Notes
on linearization of differential-algebraic equations and on optimization with differential-algebraic constraints

Roswitha März

Abstract

With this paper we intend to direct attention to linearization phenomena arising in DAEs and to the correspodence between regularity regions and linearizations. Furthermore, regarding linearization, we formulate a necessary extremal condition for optimization problems with DAE constraint. To ensure indirect optimization methods we develop algebraic criteria for the optimality DAE to be locally regular with index one.

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1 Introduction

Linearization is used throughout analysis, optimization and control - and in particular, if differential algebraic equations (DAEs) are involved. Linearized DAEs play their role as approximations of nonlinear models (e.g. [ESF98, section 1.5]). In many cases papers either apply to linear DAEs, or base their analysis on linear DAEs. Frequently even time-invariant linear DAEs are supposed. Linearized DAEs play also their role in extremal conditions (e.g. [Bac06]), as approximations of nonlinear equations in Gauß-Newton and SQP methods. Linearized DAEs and their adjoints, respectively, are utilized in the sensitivity analysis (e.g. [CLP02]), in the stability analysis (e.g. [Mär98]), and when considering observability (e.g. [Ter97]).

Even though linearization represents a generally important mathematical tool, only few papers (e.g. [Cam95, Mär95]) address general relations between a given nonlinear DAE and its linearizations along trajectories.

In general, one expects a linearized DAE to provide local information on the nonlinear DAE the linearized originates from. However, this can turn out to be an error. Linearizations of DAEs feature quite different phenomena.

With this paper we intent to direct attention to linearization phenomena of square DAEs and to the role of regularity regions. Furthermore, we apply linearization to obtain an extremal condition for an optimization problem with DAE constraints as well as an optimality DAE together with a boundary value problem as the background of indirect optimization. The associated criteria and the optimality DAE are given in terms of the original data.

The background of our analysis is a different view on DAEs. While usually the

obvious and hidden constraints come to the fore, we primarily consider the definition domain of the given DAE to decompose in several (maximal) regularity regions with inherent characteristics such as the index, which are bordered by critical points, see [Mär09, LMT11]. Solutions may remain within such a region but they also may cross the borders or stay there. We show the close correspondence between regularity regions and linearizations along reference functions standing in this regularity region.
We solely address optimization aspects in the context of linearizations. For a more general discussion we refer to [Bac06, KM06]. Nevertheless, the given below transparent algebraic criteria in terms of the original data are to enable an advanced utilization of the modeling latitude for obtaining an optimality DAE that is locally regular with index 1. This way one ensures indirect optimization methods.
The paper is organized as follows. In Section 2 we describe DAEs with properly involved derivatives and their linearizations. In Section 3, underdetermined and regular index-1 DAEs are introduced and a solvability assertion is provided, which plays its role later on when formulating extremal conditions. Section 4 provides a collection of linearization phenomena in several examples. Regularity regions are introduced in Section 5. The technical calculations of matrix function sequences and regularity regions are shifted to the Appendix. In Section 6 we consider an optimization problem comprising a DAE constraint. We provide a necessary extremal condition and algebraic conditions ensuring that the optimality DAE is locally regular with index 1 and its linearization has an inherent Hamiltonian flow. These results are specified for controlled square DAEs in Section 7.

2 Preliminaries

In the present paper we deal with DAEs of the form

$$f(\,(Dx)'(t),\, x(t),\; t) = 0, \tag{1}$$

where $f(y,x,t) \in \mathbb{R}^k,\; D(t) \in L(\mathbb{R}^m, \mathbb{R}^n),\; y \in \mathbb{R}^n,\; x \in \mathcal{D}_f \subseteq \mathbb{R}^m,\; t \in \mathcal{I}_f \subseteq \mathbb{R}$. $\mathcal{D}_f$ is open in $\mathbb{R}^m$ and $\mathcal{I}_f$ is an interval. The function f is supposed to be continuous on the definition domain $\mathbb{R}^n \times \mathcal{D}_f \times \mathcal{J}_f$, together with the partial derivatives $f_y,\; f_x$. The leading Jacobian $f_y(y,x,t)$ is everywhere singular. The matrix function D is continuously differentiable on $\mathcal{I}_f$ and has constant rank r. A solution of (1) is a function given on a certain interval $\mathcal{J} \subseteq \mathcal{J}_f$, with values in $\mathcal{D}_f$, which belongs to the function space

$$\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) := \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\},$$

and satisfies the DAE pointwise on $\mathcal{J}$.
Obviously, all solution values at time t have to belong to the so-called obvious constraint set

$$\mathcal{M}_0(t) = \{x \in \mathcal{D}_f : \exists y \in \mathbb{R}^n : f(y,x,t) = 0\}. \tag{2}$$

The extra matrix function D figuring out the derivative term entails an enhanced DAE model (e.g. [Mär06, Mär09, LMT11]). In contrast to standard form DAEs

$$\mathfrak{f}(x'(t),\, x(t),\, t) = 0, \tag{3}$$

equation (1) precisely indicates in which way derivatives of the unknown function are actually involved. Instead of the commonly used $\mathcal{C}^1$-solutions one can tolerate $\mathcal{C}^1_D$-solutions.
For standard form DAEs (3) there is mostly a singular incidence or projector matrix $D \in L(\mathbb{R}^m)$ such that the identity $\mathfrak{f}(x^1, x, t) \equiv \mathfrak{f}(Dx^1, x, t)$ is valid, and hence the standard DAE (3) can be interpreted as

$$\mathfrak{f}((Dx)'(t), x(t), t) = 0.$$

In particular, any semi-explicit system of m_1 and m_2 equations

$$x_1'(t) + b_1(x_1(t), x_2(t), t) = 0, \tag{4}$$

$$b_2(x_1(t), x_2(t), t) = 0 \tag{5}$$

can be naturally rewritten as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} (\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x)'(t) + \begin{bmatrix} b_1(x_1(t), x_2(t), t) \\ b_2(x_1(t), x_2(t), t) \end{bmatrix} = 0,$$

and also as

$$\begin{bmatrix} I \\ 0 \end{bmatrix} (\begin{bmatrix} I & 0 \end{bmatrix} x)'(t) + \begin{bmatrix} b_1(x_1(t), x_2(t), t) \\ b_2(x_1(t), x_2(t), t) \end{bmatrix} = 0. \tag{6}$$

In case of a semi-explicit DAE, one has good reason to ask for solutions having a continuously differentiable first component x_1, but accept a continuous second component x_2, as it is prescribed by the associated function space

$$\{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : x_1 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^{m_1})\}.$$

Definition 2.1 *The DAE (1) has a properly involved derivative, if there is a projector valued function $K \in \mathcal{C}^1(\mathcal{J}_f, L(\mathbb{R}^n))$ such that*

$$\ker f_y(y, x, t) = \ker K(t), \quad \operatorname{im} D(t) = \operatorname{im} K(t), \quad (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f. \tag{7}$$

The DAE is in full-rank proper form, if $n = r$ and

$$\ker f_y(y, x, t) = \{0\}, \quad \operatorname{im} D(t) = \mathbb{R}^n, \quad (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f. \tag{8}$$

The DAE has a quasi-proper involved derivative, if $r < m$ and there is a projector valued function $K \in \mathcal{C}^1(\mathcal{J}_f, L(\mathbb{R}^n))$ such that

$$\ker f_y(y, x, t) \supseteq \ker K(t), \quad \operatorname{im} D(t) = \operatorname{im} K(t), \quad (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f. \tag{9}$$

The DAE is in full-rank quasi-proper form, if $n = r < m$ and

$$\operatorname{im} D(t) = \mathbb{R}^n, \quad (y, x, t) \in \mathbb{R}^n \times \mathcal{D}_f \times \mathcal{I}_f. \tag{10}$$

In case of a properly involved derivative, the partial Jacobian $f_y(y, x, t)$ has constant rank and its nullspace is independent of y and x. In contrast, a quasi-proper involved derivative permits rank changes in $f_y(y, x, t)$.
The full-rank forms are associated with the projector $K(t) = I$. In most applications at least a full-rank quasi-proper form can be obtained by simple reformulation and interpretation, respectively.

In earlier papers, when dealing with quasi-linear DAEs, instead of the phrase *properly involved derivative* the wording *properly stated leading term* is introduced. We apply both notations also for fully nonlinear DAEs.

For each arbitrary reference function $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, with graph in $\mathcal{D}_f \times \mathcal{I}_f$, the coefficient functions

$$\begin{aligned} A_*(t) &:= f_y((Dx_*)'(t), x_*(t), t), \\ B_*(t) &:= f_x((Dx_*)'(t), x_*(t), t), \quad t \in \mathcal{I}, \end{aligned}$$

of the linear DAE

$$A_*(t)(D(t)x(t))' + B_*(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{11}$$

are continuous and the DAE (11) inherits from the nonlinear DAE the special shape of the leading term. Which further relations between the nonlinear DAE (1) and the linear DAEs (11) can we expect?

Definition 2.2 *The linear DAE (11) is called the linearization of the nonlinear DAE (1) along x_* or along the trajectory of x_*.*

It should be emphasized that the reference function $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ is not necessarily a solution of (1).
If the DAE (1) is autonomous and the reference function is a constant one, then the coefficients A_*, D, B_* are time-invariant and the resulting linear constant coefficient DAE (11) can be treated via the Kronecker canonical form of the matrix pencil $\lambda A_* D + B_*$. However, in general, the original DAE (1) is explicitly time-dependent and the reference function varies in time so that first and foremost time-varying linear DAEs (11) result.

We close this preliminary section by directing attention to the fact that all solutions $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and $\lambda \in \mathcal{C}^1_{A^T}(\mathcal{I}, \mathbb{R}^k)$ of the linear DAE (11) with $q = 0$ and its adjoint

$$-D(t)^T(A_*(t)^T\lambda(t))' + B_*^T(t)\lambda(t) = 0, \quad t \in \mathcal{I},$$

satisfy the (generalized) Lagrange Identity

$$< D(t)x(t),\ A_*(t)^T\lambda(t) >= \text{constant}, \quad t \in \mathcal{I},$$

which plays its role in optimization theory. Namely, we have

$$\begin{aligned} &\frac{d}{dt} < Dx, A_*^T\lambda >=< (Dx)', A_*^T\lambda > + < Dx, (A_*^T\lambda)' > \\ &\quad =< A_*(Dx)', \lambda > + < x, D^T(A_*^T\lambda)' >=< -B_*x, \lambda > + < x, B_*^T\lambda >= 0. \end{aligned}$$

3 Linearizations of index-1 DAEs

Definition 3.1 *Let the DAE (1) have a properly involved derivative. Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be an open connected set. If the rank condition*

$$\operatorname{rank}\{f_y(y,x,t)D(t) + f_x(y,x,t)(I - D(t)^+D(t))\} = k,\ y \in \mathbb{R}^n, (x,t) \in \mathcal{G}, \tag{12}$$

is satisfied, then the DAE (1) is said to be on $\mathcal{G}$ underdetermined with tractability index 1.
If, additionally, $k = m$, then the DAE (1) is said to be on $\mathcal{G}$ regular with tractability index 1.

The semi-explicit DAE (6) (and so system (4),(5)) is regular with index 1, exactly if the partial Jacobian $b_{2,x_2}(x_1, x_2, t)$ is nonsingular. The DAE (6) is underdetermined with index 1, if $b_{2,x_2}(x_1, x_2, t)$ is rectangular with full row-rank.
At this place it should be mentioned that a more general notion of underdetermined index-1 tractability is given in [LMT11].
For general square equations (1) with properly involved derivative, for $k = m$, condition (12) is valid if and only if the local matrix pencil $\lambda f_y(y, x, t)D(t) + f_x(y, x, t)$ is regular with Kronecker index 1 (e.g. [Mär04]).

Proposition 3.2 *Let the DAE (1) be given with properly stated leading term.*

(1) *If the DAE (1) has index 1 on the open connected set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, then each linearization (11) along a function $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ with graph in $\mathcal{G}$ is also an index-1 DAE.*

(2) *If $\mathcal{I} \subseteq \mathcal{I}_f$, $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ has values in $\mathcal{D}_f$ and the linearization (11) is an index-1 DAE, such that*

$$\operatorname{rank}\{A_*(t)D(t) + B_*(t)(I - D(t)^+D(t))\} = k, \ t \in \mathcal{I}, \tag{13}$$

then there is an open connected set $\mathcal{G}$ enclosing the graph of x_* and the DAE (1) has index 1 on $\mathcal{G}$.

Proof: The first part is a direct consequence of the definition and the second one follows from continuity arguments. □
Proposition 3.2 (2) can not be extended to apply also to DAEs with quasi-proper leading term. In Example 4.3 below, the linearization along the zero-solution has index 1, but each neighborhood contains linearizations which have index 4.

Next we add to the DAE the initial condition

$$D(t_0)x(t_0) = z_0 \tag{14}$$

and consider the solvability of initial value problems (IVPs).

Proposition 3.3 *Let the DAE (1) have a properly stated leading term, $t_0 \in \mathcal{I} \subseteq \mathcal{I}_f$ and let $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ have values in $\mathcal{D}_f$. Let the linearization (11) satisfy (13).*

(1) *Then, for each arbitrary $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$, $z_0 \in \operatorname{im} D(t_0)$, the linear IVP (11),(14) possesses at least one solution in $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$.*

(2) *For square DAEs with $k = m$, the IVP (11),(14) is uniquely solvable.*

Proof: Here we drop the argument t of the matrix functions. Denote $Q_0 := I - D^+D$, $G_{*1} := A_*D + B_*Q_0$ and introduce the reflexive generalized inverse

D^- such that $D^-D = I - Q_0$, $DD^- = K$. Since G_{*1} has full row-rank k, G_{*1}^+ is continuous, and $G_{*1}G_{*1}^+ = I$. This allows to rewrite the DAE (11) as

$$\underbrace{A_*DD^-}_{G_{*1}D^-}(Dx)' + \underbrace{B_*Q_0}_{G_{*1}Q_0}x + G_{*1}G_{*1}^+(B_*D^-Dx - q) = 0,$$

therefore

$$G_{*1}[D^-(Dx)' + Q_0x + G_{*1}^+B_*D^-Dx - G_{*1}^+q] = 0,$$

thus

$$D^-(Dx)' + Q_0x + G_{*1}^+B_*D^-Dx - G_{*1}^+q = w, \tag{15}$$

with arbitrary functions w such that $G_{*1}w = 0$. Multiplying by D and Q_0, respectively, we split equation (15) into

$$(Dx)' - K'Dx + DG_{*1}^+B_*D^-Dx - DG_{*1}^+q = Dw, \tag{16}$$

$$Q_0x + Q_0G_{*1}^+B_*D^-Dx - Q_0G_{*1}^+q = Q_0w. \tag{17}$$

Set $w = 0$. For each given $z_0 \in \operatorname{im} D(t_0)$ and continuous q, the resulting IVP (16),(14) is uniquely solvable and provides a continuously differentiable Dx. Having Dx we determine Q_0x by (17). This way we generate a solution $x = D^-Dx + Q_0x$ of the IVP (11),(14), which proves the first assertion. Taking into account that G_{*1} is also injective, if $m = k$, the unique solvability is evident, and so is the second assertion. □

The linear map $\mathfrak{L} : \mathcal{C}_D^1 \to \mathcal{C} \times \operatorname{im} D(t_0)$ being determined by

$$\mathfrak{L}x := (A_*(Dx)' + B_*x,\ D(t_0)x(t_0)),\ x \in \mathcal{C}_D^1,$$

is surjective in Proposition 3.3(1). This property of a rectangular system (1) plays its role if the DAE (1) serves as constraint in an optimization problem (cf. Section 6).

At this place it is worth mentioning that condition (13) allows a linear splitting transformation of the variable x into a state and a control part such that the resulting controlled DAE is regular with index 1 locally around the reference function (cf. [CM07] and the examples in Section 6).

Proposition 3.3(2) is related to square systems and means then the bijectivity of the map $\mathfrak{L}$. In turn, the bijectivity of the map $\mathfrak{L}$, that is, the unique solvability of the linearized problem, is most important when applying indirect optimization methods via the square optimality DAE (cf. Section 6), and also when considering the control of DAEs (cf. Section 7).

Furthermore, also famous linearization results of Perron and Lyapunov concerning asymptotical stability apply to regular index-1 DAEs (1), under certain further conditions and in slightly modified form: if the linearization along a solution $x_* \in \mathcal{C}_D^1[0, \infty)$ is strictly contractive, then the reference solution x_* is asymptotically stable ([LMT11], cf. also [Mär98]).

Altogether, in case of index-1 problems, linearization works as expected: The linearized DAE inherits relevant properties of the nonlinear DAE, and conversely, the linearized DAE provides local information on the nonlinear DAE. Unfortunately, the situation becomes different in a more general setting, as we see in the next section.

4 Linearization: Case studies

In this part we demonstrate by several case studies that linearizations may reflect properties of the original nonlinear DAE, but they may also show astonishing different properties. This is the case for reference functions being solutions of the given nonlinear DAE and also for arbitrary reference functions. In particular, a linearized DAE may

(1) misplace suggestions on stability, observability etc., see Example 4.1,

(2) show a singular flow which emerges from a singular inherent ODE, see Example 4.2/Case 1,

(3) show both lower index and higher index than the original DAE seems to have, see Examples 4.1, 4.2/Case 4, 4.3,

(4) show an index which varies on subintervals, causing serious flow singularities, see Example 4.4,

(5) show somehow harmless index changes, see Examples 4.3, 4.5,

(6) fail to remain regular at all, see Example 4.2/Case 2,

(6) reflect properties of the nonlinear DAE well, see Example 4.2/Case 3.

The regularity regions ([Mär06, Mär09, LMT11]) which we introduce in the next section are to comprehend what is going on. It seems, regularity regions actually constitute a helpfull tool for investigating the linearization phenomena.
In the following examples, we simply quote the regularity regions and their structural characteristics and as may be the case quasi-regularity regions. Later on in Section 5 we explain how those regions are determined.

Example 4.1 *The autonomous DAE [Mär98]*

$$
\begin{aligned}
x_1'(t) - x_2(t) &= 0,\\
x_1(t) - x_2(t)^3 &= 0,\\
x_3'(t) - \alpha x_3(t) &= 0,\\
-x_2(t) + x_3(t) + x_4(t) &= 0,
\end{aligned}
$$

has the form (1) with a properly stated leading term, $m = k = 4$, $r = n = 2$, *and*

$$
D(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad f(y,x,t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} -x_2 \\ x_1 - x_2^3 \\ -\alpha x_3 \\ -x_2 + x_3 + x_4 \end{bmatrix}.
$$

$\alpha < 0$ *is a real parameter,* $\mathcal{D}_f = \mathbb{R}^4$, $\mathcal{I}_f = [0, \infty)$. *The obvious constraint set is time-invariant,*

$$
\mathcal{M}_0 = \{x \in \mathbb{R}^4 : x_1 = x_2^3, x_2 = x_3 + x_4\}.
$$

Besides the obvious zero-solution, the solutions of this DAE are given by the formulae

$$
x_1(t) = (c^2 + \frac{2}{3}t)^{\frac{3}{2}},\ x_2(t) = (c^2 + \frac{2}{3}t)^{\frac{1}{2}},\ x_3(t) = e^{\alpha t} d,\ ,x_4(t) = -e^{\alpha t} d + (c^2 + \frac{2}{3}t)^{\frac{1}{2}},
$$

whereby $c, d \in \mathbb{R}$ denote the arbitrary integration constants. Obviously, all these solutions grow unboundedly if $t \to \infty$.
We take the stationary solution $x_* = 0$ as reference function. The homogeneous version of the linearized along x_* DAE is

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_*} (\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x)'(t) + \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}}_{B_*} x(t) = 0.$$

This linear constant coefficient DAE is regular with Kronecker index 2, and its general solution is given as

$$x_1(t) = x_2(t) = 0, \quad x_4(t) = -x_3(t) = -e^{\alpha t} x_3(0),$$

which shows the asymptotical stability of the linearized DAE and which induces to conjecture stability of the reference solution of the original DAE.
Since all nonzero solutions of the nonlinear DAE grow unboundedly if $t \to \infty$, the zero solution x_* of the nonlinear DAE is far from being stable, as one - by mistake - might conclude from the stability of the linearization.
The definition domain $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^4 \times [0, \infty)$ of the original DAE decomposes into the two maximal regularity regions

$$\mathcal{G}_- = \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f : x_2 < 0\} \quad \text{and}$$
$$\mathcal{G}_+ = \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} : x_2 > 0\},$$

and the border between these regions, the set $x_2 = 0$, consist of somehow critical points. Actually the DAE flow bifurcates at these critical points. Namely, for each $d \in \mathbb{R}$, there are the two solutions

$$\bar{x}(t) = \begin{bmatrix} 0 \\ 0 \\ e^{\alpha t} d \\ -e^{\alpha t} d \end{bmatrix}, \text{ and } \bar{\bar{x}}(t) = \begin{bmatrix} (\frac{2}{3}t)^{\frac{3}{2}} \\ (\frac{2}{3}t)^{\frac{1}{2}} \\ e^{\alpha t} d \\ -e^{\alpha t} d + (\frac{2}{3}t)^{\frac{1}{2}} \end{bmatrix}, \text{ such that } \bar{x}(0) = \bar{\bar{x}}(0) = \begin{bmatrix} 0 \\ 0 \\ d \\ -d \end{bmatrix}.$$

In particular, for $d = 0$, besides the zero solution x_*, there is also a nontrivial solution that satisfies the initial condition $x(0) = 0$.
Notice that our reference function resides on that critical border.
The nonlinear DAE is regular with tractability index 1 only on the two regularity regions, but not on its entire definition domain. If a reference function remains within one of the regularity regions, then the linearization along this function is also regular with index 1.

Example 4.2 *The semi-explicit DAE [LMT11, Mär09]*

$$x_1'(t) - x_3(t) = 0,$$
$$x_2(t)(1 - x_2(t)) - \frac{1}{4} + t^2 = 0,$$
$$x_1(t)x_2(t) + x_3(t)(1 - x_2(t)) - t = 0,$$

with $k = m = 3$, $r = n = 1$, $D(t) = [1\,0\,0]$, and

$$f(y,x,t) = \begin{bmatrix}1\\0\\0\end{bmatrix} y + \begin{bmatrix}-x_3\\ x_2(1-x_2)-\frac{1}{4}+t^2\\ x_1x_2+x_3(1-x_2)-t\end{bmatrix}, \quad y\in\mathbb{R},\ x\in\mathbb{R}^3,\ t\in\mathbb{R},$$

has full-rank proper form. As we will see below, its linearizations

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}1&0&0\end{bmatrix}x(t)\right)' + \begin{bmatrix}0&0&-1\\0&1-2x_{*2}(t)&0\\x_{*2}(t)&x_{*1}(t)-x_{*3}(t)&1-x_{*2}(t)\end{bmatrix}x(t) = q(t),$$

behaves manifestly different for different reference functions x_*.

Case 1: *Letting $x_{*2}(t) = \frac{1}{2}+t$ such that the second equation of the original DAE is satisfied the resulting linear DAE reads*

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\left(\begin{bmatrix}1&0&0\end{bmatrix}x(t)\right)' + \begin{bmatrix}0&0&-1\\0&-2t&0\\\frac{1}{2}+t&x_{*1}(t)-x_{*3}(t)&\frac{1}{2}-t\end{bmatrix}x(t) = q(t),$$

and in detail

$$\begin{aligned} x_1'(t)-x_3(t) =& q_1(t),\\ 2t\,x_2(t) =& q_2(t),\\ (\frac{1}{2}+t)x_1(t)+(x_{*1}(t)-x_{*3}(t))\,x_2(t)+(\frac{1}{2}-t)x_3(t) =& q_3(t). \end{aligned}$$

On all intervals that do not include neither $t=0$ nor $t=\frac{1}{2}$, this linear DAE is regular with (tractability) index 1, and its inherent ODE reads

$$x_1'(t) = -\frac{1+2t}{1-2t}x_1(t)+q_1(t)+\frac{2}{1-2t}q_3(t)-\frac{1}{(1-2t)t}q_2(t)(x_{*1}(t)-x_{*3}(t)).$$

Take a closer look at the homogeneous linear DAE, with identically vanishing right hand sides q. Now a singular homogeneous inherent ODE results, with a singularity at $t=\frac{1}{2}$. The solutions are given by $x_1(t) = (1-2t)e^tc$, $c\in\mathbb{R}$ being arbitrary, and $x_1(\frac{1}{2}) = 0$ is valid for all solutions, which indicates a singular flow at $t=\frac{1}{2}$.
The solutions of the inhomogeneous linearization may grow unboundedly, if t approaches the critical point $t=0$ or $t=\frac{1}{2}$.

Case 2: *Letting $x_{*2}(t) = \frac{1}{2}$ the linear DAE fails to be regular at all. Namely, in*

$$\begin{aligned} x_1'(t)-x_3(t) =& q_1(t),\\ 0 =& q_2(t),\\ \frac{1}{2}x_1(t)+(x_{*1}(t)-x_{*3}(t))\,x_2(t)+\frac{1}{2}x_3(t) =& q_3(t), \end{aligned}$$

the second equation is a consistency condition for q while the solution component x_2 can be fixed arbitrarily.

Case 3: *Letting $x_{*2}(t) = 0$ one arrives at the linearization*

$$\begin{aligned} x_1'(t) - x_3(t) =& q_1(t), \\ x_2(t) =& q_2(t), \\ (x_{*1}(t) - x_{*3}(t))\, x_2(t) + x_3(t) =& q_3(t), \end{aligned}$$

which is a regular DAE with index 1.

Case 4: *Letting $x_{*2}(t) = 1$ the linearization*

$$\begin{aligned} x_1'(t) - x_3(t) =& q_1(t), \\ -x_2(t) =& q_2(t), \\ x_1(t) + (x_{*1}(t) - x_{*3}(t))\, x_2(t) =& q_3(t), \end{aligned}$$

is a regular DAE with index 2.

What is going on here? The given nonlinear DAE seems to have index 1, but this is not correct. More precisely, the domain $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^3 \times \mathbb{R}$ splits into the three maximal regularity regions

$$\begin{aligned} \mathcal{G}_1 &= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ x_2 < \frac{1}{2}\Big\}, \\ \mathcal{G}_2 &= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ \frac{1}{2} < x_2 < 1\Big\}, \\ \mathcal{G}_3 &= \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ 1 < x_2\}, \end{aligned}$$

The DAE is regular with tractability index 1 on each region $\mathcal{G}_\ell$, $\ell = 1, 2, 3$. Through each point $(\bar{x}, \bar{t}) \in \mathcal{G}_\ell$ that satisfies the obvious constraint (the second and third equation of the DAE) passes a locally unique solution. Linearizations along reference functions which remain in one of the regularity regions are also regular with index 1, see Case 3. The border points indicate a critical flow behavior for the original nonlinear DAE as well as a certain degeneration of the linearization, see Cases 1,2,4, and also the figures in [LMT11, Mär09].

Example 4.3 *The DAE*

$$\begin{bmatrix} x_4(t) & 0 & 0 \\ 0 & x_4(t) & 0 \\ 0 & 0 & x_4(t) \\ 0 & 0 & 0 \end{bmatrix} (\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=:D} x)'(t) + x(t) - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma(t) \end{bmatrix} = 0$$

has a full-rank quasi-proper leading term with $m = k = 4$, $n = 3$ and

$$f(y,x,t) = \begin{bmatrix} x_4 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_4 \\ 0 & 0 & 0 \end{bmatrix} y + x - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \gamma(t) \end{bmatrix}, \quad \mathcal{D}_f = \mathbb{R}^4,\ \mathcal{I}_f = \mathbb{R},$$

and a function $\gamma \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$. This DAE is uniquely solvable. Each linearization has the form

$$\underbrace{\begin{bmatrix} x_{*4}(t) & 0 & 0 \\ 0 & x_{*4}(t) & 0 \\ 0 & 0 & x_{*4}(t) \\ 0 & 0 & 0 \end{bmatrix}}_{A_*} (Dx)'(t) + \underbrace{\begin{bmatrix} 1 & 0 & 0 & x_{*2}'(t) \\ 0 & 1 & 0 & x_{*3}'(t) \\ 0 & 0 & 1 & x_{*4}'(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B_*} x(t) = q(t).$$

On intervals where $x_{*4}(t)$ has no zeros, the resulting linearized DAE has index 4, but, on intervals where $x_{*4}(t)$ vanishes identically, the resulting linearized DAE has index 1. In particular, the linearization along the zero function is a regular index-1 DAE.
Supposing the reference function x_* to be a solution of the given nonlinear DAE, one has $x_{*4} = \gamma$. Then the index of the linearized DAE depends on the behavior of γ. Nevertheless, if q is sufficiently smooth, the linearized DAE is uniquely solvable, too.
The given nonlinear DAE has the two regularity regions

$$\begin{aligned} \mathcal{G}_+ &= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} :\ x_4 > 0\}, \\ \mathcal{G}_- &= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} :\ x_4 < 0\}. \end{aligned}$$

The nonlinear DAE is regular with tractability index 4 and characteristic values $r_0 = r_1 = r_2 = r_3 = 3$, $r_4 = 4$ on both regions $\mathcal{G}_+$ and $\mathcal{G}_-$. In contrast to other examples showing a singular flow if the reference function comes to pass a critical point, now there is no such phenomenon, supposed q is smooth enough. This kind of critical points is not recognizable in view of the flow in a smooth setting and therefore said to be harmless.
The nonlinear DAE is also quasi-regular on $\mathbb{R}^4 \times \mathbb{R}$ with $\kappa = 4$, such that all critical points are harmless.

Example 4.4 *Consider the semi-explicit DAE*

$$\begin{aligned} x_1'(t) - x_2(t) + x_3(t) &= 0, \\ x_2'(t) + x_1(t) &= 0, \\ x_1(t)^3 + \alpha(x_1(t))x_3(t) - (\sin t)^3 &= 0, \end{aligned}$$

where

$$\alpha(s) := \begin{cases} s^2 & \text{if } s > 0 \\ 0 & \text{if } s < 0 \end{cases},$$

which yield the linearizations

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_*} (\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{D} x)'(t) + \underbrace{\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ \alpha(x_{*1}(t))x_{*3}(t) + 3x_{*1}^2(t) & 0 & \alpha(x_{*1}(t)) \end{bmatrix}}_{B_*} x(t) = q(t).$$

We choose the periodic solution of the original DAE as reference function $x_*(t) = (\sin t, \cos t, 0)^T$. Then the linearization reads in detail

$$\begin{aligned} x_1'(t) - x_2(t) + x_3(t) &= q_1(t), \\ x_2'(t) + x_1(t) &= q_2(t), \\ 3(\sin t)^2 x_1(t) + \alpha(\sin t)x_3(t) &= q_3(t). \end{aligned}$$

The original DAE has the two maximal stability regions

$$\mathcal{G}_+ = \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_1 > 0\}$$

and

$$\mathcal{G}_- = \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_1 < 0\}.$$

The DAE is regular with index one on $\mathcal{G}_+$, but regular with index two on $\mathcal{G}_-$. The periodic solution x_* shuttles between the index-1 region and the index-2 region. Accordingly, the linearized along this solution DAE has index 1 on the intervals $(2j\pi, (2j+1)\pi)$, but index 2 on intervals $((2j+1)\pi, (2j+2)\pi)$. Observe that also the dynamical degree of freedom changes between two and one such that the flow becomes severely discontinuous.

Example 4.5 *Let the function α be the same as in the previous example, ε be a constant. The DAE*

$$\begin{aligned} x_1'(t) - x_2(t) &= 0, \\ x_2'(t) + x_1(t) &= 0, \\ \alpha(x_1(t))\, x_4'(t) + x_3(t) &= 0, \\ x_4(t) - \varepsilon &= 0, \end{aligned}$$

has a full-rank quasi-proper leading term, with $m = k = 4$, $n = 3$,

$$f(y,x,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha(x_1) \\ 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} -x_2 \\ x_1 \\ x_3 \\ x_4 - \varepsilon \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The linearized DAEs are

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha(x_{*1}(t)) \\ 0 & 0 & 0 \end{bmatrix}}_{A_*} (\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D} x)'(t) + \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \gamma_*(t) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B_*} x(t) = q(t).$$

with $\gamma_*(t) := \alpha_s(x_{*1}(t))x_{*1}'(t)$. The linearization reads in detail

$$\begin{aligned} x_1'(t) - x_2(t) &= q_1(t), \\ x_2'(t) + x_1(t) &= q_2(t), \\ \alpha(x_{*1}(t))\, x_4'(t) + \alpha_s(x_{*1}(t))x_{*1}'(t)\, x_1(t) + x_3(t) &= q_3(t), \\ x_4(t) &= q_4(t). \end{aligned}$$

Again we choose reference functions x_* being solutions of the original DAE, namely the stationary solution $x_*(t) = (0,0,0,\varepsilon)^T$, and the periodic solution $x_*(t) = (sint, cost, 0, \varepsilon)^T$. In the first case the resulting linearized DAE is regular with index 1. In the second case, the linearized DAE has in turn index 2 and index 1 on the intervals $(0,\pi)$, $(\pi, 2\pi)$, and so on.
The nonlinear DAE has the two regularity regions

$$\begin{aligned} \mathcal{G}_+ &= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} : x_1 > 0\}, \\ \mathcal{G}_- &= \{(x,t) \in \mathbb{R}^4 \times \mathbb{R} : x_1 < 0\}, \end{aligned}$$

The DAE is regular with tractability index 2 and $r_0 = r_1 = 3$, $r_3 = 4$ on $\mathcal{G}_+$ and regular with index 1, $r_0 = 2$, $r_1 = 4$ on $\mathcal{G}_-$. The periodic solution shuttles between these regularity regions and changes accordingly the index of the linearization.
The DAE is quasi-regular on $\mathbb{R}^4 \times \mathbb{R}$, e.g. with $\kappa = 2$ and all critical points are harmless.

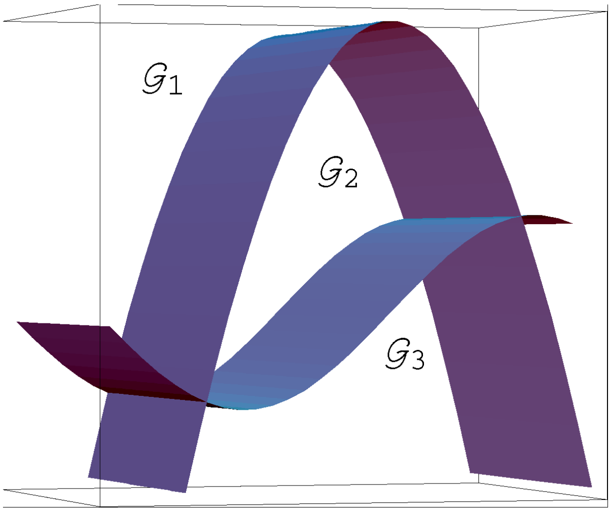

Figure 1: Regularity regions

5 Linearizations and regularity regions

Regarding our examples we do not expect a general DAE (1) to show uniform structure on its entire domain $\mathcal{D}_f \times \mathcal{I}_f$. As suggested by the case studies, and as sketched in Figure 1, it is rather natural that the domain $\mathcal{D}_f \times \mathcal{I}_f$ decomposes into several maximal regularity regions bordered by critical points. It may well happen that the structural characteristics of the DAE are different on different regularity regions. However, in each regularity region there must be uniform structural characteristics. A solution of the DAE can enter a regularity region with new characteristic values only after passing a critical point. Now it is time to explain how a regularity region is actually determined.

Turn for a moment to the matrix pencil $\lambda G + B$ given by the $m \times m$ matrices G and B. We construct a sequence of matrices by special smart projectors starting with $G_0 := G$, $B_0 := B$, $P_0 := I - Q_0$, and Q_0 being a projector matrix onto $N_0 = \ker G_0$. One can simply set $Q_0 = I - G^+ G$. Then, for $i \geq 1$, we put

$$G_i := G_{i-1} + B_{i-1} Q_{i-1}, \tag{18}$$

$$B_i := B_{i-1} P_{i-1}. \tag{19}$$

$$\text{choose a nontrivial subspace} \quad N_i \subseteq \ker G_i, \tag{20}$$

$$\text{choose } Q_i \text{ to be a projector onto } N_i, \; P_i := I - Q_i.$$

Denote further $r_i := \operatorname{rank} G_i$ and $\widehat{N_i} := N_i \cap (N_0 + \cdots + N_{i-1})$. We choose the projectors $Q_0, \ldots, Q_i$ in such a way that

$$\ker Q_i \supseteq X_i, \quad \operatorname{im} Q_i = N_i, \tag{21}$$

$$\text{with a complement } X_i \subseteq N_0 + \cdots + N_{i-1}, \quad X_i \oplus \widehat{N_i} = N_0 + \cdots + N_{i-1}.$$

Theorem 5.1 *[GM89, LMT11, Mär04] The following assertions are equivalent:*

(1) *The matrix pencil* $\lambda G + B$ *is regular with Kronecker index* μ.

(2) *Each matrix sequence (18)- (21), built in such a way that* $N_i = \ker G_i$ *is valid at each level, shows* $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$.

(3) *Each matrix sequence (18)- (21) shows* $r_0 \le \cdots \le r_{\kappa-1} < r_\kappa = m$ *with a certain number* $\kappa \ge \mu$.

Owing to the maximal choice $N_i = \ker G_i$, the numbers $r_0, \ldots, r_\mu$ in Theorem 5.1(2), characterize the detailed structure of the Weierstraß-Kronecker canonical form of the regular matrix pencil. In contrast, these numbers loose the structural meaning, if only subspaces $N_i \subset \ker G_i$ are used instead of the nullspaces itself. From this point of view the choice $N_i = \ker G_i$ is most beneficial.

Return to the DAE (1). We introduce the basic matrix functions

$$A(x^1, x, t) := f_y(D(t)x^1 + D_t(t)x,\, x, t),$$
$$B(x^1, x, t) := f_x(D(t)x^1 + D_t(t)x,\, x, t), \quad x^1 \in \mathbb{R}^m,\, x \in \mathcal{D}_f,\, t \in \mathcal{I}_f,$$

and form pointwise a sequence of continuous matrix functions analogously to (18)-(21). For clarity we drop the arguments of the matrix functions. We start by

$$G_0 := AD,\; B_0 := B,$$
$$\text{choose a projector function } Q_0 \text{ onto } N_0 := \ker D,\; P_0 := I - Q_0,\; \Pi_0 := P_0.$$

Then, for $i \ge 1$, as long as the expressions exist, we put

$$G_i := G_{i-1} + B_{i-1}Q_{i-1}, \tag{22}$$
$$\text{choose a nontrivial } \mathcal{C}\text{-subspace } N_i \subseteq \ker G_i, \tag{23}$$
$$r_i := \operatorname{rank} G_i, \quad \widehat{N_i} := N_i \cap (N_0 + \cdots + N_{i-1}),$$
$$\text{choose a projector function } Q_i \text{ such that}$$
$$\operatorname{im} Q_i = N_i, \quad \ker Q_i \supseteq X_i, \tag{24}$$
$$\text{with a complement } X_i \subseteq N_0 + \cdots + N_{i-1}, \quad X_i \oplus \widehat{N_i} = N_0 + \cdots + N_{i-1},$$
$$P_i := I - Q_i, \quad \Pi_i := \Pi_{i-1}P_i,$$
$$B_i := B_{i-1}P_{i-1} - G_i D^-(D\Pi_i D^-)' D\Pi_{i-1}. \tag{25}$$

We refer to the Appendix for examples of those matrix function sequences.
The expression $(D\Pi_i D^-)'$ in formula (25) means the total derivative in jet variables, see [LMT11, Mär09]. Formally, $D\Pi_i D^-$ may depend on $t, x, x^1, \ldots, x^i$, and then $(D\Pi_i D^-)'$ depends on $t, x, x^1, \ldots, x^i$ and x^{i+1}. The matrix function sequence (22)-(25) clearly generalizes the matrix sequence (18)-(21). The nonlinearity and time-dependence is now encoded in the new extra term in (25).
To have at each level a continuous matrix function G_i, we suppose the projector functions $Q_0, \ldots, Q_{i-1}$ to be *admissible* also in the sense, that Π_{i-1} is supposed to be continuous and $D\Pi_{i-1}D^-$ to be continuously differentiable.
By definition, a $\mathcal{C}$-subspace in $\mathbb{R}^m$ is such that the orthoprojector function onto this subpace is continuous. Any $\mathcal{C}$-subspace has constant dimension, and hence

the construction with $N_i = \ker G_i$ requires constant-rank matrix functions G_i. Mind at this point that, in the linear time-invariant case, the rank values r_i determine the structure of the Weierstraß-Kronecker canonical form if at each level $N_i = \ker G_i$. Here, in the context of general nonlinear DAEs we can benefit from the associated constant-rank conditions to describe regularity regions and to detect critical points.

Definition 5.2 *Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be open connected, further $m = k$.*

(1) *The DAE (1) with proper leading term is said to be regular on $\mathcal{G}$, if there is a number $\mu \in \mathbb{N}$, such that on $\mathcal{G}$ a matrix function sequence (22)-(25) can be formed up to level μ, with $N_i = \ker G_i$, $i = 0, \ldots, \mu - 1$, and $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.*
$\mathcal{G}$ is then named a regularity region.
The number μ is named tractability index, and the constant rank values $r_0, \ldots, r_\mu$ are said to be characteristic values of the DAE on $\mathcal{G}$.

(2) *A point $(\bar{x}, \bar{t}) \in \mathcal{D}_f \times \mathcal{I}_f$ is a regular point, if there is a neighborhood being a regularity region, and a critical point otherwise.*

(3) *The DAE (1) with quasi-proper leading term is said to be regular on $\mathcal{G}$, if it has a proper reformulation being regular on $\mathcal{G}$.*

(4) *The DAE (1) with quasi-proper leading term is said to be quasi-regular on $\mathcal{G}$, if there is a number $\kappa \in \mathbb{N}$, such that a matrix function sequence can be formed on $\mathcal{G}$ up to level κ, and G_κ is nonsingular.*
The set $\mathcal{G}$ is then called a quasi-regularity region.
A point $(\bar{x}, \bar{t}) \in \mathcal{D}_f \times \mathcal{I}_f$ is a quasi-regular point, if there is a neighborhood being a quasi-regularity region.

Roughly speaking, critical points are those points where the constant rank conditions supporting the matrix functions with N_i being the nullspace of G_i at each level fail to be valid. Critical points can cause serious flow singularities as in Examples 4.1, 4.2 and 4.4. A critical point is somehow harmless, if it is at the same time quasi-regular, as in Examples 4.3 and 4.5.

The following properties justify our regularity notion and allow a deeper comprehension at the same time. We underline once more, neither the existence of solutions nor any knowledge concerning the constraints are presupposed to determine regularity regions.

Properties of regularity regions:

(a) If $\mathcal{G}$ is a regularity region of the DAE (1), with characteristics $r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m$, then each open connected subset $\tilde{\mathcal{G}} \subset \mathcal{G}$ is a regularity region, too, and it has the same characteristics.

(b) A regularity region consists of regular points with uniform characteristics.

(c) The union of intersecting regularity regions is again a regularity region.

(d) Regularity regions, regular and critical points are unchanged, if one turns from the original DAE (1) to its perturbed version

$$f(\,(Dx)'(t)\,, x(t),\, t) = q(t). \tag{26}$$

(e) Regularity, in particular the characteristics $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, are invariant with respect to coordinate changes, to refactorizations of the leading term as well as to the special choice of the admissible projector functions Q_i, see [LMT11].

Theorem 5.3 (Linearization Theorem) *Let the DAE (1) have a proper leading term. Let $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$ be open connected. Then the following three assertions are equivalent:*

(1) *$\mathcal{G}$ is a regularity region of the DAE (1).*

(2) *Each linearization (11) of the DAE (1) along a sufficiently smooth function x_* with graph in $\mathcal{G}$ is a regular linear DAE.*

(3) *All linearizations (11) of the DAE (1) along sufficiently smooth functions x_* with graph in $\mathcal{G}$ are regular with uniform characteristics.*

Proof: The implications (1) $\rightarrow$ (3) $\rightarrow$ (2) are due to the construction. The implication (2) $\rightarrow$ (1) is proved in [LMT11] by means of so-called widely orthogonal projector functions. □

The next assertion follows from the construction.

Theorem 5.4 *If the DAE (1) is quasi-regular on the open connected set $\mathcal{G} \subseteq \mathcal{D}_f \times \mathcal{I}_f$, then each linearization (11) along a sufficiently smooth function x_* with values in $\mathcal{G}$ is also quasi-regular.*

Of course, each open connected subset of a quasi-regularity region is again a quasi-regularity region, and the points of a quasi-regularity region are quasi-regular points. However, as it is mentioned above for matrix pencils, due to the considerable arbitrariness of the subspaces $N_i \subseteq \ker G_i$, in general, the ranks of the G_i do not inherit a structural meaning, except for the maximal choice $N_i = \ker G_i$ at every level. Furthermore, now the functions G_i have variable ranks, such there are no counterparts of the meaningful characteristic values of the regularity regions. As described in Example 4.5, a quasi-regularity region may include several regularity regions having different characteristics.

6 Optimality condition

Consider the cost

$$J(x) = \int_{t_0}^{t_f} h(x(t), t)dt + g(D(t_f)x(t_f)) \tag{27}$$

to be minimized on functions $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, $\mathcal{I} = [t_0, t_f]$, subject to the constraints

$$f((Dx)'(t), x(t), t) = 0, \quad t \in \mathcal{I}, \tag{28}$$

$$D(t_0)x(t_0) = z_0 \in \mathbb{R}^n. \tag{29}$$

Let the DAE (28) have a full-rank proper leading term ($r = n$, see (8)) and let it satisfy the basic assumptions in Section 2. In particular, the DAE (28) comprises $k \leq m$ equations, usually $k < m$. Additionally, let the real functions $h(x,t)$ and $g(\eta)$ depend continuously differentiable on their arguments. Later on, in Theorem 6.3 we suppose also continuous partial derivatives h_{xx} and f_{xx}.

Theorem 6.1 (Necessary optimality condition) *Let $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ be a local solution of the optimization problem (27),(28),(29).*
Let, for each arbitrary $z_0 \in \operatorname{im} D(t_0)$, $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$, the linearized along x_* DAE

$$A_*(t)(Dx)'(t) + B_*(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{30}$$

have a solution in $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, which satisfies the initial condition (29).
Then the terminal value problem

$$-D(t)^T(A_*^T\lambda)'(t) + B_*(t)^T\lambda(t) = h_x(x_*(t), t)^T, \quad t \in \mathcal{I}, \tag{31}$$

$$D(t_f)^T A_*(t_f)^T \lambda(t_f) = (g_\eta(D(t_f)x_*(t_f))D(t_f))^T \tag{32}$$

posses a solution $\lambda_* \in \mathcal{C}^1_{A_*^T}(\mathcal{I}, \mathbb{R}^k)$.

Proof: For the case of quasilinear DAEs with $f(y,x,t) = A(x,t)y + b(x,t)$ the assertion is proved in [Bac06, pages 121-139] by applying the famous Lyusternik-Theorem [Lyu34], providing a representation of functionals on $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and then a representation of the Lagrange multiplier. The same arguments apply also in the slightly more general case discussed now. □

The required solvability concerning the linearized IVP (30),(29) is guaranteed by Proposition 3.3(1). Actually, this means the surjectivity of the linear operator

$$\mathfrak{L}x := (A_*(Dx)' + B_*x,\ D(t_0)x(t_0)) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k) \times \operatorname{im} D(t_0),\ x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m).$$

In Proposition 3.3(1), the full row-rank condition (cf. (13))

$$\operatorname{rank}[A_*(t)D(t) + B_*(t)(I - D(t)^+D(t))] = k, \quad t \in \mathcal{I}, \tag{33}$$

plays its role. The operator $\mathfrak{L}$ is surjective, exactly if condition (33) is valid. In turn, the surjectivity (closed range property) of the linear operator $\mathfrak{L}$ plays an essential part in the Lyusternik-Theorem.
Condition (33) is necessary for the existence of a solution of the terminal value problem (31),(32), as it is demonstrated in [Bac06, pages 50-52].
If the above full-rank condition is not given in a problem, then it might be a good idea to reformulate or reduce the problem so that the reduced DAE meets the condition.
An other way consists in exploiting given special structural properties with the aim to obtain surjectivity of the operator $\mathfrak{L}$ in specially adjusted function spaces, for instance, in case of controlled Hessenberg size-2 DAEs, cf. [Ger06b, Ger06a, Cal06]. Note that different function spaces may lead to different representations of the Lagrange multiplier, and hence yield another terminal value problem than (31),(32).

Corollary 6.2 *Let the DAE (28) be underdetermined index-1 tractable (Definition 3.1) such that condition (12) is valid.*
Then, if $x_* \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ is a local solution of the optimization problem (27),(28),(29), the terminal value problem (31),(32) is solvable on $\mathcal{C}^1_{A_*^T}(\mathcal{I}, \mathbb{R}^k)$.

Proof: This is a direct consequence of Theorem 6.1.□

Indirect optimization methods rely on the boundary value problem (BVP) for the composed so-called *optimality DAE*

$$f((D(t)x(t))', x(t), t) = 0, \tag{34}$$

$$-D(t)^T(f_y((D(t)x(t))', x(t), t)^T\lambda(t))' + f_x((D(t)x(t))', x(t), t)^T\lambda(t) = h_x(x(t), t)^T \tag{35}$$

completed by the boundary conditions (29) and (32). Owing to Theorem 6.1 this BVP is solvable. By introducing the new function $y = (Dx)'$ and collecting the components λ, x, y in $\tilde{x}$, the DAE (34),(35) can be put into the more prevalent form

$$\tilde{f}((\tilde{d}(\tilde{x}(t), t))', \tilde{x}(t), t) = 0,$$

with properly involved derivative and nonlinear derivative-term. Those equations are investigated in [LMT11]. Here we restrict our interest to the transparent quasi-linear case

$$f(y, x, t) = A(t)y + b(x, t), \tag{36}$$

which naturally comprises the semi-explicit systems (4),(5). For (36), the optimality DAE simplifies to

$$A(t)(Dx)'(t) + b(x(t), t) = 0, \tag{37}$$

$$-D(t)^T(A^T\lambda)'(t) + b_x(x(t), t)^T\lambda(t) = h_x(x(t), t)^T. \tag{38}$$

The optimality DAE combines $k+m$ equations for the same number of unknown functions. In view of a reliable practical treatment, when applying an indirect optimization method, it would be an great advantage to know whether the DAE is regular with index 1. For this aims we consider the linearization of the DAE (37),(38) along (λ_*, x_*), namely

$$\begin{bmatrix} A(t) & 0 \\ 0 & D(t)^T \end{bmatrix} \left(\begin{bmatrix} 0 & D(t) \\ -A(t)^T & 0 \end{bmatrix} \begin{bmatrix} \lambda(t) \\ x(t) \end{bmatrix} \right)' + \begin{bmatrix} 0 & B_*(t) \\ B_*(t)^T & -H_*(t) \end{bmatrix} \begin{bmatrix} \lambda(t) \\ x(t) \end{bmatrix} = 0. \tag{39}$$

with the continuous symmetric matrix function

$$H_*(t) := h_{xx}(x_*(t), t) - (b_x^T(x, t)\lambda_*(t))_x(x_*(t), t). \tag{40}$$

Theorem 6.3 (Properties of the optimality DAE) *Let the DAE (28) have the special form given by (36). Let the functions b and h have the additional second continuous partial derivatives b_{xx}, h_{xx}. Let $x_* \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ be a local solution of the optimization problem (27),(28),(29). Denote $B_*(t) = b_x(x_*(t), t)$, $G(t) = A(t)D(t)$, $Q_0(t) = I - D(t)^+D(t)$, $W_0(t) = I - A(t)A(t)^+$, $t \in \mathcal{I}$.*
Let the condition (33) be satisfied, that is

$$\operatorname{rank}[A(t)D(t) + B_*(t)Q_0(t)] = k, \quad t \in \mathcal{I}. \tag{41}$$

Let λ_* denote the solution of the terminal value problem (31),(32).

(1) *Then the optimality DAE (37),(38) is regular with index 1 in a neighborhood of the graph of* (λ_*, x_*), *exactly if*

$$\begin{aligned}(G(t) + W_0(t)B_*(t)Q_0(t))z = 0,&\\ H_*(t)Q_0(t)z \in \ker(G(t) + W_0(t)B_*(t)Q_0(t))^{\perp}&\\ \textit{imply} \quad z = 0, \quad \textit{for all} \quad t \in \mathcal{I}.&\end{aligned} \tag{42}$$

(2) *If condition (42) is given, then the linearized DAE (39) is selfadjoint and its inherent regular ODE has Hamiltonian structure such that*

$$\Theta' = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \mathcal{E}\,\Theta, \quad \Theta := \begin{bmatrix} Dx \\ -A^T\lambda \end{bmatrix}, \tag{43}$$

with a symmetric continuous matrix function $\mathcal{E}$ *of size* $2n \times 2n$.

(3) *If* $Q_0(t)H_*(t)Q_0(t)$ *is semi-definite for all* $t \in \mathcal{I}$, *then condition (42) simplifies to the full-rank condition*

$$\operatorname{rank} \begin{bmatrix} G(t) + W_0(t)B_*(t)Q_0(t) \\ Q_0(t)H_*(t)Q_0(t) \end{bmatrix} = m, \quad t \in \mathcal{I}. \tag{44}$$

Proof: Here we drop the argument t. The proper formulation of the leading term yields $\ker G = \ker D$ and $\operatorname{im} G = \operatorname{im} A$, therefore $Q_0 = I - D^+D = I - G^+G$, $W_0 = I - GG^+ = I - AA^+$. Introduce the $(m+k) \times (k+m)$ matrix function

$$\hat{G}_{*1} = \begin{bmatrix} 0 & G + B_*Q_0 \\ -G^T + B_*^T W_0 & -H_*Q_0 \end{bmatrix}.$$

The optimality DAE (37),(38) is regular with index 1 around the graph of (x_*, λ_*), exactly if $\hat{G}_{*1}$ is nonsingular on $\mathcal{I}$ (cf. (39) and Proposition 3.2). Compute the relations

$$\begin{aligned} G + B_*Q_0 &= (G + W_0B_*Q_0)(I + G^+B_*Q_0), \\ -G^T + B_*^T W_0 &= (-G^T + Q_0B_*^T W_0)(I - G^{+T}B_*^T W_0), \\ \operatorname{im}(G + W_0B_*Q_0) &= \operatorname{im} G \oplus \operatorname{im} W_0B_*Q_0, \\ \operatorname{im}(-G^T + Q_0B_*^T W_0) &= \operatorname{im} G^T \oplus \operatorname{im} Q_0B_*^T W_0 = \operatorname{im}(G^T + Q_0B_*^T W_0). \end{aligned}$$

From condition (41) it follows now that $\operatorname{rank}(-G^T + B_*^T W_0) = \operatorname{rank}(G + B_*Q_0) = k$, and hence $\ker(-G^T + B_*^T W_0) = \{0\}$.
The matrix function $\hat{G}_{*1}$ is nonsingular if, for $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^k$, the system

$$(G + B_*Q_0)v = 0, \tag{45}$$
$$-H_*Q_0v + (-G^T + B_*^T W_0)w = 0 \tag{46}$$

has only the trivial solution. Since $-G^T + B_*^T W_0$ has full column-rank and

$$\operatorname{im}(-G^T + B_*^T W_0) = \ker(G + W_0B_*Q_0)^{\perp},$$

equation (46) is equivalent to

$$-H_*Q_0v \in \ker(G + W_0B_*Q_0)^{\perp}, \quad w = (-G^T + B_*^T W_0)^+ H_*Q_0v.$$

Introduce $\tilde{v} = (I + G^+ b_x Q_0)v$ so that $Q_0 v = Q_0 \tilde{v}$. Now it is clear that $\hat{G}_{*1}$ is nonsingular exactly if

$$(G + W_0 B_* Q_0)\tilde{v} = 0, \tag{47}$$

$$-H_* Q_0 \tilde{v} \in \ker(G + W_0 B_* Q_0)^{\perp} \tag{48}$$

imply $\tilde{z}_1 = 0$. This proves (1).
(3): Equation (47) decomposes to $G\tilde{v} = 0$ and $W_0 b_x Q_0 \tilde{v} = 0$, thus $\tilde{v} = Q_0 \tilde{v}$ and $\tilde{v} \in \ker W_0 B_*$. Moreover, regarding condition (48), $\tilde{v}$ and $H_* \tilde{v}$ are orthogonal, $0 =< H_* \tilde{v}, \tilde{v} >=< Q_0 H_* Q_0 \tilde{v}, \tilde{v} >$. Since $Q_0 H_* Q_0$ is symmetric and semi-definite, it follows that $Q_0 H_* Q_0 v = 0$, which confirms (3).
(2): Owing to [BKM06, Theorem 4.3], the linearized DAE (39) is selfadjoint. As a selfadjoint index-1 DAE it has Hamiltonian structure due to [BKM06, Theorem 4.5]. □

Notice that the Hamiltonian structure can get lost if the leading term is properly stated but not full-rank proper (cf. Definition 2.1), as it is demonstrated in [BKM06, Example 4.7].

Example 6.4 *[Bac06, p.144-146] Minimize the cost*

$$J(x) = \frac{1}{2} \int_0^{t_f} (x_3(t)^2 + (x_4(t) - R^2)^2) dt$$

subject to the constraint

$$\begin{aligned} x_1'(t) + x_2(t) &= 0, \\ x_2'(t) - x_1(t) - x_3(t) &= 0, \\ -x_1(t)^2 - x_2(t)^2 + x_4(t) &= 0, \\ x_1(0) &= r, \\ x_2(0) &= 0, \end{aligned}$$

with constants $r > 0$, $R > 0$. If $x_3(t)$ vanishes identically, the remaining IVP has a unique solution. Then the point $(x_1(t), x_2(t))$ orbits the origin with radius r and $x_4(t) = r$. By optimizing in view of the cost, the point $(x_1(t), x_2(t)$ becomes driven to the circle of radius R, with low cost of $x_3(t)$.
We have $m = 4$, $n = 2$, $k = 3$, and

$$f(y, x, t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} x_2 \\ -x_1 - x_3 \\ -x_1^2 - x_2^2 + x_4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and condition (33) is satisfied, since

$$f_y D + f_x (I - D^+ D) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has full row-rank. The adjoint system comprises 4 equations and 3 unknown *functions* λ_1, λ_2, λ_3. *The resulting optimality DAE*

$$\begin{aligned} x_1'(t) + x_2(t) &= 0,\\ x_2'(t) - x_1(t) - x_3(t) &= 0,\\ -x_1(t)^2 - x_2(t)^2 + x_4(t) &= 0,\\ -\lambda_1'(t) - \lambda_2(t) - 2x_1(t)\lambda_3(t) &= 0,\\ -\lambda_2'(t) + \lambda_1(t) - 2x_2(t)\lambda_3(t) &= 0,\\ -\lambda_2(t) &= x_3(t),\\ \lambda_3(t) &= x_4(t) - R^2, \end{aligned}$$

has dimension 7 and is everywhere regular with index 1. Here we have

$$H_*(t) = \begin{bmatrix} 2\lambda_*(t) & 0 & 0 & 0\\ 0 & 2\lambda_*(t) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q_0(t)H_*(t)Q_0(t) = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that condition (44) applies.

Example 6.5 *Minimize the cost*

$$J(x) = \frac{1}{2}\int_0^{t_f} (\,\alpha\,(x_1(t) + x_3(t))^2 + \beta\, x_2(t)^2\,)dt + \frac{1}{2}\gamma\, x_2(t_f)^2$$

subject to the constraint

$$\begin{aligned} x_2'(t) + x_2(t) + x_3(t) &= 0,\\ x_2'(t) + \sin t &= 0,\\ x_2(0) &= 1, \end{aligned}$$

with constants α, β, $\gamma \geq 0$, $\alpha^2 + \beta^2 + \gamma^2 > 0$. *The optimal solution is*

$$x_{*1}(t) = -\sin t + \cos t, \quad x_{*2}(t) = \cos t, \quad x_{*1}(t) = \sin t - \cos t.$$

We have

$$f(y, x, t) = \begin{bmatrix} 1\\ 1 \end{bmatrix} y + \begin{bmatrix} x_2 + x_3\\ \sin t \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad f_y D + f_x Q_0 = \begin{bmatrix} 0 & 1 & 1\\ 0 & 1 & 0 \end{bmatrix},$$

so that condition (33) is satisfied for all α, β, γ. *The optimality DAE reads*

$$\begin{aligned} x_2'(t) + x_2(t) + x_3(t) &= 0,\\ x_2'(t) + \sin t &= 0,\\ -\alpha(x_1(t) + x_3(t)) &= 0,\\ -(\lambda_1(t) + \lambda_2(t))' + \lambda_1(t) - \beta x_2(t) &= 0,\\ \lambda_1(t) - \alpha(x_1(t) + x_3(t)) &= 0. \end{aligned}$$

This square DAE of dimension 5 is regular with index 1 exactly if α *does not vanish. This condition reflects condition (44). Namely, we have*

$$H_*(t) = h_{xx}(x_*(t), t) = \begin{bmatrix} \alpha & 0 & \alpha\\ 0 & \beta & 0\\ \alpha & 0 & \alpha \end{bmatrix}, \quad Q_0(t)H_*(t)Q_0(t) = \begin{bmatrix} \alpha & 0 & \alpha\\ 0 & 0 & 0\\ \alpha & 0 & \alpha \end{bmatrix},$$

$$G+W_0B_*Q_0 = \begin{bmatrix}0&1&0\\0&1&0\end{bmatrix} + \begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&1&\frac{1}{2}\\0&1&-\frac{1}{2}\end{bmatrix}.$$

In contrast, the optimality DAE fails to be regular for $\alpha = 0$.
In this example, the constraint DAE (28) consists of 2 equations and 3 unknown functions. If we fix x_3 as a control, the resulting controlled DAE (with respect to x_1, x_2) fails to be regular. In contrast, fixing x_1 as control, the resulting controlled DAE (with respect to x_2, x_3) is regular with index 1. This underlines that the only property who matters for the extremal condition is the surjectivity of the operator $\mathfrak{L}$. However, to obtain a regular index-1 optimality DAE (37),(38) the cost must be somehow consistent with the DAE describing the constraint.

Example 6.6 *Minimize the cost*

$$J(x) = \frac{1}{2}\int_0^{2\pi} (\,(x_1(t)-\sin t)^2 + (x_2(t)-\cos t)^2 + \gamma x_3(t)^2 + x_4(t)^2\,)dt$$

subject to the constraint

$$\begin{aligned} x_1'(t) - x_2(t) + x_3(t) &= 0,\\ x_2'(t) + x_1(t) &= 0,\\ x_1(t)^3 + \alpha(x_1(t))x_3(t) - (\sin t)^3 - x_4(t) &= 0,\\ x_1(0) &= 0,\\ x_2(0) &= 1, \end{aligned}$$

with the real function α given in Example 4.4 and a constant $\gamma \geq 0$. Considering x_4 as a control and letting $x_4 = 0$, the DAE from Example 4.4 reappears.
The optimal solution is

$$x_{*1}(t) = \sin t,\ x_{*2}(t) = \cos t,\ x_{*3}(t) = 0,\ x_{*4}(t) = 0.$$

We have $m = 4$, $k = 3$, $n = 2$ and

$$f(y,x,t) = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} y + \begin{bmatrix}-x_2+x_3\\x_1\\x_1^3+\alpha(x_1)x_3-(\sin t)^3-x_4\end{bmatrix}, \quad D = \begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},$$

such that the matrix function

$$f_yD + f_x(I-D^+D) = \begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&\alpha(x_1)&-1\end{bmatrix}$$

results, which has full row-rank independently of the behavior of $\alpha(x_1)$. The optimality DAE reads

$$\begin{aligned} x_1'(t) - x_2(t) + x_3(t) &= 0,\\ x_2'(t) + x_1(t) &= 0,\\ x_1(t)^3 + \alpha(x_1(t))x_3(t) - (\sin t)^3 - x_4(t) &= 0,\\ -\lambda_1'(t) + \lambda_2(t) + (3x_1(t)^2 + \alpha'(x_1(t))x_3(t))\lambda_3(t) &= x_1(t) - \sin t,\\ -\lambda_2'(t) - \lambda_1(t) &= x_2(t) - \cos t,\\ \lambda_1(t) + \alpha(x_1(t))\lambda_3(t) &= \gamma x_3(t),\\ -\lambda_3(t) &= x_4(t). \end{aligned}$$

It holds that

$$G + W_0 B_* Q_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha(x_1) & -1 \end{bmatrix}$$

and

$$H_*(t) = \begin{bmatrix} 1-6x_{*1}(t)\lambda_{*3}(t) & 0 & -\alpha'(x_{*1}(t))\lambda_{*3}(t) & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha'(x_{*1}(t))\lambda_{*3}(t) & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q_0 H_*(t) Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Condition (44) requires $\gamma + (\alpha(x_1))^2 \neq 0$. Therefore, the optimality DAE is regular with index 1 in case of $\gamma > 0$.
If $\gamma = 0$, then only the set

$$\mathcal{G}_+ = \{(z,t) \in \mathbb{R}^7 \times (t_0, t_f) : z_1 > 0\}$$

is a regularity region with characteristic $\mu = 1$. Unfortunately, the optimal solution does not remain in this index-1 region.

Alltogether, when intending to apply an indirect optimization method, it seems to be a good idea to make use of the modeling latitude to reach an *optimality DAE which is regular with index-1* or, at least, to reach the situation that the expected solution stands in a regularity region with characteristic $\mu = 1$.

7 Specification for controlled DAEs

In the present section we specify results of the previous section for the important case of constraints described by controlled DAEs. Here the DAE and the cost may depend on a pair of functions, the *state* $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ and the *control* $u \in \mathcal{C}(\mathcal{I}, \mathbb{R}^l)$. Now the DAE comprises m equations so that, for each fixed control, a square m-dimensional DAE results. Consider the cost

$$J(x,u) = \int_{t_0}^{t_f} h(x(t), u(t), t)dt + g(D(t_f)x(t_f)) \tag{49}$$

to be minimized on pairs $(x,u) \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) \times \mathcal{C}(\mathcal{I}, \mathbb{R}^l)$, subject to the constraints

$$f((Dx)'(t), x(t), u(t), t) = 0, \quad t \in \mathcal{I}, \tag{50}$$

$$D(t_0)x(t_0) = z_0 \in \mathbb{R}^n, \tag{51}$$

with $f(y,x,u,t) \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $u \in \mathbb{R}^l$, $t \in \mathcal{I} = [t_0, t_f]$, $D(t) \in L(\mathbb{R}^m, \mathbb{R}^n)$, $\operatorname{rank} D(t) = r = n$. Assume analogous smoothness as in Section 6. Moreover, as in Section 6 the DAE (50) is supposed to have a full-rank proper leading term (see (8)).
Denote

$$\begin{aligned} A_*(t) &= f_y((Dx_*)'(t), x_*(t), u_*(t), t), \\ B_*(t) &= f_x((Dx_*)'(t), x_*(t), u_*(t), t), \\ C_*(t) &= f_u((Dx_*)'(t), x_*(t), u_*(t), t), \ t \in \mathcal{I}, \end{aligned}$$

such that now the linearization along (x_*, u_*) reads

$$A_*(t)(D(t)x(t))' + B_*(t)x(t) + C_*(t)u(t) = 0, \quad t \in \mathcal{I}. \tag{52}$$

The following assertion is a straightforward consequence of Theorem 6.1.

Theorem 7.1 (Necessary optimality condition) *Let the optimization problem (49),(50),(51) have the local solution $(x_*, u_*) \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) \times \mathcal{C}(\mathcal{I}, \mathbb{R}^l)$. Then, if the full-rank condition*

$$\operatorname{rank}\,[A_*(t)D(t) + B_*(t)(I - D(t)^+D(t)),\; C_*(t)] = m, \quad t \in \mathcal{I}, \tag{53}$$

is valid, the terminal value problem

$$-D(t)^T(A_*^T\lambda)'(t) + B_*(t)^T\lambda(t) = h_x(x_*(t), u_*(t), t)^T, \tag{54}$$

$$C_*(t)^T\lambda(t) = h_u(x_*(t), u_*(t), t)^T, \quad t \in \mathcal{I} \tag{55}$$

$$D(t_f)^T A_*(t_f)^T\lambda(t_f) = D(t_f)^T(g_\eta(D(t_f)x_*(t_f)))^T \tag{56}$$

has a solution $\lambda_* \in \mathcal{C}^1_{A_*}(\mathcal{I}, \mathbb{R}^m)$.

If the controlled DAE is regular with index ≤ 1, then $A_*D + B_*(I - D^+D)$ is nonsingular such that condition (53) follows. In this connection it does not matter how C_* looks like. However, in all other cases, condition (53) entails structural restrictions concerning C_*. On the other side, no regularity conditions result for the given controlled DAE, as it is demonstrated in Example 6.5 letting $u = x_3$.
At this place we stress once more, that our criteria are clearly represented algebraic conditions, and they are given in terms of the original optimization problem. In contrast, in [KM06] an analogous optimization problem with DAE constraint

$$\mathfrak{f}(x'(t), x(t), u(t), t) = 0, \quad t \in \mathcal{I},$$

is treated by transforming this equation first into the so-called reduced form

$$x_1'(t) - \mathcal{L}(x_1(t), x_2(t), u(t), t), \quad x_2(t) = \mathcal{R}(x_1(t), u(t), t), \tag{57}$$

and not till then formulating an extremal condition and the optimality DAE in terms of (57). This pre-handling is based on demanding assumptions (e.g. [KM06, Hypothesis 1]) and it needs considerable effort.
The reduced system (57) represents a special case of a semi-explicit controlled regular index-1 DAE, such that condition (53) is given. The optimality DAE for the optimization problem with constraint DAE (57) is then the corresponding special case of the DAE (59)-(61) below.

As a consequence of Theorem 7.1, the BVP composed from the IVP (50),(51) and the terminal value problem (54)-(56) is solvable. Indirect optimization relies on this BVP. Then, for practical reasons, the question arises whether the associated *optimality DAE* is regular with index 1. We give an answer for the transparent quasi-linear case.

$$f(y, x, u, t) = A(t)y + b(x, u, t), \tag{58}$$

so that the optimality DAE simplifies to

$$A(t)(D(t)x(t))' + b(x(t),u(t),t) = 0, \tag{59}$$

$$-D(t)^T(A(t)^T\lambda(t))' + b_x(x(t),u(t),t)^T\lambda(t) = h_x(x(t),u(t),t)^T, \tag{60}$$

$$b_u(x(t),u(t),t)^T\lambda(t) = h_u(x(t),u(t),t)^T. \tag{61}$$

The optimality DAE (59)-(61) has the linearization (the argument t is dropped)

$$\begin{bmatrix} A & 0 \\ 0 & D^T \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & D & 0 \\ -A^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ x \\ u \end{bmatrix} \right)' + \begin{bmatrix} 0 & B_* & C_* \\ B_*^T & -\mathcal{W}_* & -\mathcal{S}_* \\ C_*^T & -\mathcal{S}_*^T & -\mathcal{R}_* \end{bmatrix} \begin{bmatrix} \lambda \\ x \\ u \end{bmatrix} = 0. \tag{62}$$

with continuous matrix functions

$$\begin{aligned} \mathcal{W}_*(t) &:= h_{xx}(x_*(t),t) - (b_x^T(x,t)\lambda_*(t))_x(x_*(t),t), \\ \mathcal{S}_*(t) &:= h_{xu}(x_*(t),t) - (b_x^T(x,t)\lambda_*(t))_u(x_*(t),t), \\ \mathcal{R}_*(t) &:= h_{uu}(x_*(t),t) - (b_u^T(x,t)\lambda_*(t))_u(x_*(t),t), \quad t \in \mathcal{I}. \end{aligned}$$

Theorem 7.2 (Properties of the optimality DAE) *Let the DAE (50) have the special form given by (58). Let the functions b and h have the additional second continuous partial derivatives needed. Let $(x_*, u_*) \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m) \times \mathcal{C}(\mathcal{I}, \mathbb{R}^l)$ be a local solution of the optimization problem (49),(50),(51).*
Denote $B_*(t) = b_x(x_*(t), u_*(t), t)$, $C_*(t) = b_u(x_*(t), u_*(t)t)$, $G(t) = A(t)D(t)$, $Q_0(t) = I - D(t)^+D(t)$, $W_0(t) = I - A(t)A(t)^+$, $t \in \mathcal{I}$.
Let the condition (53) be satisfied, that is

$$\operatorname{rank}\,[A(t)D(t) + B_*(t)Q_0(t), C_*(t)] = m, \quad t \in \mathcal{I}. \tag{63}$$

Let λ_* denote the solution of the terminal value problem (54)-(56).

(1) *Then the optimality DAE (59)-(61) is regular with index 1 in a neighborhood of the graph of (λ_*, x_*, u_*), exactly if*

$$\begin{aligned} &[G(t) + W_0(t)B_*(t)Q_0(t),\ W_0(t)C_*(t)]\, z = 0, \\ &\begin{bmatrix} \mathcal{W}_*(t)Q_0(t) & \mathcal{S}_*(t) \\ \mathcal{S}_*(t)^TQ_0(t) & \mathcal{R}_*(t) \end{bmatrix} z \in \ker[G(t) + W_0(t)B_*(t)Q_0(t),\ W_0(t)C_*(t)]^\perp \\ &\qquad\qquad \textit{imply} \quad z = 0, \quad \textit{for all} \quad t \in \mathcal{I}. \end{aligned} \tag{64}$$

(2) *If condition (64) is given, then the linearized DAE (62) is selfadjoint and its inherent regular ODE has Hamiltonian structure such that*

$$\Theta' = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \mathcal{E}\,\Theta, \quad \Theta := \begin{bmatrix} Dx \\ -A^T\lambda \end{bmatrix}, \tag{65}$$

with a symmetric continuous matrix function $\mathcal{E}$ of size $2n \times 2n$.

(3) *If the matrix*

$$\begin{bmatrix} Q_0(t)W_*(t)Q_0(t) & Q_0(t)S_*(t) \\ \mathcal{S}_*(t)^TQ_0(t) & \mathcal{R}_*(t) \end{bmatrix}$$

is semi-definite for all $t \in \mathcal{I}$, then condition (64) simplifies to the full-rank condition

$$\operatorname{rank} \begin{bmatrix} G(t) + W_0(t)B_*(t)Q_0(t) & W_0(t)C_*(t) \\ Q_0(t)\mathcal{W}_*(t)Q_0(t) & Q_0(t)\mathcal{S}_*(t) \\ \mathcal{S}_*(t)^TQ_0(t) & \mathcal{R}_*(t) \end{bmatrix} = m + l, \quad t \in \mathcal{I}. \tag{66}$$

Proof: All three assertions are the coresponding special cases of the assertions in Theorem 6.3.$\square$

We refer to [BKM06, Bac06, Mär05, CM07] for further index relations and for the consistency with well-known facts in the context of linear-quadratic optimal control problems.

8 Appendix to Section 4: Matrix functions

Here we provide matrix function sequences and admissible projector functions to determine the regularity regions and quasi-regularity regions of the examples in Section 4.

Example 4.1:

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B_0(x) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -3x_2^2 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix},$$

therefore

$$G_1(x) = G_0 + B_0(x)Q_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -3x_2^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The matrix function G_1 remains nonsingular where $x_2 \neq 0$, and hence the definition domain $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^4 \times [0, \infty)$ of the original DAE decomposes into the two maximal regularity regions

$$\begin{aligned} \mathcal{G}_+ &:= \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f : \; x_2 > 0\}, \\ \mathcal{G}_- &:= \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f : x_2 < 0\}, \end{aligned}$$

bordered by the critical point set $x_2 = 0$. The DAE is regular with tractability index one and characteristics $r_0 = \operatorname{rank} G_0 = 2, r_1 = \operatorname{rank} G_1 = 4$ on both regions.

Example 4.2:

$$G_0(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B_0(x) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 - 2x_2 & 0 \\ x_2 & x_1 - x_3 & 1 - x_2 \end{bmatrix},$$

therefore

$$G_1(x) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 - 2x_2 & 0 \\ 0 & x_1 - x_3 & 1 - x_2 \end{bmatrix},$$

and thus $\det G_1(x,t) = (1-2x_2)(1-x_2)$, which has the zeros $x_2 = \frac{1}{2}$ and $x_2 = 1$. Then the definition domain of the nonlinear DAE $\mathcal{D}_f \times \mathcal{I}_f = \mathbb{R}^3 \times \mathbb{R}$ splits into the three maximal regularity regions

$$\begin{aligned}
\mathcal{G}_1 &:= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ x_2 < \frac{1}{2}\Big\}, \\
\mathcal{G}_2 &:= \Big\{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ \frac{1}{2} < x_2 < 1\Big\}, \\
\mathcal{G}_3 &:= \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} :\ 1 < x_2\},
\end{aligned}$$

The DAE is regular with tractability index one on each region $\mathcal{G}_\ell$, $\ell = 1, 2, 3$. The border sets $x_2 = \frac{1}{2}$ and $x_2 = 1$ consist of critical points.

Example 4.3:
We construct the quasi-admissible matrix functions

$$G_0 = \begin{bmatrix} 0 & x_4 & 0 & 0 \\ 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 & 0 & 0 & x_2^1 \\ 0 & 1 & 0 & x_3^1 \\ 0 & 0 & 1 & x_4^1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$G_1 = \begin{bmatrix} 1 & x_4 & 0 & 0 \\ 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 0 & -x_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 0 & 0 & x_2^1 \\ 0 & 1 & 0 & x_3^1 \\ 0 & 0 & 1 & x_4^1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

further

$$G_2 = \begin{bmatrix} 1 & x_4 & 0 & 0 \\ 0 & 1 & x_4 & 0 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 & 0 & (x_4)^2 & 0 \\ 0 & 0 & -x_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 0 & 0 & x_2^1 \\ 0 & 0 & 0 & x_3^1 \\ 0 & 0 & 1 & x_4^1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$G_3 = \begin{bmatrix} 1 & x_4 & 0 & 0 \\ 0 & 1 & x_4 & 0 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_3 = \begin{bmatrix} 0 & 0 & 0 & -(x_4)^3 \\ 0 & 0 & 0 & (x_4)^2 \\ 0 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \Pi_3 = 0,\ B_3 = \begin{bmatrix} 0 & 0 & 0 & x_2^1 \\ 0 & 0 & 0 & x_3^1 \\ 0 & 0 & 0 & x_4^1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that the everywhere nonsingular matrix function

$$G_4 = \begin{bmatrix} 1 & x_4 & 0 & 0 \\ 0 & 1 & x_4 & 0 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

results. Observe that there are the two maximal regularity regions

$$\begin{aligned}
\mathcal{G}_+ &:= \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_4 > 0\}, \\
\mathcal{G}_- &:= \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f : x_4 < 0\},
\end{aligned}$$

bordered by the critical point set $x_4 = 0$. On each of these regularity regions the above matrix function sequence is even admissible, and the DAE is there regular with characteristics $r_0 = r_1 = r_3 = 3$ and $\mu = 4$.
At the same time, the DAE is quasi-regular on $\mathcal{D}_f \times \mathcal{I}_f$ since G_4 remains nonsingular. This indicates the critical points ($x_4 = 0$) to be somehow harmless.

Example 4.4:
We have for $x \in \mathcal{D}_f = \mathbb{R}^3, t \in \mathcal{I}_f = \mathbb{R}$

$$G_0(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_0(x) = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 3x_1^2 + \alpha_s(x_1)x_3 & 0 & \alpha(x_1) \end{bmatrix},$$

therefore

$$G_1(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha(x_1) \end{bmatrix}.$$

For $x_1 > 0$, it holds that $\alpha(x_1) > 0$, and therefore G_1 is nonsingular. Then it results that

$$\mathcal{G}_+ := \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_1 > 0\}$$

is a regularity region with characteristics $r_0 = 2,\ r_1 = 3$ and $\mu = 1$.
If $x_1 < 0$, then $\alpha(x_1) = 0$ and G_1 is singular, and we continue to construct the matrix function sequence by

$$G_1(x) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ B_1(x) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 3x_1^2 & 0 & 0 \end{bmatrix},$$

and

$$G_2(x) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3x_1^2 & 0 & 0 \end{bmatrix}, \quad \det G_2(x) = -3x_1^2.$$

The open set

$$\mathcal{G}_- := \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_1 < 0\}$$

is also a maximal regularity region, but now with different characteristics $r_0 = 2,\ r_1 = 2\ r_2 = 3$, and $\mu = 2$.

Example 4.5: We construct the quasi-admissible matrix functions

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha(x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \alpha_s(x_1)x_4^1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha(x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha(x_1) \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \alpha_s(x_1)x_4^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

further

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha(x_1) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the matrix function G_2 ia nonsingular everywhere, the DAE is quasi-regular on its definition domain.
Observe that $\alpha(x_1)$ remains positive on the region

$$\mathcal{G}_+ := \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f :\ x_1 > 0\}.$$

There the above projector functions Q_0, Q_1 are admissible, and the DAE is regular with characteristics $r_0 = r_1 = 3$, $r_2 = 4$ and $\mu = 2$.
In contrast, the expression $\alpha(x_1)$ disappears on

$$\mathcal{G}_- := \{(x,t) \in \mathcal{D}_f \times \mathcal{I}_f : x_1 < 0\}.$$

On this region we can turn to a proper leading term yielding

$$\tilde{G}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which says that the DAE is on $\mathcal{G}_-$ regular with index 1.

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