu - 1$, but one can do with $Q_0 x(t)$, $\mathit{\Pi}_{i-1}Q_i x(t)$, $i = 1, \dots, \mu - 1$, which corresponds to the decomposition

$$I = Q_0 + \mathit{\Pi}_0 Q_1 + \dots + \mathit{\Pi}_{\mu-2}Q_{\mu-1} + \mathit{\Pi}_{\mu-1}. \tag{45}$$

For this purpose we rearrange the system (43),(44) once again by multiplying (44) by $\mathit{\Pi}_{\mu-2}$ and (43) for $i = 1, \dots, \mu - 2$ by $\mathit{\Pi}_{i-1}$. Let us remark that, even though we scale with projectors (which are singular matrices) here, nothing of the equations gets lost. This is due to the relations

$$\begin{aligned} Q_i &= Q_i \mathit{\Pi}_{i-1} Q_i \\ &= (\mathit{\Pi}_{i-1} + (I - \mathit{\Pi}_{i-1}))Q_i \mathit{\Pi}_{i-1} Q_i \\ &= (I + (I - \mathit{\Pi}_{i-1})Q_i)\mathit{\Pi}_{i-1}Q_i, \\ \mathit{\Pi}_{i-1}Q_i &= (I - (I - \mathit{\Pi}_{i-1})Q_i)Q_i, \end{aligned} \tag{46}$$

which allow a one-to-one translation of the components $Q_i x(t)$ and $\mathit{\Pi}_{i-1}Q_i x(t)$ into each other.
With notations chosen according to the decomposition (45),

$$v_0(t) = Q_0 x(t), \quad v_i(t) := \mathit{\Pi}_{i-1}Q_i x(t), \quad i = 1, \dots, \mu - 1, \quad u(t) := \mathit{\Pi}_{i-1}x(t), \tag{47}$$

we obtain the representation resp. decomposition

$$x(t) = v_0(t) + v_1(t) + \dots + v_{\mu-1}(t) + u(t) \tag{48}$$

of the solution as well as the structured system resulting from (40), (43), (44)

$$\begin{aligned} &\left[\begin{array}{c|cccc} I & & & & \\ \hline & 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & & 0 \end{array}\right] \left[\begin{array}{c} u'(t) \\ \hline 0 \\ v_1'(t) \\ \vdots \\ v_{\mu-1}'(t) \end{array}\right] \\ &\qquad + \left[\begin{array}{c|ccc} \mathcal{W} & & & \\ \hline \mathcal{H}_0 & I & & \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ \mathcal{H}_{\mu-1} & & & I \end{array}\right] \left[\begin{array}{c} u(t) \\ \hline v_0(t) \\ \vdots \\ \vdots \\ v_{\mu-1}(t) \end{array}\right] = \left[\begin{array}{c} \mathcal{L}_d \\ \mathcal{L}_0 \\ \vdots \\ \vdots \\ \mathcal{L}_{\mu-1} \end{array}\right] q(t) \end{aligned} \tag{49}$$

with the $m \times m$-blocks

$$\begin{aligned}
\mathcal{H}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} B_\mu, \\
\mathcal{H}_i &:= \mathit{\Pi}_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} B_\mu, && i = 1, \ldots, \mu-2, \\
\mathcal{H}_{\mu-1} &:= \mathit{\Pi}_{\mu-2} Q_{\mu-1} G_\mu^{-1} B_\mu, \\
\mathcal{W} &:= \mathit{\Pi}_{\mu-1} G_\mu^{-1} B_\mu, \\
\mathcal{L}_d &:= \mathit{\Pi}_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_0 &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1}, \\
\mathcal{L}_i &:= \mathit{\Pi}_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{L}_{\mu-1} &:= \mathit{\Pi}_{\mu-2} Q_{\mu-1} G_\mu^{-1},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0 Q_1, \\
\mathcal{N}_{0j} &:= Q_0 P_1 \cdots P_{j-1} Q_j, && j = 2, \ldots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\mathit{\Pi}_{i-1} Q_i Q_{i+1}, \\
\mathcal{N}_{ij} &:= -\mathit{\Pi}_{i-1} Q_i P_{i+1} \cdots P_{j-1} Q_j, && j = i+2, \ldots, \mu-1,\ i = 1, \ldots, \mu-2.
\end{aligned}$$

System (49) almost looks like a DAE in Weierstraß-Kronecker canonical form. However, compared to the latter it is a puffed up system of dimension $(\mu+1)m$. The system (49) is equivalent to the original DAE (33) in the following sense.

Lemma 5.1 *Let $\{E, F\}$ be a pair with the characteristic values*

$$r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m.$$

(1) *If $x(.)$ is a solution of the DAE (33), then the components $u(.),\ v_0(.), \ldots, v_{\mu-1}(.)$, which are given by (47), form a solution of the puffed up system (49).*

(2) *Conversely, if the functions $u(.), v_0(.), \ldots, v_{\mu-1}(.)$ are a solution of the system (49) and if, additionally, $u(t_0) = \mathit{\Pi}_{\mu-1} u(t_0)$ holds for a $t_0 \in \mathcal{I}$, then the compound function $x(.)$ defined by (48) is a solution of the original DAE (33).*

Proof: It remains to verify (2). Due to the properties of the coefficients, for each solution of system (49) it holds that $v_i(t) = \mathit{\Pi}_{i-1} Q_i v_i(t)$, $i = 1, \ldots, \mu-1$, $v_0(t) = Q_0 v_0(t)$, which means that the components $v_i(t)$, $i = 0, \ldots, \mu-1$, belong to the desired subspaces.
The first equation in (49) is the explicit ODE $u'(t) + \mathcal{W} u(t) = \mathcal{L}_d q(t)$. Let $u_q(.)$ denote the solution fixed by the initial condition $u_q(t_0) = 0$. We have $u_q(t) = \mathit{\Pi}_{\mu-1} u_q(t)$ because of $\mathcal{W} = \mathit{\Pi}_{\mu-1} \mathcal{W}$, $\mathcal{L}_d = \mathit{\Pi}_{\mu-1} \mathcal{L}_d$. However, for each arbitrary constant $c \in \operatorname{im}(I - \mathit{\Pi}_{\mu-1})$, the function $\bar{u}(.) := c + u_q(.)$ solves this ODE but does not belong to $\operatorname{im} \mathit{\Pi}_{\mu-1}$ as we want it to.
With the initial condition $u(t_0) = u_0 \in \operatorname{im} \mathit{\Pi}_{\mu-1}$ the solution can be kept in the desired subspace, which means that $u(t) \in \operatorname{im} \mathit{\Pi}_{\mu-1}$ for all $t \in \mathcal{I}$. Now, by carrying out the decoupling procedure in reverse order and putting things together we have finished the proof. $\square$

System (49) is given in terms of the original DAE. It shows in some detail the inherent structure of that DAE. An analogous decoupling applies for time-varying linear DAEs, too.A special smart choice of the admissible projectors cancels the coefficients $\mathcal{H}_i$ in system (49) so that the second part no longer depends on the first one.

Theorem 5.2 *Let $\{E, F\}$ be a pair with characteristic values*

$$r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m.$$

Then there are admissible projectors $Q_0, \ldots Q_{\mu-1}$ such that the coupling coefficients $\mathcal{H}_0, \ldots, \mathcal{H}_{\mu-1}$ in (49) vanish, that is, (49) decouples into two independent subsystems.

Proof:
For any given sequence of admissible projectors $Q_0, \ldots Q_{\mu-1}$ the coupling coefficients can be expressed as $\mathcal{H}_0 = Q_{0*}\Pi_{\mu-1}$, $\mathcal{H}_i = \Pi_{i-1}Q_{i*}\Pi_{\mu-1}$, $i = 1, \ldots, \mu-1$, where we denote

$$\begin{aligned} Q_{0*} &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} B_0, \\ Q_{i*} &:= Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} B_0 \Pi_{i-1}, \quad i = 1, \ldots, \mu-2, \\ Q_{\mu-1*} &:= Q_{\mu-1} G_\mu^{-1} B_0 \Pi_{\mu-2}. \end{aligned}$$

We realize that $Q_{i*}Q_i = Q_i$, $i = 0, \ldots, \mu-1$, since

$$Q_{\mu-1*}Q_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B_0\Pi_{\mu-2}Q_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1}Q_{\mu-1} = Q_{\mu-1}G_\mu^{-1}G_\mu Q_{\mu-1} = Q_{\mu-1}$$

and so on. This implies $(Q_{i*})^2 = Q_{i*}$, i.e., Q_{i*} is a projector onto N_i, $i = 0, \ldots, \mu-1$. Furthermore, by construction it holds that $N_0 + \cdots + N_{i-1} \subseteq \ker Q_i$, $i = 1, \ldots, \mu-1$.

The new projectors $\bar{Q}_0 := Q_0, \ldots, \bar{Q}_{\mu-2} := Q_{\mu-2}$, $\bar{Q}_{\mu-1} := Q_{\mu-1*}$ are also admissible ones, but now, in (49) the respective coefficient $\bar{\mathcal{H}}_{\mu-1}$ disappears. Namely, the old and new sequences are related by

$$\bar{G}_i = G_i, \quad i = 0, \ldots, \mu-1, \quad \bar{G}_\mu = G_\mu^{-1} + B_{\mu-1}Q_{\mu-1*} = G_\mu Z_\mu,$$

with nonsingular $Z_\mu := I + Q_{\mu-1}Q_{\mu-1*}P_{\mu-1}$. This yields

$$\bar{Q}_{\mu-1*} := \bar{Q}_{\mu-1}\bar{G}_{\mu-1}B_0\Pi_{\mu-2} = Q_{\mu-1*}Z_\mu^{-1}G_\mu^{-1}B_0\Pi_{\mu-2} = Q_{\mu-1}G_\mu^{-1}B_0\Pi_{\mu-2} = Q_{\mu-1*} = \bar{Q}_{\mu-1}$$

because of

$$Q_{\mu-1*}Z_\mu^{-1} = Q_{\mu-1*}(I - Q_{\mu-1}Q_{\mu-1*}P_{\mu-1}) = Q_{\mu-1},$$

and hence

$$\bar{\mathcal{H}}_{\mu-1} := \bar{\Pi}_{\mu-2}\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = \Pi_{\mu-2}\bar{Q}_{\mu-1}\bar{\Pi}_{\mu-1} = 0.$$

We show by induction that the coupling coefficients disappear stepwise with an appropriate choice of admissible projectors.
Assume $Q_0, \ldots Q_{\mu-1}$ to be such that

$$\mathcal{H}_{k+1} = 0, \ \ldots, \ \mathcal{H}_{\mu-1} = 0, \tag{50}$$

or, equivalently,

$$Q_{k+1*}\Pi_{\mu-1} = 0, \ \ldots, \ Q_{\mu-1*}\Pi_{\mu-1} = 0,$$

for a certain k, $0 \leq k \leq \mu-2$.
We build a new sequence by letting $\bar{Q}_i := Q_i$, $i = 0, \ldots, k-1$ (if $k \geq 1$), and $\bar{Q}_k := Q_{k*}$.
In particular it holds that $Q_k\bar{P}_k = -\bar{Q}_k P_k$.
$\bar{Q}_0, \ldots, \bar{Q}_k$ are admissible, and the resulting two sequences are related by

$$\bar{G}_i = G_i Z_i \ \ i = 0, \ldots, k+1,$$

with factors

$$Z_0 = I,\ \dots, Z_k = I,\ \ Z_{k+1} = I + Q_k Q_{k*} P_k,\ \ Z_{k+1}^{-1} = I - Q_k Q_{k*} P_k.$$

We put $\bar{Q}_{k+1} := Z_{k+1}^{-1} Q_{k+1} Z_{k+1} = Z_{k+1}^{-1} Q_{k+1}$. $\bar{Q}_0, \dots, \bar{Q}_{k+1}$ are admissible, too. Applying Lemma 2.10 we proceed with

$$\bar{G}_j = G_j Z_j,\ \ Q_j := Z_j^{-1} Q_j Z_j,\ \ j = k+2, \dots, \mu - 1,$$

and arrive at a new sequence of admissible projectors $\bar{Q}_0, \dots, \bar{Q}_{\mu-1}$. The invertibility of Z_j is ensured by Lemma 2.10.
Put $Y_{k+1} := Z_{k+1}$. Lemma 2.10 provides us with the expressions

$$Y_j := Z_j Z_{j-1}^{-1} = I + Q_{j-1}(\bar{\varPi}_{j-2} \bar{Q}_{j-1} - \varPi_{j-2} Q_{j-1}) + \sum_{l=0}^{j-2} Q_l \bar{\varPi}_{j-2} \bar{Q}_{j-1}, \quad j \geq k+2.$$

Additionally we learn from Lemma 2.10 that the subspaces $N_0 \oplus \dots \oplus N_j$ and $\bar{N}_0 \oplus \dots \oplus \bar{N}_j$ coincide.
For our special new projectors the expression for $Y_j,\ j \geq k+2$, simplifies to

$$Y_j = I + \sum_{l=0}^{j-2} Q_l \bar{\varPi}_{j-2} \bar{Q}_{j-1} = I + \sum_{l=k}^{j-2} Q_l \bar{\varPi}_{j-2} Q_{j-1}$$

because the following relations are now valid:

$$Q_j Z_j = 0,\ \ \bar{Q}_j = Z_j^{-1} Q_j,\ \ \bar{\varPi}_{j-2} \bar{Q}_{j-1} = \bar{\varPi}_{j-2} Z_{j-1}^{-1} Q_{j-1} = \bar{\varPi}_{j-2} Q_{j-1},$$

$$Q_{j-1}(\bar{\varPi}_{j-2} \bar{Q}_{j-1} - \varPi_{j-2} Q_{j-1}) = Q_{j-1}(\bar{\varPi}_{j-2} Q_{j-1} - \varPi_{j-2} Q_{j-1}) = 0.$$

We have to verify that the new coupling coefficients $\bar{\mathcal{H}}_k$ and $\bar{\mathcal{H}}_j,\ j \geq k+1$, disappear. We compute $\bar{Q}_k Z_{k+1}^{-1} = \bar{Q}_k - \bar{Q}_k P_k = \bar{Q}_k Q_k = Q_k$ and

$$Z_{j-1} Z_j^{-1} = Y_j^{-1} = I - \sum_{l=k}^{j-2} Q_l \bar{\varPi}_{j-2} Q_{j-1}, \quad j \geq k+2. \tag{51}$$

For $j \geq k+1$ this yields

$$\bar{Q}_{j*} \bar{\varPi}_{\mu-1} = \bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} B \bar{\varPi}_{\mu-1} = Z_j^{-1} Q_j Y_{j+1}^{-1} P_{j+1} \cdots Y_{\mu-1}^{-1} P_{\mu-1} Y_\mu^{-1} B \bar{\varPi}_{\mu-1}$$

and by inserting (51) into the last expression

$$\bar{Q}_{j*} \bar{\varPi}_{\mu-1} = Z_j^{-1} Q_j (I - \sum_{l=k}^{j-1} Q_l \bar{\varPi}_{j-1} Q_j) P_{j+1} \cdots P_{\mu-1} (I - \sum_{l=k}^{\mu-2} Q_l \bar{\varPi}_{\mu-2} Q_{\mu-1}) G_\mu^{-1} B \bar{\varPi}_{\mu-1}.$$

Rearranging the terms:

$$\begin{aligned} \bar{Q}_{j*} \bar{\varPi}_{\mu-1} = (Z_j^{-1} Q_j P_{j+1} \cdots P_{\mu-1} + \mathcal{C}_{j,j+1} Q_{j+1} P_{j+2} \cdots P_{\mu-1} \\ + \cdots + \mathcal{C}_{j,\mu-2} Q_{\mu-2} P_{\mu-1} + \mathcal{C}_{j,\mu-1} Q_{\mu-1}) G_\mu^{-1} B \bar{\varPi}_{\mu-1}. \end{aligned} \tag{52}$$

The detailed expression of the coefficients $\mathcal{C}_{j,i}$ does not matter at all. With analogous arguments we derive

$$\begin{aligned}\bar{Q}_{k*}\bar{\Pi}_{\mu-1} = (Q_{k*}P_{k+1}\cdots P_{\mu-1} + \mathcal{C}_{k,j+1}Q_{k+1}P_{k+2}\cdots P_{\mu-1} \\ + \cdots + \mathcal{C}_{k,\mu-2}Q_{\mu-2}P_{\mu-1} + \mathcal{C}_{k,\mu-1}Q_{\mu-1})G_\mu^{-1}B\bar{\Pi}_{\mu-1}.\end{aligned} \tag{53}$$

Next we compute

$$\begin{aligned}\bar{\Pi}_{\mu-1} &= \Pi_{k-1}\bar{P}_k\bar{P}_{k+1}\cdots\bar{P}_{\mu-1} = \Pi_{k-1}\bar{P}_kP_{k+1}\cdots P_{\mu-1} \\ &= \Pi_{k-1}(P_k+Q_k)\bar{P}_kP_{k+1}\cdots P_{\mu-1} = \Pi_{\mu-1} - Q_k\bar{Q}_k\Pi_{\mu-1},\end{aligned}$$

and therefore

$$G_\mu^{-1}B\bar{\Pi}_{\mu-1} = G_\mu^{-1}B(\Pi_{\mu-1} - \Pi_{k-1}Q_k\bar{Q}_k\Pi_{\mu-1}) = G_\mu^{-1}B\Pi_{\mu-1} - Q_k\bar{Q}_k\Pi_{\mu-1}.$$

Now from assumption (50) and the properties of admissible projectors it follows that

$$Q_{\mu-1}G_\mu^{-1}B\bar{\Pi}_{\mu-1} = Q_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1} - Q_{\mu-1}\bar{Q}_k\Pi_{\mu-1} = Q_{\mu-1*}\Pi_{\mu-1} = 0,$$

and, for $i = k+1,\dots,\mu-2$,

$$Q_iP_{i+1}\cdots P_{\mu-1}B\bar{\Pi}_{\mu-1} = Q_iP_{i+1}\cdots P_{\mu-1}B\Pi_{\mu-1} - Q_i\bar{Q}_k\Pi_{\mu-1} = Q_{i*}\Pi_{\mu-1} = 0.$$

Furthermore, taking into account the special choice of $\bar{Q}_k$,

$$Q_kP_{k+1}\cdots P_{\mu-1}B\bar{\Pi}_{\mu-1} = Q_kP_{k+1}\cdots P_{\mu-1}B\Pi_{\mu-1} - Q_k\bar{Q}_k\Pi_{\mu-1} = (Q_{k*} - \bar{Q}_k)\Pi_{\mu-1} = 0.$$

This makes it evident that all single summands on the right hand sides of formulas (52) and (53) disappear, and thus $\bar{Q}_{j*}\bar{\Pi}_{\mu-1} = 0$ for $j = k,\dots,\mu-1$, that is, the new decoupling coefficients vanish. In consequence, starting with any admissible projectors we apply the above procedure first for $k = \mu-1$, then for $k = \mu-2$ up to $k = 0$. At each level an additional coupling coefficient is cancelled, and we finish with a complete decoupling of the two parts in (49). $\square$

Definition 5.3 *Let $\{E,F\}$ be a pair with characteristic values $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$. If all coefficients $\mathcal{H}_i,\ i = 0,\dots,\mu-1$, vanish in system (49), then the underlying admissible projectors $Q_0,\dots Q_{\mu-1}$ are called completely decoupling projectors for the DAE (33).*

Notice that for DAEs with $\mu = 1$, the completely decoupling projector Q_0 is uniquely determined. It is the projector onto N_0 along $S_0 = \{z \in \mathbb{R}^m : B_0z \in \operatorname{im} G_0\}$ (cf. Appendix 8). However, for higher index $\mu > 1$, there are many complete decouplings, as the next example shows.

Example 5.4 *Let*

$$E = G_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad F = B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and choose a projector Q_0 with a free parameter α.

$$Q_0 = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 1+\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = P_0,$$

$$Q_1 = \begin{bmatrix} 0 & -(1+\alpha) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G_2^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q_0 P_1 G_2^{-1} B_0 = Q_0,$$

i.e., Q_0, Q_1 are completely decoupling projectors for each arbitrary value α.

The completely decoupled system (49) offers as much insight as the Weierstraß-Kronecker canonical form does. Without loosing information it can be compressed back on an m-dimensional DAE. The next lemma records essential properties to be used in the compression procedure.

Lemma 5.5 *It holds that*

$$\begin{aligned} \mathcal{N}_{i,i+1} &= \mathcal{N}_{i,i+1} \Pi_i Q_{i+1}, \\ \mathcal{N}_{ij} &= \mathcal{N}_{ij} \Pi_{j-1} Q_j, \ j = i+2, \ldots, \mu-1, \\ \ker \mathcal{N}_{i,i+1} &= \ker \Pi_i Q_{i+1}, \\ \operatorname{rank} \mathcal{N}_{i,i+1} &= m - r_{i+1}, \quad i = 0, \ldots, \mu-2. \end{aligned}$$

Proof: We use the additional subspaces $S_i := \ker \mathcal{W}_i B_i \subseteq \mathbb{R}^m$ and projectors $\mathcal{W}_i \in L(\mathbb{R}^m)$ with

$$\ker \mathcal{W}_i = \operatorname{im} G_i, \ i = 0, \ldots, \mu-1.$$

Let G_i^- denote the generalized reflexive inverse of G_i with $G_i G_i^- G_i = G_i, \ \ G_i^- G_i G_i^- = G_i^-, \ G_i G_i^- = I - \mathcal{W}_i, \ G_i^- G_i = P_i$. We factorize G_{i+1} as

$$G_{i+1} = G_i + B_i Q_i = G_i + \mathcal{W}_i B_i Q_i + G_i G_i^- B_i Q_i = \mathcal{G}_{i+1} \mathcal{F}_{i+1},$$

$$\mathcal{G}_{i+1} := G_i + \mathcal{W}_i B_i Q_i, \quad \mathcal{F}_{i+1} = I + P_i G_i^- B_i Q_i.$$

Since $\mathcal{F}_{i+1}$ is invertible (cf. Lemma A.2), it follows that $\mathcal{G}_{i+1}$ has rank r_{i+1} like G_{i+1}. Furthermore, it holds that $\ker \mathcal{G}_{i+1} = N_i \cap S_i$. Namely, $\mathcal{G}_{i+1} z = 0$ means that $G_i z = 0, \ \mathcal{W}_i B_i Q_i z = 0$, i.e., $z = Q_i z, \ \mathcal{W}_i B_i z = 0$, but this is $z \in N_i \cap S_i$.
Therefore, $N_i \cap S_i$ must have dimension $m - r_{i+1}$. Next we derive the relation

$$N_i \cap S_i = \operatorname{im} Q_i Q_{i+1}. \tag{54}$$

$z \in N_i \cap S_i$ means $z = Q_i z, \ B_i z = G_i w$, which implies $(G_i + B_i Q_i)(P_i w + Q_i z) = 0$, and hence, $P_i w + Q_i z = Q_{i+1}(P_i w + Q_i z) = Q_{i+1} w$. Therefore, $z = Q_i z = Q_i Q_{i+1} w$. Taking into consideration that $(G_i + B_i Q_i) Q_{i+1} = 0$, we derive from $z = Q_i Q_{i+1} y$ that $z = Q_i z$ and $B_i z = B_i Q_i Q_{i+1} y = -G_i Q_{i+1} y$, i.e. $z \in N_i, \ z \in S_i$. Owing to (54) we have

$$\operatorname{rank} Q_i Q_{i+1} = \dim N_i \cap S_i = m - r_{i+1}. \tag{55}$$

If follows immediately that rank $\mathcal{N}_{i,i+1} = m - r_{i+1}$, and, since $\operatorname{im} P_{i+1} \subseteq \ker \mathcal{N}_{i,i+1}$, rank $P_{i+1} = r_{i+1}$, that $\operatorname{im} P_{i+1} = \ker \mathcal{N}_{i,i+1}$. □

We turn to the compression of the large system (49) on m dimensions. The projector Q_0 has rank $m - r_0$, $\Pi_{i-1}Q_i$ has rank $m - r_i$, $i = 1, \dots, \mu - 1$, and $\Pi_{\mu-1}$ has rank $d := m - \sum_{j=0}^{\mu-1}(m - r_j)$.

We introduce full-row-rank matrices $\Gamma_i \in L(\mathbb{R}^m, \mathbb{R}^{m-r_i})$, $i = 0, \dots, \mu-1$, $\Gamma_d \in L(\mathbb{R}^m, \mathbb{R}^d)$, such that

$$\begin{aligned}
\operatorname{im} \Gamma_d \Pi_{\mu-1} &= \Gamma_d \operatorname{im} \Pi_{\mu-1} &&= \mathbb{R}^d, \\
\ker \Gamma_d &= \operatorname{im}(I - \Pi_{\mu-1}) &&= N_0 + \cdots + N_{\mu-1}, \\
\Gamma_0 N_0 &= \mathbb{R}^{m-r_0}, \\
\ker \Gamma_0 &= \ker Q_0, \\
\Gamma_i \Pi_{i-1} N_i &= \mathbb{R}^{m-r_i}, \\
\ker \Gamma_i &= \ker \Pi_{i-1} Q_i, && i = 1, \dots, \mu - 1,
\end{aligned}$$

as well as generalized inverses Γ_d^-, Γ_i^-, $i = 0, \dots, \mu - 1$, such that

$$\begin{aligned}
&\Gamma_d^- \Gamma_d = \Pi_{\mu-1}, && \Gamma_d \Gamma_d^- = I, \\
&\Gamma_i^- \Gamma_i = \Pi_{i-1} Q_i, && \Gamma_i \Gamma_i^- = I, \qquad i = 1, \dots, \mu - 1, \\
&\Gamma_0^- \Gamma_0 = Q_0, && \Gamma_0 \Gamma_0^- = I.
\end{aligned}$$

If the projectors $Q_0, \dots, Q_{\mu-1}$ are widely orthogonal (cf. Proposition 2.6(6)), then the above projectors are symmetric and Γ_d^-, Γ_i^- are the Moore-Penrose generalized inverses. Denoting

$$\tilde{\mathcal{H}}_i := \Gamma_i \mathcal{H}_i \Gamma_d^-, \qquad \tilde{\mathcal{L}}_i := \Gamma_i \mathcal{L}_i, \ \ i = 0, \dots, \mu - 1, \tag{56}$$

$$\tilde{\mathcal{W}} := \Gamma_d \mathcal{W} \Gamma_d^-, \qquad \tilde{\mathcal{L}}_d := \Gamma_d \mathcal{L}_d, \tag{57}$$

$$\tilde{\mathcal{N}}_{ij} := \Gamma_i \mathcal{N}_{ij} \Gamma_j^-, \ \ j = i+1, \dots, \mu - 1, \ i = 0, \dots, \mu - 2, \tag{58}$$

and transforming the new variables

$$\tilde{u} = \Gamma_d u, \quad \tilde{v}_i = \Gamma_i v_i, \quad i = 0, \dots, \mu - 1, \tag{59}$$

$$u = \Gamma_d^- \tilde{u}, \quad v_i = \Gamma_i^- \tilde{v}_i, \quad i = 0, \dots, \mu - 1, \tag{60}$$

we compress the large system (49) without loosing information into the m-dimensional one

$$\left[\begin{array}{c|cccc}
I & & & & \\
\hline
 & 0 & \tilde{\mathcal{N}}_{01} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\
 & & \ddots & \ddots & \vdots \\
 & & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\
 & & & & 0
\end{array}\right]
\left[\begin{array}{c}
\tilde{u}'(t) \\
\hline
0 \\
\tilde{v}_1'(t) \\
\vdots \\
\tilde{v}_{\mu-1}'(t)
\end{array}\right]
+ \left[\begin{array}{c|cccc}
\tilde{\mathcal{W}} & & & & \\
\hline
\tilde{\mathcal{H}}_0 & I & & & \\
\vdots & & \ddots & & \\
\vdots & & & \ddots & \\
\tilde{\mathcal{H}}_{\mu-1} & & & & I
\end{array}\right]
\left[\begin{array}{c}
\tilde{u}(t) \\
\hline
\tilde{v}_0(t) \\
\vdots \\
\vdots \\
\tilde{v}_{\mu-1}(t)
\end{array}\right]
= \left[\begin{array}{c}
\tilde{\mathcal{L}}_d \\
\tilde{\mathcal{L}}_0 \\
\vdots \\
\vdots \\
\tilde{\mathcal{L}}_{\mu-1}
\end{array}\right] q. \tag{61}$$

As a consequence of Lemma 5.5, the blocks $\tilde{\mathcal{N}}_{i,i+1}$ have full column rank $m - r_{i+1}$ for $i = 0, \dots, \mu - 2$.

Proposition 5.6 *Let the pair $\{E, F\}$, $E, F \in L(\mathbb{R}^m)$ have characteristic values $r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m$.*

(1) *Then there are nonsingular matrices $L, K \in L(\mathbb{R}^m)$ such that*

$$LEK = \left[\begin{array}{c|cccc} I & & & & \\ \hline & 0 & \tilde{\mathcal{N}}_{01} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\ & & & & 0 \end{array}\right], \quad LFK = \left[\begin{array}{c|cccc} \tilde{\mathcal{W}} & & & & \\ \hline \tilde{\mathcal{H}}_0 & I & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \tilde{\mathcal{H}}_{\mu-1} & & & & I \end{array}\right],$$

with entries described by (56)-(58).
Each block $\tilde{\mathcal{N}}_{i,i+1}$ has full column rank $m - r_{i+1}$, $i = 0, \dots, \mu - 2$, and hence the nilpotent part in LEK has index μ.

(2) *By means of completely decoupling projectors, L and K can be built so that the coefficients $\tilde{\mathcal{H}}_0, \dots, \tilde{\mathcal{H}}_{\mu-1}$ disappear, and the DAE transforms into Weierstraß-Kronecker canonical form.*

Proof: Due to the properties

$$\begin{aligned} \mathcal{H}_i &= \mathcal{H}_i \Pi_{\mu-1} = \mathcal{H}_i \Gamma_d^- \Gamma_d, \quad i = 0, \dots, \mu - 1, \\ \mathcal{W} &= \mathcal{W} \Pi_{\mu-1} = \mathcal{W} \Gamma_d^- \Gamma_d, \\ \mathcal{N}_{ij} &= \mathcal{N}_{ij} \Pi_{j-1} Q_j = \mathcal{N}_{ij} \Gamma_j^- \Gamma_j, \ j = 1, \dots, \mu - 1, \ i = 0, \dots, \mu - 2, \end{aligned}$$

we can recover system (49) from (61) by multiplying from the left by

$$\Gamma^- := \left[\begin{array}{c|cccc} \Gamma_d^- & & & & \\ \hline & \Gamma_0^- & & & \\ & & \ddots & & \\ & & & & \Gamma_{\mu-1}^- \end{array}\right] \in L(\mathbb{R}^m, \mathbb{R}^{(\mu+1)m})$$

using transformation (60) and taking into account that $u = \Gamma_d^- \tilde{u} = \Pi_{\mu-1} u$, $\Pi_{\mu-1} u' = u'$. The matrix Γ^- is a generalized inverse of

$$\Gamma := \left[\begin{array}{c|cccc} \Gamma_d & & & & \\ \hline & \Gamma_0 & & & \\ & & \ddots & & \\ & & & & \Gamma_{\mu-1} \end{array}\right] \in L(\mathbb{R}^{(\mu+1)m}, \mathbb{R}^m), \quad \Gamma\Gamma^- = I_m,$$

$$\Gamma^-\Gamma := \left[\begin{array}{c|cccc} \Gamma_d^-\Gamma_d & & & \\ \hline & \Gamma_0^-\Gamma_0 & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1}^-\Gamma_{\mu-1} \end{array}\right] = \left[\begin{array}{c|cccc} \mathit{\Pi}_{\mu-1} & & & & \\ \hline & Q_0 & & & \\ & & \mathit{\Pi}_0 Q_1 & & \\ & & & \ddots & \\ & & & & \mathit{\Pi}_{\mu-2}Q_{\mu-1} \end{array}\right].$$

The product $K := \Gamma \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} = \begin{bmatrix} \boxed{\Gamma_d} \\ \boxed{\Gamma_0} \\ \vdots \\ \boxed{\Gamma_{\mu-1}} \end{bmatrix}$ is nonsingular.

Our decomposition means now that

$$\begin{aligned} x &= \mathit{\Pi}_{\mu-1}x + Q_0 x + \mathit{\Pi}_0 Q_1 x + \cdots + \mathit{\Pi}_{\mu-2}Q_{\mu-1}x \\ &= [I \cdots I]\Gamma^-\Gamma \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} x = [I \cdots I] \begin{bmatrix} u \\ v_0 \\ \vdots \\ v_{\mu-1} \end{bmatrix} \end{aligned}$$

and the transformation (59) reads

$$\begin{bmatrix} \tilde{u} \\ \tilde{v}_0 \\ \vdots \\ \tilde{v}_{\mu-1} \end{bmatrix} = \Gamma \begin{bmatrix} u \\ v_0 \\ \vdots \\ v_{\mu-1} \end{bmatrix} = \Gamma\Gamma^-\Gamma \begin{bmatrix} I \\ \vdots \\ \vdots \\ I \end{bmatrix} x = \Gamma \begin{bmatrix} I \\ \vdots \\ \vdots \\ I \end{bmatrix} x = Kx = \tilde{x}.$$

Thus, turning from the original DAE (33) to the DAE in the form (61) means a coordinate transformation $\tilde{x} = Kx$, with a nonsingular matrix K, combined with a scaling by

$$L := [I \cdots I]\, \Gamma^-\Gamma \begin{bmatrix} \mathit{\Pi}_{\mu-1} & & & & \\ & Q_0 P_1 \cdots P_{\mu-1} & & & \\ & & \ddots & & \\ & & & Q_{\mu-2}P_{\mu-1} & \\ & & & & Q_{\mu-1} \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ \vdots \\ I \end{bmatrix} G_\mu^{-1}.$$

L is a nonsingular matrix. Namely, $LG_\mu z = 0$ means that

$$\mathit{\Pi}_{\mu-1}z + Q_0P_1 \cdots P_{\mu-1}z + \mathit{\Pi}_0 Q_1 P_2 \cdots P_{\mu-1}z + \cdots + \mathit{\Pi}_{\mu-3}Q_{\mu-2}P_{\mu-1}z + \mathit{\Pi}_{\mu-2}Q_{\mu-1}z = 0,$$

and multiplying by $\mathit{\Pi}_{\mu-1}$ yields $\mathit{\Pi}_{\mu-1}z = 0$, multiplying by $Q_{\mu-1}$ yields $Q_{\mu-1}z = 0$, by $Q_{\mu-2}P_{\mu-1}$ yields $Q_{\mu-2}P_{\mu-2}z = 0$, and so on, hence

$$(I - \mathit{\Pi}_{\mu-1})z = Q_{\mu-1}z + Q_{\mu-2}P_{\mu-1}z + \cdots + Q_0P_1 \cdots P_{\mu-1}z = 0.$$

The original DAE (33) and the system (61) are equivalent in the usual sense, which proves the first assertion. Because of the existence of completely decoupling projectors (see Theorem 5.2), the second assertion is an immediate consequence of the first one. □

6 Characterizing matrix pencils by admissible projectors

Each regular pair of $m \times m$ matrices with Kronecker index $\mu \geq 1$ can be transformed into the Weierstraß-Kronecker canonical form (cf. Section 1).

$$\left\{ \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \right\}, \quad J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix},$$

where W is $d \times d$, J is $l \times l$, $d + l = m$, J_i is a nilpotent Jordan block of order k_i, $1 \leq k_i \leq \mu$, $\max\limits_{i=1,\dots,s} k_i = \mu$.
As in Section 1, let l_i denote the number of all Jordan blocks of order $\geq i$, Then, J has $l_\mu \geq 1$ Jordan blocks of order μ, and $l_i - l_{i+1}$ Jordan blocks of order i, $i = 1, \dots, \mu - 1$, $l_1 + \cdots + l_\mu = l$.
In the present section we show how one can get all this structural information as well as the spectrum of $-W$, that is the finite spectrum of the given matrix pencil, by means of the matrix sequence and the admissible projectors without transforming the given pair into Weierstraß-Kronecker canonical form.
Often the given matrix pair might have a large dimension m but a low Kronecker index μ so that just a few steps in the matrix sequence will do.

Theorem 6.1 *For a regular pair $\{E, F\}$ with Kronecker index $\mu \geq 1$, the matrix sequence (11) built with admissible projectors consists of singular matrices $G_0, \dots, G_{\mu-1}$, but G_μ is nonsingular.*

Proof: This is a consequence of the existence of the Weierstraß-Kronecker canonical form (cf. Proposition 1.3), Theorem 3.1 and Proposition 4.1. □

The reverse implication of this assertion is also true. If, for a given pair $\{E, F\}$, in the sequence G_i, $i \geq 0$, built with admissible projectors, there occurs a nonsingular matrix, say G_κ, and κ is the smallest such index, then $\{E, F\}$ is a regular pencil with Kronecker index κ. This was proven in [GM89] for the first time. We will obtain this result in a different way, which, from our point of view, is more transparent.

Theorem 6.2 *If the pair $\{E, F\}$, $E, F \in L(\mathbb{R}^m)$, has characteristic values*

$$r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m,$$

then it is regular with Kronecker index μ.

Proof: Let the pair $\{E, F\}$ have the characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$. By Theorem 5.2 we can choose completely decoupling projectors. Applying the decoupling and compressing procedure for the corresponding DAE (33) we arrive at an equivalent DAE of the form

$$\begin{bmatrix} I & \\ & \tilde{\mathcal{N}} \end{bmatrix} \tilde{x}' + \begin{bmatrix} \tilde{\mathcal{W}} & \\ & I \end{bmatrix} \tilde{x} = \tilde{q}. \tag{62}$$

The matrix $\tilde{\mathcal{N}}$ is nilpotent with index μ, and it has the structure

$$\tilde{\mathcal{N}} = \left[\begin{array}{ccccc} 0 & \tilde{\mathcal{N}}_{01} & \cdots & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\ & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\ & & & & 0 \end{array}\right] \begin{array}{l} \}m-r_0 \\ \\ \\ \}m-r_{\mu-2} \\ \}m-r_{\mu-1} \end{array}, \tag{63}$$

with full-column rank blocks $\tilde{\mathcal{N}}_{i,i+1}, i = 0, \dots, \mu-2$.
It turnes out that $\{E, F\}$ can be transformed into Weierstraß-Kronecker canonical form with Kronecker index μ, and hence $\{E, F\}$ is a regular pair with Kronecker index μ. $\square$

Corollary 6.3 *If $\{E, F\}$, $E, F \in L(\mathbb{R}^m)$, is a pair with characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, then the nilpotent part in its Weierstraß-Kronecker canonical form contains altogether $s = m - r_0$ Jordan blocks, among them $r_i - r_{i-1}$ Jordan chains of order i, $i = 1, \dots, \mu$. It holds that $l_i = m - r_{i-1}$, $i = 1, \dots, \mu$, $d = m - \sum_{j=1}^{\mu}(m - r_{j-1})$.*

Besides the above structural characteristics the matrix sequence provides also the finite spectrum of the matrix pencil as a part of the spectrum of the matrix $\mathcal{W} := \Pi_{\mu-1}G_\mu^{-1}B$.

Theorem 6.4 *Let $\{E, F\}$, E and $F \in L(\mathbb{R}^m)$, be regular with Kronecker index μ, and let the matrix*

$$\mathcal{W} := \Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1} = \Pi_{\mu-1}G_\mu^{-1}B$$

be generated by the matrix sequence (11) with admissible projectors. Then the following holds:

(1) *Each finite eigenvalue of $\{E, F\}$ belongs to the spectrum of $-\mathcal{W}$. More precisely, $(\lambda E + F)z = 0$, $z \neq 0$, implies $u := \Pi_{\mu-1}z \neq 0$, and $(\lambda I + \mathcal{W})u = 0$.*

(2) *If $(\lambda I + \mathcal{W})u = 0$, $\Pi_{\mu-1}u \neq 0$, then λ is a finite eigenvalue of the pair $\{E, F\}$.*

(3) *If $(\lambda I + \mathcal{W})u = 0$, $(I - \Pi_{\mu-1})u \neq 0$, then $\lambda = 0$ must hold. If, additionally, $\Pi_{\mu-1}u \neq 0$, then $\lambda = 0$ is a finite eigenvalue of the pair $\{E, F\}$.*

(4) *$(\lambda I + \mathcal{W})u = 0$, $u \neq 0$, $\lambda \neq 0$, implies $\Pi_{\mu-1}u = u$.*

(5) *If $Q_0, \dots, Q_{\mu-1}$ are completely decoupling projectors, then $\mathcal{W}$ simplifies to*

$$\mathcal{W} = G_\mu^{-1}B\Pi_{\mu-1} = G_\mu^{-1}B_\mu,$$

and $\Pi_{\mu-1}$ is the spectral projector of the matrix pair $\{E, F\}$.

Proof: Applying the decoupling procedure (see Section 5) we rewrite the equation $(\lambda E + F)z = 0$, with

$$z = u + v_0 + \cdots + v_{\mu-1}, \quad u := \Pi_{\mu-1}z,\ v_0 := Q_0 z, \dots, v_{\mu-1} := \Pi_{\mu-2}Q_{\mu-1},$$

as the decoupled system

$$\lambda u + \mathcal{W}u = 0, \tag{64}$$

$$\lambda \begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_{\mu-1} \end{bmatrix} = - \begin{bmatrix} \mathcal{H}_0 \\ \vdots \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} u. \tag{65}$$

Equation (65) leads to the representations

$$\begin{aligned} v_{\mu-1} &= -\mathcal{H}_{\mu-1}u, \\ v_{\mu-2} &= -\mathcal{H}_{\mu-2}u + \lambda\mathcal{N}_{\mu-2,\mu-1}\mathcal{H}_{\mu-1}u, \end{aligned}$$

and so on, showing the linear dependence of u, $v_i = \tilde{\mathcal{H}}_i u,\ i = 0, \ldots, \mu-1$. The property $\mathcal{H}_i = \mathcal{H}_i \Pi_{\mu-1}$ implies $\tilde{\mathcal{H}}_i = \tilde{\mathcal{H}}_i \Pi_{\mu-1}$.
If $z \neq 0$ then $u \neq 0$ must be true, since otherwise $u = 0$ would imply $v_i = 0$, $i = 0, \ldots, \mu-1$, and hence $z = 0$. Consequently, λ turns out to be an eigenvalue of $-\mathcal{W}$ and $u = \Pi_{\mu-1}z$ is the corresponding eigenvector. This proves assertion (1).
To verify (2)-(4) we consider

$$(\lambda I + \mathcal{W})\tilde{u} = 0, \quad \tilde{u} = \Pi_{\mu-1}\tilde{u} + (I - \Pi_{\mu-1})\tilde{u} \neq 0.$$

Because $\mathcal{W}(I - \Pi_{\mu-1}) = 0$ and $(I - \Pi_{\mu-1})\mathcal{W} = 0$, our equation decomposes into the following two:

$$\lambda(I - \Pi_{\mu-1})\tilde{u} = 0, \quad (\lambda I + \mathcal{W})\tilde{u} = 0. \tag{66}$$

Next, if $\Pi_{\mu-1}\tilde{u} \neq 0$, we put $\tilde{v}_i := \tilde{\mathcal{H}}_i\tilde{u} = \tilde{\mathcal{H}}_i\Pi_{\mu-1}\tilde{u},\ i = 0, \ldots, \mu-1$. Then $\tilde{z} := \Pi_{\mu-1}\tilde{u} + \tilde{v}_0 + \cdots + \tilde{v}_{\mu-1}$ is nontrivial, and it satisfies the condition $(\lambda E + F)\tilde{z} = 0$, and so assertion (2) holds true.
Furthermore, if $(I - \Pi_{\mu-1})\tilde{u} \neq 0$, then the first part of (66) yields $\lambda = 0$. Together with (2) this validates (3).
(4) is a simple consequence of (66).
(5) Compute

$$\begin{aligned} G_\mu^{-1}B_\mu - \Pi_{\mu-1}G_\mu^{-1}B_\mu &= (I - \Pi_{\mu-1})G_\mu^{-1}B\Pi_{\mu-1} \\ &= (Q_{\mu-1} + Q_{\mu-2}P_{\mu-1} + \cdots + Q_0P_1\cdots P_{\mu-1})G_\mu^{-1}B\Pi_{\mu-1} \\ &= Q_{\mu-1}\Pi_{\mu-1} + Q_{\mu-2}\Pi_{\mu-1} + \cdots + Q_0\Pi_{\mu-1} = 0. \end{aligned}$$

For the proof that $\Pi_{\mu-1}$ is the spectral projector we refer to [Mär96] □

The matrix $\mathcal{W} = \Pi_{\mu-1}G_\mu^{-1}B = \Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}$ resulting from the projector based decoupling procedure contains the finite spectrum of the pencil $\{E, F\}$. The spectrum of $-\mathcal{W}$ consists of the d finite eigenvalues of the pencil $\{E, F\}$ plus $m - d = l$ zero eigenvalues corresponding to the subspace $\operatorname{im}(I - \Pi_{\mu-1}) \subseteq \ker \mathcal{W}$.
The eigenvectors corresponding to non-zero eigenvalues of $\mathcal{W}$ necessarily belong to the subspace $\operatorname{im}\Pi_{\mu-1}$.

We now have available complete information concerning the structure of the Weierstraß-Kronecker canonical form without computing that form itself. All this information is extracted from the matrix sequence (11).
In particular, using the matrix sequence, the following characteristics of the matrix pair E, F are obtained:

- $d = m - \sum_{j=1}^{\mu}(m - r_{j-1})$, $\quad l = m - d$, $\quad \mu$ - the basic structural sizes and the Kronecker index,
- $r_{i+1} - r_i$ - the number of Jordan blocks with dimension $i+1$ in the nilpotent part,
- $m - r_i$ - the number of Jordan blocks with dimension $\geq i+1$ in the nilpotent part,
- the finite eigenvalues.

There is also an easy regularity criterion provided by the matrix sequence (11).

Proposition 6.5 *The pair $\{E, F\}$, $E, F \in L(\mathbb{R}^m)$, is singular if and only if there is a nontrivial subspace among the intersections*

$$N_i \cap N_{i-1}, \quad \widehat{N_i} = N_i \cap (N_0 + \cdots + N_{i-1}), \quad i \geq 1, \tag{67}$$

Proof: Owing to the basic property (12) and Proposition 2.6, each nontrivial subspace among (67) indicates a singular pencil.
Conversely, let $\{E, F\}$ be singular. Then all matrices G_i must be singular, their nullspaces N_i have dimensions ≥ 1 and the ranks satisfy the inequality

$$r_0 \leq \cdots \leq r_i \leq \cdots\cdots \leq m - 1.$$

There is a maximal rank $r_{max} \leq m - 1$ and an integer κ such that $r_i = r_{max}$ for all $i \geq \kappa$. If all above intersections (67) are trivial, then it follows that

$$N_0 + \cdots + N_i = N_0 \oplus \cdots \oplus N_i, \quad \dim(N_0 \oplus \cdots \oplus N_i) \geq i + 1.$$

However, this contradicts the natural property $N_0 + \cdots + N_i \subseteq \mathbb{R}^m$. $\square$

7 Singular DAEs

As described in Section 2, the concept of admissible projectors and basic matrix sequences applies to general ordered matrix pairs $\{E, F\}$, and we expect the sequence of matrices G_j to become stationary as in Example 2.2. What can we do with this knowledge? Let us have a closer look at some simple special cases.
First we revisit Example 2.4, that is, the DAE

$$\begin{array}{rcrcl} (x_1 + x_2)' & + & x_2 & = & q_1, \\ x_4' & & & = & q_2, \\ & & x_3 & = & q_3, \\ x_3' & & & = & q_4. \end{array} \tag{68}$$

A matrix sequence and admissible projectors for the DAE (68) are given in Section 2 (see Examples 2.2, 2.4). The matrix G_0 already has maximal rank three, and hence the subspaces $\operatorname{im} G_i$ are stationary beginning with $i = 0$, but the sequence itself becomes stationary at level two. The orthoprojector along $\operatorname{im} G_0$ and the projector $\Pi_0 = P_0$ are

$$\mathcal{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Pi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

How should one interpret this DAE which is not regular? There are several different ways. The way which we prefer is the following: Consider the equation picked up by the projector $\mathcal{W}_0$, that is the third equation, as a consistency condition. The remaining system of three equations can then be seen as an explicit ODE for the components marked by the projector Π_0, i.e. for $u := x_1 + x_2$, x_3, and x_4, while x_2 can be considered as an arbitrary continuous function. This means, the DAE (68) is interpreted as having index zero (the level i where the maximal subspace $\operatorname{im} G_i$ is reached first).
Obviously, instead of x_2 we could also see the component x_1 as the free one.

There is considerable space for interpretation. Which variable should be the free one? Which equations should actually represent consistency conditions? Considering the fourth equation of (68) as consistency condition, the remaining system looks like an index one DAE for u, x_3, x_4.
Furthermore, the last two equations of (68) somehow remain an index two problem, which is mirrored by the strangeness index (cf. [KM06]) of (68) having index one .

Consider now the underdetermined DAE

$$\begin{array}{ccccc} x_2' & + & x_1 & = & q_1, \\ x_3' & + & x_2 & = & q_2, \\ x_4' & + & x_3 & = & q_3, \end{array} \tag{69}$$

with

$$G_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \Pi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix G_0 has full row rank, and no equation should be seen as a consistency condition. We treat this DAE as an index zero DAE for the components indicated by the projector Π_0, and we see the variable x_1 to be free.
The matrix sequence becomes stationary at level three.
Observe that, choosing instead x_4 to be the free component, we arrive at an index three DAE for x_1, x_2, x_3.

Finally, take a look at the so-called overdetermined DAE

$$\begin{array}{ccccc} & & x_1 & = & q_1, \\ x_1' & + & x_2 & = & q_2, \\ x_2' & + & x_3 & = & q_3, \\ x_3' & & & = & q_4 \end{array} \tag{70}$$

for which the first matrix G_0 is injective, and thus the matrix sequence is stationary at the beginning. Seeing the first equation in (70) as a consistency condition, the other three equations in (70) can be treated as an index-zero DAE for x_1, x_2, x_3.
On the other hand, considering the last equation to be the consistency condition one arrives at an index three DAE for x_1, x_2, x_3. Note that (70) has strangeness index-three, while the tractability index is zero.
We stress once again the large space for different interpretations.

8 Comments

As we have seen in this chapter, the Weierstraß-Kronecker canonical form of a regular matrix pencil is very helpful for understanding the structure of a linear constant coefficient DAE, and, obviously, DAEs and matrix pencils are closely related.
Ever since Weierstraß and Kronecker ([Wei68, Kro90]) discovered the canonical forms of matrix pencils, and Gantmacher ([Gan53]) pointed out their connection with differential equations, matrix pencils have attracted much interest over and again for many years. There are untold publications on this topic; we only mention a few of them and refer to the sources therein.

A large part of the developments concerning matrix pencils and the accompanying differential equations can be found in the rich literature on control and system theory, where the resulting differential equations are called *singular systems* and *descriptor systems* rather than DAEs (e.g. [Cam80, Dai89, Lew86, Lue77]).
On the other hand, there are important contributions coming from the area of generalized eigenvalue problems and generalized matrix inverses in linear algebra (e.g. [Cam82, Boy80]). In particular, the *Drazin inverse* and spectral projections were applied to obtain expressions for the solution (cf. also [GM86]). However, it seems, that this was a blind alley in the search for a possible treatment of more general DAEs.

About half a century ago, Gantmacher ([Gan53]) and Dolezal [Dol60] first considered models describing linear time-invariant mechanical systems and electrical circuits by linear constant coefficient DAEs. Today, multibody systems and circuit simulation represent the most traditional DAE application fields (e.g. [ESF98, FS90, GF99]). In between, in about 1980, due to unexpected phenomena in numerical computations (e.g. [SEYE81, Pet82]), DAEs (descriptor systems) became an actual and challenging topic in applied mathematics.

Unfortunately, the transformation to Weierstraß-Kronecker canonical form as well as the Drazin inverse approaches do not allow for modifications appropriate to the treatment of time-varying and nonlinear DAEs. A development with great potential for suitable generalizations is given by the derivative array approach due to Campbell ([Cam87]). Following this proposal, we consider, in addition to the given DAE

$$Ex'(t) + Fx(t) = q(t), \tag{71}$$

the extended system

$$\underbrace{\begin{bmatrix} E & 0 & . & . & . & 0 \\ F & E & 0 & . & . & . \\ 0 & F & E & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & F & E \end{bmatrix}}_{\mathcal{E}_\mu} \begin{bmatrix} x'(t) \\ x''(t) \\ . \\ . \\ x^{\mu+1}(t) \end{bmatrix} = - \begin{bmatrix} F \\ 0 \\ . \\ . \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} q(t) \\ q'(t) \\ . \\ . \\ q^{(\mu)(t)} \end{bmatrix}. \tag{72}$$

which results from (71) by differentiating this equation μ times and collecting all these equations. If the $(\mu + 1) \times m$ matrix $\mathcal{E}_\mu$ is 1-full, or in other words, if there exists a nonsingular matrix $\mathcal{R}$ such that

$$\mathcal{R}\mathcal{E}_\mu = \begin{bmatrix} I_m & 0 \\ 0 & \mathcal{K} \end{bmatrix},$$

then an explicit ODE, the *completion ODE*, can be extracted from the derivative array system (72), say

$$x'(t) = \mathcal{C}x(t) + \sum_{j=0}^{\mu} \mathcal{D}_j q^{(j)}(t). \tag{73}$$

The solutions of the DAE (71) are embedded into the solutions of the explicit ODE (73). If $\{E, F\}$ forms a regular matrix pair with Kronecker index μ, then $\mathcal{E}_\mu$ is 1-full (cf. [Cam85]). Conversely, if μ is the smallest index such that $\mathcal{E}_\mu$ is 1-full, then $\{E, F\}$ is regular with Kronecker index μ. In this context, applying our sequence of matrices built using admissible projectors, we find that the 1-fullness of $\mathcal{E}_\mu$ implies that G_μ is nonsingular, and, then using completely decoupling projectors, we obtain a special representation of the scaling matrix $\mathcal{R}$. We demonstrate this just for $\mu = 1, 2$.
Case $\mu = 1$: Let $\mathcal{E}_1$ be 1-full, and consider z with $G_1 z = 0$, i.e. $Ez + FQ_0 z = 0$, and so

$$\begin{bmatrix} E & 0 \\ F & E \end{bmatrix} \begin{bmatrix} Q_0 z \\ z \end{bmatrix} = 0,$$

but then, due to the 1-fullness, it follows that $Q_0 z = 0$. This, in turn, gives $Ez = 0$ and then $z = 0$. Therefore, G_1 is nonsingular. Taking the completely decoupling projector Q_0 such that $Q_0 = Q_0 G_1^{-1} F$ holds true, we obtain

$$\underbrace{\begin{bmatrix} P_0 & Q_0 \\ -P_0 G_1^{-1} F & P_0 \end{bmatrix} \begin{bmatrix} G_1^{-1} & 0 \\ 0 & G_1^{-1} \end{bmatrix}}_{\mathcal{R}} \begin{bmatrix} E & 0 \\ F & E \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & P_0 \end{bmatrix}. \tag{74}$$

Case $\mu = 2$: Let $\mathcal{E}_2$ be 1-full, and consider z with $G_2 z = 0$, i.e. $Ez + FQ_0 z + FP_0 Q_1 z = 0$. Because $(E + FQ_0)Q_1 = G_1 Q_1 = 0$ we find that $E(Q_0 + P_0 Q_1)z = EQ_1 z = -FQ_0 Q_1 z$, and therefore

$$\begin{bmatrix} E & 0 & 0 \\ F & E & 0 \\ 0 & F & E \end{bmatrix} \begin{bmatrix} Q_0 Q_1 z \\ (Q_0 + P_0 Q_1)z \\ z \end{bmatrix} = 0.$$

Now, the 1-fullness of $\mathcal{E}_2$ implies $Q_0 Q_1 z = 0$, but this yields $EP_0 Q_1 = 0$, so that $P_0 Q_1 z = 0$, and therefore $Q_1 z = 0$ and $FQ_0 z + Ez = 0$. Finally, we conclude that $z = Q_1 z = 0$, which means that G_2 is nonsingular. With completely decoupling projectors Q_0, Q_1 we compute

$$\begin{bmatrix} P_0 P_1 & Q_0 P_1 + P_0 Q_1 & Q_0 Q_1 \\ Q_0 P_1 + P_0 Q_1 & Q_0 Q_1 & P_0 P_1 \\ -P_0 P_1 G_2^{-1} F & P_0 P_1 & P_0 Q_1 \end{bmatrix} \begin{bmatrix} G_2^{-1} & 0 & 0 \\ 0 & G_2^{-1} & 0 \\ 0 & 0 & G_2^{-1} \end{bmatrix} =: \mathcal{R},$$

$$\mathcal{R} \begin{bmatrix} E & 0 & 0 \\ F & E & 0 \\ 0 & F & E \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & P_0 P_1 G_2^{-1} F & P_0 P_1 \\ 0 & P_0 & 0 \end{bmatrix}.$$

The resulting completion ODE (cf. (73)) is

$$x'(t) + P_0P_1G_2^{-1}Fx(t) = P_0P_1G_2^{-1}q(t) + (Q_0P_1 + P_0Q_1)G_2^{-1}q'(t) + Q_0Q_1G_2^{-1}q''(t), \quad (75)$$

and it decomposes into the three parts

$$\begin{aligned} P_0P_1x'(t) + P_0P_1G_2^{-1}FP_0P_1x(t) =& P_0P_1G_2^{-1}q(t), \\ P_0Q_1x'(t) =& P_0Q_1G_2^{-1}q'(t), \\ Q_0x'(t) =& P_0Q_1G_2^{-1}q'(t) + Q_0Q_1G_2^{-1}q''(t), \end{aligned}$$

while the decoupling procedure described in Section 5 yields

$$\begin{aligned} (P_0P_1x)'(t) + P_0Q_1G_2^{-1}FP_0P_1x(t) =& P_0P_1G_2^{-1}q(t), \\ P_0Q_1x(t) =& P_0Q_1G_2^{-1}q(t), \\ Q_0x(t) =& P_0Q_1G_2^{-1}q(t) + Q_0Q_1(P_0Q_1G_2^{-1}q)'(t). \end{aligned}$$

A comparison shows consistency but also differences. In order to recover the DAE solutions from the solutions of the explicit ODE (75) one obviously needs consistent initial values. Naturally, more smoothness has to be given when using the derivative array and the completion ODE. Applying derivative array approaches to time-varying linear or nonlinear DAEs one has to ensure the existence of all the higher derivatives occurring when differentiating the original DAE again and again, and in practice one has to provide these derivatives.

The matrix sequence (11) was first introduced in [Mär87]. However, this paper was not accepted for publication since the corresponding referees did not believe that the approach would work for time-varying linear DAEs. Part of the material of [Mär87] is included in [GM89]. The completely decoupling projectors, formerly called *canonical projectors* are provided in [Mär96]. They are applied for Lyapunov type stability criteria e.g. in [Mär94, Mär98].
We stress that, in these earlier papers, the sum spaces $N_0 + ... + N_j$ do not yet play their important role as they do in the present material. The central role of these sum spaces is only pointed out in [Mär04] where linear time-varying DAEs are analyzed. In the same paper, admissible projectors are introduced for regular DAEs. Since we now allow for general rectangular systems, the notion of admissible projectors given here generalizes the previous definition and accepts nontrivial intersections $\widehat{N_i}$ while the demand for trivial intersections $\widehat{N_i}$ is included in the former notion (aiming just for regular DAEs).

Appendix A

Linear Algebra – Basics

In this appendix we collect and complete well-known facts concerning projectors and subspaces of $\mathbb{R}^m$ (Section A), and generalized inverses (Section B).

A Projectors and subspaces

We collect some basic and useful properties of projectors and subspaces.

Definition A.1 (1) *A linear mapping $Q \in L(\mathbb{R}^m)$ is called a* projector*, if $Q^2 = Q$.*

(2) *A projector $Q \in L(\mathbb{R}^m)$ is called a projector* onto *$S \subseteq \mathbb{R}^m$ if $\operatorname{im} Q = S$.*

(3) *A projector $Q \in L(\mathbb{R}^m)$ is called a projector* along *$S \subseteq \mathbb{R}^m$ if $\ker Q = S$.*

(4) *A projector $Q \in L(\mathbb{R}^m)$ is called an* orthogonal *projector if $Q = Q^*$.*

Example: The m-dimensional matrix $Q = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{bmatrix}$ with arbitrary entries for $*$ becomes a projector onto the one-dimensional subspace spanned by the first column of Q along the $(m-1)$-dimensional subspace $\{v : v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, v_1 = 0\}$.

Lemma A.2 *Let P and $\bar{P}$ be projectors, and $Q := I - P$, $\bar{Q} := I - \bar{P}$ the complementary projectors . Then the following properties hold:*

(1) $z \in \operatorname{im} Q \;\Leftrightarrow\; z = Qz$.

(2) *If Q and $\bar{Q}$ project onto the same subspace S, then $\bar{Q} = Q\bar{Q}$ and $Q = \bar{Q}Q$ are valid.*

(3) *If P and $\bar{P}$ project along the same subspace S, then $\bar{P} = \bar{P}P$ and $P = P\bar{P}$ are true.*

(4) *Q projects onto S iff $P := I - Q$ projects along S.*

(5) *Each matrix of the form $I + PZQ$, with arbitrary matrix Z, is nonsingular and its inverse is $I - PZQ$.*

(6) *Each projector P is diagonalizable. Its eigenvalues are 0 and 1. The multiplicity of the eigenvalue 1 is $r = \operatorname{rank} P$.*

Proof:

1. $z = Qy \quad \rightarrow \quad Qz = Q^2 y = Qy = z.$

2. $\bar{Q}z \in \operatorname{im} \bar{Q} = S = \operatorname{im} Q$, also $\bar{Q}z = Q\bar{Q}z \ \forall z$.

3. $\bar{P}P = (I - \bar{Q})(I - Q) = I - \bar{Q} - Q + \bar{Q}Q = I - \bar{Q} = \bar{P}$.

4. $P^2 = P \quad \Leftrightarrow \quad (I - Q)^2 = I - Q \quad \Leftrightarrow \quad -Q + Q^2 = 0 \quad \Leftrightarrow \quad Q^2 = Q$ and $z \in \ker P \ \Leftrightarrow \ Pz = 0 \Leftrightarrow \ z = Qz \ \Leftrightarrow \ z \in \operatorname{im} Q$.

5. Multiplying $(I + PZQ)z = 0$ by $Q \Rightarrow Qz = 0$. Now with $(I + PZQ)z = 0$ follows $z = 0$.
$(I + PZQ)(I - PZQ) = I - PZQ + PZQ = I$.

6. Let $\bar{P}_1$ be a matrix of the r linearly independent columns of P and $\bar{Q}_2$ a matrix of the $m - r$ linearly independent columns of $I - P$. Then by construction $P \begin{bmatrix} \bar{P}_1 & \bar{Q}_2 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix}$. Because of the nonsingularity of $\begin{bmatrix} \bar{P}_1 & \bar{Q}_2 \end{bmatrix}$ we have the structure $P = \begin{bmatrix} \bar{P}_1 & \bar{Q}_2 \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix} \begin{bmatrix} \bar{P}_1 & \bar{Q}_2 \end{bmatrix}^{-1}$. The columns of $\bar{P}_1$ resp. $\bar{Q}_2$ are the eigenvectors to the eigenvalues 1 resp. 0. $\square$

Lemma A.3 *Let $A \in L(\mathbb{R}^n, \mathbb{R}^k)$, $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, $r := \operatorname{rank}(AD)$. Then the following two implications are valid:*

(1) $\ker A \cap \operatorname{im} D = 0$, $\operatorname{im}(AD) = \operatorname{im} A \Rightarrow \ker A \oplus \operatorname{im} D = \mathbb{R}^n$.

(2) $\ker A \oplus \operatorname{im} D = \mathbb{R}^n \Rightarrow$

- $\ker A \cap \operatorname{im} D = \{0\}$,
- $\operatorname{im} AD = \operatorname{im} A$,
- $\ker AD = \ker D$,
- $\operatorname{rank} A = \operatorname{rank} D = r$.

Proof: (1) Because of $\operatorname{im}(AD) = \operatorname{im} A$, the matrix A has rank r and $\ker A$ has dimension $n - r$. Moreover, $\operatorname{rank} D \geq r$ must be true. The direct sum $\ker A \oplus \operatorname{im} D$ is well-defined, and it has dimension $n - r + \operatorname{rank} D \leq n$. This means that D has rank r. We are done with (1).
(2) The first relation is an inherent property of the direct sum. Let $R \in L(\mathbb{R}^n)$ denote the projector onto $\operatorname{im} D$ along $\ker A$. By means of suitable generalized inverses D^- and A^- of D and A we may write (Appendix B) $R = A^-A = DD^-$, $D = RD$, $A = AR$. This leads to

$$\begin{aligned} \operatorname{im} AD &\subseteq \operatorname{im} A = \operatorname{im} ADD^- \subseteq \operatorname{im} AD, \\ \ker AD &\subseteq \ker A^-AD = \ker D \subseteq \ker AD. \end{aligned}$$

The remaining rank property follows now from (1). $\square$

Lemma A.4 *[GvL91, Ch. 12.4.2] Given are matrices G, Π, $\mathcal{N}$, $\mathcal{W}$ of suitable sizes such that*

$$\begin{aligned} \ker G &= \operatorname{im} \mathcal{N}, \\ \ker \Pi\mathcal{N} &= \operatorname{im} \mathcal{W}. \end{aligned}$$

Then it holds that

$$\ker G \cap \ker \mathit{\Pi} = \ker \mathcal{N}\mathcal{W}.$$

Proof: For $x \in \ker G \cap \ker \mathit{\Pi}$ we find $x = \mathcal{N}y, \mathit{\Pi}x = 0$, further $\mathit{\Pi}\mathcal{N}y = 0$, and hence $y = \mathcal{W}z$, $x = \mathcal{N}\mathcal{W}z \in \operatorname{im}\mathcal{N}\mathcal{W}$.
Conversely, each $x = \mathcal{N}\mathcal{W}z$ belongs obviously to $\ker G$, and $\mathit{\Pi}x = \mathit{\Pi}\mathcal{N}\mathcal{W}z = 0$. □

Lemma A.5 *$N, M \subseteq \mathbb{R}^m$ subspaces $\Rightarrow (N+M)^\perp = N^\perp \cap M^\perp$.*

Proof:

$$\begin{aligned} (N+M)^\perp &= \{z \in \mathbb{R}^m : \forall w \in N+M : \langle z, w\rangle = 0\} \\ &= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N + w_M\rangle = 0\} \\ &= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N\rangle = 0,\ \langle z, w_M\rangle = 0\} \\ &= N^\perp \cap M^\perp. \end{aligned}$$

□

Lemma A.6 (1) *Given two subspaces $N, X \subseteq \mathbb{R}^m$, $N \cap X = \{0\}$. Then $\dim N + \dim X \leq m$, and there is a projector $Q \in L(\mathbb{R}^m)$ such that $\operatorname{im} Q = N$, $\ker Q \supseteq X$.*

(2) *Given two subspaces $S, N \subseteq \mathbb{R}^m$. If the decomposition*

$$\mathbb{R}^m = S \oplus N$$

holds true, i.e. S and N are transversal , then there is a uniquely determined projector $P \in L(\mathbb{R}^m)$ such that $\operatorname{im} P = S$, $\ker P = N$.

(3) *An orthoprojector P projects onto $S := \operatorname{im} P$ along $S^\perp = \ker P$.*

(4) *Given the subspaces $K, N \subseteq \mathbb{R}^m$, $\widehat{N} := N \cap K$. If a further subspace $X \subseteq \mathbb{R}^m$ is a complement of $\widehat{N}$ in K, that means $K = \widehat{N} \oplus X$, then there is a projector $Q \in L(\mathbb{R}^m)$ onto N such that*

$$X \subseteq \ker Q. \tag{76}$$

Let d_K, d_N, u denote the dimensions of the subspaces $K, N, \widehat{N}$, respectively, then

$$d_K + d_N \leq m + u \tag{77}$$

holds.

(5) *If the subspace K in (4) is the nullspace of a certain projector $\mathit{\Pi} \in L(\mathbb{R}^m)$, that is $K = \ker \mathit{\Pi} = \operatorname{im}(I - \mathit{\Pi})$, then*

$$\mathit{\Pi} Q (I - \mathit{\Pi}) = 0 \tag{78}$$

becomes true.

(6) *Given are the two projectors* $\Pi, Q \in L(\mathbb{R}^m)$, *further* $P := I - Q$, $N := \operatorname{im} Q$, $K := \ker \Pi$. *Then, supposed* (78) *is valid, the products* ΠP, ΠQ, $P\Pi P$, $P(I-\Pi)$, $Q(I-\Pi)$ *are projectors, too. The relation*

$$\ker \Pi P = \ker P\Pi P = N + K \tag{79}$$

holds true, and the subspace $X := \operatorname{im} P(I-\Pi)$ *is the complement of* $\widehat{N} := N \cap K$ *in* K, *such that* $K = \widehat{N} \oplus X$.
Moreover, the decomposition

$$\mathbb{R}^m = (N+K) \oplus \operatorname{im} P\Pi P = N \oplus \underbrace{X \oplus \operatorname{im} P\Pi P}_{\operatorname{im} P}$$

is valid.

(7) *If the projectors* Π, Q *in (6) are such that* $\Pi^* = \Pi$, $(\Pi P)^* = \Pi P$, $(P(I-\Pi))^* = P(I-\Pi)$ *and* $Q\Pi P = 0$, *then it follows that*

$$X = K \cap \widehat{N}^{\perp}, \ \operatorname{im} P = X \oplus (N+K)^{\perp}.$$

Proof: (1): Let $x_1, \ldots, x_r \in \mathbb{R}^m$ and $n_1, \ldots, n_t \in \mathbb{R}^m$ be basises of X and N. Because of $X \cap N = \{0\}$ the matrix

$$F := [x_1 \ldots x_r n_1 \ldots n_t]$$

has full column rank and $r + t = \dim X + \dim N \leq m$. The matrix F^*F is invertible, and

$$Q := F \begin{array}{c} \begin{bmatrix} 0 & \\ & I \end{bmatrix} \\ r \;\; t \end{array} (F^*F)^{-1}F^*$$

is a projector we looked for. Namely,

$$Q^2 = F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1}F^*F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1}F^* = Q, \quad \operatorname{im} Q = \operatorname{im} F \begin{bmatrix} 0 & \\ & I \end{bmatrix} = N,$$

and $z \in X$ implies that it has to have the structure $z = F \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \begin{matrix} \}r \\ \}t \end{matrix}$, which leads to $Qz = 0$.
(2): For transversal subspaces S and N we apply Assertion (1) with $t = m - r$, i.e. F is square. We have to show that P is unique. Supposed that there are two projectors P, $\bar{P}$ such that $\ker P = \ker \bar{P} = N$, $\operatorname{im} P = \operatorname{im} \bar{P} = S$, we immediately have $P = (\bar{P} + \bar{Q})P = \bar{P}P + \bar{Q}P = \bar{P}P = \bar{P}$.
(3): Let $S := \operatorname{im} P$ and $N := \ker P$. We choose a $v \in N$ and $y \in S$. Lemma A.2 (1) implies $y = Py$, therefore $\langle v, y \rangle = \langle v, Py \rangle = \langle P^*v, y \rangle$. With the symmetry of P we obtain $\langle P^*v, y \rangle = \langle Pv, y \rangle = 0$, i.e. $N = S^{\perp}$.
(4): X has dimension $d_K - u$. Since the sum space $K + N = X \oplus N \subseteq \mathbb{R}^m$ may have at most dimension m, it results that $\dim(K+N) = \dim X + \dim N = d_K - u + d_N \leq m$, and assertion (1) provides Q.
(5): Take an arbitrary $z \in \operatorname{im}(I - \Pi) = K$ and decompose $z = z_{\widehat{N}} + z_X$. It follows that

$\Pi Qz = \Pi Qz_{\widehat{N}} + \Pi \underbrace{Qz_X}_{=0} = \Pi z_{\widehat{N}} = 0$, and hence (78) is true.
(6): (78) means $\Pi Q = \Pi Q \Pi$ and hence

$$\begin{aligned}
\Pi Q \Pi Q &= \Pi QQ = \Pi Q, \\
\Pi P \Pi P &= \Pi(I - Q)\Pi P = \Pi P - \Pi \underbrace{Q \Pi P}_{=0} = \Pi P, \\
(P \Pi P)^2 &= P \Pi P \Pi P = P \Pi P, \\
(P(I - \Pi))^2 &= P(I - \Pi)(I - Q)(I - \Pi) = P(I - \Pi) - P(I - \Pi)Q(I - \Pi) \\
&= P(I - \Pi) + P \underbrace{\Pi Q(I - \Pi)}_{=0}, \\
(Q(I - \Pi))^2 &= Q(I - \Pi) - Q \Pi Q(I - \Pi) = Q(I - \Pi).
\end{aligned}$$

The representation $I - \Pi = Q(I - \Pi) + P(I - \Pi)$ corresponds to the decomposition $K = \widehat{N} \oplus X$.
Next we verify (79). The inclusion $\ker \Pi P \subseteq \ker P \Pi P$ is trivial. On the other side, $P \Pi P z = 0$ implies $\Pi P \Pi P z = 0$ and hence $\Pi P z = 0$, and it follows $\ker \Pi P = \ker P \Pi P$. Now it is evident that $K + N \subseteq \ker \Pi P$. Finally, $\Pi P z = 0$ implies $Pz \in K$, $z = Qz + Pz \in N + K$.
(7): From $Q \Pi P = 0$ and the symmetry of ΠP we know that $P \Pi P = \Pi P$, $\operatorname{im} P \Pi P = (N + K)^\perp$, $\operatorname{im} P = X \oplus (N + K)^\perp$. Next using Lemma A.5, compute $\widehat{N}^\perp = N^\perp + K^\perp$, and further

$$\begin{aligned}
K \cap \widehat{N}^\perp &= K \cap (N^\perp + K^\perp) = \{z \in \mathbb{R}^m : \Pi z = 0, z = z_{N^\perp} + z_{K^\perp}, z_{N^\perp} \in N^\perp, z_{K^\perp} \in K^\perp\} \\
&= \{z \in \mathbb{R}^m : z = (I - \Pi) z_{N^\perp}, z_{N^\perp} \in N^\perp\} = (I - \Pi) N^\perp \\
&= \operatorname{im} (I - \Pi) P^* = \operatorname{im} (P(I - \Pi))^* = \operatorname{im} P(I - \Pi) = X.
\end{aligned}$$

□

Lemma A.7 *Let $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, $M \subseteq \mathbb{R}^m$ be a subspace. $D^+ \in L(\mathbb{R}^n, \mathbb{R}^m)$ be the Moore-Penrose inverse of D. Then,*

(1) $\ker D^* = \operatorname{im} D^\perp$, $\operatorname{im} D = \ker D^{*\perp}$, $\ker D = \ker D^{+*}$, $\operatorname{im} D = \operatorname{im} D^{+*}$.

(2) $\ker D \subseteq M \Rightarrow (DM)^\perp = (\operatorname{im} D)^\perp \oplus D^{+*} M^\perp$.

(3) $\ker D \subseteq M \Rightarrow M^\perp = D^*(DM)^\perp$.

Proof: (1) The first two identities are shown in [BIG03] (Theorem 1, p.12).
If $z \in \ker D = \operatorname{im} I - D^+ D$ with Lemma A.2(1) it is valid that $z = (I - D^+ D)z$ or $D^+ D z = 0$. With (86) it holds $0 = D^+ D z = (D^+ D)^* z = D^* D^{+*} z \Leftrightarrow D^{+*} z = 0$ because of (83) for D^* and we have that $z \in \ker D^{+*}$. We prove $\operatorname{im} D = \operatorname{im} D^{+*}$ analogously.
(2) Let $T \in L(\mathbb{R}^m)$ be the orthoprojector onto M, i.e. $\operatorname{im} T = M$, $\ker T = M^\perp$, $T^* = T$.
$\Rightarrow DM = \operatorname{im} DT$,

$$\begin{aligned}
(DM)^\perp &= (\operatorname{im} DT)^\perp = \ker (DT)^* = \ker T D^* = \{z \in \mathbb{R}^n : D^* z \in M^\perp\} \\
&= \underbrace{\ker D^*}_{= \operatorname{im} D^\perp} \oplus \{v \in \operatorname{im} D : D^* v \in M^\perp\}.
\end{aligned}$$

It remains to show that

$$\{v \in \operatorname{im} D : D^* v \in M^\perp\} = D^{+*} M^\perp.$$

From $v \in \operatorname{im} D = \operatorname{im} DD^+$ we get with Lemma A.2(1) $v = DD^+ v = (DD^+)^* v = D^{+*} D^* v$. Because of $D^* v \in M^\perp$ it holds $v \in D^{+*} M^\perp$. Conversely with Lemma A.2(4), $u \in D^{+*} M^\perp = \operatorname{im} D^{+*}(I - T)$ implies $u \in \operatorname{im} D^{+*} = \operatorname{im} D$, and $\exists w : u = D^{+*}(I - T)w$, $D^* u = D^* D^{+*}(I - T)w = D^+ D(I - T)w$. Since $\operatorname{im}(I - T) = M^\perp \subseteq \ker D^\perp = \ker D^+ D^\perp = \operatorname{im}(D^+ D)^* = \operatorname{im} D^+ D$, it holds that $D^+ D(I - T) = I - T$, hence $D^* u = (I - T)w \in M^\perp$.

(3) This is a consequence of (2), because of

$$D^*(DM)^\perp = D^*[(\operatorname{im} D)^\perp \oplus D^{+*} M^\perp] = D^* D^{+*} M^\perp = D^+ D M^\perp = M^\perp.$$

□

Lemma A.8 *([GM86], AppendixA, Theorem 13)*
Let $A, B \in L(\mathbb{R}^m)$, $\operatorname{rank} A = r < m$, $N := \ker A$, $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\}$. The following statements are equivalent:

(1) *Multiplication by a nonsingular $E \in L(\mathbb{R}^m)$ such that*

$$EA = \begin{bmatrix} \bar{A}_1 \\ 0 \end{bmatrix}, \quad EB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \operatorname{rank} \bar{A}_1 = r,$$

yields a nonsingular $\begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix}$.

(2) $N \cap S = \{0\}$.

(3) *$A + BQ$ is nonsingular for each projector Q onto N.*

(4) $N \oplus S = \mathbb{R}^m$.

(5) *The pair $\{A, B\}$ is regular with Kronecker index one.*

(6) *The pair $\{A, B + AW\}$ is regular with Kronecker index one for each arbitrary $W \in L(\mathbb{R}^m)$.*

Proof: (1) ⇒ (2): With $\bar{N} := \ker \bar{A}_1 = \ker EA = \ker A = N$,

$$\bar{S} := \ker \bar{B}_2 = \{z \in \mathbb{R}^m : EBz \in \operatorname{im} EB\} = S,$$

we have

$$0 = \ker \begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix} = \bar{N} \cap \bar{S} = N \cap S.$$

(2) ⇒ (3): $(A + BQ)z = 0$ implies $BQz = -Az$, that is $Qz \in N \cap S$, thus $Qz = 0$, $Az = 0$, therefore $z = Qz = 0$.

(3) ⇒ (4): Fix any projector $Q \in L(\mathbb{R}^m)$ onto N and introduce $Q_* := Q(A + BQ)^{-1}B$. We show Q_* to be a projector with $\operatorname{im} Q_* = N$, $\ker Q_* = S$ so that the assertion follows. Compute

$$Q_* Q = Q(A + BQ)^{-1} BQ = Q(A + BQ)^{-1}(A + BQ)Q = Q,$$

hence $Q_*^2 = Q_*$, $\operatorname{im} Q_* = N$. Further, $Q_* z = 0$ implies $(A+BQ)^{-1}Bz = (I-Q)(A+BQ)^{-1}Bz$, thus

$$Bz = (A+BQ)(I-Q)(A+BQ)^{-1}Bz = A(A+BQ)^{-1}Bz,$$

that is, $z \in S$. Conversely, $z \in S$ leads to $Bz = Aw$ and

$$Q_* z = Q(A+BQ)^{-1}Bz = Q(A+BQ)^{-1}Aw = Q(A+BQ)^{-1}(A+BQ)(I-Q)w = 0.$$

This proves the relation $\ker Q_* = S$.
(4) $\Rightarrow$ (5): Let Q_* denote the projector onto N along S, $P_* := I - Q_*$. Since $N \cap S = 0$ we know already that $G_* := A + BQ_*$ is nonsingular as well as the representation $Q_* = Q_* G_*^{-1} B$. It holds that

$$\begin{aligned} G_*^{-1}A &= G_*^{-1}(A+BQ_*)P_* = P_*, \\ G_*^{-1}B &= G_*^{-1}BQ_* + G_*^{-1}BP_* = G_*^{-1}(A+BQ_*)Q_* + G_*^{-1}BP_* = Q_* + G_*^{-1}BP_*. \end{aligned}$$

Consider the equation $(\lambda A + B)z = 0$, or the equivalent one $(\lambda G_*^{-1}A + G_*^{-1}B)z = 0$, i.e.

$$(\lambda P_* + G_*^{-1}BP_* + Q_*)z = 0. \tag{80}$$

Multiplying (80) by Q_* and taking into account that $Q_* G_*^{-1} B P_* = Q_* P_* = 0$ we find $Q_* z = 0$, $z = P_* z$. Now (80) writes

$$(\lambda I + G_*^{-1}B)z = 0.$$

If λ does not belong to the spectrum of the matrix $-G_*^{-1}B$, then it follows that $z = 0$. This means, $\lambda A + B$ is nonsingular except for a finite number of values λ, hence the pair $\{A, B\}$ is regular.
Transform $\{A, B\}$ into Weierstraß-Kronecker canonical form (cf. Section 1):

$$\bar{A} := EAF = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \quad \bar{B} := EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}, \quad J^\mu = 0,\ J^{\mu-1} \neq 0.$$

We derive further

$$\begin{aligned} &\bar{N} := \ker \bar{A} = F^{-1}\ker A, \quad \bar{S} := \{z \in \mathbb{R}^m : \bar{B}z \in \operatorname{im} \bar{A}\} = F^{-1}S, \\ &\bar{N} \cap \bar{S} = F^{-1}(N \cap S) = \{0\}, \text{ and} \\ &\bar{N} \cap \bar{S} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^m : z_1 = 0,\ Jz_2 = 0,\ z_2 \in \operatorname{im} J \right\}. \end{aligned}$$

Now it follows that $J = 0$ must be true since otherwise $\bar{N} \cap \bar{S}$ would be nontrivial.
(5) $\Rightarrow$ (1): This follows from $\bar{A} = EAF = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{B} = EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$, $\bar{N} \cap \bar{S} = 0$ and $\bar{N} \cap \bar{S} = F^{-1}(N \cap S) = \{0\}$.
(6) $\Rightarrow$ (5) is trivial.
(2) $\Rightarrow$ (6): Set $\tilde{B} := B + AW, \tilde{S} := \{z \in \mathbb{R}^m : \tilde{B}z \in \operatorname{im} A\} = S$. Because of $\tilde{S} \cap N = S \cap N = \{0\}$, and the equivalence of assertion (2) and (5), which is proved already, the pair $\{A, \tilde{B}\}$ is regular with Kronecker index 1. □

Lemma A.9 *Let $A, B \in L(\mathbb{R}^m)$ be given, A singular, $N := \ker A$, $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\}$, and $N \oplus S = \mathbb{R}^m$. Then the projector Q onto N along S satisfies the relation*

$$Q = Q(A + BQ)^{-1}B. \tag{81}$$

Proof: First we notice that Q is uniquely determined. $A + BQ$ is nonsingular due to Lemma A.8. The arguments used in that lemma apply to show $Q(A+BQ)^{-1}B$ to be the projector onto N along S so that (81) becomes valid. $\square$

For any matrix $A \in L(\mathbb{R}^m)$ there exists an integer k such that

$$\begin{aligned} \mathbb{R}^m &= \operatorname{im} A^0 \supset \operatorname{im} A \supset \ldots \supset \operatorname{im} A^k = \operatorname{im} A^{k+1} = \ldots, \\ \{0\} &= \ker A^0 \subset \ker A \subset \ldots \subset \ker A^k = \ker A^{k+1} = \ldots, \end{aligned}$$

and $\operatorname{im} A^k \oplus \ker A^k = \mathbb{R}^m$. This integer $k \in \mathbb{N} \cup \{0\}$ is said to be the index of A , and we write $k = \operatorname{ind} A$.

Lemma A.10 *([GM86], Appendix A, Theorem 4)*
Let $A \in L(\mathbb{R}^m)$ be given, $k = \operatorname{ind} A$, $r = \operatorname{rank} A^k$, and let $s_1, \ldots, s_r \in \mathbb{R}^m$ and $s_{r+1}, \ldots, s_m \in \mathbb{R}^m$ be basises of $\operatorname{im} A^k$ and $\ker A^k$, respectively. Then, for $S = [s_1 \ldots s_m]$ the product $S^{-1}AS$ has the special structure

$$S^{-1}AS = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

where $M \in L(\mathbb{R}^r)$ is nonsingular and $N \in L(\mathbb{R}^{m-r})$ is nilpotent, $N^k = 0$, $N^{k-1} \neq 0$.

Proof: For $i \leq r$, it holds that $As_i \in A \operatorname{im} A^k = \operatorname{im} A^{k+1} = \operatorname{im} A^k$, therefore $As_i = \sum_{j=1}^{r} s_j m_{ji}$. For $i \geq r+1$, it holds that $As_i \in \ker A^{k+1} = \ker A^k$, thus $As_i = \sum_{j=r+1}^{m} s_j n_{ji}$. This yields the representations $A[s_1 \ldots s_r] = [s_1 \ldots s_r]M$ with $M = (m_{ij})_{i,j=1}^r$, and $A[s_{r+1} \ldots s_m] = [s_{r+1} \ldots s_m]N$, with $N = (n_{ij})_{i,j=r+1}^m$. The block M is nonsingular. Namely, for a $z \in \mathbb{R}^r$ with $Mz = 0$, we have $A[s_1 \ldots s_r]z = 0$, that is,

$$\sum_{j=1}^{r} z_j s_j \in \operatorname{im} A^k \cap \ker A \subseteq \operatorname{im} A^k \cap \ker A^k = \{0\},$$

which shows the matrix M to be nonsingular. It remains to verify the nilpotency of N. We have $AS = S \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$, hence $A^\ell S = S \begin{bmatrix} M^\ell & 0 \\ 0 & N^\ell \end{bmatrix}$. From $A^k s_i = 0$, $i \geq r+1$ it follows that $N^k = 0$ must be valid. It remains to prove the fact that $N^{k-1} \neq 0$. Since $\ker A^{k-1}$ is a proper subspace of $\ker A^k$ there is an index $i_* \geq r+1$ such that the basis element $s_{i_*} \in \ker A^k$ does not belong to $\ker A^{k-1}$. Then, $S \begin{bmatrix} M^{k-1} & 0 \\ 0 & N^{k-1} \end{bmatrix} e_{i_*} = A^{k-1} s_{i_*} \neq 0$, that is, $N^{k-1} \neq 0$. $\square$

B Generalized inverses

In [BIG03] we find a detailed collection of properties of generalized inverses for theory and application. We will here report the definitions and relations of generalized inverses we need for our considerations.

Definition B.1 *For a matrix $Z \in L(\mathbb{R}^n, \mathbb{R}^m)$, we call the matrix $Z^- \in L(\mathbb{R}^m, \mathbb{R}^n)$ a reflexive generalized inverse, if it fulfills*

$$ZZ^-Z = Z \quad \text{and} \tag{82}$$

$$Z^-ZZ^- = Z^-. \tag{83}$$

Z^- is called a $\{1,2\}$-inverse of Z in [BIG03].

The products $ZZ^- \in L(\mathbb{R}^m)$ and $Z^-Z \in L(\mathbb{R}^n)$ are projectors (cf. Appendix A). We have $(ZZ^-)^2 = ZZ^-ZZ^- = ZZ^-$ and $(Z^-Z)^2 = Z^-ZZ^-Z = Z^-Z$. We know that the rank of a product of matrices does not exceed the rank of any factor. Let Z has rank r_z. From (82) we obtain rank $r_z \leq$ rank r_{z^-} and from (83) the opposite, i.e. that both Z and Z^- and also the projectors ZZ^- and Z^-Z have the same rank.
Let $R \in L(\mathbb{R}^n)$ be any projector onto $\operatorname{im} Z$ and $P \in L(\mathbb{R}^m)$ any projector along $\ker Z$.

Lemma B.2 *With (82), (83) and the conditions*

$$Z^-Z = P \quad \text{and} \tag{84}$$

$$ZZ^- = R \tag{85}$$

the reflexive inverse Z^- is uniquely determined.

Proof: Let Y be a further matrix fulfilling (82), (83), (84) and (85).

$$\begin{aligned} Y &\overset{(83)}{=} YZY \overset{(82)}{=} YZZ^-ZY \overset{(85)}{=} YRZY \\ &\overset{(85)}{=} YR \overset{(85)}{=} YZZ^- \overset{(84)}{=} PZ^- \overset{(83)}{=} Z^-. \end{aligned}$$

□

If we choose for the projectors P and R the orthogonal ones the conditions (84) and (85) could be replaced by

$$Z^-Z = (Z^-Z)^*, \tag{86}$$

$$ZZ^- = (ZZ^-)^*. \tag{87}$$

The resulting generalized inverse is called the Moore-Penrose-inverse and denoted by Z^+.

To represent the generalized reflexive inverse Z^- we want to use a decomposition of

$$Z = U \begin{bmatrix} S & \\ & 0 \end{bmatrix} V^{-1}$$

with nonsingular matrices U, V and S. Such a decomposition is e.g. available using an SVD or a Householder decomposition of Z.
A generalized reflexive inverse is given by

$$Z^- = V \begin{bmatrix} S^{-1} & M_2 \\ M_1 & M_1SM_2 \end{bmatrix} U^{-1} \tag{88}$$

with M_1 and M_2 being matrices of free parameters that fulfill

$$P = Z^- Z = V \begin{bmatrix} I & 0 \\ M_1 S & 0 \end{bmatrix} V^{-1}$$

and

$$R = Z Z^- = U \begin{bmatrix} I & S M_2 \\ 0 & 0 \end{bmatrix} U^{-1}$$

(cf. also [Zie79]). There are two ways in looking at the parameter matrices M_1 and M_2. We can compute an arbitrary Z^- with fixed M_1 and M_2. Then also the projectors P and R are fixed by these parameter matrices. Or we provide the projectors P and R, then M_1 and M_2 are given and Z^- is fixed, too.

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