Projector Based Analysis of Linear Differential Algebraic Equations

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Abstract

We provide a comprehensive analysis of linear DAEs with continuous coefficients and properly stated leading term. We are mainly interested in so-called regular DAEs, but we address also under- and overdetermined DAEs.
In particular, we describe the structured characteristic of DAEs, explain how to formulate consistent initial conditions, investigate the flow asymptotics and admissible excitations. Also, critical points are touched.
We specify the main results for linear DAEs in standard form and discuss several canonical forms. We show that the constant rank conditions supporting the tractability index coincide with those applied in the strangeness index concept.

KeyWords: DAE, differential algebraic equation, admissible projector function, tractability index, fine decoupling, regularity, DAE flow, DAE structure, Lyapunov stability, canonical form, index reduction

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Contents

Linear DAEs with variable coefficients

1 Introduction

For constant coefficient DAEs

$$\bar{E}\bar{x}'(t) + \bar{F}\bar{x}(t) = \bar{q}(t), \tag{1}$$

the Kronecker index and regularity are well defined via the properties of the matrix pencil $\{\bar{E}, \bar{F}\}$, and these characteristics are of particular importance in view of an appropriate numerical treatment.
From about 1970, challenged by circuit simulation problems, numerical analysts and experts in circuit simulation begun to devote much work to the numerical integration of larger systems of implicit ODEs and DAEs (e.g. [Gea71], [EMR77], [SEYE81], [GHP81]). In particular, linear variable coefficient DAEs

$$\bar{E}(t)\bar{x}'(t) + \bar{F}(t)\bar{x}(t) = \bar{q}(t) \tag{2}$$

were tackled by the implicit Euler method

$$\bar{E}(t_l)\frac{1}{h}(\bar{x}_l - \bar{x}_{l-1}) + \bar{F}(t_l)\bar{x}_l = \bar{q}(t_l).$$

Obviously, for the method to be just feasible, the matrix $\frac{1}{h}\bar{E}(t_l) + \bar{F}(t_l)$ must be nonsingular, but this can be guaranteed for all steps t_l and all sufficiently small stepsizes h, if one requires the so-called *local matrix pencils* $\{\bar{E}(t), \bar{F}(t)\}$ to be regular on the given interval (We mention at this place, that feasibility is by far not sufficient for a numerical integration method to work well). However, as it was discovered already in [GP83], the local pencils are not at all relevant characteristics of more general DAEs than those being linear with constant coefficients. Except for the regular index one case, local matrix pencils may change their index and loose their regularity under smooth regular transformations of the variables. That means, the local matrix pencils $\{E(t), F(t)\}$ of the DAE

$$E(t)x'(t) + F(t)x(t) = q(t), \tag{3}$$

which results from transforming $\bar{x}(t) = K(t)x(t)$ in the DAE (2), with a pointwise nonsingular continuously differentiable matrix function K, may have completely different characteristics than the local pencils $\{\bar{E}(t), \bar{F}(t)\}$. Nevertheless, the DAEs are equivalent, and hence, the local matrix pencils are irrelevant for determining the characteristics of a DAE. The coefficients of the equivalent DAEs (2) and (3) are related by the formulae $E(t) = \bar{E}(t)K(t),\ F(t) = \bar{F}(t)K(t) + \bar{E}(t)K'(t)$, which gives the impression that one can manipulate the resulting local pencil almost arbitrarily by choosing different transforms K.
In DAEs with properly stated leading term

$$\bar{A}(t)(\bar{D}(t)\bar{x}(t))' + \bar{B}(t)\bar{x}(t) = \bar{q}(t), \tag{4}$$

the transformation $\bar{x}(t) = K(t)x(t)$ leads to the equivalent DAE

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{5}$$

that has also a properly stated leading term. The coefficients are related by $A(t) = \bar{A}(t)$, $D(t) = \bar{D}(t)K(t)$ and $B(t) = \bar{B}(t)K(t)$, and the local pencils $\{\bar{A}(t)\bar{D}(t), \bar{B}(t)\}$ and $\{A(t)D(t), B(t)\} = \{\bar{A}(t)\bar{D}(t)K(t), \bar{B}(t)K(t)\}$ are now equivalent. However, we do not consider this to justify the local pencils as relevant carriers of DAE essentials. For DAEs with properly stated leading terms, also so-called *refactorizations of the leading term* yield equivalent DAEs, and any serious concept incorporates this fact. For instance, inserting $(Dx)' = (DD^+Dx)' = D(D^+Dx)' + D'D^+Dx$ does not really change the DAE (5), however, the local matrix pencils may change their nature as the next example demonstrates. This rules out the local pencils again.

Example 1.1 *The constant coefficient DAE*

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\bar{E}} \bar{x}'(t) + \bar{x}(t) = q(t), \quad t \in \mathbb{R},$$

has Weierstraß-Kronecker canonical form, and its matrix pencil $\{\bar{E}, I\}$ is regular with Kronecker index three. We transform $\bar{x}(t) = K(t)x(t)$ by means of the smooth matrix function K,

$$K(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}, \quad t \in \mathbb{R},$$

being everywhere nonsingular. This yields a DAE (3) with variable coefficients

$$E(t) = \bar{E}K(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F(t) = K(t) + \bar{E}K'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}.$$

We expect the new DAE to inherit regularity with index three owing to the equivalence. However, for each t, the characteristic polynomial $\det(\lambda E(t) + F(t))$ vanishes identically, that is, the pencil $\{E(t), F(t)\}$ is singular.
By means of the simple factorization

$$\bar{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =: \bar{A}\bar{D}$$

we rewrite the original DAE as the following DAE with properly stated leading term:

$$\bar{A}(\bar{D}\bar{x}(t))' + \bar{x}(t) = q(t), \quad t \in \mathbb{R}.$$

Applying the transformation $\bar{x}(t) = K(t)x(t)$ to this DAE we arrive now at

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{A}} (\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{D}(t)} x(t))' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{\tilde{B}(t)} x(t) = q(t), \quad t \in \mathbb{R}. \tag{6}$$

Owing to $\det(\lambda\tilde{A}\tilde{D}(t)+\tilde{B}(t)) \equiv 1$, *the local pencil* $\{\tilde{A}\tilde{D}(t), \tilde{B}(t)\}$ *is regular with index three. However, deriving*

$$(\tilde{D}(t)x(t))' = (\tilde{D}(t)\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}x(t))' = \tilde{D}(t)(\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}x(t))' + \tilde{D}'(t)\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}x(t)$$

yields the further equivalent DAE

$$\underbrace{\begin{bmatrix}0&1&0\\0&-t&1\\0&0&0\end{bmatrix}}_{A(t)}(\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{D}x(t))' + \underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&-t&1\end{bmatrix}}_{B(t)}x(t) = q(t), \quad t\in\mathbb{R}, \tag{7}$$

the local matrix pencils $\{A(t)D, B(t)\} = \{E(t), F(t)\}$ *of which are singular for all* $t \in \mathbb{R}$.

We see, aiming for the characterization of a variable coefficient DAE, it does not make sense to check regularity and index of the local pencils, neither for standard form DAEs nor for DAEs with properly stated leading term.

In this paper we provide a comprehensive analysis of linear DAEs (5) with continuous coefficients and properly stated leading term by taking up the ideas of the projector based decoupling described for constant coefficient DAEs in [LMT11b]. To handle the time-varying case, we proceed pointwise on the given interval and generate sequences of matrix functions $G_i(.) = G_{i-1}(.) + B_{i-1}(.)Q_{i-1}(.)$ and projector functions $Q_i(.)$ instead of the former sequences of matrices and projectors. Thereby we incorporate into $B_i(.)$ an additional term that somehow comprises the variations in time. This term is the crucial one of the generalization, since without it we would be back to the local matrix pencils. Aside from the higher technical amount in the proofs, the decoupling concept applies precisely in the same way as for constant coefficient DAEs, and most results take the same or slightly modified form.

In contrast to [LMT11b] which is devoted to square DAE systems, the present paper is basically valid for arbitrary, possible rectangular DAEs. Following the arguments e.g. in [KM06], rectangular systems may play their role in optimization and control. However, we underline, our interest is mainly directed to regular DAEs being square by definition. In Sections 2, 3, and 5 we provide the basic matrix function sequences and admissible projectors together with their main properties. This part follows the lines of [Mär02], [Mär04b]. While [Mär02], [Mär04b] are devoted to regular square DAEs, we give now an adequate generalization for systems being not necessarily square.
We begin to preliminary rearrange the DAE terms for better structural insight in Section 4. Later on we resume this topic twice: in Section 6 (Subsections 6.1 and 6.2) for regular DAEs , and in Section 10 for over- and underdetermined DAEs.
The main objective of this paper constitutes in a comprehensive characterization of *regular DAEs* in Section 6, in particular, in their decoupling into the *inherent regular explicit ODE* (53) and the subsystem (64) which comprises the *inherent differentiations.* We consider the constructive existence proof of *fine* and *complete* decoupling projector functions (Theorem 6.18) to be the most important special result which exposes the DAE

structure as the basis of the further investigations. The Subsections 6.3 and 6.5 are then devoted to the intrinsic DAE theory, they offer solvability results, flow properties, and the *T-canonical form* being an appropriate generalization of the Weierstraß-Kronecker form. Several specifications for regular standard form DAEs are recorded in Subsection 6.4. The discussion of regular DAEs follows in essence the lines of [Mär04b], and [Mär04a], while the material on over- and underdetermined DAEs is to a large extend new.
Section 7 reflects aspects of the critical point discussion from [MR06], [MR07], [Ria08].
Section 9 provides *widely orthogonal* projector functions, a special sort of admissible projector functions which proves their value in theory and praxis (see [LMT11a]).
In Section 8 we explain by means of canonical forms and reduction steps how the *strangeness* and the tractability index concepts are related to each other. Thereby we concentrate on the constant rank requirements supporting these concepts. We show good reasons to conjecture these rank conditions to be fully equivalent. We prove the conditions associated with the regular strangeness index ζ to imply regularity with tractability index $\mu = \zeta - 1$. As a byproduct in this section, we offer a projector based new reduction procedure.

2 The basic matrix function sequences

We study the equation

$$A(Dx)' + Bx = q, \tag{8}$$

with continuous coefficients

$$A \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^k)), \quad D \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^n)), \quad B \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^k)),$$

and an excitation $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^k)$. $\mathcal{I} \in \mathbb{R}$ is an interval. The coefficients A and D are supposed to be well matched. Roughly speaking this means that there is no gap and no overlap of the factors within the product AD. We use the two coefficients A and D to figure out precisely all those components of the unknown function which are involved in (8) with their first derivatives.

Definition 2.1 *The leading term in equation (8) is said to be* <u>*properly stated*</u>*, if $A(t)$ and $D(t)$ have constant rank r on $\mathcal{I}$, and it holds that*

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, \; t \in \mathcal{I}, \tag{9}$$

and, additionally, there are functions $\vartheta_i \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$, $i = 1, \ldots, n$, such that

$$\operatorname{im} D = \operatorname{span}\{\vartheta_1, \ldots, \vartheta_r\}, \quad \ker A = \operatorname{span}\{\vartheta_{r+1}, \ldots, \vartheta_n\}.$$

The projector function $R \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$ given by

$$R := [\vartheta_1 \ldots \vartheta_n] \begin{bmatrix} \underbrace{I}_{r} & \\ & 0 \end{bmatrix} [\vartheta_1 \ldots \vartheta_n]^{-1} \tag{10}$$

is named the <u>*border projector*</u> *of A and D, and of the DAE.*

If A and D form a properly stated leading term, then the relations

$$\operatorname{im} AD = \operatorname{im} A, \quad \ker AD = \ker D, \quad \operatorname{rank} A = \operatorname{rank} AD = \operatorname{rank} D$$

are valid (cf. Lemma A.3), and A, AD and D have common constant rank r on $\mathcal{I}$. Besides the coefficients A, D and the projector R we use a continuous pointwise generalized inverse $D^- \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^m))$ of D satisfying the relations

$$DD^-D = D, \quad D^-DD^- = D^-, \quad DD^- = R. \tag{11}$$

Such a generalized inverse exists owing to the constant rank of D. Namely, the orthogonal projector P_D onto $\ker D^\perp$ along $\ker D$ is continuous (Lemma C.2). If we added the fourth condition $D^-D = P_D$ to (11), then the resulting D^- would be uniquely determined and continuous (Proposition C.4), and this shows the existence of a continuous generalized inverses satisfying (11). By fixing only the three conditions (11), we have in mind some more flexibility.
Here $D^-D =: P_0$ is always a continuous projector function such that $\ker P_0 = \ker D = \ker AD$. On the other side, prescribing P_0 we fix at the same time D^-.

Now we are ready for composing the basic sequence of matrix functions and subspaces to work with. Put

$$G_0 := AD, \quad B_0 := B, \quad N_0 := \ker G_0 \tag{12}$$

and choose projector functions $P_0, Q_0, \mathit{\Pi}_0 \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ such that

$$\mathit{\Pi}_0 = P_0 = I - Q_0, \quad \operatorname{im} Q_0 = N_0.$$

For $i \geq 0$, as long as the expressions exist, we form

$$\begin{aligned} G_{i+1} &= G_i + B_i Q_i, && (13)\\ N_{i+1} &= \ker G_{i+1}, && (14) \end{aligned}$$

choose projector functions P_{i+1}, Q_{i+1} such that $P_{i+1} = I - Q_{i+1}, \quad \operatorname{im} Q_{i+1} = N_{i+1}$, and put

$$\begin{aligned} \mathit{\Pi}_{i+1} &:= \mathit{\Pi}_i P_{i+1}, \\ B_{i+1} &:= B_i P_i - G_{i+1} D^-(D\mathit{\Pi}_{i+1}D^-)' D\mathit{\Pi}_i. && (15) \end{aligned}$$

We emphasize that B_{i+1} contains the derivative of $D\mathit{\Pi}_{i+1}D^-$, that is, this term comprises the variation in time. This term disappears in the constant coefficient case (see [LMT11b]). The specific form of the new term is motivated in Section 4 below, where we consider similar decoupling rearrangements for the DAE (8) as in [LMT11b] for the constant coefficient case.
We are most interested in continuous matrix functions G_{i+1}, B_{i+1}, in particular we have to take care for $D\mathit{\Pi}_{i+1}D^-$ to be smooth enough.
Important characteristic values of the given DAE emerge from the rank functions

$$r_j := \operatorname{rank} G_j, \ j \geq 0.$$

Example 2.2 *Write the semi-explicit DAE*

$$\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1, \\ B_{21}x_1 + B_{22}x_2 &= q_2, \end{aligned}$$

with $m_1 + m_2 = m$ equations in the form (8) with properly stated leading term by

$$A = \begin{bmatrix} I \\ 0 \end{bmatrix}, \; D = \begin{bmatrix} I & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \; D^- = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Then we have

$$G_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \; G_1 = \begin{bmatrix} I & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

If B_{22} is nonsingular on the given interval, then so is G_1. It results that $Q_1 = 0$, thus $G_2 = G_1$ and so on. The sequence becomes stationary. All rank functions r_i are constant, in particular $r_0 = m_1$, $r_1 = m$.
Take also a look to the case if $B_{22} = 0$, but the product $B_{21}B_{12}$ remains nonsingular. We denote by Ω a projector function onto $\operatorname{im} B_{12}$, and by B_{12}^- a reflexive generalized inverse such that $B_{12}B_{12}^- = \Omega$, $B_{12}^-B_{12} = I$. The matrix function G_1 has now rank $r_1 = m_1$, and a nontrivial nullspace. We choose the next projector functions Q_1 and the resulting $D\Pi_1D^-$, say as

$$Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad D\Pi_1 D^- = I - \Omega.$$

This makes clear, for a continuously differentiable $D\Pi_1D^-$, we have to assume the range of B_{12} to be a $\mathcal{C}^1$-subspace (cf. D). Then we form the matrix functions

$$B_1 = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix} - \begin{bmatrix} -\Omega' & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} I + (B_{11} + \Omega')\Omega & B_{12} \\ B_{21}\Omega & 0 \end{bmatrix},$$

and consider the nullspace of G_2.
$G_2 z = 0$ means

$$z_1 + (B_{11} + \Omega')\Omega z_1 + B_{12}z_2 = 0, \quad B_{21}\Omega z_1 = 0.$$

The second equations means $B_{21}B_{12}B_{12}^- z_1 = 0$, thus $B_{12}^- z_1 = 0$, and hence $\Omega z_1 = 0$. Now the first equation simplifies to $z_1 + B_{12}z_2 = 0$. Multiplication by B_{12}^- gives $z_2 = 0$, and then $z_1 = 0$. Therefore, the matrix function G_2 is nonsingular, and again the sequence becomes stationary.

Example 2.3 *We construct a matrix function sequence for the DAE (7) obtained in Example 1.1. The DAE is expected to be regular with index three, as its equivalent constant coefficient counterpart. We have*

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; D(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_0(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and $R(t) = D(t)$. Set $D(t)^- = D(t)$ and $\Pi_0(t) = P_0(t) = D(t)$. Next we compute $G_1(t) = G_0(t) + B(t)Q_0(t)$ as well as a projector $Q_1(t)$ onto $\ker G_1(t) = N_1(t)$:

$$G_1(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1(t) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}.$$

This leads to

$$\Pi_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; B_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_2(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A suitable projector function Q_2 and the resulting B_2 and G_3 are:

$$Q_2(t) = \begin{bmatrix} 0 & -t & 1 \\ 0 & t & -1 \\ 0 & -t(1-t) & 1-t \end{bmatrix}, \; \Pi_2(t) = 0, \; B_2(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}, \; G_3(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & -t & 1 \end{bmatrix}.$$

Here the matrix functions G_i, $i = 0, 1, 2$ are singular with constant ranks, and G_3 is the first matrix function being nonsingular. This is typical for regular index three DAEs (cf. Definition 10.1 below), and meets our expectation in comparison with the constant coefficient case (see [LMT11b]). Observe that the nullspaces and projectors fulfill the relations

$$N_0(t) \cap N_1(t) = \{0\}, \quad (N_0(t) + N_1(t)) \cap N_2(t) = \{0\},$$
$$Q_1(t)Q_0(t) = 0, \quad Q_2(t)Q_0(t) = 0, \quad Q_2(t)Q_1(t) = 0.$$

The matrix functions G_i as well as the projector functions Q_i are continuous here, and it holds that $\operatorname{im} G_0 = \operatorname{im} G_1 = \operatorname{im} G_2 \subset \operatorname{im} G_3$.

The matrix function sequence (12)-(15) generates subspaces

$$\operatorname{im} G_0 \subseteq \cdots \subseteq \operatorname{im} G_i \subseteq \operatorname{im} G_{i+1}$$

of nondecreasing dimensions.
To show several usefull properties we introduce the additional projector functions $\mathcal{W}_j : \mathcal{I} \to L(\mathbb{R}^k)$ and generalized inverses $G_j^- : \mathcal{I} \to L(\mathbb{R}^k, \mathbb{R}^m)$ of G_j such that

$$\ker \mathcal{W}_j = \operatorname{im} G_j, \tag{16}$$

$$G_j G_j^- G_j = G_j, \;\; G_j^- G_j G_j^- = G_j^-, \;\; G_j^- G_j = P_j, \;\; G_j G_j^- = I - \mathcal{W}_j. \tag{17}$$

Proposition 2.4 *Let the DAE (8) have a properly stated leading term. Then, for each matrix function sequence (12)-(15) the following relations are satisfied:*

(1) $\ker \Pi_i \subseteq \ker B_{i+1}$,

(2) $\mathcal{W}_{i+1}B_{i+1} = \mathcal{W}_{i+1}B_i = \cdots = \mathcal{W}_{i+1}B_0 = \mathcal{W}_{i+1}B$,
$\mathcal{W}_{i+1}B_{i+1} = \mathcal{W}_{i+1}B_0P_0 \cdots P_i = \mathcal{W}_{i+1}B_0\Pi_i$,

(3) $G_{i+1} = (G_i + \mathcal{W}_iBQ_i)F_{i+1}$ *with* $F_{i+1} = I + G_i^- B_i Q_i$ *and* $\operatorname{im} G_{i+1} = \operatorname{im} G_i \oplus \operatorname{im} \mathcal{W}_iBQ_i$,

(4) $N_i \cap \ker B_i = N_i \cap N_{i+1} \subseteq N_{i+1} \cap \ker B_{i+1}$,

(5) $N_{i-1} \cap N_i \subseteq N_i \cap N_{i+1}$,

(6) $\operatorname{im} G_i + \operatorname{im} B_i \subseteq \operatorname{im} [AD, B] = \operatorname{im} [G_0, B_0]$.

Proof: (1) From (15) we successively derive an expression for B_{i+1} being

$$\begin{aligned} B_{i+1} &= (B_{i-1}P_{i-1} - G_iD^-(D\Pi_iD^-)'D\Pi_{i-1})P_i - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ &= B_{i-1}P_{i-1}P_i - \sum_{j=i}^{i+1} G_jD^-(D\Pi_jD^-)'D\Pi_i, \end{aligned}$$

hence

$$B_{i+1} = B_0\Pi_i - \sum_{j=1}^{i+1} G_jD^-(D\Pi_jD^-)'D\Pi_i, \tag{18}$$

but this immediately verifies assertion (1).
(2) Because of $\operatorname{im} G_j \subseteq \operatorname{im} G_{i+1}$ for $j \leq i+1$, we have $\mathcal{W}_{i+1}B_{i+1} = \mathcal{W}_{i+1}B_0\Pi_i$ due to (18). Taking into account also the inclusion $\operatorname{im} B_jQ_j = \operatorname{im} G_{j+1}Q_j \subseteq \operatorname{im} G_{j+1} \subseteq \operatorname{im} G_{i+1}$, for $j \leq i$, we obtain from (15) that $\mathcal{W}_{i+1}B_{i+1} = \mathcal{W}_{i+1}B_iP_i = \mathcal{W}_{i+1}B_i - \mathcal{W}_{i+1}B_iQ_i = \mathcal{W}_{i+1}B_i = \mathcal{W}_{i+1}B_{i-1}P_{i-1} = \mathcal{W}_{i+1}B_{i-1} = \cdots = \mathcal{W}_{i+1}B_0$, which proves assertion (2).
(3) We rearrange G_{i+1} as

$$G_{i+1} = G_i + G_iG_i^-B_iQ_i + (I - G_iG_i^-)B_iQ_i = G_i((I + G_i^-B_iQ_i) + \mathcal{W}_iB_iQ_i.$$

Because of $Q_iG_i^- = Q_iP_iG_i^- = 0$ the matrix function $F_{i+1} := I + G_i^-B_iQ_i$ remains nonsingular (see Lemma A.2) and the factorization

$$G_{i+1} = (G_i + \mathcal{W}_iB_iQ_i)F_{i+1} = (G_i + \mathcal{W}_iBQ_i)F_{i+1}$$

holds true. This yields assertion (3).
(4) $z \in N_i \cap \ker B_i$, i.e., $G_iz = 0,\ B_iz = 0$, leads to $z = Q_iz$ and $G_{i+1}z = B_iQ_iz = B_iz = 0$, thus $z \in N_i \cap N_{i+1}$. Conversely, $z \in N_i \cap N_{i+1}$ yields $z = Q_iz,\ B_iz = B_iQ_iz = G_{i+1}z = 0$, i.e., $z \in N_i \cap \ker B_i$ and we are done with assertion (4).
(5) From $z \in N_{i-1} \cap N_i$ it follows that $z = Q_{i-1}z$ and $B_iz = B_iQ_{i-1}z = B_iP_{i-1}Q_{i-1}z = 0$ because of $B_i = B_iP_{i-1}$ (cf. (18)), hence $z \in N_i \cap \ker B_i = N_i \cap N_{i+1}$.
(6) follows from $\operatorname{im} G_0 + \operatorname{im} B_0 = \operatorname{im}[G_0, B_0]$ by induction. Namely, $\operatorname{im} G_i + \operatorname{im} B_i \subseteq \operatorname{im}[G_0, B_0]$ implies $\operatorname{im} B_iQ_i \subseteq \operatorname{im}[G_0, B_0]$, hence $\operatorname{im} G_{i+1} \subseteq \operatorname{im}[G_i, B_0Q_i] \subseteq \operatorname{im}[G_0, B_0]$, and further $\operatorname{im} B_{i+1} \subseteq \operatorname{im}[G_{i+1}, B_i] \subseteq \operatorname{im}[G_0, B_0]$. □

3 Admissible projector functions and characteristic values

In [LMT11b] on constant coefficient DAEs, useful decoupling properties are obtained by restricting the variety of possible projectors Q_i and choosing somehow smart ones. Here we take up this idea again, and we incorporate conditions concerning ranks and dimensions to ensure the continuity of our matrix functions. Possible rank changes will be treated as critical points.

Definition 3.1 *Given are a DAE (8) with properly stated leading term, and a $\kappa \in \mathbb{N}$.*

(1) *Each continuous projector function Q_0 onto* $\ker D$ *is named* <u>*admissible*</u>.

(2) *The projector functions $Q_0, \dots, Q_\kappa$ are said to be* <u>*admissible*</u> *on $\mathcal{I}$ for the DAE (8), if*

  (a) *G_i has constant rank r_i on $\mathcal{I}$, $i = 0, \dots, \kappa$,*

  (b) *the intersection*

$$\widehat{N_i} := N_i \cap (N_0 + \cdots + N_{i-1})$$

  has constant dimension $u_i := \dim \widehat{N_i}$ on $\mathcal{I}$, and Q_i satisfies there the condition

$$X_i := (N_0 + \cdots + N_{i-1}) \ominus \widehat{N_i} \subseteq \ker Q_i, \quad i = 1, \dots, \kappa,$$

  (c) *Π_i is on $\mathcal{I}$ continuous and $D\Pi_i D^-$ is continuously differentiable, $i = 0, \dots, \kappa$.*

(3) *If the projector functions $Q_0, \dots, Q_\kappa$ are admissible, then the corresponding matrix function sequence (12)-(15) is said to be* <u>*admissible up to level κ.*</u>

(4) *If $Q_0, \dots, Q_\kappa$ are admissible with trivial intersections $\widehat{N_1}, \dots, \widehat{N_\kappa}$, then they are named* <u>*regular admissible*</u> *.*

For a DAE (8) with properly stated leading term, all projectors $Q_0 = I - P_0$, $P_0 = D^- D$ built by a continuous generalized inverse D^-, are admissible, and $r_0 = \operatorname{rank} D(\cdot) = r$.
If $Q_0, \dots, Q_\kappa$ are admissible, besides the nullspaces $N_0, \dots, N_\kappa$ and the intersection spaces $\widehat{N_1}, \dots, \widehat{N_\kappa}$ also the sums $N_0 + \cdots + N_i$, $i = 1, \dots, \kappa$ and the complements $X_1, \dots, X_\kappa$ have constant dimension. Namely, the construction yields

$$N_0 + \cdots + N_{i-1} = X_i \oplus \widehat{N_i}, \quad N_0 + \cdots + N_i = X_i \oplus N_i, \quad i = 1, \dots \kappa,$$

and hence

$$\begin{aligned} \dim N_0 &= m - r_0, \\ \dim(N_0 + \cdots + N_{i-1}) &= \dim X_i + u_i, \\ \dim(N_0 + \cdots + N_i) &= \dim X_i + m - r_i, \quad i = 1, \dots, \kappa. \end{aligned}$$

It follows that

$$\begin{aligned} \dim(N_0 + \cdots + N_i) &= \underbrace{\dim(N_0 + \cdots + N_{i-1}) - u_i}_{\dim X_i} + \underbrace{m - r_i}_{\dim N_i} \\ &= \sum_{j=0}^{i-1}(m - r_j - u_{j+1}) + m - r_i = \sum_{j=0}^{i}(m - r_j) - \sum_{j=0}^{i-1} u_{j+1}. \end{aligned}$$

We are most interested in the case of trivial intersections $\widehat{N_i}$, yielding $X_i = N_0 + \cdots + N_{i-1}$, and $u_i = 0$. In particular, all so-called regular DAEs in Section 6 belong to this latter class. Due to the trivial intersection $\widehat{N_i} = \{0\}$, the subspace $N_0 + \cdots + N_i$ has dimension $\dim(N_0 + \cdots + N_{i-1}) + \dim N_i$, that is, its increase is maximal at each level.
For instance, the projector functions Q_0, Q_1, Q_2 constructed in Example 2.3 are regular admissible.

The next proposition collects benefits from admissible projector functions. Comparing with [LMT11b, Proposition 2.6] we recognize a far-reaching conformity. The most important benefit seems to be the fact that Π_i is a projector function along the sum space $N_0 + \cdots + N_i$ which now appears to be a $\mathcal{C}$-subspace.

Proposition 3.2 *Given are a DAE (8) with properly stated leading term, and an integer $\kappa \in \mathbb{N}$.*
If $Q_0, \ldots, Q_\kappa$ are admissible projector functions, then the following eight relations become true for $i = 1, \ldots, \kappa$:

(1) $\ker \Pi_i = N_0 + \cdots + N_i$,

(2) *the products $\Pi_i = P_0 \cdots P_i$ and $\Pi_{i-1}Q_i = P_0 \cdots P_{i-1}Q_i$, as well as $D\Pi_i D^-$ and $D\Pi_{i-1}Q_i D^-$, are projector valued functions, too,*

(3) $N_0 + \cdots + N_{i-1} \subseteq \ker \Pi_{i-1}Q_i$,

(4) $B_i = B_i \Pi_{i-1}$,

(5) $\widehat{N_i} \subseteq N_i \cap N_{i+1}$, *and hence* $\widehat{N_i} \subseteq \widehat{N_{i+1}}$,

(6) $G_{i+1}Q_j = B_j Q_j,\ 0 \le j \le i$,

(7) $D(N_0 + \cdots + N_i) = \operatorname{im} DP_0 \cdots P_{i-1}Q_i \oplus \operatorname{im} DP_0 \cdots P_{i-2}Q_{i-1} \oplus \cdots \oplus \operatorname{im} DP_0Q_1$,

(8) *the products $Q_i(I - \Pi_{i-1})$ and $P_i(I - \Pi_{i-1})$ are projector functions onto $\widehat{N_i}$ and X_i, respectively.*

Additionally, the matrix functions $G_1, \ldots, G_\kappa$, and $G_{\kappa+1}$ are continuous.
If $Q_0, \ldots, Q_\kappa$ are regular admissible then it holds for $i = 1, \ldots, \kappa$ that

$$\ker \Pi_{i-1}Q_i = \ker Q_i, \quad \textit{and} \quad Q_iQ_j = 0, \quad j = 0, \ldots, i-1.$$

Proof: (1) See the proof of [LMT11b, Proposition 2.6] (1).
(2) Due to assertion (1) it holds that $\ker \Pi_i = N_0 + \cdots + N_i$, which means $\Pi_i Q_j = 0$, $j = 0, \ldots, i$. With $0 = \Pi_i Q_j = \Pi_i(I - P_j)$, we obtain $\Pi_i = \Pi_i P_j,\ j = 0, \ldots, i$, which yields $\Pi_i \Pi_i = \Pi_i$. Derive further

$$(\Pi_{i-1}Q_i)^2 = (\Pi_{i-1} - \Pi_i)(\Pi_{i-1} - \Pi_i) = \Pi_{i-1} - \underbrace{\Pi_{i-1}\Pi_i}_{=\Pi_{i-1}P_i} - \underbrace{\Pi_i\Pi_{i-1}}_{=\Pi_i} + \Pi_i = \Pi_{i-1}Q_i,$$

$$(D\Pi_i D^-)^2 = D\Pi_i \underbrace{D^-D}_{=P_0} \Pi_i D^- = D\Pi_i D^-,$$

$$(D\Pi_{i-1}Q_i D^-)^2 = D\Pi_{i-1}Q_i \underbrace{D^-D}_{=P_0} \Pi_{i-1}Q_i D^- = D(\Pi_{i-1}Q_i)^2 D^- = D\Pi_{i-1}Q_i D^-.$$

(3) See the proof of [LMT11b, Proposition 2.6] (3)
(4) The detailed structure of B_i given in (18) and the projector property of Π_{i-1} (cf. (1)) proves the statement.
(5) $z \in N_i \cap (N_0 + \cdots + N_{i-1})$ means that $z = Q_i z,\ \Pi_{i-1} z = 0$, hence

$$G_{i+1}z = G_i z + B_i Q_i z = B_i z = B_i \Pi_{i-1} z = 0.$$

(6) For $0 \le j \le i$, it follows with (4) from

$$\begin{aligned} G_{i+1} = G_i + B_iQ_i &= G_0 + B_0Q_0 + B_1Q_1 + \cdots + B_iQ_i \\ &= G_0 + B_0Q_0 + B_1P_0Q_1 + \cdots + B_iP_0 \cdots P_{i-1}Q_i \end{aligned}$$

that

$$G_{i+1}Q_j = (G_0 + B_0Q_0 + \cdots + B_jP_0 \cdots P_{j-1}Q_j)Q_j = (G_j + B_jQ_j)Q_j = B_jQ_j.$$

(7) From $\ker P_0 \cdots P_i = N_0 + \cdots + N_i$ it follows that

$$\begin{aligned} D(N_0 + \cdots + N_i) &= D\,\mathrm{im}\,(I - P_0 \cdots P_i) = D\,\mathrm{im}\,(Q_0 + P_0Q_1 + \cdots + P_0 \cdots P_{i-1}Q_i) \\ &= D\{\mathrm{im}\,Q_0 \oplus \mathrm{im}\,P_0Q_1 \oplus \cdots \oplus \mathrm{im}\,P_0 \cdots P_{i-1}Q_i\} \\ &= \mathrm{im}\,DP_0Q_1 \oplus \cdots \oplus \mathrm{im}\,DP_0 \cdots P_{i-1}Q_i. \end{aligned}$$

This proves assertion (7).
(8) We have (cf. (3))

$$Q_i(I - \Pi_{i-1})Q_i(I - \Pi_{i-1}) = (Q_i - Q_i \underbrace{\Pi_{i-1}Q_i)(I - \Pi_{i-1})}_{=0} = Q_i(I - \Pi_{i-1}).$$

Further, $z = Q_i(I - \Pi_{i-1})z$ implies $z \in N_i$, $\Pi_{i-1}z = \Pi_{i-1}Q_i(I - \Pi_{i-1})z = 0$, and hence $z \in \widehat{N_i}$.
Conversely, from $z \in \widehat{N_i}$ it follows that $z = Q_iz$ and $z = (I - \Pi_{i-1})z$, thus $z = Q_i(I - \Pi_{i-1})z$. Similarly, we compute

$$P_i(I - \Pi_{i-1})P_i(I - \Pi_{i-1}) = P_i(I - \Pi_{i-1}) - P_i(I - \Pi_{i-1})Q_i(I - \Pi_{i-1}) = P_i(I - \Pi_{i-1}).$$

From $z = P_i(I - \Pi_{i-1})z$ it follows that $Q_iz = 0$, $\Pi_{i-1}z = \Pi_i(I - \Pi_{i-1})z = 0$, therefore $z \in X_i$.
Conversely, $z \in X_i$ yields $z = P_iz$, $z = (I - \Pi_{i-1})z$, and hence $z = P_i(I - \Pi_{i-1})z$. This verifies (8).
Next we verify the continuity of the matrix functions G_i. Applying the representation (18) of the matrix function B_i we express

$$G_{i+1} = G_i + B_0\Pi_{i-1}Q_i - \sum_{j=1}^{i} G_jD^-(D\Pi_jD^-)'D\Pi_{i-1}Q_i,$$

which shows that, supposed the previous matrix functions $G_0, \ldots, G_i$ are continuous, the continuity of $\Pi_{i-1}Q_i = \Pi_{i-1} - \Pi_i$ implies G_{i+1} to be also continuous.
Finally, let $Q_0, \ldots, Q_\kappa$ be regular admissible. $\Pi_{i-1}Q_iz = 0$ implies $Q_iz = (I - \Pi_{i-1})Q_iz \in N_0 + \cdots + N_{i-1}$, hence $Q_iz \in \widehat{N_i}$, therefore $Q_iz = 0$. It remains to apply (3). □

As in the constant coefficient case, there is a great variety of admissible projectors, and the matrix functions G_i clearly depend on the special choice of the projectors Q_j, included the way how the complements X_j in the decomposition of $N_0 + \cdots + N_{j-1}$ are chosen. However, there are invariants, in particular invariant subspaces and their dimensions, as shown by the next assertion.

Theorem 3.3 *Let the DAE (8) have a properly stated leading term. Then, for a given $\kappa \in \mathbb{N} \cup \{0\}$, if admissible projector functions up to level κ do at all exist, then the subspaces*

$$\operatorname{im} G_j, \quad N_0 + \cdots + N_j, \quad S_j := \ker \mathcal{W}_j B, \quad j = 0, \ldots, \kappa + 1,$$

as well as the numbers

$$r_j := \operatorname{rank} G_j,\ j = 0, \ldots, \kappa, \quad u_j := \dim \widehat{N}_j,\ j = 1, \ldots, \kappa,$$

and the functions $r_{\kappa+1} : \mathcal{I} \to \mathbb{N} \cup \{0\}$, $\quad u_{\kappa+1} : \mathcal{I} \to \mathbb{N} \cup \{0\}$ are independent of the special choice of admissible projector functions $Q_0, \ldots, Q_\kappa$.

Proof: These assertions are immediate consequences of Lemma 3.7 below at the end of the present section. □

Definition 3.4 *If the DAE* (8) *has admissible projector functions up to level κ, then the integers*

$$r_j := \operatorname{rank} G_j,\ j = 0, \ldots, \kappa, \quad u_j := \dim \widehat{N}_j,\ j = 1, \ldots, \kappa,$$

are named the <u>characteristic values</u> of the DAE.

The characteristic values prove to be invariant under regular transformations and refactorizations (cf. Section 5, Theorems 5.1 and 5.3), which justifies this notation. As detailed in [LMT11b], for constant regular matrix pairs, these characteristic values describe the infinite eigenstructure [LMT11b, Corollary 6.3].
The associated subspace $S_0 = \ker \mathcal{W}_0 B$ has its special meaning. At given $t \in \mathcal{I}$, the subspace

$$S_0(t) = \ker \mathcal{W}_0(t) B(t) = \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} G_0(t) = \operatorname{im} A(t)\}$$

contains all solution values $x(t)$ of the solutions of the homogeneous equation $A(Dx)' + Bx = 0$. As we will see later, for so-called regular index-one DAEs, the subspace $S_0(t)$ consists at all of those solution values, that means, for each element of $S_0(t)$ there exists a solution passing through. For regular DAEs with a higher index, the sets of corresponding solution values form proper subspaces of $S_0(t)$.
In general, the associated subspaces satisfy the relations

$$S_{i+1} = S_i + N_i = S_i + N_0 + \cdots + N_i = S_0 + N_0 + \cdots + N_i, \;\; i = 0, \ldots, \kappa.$$

Namely, because of $\operatorname{im} G_i \subseteq \operatorname{im} G_{i+1}$, it holds that $\mathcal{W}_{i+1} = \mathcal{W}_{i+1}\mathcal{W}_i$, hence $S_{i+1} = \ker \mathcal{W}_{i+1} B = \ker \mathcal{W}_{i+1}\mathcal{W}_i B \supseteq \ker \mathcal{W}_i B = S_i$, and Proposition 2.4 (2) yields $S_{i+1} = \ker \mathcal{W}_{i+1} B_{i+1} \supseteq \ker B_{i+1} \supseteq N_0 + \cdots + N_i$.

Summarizing, the following three sequences of subspaces are associated with each sequence of admissible projector functions:

$$\operatorname{im} G_0 \subseteq \operatorname{im} G_1 \subseteq \ldots \subseteq \operatorname{im} G_i \subseteq \ldots \subseteq \operatorname{im} [AD \;\; B] \subseteq \mathbb{R}^k, \tag{19}$$

$$N_0 \subseteq N_0 + N_1 \subseteq \ldots \subseteq N_0 + \ldots + N_i \subseteq \ldots \subseteq \mathbb{R}^m, \tag{20}$$

and

$$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_i \subseteq \ldots \subseteq \mathbb{R}^m. \tag{21}$$

All of these subspaces are independent of the special choice of the admissible projector functions. In all three cases, the dimension does not decrease if the index increases. We are looking for criteria indicating that a certain G_μ has already the maximal possible rank. For instance, if we meet an injective matrix G_μ as it is the case in Example 2.3, then the sequence becomes stationary with $Q_\mu = 0$, $G_{\mu+1} = G_\mu$ and so on. Therefore, the smallest index μ such that the matrix function G_μ is injective, indicates at the same time that $\operatorname{im} G_\mu$ is maximal, but $\operatorname{im} G_{\mu-1}$ is a proper subspace, if $\mu \geq 1$. The general case is more subtle. It may happen that no injective G_μ exists. Eventually one reaches

$$\operatorname{im} G_\mu = \operatorname{im}\,[AD\ \ B], \tag{22}$$

however, this is not necessarily the case, as the next example shows.

Example 3.5 *Set $m = k = 3$, $n = 2$, and consider the DAE*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \right)' + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x = q. \tag{23}$$

Here we have $\operatorname{im}\,[AD\ \ B] = \mathbb{R}^3$. *Compute successively*

$$G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We read off $N_0 = \{z \in \mathbb{R}^3 : z_1 = z_2 = 0\}$, $N_1 = \{z \in \mathbb{R}^3 : z_2 = 0, z_1 + z_3 = 0\}$ and $N_2 = \{z \in \mathbb{R}^3 : z_2 = 0, 2z_1 + z_3 = 0\}$. The intersection $N_0 \cap N_1$ is trivial, and the condition $Q_1 Q_0 = 0$ is fulfilled. We have further

$$N_0 + N_1 = \{z \in \mathbb{R}^3 : z_2 = 0\}, \quad (N_0 + N_1) \cap N_2 = \widehat{N_2} = N_2 \subseteq N_0 + N_1,$$
$$\textit{thus } N_0 + N_1 = N_0 + N_1 + N_2 \textit{ and } N_0 + N_1 = N_2 \oplus N_0.$$

We can put $X_2 = N_0$, and compute

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \textit{ with } X_2 \subseteq \ker Q_2, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The projectors Q_0, Q_1, Q_2 are admissible. It holds that $B_2 Q_2 = 0$, $G_3 = G_2$, $N_3 = N_2$, and $\Pi_2 = \Pi_1$, further

$$S_0 = \{z \in \mathbb{R}^3 : z_2 = 0\}, \quad S_0 = S_1 = S_2 = S_3.$$

We continue the matrix function sequence by $Q_3 := Q_2$, $B_3 = B_2$, $B_3Q_3 = 0$, $G_4 = G_3$ and so on. It results that no G_i is injective, and

$$\begin{aligned}
&\operatorname{im} G_0 = \ldots = \operatorname{im} G_i = \ldots = \mathbb{R}^2 \times \{0\} \subset \operatorname{im}[AD\ \ B] = \mathbb{R}^3,\\
&S_0 = \ldots = S_i = \ldots = \mathbb{R} \times \{0\} \times \mathbb{R},\\
&N_0 \subset N_0 + N_1 = N_0 + N_1 + N_2 = \ldots = \mathbb{R} \times \{0\} \times \mathbb{R},
\end{aligned}$$

and the maximal range is already $\operatorname{im} G_0$. A closer look at the DAE (23) gives

$$\begin{aligned}
x_1' + x_1 + x_3 &= q_1,\\
x_2' + x_2 &= q_2,\\
x_2 &= q_3.
\end{aligned}$$

This model is somehow dubious. It is in parts over- and underdetermined, and much place for interpretations is left (cf. also [LMT11b, Section 7]). In Section 10 below, this system is considered as an explicit ODE for the component $Dx = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with $Q_0x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$ to be chosen arbitrarily, accompanied by the consistency condition $\mathcal{W}_0Bx = \mathcal{W}_0q$, i.e. $x_2 = q_3$.

We take a closer look at problems of this kind in Section 10. Our next example is much nicer and more important with respect to applications. It is a so-called *Hessenberg form size three DAE* and might be considered as the linear prototype of a system describing constrained mechanical motion.

Example 3.6 *Hessenberg size three DAEs are relevant for the simulation of constrained mechanical motion. Consider the system*

$$\begin{bmatrix} x_1' \\ x_2' \\ 0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{24}$$

with $m = m_1 + m_2 + m_3$ equations, $m_1 \geq m_2 \geq m_3 \geq 1$, $k = m$ components, and a nonsingular product $B_{32}B_{21}B_{13}$. Put $n = m_1 + m_2$,

$$A = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix},$$

and write this DAE in the form (8).
Owing to the nonsingularity of the $m_3 \times m_3$ matrix function product $B_{32}B_{21}B_{13}$, the matrix functions B_{13} and $B_{21}B_{13}$ have full column rank m_3 each, and B_{32} has full row rank m_3. This yields $\operatorname{im}[AD\ \ B] = \mathbb{R}^m$. Further, since B_{13} and $B_{21}B_{13}$ have constant rank, there are continuous reflexive generalized inverses B_{13}^- and $(B_{21}B_{13})^-$ such that (see Proposition C.4)

$$\begin{aligned}
B_{13}^-B_{13} = I, \qquad & \Omega_1 := B_{13}B_{13}^- && \textit{is a projector onto} \quad \operatorname{im} B_{13},\\
(B_{21}B_{13})^-B_{21}B_{13} = I, \qquad & \Omega_2 := B_{21}B_{13}(B_{21}B_{13})^- && \textit{is a projector onto} \quad \operatorname{im} B_{21}B_{13}.
\end{aligned}$$

Let the coefficient function B be smooth enough so that the derivatives used below do exist. In particular, Ω_1 and Ω_2 are assumed to be continuously differentiable. We start constructing the matrix function sequence by

$$G_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} I & 0 & B_{13} \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$N_0 = \{z \in \mathbb{R}^m : z_1 = 0,\ z_2 = 0\}, \quad N_1 = \{z \in \mathbb{R}^m : z_1 + B_{13} z_3 = 0,\ z_2 = 0\},$$

$$\widehat{N_1} = N_0 \cap N_1 = \{0\}, \quad X_1 = N_0, \quad N_0 + N_1 = N_0 \oplus N_1 = \{z \in \mathbb{R}^m : z_2 = 0,\ z_1 \in \operatorname{im} B_{13}\}.$$

The matrix functions G_0 and G_1 have constant rank, $r_0 = r_1 = n$. Compute the projector functions

$$Q_1 = \begin{bmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ B_{13}^- & 0 & 0 \end{bmatrix}, \quad D\Pi_1 D^- = \begin{bmatrix} I - \Omega_1 & 0 \\ 0 & I \end{bmatrix},$$

such that $\operatorname{im} Q_1 = N_1$ *and* $Q_1 Q_0 = 0$, *that is* $\ker Q_1 \supseteq X_1$. *Q_1 is continuous, and $D\Pi_1 D^-$ is continuously differentiable. In consequence, Q_0, Q_1 are admissible. Next we form*

$$B_1 = \begin{bmatrix} B_{11} + \Omega_1' & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & B_{32} & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} I + (B_{11} + \Omega_1')\Omega_1 & 0 & B_{13} \\ B_{21}\Omega_1 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $z \in \mathbb{R}^{m_1+m_2+m_3}$ *with* $z_1 \in \ker \Omega_1$ *it holds that* $\operatorname{im} G_2 = \begin{bmatrix} z_1 + B_{13} z_3 \\ z_2 \\ 0 \end{bmatrix}$. *This proves the inclusion*

$$\operatorname{im} G_2 \subseteq \mathbb{R}^n \times \{0\} = \{G_2 z : z \in \mathbb{R}^{m_1+m_2+m_3}, z_1 \in \ker \Omega_1\} \subseteq \operatorname{im} G_2,$$

and we obtain $\operatorname{im} G_2 = \mathbb{R}^n \times \{0\}$, *and* $r_2 = \operatorname{rank} G_2 = m_1 + m_2 = n$. *Then we investigate the nullspace of G_2. If $z \in \mathbb{R}^m$ satisfies $G_2 z = 0$, then*

$$z_1 + (B_{11} + \Omega_1')\Omega_1 z_1 + B_{13} z_3 = 0, \tag{25}$$

$$B_{21}\Omega_1 z_1 + z_2 = 0. \tag{26}$$

In turn, equation (25) decomposes into

$$\begin{aligned} &(I - \Omega_1) z_1 + (I - \Omega_1)(B_{11} + \Omega_1')\Omega_1 z_1 = 0, \\ &B_{13}^-(I + B_{13}^-(B_{11} + \Omega_1'))\Omega_1 z_1 + z_3 = 0. \end{aligned}$$

Similarly, considering that $\operatorname{im} B_{21} B_{13} = \operatorname{im} B_{21} B_{13} B_{13}^-$ *is valid, we derive from (26) the relations*

$$z_2 = \Omega_2 z_2, \quad B_{13}^- z_1 = -(B_{21} B_{13})^- z_2.$$

Altogether this yields

$$N_2 = \{z \in \mathbb{R}^m : z_2 = \Omega_2 z_2,\ z_1 = \mathcal{E}_1\Omega_2 z_2,\ z_3 = \mathcal{E}_3\Omega_2 z_2\}, \quad \widehat{N_2} = \{0\}, \quad X_2 = N_0 + N_1,$$

with

$$\begin{aligned}\mathcal{E}_1 :&= -(I - (I-\Omega_1)(B_{11} + \Omega_1')\Omega_1)B_{13}(B_{21}B_{13})^- \\ &= -(I - (I-\Omega_1)(B_{11} + \Omega_1'))B_{13}(B_{21}B_{13})^-, \\ \mathcal{E}_3 :&= -B_{13}^-(I + (B_{11} + \Omega_1'))B_{13}(B_{21}B_{13})^-.\end{aligned}$$

Notice that $\mathcal{E}_1 = \mathcal{E}_1\Omega_2,\ \mathcal{E}_3 = \mathcal{E}_3\Omega_2$*. The projector functions*

$$Q_2 = \begin{bmatrix}0 & \mathcal{E}_1 & 0\\ 0 & \Omega_2 & 0\\ 0 & \mathcal{E}_3 & 0\end{bmatrix}, \quad D\Pi_2 D^- = \begin{bmatrix}I-\Omega_1 & -(I-\Omega_1)\mathcal{E}_1\\ 0 & I-\Omega_2\end{bmatrix},$$

fulfill the required admissibility conditions, in particular, $Q_2Q_0 = 0$*,* $Q_2Q_1 = 0$*, and hence* Q_0*,* Q_1*,* Q_2 *are admissible. The resulting* B_2*,* G_3 *have the form:*

$$B_2 = \begin{bmatrix}\mathcal{B}_{11} & \mathcal{B}_{12} & 0\\ \mathcal{B}_{21} & \mathcal{B}_{22} & 0\\ 0 & B_{32} & 0\end{bmatrix}, \quad G_3 = \begin{bmatrix}I + (B_{11}+\Omega_1')\Omega_1 & \mathcal{B}_{11}\mathcal{E}_1 + \mathcal{B}_{12}\Omega_2 & B_{13}\\ B_{21}\Omega_1 & I + \mathcal{B}_{21}\mathcal{E}_1 + \mathcal{B}_{22}\Omega_2 & 0\\ 0 & B_{32}\Omega_2 & 0\end{bmatrix}.$$

The detailed form of the entries $\mathcal{B}_{ij}$ *does not matter in this context. We show* G_3 *to be nonsingular. Namely,* $G_3 z = 0$ *implies* $B_{32}\Omega_2 z_2 = 0$*, thus* $\Omega_2 z_2 = 0$*, and further* $B_{21}\Omega_1 z_1 + z_2 = 0$*. The latter equation yields* $(I - \Omega_2)z_2 = 0$ *and* $B_{21}\Omega_1 z_1 = 0$*, and this gives* $\Omega_1 z_1 = 0,\ z_2 = 0$*. Now, the first line of the system* $G_3 z = 0$ *simplifies to* $z_1 + B_{13}z_3 = 0$*. In turn,* $(I-\Omega_1)z_1 = 0$ *follows, and hence* $z_1 = 0,\ z_3 = 0$*. The matrix function* G_3 *is nonsingular in fact, and we stop the construction.*
In summary, our basic subspaces behaves as

$$\begin{aligned}\operatorname{im} G_0 = \operatorname{im} G_1 = \operatorname{im} G_2 \subset \operatorname{im} G_3 = \operatorname{im}\,[AD\ \ B] = \mathbb{R}^m,\\ N_0 \subset N_0 + N_1 \subset N_0 + N_1 + N_2 = N_0 + N_1 + N_2 + N_3 \subset \mathbb{R}^m.\end{aligned}$$

The additionally associated projector functions $\mathcal{W}_i$ *onto* $\operatorname{im} G_i$ *and the subspaces* $S_i = \ker \mathcal{W}_i B$ *are here:*

$$\mathcal{W}_0 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I\end{bmatrix}, \quad \mathcal{W}_0 = \mathcal{W}_1 = \mathcal{W}_2, \quad \mathcal{W}_3 = 0,$$

and

$$S_0 = \{z \in \mathbb{R}^m : B_{32}z_2 = 0\}, \quad S_0 = S_1 = S_2 \subset S_3 = \mathbb{R}^m.$$

This special subspace behavior is typical for the large class of DAEs named Hessenberg form DAEs. While $\operatorname{im} G_3$ *and* S_3 *reach the maximal dimension* m*, the dimension of the resulting maximal subspace* $N_0 + N_1 + N_2$ *is less than* m*.*
Notice that the relation $\mathcal{W}_0 B Q_0 = 0$ *indicates that* $\operatorname{im} G_0 = \operatorname{im} G_1$ *holds true, and we can recognize this fact before explicitly computing* G_1 *(cf. Proposition 2.4(3)). Similarly,* $\mathcal{W}_1 B Q_1 = 0$ *indicates that* $\operatorname{im} G_1 = \operatorname{im} G_2$*. Furthermore, we know that* $r_3 = r_2 + \operatorname{rank}(\mathcal{W}_2 B Q_2) = n + m_3 = m$ *before we compute* G_3*.*

Now we come to an important auxiliary result which stands behind Theorem 3.3, and which generalizes [LMT11b, Lemma 2.10].

Lemma 3.7 *If there are two projector function sequences $Q_0,\dots,Q_\kappa$ and $\bar{Q}_0,\dots,\bar{Q}_\kappa$, both admissible on $\mathcal{I}$ for the DAE (8), then the corresponding matrix functions and subspaces are related by the following properties:*

(1) $\ker \bar{\Pi}_j = \bar{N}_0+\dots+\bar{N}_j = N_0+\dots+N_j = \ker \Pi_j,\ j=0,\dots,\kappa,$

(2) $\bar{G}_j = G_j Z_j,$
$\bar{B}_j = B_j - G_j Z_j \bar{D}^-(D\bar{\Pi}_j\bar{D}^-)'D\Pi_j + G_j\sum_{l=0}^{j-1} Q_l\mathfrak{A}_{jl},\ j=1,\dots,\kappa,$
with nonsingular matrix functions $Z_0,\dots,Z_{\kappa+1}$ given by
$Z_0 := I,\ Z_{i+1} := Y_{i+1}Z_i,\ i=0,\dots,\kappa,$

$$\begin{aligned} Y_1 &:= I + Q_0(\bar{Q}_0 - Q_0) = I + Q_0\bar{Q}_0P_0,\\ Y_{i+1} &:= I + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + \sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i,\ i=1,\dots,\kappa, \end{aligned}$$

and certain continuous coefficients $\mathfrak{A}_{il}$ that satisfy condition $\mathfrak{A}_{il} = \mathfrak{A}_{il}\bar{\Pi}_{i-1}$,

(3) $Z_i(\bar{N}_i\cap(\bar{N}_0+\dots+\bar{N}_{i-1})) = N_i\cap(N_0+\dots+N_{i-1}),\quad i=1,\dots,\kappa,$

(4) $\bar{G}_{\kappa+1} = G_{\kappa+1}Z_{\kappa+1},\quad \bar{N}_0+\dots+\bar{N}_{\kappa+1} = N_0+\dots+N_{\kappa+1},$
$Z_{\kappa+1}(\bar{N}_{\kappa+1}\cap(\bar{N}_0+\dots+\bar{N}_\kappa)) = N_{\kappa+1}\cap(N_0+\dots+N_\kappa).$

Proof: We have $G_0 = AD = \bar{G}_0,\quad B_0 = B = \bar{B}_0,\quad \ker P_0 = N_0 = \bar{N}_0 = \ker\bar{P}_0$, hence $P_0 = P_0\bar{P}_0,\ \bar{P}_0 = \bar{P}_0P_0$.
The generalized inverses D^- and $\bar{D}^-$ of D satisfy the properties $DD^- = D\bar{D}^- = R$, $D^-D = P_0,\ \bar{D}^-D = \bar{P}_0$, and therefore $\bar{D}^- = \bar{D}^-D\bar{D}^- = \bar{D}^-DD^- = \bar{P}_0D^-,\ D^- = P_0\bar{D}^-$.
Compare $G_1 = G_0 + B_0Q_0$ and

$$\begin{aligned} \bar{G}_1 &= \bar{G}_0 + \bar{B}_0\bar{Q}_0 = G_0 + B_0\bar{Q}_0 = G_0 + B_0Q_0\bar{Q}_0\\ &= (G_0 + B_0Q_0)(P_0 + \bar{Q}_0) = G_1Z_1, \end{aligned}$$

where $Z_1 := Y_1 := P_0 + \bar{Q}_0 = I + Q_0\bar{Q}_0P_0 = I + Q_0(\bar{Q}_0 - Q_0)$. Z_1 is invertible, it has the inverse $Z_1^{-1} = I - Q_0\bar{Q}_0P_0$.
The nullspaces N_1 and $\bar{N}_1$ are, due to $\bar{G}_1 = G_1Z_1$, related by $\bar{N}_1 = Z_1^{-1}N_1 \subseteq N_0 + N_1$. This implies $\bar{N}_0+\bar{N}_1 = N_0+(Z_1^{-1}N_1) \subseteq N_0+N_1$. From $N_1 = Z_1\bar{N}_1 \subseteq N_0+\bar{N}_1 = \bar{N}_0+\bar{N}_1$, we obtain $\bar{N}_0 + \bar{N}_1 = N_0 + N_1$.

Since the projectors P_0P_1 and $\bar{P}_0\bar{P}_1$ have the common nullspace $N_0 + N_1 = \bar{N}_0 + \bar{N}_1$, we may now derive

$$\begin{aligned} D\bar{P}_0\bar{P}_1\bar{D}^- &= D\bar{P}_0\bar{P}_1 \overbrace{P_0P_1}^{=P_0P_1P_0} \bar{P}_0D^- = D\bar{P}_0\bar{P}_1P_0P_1D^- = D\bar{P}_0\bar{P}_1\bar{D}^-DP_0P_1D^-,\\ DP_0P_1D^- &= DP_0P_1D^-D\bar{P}_0\bar{P}_1\bar{D}^-. \end{aligned}$$

Taking into account the relation $0 = \bar{G}_1\bar{Q}_1 = G_1\bar{Q}_1 + G_1(Z_1 - I)\bar{Q}_1$, thus $G_1\bar{Q}_1 = -G_1(Z_1 - I)\bar{Q}_1$ we obtain (cf. Appendix B for details)

$$\bar{B}_1 = B_1 - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1D^-)'D.$$

This gives the basis for proving our assertion by induction. The proof is carried out in detail in Appendix B. A technically easier version for the time-invariant case is given in [LMT11b]. $\square$.

4 Preliminary decoupling rearrangements

In this section we use admissible projector functions $Q_0, \dots, Q_\kappa$ to rearrange terms in the DAE (8) in a similar way as it is done in [LMT11b] on constant coefficient DAEs for obtaining decoupled systems. The objective of the rearrangements is to place a matrix function G_κ in front of the derivative component $(D\Pi_\kappa x)'$, the rank of which is as large as possible, and at the same time to separate terms living in $N_0 + \cdots + N_\kappa$.
We emphasize that we do not change at all the given DAE, and do not transform the variables. We work just with the given DAE and its unknown. What we do are *rearrangements of terms and separations or decouplings of solution components* by means of projector functions. We proceed stepwise. Within this procedure, the special form of the matrix functions B_i in (15) become appearend to make good sense.
Later on (see Definitions 6.2 and 10.1) the tractability index of the DAE is assigned to the smallest integer μ such that the rank r_μ is maximal. This is valid for general, possibly rectangular DAEs. The rearranged DAE versions serve then as the basis for the further decouplings and solutions.

Rewrite first (8) as

$$G_0 D^-(Dx)' + B_0 x = q, \tag{27}$$

and then as

$$G_0 D^-(Dx)' + B_0(Q_0 x + P_0 x) = q$$

and rearranging this in order to increase the rank of the leading coefficient to

$$(G_0 + B_0 Q_0)(D^-(Dx)' + Q_0 x) + B_0 P_0 x = q,$$

or

$$G_1 D^-(Dx)' + B_0 P_0 x + G_1 Q_0 x = q. \tag{28}$$

Compute

$$\begin{aligned} P_1 D^-(Dx)' &= P_0 P_1 D^-(Dx)' + Q_0 P_1 D^-(Dx)' \\ &= D^- D P_0 P_1 D^-(Dx)' + Q_0 P_1 D^-(Dx)' \\ &= D^-(D P_0 P_1 x)' - D^-(D P_0 P_1 D^-)' D x + Q_0 P_1 D^-(Dx)' \\ &= D^-(D P_0 P_1 x)' - D^-(D P_0 P_1 D^-)' D x - (I - P_0) Q_1 D^-(Dx)' \\ &= D^-(D\Pi_1 x)' - D^-(D\Pi_1 D^-)' D x - (I - \Pi_0) Q_1 D^-(D\Pi_0 x)', \end{aligned}$$

hence

$$G_1 D^-(Dx)' = G_1 D^-(D\Pi_1 x)' - G_1 D^-(D\Pi_1 D^-)' D P_0 x - G_1 (I - \Pi_0) Q_1 D^-(D\Pi_0 x)'.$$

Inserting this into (28) yields

$$\begin{aligned} G_1 D^-(D\Pi_1 x)' &+ (B_0 P_0 - G_1 D^-(D\Pi_1 D^-)' D P_0) x \\ &+ G_1\{Q_0 x - (I - \Pi_0) Q_1 D^-(Dx)'\} = q, \end{aligned}$$

and, regarding the definition of the matrix function B_1,

$$G_1 D^-(D\Pi_1 x)' + B_1 x + G_1\{Q_0 x - (I - \Pi_0)Q_1 D^-(Dx)'\} = q. \tag{29}$$

Note that, if $N_0 \cap N_1 = 0$, then the derivative $(DP_0P_1x)'$ is no more involved in the term

$$Q_1 D^-(Dx)' = Q_1 D^- DP_0 Q_1 D^-(Dx)' = Q_1 D^-(DP_0 Q_1 x)' - Q_1 D^-(DP_0 Q_1 D^-)' Dx.$$

In the next step we move a part of the term $B_1 x$ in (29) to the leading term, and so on. Proposition 4.1 describes the result of these systematic rearrangements.

Proposition 4.1 *Let the DAE (8) with properly stated leading term have the admissible projectors $Q_0, \ldots, Q_\kappa$, where $\kappa \in \mathbb{N} \cup \{0\}$.*

(1) *Then this DAE can be rewritten in the form*

$$G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x + G_\kappa \sum_{l=0}^{\kappa-1}\{Q_l x + (I - \Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} = q. \tag{30}$$

(2) *If, additionally, all intersections $\widehat{N_i}$, $i = 1, \ldots, \kappa$, are trivial, then the DAE (8) rewrites as*

$$\begin{aligned} &G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x \\ &\qquad + G_\kappa \sum_{l=0}^{\kappa-1}\{Q_l x - (I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + V_l D\Pi_l x\} = q, \end{aligned} \tag{31}$$

with coefficients

$$V_l = (I - \Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l D^-, \; l = 0, \ldots, \kappa - 1.$$

Comparing with the rearranged DAE obtained in the constant coefficient case (cf. [LMT11b, (38)]), now we observe the extra terms V_l caused by time-dependent movements of certain subspaces. They disappear in the time-invariant case.

Proof of Proposition 4.1:
(1) In case of $\kappa = 0$, equation (27) is just a trivial reformulation of (8). For $\kappa = 1$ we are done by considering (29). For applying induction, we suppose for $i + 1 \le \kappa$, that (8) rewrites as

$$G_i D^-(D\Pi_i x)' + B_i x + G_i \sum_{l=0}^{i-1}\{Q_l x + (I - \Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)'\} = q. \tag{32}$$

Represent $B_i x = B_i P_i x + B_i Q_i x = B_i P_i x + G_{i+1}Q_i x$ and derive

$$\begin{aligned} G_i D^-(D\Pi_i x)' &= G_{i+1}P_{i+1}P_i D^-(D\Pi_i x)' \\ &= G_{i+1}\{\Pi_{i+1}P_i D^-(D\Pi_i x)' + (I - \Pi_i)P_{i+1}P_i D^-(D\Pi_i x)'\} \\ &= G_{i+1}\{D^- D\Pi_{i+1}D^-(D\Pi_i x)' + (I - \Pi_i)P_{i+1}P_i D^-(D\Pi_i x)'\} \\ &= G_{i+1}D^-(D\Pi_{i+1}x)' - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i x \\ &\quad + G_{i+1}(I - \Pi_i)(P_i - Q_{i+1}P_i)D^-(D\Pi_i x)'). \end{aligned}$$

Taking into account that $(I-\Pi_i) = Q_0P_1\cdots P_i + \cdots + Q_{i-1}P_i + Q_i$ and $G_iQ_l = G_{i+1}Q_l$, $l = 0,\ldots,i-1$, we realize that (32) can be reformulated to

$$\begin{aligned}
G_{i+1}D^-(D\Pi_{i+1}x)' &+ (B_iP_i - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i)x \\
&\qquad + G_{i+1}Q_ix + G_{i+1}\sum_{l=0}^{i-1}\{Q_lx + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lx)'\} \\
&\qquad + G_{i+1}(I-\Pi_i)(P_i - Q_{i+1}P_i)D^-(D\Pi_ix)' = q.
\end{aligned}$$

We obtain in fact

$$G_{i+1}D^-(D\Pi_{i+1}x)' + B_{i+1}x + G_{i+1}\sum_{l=0}^{i}\{Q_lx + (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lx)'\} = q$$

as we tried for.
(2) Finally assuming $\widehat{N}_i = \{0\}$, $i = 1,\ldots,\kappa$, and taking into account Proposition 3.2, we compute the part being in question as

$$\begin{aligned}
\mathcal{F} :=& \sum_{l=0}^{k-1}(I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lx)' = \sum_{l=0}^{k-1}(I-\Pi_l)(P_l - Q_{l+1})D^-(D\Pi_lx)' \\
=& \sum_{l=0}^{k-1}(I-\Pi_l)[P_lD^-(D\Pi_lx)' - Q_{l+1}D^-D\Pi_lQ_{l+1}D^-(D\Pi_lx)'].
\end{aligned}$$

Applying the relations

$$\begin{aligned}
(D\Pi_lx)' &= (D\Pi_lD^-)'(D\Pi_lx) + D\Pi_lD^-(D\Pi_lx)', \\
(I-\Pi_l)P_lD^-D\Pi_lD^- &= (I-\Pi_l)P_l\Pi_lD^- = 0, \\
D\Pi_lQ_{l+1}D^-(D\Pi_lx)' &= (D\Pi_lQ_{l+1}x)' - (D\Pi_lQ_{l+1}D^-)'D\Pi_lx, \\
Q_{l+1}(D\Pi_lQ_{l+1}D^-)'D\Pi_l &= Q_{l+1}(D\Pi_lD^-)'D\Pi_l - Q_{l+1}(D\Pi_{l+1}D^-)'D\Pi_l \\
&= -Q_{l+1}(D\Pi_{l+1}D^-)'D\Pi_l,
\end{aligned}$$

we obtain, with the coefficients V_l described by the assertion,

$$\begin{aligned}
\mathcal{F} = \sum_{l=0}^{k-1}(I-\Pi_l)&[P_lD^-(D\Pi_lD^-)'D\Pi_lx + Q_{l+1}D^-(D\Pi_lQ_{l+1}D^-)'D\Pi_lx \\
&- Q_{l+1}D^-(D\Pi_lQ_{l+1}x)'] = \sum_{l=0}^{k-1}[V_lD\Pi_lx - (I-\Pi_l)Q_{l+1}D^-(D\Pi_lQ_{l+1}x)'],
\end{aligned}$$

and this completes the proof. □

How can one take use of the rearranged version of the DAE (8) and the structural information included in this version? We discuss this question in Section 6 for the case of regular DAEs, that is for $m = k$, and if a nonsingular G_μ exists. We study the general case in Section 10. At the moment, to gain a first impression, we cast a look on the simplest situation, if already G_0 has maximal rank. Then the DAE (27) splits into the two parts

$$G_0D^-(Dx)' + G_0G_0^-B_0x = G_0G_0^-q, \quad \mathcal{W}_0B_0x = \mathcal{W}_0q. \tag{33}$$

Since $\operatorname{im} G_0$ is maximal, it holds that $\operatorname{im} B_0Q_0 \subseteq \operatorname{im} G_1 = \operatorname{im} G_0$, hence $\mathcal{W}_0B_0 = \mathcal{W}_0B_0P_0$. Further, since $DG_0^-G_0 = D$, we find the DAE (27) to be equivalent to the system

$$(Dx)' - R'Dx + DG_0^-B_0D^-Dx + DG_0^-B_0Q_0x = DG_0^-q, \quad \mathcal{W}_0B_0D^-Dx = \mathcal{W}_0q, \tag{34}$$

the solution of which decomposes as $x = D^-Dx + Q_0x$. It becomes clear, this DAE comprises an explicit ODE for Dx, that has an undetermined part Q_0x to be chosen arbitrarily. The ODE for Dx is accompanied by a consistency condition applied to Dx and q. If G_0 is surjective, the consistency condition disappears. If G_0 is injective, then the undetermined component Q_0x disappears. If G_0 is nonsingular, what happens just for $m = k$, then the DAE is nothing else a regular implicit ODE with respect to Dx. Later on we assign the *tractability index zero* to each DAE whose matrix functions G_0 have already maximal rank .
Of course, if the tractability index is greater than zero, things become much more subtle. We refer once again to the discussion in Sections 10 and 6.

5 Invariants under transformations and refactorizations

Given is a DAE (8) with properly stated leading term. We premultiply this equation by a nonsingular matrix function $L \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^k))$ and transform the unknown function $x = K\bar{x}$ by means of a nonsingular function $K \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ such that the DAE

$$\bar{A}(\bar{D}\bar{x})' + \bar{B}\bar{x} = \bar{q} \tag{35}$$

results, where $\bar{q} := Lq$, and

$$\bar{A} := LA, \quad \bar{D} := DK, \quad \bar{B} := LBK. \tag{36}$$

The new coefficients are continuous, too. $\bar{A}$ and $\bar{D}$ inherit from A and D the constant ranks, and the leading term of (35) is properly stated (cf. Definition 2.1) with the same border projector $\bar{R} = R$ as $\ker \bar{A} = \ker A$, $\operatorname{im} \bar{D} = \operatorname{im} D$.
Suppose that the original DAE (8) has admissible projectors $Q_0, \ldots, Q_\kappa$. We form a corresponding matrix function sequence for the transformed DAE (35) starting with

$$\begin{aligned} \bar{G}_0 &= \bar{A}\bar{D} = LADK = LG_0K, \quad \bar{B}_0 = \bar{B} = LB_0K, \\ \bar{Q}_0 &:= K^{-1}Q_0K, \quad \bar{D}^- = K^{-1}D^-, \quad \bar{P}_0 = K^{-1}P_0K, \end{aligned}$$

such that $\bar{D}\bar{D}^- = DD^- = R$, $\bar{D}^-\bar{D} = \bar{P}_0$, and

$$\bar{G}_1 = \bar{G}_0 + \bar{B}_0\bar{Q}_0 = L(G_0 + B_0Q_0)K = LG_1K.$$

This yields $\bar{N}_0 = K^{-1}N_0$, $\bar{N}_1 = K^{-1}N_1$, $\bar{N}_0 \cap \bar{N}_1 = K^{-1}(N_0 \cap N_1)$. Choose $\bar{Q}_1 := K^{-1}Q_1K$ what corresponds to $\bar{X}_1 := K^{-1}X_1$. Proceeding in this way at each level, $i = 1, \ldots, \kappa$, with

$$\bar{Q}_i := K^{-1}Q_iK$$

it results that $\bar{\Pi}_i = K^{-1}\Pi_iK$, $\bar{D}\bar{\Pi}_i\bar{D}^- = D\Pi_iD^-$, $\bar{X}_i = K^{-1}X_i$, $\overline{N\mathcal{U}}_i = K^{-1}(\widehat{N_i})$, and

$$\bar{G}_{i+1} = LG_{i+1}K, \quad \bar{B}_{i+1} = LB_{i+1}K.$$

This shows that $\bar{Q}_0, \ldots, \bar{Q}_\kappa$ are admissible for (35), and the following assertion becomes evident.

Theorem 5.1 *If the DAE* (8) *has admissible projectors up to level* $\kappa \in \mathbb{N}$, *and characteristic values* r_i, u_i, $i = 0, \ldots, \kappa$, *then the transformed equation* (35) *has also admissible projectors up to level* κ, *and the characteristic values coincide, i.e.* $\bar{r}_i = r_i$, $\bar{u}_i = u_i$, $i = 1, \ldots, \kappa$.

By Theorem 5.1 the characteristic values and the tractability index are invariant under transformations of the unknown function as well as under premultiplications of the DAE. This feature seems to be rather trivial.
The invariance with respect to refactorizations of the leading term, which we verify next, is more subtle. For a given DAE (8) with properly stated leading term, we consider the product AD to represent a *factorization of the leading term* and we ask whether we can turn to a different factorization $AD = \bar{A}\bar{D}$ such that $\ker \bar{A}$ and $\operatorname{im} \bar{D}$ are again transversal $\mathcal{C}^1$-subspaces. For instance, in Example 1.1, equation (7) results from equation (6) by taking a different factorization.

In general, we describe the change to a different factorization as follows:
Let $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^n)$ be given together with a generalized inverse $H^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^s))$ such that $H^-HH^- = H^-$, $HH^-H = H$, and

$$RHH^-R = R. \tag{37}$$

H has constant rank greater or equal the rank of the border projector R. In particular, one can use any nonsingular $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$. However, we do not restrict ourselves to square nonsingular matrix functions H.
Due to $AR = ARHH^-R$ we may write

$$\begin{aligned} A(Dx)' = ARHH^-R(Dx)' &= ARH(H^-RDx)' - ARH(H^-R)'Dx \\ &= AH(H^-Dx)' - AH(H^-R)'Dx. \end{aligned}$$

This leads to the new DAE

$$\bar{A}(\bar{D}x)' + \bar{B}x = q \tag{38}$$

with the continuous coefficients

$$\bar{A} := AH, \quad \bar{D} := H^-D, \quad \bar{B} := B - ARH(H^-R)'D. \tag{39}$$

Because of $\bar{A}\bar{D} = AD$ we call this procedure that changes (8) to (38) a *refactorization of the leading term*. It holds that

$$\ker \bar{A} = \ker AH = \ker RH, \quad \operatorname{im} \bar{D} = \operatorname{im} H^-D = \operatorname{im} H^-R,$$

further $(H^-RH)^2 = H^-RHH^-RH = H^-RH$. It becomes clear that $H^-RH \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s))$ is actually the border projector corresponding to the new DAE (38), and (38) has a properly stated leading term.
We emphasize that the old border space $\mathbb{R}^n$ and the new one $\mathbb{R}^s$ may actually have different dimensions, and this is accompanied by different sizes of the involved matrix functions. Here, the only restriction is $n, s \geq r$. The next example underlines the need of those changes.

Example 5.2 *The semi-explicit DAE*

$$\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1, \\ B_{21}x_1 + B_{22}x_2 &= q_2, \end{aligned}$$

with $m_1 + m_2 = m = k$ equations can be put into the form (8) in different ways.
a) Choose $n = m$,

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad D^- = D.$$

b) Choose $n = m_1$, $\bar{A} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\bar{D} = \begin{bmatrix} I & 0 \end{bmatrix}$, and $\bar{B} = B$, $\bar{D}^- = \begin{bmatrix} I \\ 0 \end{bmatrix}$.
In both cases, it results that

$$G_0 = \bar{G}_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_0 = \bar{Q}_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad G_1 = \bar{G}_1 = \begin{bmatrix} I & B_{12} \\ 0 & B_{22} \end{bmatrix}.$$

Observe that with $H = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $H^- = \begin{bmatrix} I & 0 \end{bmatrix}$ we can write $\bar{A} = AH$, $\bar{D} = H^-D$, and $\bar{A}\bar{D} = AHH^-D = AD$. The condition (37) is fulfilled. Therefore, the DAE in b) results from a refactorization of the DAE in a).

Theorem 5.3 *Let the DAE* (8) *have a properly stated leading term and admissible projectors up to level $\kappa \in \mathbb{N}$ as well as characteristic values $r_0, \ldots, r_\kappa$, $u_0, \ldots, u_\kappa$.*

(a) *Then the refactorized DAE* (38) *has also a properly stated leading term and admissible projectors up to level κ. Its characteristic values coincide with that of* (8)*.*

(b) *The subspaces* $\operatorname{im} G_i$, $N_0 + \ldots + N_i$, $i = 0, \ldots, \kappa$, *are invariant.*

Proof: Put $F_1 := I$.
We use induction to show that the following relations are valid:

$$\bar{G}_i = G_i F_i \cdots F_1, \tag{40}$$

$$\bar{Q}_i := (F_i \cdots F_1)^{-1} Q_i F_i \cdots F_1, \quad \bar{\Pi}_{i-1}\bar{Q}_i = \Pi_{i-1}Q_i, \quad \bar{\Pi}_i = \Pi_i, \tag{41}$$

$$\bar{B}_i = B_i - G_i D^- H (H^- R)' D \Pi_i + G_i \sum_{j=0}^{i-1} Q_j Z_{ij} \Pi_{i-1}, \tag{42}$$

with nonsingular

$$F_i := I + P_{i-1} \sum_{j=0}^{i-2} Q_j Z_{i-1j} \Pi_{i-2} Q_{i-1}, \quad i = 1, \ldots, \kappa.$$

The coefficients $Z_{\ell j}$ are continuous matrix functions which special form does not matter at all.
Since $\bar{G}_0 = \bar{A}\bar{D} = AD = G_0$ we may choose $\bar{D}^- = D^-H$, $\bar{Q}_0 = Q_0$. It results that $\bar{\Pi}_0 = \Pi_0$, $\bar{B}_0 = \bar{B} = B - ARH(H^-R)'D$ and $\bar{B}_0\bar{Q}_0 = BQ_0 = B_0Q_0$, hence

$\bar{G}_1 = \bar{G}_0 + \bar{B}_0\bar{Q}_0 = G_0 + B_0Q_0 = G_1 = G_1F_1$. Choose $\bar{Q}_1 = Q_1 = F_1^{-1}Q_1$ such that $\bar{\Pi}_1 = \Pi_1$, $\bar{\Pi}_0\bar{Q}_1 = \Pi_0Q_1$, $\bar{D}\bar{\Pi}_1\bar{D}^- = H^-D\Pi_1D^-H$, and further

$$\begin{aligned}
\bar{B}_1 &= \bar{B}_0\bar{P}_0 - \bar{G}_1\bar{D}^-(\bar{D}\bar{\Pi}_1\bar{D}^-)'\bar{D}\bar{\Pi}_0 \\
&= B_0P_0 - ARH(H^-R)'D - G_1D^-H(H^-D\Pi_1D^-H)'H^-D\Pi_0 \\
&= B_0P_0 - G_1D^-(D\Pi_1D^-)'D\Pi_0 + G_1D^-(D\Pi_1D^-)'D\Pi_0 \\
&\quad -ARH(H^-R)'D - G_1D^-H(H^-RD\Pi_1D^-RH)'H^-D\Pi_0 \\
&= B_1 + G_1D^-(D\Pi_1D^-)'D\Pi_0 - ARH(H^-R)'D - G_1D^-H\{(H^-R)'D\Pi_1D^-RH \\
&\quad +H^-R(D\Pi_1D^-)'RH + H^-RD\Pi_1D^-(RH)'\}H^-D \\
&= B_1 - ARH(H^-R)'D - G_1D^-H(H^-R)'D\Pi_1 - G_1\Pi_1D^-(RH)'H^-RD \\
&= B_1 - G_1D^-H(H^-R)'D\Pi_1 - ARH(H^-R)'D + G_1\Pi_1D^-RH(H^-R)'D.
\end{aligned}$$

In the last expression we have used that

$$D^-(RHH^-R)'D = D^-R'D = 0.$$

Compute $G_1\Pi_1D^-RH(H^-R)'D - ARH(H^-R)'D = G_1(\Pi_1 - I)D^-RH(H^-R)'D$ and

$$\begin{aligned}
G_1(\Pi_1 - I) &= G_1((I-Q_0)(I-Q_1) - I) = G_1(-Q_0 - Q_1 + Q_0Q_1) \\
&= G_1(-Q_0 + Q_0Q_1) = -G_1Q_0P_1.
\end{aligned}$$

This yields the wanted expression

$$\bar{B}_1 = B_1 - G_1D^-H(H^-R)'D\Pi_1 + G_1Q_0Z_{10}\Pi_0$$

with $Z_{10} := -Q_0P_1D^-RH(H^-R)'D$.
Next, supposing the relations (40)–(42] to be given up to i we show their validity for $i+1$. Derive

$$\begin{aligned}
\bar{G}_{i+1} &= \bar{G}_i + \bar{B}_i\bar{Q}_i = \{G_i + \bar{B}_i(F_i\cdots F_1)^{-1}Q_i\}F_i\cdots F_1 \\
&= \{G_i + \bar{B}_i\Pi_{i-1}(F_i\cdots F_1)^{-1}Q_i\}F_i\cdots F_1,
\end{aligned}$$

and, because of $\Pi_{i-1}F_1^{-1}\cdots F_i^{-1} = \Pi_{i-1}$, we obtain further

$$\begin{aligned}
\bar{G}_{i+1} &= \Big\{G_i + B_iQ_i - G_iD^-H(H^-R)'D\Pi_iQ_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1 \\
&= \Big\{G_{i+1} + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1 \\
&= G_{i+1}\Big\{I + P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i\Big\}F_i\cdots F_1 \\
&= G_{i+1}F_{i+1}F_i\cdots F_1,
\end{aligned}$$

with nonsingular matrix functions

$$F_{i+1} = I + P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i, \quad F_{i+1}^{-1} = I - P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\Pi_{i-1}Q_i.$$

Put $\bar{Q}_{i+1} := (F_{i+1}\cdots F_1)^{-1}Q_{i+1}F_{i+1}\cdots F_1$, and compute

$$\begin{aligned}\bar{\varPi}_i\bar{Q}_{i+1} &= \varPi_i\bar{Q}_{i+1} = \varPi_i F_1^{-1}\cdots F_{i+1}^{-1}Q_{i+1}F_{i+1}\cdots F_1\\ &= \varPi_i Q_{i+1}F_{i+1}\cdots F_1 = \varPi_i Q_{i+1}\varPi_i F_{i+1}\cdots F_1 = \varPi_i Q_{i+1}\varPi_i = \varPi_i Q_{i+1},\\ \bar{\varPi}_{i+1} &= \bar{\varPi}_i - \bar{\varPi}_i\bar{Q}_{i+1} = \varPi_i - \varPi_i Q_{i+1} = \varPi_{i+1}.\end{aligned}$$

It remains to verify the expression for $\bar{B}_{i+1}$. We derive

$$\begin{aligned}\bar{B}_{i+1} &= \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(\bar{D}\bar{\varPi}_{i+1}\bar{D}^-)'\bar{D}\bar{\varPi}_i\\ &= \bar{B}_i\varPi_i - G_{i+1}F_{i+1}\cdots F_1 D^-H(H^-D\varPi_{i+1}D^-H)'H^-D\varPi_i,\end{aligned}$$

and

$$\begin{aligned}\bar{B}_{i+1} = \; & \Big\{B_i - G_iD^-H(H^-R)'D\varPi_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_{i-1}\Big\}\varPi_i\\ & -G_{i+1}(F_{i+1}\cdots F_1 - I)D^-H(H^-D\varPi_{i+1}D^-H)'H^-D\varPi_i\\ & -G_{i+1}D^-H\{(H^-R)'RD\varPi_{i+1}D^-RH + H^-R(D\varPi_{i+1}D^-)'RH\\ & +H^-RD\varPi_{i+1}D^-(RH)'\}H^-D\varPi_i,\end{aligned}$$

and

$$\begin{aligned}\bar{B}_{i+1} = \; & B_iP_i - G_iD^-H(H^-R)'D\varPi_i + G_i\sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_i\\ & -G_{i+1}D^-H(H^-R)'D\varPi_{i+1} - G_{i+1}D^-(D\varPi_{i+1}D^-)'D\varPi_i\\ & -G_{i+1}\varPi_{i+1}D^-(RH)'H^-RD\varPi_i\\ & -G_{i+1}(F_{i+1}\cdots F_1 - I)D^-H(H^-D\varPi_{i+1}D^-H)'H^-D\varPi_i,\end{aligned}$$

and

$$\begin{aligned}\bar{B}_{i+1} = \; & B_{i+1} - G_{i+1}D^-H(H^-R)'D\varPi_{i+1} - G_{i+1}P_iD^-H(H^-R)'D\varPi_i\\ & +G_{i+1}\varPi_{i+1}D^-H(H^-R)'D\varPi_i + G_{i+1}P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_i\\ & -G_{i+1}(F_{i+1}\cdots F_i - I)D^-H(H^-D\varPi_{i+1}D^-H)'H^-D\varPi_i.\end{aligned}$$

Finally, decomposing

$$P_i\sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_i = \sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_i - Q_i\sum_{j=0}^{i-1}Q_jZ_{ij}\varPi_i,$$

and expressing

$$F_{i+1}\cdots F_1 - I = \sum_{j=0}^{i}Q_j\mathfrak{A}_{i+1,j},$$

and taking into account that

$$G_{i+1}\{\varPi_{i+1} - P_i\}D^-H(H^-R)'D\varPi_i = G_{i+1}\sum_{j=0}^{i}Q_i\mathfrak{B}_{i+1,j}D^-H(H^-R)'D\varPi_i$$

we obtain

$$\bar{B}_{i+1} = B_{i+1} - G_{i+1}D^- H(H^- R)' D\Pi_{i+1} + \sum_{j=0}^{i} Q_j Z_{i+1,j} D\Pi_i. \qquad \square$$

By Theorem 5.3, the characteristic values and the tractability index are invariant under refactorizations of the leading term. Thereby, the size of A and D may change or not (cf. Examples 1.1 and 5.2).

6 Regular DAEs

6.1 Regularity and basic decoupling

We define regularity for DAEs after the model of classical ODE theory, where the system

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{43}$$

with continuous coefficients, is named an implicit regular ODE or an ODE having solely regular line elements, if the matrix $A(t)$ remains nonsingular on the given interval. Roughly speaking, in our view, regular DAEs should be such that the corresponding homogeneous versions have finite-dimensional solution spaces, and no consistency conditions related to the excitations q will arise for inhomogeneous equations. This rules out the DAEs being non-square. Additionally, each restriction of the DAE to a subinterval should inherit also the space of admissible excitations.
In case of constant coefficients, regularity of DAEs is bound to regular pairs of square matrices. In turn, regularity of matrix pairs can be characterized by means of matrix sequences built by admissible projectors, and the associated characteristic values, as described in [LMT11b, Section 5]. A pair of $m \times m$ matrices is regular, if and only if there is a characteristic value $r_\mu = m$. Then the Kronecker index of the given matrix pair results as the smallest such index μ. The same idea applies to DAEs with time-varying coefficients, too. However, in distinction to the case of constant matrices in [LMT11b], we are now facing matrix functions. While, in case of constant coefficients, admissible projectors do always exist, their existence is now tied to several rank conditions. These rank conditions do not represent a mistake in the construction, but they are indeed relevant for the problem. In particular, in case of the implicit ODE (43), each point at which the matrix $A(t)$ becomes singular is a critical point, and different kind of singularities may arise (e.g. [KKW01]).

We turn back to equation (8), i.e.,

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{I}. \tag{44}$$

We are looking for solutions in the function space

$$\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m) = \{x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m) : Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)\}.$$

If $x_q \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ denotes a solution corresponding to the excitation q, and $x_{hom} \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ satisfies the homogenous DAE

$$A(t)(D(t)x(t))' + B(t)x(t) = 0, \quad t \in \mathcal{I}, \tag{45}$$

then the sum $x_q + x_{hom}$ is also a solution of the excited DAE (44). This linearity property motivates special attention to the solution structure of the homogenous DAE.

Definition 6.1 *The subspace*

$$S_{can}(\bar{t}) := \{z \in \mathbb{R}^m : \exists x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m),\ A(Dx)' + Bx = 0,\ x(\bar{t}) = z\},\ \bar{t} \in \mathcal{I},$$

is said to be a canonical subspace of the DAE (44).

The canonical subspace $S_{can}(t)$ represents the geometric locus of all solution values of the homogenous DAE at time t. For the implicit regular ODE (43), $S_{can}(t) = \mathbb{R}^m$ is simply the entire time-invariant state space. In contrast, for DAEs, the inclusion $S_{can}(t) \subseteq S_0(t)$ is valid. While $S_0(t)$ is the *obvious constraint* associated to the homogenous DAE (43), the canonical subspace represents the final constraint which includes all hidden ones.
In particular, for the semi-explicit DAE in Example 2.2, the resulting

$$S_{can}(t) = \{z \in \mathbb{R}^{m_1+m_2} : z_2 = -B_{22}(t)^{-1}B_{21}(t)z_1\} = S_0(t)$$

is a m_1-dimensional time-varying subspace of $\mathbb{R}^m$, supposed $B_{22}(t)$ remains nonsingular. If $B_{22}(t) \equiv 0$, but $B_{21}(t)B_{12}(t)$ remains nonsingular, then

$$\begin{aligned} S_{can}(t) = \{z \in \mathbb{R}^{m_1+m_2} :& B_{21}(t)z_1 = 0,\\ & z_2 = -[(B_{21}B_{21})^{-1}B_{21}(B_{11} - (B_{12}((B_{21}B_{21})^{-1}B_{21})')](t)z_1\} \end{aligned}$$

is a proper subspace of the obvious constraint $S_0(t) = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}(t)z_1 = 0\}$. Example 1.1 confronts us with a zero-dimensional subspace $S_{can}(t) = \{0\}$.
Except for those simpler cases, the canonical subspace S_{can} is not easy of access. It coincides with the finite eigenspace of the matrix pencil for regular linear time-invariant DAEs. Theorem 6.15 below provides a description for general regular DAEs (44) by projector functions.

Omitting the argument t, we write (44) also in the form

$$G_0D^-(Dx)' + B_0x = q, \tag{46}$$

where the begin of our matrix function sequence (12) is already included.

Definition 6.2 *The DAE (44) with properly stated leading term and $m = k$ is said to be*

(1) *regular with tractability index zero if $r_0 = m$,*

(2) *regular with tractability index $\mu \geq 1$ if there are admissible projector functions $Q_0, \dots, Q_{\mu-1}$ such that $r_{\mu-1} < r_\mu = m$,*

(3) *regular if the DAE is regular with any tractability index μ (i.e. case (1) or (2) apply).*

The numbers $r_0, \dots, r_\mu$ defined by the matrix function sequence (12)-(15) are called characteristic valu of the DAE (44).
The subspace $N_{can} := N_0 + \dots + N_{\mu-1}$ is said to be a canonical subspace of the DAE.

These notions are well-defined in the sense they do not at all depend on the special choice of the admissible projector functions, which is guaranteed by Theorem 3.3.
Since for a regular DAE the matrix function G_μ is nonsingular, all intersections $\widehat{N}_i = N_i \cap (N_0 + \cdots + N_{i-1})$ are trivial, as a consequence of Proposition 3.2. Then it holds that

$$X_i = (N_0 + \cdots + N_{i-1}) \ominus \widehat{N}_i = N_0 + \cdots + N_{i-1} = N_0 \oplus \cdots \oplus N_{i-1} \subseteq \ker Q_i, \; i = 1, \ldots, \mu - 1,$$

thus $Q_i(I - \Pi_{i-1}) = 0$, and, equivalently,

$$Q_i Q_j = 0, \quad 0 \leq j \leq i-1, \;\; i = 1, \ldots, \mu - 1. \tag{47}$$

Additionally, Proposition 3.2 (4) yields $G_\mu Q_j = B_j Q_j$, thus

$$Q_j = G_\mu^{-1} B_j \Pi_{j-1} Q_j, \quad j = 1, \ldots, \mu - 1. \tag{48}$$

While, in the general Definition 3.1, only the part $\Pi_{j-1} Q_j = \Pi_{j-1} - \Pi_j$ of an admissible projector function Q_j is required to be continues, for a regular DAE, the admissible projector functions are continuous in all their components, as it follows from the representation(48).
We underline once again, for regular DAEs, the admissible projector functions are always *regular* admissible, and they are continuous in all components. At this place, we draw the readers attention to the fact that, in papers dealing exclusively with regular DAEs, the requirements for trivial intersections $\widehat{N}_i$ and the continuity of Q_i are usually incorporated already into the admissibility notion (e.g. [Mär04b]) or into the regularity notion (e.g. [Mär02], [Lam05]). Then, the relations (48) are constituent parts of the definitions (see also the recent monograph [Ria08]).

Here is a further special quality of regular DAEs: The associated subspaces (cf. Section 3)

$$S_i = \ker \mathcal{W}_i B = \{z \in \mathbb{R}^m : B_i z \in \operatorname{im} G_i\} = S_{i-1} + N_{i-1}$$

are now $\mathcal{C}$-subspaces, too. They have the constant dimensions r_i. This can be immediately checked. By Lemma A.8, the nonsingularity of G_μ implies the decomposition $N_{\mu-1} \oplus S_{\mu-1} = \mathbb{R}^m$, thus $\dim S_{\mu-1} = r_{\mu-1}$. Regarding the relation $\ker(G_{\mu-2} + \mathcal{W}_{\mu-2} B_{\mu-2} Q_{\mu-2}) = N_{\mu-2} \cap S_{\mu-2}$, we conclude by Proposition 2.4 (3) that $N_{\mu-2} \cap S_{\mu-2}$ has the same dimension as $N_{\mu-1}$ has. This means $\dim N_{\mu-2} \cap S_{\mu-2} = m - r_{\mu-1}$. Next, the representation $S_{\mu-1} = S_{\mu-2} + N_{\mu-2}$ leads to $r_{\mu-1} = \dim S_{\mu-2} + (m - r_{\mu-2}) - (m - r_{\mu-1})$, therefore $\dim S_{\mu-2} = r_{\mu-2}$, and so on.

We decouple the regular DAE (44) into its characteristic components, in a similar way as we did with constant coefficient DAEs in [LMT11b, Section 5]. Since G_μ is nonsingular, by introducing $Q_\mu = 0, \; P_\mu = I, \; \Pi_\mu = \Pi_{\mu-1}$, the sequence $Q_0, \ldots, Q_{\mu-1}, Q_\mu$ is admissible, and we can apply Proposition 4.1. The DAE (44) rewrites to

$$G_\mu D^-(D\Pi_{\mu-1}x)' + B_\mu x + G_\mu \sum_{l=0}^{\mu-1} \{Q_l x - (I - \Pi_l) Q_{l+1} D^-(D\Pi_l Q_{l+1} x)' + V_l D\Pi_l x\} = q. \tag{49}$$

If the coefficients were constant, we would have $D^-(D\Pi_{\mu-1}x)' = (D^-D\Pi_{\mu-1}x)' = (\Pi_{\mu-1}x)'$, further $D^-(D\Pi_l Q_{l+1}x)' = (\Pi_l Q_{l+1}x)'$, and $V_l = 0$. This means that formula (49) precisely generalizes formula [LMT11b, (38)] obtained for constant coefficients. The new formula (49) contains the extra terms V_l which arise from subspaces moving with time. They disappear in the time-invariant case.

In [LMT11b, Section 5], the decoupled version of the DAE is generated by the scaling with G_μ^{-1}, and then by the splitting by means of the projectores $\Pi_{\mu-1}$ and $I - \Pi_{\mu-1}$. Here we go a slightly different way and use $D\Pi_{\mu-1}$ instead of $\Pi_{\mu-1}$. Since $\Pi_{\mu-1}$ can be recovered from $D\Pi_{\mu-1}$ due to $\Pi_{\mu-1} = D^-D\Pi_{\mu-1}$, no information gets lost.
The equation (49) scaled by G_μ^{-1} reads

$$D^-(D\Pi_{\mu-1}x)' + G_\mu^{-1}B_\mu x + \sum_{l=0}^{\mu-1}\{Q_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + V_l D\Pi_l x\} = G_\mu^{-1}q. \tag{50}$$

The detailed expression for V_l (Proposition 4.1) is

$$V_l = (I-\Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l D^-.$$

This yields $D\Pi_{\mu-1}V_l = 0,\ l = 0,\dots,\mu-1$, and multiplying (50) by $D\Pi_{\mu-1}$ results in the equation

$$D\Pi_{\mu-1}D^-(D\Pi_{\mu-1}x)' + D\Pi_{\mu-1}G_\mu^{-1}B_\mu x = D\Pi_{\mu-1}G_\mu^{-1}. \tag{51}$$

Applying the $\mathcal{C}^1$ property of the projector $D\Pi_{\mu-1}D^-$, and recognizing that $B_\mu = B_\mu\Pi_{\mu-1} = B_\mu D^-D\Pi_{\mu-1}$, we get

$$(D\Pi_{\mu-1}x)' - (D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}x + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^-D\Pi_{\mu-1}x = D\Pi_{\mu-1}G_\mu^{-1}q. \tag{52}$$

Definition 6.3 *For the regular DAE (44) with tractability index μ, and admissible projector functions $Q_0,\dots,Q_{\mu-1}$, the resulting explicit regular ODE*

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^-u = D\Pi_{\mu-1}G_\mu^{-1}q \tag{53}$$

is called an inherent explicit regular ODE (IERODE) of the DAE.

It should be pointed out that there is a great variety of admissible projector functions. In consequence, there are various projector functions $\Pi_{\mu-1}$, and the IERODE (53) is not unique, except for the index one case. So far, we know the nullspace $N_0 + \cdots + N_{\mu-1}$ of the projector function $\Pi_{\mu-1}$ to be independent of the choice of the admissible projector functions $Q_0,\dots,Q_{\mu-1}$, that means the subspace $N_0+\cdots+N_{\mu-1}$ is unique; it is determined by the DAE coefficients only (Theorem 3.3). Later on we introduce advanced *fine decouplings* which make the corresponding IERODE unique. This means, then the IERODE coefficients are fully determined by the problem data, and do not depend on the special choice of fine decoupling projector functions.

Lemma 6.4 *If the DAE (44) is regular with index μ, and $Q_0,\dots,Q_{\mu-1}$ are admissible, then the subspace $\operatorname{im} D\Pi_{\mu-1}$ is an invariant subspace for the IERODE (53), that is, for the solutions $u \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n)$ of the ODE (53) the following assertion is valid:*

$$u(t_*) \in \operatorname{im}(D\Pi_{\mu-1})(t_*),\ \textit{with a certain}\ t_* \in \mathcal{I} \iff u(t) \in \operatorname{im}(D\Pi_{\mu-1})(t)\ \textit{for all}\ t\in\mathcal{I}.$$

Proof: Let $\bar{u} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$ denote a solution of (53) with $\bar{u}(t_*) = (D\Pi_{\mu-1}D^-)(t_*)\bar{u}(t_*)$. We multiply the identity

$$\bar{u}' - (D\Pi_{\mu-1}D^-)'\bar{u} + D\Pi_{\mu-1}G_\mu^{-1}D^-\bar{u} = D\Pi_{\mu-1}G_\mu^{-1}q$$

by $I - D\Pi_{\mu-1}D^-$, and introduce the function $\bar{v} := (I - D\Pi_{\mu-1}D^-)\bar{u} \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$. This gives

$$(I - D\Pi_{\mu-1}D^-)\bar{u}' - (I - D\Pi_{\mu-1}D^-)(D\Pi_{\mu-1}D^-)'\bar{u} = 0,$$

further,

$$\bar{v}' - (I - D\Pi_{\mu-1}D^-)'\bar{u} - (I - D\Pi_{\mu-1}D^-)(D\Pi_{\mu-1}D^-)'\ \bar{u} = 0,$$

and

$$\bar{v}' - (I - D\Pi_{\mu-1}D^-)'\bar{v} = 0.$$

Because of $\bar{v}(t_*) = 0$, $\bar{v}$ must vanish identically, and hence $\bar{u} = D\Pi_{\mu-1}D^-\bar{u}$ holds true. □

We leave the IERODE for a while, and turn back to the scaled version (50) of the DAE (44). Now we consider the other part of this equation, which results from multiplication by the projector function $I - \Pi_{\mu-1}$. First we express

$$\begin{aligned}
(I-&\Pi_{\mu-1})D^-(D\Pi_{\mu-1}x)' + (I - \Pi_{\mu-1})G_\mu^{-1}B_\mu x \\
&= (I - \Pi_{\mu-1})G_\mu^{-1}\{G_\mu D^-(D\Pi_{\mu-1}x)' + B_{\mu-1}P_{\mu-1}x - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}x\} \\
&= (I - \Pi_{\mu-1})G_\mu^{-1}\{B_{\mu-1}P_{\mu-1}x + G_\mu D^-D\Pi_{\mu-1}D^-(D\Pi_{\mu-1}x)'\} \\
&= (I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}x,
\end{aligned}$$

and obtain then the equation

$$\begin{aligned}
(I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}x + \sum_{l=0}^{\mu-1}\{Q_l x + V_l D\Pi_l x\}& \\
- \sum_{l=0}^{\mu-2}(I - \Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' &= (I - \Pi_{\mu-1})G_\mu^{-1}q,
\end{aligned} \tag{54}$$

which is the precise counterpart of equation [LMT11b, (41)]. Again, the extra terms V_l comprise the time variation. By means of the decompositions

$$\begin{aligned}
D\Pi_l x &= D\Pi_l(\Pi_{\mu-1} + I - \Pi_{\mu-1})x = D\Pi_{\mu-1}x + D\Pi_l(I - P_{l+1}\cdots P_{\mu-1})x \\
&= D\Pi_{\mu-1}x + D\Pi_l(Q_{l+1} + P_{l+1}Q_{l+2} + \cdots + P_{l+1}\cdots P_{\mu-2}Q_{\mu-1})x \\
&= D\Pi_{\mu-1}x + D\Pi_l(Q_{l+1} + \cdots + D\Pi_{\mu-2}Q_{\mu-1})x,
\end{aligned}$$

we rearrange the terms in (54) once more to

$$\sum_{l=0}^{\mu-1} Q_l x - \sum_{l=0}^{\mu-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + \sum_{l=0}^{\mu-2}\mathcal{M}_{l+1}D\Pi_l Q_{l+1}x + \mathcal{K}\Pi_{\mu-1}x = (I-\Pi_{\mu-1})G_\mu^{-1}q, \tag{55}$$

with the continuous coefficients

$$\begin{aligned}
\mathcal{K} :&= (I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} + \sum_{l=0}^{\mu-1} V_l D\Pi_{\mu-1} \qquad (56)\\
&= (I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} + \sum_{l=0}^{\mu-1}(I-\Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_{\mu-1}\\
&= (I - \Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} + \sum_{l=1}^{\mu-1}(I-\Pi_{l-1})(P_l - Q_l)(D\Pi_l D^-)'D\Pi_{\mu-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_{l+1} :&= \sum_{j=0}^{l} V_j D\Pi_l Q_{l+1} D^- \qquad (57)\\
&= \sum_{j=0}^{l}(I-\Pi_l)\{P_l D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l Q_{l+1}D^-,\\
l &= 0, \dots, \mu-2.
\end{aligned}$$

The coefficients $\mathcal{M}_{l+1}$ vanish together with the V_j in the constant coefficient case.

Next we provide a further splitting of the subsystem (55) according to the decomposition

$$I - \Pi_{\mu-1} = Q_0 P_1 \cdots P_{\mu-1} + \cdots + Q_{\mu-2}P_{\mu-1} + Q_{\mu-1}$$

into μ parts. Notice that the products $Q_i P_{i+1} \cdots P_{\mu-1}$ are also continuous projectors. To prepare the further decoupling we provide some useful properties of our projectors and coefficients.

Lemma 6.5 *For the regular DAE (44) with tractability index μ, and admissible projector functions $Q_0, \dots, Q_{\mu-1}$, the following relations become true:*

(1) $Q_i P_{i+1} \cdots P_{\mu-1}(I - \Pi_l) = 0,\ \ l = 0, \dots, i-1,\ i = 1, \dots, \mu-2,$
$Q_{\mu-1}(I - \Pi_l) = 0,\ \ l = 0, \dots, \mu-2,$

(2) $Q_i P_{i+1} \cdots P_{\mu-1}(I - \Pi_i) = Q_i,\ \ i = 0, \dots, \mu-2,$
$Q_{\mu-1}(I - \Pi_{\mu-1}) = Q_{\mu-1},$

(3) $Q_i P_{i+1} \cdots P_{\mu-1}(I - \Pi_{i+s}) = Q_i P_{i+1} \cdots P_{i+s},\ \ s = 1, \dots, \mu-1-i,\ i = 0, \dots, \mu-2,$

(4) $Q_i P_{i+1} \cdots P_{\mu-1}\mathcal{M}_{l+1} = 0,\ \ l = 0, \dots, i-1,\ i = 0, \dots, \mu-2,$
$Q_{\mu-1}\mathcal{M}_{l+1} = 0,\ \ l = 0, \dots, \mu-2,$

(5) $Q_i P_{i+1} \cdots P_{\mu-1} Q_s = 0$ *if* $s \neq i,\ s = 0, \dots, \mu-1,$
$Q_i P_{i+1} \cdots P_{\mu-1} Q_i = Q_i,\ \ i = 0, \dots, \mu-2,$

(6) $\mathcal{M}_j = \sum_{l=1}^{j-1}(I - \Pi_{l-1})(P_l - Q_l)D^-(D\Pi_{j-1}Q_j D^-)'D\Pi_{j-1}Q_j D^-,\ j = 1, \dots, \mu-1,$

(7) $\Pi_{\mu-1}G_\mu^{-1}B_\mu = \Pi_{\mu-1}G_\mu^{-1}B_0\Pi_{\mu-1}$*, and hence* $D\Pi_{\mu-1}G_\mu^{-1}B_\mu D^- = D\Pi_{\mu-1}G_\mu^{-1}BD^-$.

Proof: (1) The first part of the assertion results from the relation $Q_iP_{i+1}\cdots P_{\mu-1} = Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{i-1}$, and the inclusion $\operatorname{im}(I-\Pi_l) \subseteq \ker \Pi_{i-1},\ l \le i-1$. The second part is a consequence of the inclusion $\operatorname{im}(I-\Pi_l) \subseteq \ker Q_{\mu-1},\ l \le \mu-2$.
(2) This is a consequence of the relations $P_{i+1}\cdots P_{\mu-1}(I-\Pi_i) = (I-\Pi_i)$ and $Q_i(I-\Pi_i) = Q_i$.
(3) We have

$$Q_iP_{i+1}\cdots P_{\mu-1}\Pi_{\mu-1} = 0, \quad \text{thus} \quad Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_{\mu-1}) = Q_iP_{i+1}\cdots P_{\mu-1}.$$

Taking into account that $Q_j(I-\Pi_{i+s}) = 0$ for $j > i+s$, we find

$$\begin{aligned} Q_iP_{i+1}\cdots P_{\mu-1}(I-\Pi_{i+s}) &= Q_iP_{i+1}\cdots P_{i+s}P_{i+s+1}\cdots P_{\mu-1}(I-\Pi_{i+s}) \\ &= Q_iP_{i+1}\cdots P_{i+s}P_{i+s+1}\cdots P_{\mu-1}(I-\Pi_{i+s}) \\ &= Q_iP_{i+1}\cdots P_{i+s}(I-\Pi_{i+s}) = Q_iP_{i+1}\cdots P_{i+s}. \end{aligned}$$

(4) This is a consequence of (1).
(5) This is evident.
(6) We derive

$$\begin{aligned} \mathcal{M}_j = &\sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-(D\Pi_lD^-)'D\Pi_{j-1}Q_jD^- \\ &\qquad - \sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{j-1}Q_jD^- \\ = &\sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-\{(D\Pi_{j-1}Q_jD^-)' - D\Pi_lD^-(D\Pi_{j-1}Q_jD^-)'\}D\Pi_{j-1}Q_jD^- \\ &- \sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-\{(D\Pi_{j-1}Q_jD^-)' - D\Pi_{l+1}D^-(D\Pi_{j-1}Q_jD^-)'\}D\Pi_{j-1}Q_jD^- \\ = &\sum_{l=1}^{j-1}(I-\Pi_l)P_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^- \\ &\qquad - \sum_{l=0}^{j-2}(I-\Pi_l)Q_{l+1}D^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^- \\ = &\sum_{l=1}^{j-1}(I-\Pi_{l-1})P_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^- \\ &\qquad - \sum_{l=1}^{j-1}(I-\Pi_{l-1})Q_lD^-(D\Pi_{j-1}Q_jD^-)'D\Pi_{j-1}Q_jD^-. \end{aligned}$$

(7) Owing to $P_\mu = I$, it holds that

$$B_\mu = B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_\mu D^-)'D\Pi_{\mu-1} = B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.$$

We compute

$$\begin{aligned} \Pi_{\mu-1}G_\mu^{-1}B_\mu &= \Pi_{\mu-1}G_\mu^{-1}\{B_{\mu-1}P_{\mu-1} - G_\mu D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\} \\ &= \Pi_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} - \underbrace{\Pi_{\mu-1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}}_{=0}. \end{aligned}$$

The next step is

$$\begin{aligned}\mathit{\Pi}_{\mu-1}G_\mu^{-1}B_{\mu-1}\mathit{\Pi}_{\mu-1} &= \mathit{\Pi}_{\mu-1}G_\mu^{-1}\{B_{\mu-2}P_{\mu-2} - G_{\mu-1}D^-(D\mathit{\Pi}_{\mu-1}D^-)'D\mathit{\Pi}_{\mu-2}\}\mathit{\Pi}_{\mu-1}\\ &= \mathit{\Pi}_{\mu-1}G_\mu^{-1}B_{\mu-2}\mathit{\Pi}_{\mu-1} - \underbrace{\mathit{\Pi}_{\mu-1}P_{\mu-1}D^-(D\mathit{\Pi}_{\mu-1}D^-)'D\mathit{\Pi}_{\mu-1}}_{=0},\end{aligned}$$

and so on. □

As announced before we split the subsystem (55) into μ parts. Multiplying by the projector functions $Q_iP_{i+1}\cdots P_{\mu-1}$, $i=0,\dots,\mu-2$, and $Q_{\mu-1}$, and regarding Lemma 6.5 one attains the system

$$\begin{aligned}Q_ix - Q_iQ_{i+1}D^-(D\mathit{\Pi}_iQ_{i+1}x)' - \sum_{l=i+1}^{\mu-2} Q_iP_{i+1}\cdots P_lQ_{l+1}D^-(D\mathit{\Pi}_lQ_{l+1}x)'&\\ + \sum_{l=i}^{\mu-2} Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{M}_{l+1}D\mathit{\Pi}_lQ_{l+1}x&\\ = -Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{K}\mathit{\Pi}_{\mu-1}x + Q_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}q, \qquad i=0,\dots,\mu-2.&\end{aligned} \tag{58}$$

as well as

$$Q_{\mu-1}x = -Q_{\mu-1}\mathcal{K}\mathit{\Pi}_{\mu-1}x + Q_{\mu-1}G_\mu^{-1}q. \tag{59}$$

The equation (59) determines $Q_{\mu-1}x$ in terms of q and $\mathit{\Pi}_{\mu-1}x$. The i-th equation in (58) determines Q_ix in terms of q, $\mathit{\Pi}_{\mu-1}x$, $Q_{\mu-1}x,\dots,Q_{i+1}x$, and so on, that is, the system (58), (59) successively determines all components of $I-\mathit{\Pi}_{\mu-1} = Q_0 + \mathit{\Pi}_0Q_1 + \cdots + \mathit{\Pi}_{\mu-2}Q_{\mu-1}$ in a unique way. Comparing with the constant coefficient case, we recognize the system (58), (59) to generalize the system [LMT11b, (43)-(44)].
So far, the regular DAE (44) decouples into the IERODE (53) and the subsystem (58), (59) by means of each arbitrary sequence of admissible projector functions. The solutions of the DAE can be expressed as

$$x = \mathit{\Pi}_{\mu-1}x + (I-\mathit{\Pi}_{\mu-1}x) = D^-u + (I-\mathit{\Pi}_{\mu-1})x,$$

whereby $(I-\mathit{\Pi}_{\mu-1})x$ is determined by the subsystem (58), (59), and $u = D\mathit{\Pi}_{\mu-1}D^-u$ is a solution of the IERODE, which belongs to its invariant subspace.

The property

$$\ker Q_i = \ker \mathit{\Pi}_{i-1}Q_i, \quad i=1,\dots,\mu-1, \tag{60}$$

is valid, since we may represent $Q_i = (I + (I-\mathit{\Pi}_{i-1})Q_i)\mathit{\Pi}_{i-1}Q_i$ with the nonsingular factor $I+(I-\mathit{\Pi}_{i-1})Q_i$, $i=1,\dots,\mu-1$. This allows to compute Q_ix from $\mathit{\Pi}_{i-1}Q_ix$ and vice versa. We take advantage of this in the following rather cosmetic changes.

Denote

$$v_0 := Q_0x, \quad v_i := \mathit{\Pi}_{i-1}Q_ix, \quad i=1,\dots,\mu-1, \tag{61}$$

$$u := D\mathit{\Pi}_{\mu-1}x, \tag{62}$$

such that we have the solution expression

$$x = v_0 + v_1 + \cdots + v_{\mu-1} + D^- u. \tag{63}$$

Multiply the equation (59) by $\Pi_{\mu-2}$, and, if $i \geq 1$, the i-th equation in (58) by Π_{i-1}. This yields the following system which determines the functions $v_{\mu-1}, \ldots, v_0$ in terms of q and u:

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} (\mathcal{D} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix})' \tag{64}$$

$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^- u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q.$$

The matrix function $\mathcal{D} := (\mathcal{D}_{ij})_{i,j=0}^{\mu-1}$ has as entries the blocks $\mathcal{D}_{ii} = D\Pi_{i-1}Q_i,\ i = 1, \ldots, \mu-1,\ \ \mathcal{D}_{00} = 0$, and $\mathcal{D}_{ij} = 0$, if $i \neq j$. This matrix function is block-diagonal if $n = m$. The further coefficients in (64) are also continuous, their detailed form is

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0Q_1D^- \\
\mathcal{N}_{0j} &:= -Q_0P_1 \cdots P_{j-1}Q_jD^-, && j = 2, \ldots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1}Q_iQ_{i+1}D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1}Q_iP_{i+1} \cdots P_{j-1}Q_jD^-, && j = i+2, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{M}_{0j} &:= Q_0P_1 \cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j = 1, \ldots, \mu-1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j = i+1, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{L}_0 &:= Q_0P_1 \cdots P_{\mu-1}G_\mu^{-1}, \\
\mathcal{L}_i &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}G_\mu^{-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}G_\mu^{-1}, \\
\mathcal{H}_0 &:= Q_0P_1 \cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}, \\
\mathcal{H}_i &:= \Pi_{i-1}Q_iP_{i+1} \cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}, && i = 1, \ldots, \mu-2, \\
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}\mathcal{K}\Pi_{\mu-1}.
\end{aligned}$$

Introducing the matrix functions $\mathcal{N},\ \mathcal{M},\ \mathcal{H},\ \mathcal{L}$ of appropriate sizes according to (64), we write this subsystem as

$$\mathcal{N}(\mathcal{D}v)' + \mathcal{M}v + \mathcal{H}D^-u = \mathcal{L}q, \tag{65}$$

whereby the vector function v contains the entries $v_0, \ldots, v_{\mu-1}$.

Again, we draw the attention to the great consistency with [LMT11b, (49)]. The difficulties caused by the time-variations are now hidden in the coefficients $\mathcal{M}_{ij}$ which disappear for constant coefficients.
We emphasize that the system (64) is nothing else a more transparent reformulation of the former subsystem (58), (59). The next proposition records important properties.

Proposition 6.6 *Let the DAE (44) be regular with tractability index μ, and let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions. Then the coefficient functions in (64) have the further properties:*

(1) $\mathcal{N}_{ij} = \mathcal{N}_{ij} D\Pi_{j-1} Q_j D^-$ *and* $\mathcal{N}_{ij} D = \mathcal{N}_{ij} D\Pi_{j-1} Q_j$, *for* $j = 1, \ldots, \mu-1$, $i = 0, \ldots, \mu-2$.

(2) $\operatorname{rank} \mathcal{N}_{i,i+1} = \operatorname{rank} \mathcal{N}_{i,i+1} D = m - r_{i+1}$, *for* $i = 0, \ldots, \mu-2$.

(3) $\ker \mathcal{N}_{i,i+1} = \ker D\Pi_i Q_{i+1} D^-$, *and* $\ker \mathcal{N}_{i,i+1} D = \ker \Pi_i Q_{i+1}$, *for* $i = 0, \ldots, \mu-2$.

(4) *The subsystem (64) is a DAE with properly stated leading term.*

(5) *The square matrix function $\mathcal{N}\mathcal{D}$ is pointwise nilpotent with index μ, more precisely,* $(\mathcal{N}\mathcal{D})^\mu = 0$ *and* $\operatorname{rank} (\mathcal{N}\mathcal{D})^{\mu-1} = m - r_{\mu-1} > 0$.

(6) $\mathcal{M}_{i,i+1} = 0, \quad i = 0, \ldots, \mu-2$.

Proof: (1) This is given by the construction.
(2) Because of $\mathcal{N}_{i,i+1} = \mathcal{N}_{i,i+1} DD^-$, the matrix functions $\mathcal{N}_{i,i+1}$ and $\mathcal{N}_{i,i+1} D$ have equal rank. To show that this is precisely $m - r_{i+1}$ we apply the same arguments as for [LMT11b, Lemma 5.5]. First we validate the relation

$$\operatorname{im} Q_i Q_{i+1} = N_i \cap S_i.$$

Namely, $z \in N_i \cap S_i$ implies $z = Q_i z$ and $B_i z = G_i w$, therefore, $(G_i + B_i Q_i)(P_i w + Q_i z) = 0$, further $(P_i w + Q_i z) = Q_{i+1}(P_i w + Q_i z) = Q_{i+1} w$, $Q_i z = Q_i Q_{i+1} w$, and hence $z = Q_i z = Q_i Q_{i+1} w$.
Conversely, $z \in \operatorname{im} Q_i Q_{i+1}$ yields $z = Q_i z$, $z = Q_i Q_{i+1} w$. Then the identity $(G_i + B_i Q_i) Q_{i+1} = 0$ leads to $B_i z = B_i Q_i Q_{i+1} w = -G_i Q_{i+1} w$, thus $z \in N_i \cap S_i$.
The intersection $N_i \cap S_i$ has the same dimension as N_{i+1}, so that we attain $\dim \operatorname{im} Q_i Q_{i+1} = \dim N_{i+1} = m - r_{i+1}$.
(3) From (1) we derive the inclusions

$$\ker D\Pi_i Q_{i+1} D^- \subseteq \ker \mathcal{N}_{i,i+1}, \quad \ker \Pi_i Q_{i+1} \subseteq \ker \mathcal{N}_{i,i+1} D.$$

Because of $\Pi_i Q_{i+1} = D^-(D\Pi_i Q_{i+1} D^-) D$, and $\ker \Pi_i Q_{i+1} = \ker Q_{i+1}$, the assertion becomes true for reasons of dimensions.
(4) We provide the subspaces

$$\ker \mathcal{N} = \{z = \begin{bmatrix} z_0 \\ \vdots \\ z_{\mu-1} \end{bmatrix} \in \mathbb{R}^{n\mu} : z_i \in \ker \Pi_{i-1} Q_i, \ i = 1, \ldots, \mu-1\}$$

and

$$\operatorname{im} \mathcal{D} = \{z = \begin{bmatrix} z_0 \\ \vdots \\ z_{\mu-1} \end{bmatrix} \in \mathbb{R}^{n\mu} : z_i \in \operatorname{im} \Pi_{i-1} Q_i, \ i = 1, \ldots, \mu-1\}$$

which obviously fulfill the condition $\ker \mathcal{N} \oplus \operatorname{im} \mathcal{D} = \mathbb{R}^{n\mu}$. The border projector is $\mathcal{R} = \operatorname{diag}(0, D\Pi_0 Q_1 D^-, \ldots, D\Pi_{\mu-2} Q_{\mu-1} D^-)$, and it is continuously differentiable.

(5) The matrix function $\mathcal{ND}$ is by nature strictly block upper triangular, and its main entries $(\mathcal{ND})_{i,i+1} = \mathcal{N}_{i,i+1}D$ have constant rank $m - r_{i+1}$, for $i = 0, \dots, \mu - 2$.
The matrix function $(\mathcal{ND})^2$ has zero-entries on the block positions $(i, i+1)$, and the dominating entries are

$$((\mathcal{ND})^2)_{i,i+2} = \mathcal{N}_{i,i+1}D\mathcal{N}_{i+1,i+2}D = \Pi_{i-1}Q_iQ_{i+1}\Pi_iQ_{i+1}Q_{i+2} = \Pi_{i-1}Q_iQ_{i+1}Q_{i+2},$$

which have rank $m - r_{i+2}$, and so on.
In $(\mathcal{ND})^{\mu-1}$ there remains exactly one nontrivial block in the upper right corner, $((\mathcal{ND})^{\mu-1})_{0,\mu-1} = (-1)^{\mu-1}Q_0Q_1 \cdots Q_{\mu-1}$, and it has rank $m - r_{\mu-1}$.
(6) This property is a direct consequence of the representation of $\mathcal{M}_{i+1}$ in Lemma 6.5 (6) and Lemma 6.5 (1). □

By this proposition, the subsystem (64) is in turn a regular DAE with tractability index μ and transparent structure. Property (6) slightly improves the structure of (64). We underline that the DAE (64) lives in $\mathbb{R}^{m\mu}$. The solutions belong to the function space $\mathcal{C}^1_{\mathcal{D}}(\mathcal{I}, \mathbb{R}^{m\mu})$. Owing to the special form of the matrix function $\mathcal{L}$ on the right hand side, each solution of (64) satisfies the conditions $v_0 = Q_0v_0$ and $v_i = \Pi_{i-1}Q_iv_i$, for $i = 1, \dots, \mu - 1$.
We formulate now the main result concerning the basic decoupling:

Theorem 6.7 *Let the DAE (44) be regular with tractability index μ, and let $Q_0, \dots, Q_{\mu-1}$ be admissible projector functions. Then the DAE is equivalent via (61)-(63) to the system consisting of the IERODE (53) related to its invariant subspace* $\operatorname{im} D\Pi_{\mu-1}$*, and the subsystem (64).*

Proof: If $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ is a solution of the DAE, then the component $u := D\Pi_{\mu-1}x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ satisfies the IERODE (53) and belongs to the invariant subspace $\operatorname{im} \Pi_{\mu-1}$. The functions $v_0 := Q_0x \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, $v_i := \Pi_{i-1}Q_ix \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, $i = 1, \dots, \mu - 1$, form the unique solution of the system (64) corresponding to u. Thereby, we recognize that $D\Pi_{\mu-1}x = D\Pi_{\mu-1}D^-Dx$, $Dv_i := D\Pi_{i-1}Q_ix = D\Pi_{i-1}Q_iD^-Dx$, $i = 1, \dots, \mu - 1$, are continuously differentiable functions since Dx and the used projectors are so.
Conversely, let $u = D\Pi_{\mu-1}x$ denote a solution of the IERODE, and let $v_0, \dots, v_{\mu-1}$ form a solution of the subsystem (64). Then, it holds that $v_i = \Pi_{i-1}Q_iv_i$, for $i = 1, \dots, \mu - 1$, and $v_0 = Q_0v_0$. The functions u and $Dv_i = D\Pi_{i-1}Q_iv_i$, $i = 1, \dots, \mu - 1$ are continuously differentiable. The composed function $x := D^-u + v_0 + v_1 + \cdots + v_{\mu-1}$ is continuous and has a continuously part Dx. It remains to insert x into the DAE, and to recognize that x fulfills the DAE. □

The coefficients of the IERODE and the system (64) are determined in terms of the DAE coefficients and the resulting from these coefficients projector functions. We can take use of these equations unless supposing that there is a solution of the DAE. Considering the IERODE (53) and the system (64) as equations with unknown functions $u \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n)$, $v_0 \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, $v_i \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, $i = 1, \dots, \mu - 1$, we may solve these equations and construct continuous functions $x := D^-u + v_0 + v_1 + \cdots + v_{\mu-1}$ with $Dx = DD^-u + Dv_1 + \cdots + Dv_{\mu-1}$ being continuously differentiable, such that x satisfies the DAE. Thereby we restrict our interest to those solutions u of the IERODE that have the property $u = D\Pi_{\mu-1}D^-u$. This way we could prove the existence of DAE solutions, if the excitation and the coefficients are sufficiently smooth.

We finish this subsection by a closer look to the special case of index one DAEs. Let the DAE (44) be regular with tractability index one. The matrix function $G_0 = AD$ is singular with constant rank. We take an arbitrary continuous projector function Q_0. The resulting matrix function $G_1 = G_0 + BQ_0$ is nonsingular. It follows that $Q_1 = 0$, $\Pi_1 = \Pi_0$ and $V_0 = 0$ (cf. Prop. 4.1), further $B_1 = BP_0 - G_1 D^-(D\Pi_0 D^-)' D\Pi_0 = BP_0$. The DAE scaled by G_1^{-1} is (cf. (50)) now

$$D^-(D\Pi_0 x)' + G_1^{-1}BP_0 x + Q_0 x = G_1^{-1}q.$$

Multiplication by $D\Pi_0 = D$ and $I - \Pi_0 = Q_0$ leads to the system

$$(Dx)' - R'Dx + DG_1^{-1}BD^-Dx = DG_1^{-1}q, \tag{66}$$

$$Q_0 x + Q_0 G_1^{-1}BD^-Dx = Q_0 G_1^{-1}q, \tag{67}$$

and the solution expression $x = D^-Dx + Q_0 x$. Equation (67) stands for the subsystem (64), i.e. for

$$Q_0 x + \mathcal{H}_0 D^-Dx = \mathcal{L}_0 q, \quad \text{with } \mathcal{H}_0 = Q_0 \mathcal{K}\Pi_0 = Q_0 G_1^{-1}B\Pi_0 = Q_0 G_1^{-1}BP_0,\ \mathcal{L}_0 = Q_0 G_1^{-1}.$$

The nonsingularity of G_1 implies the decomposition $S_0 \oplus N_0 = \mathbb{R}^m$ (cf. Lemma A.8), and the matrix function $Q_0 G_1^{-1}B$ is a representation of the projector function onto N_0 along S_0.

We can choose Q_0 to be the special projector function onto N_0 along S_0 at the beginning. The benefit from this choice consists in the property $\mathcal{H}_0 = Q_0 G_1^{-1}BP_0 = 0$, that is, the subsystems (67) uncouples from (66).

Example 6.8 *We reconsider the semi-explicit DAE from Example 2.2*

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = q$$

with nonsingular B_{22}. Here we have the subspaces

$$N_0 = \{z \in \mathbb{R}^{m_1+m_2} : z_1 = 0\} \quad \textit{and} \quad S_0 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}z_1 + B_{22}z_2 = 0\},$$

and the projector function onto N_0 along S_0 is given by

$$Q_0 = \begin{bmatrix} 0 & 0 \\ B_{22}^{-1}B_{21} & I \end{bmatrix}.$$

We know this projector to be reasonable, although it is far from being orthogonal. This choice leads to the matrix functions

$$D^- = \begin{bmatrix} I \\ -B_{22}^{-1}B_{21} \end{bmatrix},\ G_1 = \begin{bmatrix} I + B_{12}B_{22}^{-1}B_{21} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},\ G_1^{-1} = \begin{bmatrix} I & -B_{12}B_{22}^{-1} \\ -B_{22}^{-1}B_{21} & (I + B_{22}^{-1}B_{21}B_{12}) \end{bmatrix},$$

and the IERODE

$$x_1' + (B_{11} - B_{12}B_{22}^{-1}B_{21})x_1 = q_1 - B_{12}B_{22}^{-1}q_2.$$

Notice that in Example 2.2, Q_0 is chosen to be the orthoprojector. Precisely the same IERODE results for this choice, which appears to be typical for regular index one DAEs, and for fine decouplings of general regular DAEs.

Proposition 6.9 *Let the DAE (44) be regular with index one. Then its IERODE*

$$u' - R'u + DG_1^{-1}BD^-u = DG_1^{-1}q$$

is actually independent of the choice of the continuous projector function Q_0.

Proof: We compare the IERODEs built for two different projector functions Q_0 and $\bar{Q}_0$. It holds that $\bar{G}_1 = G_0 + B\bar{Q}_0 = G_0 + BQ_0\bar{Q}_0 = G_1(P_0 + \bar{Q}_0) = G_1(I + Q_0\bar{Q}_0P_0)$ and $\bar{D}^- = \bar{D}^-D\bar{D}^- = \bar{D}^-R = \bar{D}^-DD^- = \bar{P}_0D^-$, therefore $D\bar{G}_1^{-1} = DG_1^{-1}$, $D\bar{G}_1^{-1}B\bar{D}^- = DG_1^{-1}B(I - \bar{Q}_0)D^- = DG_1^{-1}B(I - Q_0\bar{Q}_0)D^- = DG_1^{-1}BD^-$. □

6.2 Fine and complete decouplings

In this subsection we advance the decoupling of the subsystem (64) of the regular DAE (44). As benefits of such a refined decoupling we get further natural information on the DAE, that is, information being independent of the choice of projectors in the given context. In particular, we arrive at a natural IERODE.
As discussed at the end of the previous subsection, regular index one DAEs are transparent and simple, and the coefficients of their IERODEs are *always* independent of the projector choice. However, higher index DAEs are different. We take a closer look to the simplest class among them, the regular DAEs with tractability index $\mu = 2$.

Let the DAE (44) be regular with tractability index $\mu = 2$, then the IERODE (53) and the subsystem (64) reduce to

$$u' - (D\Pi_1D^-)'u + D\Pi_1G_2^{-1}B_1D^-u = D\Pi_1G_2^{-1}q,$$

and

$$\begin{bmatrix} 0 & -Q_0Q_1D^- \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & D\Pi_0Q_1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right)' + \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{bmatrix} D^-u = \begin{bmatrix} Q_0P_1G_2^{-1} \\ \Pi_0Q_1G_2^{-1} \end{bmatrix} q,$$

with

$$\begin{aligned} \mathcal{H}_0 &= Q_0P_1\mathcal{K}\Pi_1 = Q_0P_1G_2^{-1}B_1\Pi_1 + Q_0P_1(D\Pi_1D^-)'D\Pi_1 \\ \mathcal{H}_1 &= \Pi_0Q_1\mathcal{K}\Pi_1 = \Pi_0Q_1G_2^{-1}B_1\Pi_1. \end{aligned}$$

Owing to the nonsingularity of G_2, the decomposition (cf. Lemma A.8)

$$N_1 \oplus S_1 = \mathbb{R}^m$$

is given, and the expression $Q_1G_2^{-1}B_1$ appearing in $\mathcal{H}_1$ reminds of the representation of the special projector function onto N_1 along S_1 (cf. Lemma A.9) which is uniquely determined. In fact, $Q_1G_2^{-1}B_1$ is this projector function. The subspaces N_1 and S_1 are given before one has to choose the projector function Q_1, and hence one can settle on the projector function Q_1 onto N_1 along S_1 at the beginning. Thereby, the necessary admissibility condition $N_0 \subseteq \ker Q_1$ is fullfilled because of $N_0 \subseteq S_1 = \ker Q_1$. It follows that

$$Q_1G_2^{-1}B_1\Pi_1 = Q_1G_2^{-1}B_1P_1 = Q_1P_1 = 0, \quad \mathcal{H}_1 = \Pi_0Q_1G_2^{-1}B_1\Pi_1 = 0.$$

With the next example we demonstrate that there are various different resulting projector functions $D\Pi_{\mu-1}D^-$, and hence different IERODEs.

Example 6.10 *Consider once again the so-called Hessenberg size two DAE*

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} x = q, \tag{68}$$

with the nonsingular product $B_{21}B_{12}$. *Suppose the subspaces* $\operatorname{im} B_{12}$ *and* $\ker B_{21}$ *to be* $\mathcal{C}^1$-*subspaces. As it is shown in Example 2.2, this DAE is regular with index two, and the projector functions*

$$Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad \Omega := B_{12}B_{12}^-, \tag{69}$$

are admissible, for each arbitrary reflexive inverse B_{12}^- *such that* Ω *is continuously differentiable. We have further* $D\Pi_1 D^- = I - \Omega$ *and*

$$S_0 = S_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}z_1 = 0\}.$$

(a) *Set* $B_{12}^- := B_{12}^+ = (B_{12}^* B_{12})^{-1} B_{12}^*$ *in* Ω. *Then* Ω *projects* $\mathbb{R}^{m_1}$ *onto* $\operatorname{im} B_{12}$ *along* $\ker B_{12}^* = \operatorname{im} B_{12}^{\perp}$, *and* Q_1 *projects* $\mathbb{R}^m$ *onto* N_1 *along*

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{12}^* z_1 = 0\} = (N_0 \oplus N_1)^{\perp} \oplus N_0.$$

It results that $D\Pi_1 D^- = I - \Omega$ *is symmetric. These projector functions* Q_0, Q_1 *are widely orthogonal in the sense of Definition 9.1. Notice that, for this construction we could dispense with the* $\mathcal{C}^1$ *property of the subspace* $\ker B_{21}$.

(b) *Set* $B_{12}^- := (B_{21}B_{12})^{-1} B_{21}$ *in (68). Then* Ω *projects* $\mathbb{R}^{m_1}$ *onto* $\operatorname{im} B_{12}$ *along* $\ker B_{21}$, *and* Q_1 *projects* $\mathbb{R}^m$ *onto* N_1 *along*

$$\ker Q_1 = \{z \in \mathbb{R}^{m_1+m_2} : B_{21}z_1 = 0\} = S_1.$$

Except for the special case, if $\ker B_{12}^* = \ker B_{21}$, *a nonsymmetric projector function* $D\Pi_1 D^- = I - \Omega = I - B_{12}(B_{21}B_{12})^{-1}B_{21}$ *results. As we already know, this choice has the advantage of a vanishing coupling coefficient* $\mathcal{H}_1$.

In contrast to (69) *the projector functions*

$$Q_0 = \begin{bmatrix} 0 & 0 \\ B_{12}^-(B_{11} - \Omega')(I - \Omega) & I \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \Omega & 0 \\ -B_{12}^- & 0 \end{bmatrix}, \quad \Omega := B_{12}B_{12}^-, \tag{70}$$

form a further pair of admissible projector functions yielding again $D\Pi_1 D^- = I - \Omega$. *If* $B_{12}^- := (B_{21}B_{12})^{-1}B_{21}$, *then this choice forces both coefficients* $\mathcal{H}_1$ *and* $\mathcal{H}_0$ *to disappear, and the subsystem (64) uncouples from the IERODE. Notice that the resulting IERODE coincides with that from* (b).

As mentioned before, the index two case has the simplest higher index structure. The higher the index, the greater the variety of admissible projector functions. We remind [LMT11b, Example 5.4] which shows several completely decoupling projectors for a time-invariant regular matrix pair with Kronecker index two.

Definition 6.11 *Let the DAE (44) be regular with tractability index μ, and let $Q_0, \dots, Q_{\mu-1}$ denote admissible projector functions.*

(1) *If the $\mu - 1$ coupling coefficients $\mathcal{H}_1, \dots, \mathcal{H}_{\mu-1}$ of the subsystem (64) vanish, then we speak of fine decoupling projector functions $Q_0, \dots, Q_{\mu-1}$, and of a fine decoupling.*

(2) *If all the μ coupling coefficients $\mathcal{H}_0, \dots, \mathcal{H}_{\mu-1}$ of the subsystem (64) vanish, then we speak of complete decoupling projector functions $Q_0, \dots, Q_{\mu-1}$, and of a complete decoupling.*

In the sense of this definition, fine and complete decoupling projector functions Q_0, Q_1 are given in Example 6.10(b) and (70).

In general, if the DAE (44) is regular with tractability index μ, and $Q_0, \dots, Q_{\mu-1}$ are admissible projector functions, then the decomposition

$$N_{\mu-1} \oplus S_{\mu-1} = \mathbb{R}^m$$

holds true (cf. Lemma A.8). If the last projector function $Q_{\mu-1}$ is chosen such that the associated subspace $S_{\mu-1} \supseteq N_0 \oplus \cdots \oplus N_{\mu-2}$ becomes its nullspace, that is $\ker Q_{\mu-1} = S_{\mu-1}$, $\operatorname{im} Q_{\mu-1} = N_{\mu-1}$, then it follows (cf. Lemma A.9) that $Q_{\mu-1} = Q_{\mu-1} G_{\mu-1}^{-1} B_{\mu-1}$, and hence (cf. (56))

$$\begin{aligned}
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \Pi_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} \mathcal{K} \\
&= \Pi_{\mu-2} \underbrace{Q_{\mu-1}(I - \Pi_{\mu-1})}_{=Q_{\mu-1}} G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} \\
&\qquad + \sum_{l=0}^{\mu-1} \Pi_{\mu-2} \underbrace{Q_{\mu-1}(I - \Pi_l)}_{=0} (P_l - Q_l)(D\Pi_l D^-)' D\Pi_{\mu-1} \\
&= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} = \Pi_{\mu-2} Q_{\mu-1} \Pi_{\mu-1} = 0.
\end{aligned}$$

So far one can prevail on the coefficients $\mathcal{H}_{\mu-1}$ to vanish by determining $\ker Q_{\mu-1} = S_{\mu-1}$. This confirms the existence of complete decoupling projector functions for regular index one DAEs, and the existence of fine decoupling projector functions for regular index two DAEs.
Remember that, for regular constant coefficient DAEs with arbitrary index, complete decoupling projectors are provided by [LMT11b, Theorem 5.2]. We follow the lines of [Mär04a] to prove a similar result for general regular DAEs (44).

The following additional description of the coupling coefficients $\mathcal{H}_0, \dots, \mathcal{H}_{\mu-1}$ in the subsystem (64), which tie the solution u of the IERODE in this subsystem, supports the idea of an advanced decoupling. We draw the reader's attention to the consistency with [LMT11b, Theorem 5.2] which provides the easier time-invariant counterpart of a complete decoupling.

Lemma 6.12 *Let the DAE (44) be regular with tractability index μ. Let $Q_0, \dots, Q_{\mu-1}$ denote admissible projector functions, and*

$$\begin{aligned}
Q_{0*} &:= Q_0 P_1 \cdots P_{\mu-1} G_\mu^{-1} \{B_0 + G_0 D^-(D\Pi_{\mu-1} D^-)' D\}, \\
Q_{k*} &:= Q_k P_{k+1} \cdots P_{\mu-1} G_\mu^{-1} \{B_k + G_k D^-(D\Pi_{\mu-1} D^-)' D\Pi_{k-1}\},\ k = 1, \dots, \mu-2, \\
Q_{\mu-1*} &:= Q_{\mu-1} G_\mu^{-1} B_{\mu-1},
\end{aligned}$$

(1) *Then the coupling coefficients of the subsystem (64) have the representations*

$$\begin{aligned}\mathcal{H}_0 &= Q_{0*}\Pi_{\mu-1},\\ \mathcal{H}_k &= \Pi_{k-1}Q_{k*}\Pi_{\mu-1},\ k=1,\dots,\mu-2,\\ \mathcal{H}_{\mu-1} &= \Pi_{\mu-2}Q_{\mu-1*}\Pi_{\mu-1}.\end{aligned}$$

(2) *The* $Q_{0*},\dots,Q_{\mu-1*}$ *are also continuous projector functions onto the subspaces* $N_0,\dots,N_{\mu-1}$*, and it holds that* $Q_{k*}=Q_{k*}\Pi_{k-1}$ *for* $k=1,\dots,\mu-1$.

Proof: (1) For $k=0,\dots,\mu-2$, we express

$$\begin{aligned}\mathcal{A}_k &:= Q_kP_{k+1}\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1} \quad \text{(cf. (56) for } \mathcal{K} \text{ and Prop. 4.1 for } V_l)\\ &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1}+Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=0}^{\mu-1}V_lD\Pi_{\mu-1}.\end{aligned}$$

Regarding the identity $\Pi_lD^-(D\Pi_lD^-)'D\Pi_l=0$ we derive first

$$\begin{aligned}&\Pi_{k-1}\sum_{l=0}^{\mu-1}V_lD\Pi_{\mu-1}=\Pi_{k-1}\sum_{l=k}^{\mu-1}V_lD\Pi_{\mu-1}\\ &=\Pi_{k-1}\sum_{l=k}^{\mu-1}\{\underbrace{(I-\Pi_l)P_l}_{P_l-\Pi_l}D^-(D\Pi_lD^-)'D\Pi_{\mu-1}-(I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{\mu-1}\}\\ &=\Pi_{k-1}\sum_{l=k}^{\mu-1}\{P_lD^-(D\Pi_lD^-)'-(I-\Pi_l)Q_{l+1}D^-(D\Pi_{l+1}D^-)'D\Pi_{\mu-1}D^-\}D\Pi_{\mu-1}\\ &=\Pi_{k-1}\sum_{l=k}^{\mu-1}\{P_lD^-(D\Pi_lD^-)'-(I-\Pi_l)Q_{l+1}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}D^-\}D\Pi_{\mu-1}.\end{aligned}$$

Then, taking into account that $Q_\mu=0$, as well as the properties

$$\begin{aligned}&Q_kP_{k+1}\cdots P_{\mu-1}=Q_kP_{k+1}\cdots P_{\mu-1}\Pi_{k-1},\quad Q_kP_{k+1}\cdots P_{\mu-1}P_k=Q_kP_{k+1}\cdots P_{\mu-1}\Pi_k,\\ &Q_kP_{k+1}\cdots P_{\mu-1}Q_l=0,\ \text{ if }\ l\geq k+1,\end{aligned}$$

we compute

$$\begin{aligned}&Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=0}^{\mu-1}V_lD\Pi_{\mu-1}\\ &=Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1}D^-(D\Pi_lD^-)'D\Pi_{\mu-1}\\ &\qquad\qquad+Q_kP_{k+1}\cdots P_{\mu-1}\underbrace{\sum_{l=k}^{\mu-1}\Pi_lQ_{l+1}}_{\Pi_k-\Pi_{\mu-1}}D^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\ &=Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1}D^-(D\Pi_lD^-)'D\Pi_{\mu-1}+Q_kP_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.\end{aligned}$$

This leads to

$$\mathcal{A}_k = Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\Big\{B_k\Pi_{\mu-1} - \sum_{j=k+1}^{\mu-1} G_jD^-(D\Pi_jD^-)'D\Pi_{\mu-1}\Big\}$$
$$+ Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_lD^-)'D\Pi_{\mu-1} + Q_kP_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}.$$

Due to $Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}G_j = Q_kP_{k+1}\cdots P_{\mu-1}$, for $j \geq k+1$, it follows that

$$\begin{aligned}\mathcal{A}_k &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_k\Pi_{\mu-1} - Q_kP_{k+1}\cdots P_{\mu-1}\sum_{j=k+1}^{\mu-1} D^-(D\Pi_jD^-)'D\Pi_{\mu-1}\\ &+ Q_kP_{k+1}\cdots P_{\mu-1}\sum_{l=k+1}^{\mu-1} D^-(D\Pi_lD^-)'D\Pi_{\mu-1} + Q_kP_{k+1}\cdots P_{\mu-1}P_k(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\ &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_k\Pi_{\mu-1} + Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\\ &= Q_{k*}\Pi_{\mu-1},\end{aligned}$$

which proves the relations $\mathcal{H}_0 = Q_0P_1\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1} = Q_{0*}\Pi_{\mu-1}$, and $\mathcal{H}_k = \Pi_{k-1}Q_kP_{k+1}\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1} = \Pi_{k-1}\mathcal{A}_k\Pi_{\mu-1} = Q_{k*}\Pi_{\mu-1},\ \ k=1,\dots,\mu-2$. Moreover, it holds that $\mathcal{H}_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1}\mathcal{K} = Q_{\mu-1}G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1*}\Pi_{\mu-1}$.
(2) Derive

$$\begin{aligned}Q_{k*}Q_k &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\{B_k + G_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\}Q_k\\ &= Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}B_kQ_k + Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}Q_k\\ &= \underbrace{Q_kP_{k+1}\cdots P_{\mu-1}Q_k}_{=Q_k} - \underbrace{Q_kP_{k+1}\cdots P_{\mu-1}P_kD^-(D\Pi_{\mu-1}}_{=0}D^-)(D\Pi_{k-1}Q_kD^-)'D.\end{aligned}$$

Then, $Q_{k*}Q_{k*} = Q_{k*}$ follows. The remaining part is evident. □

Later on we prove the existence of fine and complete decouplings. Beforehand we present several benefits coming along with fine decouplings.

Applying fine decoupling projector functions $Q_0,\dots,Q_{\mu-1}$, the subsystem (64) corresponding to the homogeneous DAE (45) simplifies to

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1}\\ & 0 & \ddots & \vdots\\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1}\\ & & & 0\end{bmatrix}\left(\mathcal{D}\begin{bmatrix} v_0\\ v_1\\ \vdots\\ v_{\mu-1}\end{bmatrix}\right)' + \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1}\\ & I & \ddots & \vdots\\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1}\\ & & & I\end{bmatrix}\begin{bmatrix} v_0\\ v_1\\ \vdots\\ v_{\mu-1}\end{bmatrix} \tag{71}$$
$$+ \begin{bmatrix}\mathcal{H}_0\\ 0\\ \vdots\\ 0\end{bmatrix} D^-u = \begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}.$$

For given u, its solution components are determined successively as

$$v_{\mu-1} = 0, \ \ldots, \ v_1 = 0, \ v_0 = -\mathcal{H}_0 D^- u,$$

and hence each solution $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ of the homogeneous DAE (45) possesses the representation

$$x = D^- u + v_0 = (I - \mathcal{H}_0)D^- u = (I - Q_{0*}\Pi_{\mu-1})D^- D\Pi_{\mu-1}D^- u = (I - Q_{0*})\Pi_{\mu-1}D^- u,$$

whereby $u = D\Pi_{\mu-1}D^- u$ is a solution of the homogeneous IERODE

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}BD^- u = 0.$$

Owing to the relations $P_0 Q_{0*} = 0$, the continuous matrix function $(I - Q_{0*})\Pi_{\mu-1}$ is also a projector function, and the nullspace is easily checked to be

$$\ker (I - Q_{0*})\Pi_{\mu-1} = N_{can}.$$

Since each solution of the homogeneous DAE can be represented in this way, the inclusion

$$S_{can} \subseteq \operatorname{im} (I - Q_{0*})\Pi_{\mu-1}$$

is valid. On the other side, through each element of $\operatorname{im}((I - Q_{0*}(t))\Pi_{\mu-1}(t))$, at time t, passes a DAE solution, and we obtain

$$\operatorname{im} (I - Q_{0*})\Pi_{\mu-1} = S_{can}.$$

In fact, fixing an arbitrary pair $x_0 \in \operatorname{im}((I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0)), \ \ t_0 \in \mathcal{I}$, we determine the unique solution u of the standard IVP

$$u' - (D\Pi_{\mu-1}D^-)'u + D\Pi_{\mu-1}G_\mu^{-1}BD^- u = 0, \quad u(t_0) = D(t_0)\Pi_{\mu-1}(t_0)x_0,$$

and then the DAE solution $x := (I - Q_{0*})\Pi_{\mu-1}D^- u$. It results that $x(t_0) = (I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0)x_0 = x_0$. In consequence, the DAE solution passes through $x_0 \in \operatorname{im}((I - Q_{0*}(t_0))\Pi_{\mu-1}(t_0))$.
Owing to the projector properties, the decomposition

$$N_{can}(t) \oplus S_{can}(t) = \mathbb{R}^m, \quad t \in \mathcal{I}, \tag{72}$$

becomes valid. Moreover, now we see S_{can} is a $\mathcal{C}$-subspace of dimension $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$.

Definition 6.13 *For a regular DAE (44) with tractability index μ, which has a fine decoupling, the projector function $\Pi_{can} \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$ being uniquely determined by*

$$\operatorname{im} \Pi_{can} = S_{can}, \quad \ker \Pi_{can} = N_{can}$$

is named the canonical projector function of the DAE.

We underline, both canonical subspaces S_{can} and N_{can}, and the canonical projector function Π_{can} depend on the index μ. Sometimes it is reasonable to indicate this writing $S_{can\ \mu}$, $N_{can\ \mu}$ and $\Pi_{can\ \mu}$.
The canonical projector plays the same role as the spectral projector does in the time-invariant case.

Remark 6.14 *In earlier papers also the subspaces S_i (e.g. [Mär89b]) and the single projector functions $Q_0, \dots, Q_{\mu-1}$ forming a fine decoupling (e.g. [Mär89a], [Mär96]) are named canonical. This applies, in particular, to projector functions $Q_{\mu-1}$ onto $N_{\mu-1}$ along $S_{\mu-1}$. We do not carry on this notation. We know the canonical projector function in Definition 6.13 to be unique, however, for higher index cases, the single Q_i behind are not uniquely determined as it is demonstrated by [LMT11b, Example 5.4].*

Now we are in the position to gather the fruit of the construction.

Theorem 6.15 *Let the DAE (44) be regular with tractability index μ, and let $Q_0, \dots, Q_{\mu-1}$ be fine decoupling projector functions.*

(1) *Then the canonical subspaces S_{can} and N_{can} are $\mathcal{C}$-subspaces of dimensions $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$ and $m - d$.*

(2) *The decomposition (72) is valid, and the canonical projector function has the representation*

$$\Pi_{can} = (I - Q_{0*})\Pi_{\mu-1}.$$

(3) *The coefficients of the IERODE (53) are independent of the choice of the fine decoupling projector functions.*

Proof: It remains to verify (3). Let two sequences of fine decoupling projector functions $Q_0, \dots, Q_{\mu-1}$ and $\bar{Q}_0, \dots, \bar{Q}_{\mu-1}$ be given. Then the canonical projector function has the representations $\Pi_{can} = (I - Q_{0*})\Pi_{\mu-1}$ and $\Pi_{can} = (I - \bar{Q}_{0*})\bar{\Pi}_{\mu-1}$. Taking into account that $\bar{D}^- = \bar{P}_0 D^-$ we derive

$$D\Pi_{\mu-1}D^- = D\Pi_{can}D^- = D\bar{\Pi}_{\mu-1}D^- = D\bar{\Pi}_{\mu-1}\bar{D}^-.$$

Then, with the help of Lemma 3.7 yielding the relation $\bar{G}_\mu = G_\mu Z_\mu$, we arrive at

$$D\bar{\Pi}_{\mu-1}\bar{G}_\mu^{-1} = D\Pi_{\mu-1}D^- D Z_\mu^{-1} G_\mu^{-1} = D\Pi_{\mu-1}G_\mu^{-1},$$

$$D\bar{\Pi}_{\mu-1}\bar{G}_\mu^{-1}B\bar{D}^- = D\Pi_{\mu-1}G_\mu^{-1}B\bar{D}^- = D\Pi_{\mu-1}G_\mu^{-1}B(I - \bar{Q}_0)D^- = D\Pi_{\mu-1}G_\mu^{-1}BD^-,$$

and this proves the assertion. □.

For regular index one DAEs, each continuous projector function Q_0 generates already a fine decoupling. Therefore, Proposition 6.9 is now a special case of Theorem 6.15 (3).

For regular index two DAEs, the admissible pair Q_0, Q_1 provides a fine decoupling, if Q_1 is chosen such that $\ker Q_1 = S_1$. This is accompanied by the requirement that $\operatorname{im} D\Pi_1 D^- = DS_1$ is a $\mathcal{C}^1$-subspace. We point out that, for fine decouplings, we need some additional smoothness with respect to the regularity notion. While regularity with index two comprises the *existence* of an arbitrary $\mathcal{C}^1$ decomposition (i.e. the existence of a continuously differentiable projector function $D\Pi_1 D^-$)

$$\operatorname{im} D\Pi_1 D^- \oplus \underbrace{\operatorname{im} D\Pi_0 Q_1 D^-}_{=DN_1} \oplus \ker A = \mathbb{R}^n,$$

one needs for fine decouplings that the *special* decomposition

$$DS_1 \oplus DN_1 \oplus \ \ker A = \mathbb{R}^n,$$

consists of $\mathcal{C}^1$ subspaces. For instance, the semi-explicit DAE in Example 6.10 possesses the fine decoupling projector functions described in (b), if both subspaces $\operatorname{im} B_{12}$ and $\ker B_{21}$ are continuously differentiable. However, for regularity, it is enough if $\operatorname{im} B_{12}$ is a $\mathcal{C}^1$-subspace, as it is shown in (a).
Assuming the coefficients A, D, B to be $\mathcal{C}^1$, and choosing a continuously differentiable projector function Q_0, the resulting DN_1 and DS_1 are always $\mathcal{C}^1$-subspaces. However, we do not feel comfortable with such a generous sufficient smoothness assumption, though it is less demanding than that in derivative array approaches, where one necessarily has to require $A, D, B \in \mathcal{C}^2$ for the treatment of an index two problem.
We underline, here only certain continuous subspaces are additionally assumed to belong to the class $\mathcal{C}^1$. Since the precise description of these subspaces is somewhat cumbersome, we use instead the wording *the coefficients of the DAE are sufficiently smooth* just to indicate the smoothness problem.
In essence, the additional smoothness requirements are related to the coupling coefficients $\mathcal{H}_1, \ldots, \mathcal{H}_{\mu-1}$ in the subsystem (64), and in particular to the special projectors introduced in Lemma 6.12. It turns out that, for a fine decoupling of a regular index μ DAE, certain parts of the coefficients A, D, B have to be continuously differentiable up to degree $\mu - 1$. This meets the common understanding of index μ DAEs, and it is closely related to solvability conditions. We present an example for more clarity.

Example 6.16 *Consider the DAE*

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}}_{A}(\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}}_{D}x)' + \underbrace{\begin{bmatrix}0&0&0&0\\0&0&0&-1\\0&-1&0&0\\\alpha&0&-1&0\end{bmatrix}}_{B}x = 0.$$

on the interval $\mathcal{I} = [0,1]$. According to the basic continuity assumption, B is continuous, that is, $\alpha \in \mathcal{C}([0,1])$. Taking a look at the solution satisfying the initial condition $x_1(0) = 1$, that is

$$x_1(t) = 1,\ x_3(t) = \alpha(t),\ x_2(t) = x_3'(t) = \alpha'(t),\ x_4(t) = x_3''(t) = \alpha''(t)$$

we recognize that we must more reasonably assume $\alpha \in \mathcal{C}^2([0,1])$. We demonstrate by constructing a fine decoupling sequence that precisely this is the smoothness we need.
The first elements of the matrix function sequence can be chosen resp. computed as

$$Q_0 = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix},\ G_1 = \begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&0\end{bmatrix},\ Q_1 = \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix},\ G_2 = \begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&-1&1&0\\0&0&0&0\end{bmatrix}.$$

We could continue with

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

which shows the DAE to be regular with tractability index three, and Q_0, Q_1, Q_2 to be admissible, if $\alpha \in \mathcal{C}([0,1])$. However, we dismiss this choice of Q_2 and compute it instead in correspondence with the decomposition

$$N_2 \oplus S_2 = \{z \in \mathbb{R}^4 : z_1 = 0, z_2 = z_3 = z_4\} \oplus \{z \in \mathbb{R}^4 : \alpha z_1 = z_3\} = \mathbb{R}^4.$$

This leads to

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ \alpha & 0 & 1 & 0 \\ \alpha & 0 & 1 & 0 \end{bmatrix}, \; D\Pi_2 D^- = \Pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and hence, for these Q_0, Q_1, Q_2 to be admissible, the function α is required to be continuously differentiable. The coupling coefficients related to the present projector functions are

$$\mathcal{H}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{H}_2 = 0.$$

If α' does not vanish identically, we have not yet reached a fine decoupling. In the next round we set $\bar{Q}_0 = Q_0$ such that $\bar{G}_1 = G_1$, but then we put

$$\bar{Q}_1 := Q_{1*} := Q_1 P_2 G_3^{-1}\{B_1 + G_1 D^-(D\Pi_2 D^-)' D\Pi_0\} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha' & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha' & 1 & 0 & 0 \end{bmatrix}.$$

in accordance with Lemma 6.12 (see also Lemma 6.17 below). It follows that

$$D\bar{\Pi}_1 D^- = \bar{\Pi}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; \bar{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -\alpha' & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we see, to ensure that $D\bar{\Pi}_1 D^-$ becomes continuously differentiable, and $\bar{Q}_0, \bar{Q}_1$ admissible, we need a two times continuously differentiable function α. Then we have $\bar{N}_2 = N_2$, which allows for the choice $\bar{Q}_2 = Q_2$. The resulting $\bar{Q}_0, \bar{Q}_1, \bar{Q}_2$ are fine decoupling projector functions.

Construction of fine decoupling projector functions.

Now we construct fine decoupling projector functions for the general regular DAE (44).

As in Example 6.16, we successively improve the decoupling with the help of Lemma 6.12 in several rounds. We begin forming arbitrary admissible projector functions $Q_0,\dots,Q_{\mu-2}$ and $G_{\mu-1}$. Then we determine $Q_{\mu-1}$ by $\ker Q_{\mu-1}=S_{\mu-1}$ and $\operatorname{im} Q_{\mu-1}=N_{\mu-1}$. This yields $G_\mu=G_{\mu-1}+B_{\mu-1}Q_{\mu-1}$ as well as

$$Q_{\mu-1}=Q_{\mu-1}G_\mu^{-1}B_{\mu-1}=Q_{\mu-1*},\quad \text{and}\quad \mathcal{H}_{\mu-1}=\Pi_{\mu-2}Q_{\mu-1*}\Pi_{\mu-1}=\Pi_{\mu-2}Q_{\mu-1}\Pi_{\mu-1}=0.$$

If $\mu=2$ we have already a fine decoupling. If $\mu\geq 3$, we assume $D\Pi_{\mu-3}Q_{\mu-2*}D^-$ which is a priori continuous to be even continuously differentiable, and compose a new sequence from the previous one. We set

$$\bar{Q}_0:=Q_0,\dots,\bar{Q}_{\mu-3}=Q_{\mu-3},\quad \text{and}\quad \bar{Q}_{\mu-2}=Q_{\mu-2*}.$$

$D\bar{\Pi}_{\mu-2}D^-=D\Pi_{\mu-3}D^--D\Pi_{\mu-3}Q_{\mu-2*}D^-$ is continuously differentiable, and the projector functions $\bar{Q}_0,\dots,\bar{Q}_{\mu-2}$ are admissible. Further, some technical calculations yield

$$\bar{G}_{\mu-1}=G_{\mu-1}\underbrace{\{I+\bar{Q}_{\mu-2}P_{\mu-2}+(I-\Pi_{\mu-3})Q_{\mu-2}D^-(D\bar{\Pi}_{\mu-2}D^-)'D\Pi_{\mu-3}\bar{Q}_{\mu-2}\}}_{Z_{\mu-1}}.$$

The matrix function $Z_{\mu-1}$ remains nonsingular, it has the pointwise inverse

$$Z_{\mu-1}^{-1}=I-\bar{Q}_{\mu-2}P_{\mu-2}-(I-\Pi_{\mu-3})Q_{\mu-2}D^-(D\bar{\Pi}_{\mu-2}D^-)'D\Pi_{\mu-3}Q_{\mu-2}.$$

We complete the current sequence by

$$\bar{Q}_{\mu-1}:=Z_{\mu-1}^{-1}Q_{\mu-1}Z_{\mu-1}=Z_{\mu-1}^{-1}Q_{\mu-1}.$$

It results that $\bar{Q}_{\mu-1}\bar{Q}_{\mu-2}=Z_{\mu-1}^{-1}Q_{\mu-1}Q_{\mu-2*}=0$ and $\bar{Q}_{\mu-1}\bar{Q}_i=Z_{\mu-1}^{-1}Q_{\mu-1}Q_i=0$ for $i=0,\dots,\mu-3$. Applying several basic properties (e.g. $\bar{\Pi}_{\mu-2}=\bar{\Pi}_{\mu-2}\Pi_{\mu-2}$) we find the representation $D\bar{\Pi}_{\mu-1}D^-=(D\bar{\Pi}_{\mu-2}D^-)(D\Pi_{\mu-1}D^-)$ which shows the continuous differentiability of $D\bar{\Pi}_{\mu-1}D^-$. Our new sequence $\bar{Q}_0,\dots,\bar{Q}_{\mu-1}$ is admissible. We have further $\operatorname{im}\bar{G}_{\mu-1}=\operatorname{im}G_{\mu-1}$, thus

$$\bar{S}_{\mu-1}=S_{\mu-1}=\ker\mathcal{W}_{\mu-1}B=\ker\mathcal{W}_{\mu-1}BZ_{\mu-1}=Z_{\mu-1}^{-1}S_{\mu-1}.$$

This makes clear, $\bar{Q}_{\mu-1}=Z_{\mu-1}^{-1}Q_{\mu-1}$ projects onto $\bar{N}_{\mu-1}=Z_{\mu-1}^{-1}N_{\mu-1}$ along $\bar{S}_{\mu-1}=Z_{\mu-1}^{-1}S_{\mu-1}$, and therefore the new coupling coefficient satisfies $\bar{\mathcal{H}}_{\mu-1}=0$. Additionally, making further technical efforts one attains $\bar{\mathcal{H}}_{\mu-2}=0$.
If $\mu=3$, a fine decoupling is reached. If $\mu\geq 4$, we built the next sequence analogously as

$$\bar{\bar{Q}}_0:=\bar{Q}_0,\dots,\bar{\bar{Q}}_{\mu-4}:=\bar{Q}_{\mu-4},\ \bar{\bar{Q}}_{\mu-3}:=\bar{Q}_{\mu-3*},$$
$$\bar{\bar{Q}}_{\mu-2}:=\bar{Z}_{\mu-2}^{-1}\bar{Q}_{\mu-2}\bar{Z}_{\mu-2},\ \bar{\bar{Q}}_{\mu-1}:=\bar{Z}_{\mu-1}^{-1}\bar{Q}_{\mu-1}\bar{Z}_{\mu-1}.$$

Supposing $D\bar{\Pi}_{\mu-4}\bar{Q}_{\mu-3*}D^-$ to be continuously differentiable, we prove the new sequence to be admissible, and to generate the coupling coefficients

$$\bar{\bar{\mathcal{H}}}_{\mu-1}=0,\quad \bar{\bar{\mathcal{H}}}_{\mu-2}=0,\quad \bar{\bar{\mathcal{H}}}_{\mu-3}=0.$$

And so on. Lemma 6.17 below guarantees the procedure to reach its goal.

Lemma 6.17 *Let the DAE (44) with sufficiently smooth coefficients be regular with tractability index $\mu \geq 3$, and let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions.*
Let $k \in \{1, \ldots, \mu-2\}$ be fixed, and let $\bar{Q}_k$ be an additional continuous projector function onto $N_k = \ker G_k$ such that $D\Pi_{k-1}\bar{Q}_k D^-$ is continuously differentiable and the inclusion $N_0 + \cdots + N_{k-1} \subseteq \ker \bar{Q}_k$ is valid. Then the following becomes true:

(1) *The projector function sequence*

$$\begin{aligned}&\bar{Q}_0 := Q_0, \ldots, \bar{Q}_{k-1} := Q_{k-1},\\ &\bar{Q}_k,\\ &\bar{Q}_{k+1} := Z_{k+1}^{-1} Q_{k+1} Z_{k+1}, \ldots, \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1} Q_{\mu-1} Z_{\mu-1},\end{aligned}$$

with the determined below continuous nonsingular matrix functions $Z_{k+1}, \ldots, Z_{\mu-1}$, is also admissible.

(2) *If, additionally, the projector functions $Q_0, \ldots, Q_{\mu-1}$ provide an advanced decoupling in the sense that the conditions (cf. Lemma 6.12)*

$$Q_{\mu-1*}\Pi_{\mu-1} = 0, \ \ldots, \ Q_{k+1*}\Pi_{\mu-1} = 0$$

are given, then also the relations

$$\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = 0, \ \ldots, \ \bar{Q}_{k+1*}\bar{\Pi}_{\mu-1} = 0, \tag{73}$$

are valid, and further

$$\bar{Q}_{k*}\bar{\Pi}_{\mu-1} = (Q_{k*} - \bar{Q}_k)\Pi_{\mu-1}. \tag{74}$$

The matrix functions Z_i are consistent with those given in Lemma 3.7, however, for an easier reading we do not access this general lemma in the proof below. In the special case given here, Lemma 3.7 yields simply $Z_0 = I, Y_1 = Z_1 = I, \ldots, Y_k = Z_k = I$, and further

$$\begin{aligned} Y_{k+1} &= I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} \bar{Q}_k = (I + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{kl} Q_k)(I + Q_k(\bar{Q}_k - Q_k)),\\ Z_{k+1} &= Y_{k+1},\\ Y_j &= I + \sum_{l=0}^{j-2} Q_l \mathfrak{A}_{j-1\,l} Q_{j-1}, \quad Z_j = Y_j Z_{j-1}, \quad j = k+2, \ldots, \mu. \end{aligned}$$

Besides the general property $\ker \bar{\Pi}_j = \ker \Pi_j$, $j = 0, \ldots, \mu-1$, which follows from Lemma 3.7, now it additionally holds that

$$\operatorname{im} \bar{Q}_k = \operatorname{im} Q_k, \quad \text{but} \quad \ker \bar{Q}_j = \ker Q_j, \quad j = k+1, \ldots, \mu-1.$$

We refer to the Appendix B for the extensive calculations proving this lemma.

Lemma 6.17 guarantees the existence of fine decoupling projector functions, and it confirms the procedure sketched above to be reasonable.
The following theorem is the time-varying counterpart of [LMT11b, Theorem 5.2] on constant coefficient DAEs.

Theorem 6.18 *Let the DAE (44) be regular with tractability index μ.*

(1) *If the coefficients of the DAE are sufficiently smooth, then a fine decoupling exists.*

(2) *If there is a fine decoupling, then there is also a complete decoupling.*

Proof: (1) The first assertion is a consequence of Lemma 6.17 and the procedure described above.
(2) Let fine decoupling projectors $Q_0, \ldots, Q_{\mu-1}$ be given. We form the new sequence

$$\bar{Q}_0 := Q_{0*},\ \bar{Q}_1 := Z_1^{-1} Q_1 Z_1,\ \ldots,\ \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1} Q_{\mu-1} Z_{\mu-1},$$

with the matrix functions Z_j from Lemma 3.7, in particular $Z_1 = I + \bar{Q}_0 P_0$. It holds that $\bar{D}^- = \bar{P}_0 D^-$. Owing to the special form of Z_j, the relations $\Pi_{j-1} Z_j = \Pi_{j-1}$, $\Pi_{j-1} Z_j^{-1} = \Pi_{j-1}$ are given for $j \leq i-1$. This yields $\bar{Q}_i \bar{Q}_j = \bar{Q}_i Z_j^{-1} Q_j Z_j = \bar{Q}_i \underbrace{\Pi_{i-1} Z_j^{-1} Q_j}_{=0} Z_j = 0$.
Expressing $D\bar{\Pi}_1 \bar{D}^- = D\bar{P}_0 Z_1^{-1} P_1 Z_1 \bar{P}_0 D^- = D \underbrace{P_0 Z_1^{-1} P_1}_{\Pi_1} Z_1 \bar{P}_0 D^- = D\Pi_1 D^-$, and successively,

$$\begin{aligned} D\bar{\Pi}_i \bar{D}^- &= D\bar{\Pi}_{i-1} Z_i^{-1} P_i Z_i \bar{P}_D^- \\ &= D\bar{\Pi}_{i-1} \bar{D}^- D Z_i^{-1} P_i Z_i \bar{P}_D^- = D \underbrace{\Pi_{i-1} D^- D Z_i^{-1} P_i}_{\Pi_i} Z_i \bar{P}_D^- = D\Pi_i D^-, \end{aligned}$$

we see the new sequence of projector functions $\bar{Q}_0, \ldots, \bar{Q}_{\mu-1}$ to be admissible, too. Analogously to Lemma 6.17, one shows

$$\bar{\mathcal{H}}_{\mu-1} = 0, \ldots, \bar{\mathcal{H}}_1 = 0, \quad \bar{\mathcal{H}}_0 = (Q_{0*} - \bar{Q}_0)\Pi_{\mu-1},$$

and this completes the proof. □

6.3 Solvability and flow

Here we continue to investigate regular DAEs (44) which have tractability index μ and fine decoupling projector functions $Q_0, \ldots, Q_{\mu-1}$. It is worth emphasizing once more that Theorem 6.18 guarantees the existence of a fine decoupling for all regular DAEs with sufficiently smooth coefficients. By Theorem 6.15 (cf. also Lemma 6.12),

$$\Pi_{can} = (I - Q_{0*})\Pi_{\mu-1} = (I - \mathcal{H}_0)\Pi_{\mu-1}$$

is the canonical projector function onto S_{can} along N_{can}, and hence

$$D\Pi_{can} = D\Pi_{\mu-1}, \quad D\Pi_{can} D^- = D\Pi_{\mu-1} D^-, \quad \text{and} \quad \operatorname{im} D\Pi_{\mu-1} = \operatorname{im} D\Pi_{can} = DS_{can}.$$

Taking into account also Lemma 6.5 (7), the IERODE can now be written as

$$u' - (D\Pi_{can} D^-)' u + D\Pi_{can} G_\mu^{-1} B D^- u = D\Pi_{can} G_\mu^{-1} q, \tag{75}$$

and, by Lemma 6.4, DS_{can} is a time-varying invariant subspace for its solutions, that means, $u(t_0) \in D(t_0)S_{can}(t_0)$ implies $u(t) \in D(t)S_{can}(t)$ for all $t \in \mathcal{I}$. This invariant subspace applies also to the homogeneous version of the IERODE. The IERODE is unique, its coefficients are independent of the special choice of the fine decoupling projector functions, as it is pointed out in the previous subsection.
With regard to the fine decoupling, the Proposition 6.6 (6), and the fact that $v_i = \Pi_{i-1}Q_i v_i$ holds true for $i = 1, \ldots, \mu - 1$, the subsystem (64) has now the slightly simpler form

$$v_0 = -\sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l}\, v_l - \mathcal{H}_0 D^- u + \mathcal{L}_0 q, \tag{76}$$

$$v_i = -\sum_{l=i+1}^{\mu-1} \mathcal{N}_{il}(Dv_l)' - \sum_{l=i+2}^{\mu-1} \mathcal{M}_{il}\, v_l + \mathcal{L}_i q, \quad i = 1, \ldots, \mu - 3, \tag{77}$$

$$v_{\mu-2} = -\mathcal{N}_{\mu-2,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2} q, \tag{78}$$

$$v_{\mu-1} = \mathcal{L}_{\mu-1} q. \tag{79}$$

By Theorem 6.7, the DAE (44) is equivalent to the system consisting of the IERODE and the subsystem (76)-(79).

6.3.1 Homogeneous DAEs

The following solvability assertion is a simple consequence of the above.

Theorem 6.19 *Let the homogeneous DAE (45) be regular, and let the coefficients be smooth enough for the existence of a fine decoupling. Then,*

(1) *for each arbitrary $x^0 \in \mathbb{R}^m$, the IVP*

$$A(Dx)' + Bx = 0, \quad x(t_0) - x^0 \in N_{can}(t_0), \tag{80}$$

is uniquely solvable in $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$,

(2) *the homogeneous IVP*

$$A(Dx)' + Bx = 0, \quad x(t_0) \in N_{can}(t_0),$$

has the trivial solution only, and

(3) *through each $x_0 \in S_{can}(t_0)$ passes exactly one solution.*

Remark 6.20 *Sometimes is seems to be more comfortable to describe the initial condition in (80) by an equation, for instance, as*

$$\Pi_{can}(t_0)(x(t_0) - x^0) = 0, \tag{81}$$

and as

$$C(x(t_0) - x^0) = 0, \tag{82}$$

by any matrix C such that $\ker C = \ker \Pi_{can}(t_0) = N_{can}(t_0)$. For instance, taking arbitrary admissible projector functions $\tilde{Q}_0, \ldots, \tilde{Q}_{\mu-1}$, one can choose C such that $C = C\tilde{\Pi}_{can}(t_0)$.

Proof: (2) The initial condition yields $u(t_0) = D(t_0)\Pi_{can}(t_0)x(t_0) = 0$. Then, the resulting homogeneous IVP for the IERODE admits the trivial solution $u = 0$ only. Therefore, the DAE solution $x = \Pi_{can}D^- u$ vanishes identically, too.
(1) We provide the solution u of the homogeneous IERODE which satisfies the initial condition $u(t_0) = D(t_0)\Pi_{can}(t_0)x^0$. Then we form the DAE solution $x = \Pi_{can}D^- u$, and check the initial condition to be met:

$$\begin{aligned} x(t_0) - x^0 &= \Pi_{can}(t_0)D(t_0)^- u(t_0) - x^0 = \Pi_{can}(t_0)D(t_0)^- D(t_0)\Pi_{can}(t_0)x^0 - x^0 \\ &= -(I - \Pi_{can}(t_0))x^0 \in N_{can}(t_0). \end{aligned}$$

Owing to (2) this is the only solution of the IVP.
(3) We provide the IVP solution as in (1), with x^0 replaced by x_0. This leads to

$$x(t_0) = \Pi_{can}(t_0)D(t_0)^- u(t_0) = \Pi_{can}(t_0)D(t_0)^- D(t_0)\Pi_{can}(t_0)x_0 = \Pi_{can}(t_0)x_0 = 0.$$

The uniqueness is ensured by (2). □

As it is common in ODE theory we denote by $x(., t_0, x^0)$ the solution of the IVP (80). In contrast to the value x^0 being not necessarily consistent, we indicate by x_0 a consistent value. As for regular time varying ODEs (e.g. [Gaj99]), we may also consider the qualitative behavior of solutions.

Definition 6.21 *Let the homogeneous regular DAE (45) be given on the infinite interval $\mathcal{I} = [0, \infty)$, and let the coefficients be smooth enough for fine decouplings. The homogeneous DAE is said to be*

(1) *stable, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$ a value $\delta(\varepsilon, t_0) > 0$ exists, such that $x_0, \bar{x}_0 \in S_{can}(t_0)$, $|x_0 - \bar{x}_0| < \delta(\varepsilon, t_0)$ imply the existence of solutions $x(., t_0, x_0)$, $x(., t_0, \bar{x}_0) \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, as well as $|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| < \varepsilon$, $t \geq t_0$,*

(2) *uniformly stable, if $\delta(\varepsilon, t_0)$ in (1) is independent of t_0,*

(3) *asymptotically stable, if (1) holds true and*

$$|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| \xrightarrow[t\to\infty]{} 0 \quad \textit{for all} \quad x_0, \bar{x}_0 \in S_{can}(t_0),\ t_0 \in \mathcal{I},$$

(4) *uniformly asymptotically stable, if the limit in (3) is uniform with respect to t_0.*

6.3.2 Fundamental solution matrices

By Theorem 6.19, regular homogeneous DAEs are close to regular homogeneous ODEs. This applies also to their fundamental solution matrices.
Denote by $U(t, t_0)$ the classical fundamental solution matrix of the IERODE, that is, of the explicit ODE (75), which is normalized at $t_0 \in \mathcal{I}$, i.e. $U(t_0, t_0) = I$.
For each arbitrary initial value $u_0 \in D(t_0)S_{can}(t_0)$, the solution of the homogeneous IERODE passing through remains for ever in this invariant subspace, which means $U(t, t_0)u_0 \in D(t)S_{can}(t)$ for all $t \in \mathcal{I}$, and hence

$$U(t, t_0)D(t_0)\Pi_{can}(t_0) = D(t)\Pi_{can}(t)D(t)^- U(t, t_0)D(t_0)\Pi_{can}(t_0), \quad t \in \mathcal{I}. \tag{83}$$

Each solution of the homogeneous DAE (45) can now be expressed as

$$x(t) = (I - \mathcal{H}_0(t))D(t)^-U(t,t_0)u_0 = \Pi_{can}(t)D(t)^-U(t,t_0)u_0, \quad t \in \mathcal{I},\ u_0 \in D(t_0)S_{can}(t_0), \tag{84}$$

and also as

$$x(t) = \underbrace{\Pi_{can}(t)D(t)^-U(t,t_0)D(t_0)\Pi_{can}(t_0)}_{X(t,t_0)}x^0, \quad t \in \mathcal{I}, \quad \text{with} \quad x^0 \in \mathbb{R}^m. \tag{85}$$

If $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ fulfills the homogeneous DAE (45), then there is exactly one $u_0 \in D(t_0)S_{can}(t_0)$ such that the expression (84) is valid, and there are elements $x^0 \in \mathbb{R}^m$ such that (85) applies. Except for the index zero case, x^0 is not unique.
Conversely, for each arbitrary $x^0 \in \mathbb{R}^m$, formula (85) provides a solution of (45). We know, the solution values of the homogeneous DAE lie in the d-dimensional canonical subspace S_{can}, in particular $x(t_0) \in S_{can}(t_0)$. Therefore, starting from an arbitrary $x^0 \in \mathbb{R}^m$, the consistency of $x(t_0)$ with x^0 can not be expected. What we always attain is the relation

$$x(t_0) = \Pi_{can}(t_0)x^0,$$

but the condition $x(t_0) = x_0$ is exclusively reserved for x_0 belonging to $S_{can}(t_0)$.

The composed matrix function

$$X(t,t_0) := \Pi_{can}(t)D(t)^-U(t,t_0)D(t_0)\Pi_{can}(t_0), \quad t \in \mathcal{I}, \tag{86}$$

arising in the solution expression (85) plays the role of a fundamental solution matrix of the DAE (44). In comparison with the (regular) ODE theory, there are several differences to be considered. By construction, it holds that $X(t_0,t_0) = \Pi_{can}(t_0)$ and

$$\operatorname{im} X(t,t_0) \subseteq S_{can}(t), \quad N_{can}(t_0) \subseteq \ker X(t,t_0), \quad t \in \mathcal{I}, \tag{87}$$

so that $X(t,t_0)$ is a *singular* matrix, except for the case $\mu = 0$. $X(.,t_0)$ is continuous, and $DX(.,t_0) = D\Pi_{can}D^-U(.,t_0)D(t_0)\Pi_{can}(t_0)$ is continuously differentiable, thus the columns of $X(.,t_0)$ are functions belonging to $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$.
We show that $X(t,t_0)$ has constant rank d. Fix an arbitrary $t \neq t_0$ and investigate the nullspace of $X(t,t_0)$. $X(t,t_0)z = 0$ means $U(t,t_0)D(t_0)\Pi_{can}(t_0)z \in \ker \Pi_{can}(t)D(t)^-$, and with regard of (83) this yields $U(t,t_0)D(t_0)\Pi_{can}(t_0)z = 0$, thus $D(t_0)\Pi_{can}(t_0)z = 0$, and further $\Pi_{can}(t_0)z = 0$. Owing to (87), and for reasons of dimensions, it follows that

$$\operatorname{im} X(t,t_0) = S_{can}(t), \quad N_{can}(t_0) = \ker X(t,t_0), \quad \operatorname{rank} X(t,t_0) = d, \quad t \in \mathcal{I}. \tag{88}$$

Lemma 6.22 *The matrix function*

$$X(t,t_0)^- = \Pi_{can}(t_0)D(t_0)^-U(t,t_0)^{-1}D(t)\Pi_{can}(t), \quad t \in \mathcal{I},$$

is the reflexive generalized inverse of $X(t,t_0)$ determined by

$$XX^-X = X, \quad X^-XX^- = X^-, \quad X^-X = \Pi_{can}(t_0), \quad XX^- = \Pi_{can}.$$

Proof: Applying the invariance (83), we derive

$$\begin{aligned}
X^-X &= \mathit{\Pi}_{can}(t_0)D(t_0)^-U^{-1}D\mathit{\Pi}_{can}\mathit{\Pi}_{can}D^-UD(t_0)\mathit{\Pi}_{can}(t_0) \\
&= \mathit{\Pi}_{can}(t_0)D(t_0)^-U^{-1}\underbrace{D\mathit{\Pi}_{can}D^-UD(t_0)\mathit{\Pi}_{can}(t_0)}_{UD(t_0)\mathit{\Pi}_{can}(t_0)} = \mathit{\Pi}_{can}(t_0),
\end{aligned}$$

and $X^-XX^- = (X^-X)X^- = X^-$, $XX^-X = X(X^-X) = X$.
Next we verify the relation

$$U^{-1}D\mathit{\Pi}_{can} = D(t_0)\mathit{\Pi}_{can}(t_0)D(t_0)^-U^{-1}D\mathit{\Pi}_{can}, \tag{89}$$

which in turn implies

$$\begin{aligned}
XX^- &= \mathit{\Pi}_{can}D^-UD(t_0)\mathit{\Pi}_{can}(t_0)\mathit{\Pi}_{can}(t_0)D(t_0)^-U^{-1}D\mathit{\Pi}_{can} \\
&= \mathit{\Pi}_{can}D^-U\underbrace{D(t_0)\mathit{\Pi}_{can}(t_0)D(t_0)^-U^{-1}D\mathit{\Pi}_{can}}_{U^{-1}D\mathit{\Pi}_{can}} = \mathit{\Pi}_{can}.
\end{aligned}$$

From

$$U' - (D\mathit{\Pi}_{can}D^-)'U + D\mathit{\Pi}_{can}G_\mu^{-1}BD^-U = 0, \quad U(t_0) = 0,$$

it follows that

$$U^{-1\prime} + U^{-1}(D\mathit{\Pi}_{can}D^-)' - U^{-1}D\mathit{\Pi}_{can}G_\mu^{-1}BD^- = 0.$$

Multiplication by $D\mathit{\Pi}_{can}D^-$ from the right results in the explicit ODE

$$V' = V(D\mathit{\Pi}_{can}D^-)' + VD\mathit{\Pi}_{can}G_\mu^{-1}BD^-$$

for the matrix function $V = U^{-1}D\mathit{\Pi}_{can}D^-$. Then, the matrix function $\tilde{V} := (I - D(t_0)\mathit{\Pi}_{can}(t_0)D(t_0)^-)V$ vanishes identically as the solution of the classical homogeneous IVP

$$\tilde{V}' = \tilde{V}(D\mathit{\Pi}_{can}D^-)' + \tilde{V}D\mathit{\Pi}_{can}G_\mu^{-1}BD^-, \quad \tilde{V}(t_0) = 0,$$

and this proves (89). □

The columns of $X(.,t_0)$ are solutions of the homogeneous DAE (45), and the matrix function $X(.,t_0)$ itself satisfies the equation

$$A(DX)' + BX = 0, \tag{90}$$

as well as the initial condition

$$X(t_0,t_0) = \mathit{\Pi}_{can}(t_0), \tag{91}$$

or, equivalently,

$$\mathit{\Pi}_{can}(t_0)(X(t_0,t_0) - I) = 0. \tag{92}$$

Definition 6.23 *Let the DAE (44) be regular with fine decoupling projector functions. The matrix function $Y \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^m))$, $d \le s \le m$, is said to be a fundamental solution matrix of the DAE, if its columns belong to $\mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$, the equation*

$$A(DY)' + BY = 0$$

is fulfilled, and the condition $\operatorname{im} Y = S_{can}$ *is valid.*
A fundamental solution matrix is named minimal of size, *if* $s = d$, *and* maximal of size, *if* $s = m$.
A maximal size fundamental solution matrix Y *is said to be* normalized at t_0, *if* $\Pi_{can}(t_0)(Y(t_0) - I) = 0$.

In this sense, the above matrix function $X(., t_0)$ (cf. 86) is a maximal size fundamental solution normalized at t_0.

Remark 6.24 *Concerning fundamental solution matrices of DAEs, there is no common agreement in the literature. Minimal and maximal size fundamental solution matrices, as well as relations among them, are first described in [BM00] for standard form index one DAEs. A comprehensive analysis for regular lower index DAEs, both in standard form and with properly stated leading term, is given in [Bal04]. This analysis applies analogously to regular DAEs with arbitrary index.*
Roughly speaking, minimal size fundamental solution matrices have a certain advantage in view of computational aspects, since they have full column rank. For instance, the Moore-Penrose inverse can be easily computed. In contrast, the benefits from maximal size fundamental solution matrices are a natural normalization and useful group properties as pointed out e.g. in [BM02], [Bal04].
If $X(t, t_0)$ *is the maximal size fundamental solution matrix normalized at* $t_0 \in \mathcal{I}$, *and* $X(t, t_0)^-$ *is the generalized inverse described by Lemma 6.22, then it holds for all* $t, t_0, t_1 \in \mathcal{I}$ *that*

$$X(t, t_1)X(t_1, t_0) = X(t, t_0), \quad \text{and} \quad X(t, t_0)^- = X(t_0, t),$$

as immediate consequences of the construction, and Lemma 6.22.

Applying normalized maximal size fundamental solution matrices one can modify results on flow properties of explicit ODEs (e.g. [Gaj99] to be considered for DAEs.

Proposition 6.25 *Let the homogeneous DAE (45) be regular with sufficiently smooth coefficients so that fine decoupling projector functions exist. Then the following assertions hold true with positive constants* K_{t_0}, K *and* α:

(1) *If* $|X(t, t_0)| \leq K_{t_0}$, $t \geq t_0$, *then the DAE is stable.*

(2) *If* $|X(t, t_0)| \xrightarrow[t\to\infty]{} 0$, *then the DAE is asymptotically stable.*

(3) *If* $|X(t, t_0)X(s, t_0)^-| \leq K$, $t \geq s \geq t_0$, *then the DAE is uniformly stable.*

(4) *If* $|X(t, t_0)X(s, t_0)^-| \leq K\mathrm{e}^{-\alpha(t-s)}$, $t \geq s \geq t_0$, *then the DAE is uniformly asymptotically stable.*

Proof: (1) It suffices to put $\delta(t_0, \varepsilon) = \varepsilon / K_{t_0}$.
(2) This is now obvious.
(4) Take $x_0, \bar{x}_0 \in S_{can}(t_0)$, $z_0 := x_0 - \bar{x}_0 \neq 0$ such that $X(t, t_0)z_0$ has no zeros. then we compute for $t \geq s$

$$\begin{aligned}\frac{|X(t,t_0)z_0|}{|X(s,t_0)z_0|} = \frac{|X(t,t_0)\Pi_{can}z_0|}{|X(s,t_0)z_0|} &= \frac{|X(t,t_0)X(s,t_0)^-X(s,t_0)z_0|}{|X(s,t_0)z_0|} \\ &\leq |X(t,t_0)X(s,t_0)^-| \leq K\mathrm{e}^{-\alpha(t-s)}.\end{aligned}$$

This implies

$$|x(t,t_0,x_0) - x(t,t_0,\bar{x}_0)| = |X(t,t_0)z_0| \leq K\mathrm{e}^{-\alpha(t-s)}|x(s,t_0,x_0) - x(s,t_0,\bar{x}_0)|.$$

(3) This proves as (4), with $\alpha = 0$. $\square$

Definition 6.26 *The regular DAE* (45) *with fine decoupling is said to be* <u>*dichotomic*</u>*, if there are constants $K, \alpha, \beta \geq 0$, and a nontrivial projector (not equal to the zero or identity matrix) $P_{dich} \in L(\mathbb{R}^m)$ such that $P_{dich} = \Pi_{can}(t_0)P_{dich} = P_{dich}\Pi_{can}(t_0)$, and the following inequalities apply for all $t,s \in \mathcal{I}$:*

$$\begin{aligned} |X(t,t_0)P_{dich}X(s,t_0)^-| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ |X(t,t_0)(I - P_{dich})X(s,t_0)^-| &\leq Ke^{-\beta(s-t)}, \quad t \leq s. \end{aligned}$$

If $\alpha\beta > 0$, then one speaks of an <u>*exponential dichotomy*</u>*.*

Sometimes it is reasonable writing the last inequality in the form

$$|X(t,t_0)(\Pi_{can}(t_0) - P_{dich})X(s,t_0)^-| \leq Ke^{-\beta(s-t)}, \quad t \leq s.$$

It should be pointed out that dichotomy is actually independent of the reference point t_0. Namely, for $t_1 \neq t_0$, with $P_{dich,t_1} := X(t_1,t_0)P_{dich}X(t_1,t_0)^-$ we have a projector such that $P_{dich,t_1} = \Pi_{can}(t_1)P_{dich,t_1} = P_{dich,t_1}\Pi_{can}(t_1)$ and

$$\begin{aligned} |X(t,t_1)P_{dich,t_1}X(s,t_1)^-| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ |X(t,t_1)(\Pi_{can}(t_1) - P_{dich,t_1})X(s,t_1)^-| &\leq Ke^{-\beta(s-t)}, \quad t \leq s. \end{aligned}$$

Analogously to the ODE case, the flow of a dichotomic regular DAE is divided into two parts, one containing in certain sense nonincreasing solution, the other with nondecreasing ones. More precisely, for a nontrivial $x_0 \in \operatorname{im} P_{dich} \subseteq S_{can}(t_0)$, the DAE solution $x(t,t_0,x_0) = X(t,t_0)x_0$ has no zeros, and it satisfies for $t \geq s$ the inequalities

$$\begin{aligned} \frac{|x(t,t_0,x_0)|}{|x(s,t_0,x_0)|} &= \frac{|X(t,t_0)x_0|}{|X(s,t_0)x_0|} = \frac{|X(t,t_0)P_{dich}\Pi_{can}(t_0)x_0|}{|X(s,t_0)x_0|} \\ &= \frac{|X(t,t_0)P_{dich}X(s,t_0)^-X(s,t_0)x_0|}{|X(s,t_0)x_0|} \leq |X(t,t_0)P_{dich}X(s,t_0)^-| \leq Ke^{-\alpha(t-s)}. \end{aligned}$$

For solutions $x(t,t_0,x_0) = X(t,t_0)x_0$ with $x_0 \in \operatorname{im}(I - P_{dich})\Pi_{can} \subseteq S_{can}(t_0)$ we show analogously, for $t \leq s$,

$$\begin{aligned} \frac{|x(t,t_0,x_0)|}{|x(s,t_0,x_0)|} &= \frac{|X(t,t_0)x_0|}{|X(s,t_0)x_0|} = \frac{|X(t,t_0)(I - P_{dich})\Pi_{can}(t_0)x_0|}{|X(s,t_0)x_0|} \\ &= \frac{|X(t,t_0)(I - P_{dich})X(s,t_0)^-X(s,t_0)x_0|}{|X(s,t_0)x_0|} \\ &\leq |X(t,t_0)(I - P_{dich})X(s,t_0)^-| \leq Ke^{-\beta(s-t)}. \end{aligned}$$

The canonical subspace of the dichotomic DAE decomposes into

$$S_{can}(t) = \operatorname{im} X(t,t_0) = \operatorname{im} X(t,t_0)P_{dich} \oplus \operatorname{im} X(t,t_0)(I - P_{dich}) =: S^-_{can}(t) \oplus S^+_{can}(t).$$

The following two inequalities result for $t \geq s$, and they characterize the subspaces S^-_{can} and S^+_{can} as those containing nonincreasing and nondecreasing solutions, respectively:

$$
\begin{aligned}
|x(t,t_0,x_0)| \leq K e^{-\alpha(t-s)} |x(s,t_0,x_0)|, \quad &\text{if} \quad x_0 \in S^-_{can},\\
\frac{1}{K} e^{\beta(t-s)} |x(s,t_0,x_0)| \leq |x(t,t_0,x_0)|, \quad &\text{if} \quad x_0 \in S^+_{can}.
\end{aligned}
$$

In particular, for $s = t_0$ it results that

$$
\begin{aligned}
|x(t,t_0,x_0)| \leq K e^{-\alpha(t-t_0)} |x_0|, \quad &\text{if} \quad x_0 \in S^-_{can},\\
\frac{1}{K} e^{\beta(t-t_0)} |x_0| \leq |x(t,t_0,x_0)|, \quad &\text{if} \quad x_0 \in S^+_{can}.
\end{aligned}
$$

If $\alpha > 0$, and $\mathcal{I} = [t_0, \infty)$, then $|x(t,t_0,x_0)|$ tends to zero for t tending to ∞, if x_0 belongs to $S^-_{can}(t_0)$. If $\beta > 0$ and $x_0 \in S^+_{can}(t_0)$, then $x(t,t_0,x_0)$ growths unboundedly with increasing t.

As for explicit ODEs, dichotomy makes good sense on infinite intervals I. The growth behavior of fundamental solutions is also important for the condition of boundary value problems stated on compact intervals (e.g. [AMR88] for explicit ODEs, also [LM90] for index one DAEs). Dealing with compact intervals one supposes a constant K of moderate size.

Example 6.27 *Consider the semi-explicit DAE*

$$
\begin{bmatrix} I \\ 0 \end{bmatrix} \left(\begin{bmatrix} I & 0 \end{bmatrix} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} x = 0,
$$

consisting of three equations, $m_1 = 2, m_2 = 1, n = 2$. Let B_{22} have no zeros, let the coefficients be such that

$$
B_{11} + B_{12} \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}, \quad \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} := -B_{22}^{-1} B_{21},
$$

with constants $\alpha, \beta \geq 0$. Then, the canonical projector function and the IERODE have the form (cf. Example 6.8)

$$
\Pi_{can} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & 0 \end{bmatrix}, \quad \text{and} \quad u' + \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} u = 0.
$$

The IERODE is obviously dichotomic. Compute the fundamental solution matrix of the DAE and its generalized inverse:

$$
X(t,t_0) = \begin{bmatrix} e^{-\alpha(t-t_0)} & 0 & 0 \\ 0 & e^{\beta(t-t_0)} & 0 \\ \gamma_1(t) e^{-\alpha(t-t_0)} & \gamma_2(t) e^{\beta(t-t_0)} & 0 \end{bmatrix}, \; X(t,t_0)^- = \begin{bmatrix} e^{\alpha(t-t_0)} & 0 & 0 \\ 0 & e^{-\beta(t-t_0)} & 0 \\ \gamma_1(t_0) e^{\alpha(t-t_0)} & \gamma_2(t_0) e^{-\beta(t-t_0)} & 0 \end{bmatrix}.
$$

The projector

$$
P_{dich} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1(t_0) & 0 & 0 \end{bmatrix}, \quad \Pi_{can}(t_0) - P_{dich} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma_2(t_0) & 0 \end{bmatrix},
$$

meets the condition of Definition 6.26, and it results that

$$X(t,t_0)P_{dich}X(t,t_0)^- = \mathrm{e}^{-\alpha(t-t_0)}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_1(t) & 0 & 0 \end{bmatrix}, \quad \textit{and} \quad S^-_{can}(t) = \operatorname{span}\begin{bmatrix} 1 \\ 0 \\ \gamma_1(t) \end{bmatrix},$$

$$X(t,t_0)(\mathit{\Pi}_{can}(t_0) - P_{dich})X(t,t_0)^- = \mathrm{e}^{\beta(t-t_0)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma_2(t) & 0 \end{bmatrix}, \quad \textit{and} \quad S^+_{can}(t) = \operatorname{span}\begin{bmatrix} 0 \\ 1 \\ \gamma_2(t) \end{bmatrix},$$

If both γ_1 and γ_2 are bounded functions, then this DAE is dichotomic. If, additionally, α and β are positive, the DAE has an exponential dichotomy. We see, if the entries of the canonical projector remain bounded, then the dichotomy of the IERODE is passed over to the DAE. In contrast, if the functions γ_1, γ_2 growth unboundedly, the situation within the DAE may change. For instance, if $\alpha = 0$ and $\beta > 0$, then the fundamental solution

$$X(t,t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{e}^{\beta(t-t_0)} & 0 \\ \gamma_1(t) & \gamma_2(t)\mathrm{e}^{\beta(t-t_0)} & 0 \end{bmatrix}$$

indicates each nontrivial solution to growth unboundedly though the IERODE is dichotomic.

The last example is somewhat too simple in the sense that $DS_{can} = \operatorname{im} D = \mathbb{R}^n$ is valid, which happens only for regular index one DAEs, if A has full column rank, and D has full row rank. In general, DS_{can} is a time-varying subspace of $\operatorname{im} D$, and the IERODE at the whole does not comprise an exponential dichotomy. Here the question is, whether the IERODE shows a dichotomic behavior along its (time-varying) invariant subspace DS_{can}. We do not go in more details in this direction.

6.3.3 Inhomogeneous DAEs with admissible excitations

Turn to inhomogeneous DAEs, first supposing the excitation to be such that a solution exists. Before long, in the next part, we characterize the classes of admissible functions in detail.

Definition 6.28 *The function $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ is named an admissible excitation for the DAE (44), if the DAE is solvable for this q, i.e., if a solution $x_q \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ exists such that $A(Dx_q)' + Bx_q = q$.*

Proposition 6.29 *Let the DAE (44) be regular with tractability index μ, and let a fine decoupling be given.*

(1) *Then, $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ is an admissible excitation, if and only if the IVP*

$$A(Dx)' + Bx = q, \quad x(t_0) \in N_{can}(t_0), \tag{93}$$

admits a unique solution.

(2) *Each $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$, which for $\mu \geq 2$ fulfills the condition $q = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} q$, is an admissible excitation.*

Proof: (1) Let q be admissible and x_q the associated solution. Then the function $\tilde{x}(t) := x_q(t) - X(t,t_0)x_q(t_0),\ t \in \mathcal{I}$, satisfies the IVP (93). The uniqueness results from Theorem 6.19 (2).
The reverse is trivial.
(2) From the condition $q = G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1} q$ it follows that

$$\begin{aligned}\mathcal{L}_i q &= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} q \\ &= \Pi_{i-1} Q_i P_{i+1} \cdots P_{\mu-1} P_1 \cdots P_{\mu-1} G_\mu^{-1} q = 0, \quad i = 1, \dots \mu - 2, \\ \mathcal{L}_{\mu-1} q &= \Pi_{\mu-2} Q_{\mu-1} G_\mu^{-1} q = \Pi_{\mu-2} Q_{\mu-1} P_1 \cdots P_{\mu-1} G_\mu^{-1} q = 0.\end{aligned}$$

In consequence, the subsystem (77)-(79) yields successively $v_{\mu-1}, \dots, v_1 = 0$. The IERODE (75) is solvable for each arbitrary continuous excitation. Denote by u_* an arbitrary solution corresponding to q. Then, the function

$$v_0 = -\mathcal{H}_0 D^- u_* + \mathcal{L}_0 q = -\mathcal{H}_0 D^- u_* + Q_0 G_\mu^{-1} q$$

results from equation (76), and

$$x := D^- u_* + v_0 = \Pi_{can} D^- u_* + Q_0 G_\mu^{-1} q$$

is a solution of the DAE (44) corresponding to this excitation q. □

For a regular index one DAE, all continuous functions q are admissible. For regular higher index DAEs, the additional projector function $G_\mu P_1 \cdots P_{\mu-1} G_\mu^{-1}$ cuts away the "dangerous" parts of a function, and ensures that only the zero function is differentiated within the subsystem (76)-(79). For higher index DAEs, general admissible excitations have certain smoother components. We turn back to this problem before long.

Example 6.30 *Consider the DAE*

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\left(\begin{bmatrix}1 & \alpha & 0\\0 & 1 & 0\end{bmatrix} x\right)' + \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix} x = q.$$

Here, α is a continuous scalar function. Set and derive

$$D^- = \begin{bmatrix}1 & -\alpha\\0 & 1\\0 & 0\end{bmatrix},\ G_0 = \begin{bmatrix}1 & \alpha & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix},\ Q_0 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix},\ G_1 = \begin{bmatrix}1 & \alpha & 0\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix},$$

and further

$$Q_1 = \begin{bmatrix}0 & -\alpha & 0\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix},\ Q_1 Q_0 = 0,\ D\Pi_1 D^- = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\ G_2 = \begin{bmatrix}1 & \alpha & 0\\0 & 1 & -1\\0 & 1 & 0\end{bmatrix}$$

The projector functions Q_0, Q_1 are admissible, G_2 is nonsingular, and hence the DAE is regular with tractability index two. The given property $\ker Q_1 = S_1 = \{z \in \mathbb{R}^3 : z_2 = 0\}$ *indicates that Q_0, Q_1 already provide a fine decoupling. Compute additionally*

$$\Pi_{can} = \Pi_1 = \begin{bmatrix}1 & \alpha & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},\ G_2^{-1} = \begin{bmatrix}1 & 0 & -\alpha\\0 & 0 & 1\\0 & -1 & 1\end{bmatrix},\ G_2 P_1 G_2^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

A closer look to the detailed equations makes clear, each admissible excitation q must have a continuously differentiable component q_3. By condition $q = G_2 P_1 G_2^{-1} q$, the third component of q is put to be zero.

Theorem 6.31 *Let the DAE (44) be regular with sufficiently smooth coefficients so that a fine decoupling exists. Let $q \in \mathcal{C}(\mathcal{I}, \mathbb{R}^m)$ be an admissible excitation, and let the matrix $C \in L(\mathbb{R}^m, \mathbb{R}^s)$ have the nullspace $\ker C = N_{can}(t_0)$.*

(1) *Then, for each $x^0 \in \mathbb{R}^m$, the IVP*

$$A(Dx)' + Bx = q, \quad C(x(t_0) - x^0) = 0, \tag{94}$$

admits exactly one solution.

(2) *The solution of the IVP (94) can be expressed as*

$$x(t, t_0, x^0) = X(t, t_0)x^0 + x_q(t),$$

whereby $x_q \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ is the unique solution of the IVP

$$A(Dx)' + Bx = q, \quad Cx(t_0) = 0, \tag{95}$$

Proof: (1) It holds that $C = C\Pi_{can}(t_0)$. Since q is admissible, by Proposition 6.29(1), the solution x_q exists and is unique. Then the function $x_* := X(., t_0)x^0 + x_q$ belongs to the function space $\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ and satisfies the DAE. Further, x_* meets the initial condition

$$C(x_*(t_0) - x^0) = C\Pi_{can}(t_0)(x_*(t_0) - x^0) = C\Pi_{can}(t_0)(\Pi_{can}(t_0)x^0 + x_q(t_0) - x^0) = 0,$$

and hence, x_* satisfies the IVP (94). By Theorem 6.19, x_* is the only IVP solution. This proves at the same time (2). □

We take a further look to the structure of the DAE solutions x_q and $x(., t_0, x^0)$. To the given admissible excitation q, we denote

$$v := v_1 + \cdots + v_{\mu-1} + \mathcal{L}_0 q - \sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l} v_l, \tag{96}$$

whereby $v_1, \ldots, v_{\mu-1} \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$ are determined by equations (77)-(79) in dependence of q. All needed derivatives exist due to the admissibility of q. If q vanishes identically, so does v. By construction, $v(t) \in N_{can}(t),\ t \in \mathcal{I}$, and $Dv = Dv_1 + \cdots + Dv_{\mu-1}$, thus $v \in \mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^m)$. The function v is fully determined by q and the coefficients of the subsystem (76)-(79). It does not depend neither of the initial condition nor of the IERODE solution.
Introduce further the continuously differentiable function u_q as

$$\begin{aligned} u_q(t) :=& \int_{t_0}^{t} U(t, t_0)U(s, t_0)^{-1} D(s)\Pi_{can}(s)G_\mu^{-1}(s)q(s)ds \\ =& U(t, t_0)\int_{t_0}^{t} X(s, t_0)^{-} G_\mu^{-1}(s)q(s)ds, \quad t \in \mathcal{I}, \end{aligned}$$

that is, as the solution of the inhomogeneous IERODE completed by the homogeneous initial condition $u(t_0) = 0$. Now the solution x_q and, in particular, its value at t_0 can be expressed as

$$\begin{aligned} x_q(t) &= D(t)^- u_q(t) - \mathcal{H}_0(t) D(t)^- u_q(t) + v(t) = \mathit{\Pi}_{can}(t) D(t)^- u_q(t) + v(t), \\ x_q(t_0) &= v(t_0) \in N_{can}(t_0). \end{aligned}$$

The solution of the IVP (94) and its value at t_0 can be written in the form

$$x(t, t_0, x^0) = X(t, t_0) x^0 + \mathit{\Pi}_{can}(t) D(t)^- u_q(t) + v(t), \tag{97}$$

$$x(t_0, t_0, x^0) = \mathit{\Pi}_{can}(t_0) x^0 + v(t_0), \tag{98}$$

but also as

$$\begin{aligned} x(t, t_0, x^0) &= \mathit{\Pi}_{can}(t) D(t)^- U(t, t_0) D(t_0) \mathit{\Pi}_{can}(t_0) x^0 + \mathit{\Pi}_{can}(t) D(t)^- u_q(t) + v(t) \\ &= \mathit{\Pi}_{can}(t) D(t)^- \underbrace{\{U(t, t_0) D(t_0) \mathit{\Pi}_{can}(t_0) x^0 + u_q(t)\}}_{u(t, t_0, D(t_0) \mathit{\Pi}_{can}(t_0) x^0)} + v(t). \end{aligned}$$

The last representation

$$x(t, t_0, x^0) \quad = \quad \underbrace{\mathit{\Pi}_{can}(t) D(t)^-}_{\substack{\Uparrow \\ \text{wrapping}}} \quad \underbrace{u(t, t_0, D(t_0) \mathit{\Pi}_{can}(t_0) x^0)}_{\substack{\Uparrow \\ \text{inherent flow}}} \quad + \quad \underbrace{v(t)}_{\substack{\Uparrow \\ \text{perturbation}}}$$

unveils the general solution structure of regular linear DAEs to be the perturbed and wrapped flow of the IERODE along the invariant subspace DS_{can}. If the wrapping is thin (bounded) and the perturbation disappears, then the situation is close to regular ODEs. However, it may well happen that wrapping and perturbation dominate (cf. Example 6.27). In extreme cases, it holds that $S_{can} = \{0\}$, thus the inherent flow vanishes, and the perturbation term only remains (cf. Example 2.3).

From Theorem 6.31, and the representation (97), it follows that, for each given admissible excitation, the set

$$\mathcal{M}_{can,q}(t) := \{z + v(t) : z \in S_{can}(t)\}, \quad t \in \mathcal{I}, \tag{99}$$

is occupied with solution values at time t, and all solution values at time t belong to this set. In particular, for $x_0 \in \mathcal{M}_{can,q}(t_0)$ it results that $x_0 = z_0 + v(t_0)$, $z_0 \in S_{can}(t_0)$, further $\mathit{\Pi}_{can}(t_0) x_0 = z_0$ and

$$x(t_0, t_0, x_0) = \mathit{\Pi}_{can}(t_0) x_0 + v(t_0) = z_0 + v(t_0) = x_0.$$

By construction, the inclusions

$$\begin{aligned} S_{can}(t) &\subseteq S_0(t) = \{z \in \mathbb{R}^m : B(t) z \in \operatorname{im} A(t)\} = \ker \mathcal{W}_0(t) B(t), \\ \mathcal{M}_{can,q}(t) &\subseteq \mathcal{M}_0(t) = \{x \in \mathbb{R}^m : B(t) x - q(t) \in \operatorname{im} A(t)\} \end{aligned}$$

are valid, whereby $\mathcal{W}_0(t)$ is again a projector along $\operatorname{im} A(t) = \operatorname{im} G_0(t)$. Recall that $S_{can}(t)$ and $S_0(t)$ have the dimensions $d = m - \sum_{j=0}^{\mu-1}(m-r_j) = r_0 - \sum_{j=1}^{\mu-1}(m-r_j)$ and r_0, respectively. Representing the obvious constraint set as

$$\begin{aligned}\mathcal{M}_0(t) &= \{x \in \mathbb{R}^m : \mathcal{W}_0(t)B(t)x = \mathcal{W}_0(t)q(t)\}\\ &= \{z + (\mathcal{W}_0(t)B(t))^- \mathcal{W}_0(t)q(t) : z \in S_0(t)\}\end{aligned}$$

we know that $\mathcal{M}_0(t)$, as an affine space, inherits its dimension from $S_0(t)$, while $\mathcal{M}_{can,q}(t)$ has the same dimension d as $S_{can}(t)$.
Since $d = r_0$ if $\mu = 1$, and $d < r_0$ if $\mu > 1$, $\mathcal{M}_{can,q}(t)$ coincides with $\mathcal{M}_0(t)$ for index-1 DAEs, however, for higher index DAEs, $\mathcal{M}_{can,q}(t)$ is merely a proper subset of $\mathcal{M}_0(t)$. $\mathcal{M}_{can,q}(t)$ is the set of consistent values at time t. The knowledge of this set gives rise for an adequate modification of the stability notions given in Definition 6.21 for homogeneous DAEs. As pointed out in [Bal04] for lower index cases, in general, $\mathcal{M}_{can,q}$ is a time-varying affine linear subspace of dimension d.

Definition 6.32 *Let the regular DAE (44) be given on the infinite interval $\mathcal{I} = [0, \infty)$, and let the coefficients be smooth enough for fine decouplings. Let the excitation q be admissible. The DAE is said to be*

(1) *<u>stable</u>, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$ a value $\delta(\varepsilon, t_0) > 0$ exists, such that $x_0, \bar{x}_0 \in \mathcal{M}_{can,q}(t_0)$, $|x_0 - \bar{x}_0| < \delta(\varepsilon, t_0)$ imply the existence of solutions $x(., t_0, x_0)$, $x(., t_0, \bar{x}_0) \in C^1_D(\mathcal{I}, \mathbb{R}^m)$ as well as $|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| < \varepsilon$, $t_0 \le t$,*

(2) *<u>uniformly stable</u>, if $\delta(\varepsilon, t_0)$ in (1) is independent of t_0,*

(3) *<u>asymptotically stable</u>, if (1) holds true, and*

$$|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| \xrightarrow[t\to\infty]{} 0 \quad \textit{for all} \quad x_0, \bar{x}_0 \in \mathcal{M}_{can,q}(t_0),\ t_0 \in \mathcal{I},$$

(4) *<u>uniformly asymptotically stable</u>, if the limit in (3) is uniform with respect to t_0.*

Remark 6.33 *We can dispense with the explicit use of the set $\mathcal{M}_{can,q}(t_0)$ within the stability notion by turning to appropriate IVPs (cf. Theorem 6.31). This might be more comfortable from the practical point of view.*
Let $C \in L(\mathbb{R}^m, \mathbb{R}^s)$ denote a matrix that has precisely $N_{can}(t_0)$ as nullspace, for instance $C = \Pi_{\mu-1}(t_0)$ or $C = \Pi_{can}(t_0)$.
The DAE (44) is stable, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$, there exists a value $\delta_C(\varepsilon, t_0) > 0$ such that the IVPs

$$\begin{aligned}A(Dx)' + Bx = q, &\quad C(x(t_0) - x^0) = 0,\\ A(Dx)' + Bx = q, &\quad C(x(t_0) - \bar{x}^0) = 0,\end{aligned}$$

with x^0, $\bar{x}^0 \in \mathbb{R}^m$, $|C(x^0 - \bar{x}^0)| < \delta_C(\varepsilon, t_0))$, have solutions $x(., t_0, x^0)$, $x(., t_0, \bar{x}^0) \in C^1_D(\mathcal{I}, \mathbb{R}^m)$, and it holds that $|x(., t_0, x^0) - x(., t_0, \bar{x}^0)| < \varepsilon$, for $t \ge t_0$.

This notion is equivalent with the previous one. Namely, denoting by C^- a generalized reflexive inverse of C such that $C^-C = \Pi_{can}(t_0)$, and considering the relation

$$\begin{aligned} C^-C(x^0 - \bar{x}^0) &= \Pi_{can}(t_0)x^0 - \Pi_{can}(t_0)\bar{x}^0 \\ &= \underbrace{\Pi_{can}(t_0)x^0 + v(t_0)}_{=x_0 \in \mathcal{M}_0(t_0)} - \underbrace{(\Pi_{can}(t_0)\bar{x}^0 + v(t_0))}_{=\bar{x}_0 \in \mathcal{M}_0(t_0)} = x_0 - \bar{x}_0, \end{aligned}$$

we know that the existence of $\delta(\varepsilon, t_0)$ in Definition 6.32 implies the existence of $\delta_C(\varepsilon, t_0) = |C|\delta(\varepsilon, t_0)$. Conversely, having $\delta_C(\varepsilon, t_0)$ we may put $\delta(\varepsilon, t_0) = |C^-|\delta_C(\varepsilon, t_0)$.

Making use of the linearity,

$$x(t, t_0, x_0) - x(t, t_0, \bar{x}_0) = X(t, t_0)(x_0 - \bar{x}_0) = X(t, t_0)(z_0 - \bar{z}_0) \tag{100}$$

we trace back the stability questions to the growth behavior of the fundamental solution matrices.

Theorem 6.34 *Let the DAE (44) be regular with sufficiently smooth coefficients so that fine decoupling projector functions exist. Then, for each admissible excitation q, the following assertions hold true with positive constants K_{t_0}, K and α:*

(1) *If $|X(t, t_0)| \le K_{t_0},\ t \ge t_0$, then the DAE is stable.*

(2) *If $|X(t, t_0)| \xrightarrow[t \to \infty]{} 0$, then the DAE is asymptotically stable.*

(3) *If $|X(t, t_0)X(s, t_0)^-| \le K,\ t_0 \le s \le t$, then the DAE is uniformly stable.*

(4) *If $|X(t, t_0)X(s, t_0)^-| \le K\mathrm{e}^{-\alpha(t-s)},\ t_0 \le s \le t$, then the DAE is uniformly asymptotically stable.*

Proof: With regard of (100), the proof of Proposition 6.25 applies. □

6.3.4 Characterizing admissible excitations

The fine decoupled version of a regular DAE into the IERODE (53) and the subsystem (64) allows for a precise and detailed description of admissible excitations. The IERODE is solvable for each arbitrary continuous inhomogeneity, therefore, additional smoothness requirements may occur from the subsystem (64), and for $\mu > 1$ only. We recall the version (76)-(79) of the subsystem, which is already specified for fine decouplings:

$$v_0 = -\sum_{l=1}^{\mu-1} \mathcal{N}_{0l}(Dv_l)' - \sum_{l=2}^{\mu-1} \mathcal{M}_{0l}\, v_l - \mathcal{H}_0 D^- u + \mathcal{L}_0 q, \tag{101}$$

$$v_i = -\sum_{l=i+1}^{\mu-1} \mathcal{N}_{il}(Dv_l)' - \sum_{l=i+2}^{\mu-1} \mathcal{M}_{il}\, v_l + \mathcal{L}_i q, \quad i = 1, \ldots, \mu - 3, \tag{102}$$

$$v_{\mu-2} = -\mathcal{N}_{\mu-2,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2} q, \tag{103}$$

$$v_{\mu-1} = \mathcal{L}_{\mu-1} q, \tag{104}$$

as well as the coefficients

$$
\begin{aligned}
\mathcal{N}_{01} &:= -Q_0Q_1D^-, \\
\mathcal{N}_{0j} &:= -Q_0P_1\cdots P_{j-1}Q_jD^-, && j=2,\ldots,\mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1}Q_iQ_{i+1}D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1}Q_iP_{i+1}\cdots P_{j-1}Q_jD^-, && j=i+2,\ldots,\mu-1,\ i=1,\ldots,\mu-2, \\
\mathcal{M}_{0j} &:= Q_0P_1\cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j=1,\ldots,\mu-1, \\
\mathcal{M}_{ij} &:= \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j=i+1,\ldots,\mu-1,\ i=1,\ldots,\mu-2, \\
\mathcal{L}_0 &:= Q_0P_1\cdots P_{\mu-1}G_\mu^{-1}, \\
\mathcal{L}_i &:= \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}, && i=1,\ldots,\mu-2, \\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}G_\mu^{-1}, \\
\mathcal{H}_0 &:= Q_0P_1\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}.
\end{aligned}
$$

For the detailed form of $\mathcal{K}$ and $\mathcal{M}_j$ we refer to (56) and (57), respectively. All these coefficients are continuous. This allows to introduce the following linear function space, if $\mu \geq 2$:

$$
\begin{aligned}
\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) := \{q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m) :& \\
\nu_{\mu-1} &:= \mathcal{L}_{\mu-1}q, && D\nu_{\mu-1}\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
\nu_{\mu-2} &:= -\mathcal{N}_{\mu-2,\mu-1}(D\nu_{\mu-1})' + \mathcal{L}_{\mu-2}q, && D\nu_{\mu-2}\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
\nu_i &:= -\sum_{l=i+1}^{\mu-1}\mathcal{N}_{il}(D\nu_l)' - \sum_{l=i+2}^{\mu-1}\mathcal{M}_{il}\,\nu_l + \mathcal{L}_iq, && D\nu_i\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n),\ i=1,\ldots,\mu-3\}.
\end{aligned}
\tag{105}
$$

This function space makes sense without any further smoothness assumptions concerning the coefficients. $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ contains, in particular, all functions q that satisfy the condition $q = G_\mu P_1\cdots P_{\mu-1}G_\mu^{-1}q$ (cf. Proposition 6.29), which corresponds to $\nu_1 = 0,\ldots,\nu_{\mu-1}=0$. $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ is always a proper subset of the continuous function space $\mathcal{C}(\mathcal{I},\mathbb{R}^m)$. Here are the particular cases $\mu=2$ and $\mu=3$:

$$
\begin{aligned}
\mathcal{C}^{ind\,2}(\mathcal{I},\mathbb{R}^m) :=& \{q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m) : \nu_1 := \mathcal{L}_1q,\ D\nu_1\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)\} \\
=& \{q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m) : D\Pi_0Q_1G_2^{-1}q\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^m)\} = \mathcal{C}^1_{D\Pi_0Q_1G_2^{-1}}(\mathcal{I},\mathbb{R}^m),
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
\mathcal{C}^{ind\,3}(\mathcal{I},\mathbb{R}^m) :=& \{q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m) : \nu_2 := \mathcal{L}_2q,\ D\nu_2\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
&\qquad \nu_1 := -\mathcal{N}_{12}(D\nu_2)' + \mathcal{L}_1q,\ D\nu_1\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)\} \\
=& \{q\in\mathcal{C}(\mathcal{I},\mathbb{R}^m) : \nu_2 := \Pi_1Q_2G_3^{-1}q,\ D\nu_2\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n), \\
&\qquad \nu_1 := \Pi_0Q_1Q_2D^-(D\nu_2)' + \Pi_0Q_1P_2G_3^{-1}q,\ D\nu_1\in\mathcal{C}^1(\mathcal{I},\mathbb{R}^n)\}.
\end{aligned}
\tag{107}
$$

If the interval $\mathcal{I}$ is compact, we may equip the space $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ with its natural norm

$$
\|q\|_{ind\,\mu} := \|q\|_\infty + \|(D\nu_{\mu-1})'\|_\infty + \cdots + \|(D\nu_1)'\|_\infty,
$$

which means for the special cases $\mu=2$ and $\mu=3$:

$$
\|q\|_{ind\,2} := \|q\|_\infty + \|(D\nu_1)'\|_\infty = \|q\|_\infty + \|(D\Pi_0Q_1G_2^{-1}q)'\|_\infty, \tag{108}
$$

$$
\begin{aligned}
\|q\|_{ind\,3} :=& \|q\|_\infty + \|(D\nu_2)'\|_\infty + \|(D\nu_1)'\|_\infty \\
=& \|q\|_\infty + \|(D\Pi_1Q_2G_3^{-1}q)'\|_\infty \\
&+ \|(D\Pi_0Q_1Q_2D^-(D\Pi_1Q_2G_3^{-1}q)' + D\Pi_0Q_1P_2G_3^{-1}q)'\|_\infty.
\end{aligned}
\tag{109}
$$

We introduce now the linear operator $L : \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m) \to \mathcal{C}(\mathcal{I},\mathbb{R}^m)$,

$$Lx := A(Dx)' + Bx, \quad x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m), \tag{110}$$

so that the DAE (44) is represented by the equation $Lx = q$, and the excitation q is admissible, exactly if it belongs to the range $\operatorname{im} L$ of the operator L.

Proposition 6.35 *If the DAE (44) is regular with tractability index $\mu \in \mathbb{N}$, and its coefficients are smooth enough for a fine decoupling, then the linear operator L has the range*

$$\begin{aligned} &\operatorname{im} L = \mathcal{C}(\mathcal{I},\mathbb{R}^m), && \text{if} \quad \mu = 1,\\ &\operatorname{im} L = \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m), && \text{if} \quad \mu \geq 2. \end{aligned}$$

Proof: The index one case is already known from Proposition 6.29 and the definition of L. Turn to the case $\mu \geq 2$. By means of the decoupled version, to each excitation $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, we find a solution $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of the DAE, so that the inclusion $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) \subseteq \operatorname{im} L$ follows. Namely, owing to the properties of q (cf. (105), there is a solution $v_{\mu-1} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of the equation (104), then a solution $v_{\mu-2} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of (103), and solutions $v_i \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of (102), successively for $i = \mu-3,\ldots,1$. Furthermore, compute a solution u of the IERODE, and v_0 from the equation (101). Finally put $x := D^-u + v_0 + \cdots + v_{\mu-1}$.
To show the reverse inclusion $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m) \supseteq \operatorname{im} L$ we fix an arbitrary $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ and investigate the resulting $q := A(Dx)' + Bx$. We apply again the decoupling. Denote $v_0 := Q_0 x$, and $v_i := \Pi_{i-1}Q_i x$, for $i = 1,\ldots,\mu-1$. Since the projector functions $D\Pi_{i-1}Q_iD^-$, $i = 1,\ldots,\mu-1$ and the function Dx are continuously differentiable, so are the functions $Dv_i = D\Pi_{i-1}Q_iD^-Dx$, $i = 1,\ldots,\mu-1$. Now the equation (104) yields $\nu_{\mu-1} := \mathcal{L}_{\mu-1}q = v_{\mu-1} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, the equation (103) gives
$\nu_{\mu-2} := -\mathcal{N}_{\mu-2\,\mu-1}(Dv_{\mu-1})' + \mathcal{L}_{\mu-2}q = v_{\mu-2} \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$, and so on. □

At this place, the reader's attentions should be directed to the fact that the linear function space $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ does not necessarily contain all continuously differentiable functions. For instance, if D is continuous, but fails to be continuously differentiable, then there are constant functions x_{const} such that Dx_{const} fails to be continuously differentiable, and hence x_{const} does not belong to $\mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$. In contrast, if D is continuously differentiable and its nullspace is nontrivial, then the proper inclusion

$$\mathcal{C}^1(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$$

is valid. Similar aspects are to be considered if one deals with the space $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$ containing the admissible excitations. Only if the involved coefficients $\mathcal{L}_i$, $\mathcal{N}_{ij}$ and $\mathcal{M}_{ij}$ are supposed to be sufficiently smooth, the inclusion

$$\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m).$$

holds true.

Theorem 6.36 *Let the DAE (44) be regular with tractability index μ, and let the coefficients be smooth enough for the existence of a fine decoupling. Then the following assertions are true:*

(1) *The excitation q is admissible, if and only if it belongs to $\mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$.*

(2) *For each pair $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, $x^0 \in \mathbb{R}^m$, the solution $x \in \mathcal{C}^1_D(\mathcal{I},\mathbb{R}^m)$ of the IVP (94) satisfies the inequality*

$$\begin{aligned} \|x\|_\infty &\le c_1|\mathit{\Pi}_{can}x^0| + c_2\|q\|_\infty + c_3\|q\|_{ind\,\mu} \\ &\le c\,\{|\mathit{\Pi}_{can}x^0| + \|q\|_{ind\,\mu}\}, \end{aligned} \tag{111}$$

whereby the function norms can be related to each arbitrary compact subinterval of $\mathcal{I}$, which contains t_0. The constants c and c_i depend only on this subinterval.

(3) *If the DAE coefficients are so smooth that $\mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m) \subset \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, and*

$$\|q\|_{ind\,\mu} \le c_0\{\|q\|_\infty + \sum_{l=1}^{\mu-1}\|q^{(l)}\|_\infty\},\; q \in \mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m),$$

then, for each pair $q \in \mathcal{C}^{\mu-1}(\mathcal{I},\mathbb{R}^m)$, $x^0 \in \mathbb{R}^m$, it holds that

$$\|x\|_\infty \le K\{|\mathit{\Pi}_{can}x^0| + \|q\|_\infty + \sum_{l=1}^{\mu-1}\|q^{(l)}\|_\infty\}. \tag{112}$$

Proof: (1) is a consequence of Proposition 6.35, and (3) results from (2). It remains to verify (2). We apply the solution representation (97). First we consider the function v given by (96). For a given $q \in \mathcal{C}^{ind\,\mu}(\mathcal{I},\mathbb{R}^m)$, one has in detail

$$\begin{aligned} &v_{\mu-1} = \mathcal{L}_{\mu-1}q = \nu_{\mu-1}, \quad \text{thus} \quad \|v_{\mu-1}\|_\infty \le \bar{c}_{\mu-1}\|q\|_{ind\,\mu}, \\ &v_{\mu-2} = \mathcal{L}_{\mu-2}q - \mathcal{N}_{\mu-2\,\mu-1}(D\nu_{\mu-1})' = \nu_{\mu-2}, \quad \text{thus} \quad \|v_{\mu-2}\|_\infty \le \bar{c}_{\mu-2}\|q\|_{ind\,\mu}, \end{aligned}$$

and so on, such that

$$\|v_i\|_\infty \le \bar{c}_i\|q\|_{ind\,\mu}, \quad i = \mu-3,\dots,1,$$

with certain constants $\bar{c}_i$. Then, with a suitable constant $\bar{c}$, it results that

$$\|v\|_\infty \le \bar{c}\|q\|_{ind\,\mu}.$$

Now the representation (97) leads to the first part of (111) with c_1 being a bound of the fundamental solution matrix $X(t,t_0)$, $c_3 := \bar{c}$ and c_2 resulting as a bound of the term $X(t,t_0)X(s,t_0)^- G_\mu^{-1}(s)$, whereby s varies between t_0 and t. We finish the proof by letting $c := \max\{c_1, c_2 + c_3\}$. □

The inequality (112) indicates that the DAE has *perturbation index μ*.

6.4 Regular standard form DAEs

At present, most of the literature on DAEs is devoted to standard form DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{113}$$

where E and F are smooth square matrix functions, and $E(t)$ has constant rank on the given interval.

As proposed in [GM86] one can treat (113) as

$$E(t)(P(t)x(t))' + (F(t) - E(t)P'(t))x(t) = q(t), \quad t \in \mathcal{I}, \tag{114}$$

by means of such a continuously projector function P that $\ker P = \ker E$. The DAE (114) has a priori a properly stated leading term, and all results of the previous sections apply. In particular, we build the matrix function sequence beginning with

$$A := E,\ D := P,\ R = P,\ B := F - EP',\ G_0 = E,\ B_0 := B,$$

develop decouplings etc. However, now the new question arises which effects are caused by a change from one projector function P to another one. Clearly, the matrix function sequence depends on the projector function P.
Suppose P and $\tilde{P}$ to be two continuously differentiable projector functions such that

$$\ker E = \ker P = \ker \tilde{P}.$$

Besides (114) we consider

$$E(t)(\tilde{P}(t)x(t))' + (F(t) - E(t)\tilde{P}'(t))x(t) = q(t), \quad t \in \mathcal{I}. \tag{115}$$

The function spaces $\mathcal{C}^1_P(\mathcal{I}, \mathbb{R}^m)$ and $\mathcal{C}^1_{\tilde{P}}(\mathcal{I}, \mathbb{R}^m)$ to coincide. Furthermore, the DAE (115) results from the DAE (114) by a *refactorization of the leading term*. Namely, set

$$A := E,\ D := P,\ R := P,\ B := F - EP', \quad \text{and} \quad H := \tilde{P},\ H^- := \tilde{P}.$$

Then, condition (37) is satisfied with $RHH^-R = P\tilde{P}P = P = R$, and the refactorized DAE (38) coincides with (115) because of (cf. (39))

$$\begin{aligned}
&\bar{A} = AH = E\tilde{P} = E,\\
&\bar{D} = H^-D = \tilde{P}P = \tilde{P},\\
&\bar{B} = B - ARH(H^-R)'D = F - EP' - E\tilde{P}'P = F - E\tilde{P}P' - E\tilde{P}'P = F - E(\tilde{P}P)'\\
&\quad = F - E\tilde{P}'.
\end{aligned}$$

In consequence, by Theorem 5.3 on refactorizations, the subspaces $\operatorname{im} G_i$, S_i, and $N_0 + \cdots + N_i$, as well as the characteristic values r_i, are independent of the special choice of P. This justifies the following regularity notion for standard form DAEs which traces back the problem to Definition 6.2 for DAEs with properly stated leading terms.

Definition 6.37 *The standard form DAE (113) is regular with tractability index μ, if the properly stated version (114) is so for one (or, equivalently, for each) continuously differentiable projector function P , $\ker P = \ker E$.*
The characteristic values of (114) are named characteristic values of (113).
The canonical subspaces S_{can} and N_{can} of (114) are called canonical subspaces of (113).

While the canonical subspaces S_{can} and N_{can} are independent of the special choice of P, the IERODE resulting from (114) obviously depends on P:

$$u' - (P\Pi_{\mu-1})'u + P\Pi_{\mu-1}G_\mu^{-1}Bu = P\Pi_{\mu-1}G_\mu^{-1}q, \quad u \in \operatorname{im} P\Pi_{\mu-1}. \tag{116}$$

This is a natural consequence of the standard formulation.

When dealing with standard form DAEs, the choice $P_0 := P$, $D^- = P$ suggests itself to begin the matrix function sequence with. In fact, this is done in the related previous work. Then the accordingly specialized sequence is

$$\begin{aligned} G_0 &= E, & B_0 &= F - EP_0' = F - G_0\Pi_0' \\ G_{i+1} &= G_i + B_iQ_i, & B_{i+1} &= B_iP_i - G_{i+1}P_0\Pi_{i+1}'\Pi_i, \quad i \geq 0. \end{aligned} \tag{117}$$

In this context, the projector functions $Q_0, \dots, Q_\kappa$ are *regular admissible*, if

(a) the projector functions $G_0, \dots, G_\kappa$ have constant ranks,

(b) the relations $Q_iQ_j = 0$ are valid for $j = 0, \dots, i-1$, $i = 1, \dots, \kappa$,

(c) and $\Pi_0, \dots, \Pi_\kappa$ are continuously differentiable.

Then, it holds that $P\Pi_i = \Pi_i$, and the IERODE of a regular DAE (113) is

$$u' - \Pi_{\mu-1}'u + \Pi_{\mu-1}G_\mu^{-1}Bu = \Pi_{\mu-1}G_\mu^{-1}q, \quad u \in \operatorname{im}\Pi_{\mu-1}. \tag{118}$$

In previous papers devoted to regular DAEs exclusively, some higher smoothness is supposed to Q_i, and these projector functions are simply called admissible, without the addendum *regular*. A detailed description of the decoupling supported by the specialized matrix function (117) can be found in [Ria08].

Remark 6.38 *In earlier papers (e.g. [Mär89a], [Mär89b], [Han90], [Mär92]) the matrix function sequence*

$$G_{i+1} = G_i + B_iQ_i, \quad B_{i+1} = B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i, \quad i \geq 0, \tag{119}$$

is used, which is slightly different from (117). While [Mär89a], [Mär89b] provide solvability results and decouplings for regular index two and index three DAEs, [Han90] deserves well of proving the invariance of the tractability index $\mu \in \mathbb{N}$ with respect to transformations (see also [Mär92], but notice that, unfortunately, there is a misleading misprint in the sequence on page 158). In these earlier papers the famous role of the sum spaces $N_0 + \cdots + N_i$ was not yet discovered, so that the reasoning is less transparent and needs patient readers.
In [Mär02, Remark 2.6] it is thought that the sequence (117) coincides with the sequence (119), however this is not fully correct. Because of

$$\begin{aligned} B_{i+1} &= B_iP_i - G_{i+1}P_0\Pi_{i+1}'\Pi_i = B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i + G_{i+1}Q_0\underbrace{\Pi_{i+1}'}_{(P_0\Pi_{i+1})'}\Pi_i \\ &= B_iP_i - G_{i+1}\Pi_{i+1}'\Pi_i + G_{i+1}Q_0P_0'\Pi_{i+1}, \end{aligned}$$

both matrix function sequences coincide in fact, if $Q_0P_0' = 0$. One can always arrange that $Q_0P_0' = 0$ is locally valid. Namely, for each fixed $t_* \in \mathcal{I}$, we find a neighborhood $\mathcal{N}_{t_*}$ such that $\ker E(t) \oplus \ker E(t_*)^\perp = \mathbb{R}^m$ holds true for all $t \in \mathcal{N}_{t_*}$. The projector function Q_0 onto $\ker E(t)$ along $\ker E(t_*)^\perp$ has the wanted property

$$Q_0P_0' = Q_0(P_0(t_*)P_0)' = Q_0P_0(t_*)P_0' = 0.$$

Owing to the independence of the choice of the projector function $P_0 = P$, the regularity notions for (113), defined by means of (117) or by (119), are actually consistent, and the sum subspaces, the canonical subspaces, and the characteristic values are precisely the same.
Several papers on lower index DAEs use subspace properties rather than rank conditions for the index definition. For instance, in [Mär95], an index-two tractable DAE is characterized by a constant-dimensional nontrivial nullspace N_1, together with the transversality condition $N_1 \oplus S_1 = \mathbb{R}^m$. Owing to Lemma A.8, this is equivalent to the condition for G_1 to have constant rank lower than m, and the requirement for G_2 to remain nonsingular.

Theorem 6.39 *Let the DAE (113) be regular with tractability index μ, and let the coefficients be sufficiently smooth for the existence of a fine decoupling.*
Let the matrix $C \in L(\mathbb{R}^m, \mathbb{R}^s)$ be such that $\ker C = N_{can}(t_0)$.

(1) *Then, the IVP*

$$Ex' + Fx = 0, \quad Cx(t_0) = 0,$$

has the zero solution only.

(2) *For each admissible excitation q, and each $x^0 \in \mathbb{R}^m$, the IVP*

$$Ex' + Fx = q, \quad C(x(t_0) - x^0) = 0,$$

has exactly one solution in $\mathcal{C}^1_P(\mathcal{I}.\mathbb{R}^m)$.

(3) *For each given admissible excitation q, the set of consistent initial values at time t_0 is*

$$\mathcal{M}_{can,q}(t_0) = \{z + v(t_0) : z \in S_{can}(t_0)\},$$

whereby v is constructed as in (96) by means of fine decoupling projector functions.

(4) *If the coefficients of the DAE are sufficiently smooth, then each $q \in \mathcal{C}^{\mu-1}(\mathcal{I}, \mathbb{R}^m)$ is admissible. If the interval $\mathcal{I}$ is compact, then for the IVP solution from (2), the inequality*

$$\|x\| \leq K\{|\Pi_{can}(t_0)x^0| + \|q\|_\infty + \sum_{l=1}^{\mu-1} \|q^{(l)}\|_\infty\} \tag{120}$$

is valid with a constant K independent of q and x^0.

Proof: (1) and (2) are consequences of Theorem 6.19(2) and Theorem 6.31(1), respectively. Assertion (4) follows from Theorem 6.36(3). Assertion (3) results from the representations (96) and (99), with $D = D^- = P$. □
The inequality (120) indicates that the DAE has *perturbation index μ*.

6.5 The T-canonical form

Definition 6.40 *The structured continuous coefficient DAE with properly stated leading term*

$$
\left[\begin{array}{c|cccc}
I_d & & & & \\
\hline
 & 0 & \tilde{\mathcal{N}}_{0,1} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\
 & & \ddots & \ddots & \vdots \\
 & & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\
 & & & & 0
\end{array}\right]
\Bigg(
\left[\begin{array}{c|cccc}
I_d & & & & \\
\hline
 & 0 & & & \\
 & & I_{m-r_1} & & \\
 & & & \ddots & \\
 & & & & \ddots \\
 & & & & I_{m-r_{\mu-1}}
\end{array}\right]
\tilde{x}\Bigg)'
+
\left[\begin{array}{c|cccc}
\tilde{\mathcal{W}} & & & & \\
\hline
\tilde{\mathcal{H}}_0 & I_{m-r_0} & & & \\
\vdots & & \ddots & & \\
\vdots & & & \ddots & \\
\tilde{\mathcal{H}}_{\mu-1} & & & & I_{m-r_{\mu-1}}
\end{array}\right]
\tilde{x} = \tilde{q}, \tag{121}
$$

$m = d + \sum_{j=0}^{\mu-1}(m - r_j)$*, as well as its counterpart in standard form*

$$
\begin{bmatrix} I_d & 0 \\ 0 & \tilde{\mathcal{N}} \end{bmatrix} \tilde{x}' + \begin{bmatrix} \tilde{\mathcal{W}} & 0 \\ \tilde{\mathcal{H}} & I_{m-d} \end{bmatrix} \tilde{x} = \tilde{q}, \tag{122}
$$

with

$$
\tilde{\mathcal{N}} = \begin{bmatrix} 0 & \tilde{\mathcal{N}}_{0,1} & \cdots & \tilde{\mathcal{N}}_{0,\mu-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \tilde{\mathcal{N}}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix},
$$

are said to be in *T-canonical form* *(T indicates tractability), if the entries* $\tilde{\mathcal{N}}_{0,1}, \dots, \tilde{\mathcal{N}}_{\mu-2,\mu-1}$ *are full column rank matrix functions, that is* $\operatorname{rank} \mathcal{N}_{i-1,i} = m - r_i$*, for* $i = 1, \dots, \mu - 1$.

The subscript μ indicates the tractability index μ, and at the same time the uniform nilpotency index of the block upper triangular matrix function $\tilde{\mathcal{N}}$. $\tilde{\mathcal{N}}^{\mu}$ vanishes identically, and $\tilde{\mathcal{N}}^{\mu-1}$ has the only nontrivial entry $\tilde{\mathcal{N}}_{0,1}\tilde{\mathcal{N}}_{1,2} \cdots \tilde{\mathcal{N}}_{\mu-2,\mu-1}$ of rank $m - r_{\mu-1}$ in the right upper corner. If the coefficients $\tilde{\mathcal{H}}_0, \dots, \tilde{\mathcal{H}}_{\mu-1}$ vanish, the T-canonical form (122) looks precisely as the Weierstraß-Kronecker canonical form for constant matrix pencils does.
Generalizing [LMT11b, Proposition 5.6], we show that a DAE (44) is regular with tractability index μ if and only if it can be brought into T-canonical form by a regular multiplication, a regular transformations of the unknown function, and a refactorization of the leading term as described in Section 5. This justifies the attribute *canonical*. The structural sizes $r_0, \dots, r_{\mu-1}$ coincide with the characteristic values from the tractability index framework.

Theorem 6.41 (1) *The DAE (44) is regular with tractability index* μ *and characteristic values* $r_0 \le \cdots \le r_{\mu-1} < r_\mu = m$*, if and only if there are pointwise regular* *matrix functions $L, K \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m))$, and a constant rank refactorization matrix function $H \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^s, \mathbb{R}^n))$, $RHH^-R = R$, such that premultiplication by L, the transformation $x = K\tilde{x}$, and the refactorization of the leading term by H yield a DAE in T-canonical form, whereby the entry $\tilde{\mathcal{N}}_{i-1,i}$ has size $(m - r_{i-1}) \times (m - r_i)$ and*

$$\operatorname{rank} \tilde{\mathcal{N}}_{i-1,i} = m - r_i, \quad \textit{for} \quad i = 1, \ldots, \mu - 1.$$

(2) *If the DAE (44) is regular with tractability index μ, and its coefficients are smooth enough for the existence of completely decoupling projector functions, then the DAE is equivalent to a T-canonical form with zero coupling coefficients $\tilde{\mathcal{H}}_0, \ldots, \tilde{\mathcal{H}}_{\mu-1}$.*

Proof: (1) If the DAE has T-canonical form, one can construct a matrix function sequence and admissible projector functions in the same way as described in [LMT11b, Section 4] for constant matrix pencils, and this shows regularity and confirms the characteristic values.
The reverse implication is more difficult. Let the DAE (44) be regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$. Let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions. As explained in Subsection 6.1, the DAE decomposes into equation (51) being a pre-version of the IERODE and subsystem (65), together

$$\underbrace{\begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \mathcal{N} \end{bmatrix}}_{\mathfrak{A}} (\underbrace{\begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \mathcal{D} \end{bmatrix}}_{\mathfrak{D}} \begin{bmatrix} u \\ v \end{bmatrix})' + \underbrace{\begin{bmatrix} \mathcal{W} & 0 \\ \mathcal{H}D^- & \mathcal{M} \end{bmatrix}}_{\mathfrak{B}} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathcal{L}_d \\ \mathcal{L} \end{bmatrix} q. \tag{123}$$

This is an inflated system in $\mathbb{R}^{m(\mu+1)}$, with $\mathcal{W} := D\Pi_{\mu-1}G_\mu^{-1}BD^-$, further coefficients given in Subsection 6.1, and the unknown functions

$$\begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} u \\ v_0 \\ \vdots \\ v_{\mu-1} \end{bmatrix} := \begin{bmatrix} D\Pi_{\mu-1} \\ Q_0 \\ \Pi_0 Q_1 \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1} \end{bmatrix} x.$$

We condense this inflated system back to $\mathbb{R}^m$ in a similar way as in [LMT11b, Proposition 5.6]. The projector functions $D\Pi_{\mu-1}D^-$ and $D\Pi_{i-1}Q_iD^-$ are continuously differentiable, and so are their ranges and nullspaces. The $\mathcal{C}^1$-subspace $\operatorname{im}(D\Pi_{\mu-1}D^-)^*$ has dimension $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$, and it is spanned by continuously differentiable basis functions, which means that there is a matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^d))$ such that

$$\operatorname{im}(D\Pi_{\mu-1}D^-)^* = \operatorname{im}\Gamma_d^*, \quad \ker\Gamma_d^* = \{0\},$$

and hence

$$\operatorname{im}\Gamma_d = \mathbb{R}^d, \quad \ker\Gamma_d = (\operatorname{im}(D\Pi_{\mu-1}D^-)^*)^\perp = \ker D\Pi_{\mu-1}D^-.$$

By Proposition C.4, there is a pointwise reflexive generalized inverse $\Gamma_d^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^d, \mathbb{R}^n))$ such that $\Gamma_d\Gamma_d^- = I_d$ and $\Gamma_d^-\Gamma_d = D\Pi_{\mu-1}D^-$. Analogously we find $\Gamma_i \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n, \mathbb{R}^{m-r_i}))$ and $\Gamma_i^- \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^{m-r_i}, \mathbb{R}^n))$ such that for $i = 1, \ldots, \mu - 1$

$$\operatorname{im}\Gamma_i = \mathbb{R}^{m-r_i}, \quad \ker\Gamma_i = \ker D\Pi_{i-1}Q_iD^-, \quad \Gamma_i\Gamma_i^- = I_{m-r_i}, \quad \Gamma_i^-\Gamma_i = D\Pi_{i-1}Q_iD^-.$$

This implies

$$\Gamma_i D = \Gamma_i D \varPi_{i-1} Q_i, \quad D^- \Gamma_i^- = \varPi_{i-1} Q_i D^- \Gamma_i^-, \quad \Gamma_i D D^- \Gamma_i^- = \Gamma_i \Gamma_i^- = I_{m-r_i}.$$

Finally we provide $\Gamma_0 \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^{m-r_0}))$ and $\Gamma_0^- \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^{m-r_0}, \mathbb{R}^m))$ such that

$$\operatorname{im} \Gamma_0 = \mathbb{R}^{m-r_0}, \quad \ker \Gamma_0 = \ker Q_0, \quad \Gamma_0 \Gamma_0^- = I_{m-r_0}, \quad \Gamma_0^- \Gamma_0 = Q_0.$$

Then we compose

$$\Gamma := \begin{bmatrix} \Gamma_d & \\ & \Gamma_{sub} \end{bmatrix}, \quad \Gamma^- := \begin{bmatrix} \Gamma_d^- & \\ & \Gamma_{sub}^- \end{bmatrix},$$

$$\Gamma_{sub} := \begin{bmatrix} \Gamma_0 & & & \\ & \Gamma_1 D & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1} D \end{bmatrix}, \quad \Gamma_{sub}^- := \begin{bmatrix} \Gamma_0^- & & & \\ & D^- \Gamma_1^- & & \\ & & \ddots & \\ & & & D^- \Gamma_{\mu-1}^- \end{bmatrix}$$

such that $\Gamma \Gamma^- = I_m, \quad \Gamma_{sub} \Gamma_{sub}^- = I_{m-d}$, and

$$\Gamma^- \Gamma = \begin{bmatrix} D \varPi_{\mu-1} D^- & & & & \\ & Q_0 & & & \\ & & \varPi_0 Q_1 & & \\ & & & \ddots & \\ & & & & \varPi_{\mu-2} Q_{\mu-1} \end{bmatrix},$$

$$\Gamma_{sub}^- \Gamma_{sub} = \begin{bmatrix} Q_0 & & & \\ & \varPi_0 Q_1 & & \\ & & \ddots & \\ & & & \varPi_{\mu-2} Q_{\mu-1} \end{bmatrix}.$$

Additionally we introduce

$$\Omega := \begin{bmatrix} 0 & & & \\ & \Gamma_1 & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1} \end{bmatrix}, \quad \Omega^- := \begin{bmatrix} 0 & & & \\ & \Gamma_1^- & & \\ & & \ddots & \\ & & & \Gamma_{\mu-1}^- \end{bmatrix},$$

such that

$$\Omega^- \Omega = \begin{bmatrix} 0 & & & \\ & D \varPi_0 Q_1 D^- & & \\ & & \ddots & \\ & & & D \varPi_{\mu-2} Q_{\mu-1} D^- \end{bmatrix}, \quad \Omega \Omega^- = \begin{bmatrix} 0 & & & \\ & I_{m-r_1} & & \\ & & \ddots & \\ & & & I_{m-r_{\mu-1}} \end{bmatrix}.$$

For the coefficients of the inflated system (123) it results that

$$\Gamma_{sub}^- \Gamma_{sub} \, \mathcal{N} = \mathcal{N} \Omega^- \Omega = \mathcal{N}, \quad \Gamma_{sub}^- \Gamma_{sub} \mathcal{M} = \mathcal{M} \Gamma_{sub}^- \Gamma_{sub}, \quad \mathcal{D} = \Omega^- \Gamma_{sub},$$

and further

$$\Gamma\mathfrak{A} = \begin{bmatrix} \Gamma_d D\Pi_{\mu-1}D^- & \\ & \Gamma_{sub}\mathcal{N} \end{bmatrix} = \begin{bmatrix} \Gamma_d & \\ & \Gamma_{sub}\mathcal{N}\Omega^-\Omega \end{bmatrix} = \begin{bmatrix} I_d & \\ & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix} \begin{bmatrix} \Gamma_d & \\ & \Omega \end{bmatrix},$$

$$\Gamma\mathfrak{B} = \begin{bmatrix} \Gamma_d\mathcal{W} & 0 \\ \Gamma_{sub}\mathcal{H}D^- & \Gamma_{sub}\mathcal{M} \end{bmatrix} = \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^-\Gamma_d & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^-\Gamma_d & \Gamma_{sub}\mathcal{M}\Gamma_{sub}^-\Gamma_{sub} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma^- & \Gamma_{sub}\mathcal{M}\Gamma_{sub} \end{bmatrix} \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix},$$

$$\mathfrak{D} = \begin{bmatrix} \Gamma_d^-\Gamma_d & 0 \\ 0 & \Omega^-\Gamma_{sub} \end{bmatrix} = \begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix} \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix}.$$

Multiplying the inflated system (123) by the condensing matrix function Γ and introducing the new variables

$$\tilde{x} := \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} := \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Gamma_{sub} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

gives

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix} \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix}}_{\bar{A}} (\underbrace{\begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix}}_{\bar{D}} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix})' + \underbrace{\begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^- & \Gamma_{sub}\mathcal{M}\Gamma_{sub}^- \end{bmatrix}}_{\bar{B}} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \Gamma \underbrace{\begin{bmatrix} \mathcal{L}_d \\ \mathcal{L} \end{bmatrix}}_{\bar{L}} q.$$

This last DAE lives in $\mathbb{R}^m$, but the border space of its leading term is $\mathbb{R}^{n(\mu+1)}$. Because of

$$\ker \bar{A} = \ker \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix} = \ker \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix}, \quad \operatorname{im} \bar{D} = \operatorname{im} \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix},$$

the refactorization of the leading term (cf. Section 5) by means of

$$H := \begin{bmatrix} \Gamma_d & 0 \\ 0 & \Omega \end{bmatrix}, \quad H^- = \begin{bmatrix} \Gamma_d^- & 0 \\ 0 & \Omega^- \end{bmatrix}$$

suggests itself. H has constant rank d, and H^- is the reflexive generalized inverse with

$$HH^- = \begin{bmatrix} I_d & 0 \\ 0 & \Omega\Omega^- \end{bmatrix}, H^-H = \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix}, \bar{R}HH^-\bar{R} = \bar{R} = \begin{bmatrix} D\Pi_{\mu-1}D^- & 0 \\ 0 & \Omega^-\Omega \end{bmatrix}.$$

This way we arrive at the DAE

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = \bar{L}q,$$

$$\tilde{A} := \begin{bmatrix} I & 0 \\ 0 & \Gamma_{sub}\mathcal{N}\Omega^- \end{bmatrix}, \tilde{D} := \begin{bmatrix} I & 0 \\ 0 & \Omega\Omega^- \end{bmatrix}, \tilde{B} := \begin{bmatrix} \Gamma_d\mathcal{W}\Gamma_d^- - \Gamma_d'\Gamma_d^- & 0 \\ \Gamma_{sub}\mathcal{H}D^-\Gamma_d^- & \tilde{B}_{22} \end{bmatrix}.$$

The entry

$$\begin{aligned} \tilde{B}_{22} :&= \Gamma_{sub}\mathcal{M}\Gamma_{sub}^- - \Gamma_{sub}\mathcal{N}\Omega^-\Omega'\Omega^- \\ &= \Gamma_{sub}\Gamma_{sub}^- + \Gamma_{sub}(\mathcal{M} - I)\Gamma_{sub} - \Gamma_{sub}\mathcal{N}\Omega^-\Omega'\Omega^- =: I + \tilde{\mathcal{M}} \end{aligned}$$

has block upper triangular form, with identity diagonal blocks. $\tilde{\mathcal{M}}$ is strictly block upper triangular, and $I+\tilde{\mathcal{M}}$ remains nonsingular. Scaling the DAE by $\operatorname{diag}(I,(I+\tilde{\mathcal{M}})^{-1})$ yields

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{\mathcal{N}} \end{bmatrix} (\begin{bmatrix} I & 0 \\ 0 & \Omega\Omega^- \end{bmatrix} \tilde{x})' + \begin{bmatrix} \tilde{\mathcal{W}} & 0 \\ \tilde{\mathcal{H}} & I \end{bmatrix} \tilde{x} = \begin{bmatrix} I & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \bar{L}q, \tag{124}$$

with coefficients

$$\tilde{\mathcal{N}} := (I+\tilde{\mathcal{M}})^{-1}\Gamma_{sub}\mathcal{N}\Omega^-, \quad \tilde{\mathcal{H}} := (I+\tilde{\mathcal{M}})^{-1}\Gamma_{sub}\mathcal{H}D^-\Gamma_d^-, \quad \tilde{\mathcal{W}} := \Gamma_d\mathcal{W}\Gamma_d^- - \Gamma_d'\Gamma_d^-.$$

The DAE (124) has T-canonical form, if the entries $\tilde{\mathcal{N}}_{i,i+1}$ have full column rank. Therefore, we take a closer look to these entries. Having in mind that $\tilde{\mathcal{M}}$ is strictly block upper triangular, we derive

$$\begin{aligned} \tilde{\mathcal{N}}_{i,i+1} &= (\Gamma_{sub}\mathcal{N}\Omega)_{i,i+1} = \Gamma_i D\mathcal{N}_{i,i+1}\Gamma_{i+1}^- = -\Gamma_i D\Pi_{i-1}Q_iQ_{i+1}D^-\Gamma_{i+1}^- \\ &= -\Gamma_i\Gamma_i^-\Gamma_i DQ_{i+1}D^-\Gamma_{i+1}^- = -\Gamma_i DQ_{i+1}D^-\Gamma_{i+1}^-. \end{aligned}$$

Then, $\tilde{\mathcal{N}}_{i,i+1}z = 0$ means $\Gamma_i D\mathcal{N}_{i,i+1}\Gamma_{i+1}^- z = 0$, thus $\mathcal{N}_{i,i+1}\Gamma_{i+1}^- z = 0$. Applying Proposition 6.6 (3) we find that $D\Pi_iQ_{i+1}D^-\Gamma_{i+1}^- z = \Gamma_{i+1}^- z \in \ker D\Pi_iQ_{i+1}D^-$, and hence $\Gamma_{i+1}^- z = 0$, therefore $z = 0$. This shows that $\tilde{\mathcal{N}}_{i,i+1}$ is injective for $i = 1,\ldots,\mu-2$. The injectivity of $\tilde{\mathcal{N}}_{0,1}$ follows analogously. We obtain a T-canonical form in fact. The resulting transformations are

$$L = \begin{bmatrix} I & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \Gamma \begin{bmatrix} \mathcal{L}_d \\ \mathcal{L} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_d D\Pi_{\mu-1} \\ \Gamma_0 Q_0 \\ \Gamma_1 D\Pi_0 Q_1 \\ \vdots \\ \Gamma_{\mu-1}D\Pi_{\mu-2}Q_{\mu-1} \end{bmatrix} G_\mu^{-1}$$

and

$$K = \Gamma \begin{bmatrix} D\Pi_{\mu-1} \\ Q_0 \\ \Pi_0 Q_1 \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1} \end{bmatrix} = \begin{bmatrix} \Gamma_d D\Pi_{\mu-1} \\ \Gamma_0 Q_0 \\ \Gamma_1 D\Pi_0 Q_1 \\ \vdots \\ \Gamma_{\mu-1}D\Pi_{\mu-2}Q_{\mu-1} \end{bmatrix}$$

Both matrix functions K and L are continuous and pointwise nonsingular. This completes the proof of (1).
The assertion (2) follows now immediately, since $\mathcal{H} = 0$ implies $\tilde{\mathcal{H}} = 0$. □

7 Critical points

Critical points attract per se much special interest and effort. In particular, to find out whether the ODE with a so-called singularity of the first kind (e.g. [KKW01])

$$x'(t) = \frac{1}{t}M(t)x(t) + q(t),$$

has bounded solutions, the standard ODE theory is of no avail, and one is in need of smarter tools using the eigenstructure of the matrix $M(0)$.

In case of DAEs, the inherent ODE might be affected by singularities. For instance, the DAEs in [KMPW10] show inherent ODEs having a singularity of the first kind. The following example is taken from [KMPW10].

Example 7.1 *The DAE*

$$\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}1 & -1\end{bmatrix} x(t)\right)' + \begin{bmatrix}2 & 0\\0 & t+2\end{bmatrix} x(t) = q(t)$$

has a properly stated leading term on $[0,1]$*. It is accompanied by the matrix functions*

$$G_0(t) = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}, \quad Q_0(t) = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \quad G_1(t) = \begin{bmatrix}2 & 0\\2+\frac{t}{2} & \frac{t}{2}\end{bmatrix},$$

such that the DAE is regular with tractability index 1 just on the interval $(0,1]$*. The inherent ODE resulting there applies to* $u(t) = x_1(t) - x_2(t)$*, and it reads*

$$u'(t) = -\frac{2}{t}(t+2)u(t) + \frac{1}{t}((t+2)q_1(t) - 2q_2(t)).$$

Observe that, in view of the closed interval $[0,1]$*, this is no longer a regular ODE but an inherent explicit singular ODE (IESODE). Given a solution* $u(\cdot)$ *of the IESODE, a DAE solution is formed by*

$$x(t) = \frac{1}{t}\begin{bmatrix}t+2\\2\end{bmatrix}u(t) + \frac{1}{t}\begin{bmatrix}-q_1(t)+q_2(t)\\-q_1(t)+q_2(t)\end{bmatrix}.$$

We refer to [KMPW10] for the specification of bounded solutions by means of boundary conditions as well as for collocation approximations.

One could presume that rank changes in G_1 would always lead to singular inherent ODEs, but the situation is much more intricate. A rank drop of the matrix function G_1 is not necessarily acompanied by a singular inherent ODE, as the next example shows.

Example 7.2 *The DAE*

$$\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}t & 1\end{bmatrix} x(t)\right)' + \begin{bmatrix}\beta(t) & 0\\0 & 1\end{bmatrix} x(t) = q(t),$$

with an arbitrary continuous real function β*, has a properly stated leading term on* $(-\infty,\infty)$*. Put*

$$G_0(t) = \begin{bmatrix}t & 1\\0 & 0\end{bmatrix}, \quad D(t)^- = \frac{1}{1+t^2}\begin{bmatrix}t\\1\end{bmatrix}, \quad Q_0(t) = \frac{1}{1+t^2}\begin{bmatrix}1 & -t\\-t & t^2\end{bmatrix},$$

and compute

$$G_1(t) = \frac{1}{1+t^2}\begin{bmatrix}\beta(t)+t+t^3 & 1+t^2-t\beta(t)\\-t & t^2\end{bmatrix}, \quad \omega_1(t) := \det G_1(t) = t(1+t^2).$$

This DAE is regular with index 1 on $(-\infty,0)$ *and* $(0,\infty)$*,* $t_* = 0$ *is a critical point, and the inherent ODE reads, with* $u(t) = tx_1(t) + x_2(t)$*,*

$$u'(t) = -\frac{\beta(t)}{t}u(t) + q_1(t) + \frac{\beta(t)}{t}q_2(t).$$

All DAE solutions have the form

$$x(t) = \frac{1}{t}\begin{bmatrix}1\\0\end{bmatrix}u(t) + \frac{1}{t}\begin{bmatrix}-q_2(t)\\tq_2(t)\end{bmatrix}.$$

Obviously, if the function β has a zero at $t_* = 0$, or if it actually vanishes identically, then there is no singularity within the inherent ODE, even though the matrix $G_1(t_*)$ becomes singular. Underscore, the determinant ω_1 does not at all depend on the coefficient β.
Turn to a special case. Set q identically zero, $\beta(t) = t^\gamma$, with an integer $\gamma \geq 0$. The inherent ODE simplifies to

$$u'(t) = -t^{\gamma-1}u(t).$$

If $\gamma = 0$, this is a singular ODE, and its solutions have the form $u(t) = \frac{1}{t}c$. All nontrivial solutions grow unboundedly, if t approaches zero. In contrast, if $\gamma \geq 1$, the ODE is regular, and it has the solutions $u(t) = e^{-\frac{1}{\gamma}t^\gamma}u(0)$ which remain bounded. However, among the resulting nontrivial DAE solutions

$$x(t) = \frac{1}{t}\begin{bmatrix}1\\0\end{bmatrix}u(t)$$

there is no bounded one, even if $\gamma \geq 1$.

As adumbrated by the above example, apart from the singularities concerning the inherent ODE, DAEs involve further sources for critical points which are unacquainted at all in explicit ODEs. In DAEs, not only the inherent ODE but also the associated subsystem (64) which constitutes the wrapping up, and which in higher index cases includes the differentiated parts, might be hit by singularities. In the previous two examples which show DAEs being almost overall index 1, a look to the solution representations supports this idea. The next example provides a first impression of a higher index case.

Example 7.3 *The DAE with properly stated leading term*

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}x(t)\right)' + \begin{bmatrix}0&0&\beta(t)\\1&1&0\\\gamma(t)&0&0\end{bmatrix}x(t) = q(t)$$

yields

$$G_0(t) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, \quad Q_0(t) = \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad G_1(t) = \begin{bmatrix}1&0&\beta(t)\\0&1&0\\0&0&0\end{bmatrix}, \quad \Pi_0(t) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

and further $\widehat{N_1}(t) = N_1(t) \cap N_0(t) = \{z \in \mathbb{R}^3 : z_1 = 0, z_2 = 0, \beta(t)z_3 = 0\}$. Supposing $\beta(t) \neq 0$, for all t, we derive

$$Q_1(t) = \begin{bmatrix}1&0&0\\0&0&0\\-\frac{1}{\beta(t)}&0&0\end{bmatrix}, \quad \Pi_0(t)Q_1(t) = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \quad G_2(t) = \begin{bmatrix}1&0&\beta(t)\\1&1&0\\\gamma(t)&0&0\end{bmatrix},$$

and $\omega_2(t) := \det G_2(t) = -\beta(t)\gamma(t)$. *The projector functions* Q_0, Q_1 *are the widely orthogonal ones. Taking a look at the following equivalent formulation of the DAE,*

$$\begin{aligned} x_1(t) &= \frac{1}{\gamma(t)} q_3(t), \\ x_2'(t) + x_2(t) &= q_2(t) - \frac{1}{\gamma(t)} q_3(t), \\ x_3(t) &= \frac{1}{\beta(t)} (q_1(t) - (\frac{1}{\gamma(t)} q_3(t))'), \end{aligned}$$

we see the correspondence of zeros of the function γ *to rank drops in* G_2*, and to a critical solution behavior.*
Observe also, if we dispense with the demand that the function β *has no zeros, and allow a zero at a certain point* t_**, then the intersection* $\widehat{N_1}(t_*)$ *is non-trivial,* $\widehat{N_1}(t_*) = N_0(t_*)$*, and the above projector function* $Q_1(t)$ *grows unboundedly, if* t *approaches* t_**. Nevertheless, since by construction* G_2 *depends just on the product* $\Pi_1 Q_2$*, we can continue forming the next matrix function* G_2 *considering the product* $\Pi_0 Q_1$ *that has a continuous extension. The zero of the function* β *also leads to a zero of* $\det G_2$.
Apart from critical points, the resulting IERODE applies to

$$u = D\Pi_1 x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$

and it reads

$$u' + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{D\Pi_1 G_2^{-1} B_1 D^-} u = \underbrace{\begin{bmatrix} 0 \\ q_2 - \frac{1}{\gamma} q_3 \end{bmatrix}}_{D\Pi_1 G_2^{-1} q}.$$

Observe the coefficient $D\Pi_1 G_2^{-1} B D^-$ *to be independent of the functions* β *and* γ*, while* $D\Pi_1 G_2^{-1}$ *does not depend on* β*. Therefore, the IERODE does not at all suffer from zeros of* β*.*
Notice that, if one restricts the interest to homogenous DAEs only, then one cannot see the singular solution behavior in this example.

Our examples clearly account for a correspondence between singular solution behavior and points at which the matrix function sequence loses one of the required properties. Roughly speaking, at all points where the matrix function sequence determining regularity can not be built, we expect a critical in some sense solution behavior. We refer to [Ria08] for a closer view onto the relevant literature. As [Ria08], we consider critical (in [Ria08] named *singular*) points to be the counterparts of regular points. Therefore, in this section, we deal with square DAEs (44) the coefficients A of which do not necessarily show constant rank.

Definition 7.4 *Let the DAE (44) be square,* $m = k$ *and let its leading term be* *almost proper in the sense that* $\operatorname{im} D$ *is a* $\mathcal{C}^1$ *subspace and there is a further* $\mathcal{C}^1$ *subspace* N_A *in* $\mathbb{R}^n$ *such that*

$$N_A(t) \subseteq \ker A(t), N_A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^n, t \in \mathcal{I},$$

$N_A(t)$ *coincides with* $\ker A(t)$ *on a dense subset of* $\mathcal{I}$.
Then, $t_* \in \mathcal{I}$ *is said to be a regular point of the DAE, if there is an open interval containing*

t_*, such that the DAE is regular on the intersection of this interval and $\mathcal{I}$. Otherwise, t_* is said to be a critical point.
Denote by $\mathcal{I}_{reg}$ the set of all $t \in \mathcal{I}$ being regular points of the DAE.

In this sense, $t_* = 0$ is the only critical point of the DAEs in Examples 7.1 and 7.2, while in Example 7.3 the set of critical points is formed by the zeros of the functions β and γ.

Any open interval, on which the DAE is regular, is called a *regularity interval*. If there are intersecting regularity intervals, then the DAE has common characteristic values on these intervals, and the union of the intervals is a regularity interval, again ([MR06], applying widely orthogonal projector functions one can simplify the proof given there). The set $\mathcal{I}_{reg} \subseteq \mathcal{I}$ is open, and it may be described as the union of disjoint open regularity intervals. By defintion, $\mathcal{I} - \mathcal{I}_{reg}$ is the set of critical points of the DAE (44).

The regularity notion (cf. Definitions 3.1, 6.2) involves several constant rank conditions. In particular, the proper leading term brings the matrix function $G_0 = AD$ with constant rank $r_0 = r$. Further, the existence of regular admissible projector functions $Q_0, \ldots, Q_{\mu-1}$ includes that, at each level $k = 1, \ldots, \mu - 1$,

(A) the matrix function G_k has constant rank r_k, and

(B) the intersection $\widehat{N_k}$ is trivial, i.e. $\widehat{N_k} = \{0\}$.

Owing to Proposition 3.2 we have $\ker \Pi_{k-1} = N_0 + \cdots + N_{k-1}$, and hence

$$\widehat{N_k} = N_k \cap (N_0 + \cdots + N_{k-1}) = \ker G_k \cap \ker \Pi_{k-1}.$$

Then, the intersection $\widehat{N_k}$ is trivial, exactly if the matrix function

$$\begin{bmatrix} G_k \\ \Pi_{k-1} \end{bmatrix} \tag{125}$$

has full column rank m. This means, condition (B) represents also a rank condition.

Supposed the coefficients A, D and B of the DAE are sufficiently smooth (at most class $\mathcal{C}^{m-1}$ will do), then, if the *algebraic rank conditions* are fulfilled, the requirements for the projector functions Π_k and $D\Pi_k D^-$ to be continuous resp. continuously differentiable, can be satisfied at one level after the other. In consequence (cf. [MR06, MR07, Ria08]), a critical point can be formally characterized as location, where the leading term fails to be properly stated, or where one of the constant rank conditions type (A) or type (B), at a level $k \geq 1$, is violated first.

Definition 7.5 *Let the DAE (44) have an almost proper leading term, and t_* be a critical point. Then, t_* is called*

(1) *a critical point of type 0, if* $\operatorname{rank} G_0(t_*) < r := \operatorname{rank} D(t_*)$,

(2) *a critical point of type A at level $k \geq 1$ (shortly, type k-A), if there are admissible projectors functions $Q_0, \ldots, Q_{k-1}$, and G_k changes its rank at t_*,*

(3) *a <u>critical point of type B at level $k \geq 1$</u> (shortly, type k-B), if there are admissible projector functions $Q_0, \ldots, Q_{k-1}$, the matrix function G_k has constant rank, but the full-rank condition for the matrix function (125) is violated at t_*.*

It is worth to be underscored that the proposed typification of critical points remains invariant with respect to transformations and refactorizations (Sections 5), and also with respect to the choice of admissible projector function (see Section 3). The DAEs in Examples 7.1, 7.2 have both the type 1-A critical point $t_* = 0$. In Example 7.3, the zeros of the function γ are type 2-A critical points of the DAE, while zeros of the function β yield type 1-B critical points. The next example shows different cases of type 0 critical points, and it reinforces once again the expectation of a critical solution behavior emerging from critical points. As before, one might be confronted with serious singularities, but in other cases, the critical behavior can be restored by more smoothness of the excitation, and then the critical points are somehow harmless.

Example 7.6 *Let the continuous scalar function α have a zero at $t_* = 0$, $\alpha(t) \neq 0$, for all $t \neq t_*$, $t \in \mathcal{I} := (-\infty, \infty)$. Then the DAE*

$$\begin{bmatrix} \alpha(t) \\ 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \right)' + \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} x(t) = q(t)$$

has a quasi-proper leading term, and t_* is a critical point of type 0. Generate

$$G_0(t) = \begin{bmatrix} 0 & \alpha(t) \\ 0 & 0 \end{bmatrix}, \quad Q_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G_1(t) = \begin{bmatrix} b_{11}(t) & \alpha(t) \\ b_{21}(t) & 0 \end{bmatrix}.$$

Case 1: Assume $b_{21}(t) = 1,\ b_{11}(t) = -1,\ b_{22}(t) = 1$.
Then G_1 inherits the rank drop from α. Apart from t_* the DAE is regular with index 1. As in Example 7.1, we are confronted with an inherent singular ODE, namely, for $u(t) = x_2(t)$,

$$u'(t) = \frac{1}{\alpha(t)} M u(t) + \frac{1}{\alpha(t)} (q_1(t) - b_{11}(t) q_2(t)),$$

with $M(t) = b_{11}(t) b_{22}(t) - b_{21}(t) = -2$. For instance, if $\alpha(t) = t$, then this IESODE is in fact an ODE with a singularity of the first kind, and all nontrivial solutions of the homogenous version grow unboundedly if t approches zero (e.g. [KKW01]).
Case 2: Assume $b_{21}(t) = 0,\ b_{11}(t) = 1,\ b_{22}(t) = 1$.
We derive

$$G_1(t) = \begin{bmatrix} 1 & \alpha(t) \\ 0 & 0 \end{bmatrix}, \quad Q_1(t) = \begin{bmatrix} 0 & -\alpha(t) \\ 0 & 1 \end{bmatrix}, \quad G_2(t) = \begin{bmatrix} 1 & \alpha(t) + b_{12}(t) \\ 0 & 1 \end{bmatrix}.$$

Obviously, $G_2(t)$ remains nonsingular. It results that $\Pi_1 = 0$, such that there is actually no inherent ODE. Even though on both subintervals $(-\infty, 0)$ and $(0, \infty)$ there are unique $\mathcal{C}_D^1$-solutions, the solution pieces can not be glued together to form a continuous solution on the entire interval $\mathcal{I}$. However, smoother excitations yield $\mathcal{C}_D^1$-solutions with regard to the entire interval. More precisely, for $q_1 \in \mathcal{C}(\mathcal{I}, \mathbb{R}), q_2 \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})$, the DAE solution belongs to $\mathcal{C}_D^1(\mathcal{I}, \mathbb{R}^2)$.

Definition 7.7 *A critical point $t_* \in \mathcal{I}$ of the DAE (44) is named <u>harmless</u>, if all DAE solutions defined on a neighborhood of t_* belong to the class $\mathcal{C}_D^1$, supposed the corresponding excitations q are sufficiently smooth.*

While the zero of the function α in the second part of Example 7.6 yields a harmless critical point, in contrast, in the first part, this zero causes a singular inherent ODE. We advert the reader to the fact that the critical points in the other examples, and in particular in Example 7.3, fail to be harmless in this sense.
Next we turn to the question how harmless critical points differ from the other ones. As it is suggested by Example 7.6, we prove the nonsingularity of the matrix function G_μ to indicate harmless critical points in general.

Let the DAE (44) have an almost proper leading term. For simplicity, let DD^* be continuously differentiable such that the widely orthogonal projector functions can be used. Assume the set of regular points $\mathcal{I}_{reg}$ to be dense in $\mathcal{I}$.
Let Q_0 be the orthogonal projector function onto $\ker D =: N_0$, which is continuous on the entire interval $\mathcal{I}$, since D has constant rank r there. Set $G_0 = AD$, $B_0 = B$, $G_1 = G_0 + BQ_0$. These functions are also continuous on $\mathcal{I}$. For all $t \in \mathcal{I}_{reg}$ it holds further that $\operatorname{rank} G_0(t) = r$. On each connected part of $\mathcal{I}_{reg}$, which is a regularity region, we construct the matrix function sequence by means of widely orthogonal projector functions up to G_μ, whereby μ denotes the lowest index such that $G_\mu(t)$ is nonsingular for all $t \in \mathcal{I}_{reg}$. In particular, $\Pi_1, \ldots, \Pi_{\mu-1}$ are defined and continuous on each part of $\mathcal{I}_{reg}$. Assume now that

$$\Pi_1, \ldots, \Pi_{\mu-1} \quad \text{have continuous extensions on } \mathcal{I}, \tag{126}$$

and we keep the same denotation for the extensions. Additionally,

$$D\Pi_1 D^-, \ldots, D\Pi_{\mu-1}D^- \quad \text{be continuously differentiable on } \mathcal{I}.$$

Then, the projector functions $\Pi_{i-1}Q_i = \Pi_{i-1} - \Pi_i$, $i = 1, \ldots, \mu - 1$, have continuous extensions, too, and the matrix function sequence (cf. (12)-(15), and Proposition 3.2)

$$\begin{aligned} B_i &= B_{i-1}\Pi_{i-1} - G_i D^-(D\Pi_i D^-)' D\Pi_{i-1}, \\ G_{i+1} &= G_i + B_i \Pi_{i-1} Q_i, \quad i = 1, \ldots, \mu - 1, \end{aligned}$$

is defined and continuous on the entire interval $\mathcal{I}$. In contrast to the regular case, where the matrix functions G_j have constant rank on the entire interval $\mathcal{I}$, now, for the time being, the projector functions Q_j are given on $\mathcal{I}_{reg}$ only, and

$$N_i(t) = \operatorname{im} Q_i(t) = \ker G_i(t), \quad \text{for all} \quad t \in \mathcal{I}_{reg}.$$

The projector function $\Pi_0 = P_0$ inherits constant rank $r = \operatorname{rank} D$ from D. On each of the regularity intervals, the rank r_0 of G_0 coincides with the rank of D, and hence we are aware of the uniform characteristic value $r_0 = r$ on all regularity intervals, that is on $\mathcal{I}_{reg}$. Owing to its continuity, the projector function Π_1 has constant rank on $\mathcal{I}$. Taking into account the relations

$$\ker \Pi_1(t) = N_0(t) \oplus N_1(t), \quad \dim N_0(t) = m - r_0, \quad \dim N_1(t) = m - r_1, \quad t \in \mathcal{I}_{reg}$$

we recognize the characteristic value $r_1 = \operatorname{rank} G_1$ to be also uniform on $\mathcal{I}_{reg}$, and so on. This way we find out that all characteristics

$$r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m \quad \text{are uniform on} \quad \mathcal{I}_{reg}.$$

In particular, the DAE has index μ on $\mathcal{I}_{reg}$.
Denote by $G_\mu(t)^{adj}$ the matrix of cofactors to $G_\mu(t)$, and introduce the determinant $\omega_\mu(t) := \det G_\mu(t)$, such that

$$\omega_\mu(t) G_\mu(t)^{-1} = G_\mu(t)^{adj}, \quad t \in \mathcal{I}_{reg}.$$

By construction, it results that $G_\mu Q_i = B_i Q_i = B_i \Pi_{i-1} Q_i$, for $i = 1, \ldots, \mu - 1$, thus

$$\omega_\mu(t) Q_i(t) = G_\mu(t)^{adj} B_i(t) \Pi_{i-1}(t) Q_i(t), \quad i = 1, \ldots, \mu - 1, \quad t \in \mathcal{I}_{reg}. \tag{127}$$

The last expression possesses a continuous extension, and hence $\omega_\mu Q_i = G_\mu^{adj} B_i \Pi_{i-1} Q_i$ is valid on $\mathcal{I}$.
Observe that a nonsingular $G_\mu(t_*)$ indicates also each of the projector functions $Q_1, \ldots, Q_{\mu-1}$ to have a continuous extension over the critical point t_*. In this case, the decoupling formulae (53),(64) keep their value for the continuous extensions, and it is evident that the critical point is a harmless one.
In contrast, if G_μ has a rank drop at the critical point t_*, then the decoupling formulae actually indicate different but singular solution phenomena. Additionally, several projector functions Q_j may suffer discontinuities, as it is the case in Example 7.3.
Next, by means of the widely orthogonal projector functions, on each regularity interval, we apply the basic decoupling (see Subsection 6.1, Theorem 6.7) of a regular DAE into the IERODE (53) and the subsystem (64). In order to safely obtain coefficients being continuous on the entire interval $\mathcal{I}$, we multiply the IERODE (53) by ω_μ, the first row of (64) by ω_μ^μ, the second by $\omega_\mu^{\mu-1}$, and so on up to the last line which we multiply by ω_μ. With regard to assumtion (126) and relation (127), the expressions $\omega_\mu G_\mu^{-1}$ and (cf. (56),(57)), $\omega_\mu \mathcal{K}$, $\omega_\mu \mathcal{M}_{l+1}$ are continuous on $\mathcal{I}$, and so are all the coefficients of the subsystem resulting from (64). Instead of the IERODE (53) we are now confronted with the equation

$$\omega_\mu u' - \omega_\mu (D\Pi_{\mu-1}D^-)' u + D\Pi_{\mu-1} G_\mu^{adj} B_\mu D^- u = D\Pi_{\mu-1} G_\mu^{adj} q, \tag{128}$$

which is rather a scalarly implicit inherent ODE or an inherent explicit singular ODE (IESODE). As it is proved for regular DAEs by Theorem 6.7, the equivalence of the DAE and the system decoupled in this way is given. We refer to [Ria08, Subsection 4.2.2] for a detailled description in a slightly different way. Here we take a look at the simplest lower index cases only.
The case $\mu = 1$ corresponds to the solution decomposition $x = D^- u + Q_0 x$, the inherent ODE

$$\omega_1 u' - \omega_1 R' u + D G_1^{adj} B_1 D^- u = D G_1^{adj} q, \tag{129}$$

and the subsystem

$$\omega_1 Q_0 x = -Q_0 G_1^{adj} B_1 D^- u + Q_0 G_1^{adj} q. \tag{130}$$

For $\mu = 2$, we apply the solution decomposition $x = D^- u + \Pi_0 Q_1 x + Q_0 x$. The inherent ODE reads

$$\omega_2 u' - \omega_2 (D\Pi_1 D^-)' u + D\Pi_1 G_2^{adj} B_1 D^- u = D\Pi_1 G_2^{adj} q, \tag{131}$$

and we have to add the subsystem

$$\begin{bmatrix} -\omega_2 Q_0 \omega_2 Q_1 D^- (D\Pi_0 Q_1 x)' \\ 0 \end{bmatrix} + \begin{bmatrix} \omega_2^2 Q_0 x \\ \omega_2 \Pi_0 Q_1 x \end{bmatrix} + \begin{bmatrix} Q_0 \omega_2 P_1 \omega_2 \mathcal{K} \Pi_1 \\ \Pi_0 Q_1 \omega_2 \mathcal{K} \Pi_1 \end{bmatrix} D^- u = \begin{bmatrix} Q_0 \omega_2 P_1 G_2^{adj} \\ \Pi_0 Q_1 G_2^{adj} \end{bmatrix} q. \tag{132}$$

A carefull inspection of our examples proves these formulae to comprise a worst case scenario. For instance, in Example 7.3, not only $D\Pi_1 G_2^{adj} B_1 D^-$ is continuous but already $D\Pi_1 G_2^{-1} B_1 D^-$ can be extended continuously. However, as in Example 7.1, the worst case can well happen.

Proposition 7.8 *Let the DAE (44) have an almost proper leading term, and DD^* be continuously differentiable. Let the set of regular points $\mathcal{I}_{reg}$ be dense in $\mathcal{I}$. If the projector functions $\Pi_1, \dots, \Pi_{\mu-1}$ associated with the widely orthogonal projector functions have continuous extensions on the entire interval $\mathcal{I}$, and $D\Pi_1 D^-, \dots, D\Pi_{\mu-1}D^-$ are continuously differentiable, then the following holds true:*

(1) *The DAE has on $\mathcal{I}_{reg}$ uniform characteristics $r_0 \leq \dots \leq r_{\mu-1} < r_\mu = m$.*

(2) *If $G_\mu(t_*)$ is nonsingular at the critical point t_*, then the widely orthogonal projector functions $Q_0, \dots, Q_{\mu-1}$ themselves have continuous extensions over t_*. If the coefficients A, D, and B are sufficiently smooth, then t_* is a harmless critical point.*

(3) *If $G_\mu(t_*)$ is nonsingular at the critical point t_*, then $G_{\mu-1}(t)$ has necessarily constant rank $r_{\mu-1}$ on a neighborhood including t_*.*

(4) *If the DAE has index 1 on $\mathcal{I}_{reg}$, then its critical points fail to be harmless.*

(5) *A critical point of type B leads necessarily to a singular G_μ, and hence it can never been harmless.*

Proof: Assertion (1) is already verified. Assertion (2) follows immediately by taking use of the decoupling. If A, D, B are smooth, then the coefficients of the subsystem (64) are also sufficiently smooth, and alow for the respective solutions.
Turn to (3). Owing to (2), $Q_{\mu-1}$ is continuous, and $\operatorname{rank} Q_{\mu-1}(t_*) = m - r_{\mu-1}$, $G_{\mu-1}(t_*)Q_{\mu-1}(t_*) = 0$ are valid, thus $\operatorname{rank} G_{\mu-1}(t_*) \leq r_{\mu-1}$. The existence of a $z \in \ker G_{\mu-1}(t_*)$, $P_{\mu-1}(t_*)z = z \neq 0$, would imply $G_{\mu-1}(t_*)z = 0$, and hence contradict the nonsingularity of $G_{\mu-1}(t_*)$.
(4) is a direct consequence of (3).
For proving Assertion (5) we remember the relations

$$\Pi_{j-1}(t)Q_j(t) = \Pi_{j-1}(t)Q_j(t)\Pi_{j-1}(t), \quad t \in \mathcal{I}_{reg}.$$

These relations keep to be valid for the continuous extensions, that is , for $t \in \mathcal{I}$. Consider a type $k - B$ critical point t_*, and a nontrivial $z \in N_k(t_*) \cap (N_0(t_*) + \dots + N_{\mu-1}(t_*))$, which means $G_k(t_*)z = 0$, $\Pi_{k-1}(t_*)z = 0$. This yields

$$G_\mu(t_*)z = G_k(t_*)z + B_k(t_*)Q_k(t_*)\Pi_{k-1}(t_*)z + \dots + B_{\mu-1}(t_*)\Pi_{\mu-2}(t_*)Q_{\mu-1}(t_*)\Pi_{k-1}(t_*)z = 0,$$

and hence, $G_\mu(t_*)$ is singular. □

8 Strangeness versus tractability

8.1 Canonical forms

Among the traditional goals of the theory of linear time-varying DAEs are appropriate generalizations of the Weierstraß-Kronecker canonical form and equivalence transformations into these canonical forms. So far, except for the T-canonical form which applies to both standard form DAEs and DAEs with properly stated leading term (cf. Subsection 6.5), reduction to canonical forms is developed for standard form DAEs (e.g. [Cam83], [BCP89], [KM94]).
While equivalence transformations for DAEs with properly stated leading term include transformations K of the unknown, scalings L and refactorizations H of the leading term (cf. Section 5), equivalence transformations for standard form DAEs combine only the transformations K of the unknowns and the scalings L.
Transforming the unknown function by $x = K\tilde{x}$ and scaling the standard form DAE (113) by L yields the equivalent DAE

$$\underbrace{LEK}_{\tilde{E}}\tilde{x}' + \underbrace{(LFK + LEK')}_{\tilde{F}}\tilde{x} = Lq.$$

Thereby the transformation matrix functions K must be continuously differentiable.

In the remaining part of this subsection we use the letters K and H also for special entries in the matrix functions describing the coefficients of the canonical forms below. No confusion will arise from this.

Definition 8.1 *The structured DAE with continuous coefficients*

$$\begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} x' + \begin{bmatrix} W & 0 \\ H & I_l \end{bmatrix} x = q, \tag{133}$$

$0 \leq l \leq m$, *is said to be in*

(1) *standard canonical form(SCF), if* $H = 0,\ K = 0$, *and* N *is strictly upper triangular,*

(2) *strong standard canonical form(SSCF), if* $H = 0,\ K = 0$, *and* N *is a constant, strictly upper triangular matrix,*

(3) *S-canonical form, if* $H = 0$, $K = [0\ K_1 \dots K_\kappa]$, *and*

$$N = \begin{bmatrix} 0 & N_{1,2} & \cdots & N_{1,\kappa} \\ & \ddots & & \vdots \\ & & \ddots & N_{\kappa-1,\kappa} \\ & & & 0 \end{bmatrix} \begin{array}{l} \}l_1 \\ \\ \}l_{\kappa-1} \\ \}l_\kappa \end{array},$$

is strictly block upper triangular with full row rank entries $N_{i,i+1},\ i = 1, \dots, \kappa-1,$

(4) *T-canonical form, if* $K = 0$ *and* N *is strictly block upper triangular with full column rank entries* $N_{i,i+1},\ i = 1, \dots, \kappa-1.$

In case of time-invariant coefficients, these four canonical forms are obviously equivalent. However, this is no longer true for time-varying coefficients.
The matrix function N is nilpotent in all four canonical forms, N has uniform nilpotency index κ in (3) and (4). N and all its powers N^k have constant rank in (2), (3) and (4). In contrast, in (1), the nilpotency index and the rank of N may vary with time. The S-canonical form is associated with DAEs with regular strangeness index $\zeta = \kappa - 1$ (cf. [KM94]), while the T-canonical form is associated with regular DAEs with tractability index $\mu = \kappa$ (cf. Subsection 6.5). The classification into SCF and SSCF goes back to [Cam83] (cf. also [BCP89]). One can treat DAEs being transformable into SCF as quasi-regular DAEs. Here we concentrate on the S-canonical form. We prove that each DAE being transformable into S-canonical form is regular with tractability index $\mu = \kappa$, and hence, each DAE with well-defined regular strangeness index ζ is a regular DAE with tractability index $\mu = \zeta + 1$. All above canonical forms are given in standard form. For the T-canonical form, a version with properly stated leading term is straightforward (cf. Definition 6.40).

The strangeness index concept applies to standard form DAEs (113) with sufficiently smooth coefficients. A reader who is not familiar with this concept finds a short introduction in the next subsection. For the moment, *we interprete DAEs with regular strangeness index as those being transformable into S-canonical form.* This is justified by an equivalence result of [KM94], which is reflected by Theorem 8.2 below.
The regular strangeness index ζ is supported by a sequence of *characteristic values* $\bar r_i, \bar a_i, \bar s_i,\ i = 0, \ldots, \zeta$, which are associated with constant rank conditions for matrix functions, and which describe the detailed size of the S-canonical form. By definition, $s_\zeta = 0$ (cf. Subsection 8.2). These characteristic values are invariant with respect to the equivalence transformations, however, they are not independent of each other.

Theorem 8.2 *Each DAE (113) with smooth coefficients, well-defined strangeness index ζ and characteristic values $\bar r_i, \bar a_i, \bar s_i,\ i = 0, \ldots, \zeta$, is equivalent to a DAE in S-canonical form with $\kappa = \zeta + 1$, $l = l_1 + \cdots + l_\kappa$, $m - l = \bar r_\zeta$, and*

$$l_1 \le \ldots \le l_\kappa, \quad l_1 = \bar s_{\kappa-2} = \bar s_{\zeta-1},\ l_2 = \bar s_{\kappa-3}, \ldots, l_{\kappa-1} = \bar s_0,\ l_\kappa = \bar s_0 + \bar a_0, .$$

Proof: This assertion comprises the regular case of [KM94, Theorem 12] which considers more general equations having also underdetermined parts (indicated by nontrivial further characteristic values $\bar u_i$). □

By the next assertion, which represents the main result of this subsection, we prove each DAE with regular strangeness index ζ to be at the same time a regular DAE with tractability index $\mu = \zeta + 1$. Therefore, the tractability index concept applies at least to the entire class of DAEs which are accessible by the strangeness index concept. Both concepts are associated with characteristic values being invariant under equivalence transformations, and, of course, we would like to know how these characteristic values are related to each other. In particular, the question arises whether the constant rank conditions supporting the strangeness index coincide with the constant rank conditions supporting the tractability index.

Theorem 8.3 (1) *Let the standard form DAE (113) have smooth coefficients, the regular strangeness index ζ and the characteristic values $\bar r_i,\ \bar a_i,\ \bar s_i,\ i = 0, \ldots, \zeta$. Then* *this DAE is regular with tractability index $\mu = \zeta + 1$ and associated characterstic values*

$$r_0 = \bar{r}_0, \quad r_j = m - \bar{s}_{j-1}, \quad j = 1, \dots, \mu.$$

(2) *Each DAE in S-canonical form with smooth coefficients can be tranformed into T-canonical form with $H = 0$.*

Proof: (1) We prove the assertion by constructing a matrix function sequence and admissible projector functions associated with the tractability index framework for the resulting S-canonical form described by Theorem 8.2.
The matrix function N within the S-canonical form has constant rank $l - l_\kappa$. Exploiting the structure of N we compose a projector function $Q_0^{[N]}$ onto $\ker N$, which is block upper triangular, too. Then we set

$$P_0 := \begin{bmatrix} I_{m-l} & KQ_0^{[N]} \\ 0 & P_0^{[N]} \end{bmatrix}, \quad \text{such that} \quad \ker P_0 = \ker \begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix}.$$

P_0 is a projector function. The DAE coefficients are supposed to be smooth enough so that P_0 is continuously differentiable. Then we can turn to the following properly stated version of the S-canonical form:

$$\begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} (P_0 x)' + \underbrace{(\begin{bmatrix} W & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} I_{m-l} & K \\ 0 & N \end{bmatrix} P_0')}_{\begin{bmatrix} W & -K'Q_0^{[N]} \\ 0 & I_l - NP_0^{[N]\prime} \end{bmatrix}} x = q, \tag{134}$$

The product $NP_0^{[N]\prime}$ is again strictly block upper triangular, and $I_l - NP_0^{[N]\prime}$ is nonsingular. Scaling the DAE by

$$\begin{bmatrix} I_{m-l} & 0 \\ 0 & (I_l - NP_0^{[N]\prime})^{-1} \end{bmatrix}$$

yields

$$\begin{bmatrix} I_{m-l} & K \\ 0 & M_0 \end{bmatrix} (P_0 x)' + \begin{bmatrix} W & -K'Q_0^{[N]} \\ & I_l \end{bmatrix} x = q, \tag{135}$$

The matrix function M_0 has the same structure as N, and $\ker M_0 = \ker N$. For the subsystem corresponding to the second line of (135)

$$M_0 (P_0^{[N]} v)' + v = q_2,$$

Proposition G.2 in Appendix D provides a matrix function sequence $G_j^{[N]}$, $j = 0, \dots, \kappa$, and admissible projector functions $Q_0^{[N]}, \dots, Q_{\kappa-1}^{[N]}$ such that this subsystem is a regular DAE with tractability index $\mu^{[N]} = \kappa$ and characteristic values

$$r_i^{[N]} = l - l_{\kappa - i}, \quad i = 0, \dots, \kappa - 1, \quad r_\kappa^{[N]} = l.$$

Now we compose a matrix function sequence and admissible projector functions for the DAE (135). We begin with $D = D^- = R = P_0$, and build successively for $i = 0, \dots, \kappa$

$$G_i = \begin{bmatrix} I_{m-l} & * \\ 0 & G_i^{[N]} \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & * \\ 0 & Q_i^{[N]} \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} I_{m-l} & * \\ 0 & \Pi_i^{[N]} \end{bmatrix}, \quad B_i = \begin{bmatrix} W & * \\ 0 & B_i^{[N]} \end{bmatrix}.$$

The coefficients are supposed to be smooth enough so that the Π_i are continuously differentiable. It results that the matrix functions G_i have constant ranks

$$r_i = m - l + r_i^{[N]} = m - l + l - l_{\kappa-i} = m - l_{\kappa-i}, \quad i = 0,\dots,\kappa-1, \quad r_\kappa = m - l + r_\kappa^{[N]} = m.$$

This confirms that the DAE is regular with tractability index $\mu = \kappa$. Applying again Theorem 8.2, we express $r_i = m - l_{\kappa-i} = \bar{s}_{i-1}$ for $i = 1,\dots,\kappa-1$, further $r_0 = m - (\bar{s}_0 + \bar{a}_0) = \bar{r}_0$, and this completes the proof of (1). (2) This is a consequence of assertion (1), and the fact that each regular DAE with tractability index μ can be transformed into T-canonical form (with $\kappa = \mu$, cf. Theorem 6.41). □

8.2 Strangeness reduction

The original strangeness index concept is a special reduction technique for standard form DAEs (113)

$$E(t)x'(t) + F(t)x(t) = q(t)$$

with sufficiently smooth coefficients on a compact interval $\mathcal{I}$. We repeat the basic reduction step from [KM94]. For more details and a comprehensive discussion of reduction techniques we refer to [KM06] and [RR02].

As mentioned before, the strangeness index is supported by several constant rank conditions. In particular, the matrix E in (113) is assumed to have constant rank $\bar{r}$. This allows to construct continuous injective matrix functions T, Z, and $\bar{T}$ such that

$$\operatorname{im} T = \ker E, \quad \operatorname{im} \bar{T} = (\ker E)^\perp, \quad \operatorname{im} Z = (\operatorname{im} E)^\perp.$$

The columns of $T, \bar{T}$, and Z are basis functions of the corresponding subspaces.
Supposing Z^*FT to have constant rank $\bar{a}$, we find a continuous injective matrix function V such that

$$\operatorname{im} V = (\operatorname{im} Z^*FT)^\perp.$$

If, additionally, $V^*Z^*F\bar{T}$ has constant rank $\bar{s}$, then one can construct pointwise nonsingular matrix functions K and L, such that the transformation $x = K\tilde{x}$ and scaling the DAE (113) by L leads to

$$\begin{bmatrix} I_{\bar{s}} & & & & \\ & I_{\bar{d}} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \bar{x}' + \begin{bmatrix} 0 & \tilde{F}_{1,2} & 0 & \tilde{F}_{1,4} & \tilde{F}_{1,5} \\ 0 & 0 & 0 & \tilde{F}_{2,4} & \tilde{F}_{2,5} \\ 0 & 0 & I_{\bar{a}} & 0 & 0 \\ I_{\bar{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \bar{x} = Lq, \tag{136}$$

with $\bar{d} := \bar{r} - \bar{s}$.
The system (136) consists of $m = \bar{s} + \bar{d} + \bar{a} + \bar{s} + \bar{u}$ equations, $\bar{u} := m - \bar{r} - \bar{a} - \bar{s}$. The construction of K and L involves three smooth factorizations of matrix functions and the solution of a classical linear IVP (see [KM06]).
The fourth equation in (136) is simply $\bar{x}_1 = (Lq)_4$, which gives rise to replace the derivative

$\bar{x}_1'$ in the first line by $(Lq)_4'$. Doing so we attain the new DAE

$$\underbrace{\begin{bmatrix} 0 & & & & \\ & I_{\bar{d}} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}}_{E_{new}} \bar{x}' + \underbrace{\begin{bmatrix} 0 & \tilde{F}_{1,2} & 0 & \tilde{F}_{1,4} & \tilde{F}_{1,5} \\ 0 & 0 & 0 & \tilde{F}_{2,4} & \tilde{F}_{2,5} \\ 0 & 0 & I_{\bar{a}} & 0 & 0 \\ I_{\bar{s}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{F_{new}} \bar{x} = Lq - \begin{bmatrix} (Lq)_4' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{137}$$

which is expected to have a lower index since the mentioned differentiation of $\bar{x}_1$ is carried out analytically.
This *reduction step* is supported by the three rank conditions

$$\operatorname{rank} E = \bar{r}, \quad \operatorname{rank} Z^* F T = \bar{a}, \quad \operatorname{rank} V^* Z^* F \bar{T} = \bar{s}. \tag{138}$$

The following proposition guarantees these constant rank conditions to be valid, if the DAE under consideration is regular in the tractability sense.

Proposition 8.4 *Let the DAE (113) be regular with tractability index μ and characteristic values $r_0 \le \ldots \le r_{\mu-1} < r_\mu$. Then the constant rank conditions (138) are valid,*

$$\bar{r} = r_0, \quad \bar{a} = r_1 - r_0, \quad \bar{s} = m - r_1,$$

so that the reduction step is feasible.

Proof: We choose symmetric projector functions $\mathcal{W}_0,\ Q_0$ and $\mathcal{W}_1$, and verify the relations

$$\operatorname{rank} Z^* B T = \operatorname{rank} \mathcal{W}_0 B Q_0 = r_1 - r_0, \quad \operatorname{rank} V^* Z^* F \bar{T} = \operatorname{rank} \mathcal{W}_1 B = m - r_1.$$

□

The reduction from $\{E, F\}$ to $\{E_{new}, F_{new}\}$ can be repeated as long as the constant rank conditions are given. This leads to an iterative reduction procedure. One starts with $\{E_0, F_0\}{:=}\{E, F\}$ and forms, for each $i \ge 0$, a new pair $\{E_{i+1}, F_{i+1}\}$ to $\{E_i, F_i\}$. This works as long as the three constant rank conditions

$$\bar{r}_i = \operatorname{rank} E_i, \quad \bar{a}_i = \operatorname{rank} Z_i^* F_i T_i, \quad \bar{s}_i = \operatorname{rank} V_i^* Z_i^* F_i \bar{T}_i, \tag{139}$$

hold true.
The *strangeness index* $\zeta \in \mathbb{N} \cup \{0\}$ is defined to be

$$\zeta := \min\{i \in \mathbb{N} \cup \{0\} : \bar{s}_i = 0\}.$$

The strangeness index is the minimal index such that the so-called strangeness disappears. ζ is named *regular strangeness index*, if there are no so-called underdetermined parts during the iteration such that $\bar{u}_i = 0$ and $\bar{r}_i + \bar{a}_i + \bar{s}_i = m$ for all $i = 0, \ldots, \zeta$.
The values $\bar{r}_i,\ \bar{a}_i,\ \bar{s}_i, \quad i \ge 0$, and several additional ones, are called *characteristic values* associated with the strangeness index concept.
If the original DAE (113) has regular strangeness index ζ, then the reduction procedure ends up with the DAE

$$\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix} \tilde{\tilde{x}}' + \begin{bmatrix} 0 & 0 \\ 0 & I_a \end{bmatrix} \tilde{\tilde{x}} = \tilde{\tilde{q}},$$

with $d = \bar{d}_\zeta,\ a = \bar{a}_\zeta$.

Remark 8.5 *Turn for a moment back to time-invariant DAEs and constant matrix pairs. If the matrix pair $\{E, F\}$ is regular with Kronecker index μ (which is the same as tractability index μ), and characteristic values $r_0 \leq \ldots \leq r_{\mu-1} < r_\mu = m$, then this pair has the regular strangeness index $\zeta = \mu - 1$. The characteristic values associated with the strangeness index can then be obtained from the $r_0, \ldots, r_\mu$ by means of the formulas*

$$\begin{aligned}\bar{r}_i &= m - \sum_{j=0}^{i}(m - r_j),\\ \bar{a}_i &= \sum_{j=0}^{i}(m - r_j) - (m - r_{i+1}),\\ \bar{s}_i &= m - r_{i+1}, \quad i = 0, \ldots, \zeta.\end{aligned}$$

The same relations apply to DAEs with time-varying coefficients, too (cf. [Lam08]).

8.3 Projector based reduction

Although linear regular higher index DAEs are well understood, they are not accessible for a direct numerical integration. Especially for this reason, different kind of index reduction have their meaning.
We formulate a reduction step for the DAE (44) with properly stated leading term, i.e.

$$A(Dx)' + Bx = q,$$

by applying the projector function $\mathcal{W}_1$ associated to the first terms of the matrix function sequence. $\mathcal{W}_1$ projects along $\operatorname{im} G_1 = \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 B Q_0$, and, because of $\operatorname{im} A \subseteq \operatorname{im} G_0 \subseteq \operatorname{im} G_1$, multiplication of the DAE by $\mathcal{W}_1$ leads to the derivative free equations

$$\mathcal{W}_1 B x = \mathcal{W}_1 q. \tag{140}$$

Emphasize these equations to be just a part of the derivative free equations, except for the case $\mathcal{W}_0 = \mathcal{W}_1$, which is given in Hessenberg systems, and in Example 8.6 below. The complete set is described by

$$\mathcal{W}_0 B x = \mathcal{W}_0 q. \tag{141}$$

We suppose the matrix function $\mathcal{W}_1$ to have constant rank $m - r_1$, which is at least ensured in regular DAEs. For regular DAEs the subspace

$$S_1 = \ker \mathcal{W}_1 B$$

is known to have dimension r_1.
Introduce a continuous reflexive generalized inverse $(\mathcal{W}_1 B)^-$, and put

$$Z_1 := I - (\mathcal{W}_1 B)^- \mathcal{W}_1 B.$$

Z_1 is a continuous projector function onto S_1. Because of $\mathcal{W}_1 B Q_0 = 0$ the following properties hold true:

$$\begin{aligned}Z_1 Q_0 &= Q_0\\ D Z_1 &= D Z_1 P_0 = D Z_1 D^- D\\ D Z_1 D^- &= D Z_1 D^- D Z_1 D^-\\ \operatorname{im} D Z_1 D^- &= \operatorname{im} D Z_1 = D S_1 = D S_0.\end{aligned}$$

DZ_1D^- is a priori a continuous projector function. Assuming the DAE coefficients to be sufficiently smooth, it becomes continuously differentiable, and we do so. In consequence, for each function $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ it follows that

$$DZ_1x = DZ_1D^-Dx \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n), \quad D(I - Z_1)x = Dx - DZ_1x \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^n),$$

which allows for writing the DAE as

$$A(DZ_1x)' + A(D(I - Z_1)x)' + Bx = q. \tag{142}$$

The equation (140) is consistent, since, for reasons of dimensions, $\operatorname{im} \mathcal{W}_1 B = \operatorname{im} \mathcal{W}_1$. It results that

$$(I - Z_1)x = (\mathcal{W}_1B)^-\mathcal{W}_1q. \tag{143}$$

This allows to remove the derivative $(D(I - Z_1)x)'$ from the DAE, and to replace it by the exact solution part derived from (140). The resulting new DAE

$$A(DZ_1x)' + Bx = q - A(D(\mathcal{W}_1B)^-\mathcal{W}_1q)'$$

has no properly stated lading term. This why we express $A(DZ_1x)' = A\{DZ_1D^-(DZ_1x)' + (DZ_1D^-)'DZ_1x\}$, and turn to the new DAE with a properly stated leading term

$$\underbrace{ADZ_1D^-}_{A_{new}}(\underbrace{DZ_1}_{D_{new}}x)' + \underbrace{(A(DZ_1D^-)'DZ_1 + B)}_{B_{new}}x = q - A(D(\mathcal{W}_1B)^-\mathcal{W}_1q)' \tag{144}$$

which has the same solutions as the original DAE (44) has, and which is expected to have a lower index.

Example 8.6 *We reconsider the DAE (7) from Example 1.1,*

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} (\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} x(t))' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B(t)} x(t) = q(t), \quad t \in \mathbb{R}.$$

A matrix function sequence and admissible projector functions for this DAE are generated in Example 2.3. This DAE is regular with tractability index three. Compute now

$$\mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_1B(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -t & 1 \end{bmatrix}.$$

Since $\mathcal{W}_1B$ is already a projector function, we can set $(\mathcal{W}_1B)^- = \mathcal{W}_1B$. This implies

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}, \quad D(t)Z_1(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix},$$

and finally the special DAE (144)

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_{new}(t)} (\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 0 \end{bmatrix}}_{D_{new}(t)} x(t))' + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{bmatrix}}_{B_{new}(t)} x(t) = \begin{bmatrix} q_1(t) \\ q_2(t) - q_3'(t) \\ q_3(t) \end{bmatrix}, \quad t \in \mathbb{R},$$

which is indeed regular with tractability index two.

For the special choice $(\mathcal{W}_1 B)^- = (\mathcal{W}_1 B)^+$, the resulting Z_1 is the orthoprojector function onto S_1. This version is the counterpart to the strangeness reduction step from Subsection 8.2.

At the first glance it seems to be somehow arbitrary to figure out just the equations (140) for reduction. However, after the explanations below it will be seen as nice option.

An analogous reduction step can be arranged by choosing the complete set of derivative free equations (141) as candidate. For regular DAEs, the subspace $\ker \mathcal{W}_0 B = S_0$ has dimension r_0, and we obtain again consistency, as well as the projector $Z_0 := I - (\mathcal{W}_0 B)^- \mathcal{W}_0 B$ onto S_0. From (141) it results that

$$(I - Z_0)x = (\mathcal{W}_0 B)^- \mathcal{W}_0 q.$$

Now we need a smoother solution x to be able to differentiate this expression. To be more transparent we assume at least D and Z_0, as well as the solution x to be continuously differentiable, and turn to the standard form

$$\underbrace{AD}_{E} x' + \underbrace{(B - AD')}_{F} x = q.$$

Here we express

$$x' = (Z_0 x)' + ((\mathcal{W}_0 B)^- \mathcal{W}_0 q)' = Z_0 x' + Z_0' x + ((\mathcal{W}_0 B)^- \mathcal{W}_0 q)',$$

such that we arrive at the new DAE

$$\underbrace{EZ_0}_{E_{new}} x' + \underbrace{(F + EZ_0')}_{F_{new}} x = q - E((\mathcal{W}_0 B)^- \mathcal{W}_0 q)'. \tag{145}$$

This kind of reduction is in essence the procedure described in [RR02]. The description in [RR02] concentrates on the coefficient pairs, and one turns to a condensed version of the pair $\{EZ_0, (I - \mathcal{W}_0)(F + EZ_0')\}$.

In the following we do not provide a precise proof of the index reduction, but explain the idea behind. Assume the DAE (44) to be regular with tractability index μ and characteristic values $r_0 \leq \ldots \leq r_{\mu-1} = r_\mu = m$, and take a furter look to the completely decoupled version consisting of the IERODE (53) and the subsystem (cf. (65))

$$\mathcal{N}(\mathcal{D}v)' + \mathcal{M}v = \mathcal{L}q. \tag{146}$$

This subsystem comprises the inherent differentiations. It reads in detail

$$\begin{bmatrix} 0 & \mathcal{N}_{0,1} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (D\Pi_0 Q_1 x)' \\ \vdots \\ (D\Pi_{\mu-2} Q_{\mu-1} x)' \end{bmatrix} \tag{147}$$

$$+ \begin{bmatrix} I & \mathcal{M}_{0,1} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} Q_0 x \\ \Pi_0 Q_1 x \\ \vdots \\ \Pi_{\mu-2} Q_{\mu-1} x \end{bmatrix} = \begin{bmatrix} \mathcal{L}_0 q \\ \mathcal{L}_1 q \\ \vdots \\ \mathcal{L}_{\mu-1} q \end{bmatrix}.$$

We see, if we replace the derivative term $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ by its exact solution part $(D\mathcal{L}_{\mu-1}q)'$ we arrive at the system

$$\mathcal{N}_{new}\begin{bmatrix} 0 \\ (D\Pi_0 Q_1 x)' \\ \vdots \\ (D\Pi_{\mu-3}Q_{\mu-2}x)' \\ 0 \end{bmatrix} + \begin{bmatrix} I & \mathcal{M}_{0,1} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix}\begin{bmatrix} Q_0 x \\ \Pi_0 Q_1 x \\ \vdots \\ \Pi_{\mu-2}Q_{\mu-1}x \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{L}_0 q - \mathcal{N}_{0,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \mathcal{L}_1 q - \mathcal{N}_{1,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \vdots \\ \mathcal{L}_{\mu-2} q - \mathcal{N}_{\mu-2,\mu-1}(\mathcal{L}_{\mu-1}q)' \\ \mathcal{L}_{\mu-1}q \end{bmatrix} \tag{148}$$

While the matrix function $\mathcal{N}$ has nilpotency index μ, the new matrix function

$$\mathcal{N}_{new} = \begin{bmatrix} 0 & \mathcal{N}_{0,1} & \cdots & \mathcal{N}_{0,\mu-2} & 0 \\ & 0 & \ddots & \vdots & 0 \\ & & \ddots & \mathcal{N}_{\mu-3,\mu-2} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

has nilpotency index $\mu - 1$ (cf. Proposition 6.6). That means, replacing the derivative $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ by the true solution term reduces the index by one. Clearly, replacing further derivatives and successively solving the subsystem for $(I - \Pi_{\mu-1})x = Q_0 x + \Pi_0 Q_1 x + \cdots + \Pi_{\mu-2}Q_{\mu-1}x$ reduces the index up to one. We keep in mind that, replacing at least the derivative $(D\Pi_{\mu-2}Q_{\mu-1}x)'$ reduces the index at least by one. However, in practice, we are not given the decoupled system. How can we otherwise make sure that this derivative is replaced?
Consider for a moment the equation

$$\mathcal{W}_{\mu-1}Bx = \mathcal{W}_{\mu-1}q \tag{149}$$

that is also a part of the derivative free equations of our DAE. Since the subspace $S_{\mu-1} = \ker \mathcal{W}_{\mu-1}$ has dimension $r_{\mu-1}$, the matrix function $\mathcal{W}_{\mu-1}B$ has constant rank $m - r_{\mu-1}$, equation (149) is consistent, we obtain with $Z_{\mu-1} := I - (\mathcal{W}_{\mu-1}B)^- \mathcal{W}_{\mu-1}B$ a continuous projector function onto $S_{\mu-1}$, and it follows that

$$(I - Z_{\mu-1})x = (\mathcal{W}_{\mu-1}B)^- \mathcal{W}_{\mu-1}q.$$

Since we use completely decoupling projector functions $Q_0, \ldots, Q_{\mu-1}$, we know that $\Pi_{\mu-2}Q_{\mu-1}$ is the projector function onto $\operatorname{im}\Pi_{\mu-2}Q_{\mu-1}$ along $S_{\mu-1}$. Therefore, with $I - Z_{\mu-1}$ and $\Pi_{\mu-2}Q_{\mu-1}$ we have two projector functions along $S_{\mu-1}$. This yields

$$I - Z_{\mu-1} = (I - Z_{\mu-1})\Pi_{\mu-2}Q_{\mu-1}, \quad \Pi_{\mu-2}Q_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1}(I - Z_{\mu-1}),$$

and therefore, by replacing $(D(I - Z_{\mu-1})x)'$ we replace at the same time $(D\Pi_{\mu-2}Q_{\mu-1}x)'$. This means, that turning from the original DAE (44) to

$$ADZ_{\mu-1}D^-(DZ_{\mu-1}x)' + (A(DZ_{\mu-1}D^-)'DZ_{\mu-1} + B)x = q - A(D(\mathcal{W}_{\mu-1}B)^- \mathcal{W}_{\mu-1}q)'$$

reduces the index by one indeed. However, the use of $Z_{\mu-1}$, is rather a theoretical option, since $\mathcal{W}_{\mu-1}$ is not easy to obtain. The point is, that working instead with (140) and Z_1 as described above, and differentiating the more components $D(I - Z_1)x$, includes the differentiation of the component $D(I - Z_{\mu-1})x$ as a part of it. By this, the reduction step from (44) to (144) seems to be a reasonable compromise from both theoretical and practical view.
At this place we underline that there are various possibilities to compose special reduction techniques.

9 Widely orthogonal projector functions

For each DAE with properly stated leading term the orthogonal projector onto N_0 is an admissible one. We can always start the matrix function sequence by choosing $Q_0 = Q_0^*$, $P_0 = P_0^*$. In the next level, applying the decomposition $\mathbb{R}^m = (N_0 \cap N_1)^\perp \oplus (N_0 \cap N_1)$ we determine X_1 in the decomposition $N_0 = X_1 \oplus (N_0 \cap N_1)$ by $X_1 = N_0 \cap (N_0 \cap N_1)^\perp$. This leads to $N_0 + N_1 = (X_1 \oplus (N_0 \cap N_1)) + N_1 = X_1 \oplus N_1$ and $\mathbb{R}^m = (N_0 + N_1)^\perp \oplus (N_0 + N_1) = (N_0 + N_1)^\perp \oplus X_1 \oplus N_1$. By this, Q_1 is uniquely determined.
On the next levels, if $Q_0, \ldots, Q_{i-1}$ are admissible, we first apply the decomposition $\mathbb{R}^m = (\widehat{N_i})^\perp \oplus \widehat{N_i}$, and choose

$$X_i = (N_0 + \cdots + N_{i-1}) \cap (\widehat{N_i})^\perp. \tag{150}$$

The resulting decompositions $N_0 + \cdots + N_i = X_i \oplus N_i$, and
$\mathbb{R}^m = (N_0 + \cdots + N_i)^\perp \oplus (N_0 + \cdots + N_i) = (N_0 + \cdots + N_i)^\perp \oplus X_i \oplus N_i$ allow for the choice

$$\operatorname{im} Q_i = N_i, \ \ \ker Q_i = (N_0 + \cdots + N_i)^\perp \oplus X_i. \tag{151}$$

Definition 9.1 *Admissible projector functions $Q_0, \ldots, Q_\kappa$ are called <u>widely orthogonal</u> if (150) and (151) are fulfilled for $i = 1, , \ldots, \kappa$.*

Notice that widely orthogonal projector functions are uniquely determined. They provide also special symmetry properties. In fact, applying widely orthogonal projector functions, the decompositions

$$x(t) = \mathit{\Pi}_i(t)x(t) + \mathit{\Pi}_{i-1}(t)Q_i(t)x(t) + \ldots + \mathit{\Pi}_0(t)Q_1(t)x(t) + Q_0(t)x(t)$$

are orthogonal ones for all t.

Proposition 9.2 *If $Q_0, \ldots, Q_\kappa$ are widely orthogonal, then $\mathit{\Pi}_i$, $i = 0, \ldots, \kappa$, and $\mathit{\Pi}_{i-1}Q_i, i = 1, \ldots, \kappa$, are symmetric projectors.*

Proof:
Let $Q_0, \ldots, Q_\kappa$ be widely orthogonal. In particular, it holds that $\mathit{\Pi}_0 = \mathit{\Pi}_0^*$, $\ker \mathit{\Pi}_0 = N_0$, $\operatorname{im} \mathit{\Pi}_0 = N_0^\perp$.
Compute $\operatorname{im} \mathit{\Pi}_1 = \operatorname{im} P_0P_1 = P_0 \operatorname{im} P_1 = P_0((N_0 + N_1)^\perp \oplus X_1) = P_0(N_0 + N_1)^\perp$
$= P_0(N_0^\perp \cap N_1^\perp) = N_0^\perp \cap N_1^\perp = (N_0 + N_1)^\perp$.
To use induction, assume that $\operatorname{im} \mathit{\Pi}_j = (N_0 + \cdots + N_j)^\perp$, $j \leq i - 1$.

Due to Proposition 3.2 (1) we know that $\ker \Pi_i = N_0 + \cdots + N_i$ is true, further $\Pi_{i-1}X_i = 0$. From (151) it follows that $\operatorname{im} \Pi_i = \Pi_{i-1}\operatorname{im} P_i = \Pi_{i-1}((N_0+\cdots+N_i)^\perp \oplus X_i) = \Pi_{i-1}(N_0 + \cdots + N_i)^\perp = \Pi_{i-1}((N_0 + \cdots + N_{i-1})^\perp \cap N_i^\perp) = (N_0 + \cdots + N_{i-1})^\perp \cap N_i^\perp = (N_0 + \cdots + N_i)^\perp$.
Since Π_i is a projector, and $\ker \Pi_i = N_0 + \cdots + N_i$, $\operatorname{im} \Pi_i = (N_0 + \cdots + N_i)^\perp$, Π_i must be the orthoprojector.
Finally, derive $(\Pi_{i-1}Q_i)^* = (\Pi_{i-1} - \Pi_{i-1}P_i)^* = \Pi_{i-1} - \Pi_{i-1}P_i = \Pi_{i-1}Q_i$. □

Proposition 9.3 *If, for the DAE (8) with properly stated leading term, there exist any admissible projector functions $Q_0, \ldots, Q_\kappa$, and if $DD^* \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^n))$, then also widely orthogonal projector functions can be chosen (do exist).*

Proof:
Let $Q_0, \ldots, Q_\kappa$ be admissible. Then, in particular the subspaces $N_0+\cdots+N_i$, $i = 0, \ldots, \kappa$ are continuous. The subspaces $\operatorname{im} D\Pi_0 Q_1, \ldots, \operatorname{im} D\Pi_{\kappa-1}Q_\kappa$ belong to the class $\mathcal{C}^1$, since the projectors $D\Pi_0Q_1D^-, \ldots, D\Pi_{\kappa-1}Q_\kappa D^-$ do so. Taking Proposition 3.2 into account we know the subspaces $D(N_0+\cdots+N_i)$, $\quad i = 1, \ldots, \kappa$, to be continuously differentiable.
Now we construct widely orthogonal projectors. Choose $\bar{Q}_0 = \bar{Q}_0^*$, and form $\bar{G}_1 = G_0 + B_0\bar{Q}_0$. Due to Lemma 3.7 (d) it holds that $\bar{G}_1 = G_1Z_1$, $\bar{N}_0 + \bar{N}_1 = N_0 + N_1$, $Z_1(\bar{N}_0 \cap \bar{N}_1) = N_0 \cap N_1$. Since Z_1 is nonsingular, $\bar{G}_1$ has constant rank r_1, and the intersection $\overline{N\mathcal{U}}_1 = \bar{N}_1 \cap \bar{N}_0$ has constant dimension u_1. Put $\bar{X}_1 = \bar{N}_0 \cap (\bar{N}_0 \cap \bar{N}_1)^\perp$ and fix the projector $\bar{Q}_1$ by means of $\operatorname{im} \bar{Q}_1 = \bar{N}_1$, $\ker \bar{Q}_1 = \bar{X}_1 \oplus (\bar{N}_0+\bar{N}_1)^\perp$. $\bar{Q}_1$ is continuous, but for the sequence $\bar{Q}_0, \bar{Q}_1$ to be admissible, $D\bar{\Pi}_1\bar{D}^-$ has to belong to the class $\mathcal{C}^1$. This projector has the nullspace $\ker D\bar{\Pi}_1\bar{D}^- = D(\bar{N}_0 + \bar{N}_1) \oplus \ker R = D(N_0 + N_1) \oplus \ker R$, which is already known to belong to $\mathcal{C}^1$. If $D\bar{\Pi}_1\bar{D}^-$ has a range that is a $\mathcal{C}^1$ subspace, then $D\bar{\Pi}_1\bar{D}^-$ itself is continuously differentiable. Derive $\operatorname{im} D\bar{\Pi}_1\bar{D}^- = \operatorname{im} D\bar{\Pi}_1 = D(\bar{N}_0 + \bar{N}_1)^\perp = D(N_0 + N_1)^\perp = DD^*(D(N_0 + N_1))^\perp$. Since $D(N_0 + N_1)$ belongs to the class $\mathcal{C}^1$, so does $(D(N_0 + N_1))^\perp$. It comes out that $D\bar{P}_0\bar{P}_1\bar{D}^-$ is in fact continuously differentiable, and hence, $\bar{Q}_0, \bar{Q}_1$ are admissible.
To use induction, assume that $\bar{Q}_0, \ldots, \bar{Q}_{i-1}$ are admissible and widely orthogonal. Lemma 3.7 (d) yields $\bar{G}_i = G_iZ_i$, $\bar{N}_0 + \cdots + \bar{N}_{i-1} = N_0 + \cdots + N_{i-1}$, $\bar{N}_0 + \cdots + \bar{N}_i = N_0 + \cdots + N_i$, $Z_i(\bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})) = N_i \cap (N_0 + \cdots + N_{i-1})$.
Since Z_i is nonsingular, it follows that $\bar{G}_i$ has constant rank r_i and the intersection $\overline{N\mathcal{U}}_i = \bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})$ has constant dimension u_i. The involved subspaces are continuous. Put

$$\bar{X}_i = (\bar{N}_0 + \cdots + \bar{N}_{i-1}) \cap ((\bar{N}_0 + \cdots + \bar{N}_{i-1}) \cap \bar{N}_i)^\perp$$

and choose $\bar{Q}_i$ to be the projector onto $\bar{N}_i$ along $(\bar{N}_0 + \cdots + \bar{N}_i)^\perp \oplus \bar{X}_i$.
$\bar{Q}_0 \ldots, \bar{Q}_{i-1}, \bar{Q}_i$ would be admissible if $D\bar{\Pi}_i\bar{D}^-$ was continuously differentiable. We know $\ker D\bar{\Pi}_i\bar{D}^- = D(N_0 + \cdots + N_i) \oplus \ker R$ to be already continuously differentiable. On the other hand, we have $\operatorname{im} D\bar{\Pi}_i\bar{D}^- = D \operatorname{im} \bar{\Pi}_i = D(N_0 + \cdots + N_i)^\perp = DD^*(D(N_0 + \cdots + N_i))^\perp$, hence $\operatorname{im} D\bar{\Pi}_i\bar{D}^-$ belongs to the class $\mathcal{C}^1$. □

The widely orthogonal projectors have the advantage that they are uniquely determined. This proves its value in theoretical investigations on necessary and sufficient regularity conditions for nonlinear DAEs, as well as for investigating critical points. Moreover, in practical calculations, in general, there might be difficulties to assure the continuity of

the projector functions Π_i. Fortunately, owing to their uniqueness the widely orthogonal projector functions are continuous a priori.
By Proposition 9.3, at least for all DAEs with properly stated leading term, and with a continuously differentiable coefficient D, we may access widely orthogonal projector functions. However, if D is just continuous, and if DD^* fails to be continuously differentiable as required, then it may happen in fact that admissible projector functions exist but the special widely orthogonal projector functions do not exist for lack of smoothness. The following example shows this situation. At this point we underscore that most DAEs are given with a smooth D, and our example is rather academic.

Example 9.4 *We reconsider Example 6.30 which is a regular DAE with tractability index two. The detailed equations are*

$$\begin{aligned} (x_1 + \alpha x_2)' &= q_1, \\ x_2' - x_3 &= q_2, \\ x_2 &= q_3. \end{aligned}$$

Written as (8) with $m = k = 3,\ n = 2$, the DAE has the coefficients.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\alpha : \mathcal{I} \to \mathbb{R}$ is a continuous function. Example 6.30 provides fine decoupling projector functions. Now we construct widely orthogonal projector functions. We start with

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D^- = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Compute further

$$N_0 \oplus N_1 = \operatorname{span}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\alpha \\ 1 \\ 1 \end{bmatrix} \}, \ (N_0 \oplus N_1)^\perp = \operatorname{span} \begin{bmatrix} 1 \\ \alpha \\ 0 \end{bmatrix}.$$

The wanted projector function onto N_1 along $N_0 \oplus (N_0 \oplus N_1)^\perp$ is

$$Q_1 = \frac{1}{1+\alpha^2} \begin{bmatrix} \alpha^2 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ -\alpha & 1 & 0 \end{bmatrix}, \quad \textit{and it results that} \quad D\Pi_1 D^- = \begin{bmatrix} 1 & 0 \\ \frac{\alpha}{1+\alpha^2} & 0 \end{bmatrix}.$$

We recognize that, in the given setting, $D\Pi_1 D^-$ is just continuous. If we additionally assume that $\alpha \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})$, then Q_0, Q_1 appear to be admissible. Notice that in this case $DD^* = \begin{bmatrix} 1+\alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}$ is continuously differentiable, which confirms Proposition 9.3 once more.
Let us stress that this special DAE is solvable for arbitrary continuous α. From this point of view there is no need for assuming α to be $\mathcal{C}^1$. □

10 Over- and underdetermined DAEs (Nonregular DAEs)

The general purpose of this monograph is the detailed analysis of *regular* DAEs. In particular we aim for regularity criteria, and we would like to assist in modeling regular DAEs in applications, and in avoiding DAE models that fail to be regular.
On the other side, several authors spend much place also to rectangular DAEs (cf. [KM06] for a summary).
In our view, more general linear DAEs (8) than regular ones are less interesting, and we tear this topic just slightly. As usually, we speak on *overdetermined* systems, if $k > m$, but on *underdetermined* ones, if $k < m$. However, this notion does not say so much, it simply indicates the relation between the numbers of equations and unknown functions. It seems to be more appropriate speaking on *nonregular DAEs*, that is, on DAEs not being regular. This option includes also the square systems (with $m = k$) which may also contain free variables and consistency conditions if the regularity conditions fail.

As in [LMT11b, Section 7], we point out the great latitude for interpretations when considering nonregular DAEs.
Turn for a moment to the overdetermined DAE

$$x' + x = q_1, \tag{152}$$
$$x = q_2, \tag{153}$$

with $k = 2$, $m = n = 1$. If one more emphasizes the algebraic equation $x = q_2$, one is led to a differentiation of q_2 as well as to a consistency condition coming from the first equation, namely

$$q_2' + q_2 - q_1 = 0.$$

Contrarily, if one puts emphasis on the differential equation $x' + x = q_1$ one can solve this equation for

$$x(t) = e^{-t}\Big(x_0 + \int_0^t e^s q_1(s)ds\Big)$$

and then consider the second equation to be responsible for consistency. This leads to the consistency condition

$$e^{-t}\Big(x_0 + \int_0^t e^s q_1(s)ds\Big) - q_2(t) = 0.$$

At a first glance this consistency condition looks quite different, but differentiation immediately yields again $q_2 - q_1 + q_2' = 0$.
The last interpretation is oriented to solve rather differential equations than algebraic ones and to differentiate. We join this point of view.

A further room of interpretation is given for the trivial underdetermined DAE

$$(x_1 + x_2)' + x_1 = q \tag{154}$$

with $k = 1$, $m = 2$, $n = 1$. Should we choose x_1 or x_2 to be free? One can also think on writing

$$(x_1 + x_2)' + (x_1 + x_2) - x_2 = q, \tag{155}$$

or

$$(x_1+x_2)' + \frac{1}{2}(x_1+x_2) + \frac{1}{2}(x_1-x_2) = q. \tag{156}$$

As described in Section 4, the special structure of the matrix function sequence (12)-(15) built by admissible projector functions allows for a systematic rearrangement of general DAEs (8), among them also rectangular ones. Section 4 ends up with a first glance at DAEs the matrix function G_0 of which has already maximal rank. We resume this discussion noting that, in the above two examples, we have the constant matrix functions

$$G_0 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \text{and} \quad G_0 = \begin{bmatrix}1 & 1\end{bmatrix},$$

and both have already maximal rank.
Recall that, in this case, the DAE (8) is equivalent to the system (34), that is to

$$(Dx)' - R'Dx + DG_0^- B_0 D^- Dx + DG_0^- B_0 Q_0 x = DG_0^- q, \quad \mathcal{W}_0 B_0 D^- Dx = \mathcal{W}_0 q, \tag{157}$$

the solution of which decomposes as $x = D^-Dx + Q_0 x$.
For the overdetermined system (152), (153), we have in detail: $D = D^- = R = 1$, $Q_0 = 0$,

$$A = \begin{bmatrix}1\\0\end{bmatrix}, \; B = \begin{bmatrix}1\\1\end{bmatrix} \; G_0 = \begin{bmatrix}1\\0\end{bmatrix}, \; G_0^- = \begin{bmatrix}1 & 0\end{bmatrix}, \; \mathcal{W}_0 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \; DG_0^- B_0 D^- = 1.$$

Inserting these coefficients we see the first equation in (157) coincides with the ODE (152), while the second equation in (157) is nothing else (153). This confirms the interpretation of the given DAE to be primarily the explicit ODE (152) subject to the consistency condition (153).

For the underdetermined DAE (154), one has $A = 1$, $D = \begin{bmatrix}1 & 1\end{bmatrix}$, $R = 1$, $B = \begin{bmatrix}1 & 0\end{bmatrix}$, $\mathcal{W}_0 = 0$, and the second equation in (157) disappears. Many different projectors Q_0 are admissible, and different choices lead to different ODEs

$$(Dx)' + DG_0^- B_0 D^- Dx + DG_0^- B_0 Q_0 x = DG_0^- q, \tag{158}$$

and solution representations $x = D^-Dx + Q_0 x$. We consider three cases:

(a) Set and compute

$$D^- = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}, \; P_0 = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}, \; Q_0 = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}, \; G_0^- = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix},$$

and further $DG_0^- B_0 D^- = \frac{1}{2}$, $DG_0^- = 1$, $DG_0^- B_0 Q_0 = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$, and we see the corresponding ODE (158) coincides with (156).

(b) Set and compute

$$D^- = \begin{bmatrix}1\\0\end{bmatrix}, \; P_0 = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \; Q_0 = \begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}, \; G_0^- = \begin{bmatrix}1\\0\end{bmatrix},$$

$DG_0^- B_0 D^- = 1$, $DG_0^- = 1$, $DG_0^- B_0 Q_0 = \begin{bmatrix}0 & -1\end{bmatrix}$. Now the equation (158) coincides with (155).

(c) Set and compute

$$D^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; P_0 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \; Q_0 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \; G_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$DG_0^- B_0 D^- = 0$, $DG_0^- = 1$, $DG_0^- B_0 Q_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and the equation (158) coincides with the version (154).

Observe that the eigenvalues of $DG_0^- B_0 D^-$ are $\frac{1}{2}$ or 1 or 0, in dependence of the choice of the admissible projector Q_0. One could restrict the variety of admissible projectors and take just the widely orthogonal ones which are uniquely determined. In our example this corresponds to (a). However, this would be an arbitrary action. We think that is it worth to be mentioned at this place, that the *inherent explicit regular ODE* of a *regular DAE* is uniquely defined by the problem data, independent of the choice of fine decoupling projectors.
The tractability index of a regular DAE is determined to be the smallest index μ such that the matrix function G_μ is nonsingular (Definition 6.2). We intend to generalize this notion, and to assign the tractability index μ to each general DAE (8) with properly stated leading term, for which admissible projector functions do exist and μ is the smallest index such that G_μ has maximal possible rank. In this sense, both above examples are tractable with index zero.
Before we formulate the detailed definition, we remember that the ranks of the matrix functions $G_0, \ldots, G_i$ form a nondecreasing sequence $r = r_0 \le r_1 \le \ldots \le r_i$, but not necessarily a strictly increasing one. It may well happen that the ranks do not change in several consecutive steps. For instance, a Hessenberg size μ DAE is characterized by the sequence $r_0 = \ldots = r_{\mu-1} < r_\mu$. This feature makes the task to recognize the maximal rank and stop constructing the matrix functions in practice somewhat more subtle as thought before. Of course, if one reaches $\operatorname{im} G_\mu = \operatorname{im}[AD\ B]$, or equivalently $\mathcal{W}_\mu B = \{0\}$, then r_μ is maximal, and one can stop owing to the basic property (19) of the matrix functions. If one obtains an injectice G_μ, then one can stop because of the resulting stationarity $G_\mu = G_{\mu+1} = \ldots = G_{\mu+i}$. Proposition 10.2(3) below gives further useful information which also applies to the case if $\operatorname{im}[AD\ B]$ can not be reached. More precisely, it says, if there are admissible projector functions $Q_0, \ldots, Q_{r+1}$, then there is an index $\bar{\mu} \le r+1$ such that the matrix function sequence can be continued up to infinity, and it is stationary at least beginning with this index, that is $G_{\bar{\mu}} = G_{\bar{\mu}+i}$ for all $i \ge 1$. This provides the upper bound $r+1$ of the index $\bar{\mu}$.
We are looking for the index μ such that the rank r_μ reaches the maximal possible value. Since μ must be always equal or less than $\bar{\mu}$, having the bound $r+1$ for the second index, we have at the same time a bound for the first one, that is $\mu \le r+1$.

Definition 10.1 *Let the DAE (8) have a properly stated leading term, and let the matrix function $\begin{bmatrix} AD & B \end{bmatrix}$ have constant rank.*

(1) *The DAE is said to be tractable on $\mathcal{I}$ with index zero ($\mu = 0$), if either $\operatorname{im} G_0 = \operatorname{im}[AD\ B]$ or admissible projector functions $Q_0, \ldots, Q_{r+1}$ exist such that $\operatorname{im} G_0 = \ldots = \operatorname{im} G_{r+1}$.*

(2) *The DAE is said to be tractable on $\mathcal{I}$ with index $\mu \in \mathbb{N}$,*

(a) *if there are admissible projector functions* $Q_0, \ldots, Q_{\mu-1}$, *and* $\operatorname{im} G_\mu = \operatorname{im}[AD \;\; B]$, *or*

(b) *if there are admissible projector functions* $Q_0, \ldots, Q_{r+1}$, *and* $\operatorname{im} G_\mu = \ldots = \operatorname{im} G_{r+2}$,

and μ *is the smallest integer of this kind.*

(3) *The DAE is regular on* $\mathcal{I}$ *with tractability index* $\mu \in \mathbb{N} \cup \{0\}$, *if it is tractable with index* μ, *and additionally* $m = k$ *and* $\operatorname{im} G_\mu = \mathbb{R}^m$.

This definition generalizes Definition 6.2. Item(3) repeats Definition 6.2 for completeness. The special examples (152), (153) and (154) show DAEs being tractable with index zero. From our point of view one should take care to attain the condition $\operatorname{im}[AD \;\; B] = \mathbb{R}^k$ during the modeling.
A particular case of interest is given if one meets matrix functions G_i being injective. This can only happen if $k \geq m$. Then, the tractability index is the smallest integer μ such that G_μ is injective, thus $r_\mu = m$. It is worth mentioning that then $u_0 = \ldots = u_{\mu-1} = 0$, i.e. the intersections $\widehat{N_i}$ are trivial.
If the complement subspace X_1 is trivial, then it holds that $G_i = G_0$ for all $i \geq 1$, and the DAE is tractable with index zero and therefore, if $X_1 = \{0\}$, then one can stop. Namely, $X_1 = \{0\}$ means $N_1 \cap N_0 = N_0$. This implies $N_0 \subseteq N_1$, and $N_0 = N_1$ because of the dimensions $\dim N_0 = m - r_0 \geq m - r_1 = \dim N_1$. Choose $Q_1 := Q_0$. The projector functions Q_0, Q_1 are admissible. It follows that $0 = G_1 Q_1 = G_0 Q_1 + B_0 Q_0 Q_1 = B_0 Q_0$, thus $G_1 = G_0$ and $G_2 = G_1 + B_1 Q_1 = G_1 + B_1 P_0 Q_1 = G_1$. Then we set $Q_2 := Q_1$ and so on. In particular, it results that $X_i = \{0\}$ for all $i \geq 1$.
Notice that, if there is a trivial complement subspace X_κ in a matrix function sequence, the all these subspaces X_i must be trivial, too.

Proposition 10.2 *Given is the DAE (8) with a properly stated leading term.*

(1) *If there are admissible projector functions* $Q_0, \ldots, Q_\kappa$, *with a* $\kappa \in \mathbb{N}$, *such that*

$$G_\kappa = G_{\kappa+1},$$

then, the projector functions $Q_0, \ldots, Q_{\kappa+i}$, *with* $Q_{\kappa+i} := Q_\kappa$ *for* $i \geq 1$, *are also admissible, and it holds that*

$$G_\kappa = G_{\kappa+i}, \quad N_0 + \ldots + N_\kappa = N_0 + \ldots + N_{\kappa+i}.$$

(2) *If there admissible projector functions* $Q_0, \ldots, Q_\kappa$, *such that*

$$N_0 + \ldots + N_{\kappa-1} = N_0 + \ldots + N_\kappa,$$

then $G_\kappa = G_{\kappa+1}$ *holds true.*

(3) *If* $Q_0, \ldots, Q_{r+1}$ *are admissible projector functions, then the sequence can be continued up to infinity, and there is an index* $\bar{\mu} \leq r+1$ *such that* $G_{\bar{\mu}+1} = G_{\bar{\mu}+i}$ *for all* $i \geq 2$.

Proof: (1) $N_\kappa = N_{\kappa+1}$ implies $N_{\kappa+1} \subseteq N_0 + \ldots + N_\kappa$, $N_0 + \ldots + N_\kappa = N_0 + \ldots + N_{\kappa+1}$, $N_0 + \ldots + N_\kappa = X_\kappa \oplus N_\kappa = X_\kappa \oplus N_{\kappa+1}$, hence, choosing $X_{\kappa+1} := X_\kappa$, $Q_{\kappa+1} := Q_\kappa$ leads to $u_{\kappa+1} = u_\kappa$, $D\Pi_{\kappa+1}D^- = D\Pi_\kappa D^-$, so that $Q_0, \ldots, Q_\kappa, Q_{\kappa+1}$ are admissible, and further $B_{\kappa+1}Q_{\kappa+1} = B_{\kappa+1}\Pi_\kappa Q_{\kappa+1} = 0$, $G_{\kappa+2} = G_{\kappa+1}$, and so on.
(2) $N_0 + \ldots + N_{\kappa-1} = N_0 + \ldots + N_\kappa$ implies $N_\kappa \subseteq N_0 + \ldots + N_{\kappa-1}$, hence $\Pi_{\kappa-1}Q_\kappa = 0$, $B_\kappa Q_\kappa = B_\kappa \Pi_{\kappa-1}Q_\kappa = 0$, $G_{\kappa+1} = G_\kappa$.
(3) Let $Q_0, \ldots, Q_{r+1}$ be admissible projector functions. Apply the decompositions $N_i = \widehat{N_i} \oplus \mathcal{Y}_i$, which is accompanied by $(N_0 + \ldots + N_{i-1}) \cap \mathcal{Y}_i = \{0\}$. Namely, $z \in (N_0 + \ldots + N_{i-1}) \cap \mathcal{Y}_i$ yields $z \in (N_0 + \ldots + N_{i-1}) \cap N_i = \widehat{N_i}$, thus $z = 0$. It results that $N_0 + \ldots + N_i = N_0 + \ldots + N_{i-1} + \mathcal{Y}_i = (N_0 + \ldots + N_{i-1}) \oplus \mathcal{Y}_i$, that is, the supplement to $N_0 + \ldots + N_{i-1}$ is exactly the subspace $\mathcal{Y}_i$, and therefore $\dim(N_0 + \ldots + N_i) = \dim(N_0 + \ldots + N_{i-1}) + \dim \mathcal{Y}_i$.
If $\dim \mathcal{Y}_i \geq 1$ for $j = 1, \ldots, r$, then

$$\dim(N_0 + \ldots + N_r) \geq \dim N_0 + r = m - r_0 + r = m.$$

In consequence, the subspaces $N_0 + \ldots + N_r$ and $N_0 + \ldots + N_{r+1}$ must coincide, and assertion (2) leads to $G_{r+1} = G_{r+2}$.

If there is an index $j_* \leq r$ such that $\dim \mathcal{Y}_i = 0$, then we have $N_{j_*} = \widehat{N}_{j_*} = N_{j_*} \cap (N_0 + \ldots + N_{j_*-1})$, and the inclusion $N_{j_*} \subseteq N_0 + \ldots + N_{j_*-1}$ is valid. This leads to $N_0 + \ldots + N_{j_*-1} = N_0 + \ldots + N_{j_*}$, and due to assertion (2), to $G_{j_*} = G_{j_*+1}$.
Owing to (1), the matrix function sequence can be continued in both cases, and there exists an index $\bar{\mu} \leq r + 1$ with $G_{\bar{\mu}} = G_{\bar{\mu}+i}$, $i \geq 1$. □

By Proposition 10.2, we know that equal subspaces $N_0 + \ldots + N_{\kappa-1} = N_0 + \ldots + N_\kappa$ in the sequence (20) indicate that the matrix functions G_i coincide with G_κ on all following levels, and we can stop constructing the matrix function sequence. However, the smallest integer κ with $N_0 + \ldots + N_{\kappa-1} = N_0 + \ldots + N_\kappa$ does not necessarily coincide with the smallest integer μ indicating that G_μ has the maximal possible rank. For instance, in Example 3.5, we have $\kappa = 2$, but $\mu = 0$.

In general, applying Proposition 10.2 we know the tractability index μ to be smaller or equal to $r + 1 = \operatorname{rank}(AD) + 1$. The inequality

$$\mu \leq \operatorname{rank}(AD) + 1 \tag{159}$$

is rigorous. This is confirmed by Example 3.6 with $m_1 = m_2 = m_3 = 1$, $r_0 = 2$, and $\mu = 3$, i.e. $\mu = r_0 + 1$.

Next we reconsider the rearranged version (30) of the DAE (8), and provide a refined form which serves below as a basis of the further decouplings.

Proposition 10.3 *Let the DAE (8) with properly stated leading term have the admissible projectors $Q_0, \ldots, Q_\kappa$, with $\kappa \in \mathbb{N}$. Then this DAE can be rewritten as*

$$\begin{aligned} G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x + G_\kappa \sum_{l=0}^{\kappa-1} \{ Q_l x - (I - \Pi_l) Q_{l+1} D^-(D\Pi_l Q_{l+1} x)' \\ + \mathcal{V}_l D\Pi_l x + \mathcal{U}_l (D\Pi_l x)' \} = q \end{aligned} \tag{160}$$

with coefficients

$$\begin{aligned}
\mathcal{U}_l &:= -(I-\Pi_l)\{Q_l + Q_{l+1}(I-\Pi_l)Q_{l+1}P_l\}\Pi_l D^-,\\
\mathcal{V}_l &:= (I-\Pi_l)\{(P_l + Q_{l+1}Q_l)D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l D^-.
\end{aligned}$$

Before we verify this assertion, we point out the coefficients $\mathcal{V}_l$ to be caused by variations in time, these coefficients vanish in the constant coefficient case.

The coefficients $\mathcal{U}_l$ disappear, if the intersections $\widehat{N}_1, \ldots, \widehat{N}_l$ are trivial.

If the intersections $\widehat{N}_1, \ldots, \widehat{N}_\kappa$ are trivial, then it results (cf. Proposition 4.1) that $\mathcal{V}_l = V_l,\ l = 1, \ldots, \kappa$.

Proof: Recall from Proposition 4.1 the general rearranged version (30) of the DAE (8):

$$G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x + G_\kappa \sum_{l=0}^{\kappa-1}\{Q_l x + (I-\Pi_l)(P_l - Q_{l+1}P_l)(D\Pi_l x)'\} = q. \qquad (161)$$

For $\kappa = 1$ we compute

$$\begin{aligned}
G_1(I-\Pi_0)(P_0 - Q_1P_0)D^-(D\Pi_0 x)' &= -G_1(I-\Pi_0)Q_1D^-(D\Pi_0 x)'\\
&= -G_1(I-\Pi_0)Q_1D^-(D\Pi_0Q_1x)' - G_1(I-\Pi_0)Q_1D^-(D\Pi_1D^-D\Pi_0x)'\\
&= -G_1(I-\Pi_0)Q_1D^-(D\Pi_0Q_1x)' + G_1\mathcal{V}_0D\Pi_0x + G_1\mathcal{U}_0(D\Pi_0x)'
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{U}_0 &= -(I-\Pi_0)Q_1\Pi_1D^- = -(I-\Pi_0)\{Q_0 + Q_1(I-\Pi_0)Q_1P_0\}\Pi_0D^-,\\
\mathcal{V}_0 &= -(I-\Pi_0)Q_1D^-(D\Pi_1D^-)'D\Pi_0D^-.
\end{aligned}$$

Set $\kappa > 1$, and take a closer look to

$$\mathcal{E}_l := (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l x)', \quad 0 \le l \le \kappa - 1.$$

Compute

$$\begin{aligned}
\mathcal{E}_l &= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-((D\Pi_lD^-)'D\Pi_lx + D\Pi_lD^-(D\Pi_lx)')\\
&= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lD^-)'D\Pi_lx + (I-\Pi_l)(-Q_l - Q_{l+1}P_l)\Pi_lD^-(D\Pi_lx)'\\
&= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lD^-)'D\Pi_lx - (I-\Pi_l)Q_l\Pi_lD^-(D\Pi_lx)'\\
&\qquad - (I-\Pi_l)Q_{l+1}\{\Pi_l + I - \Pi_l\}Q_{l+1}P_l\Pi_lD^-(D\Pi_lx)'\\
&= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_lD^-)'D\Pi_lx\\
&\qquad - (I-\Pi_l)(Q_l + Q_{l+1}\{\Pi_l + I - \Pi_l\}Q_{l+1}P_l)\Pi_lD^-(D\Pi_lx)'
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{E}_l &= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l D^-)'D\Pi_l x \\
&\quad \underbrace{-(I-\Pi_l)(Q_l + Q_{l+1}\{I-\Pi_l\}Q_{l+1}P_l)\Pi_l D^-}_{\mathcal{U}_l}(D\Pi_l x)' \\
&\quad -(I-\Pi_l)Q_{l+1}\underbrace{\Pi_l Q_{l+1}P_l\Pi_l D^-}_{\Pi_l Q_{l+1}}(D\Pi_l x)' \\
&= (I-\Pi_l)(P_l - Q_{l+1}P_l)D^-(D\Pi_l D^-)'D\Pi_l x + \mathcal{U}_l(D\Pi_l x)' \\
&\quad -(I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' + (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}D^-)'D\Pi_l x \\
&= (I-\Pi_l)\{(P_l - Q_{l+1}P_l)D^-(D\Pi_l D^-)' + Q_{l+1}D^-(D\underbrace{\Pi_l Q_{l+1}}_{\Pi_l - \Pi_{l+1}}D^-)'\}D\Pi_l x + \mathcal{U}_l(D\Pi_l x)' \\
&\quad -(I-\Pi_l)\{(P_l - Q_{l+1}P_l)D^-(D\Pi_l D^-)' \\
&= \mathcal{U}_l(D\Pi_l x)' - (I-\Pi_l)\{(P_l - Q_{l+1}P_l)D^-(D\Pi_l D^-)' \\
&\quad + \underbrace{(I-\Pi_l)\{(P_l - Q_{l+1}P_l + Q_{l+1})D^-(D\Pi_l D^-)' - Q_{l+1}D^-(D\Pi_{l+1}D^-)'\}D\Pi_l}_{\mathcal{V}_l D} x.
\end{aligned}
$$

In consequence, the representation (161) is nothing else

$$
\begin{aligned}
G_\kappa D^-(D\Pi_\kappa x)' + B_\kappa x + G_\kappa \sum_{l=0}^{\kappa-1}\{Q_l x - (I-\Pi_l)Q_{l+1}D^-(D\Pi_l Q_{l+1}x)' \\
+ \mathcal{V}_l D\Pi_l x + \mathcal{U}_l(D\Pi_l x)'\} = q,
\end{aligned}
$$

which completes the proof. □

Throughout the rest of this section the DAE (8) is supposed to be tractable with index μ, and $Q_0,\dots,Q_{\mu-1}$ denote admissible projector functions. We may take use of the rearranged version of (8) (cf. (160))

$$
\begin{aligned}
&G_{\mu-1}D^-(D\Pi_{\mu-1}x)' + B_{\mu-1}x \\
&\quad + G_{\mu-1}\sum_{\ell=0}^{\mu-2}\{Q_\ell x - (I-\Pi_\ell)Q_{\ell+1}D^-(D\Pi_\ell Q_{\ell+1}x)' + \mathcal{V}_\ell D\Pi_\ell x + \mathcal{U}_\ell(D\Pi_\ell x)'\} = q,
\end{aligned}
\tag{162}
$$

the coefficients $\mathcal{V}_\ell, \mathcal{U}_\ell$ are from Proposition 10.3. By expressing

$$
G_{\mu-1}Q_\ell = G_\mu Q_\ell,\ G_{\mu-1}\mathcal{V}_\ell = G_\mu\mathcal{V}_\ell,\ G_{\mu-1}\mathcal{U}_\ell = G_\mu\mathcal{U}_\ell,\ \ell = 0,\dots,\mu-2,
$$

$$
B_{\mu-1} = B_{\mu-1}P_{\mu-1} + B_{\mu-1}Q_{\mu-1} = B_{\mu-1}D^-D\Pi_{\mu-1} + B_{\mu-1}Q_{\mu-1},
$$

formula (162) becomes

$$
\begin{aligned}
G_\mu\Big\{&P_{\mu-1}D^-(D\Pi_{\mu-1}x)' + Q_{\mu-1}x \\
&+ \sum_{\ell=0}^{\mu-2}\{Q_\ell x - (I-\Pi_\ell)Q_{\ell+1}D^-(D\Pi_\ell Q_{\ell+1}x)' + \mathcal{V}_\ell D\Pi_\ell x + \mathcal{U}_\ell(D\Pi_\ell x)'\}\Big\} \\
&\qquad + B_{\mu-1}D^-D\Pi_{\mu-1}x = q.
\end{aligned}
\tag{163}
$$

According to the definition of the tractability index μ, the matrix function G_μ has constant rank. We find a continuous generalized inverse G_μ^-, and a projector function $\mathcal{W}_\mu = I - G_\mu G_\mu^-$ along $\operatorname{im} G_\mu$. Notice that there is no need for the resulting projector function $G_\mu^- G_\mu$ to be also admissible. The projector functions $G_\mu G_\mu^-$ and $\mathcal{W}_\mu$ split the DAE (163) into two parts. Multiplication by $\mathcal{W}_\mu$ leads to equation (165) below. Multiplication by $G_\mu G_\mu^-$ yields

$$\begin{aligned} G_\mu\Big[&P_{\mu-1}D^-(D\varPi_{\mu-1}x)' + Q_{\mu-1}x \\ &+\sum_{\ell=0}^{\mu-1}\{Q_\ell x - (I-\varPi_\ell)Q_{\ell+1}D^-(D\varPi_\ell Q_{\ell+1}x)' + \mathcal{V}_\ell D\varPi_\ell x + \mathcal{U}_\ell(D\varPi_\ell x)'\} \\ &+ G_\mu^- B_{\mu-1}D^-D\varPi_{\mu-1}x - G_\mu^- q\Big] = 0. \end{aligned}$$

This equation $G_\mu[\ ] = 0$ may be rewritten as $[\ ] = y$, where y is an arbitrary continuous function such that $G_\mu y = 0$. Together this leads to the system

$$\begin{aligned} P_{\mu-1}D^-(D\varPi_{\mu-1}x)' + Q_{\mu-1}x + \sum_{\ell=0}^{\mu-2}\{Q_\ell x - (I-\varPi_\ell)Q_{\ell+1}D^-(D\varPi_\ell Q_{\ell+1}x)' \\ +\mathcal{V}_\ell D\varPi_\ell x + \mathcal{U}_\ell(D\varPi_\ell x)'\} + y = G_\mu^-(q - B_{\mu-1}D^-D\varPi_{\mu-1}x), \qquad (164) \\ \mathcal{W}_\mu B_{\mu-1}D^-D\varPi_{\mu-1}x = \mathcal{W}_\mu q, \qquad (165) \end{aligned}$$

where y can be chosen arbitrarily such that $G_\mu y = 0$. Thereby, the relation

$$\ker G_\mu = (I - G_{\mu-1}^- B_{\mu-1}Q_{\mu-1})(N_{\mu-1}\cap S_{\mu-1}) \qquad (166)$$

might be helpful. The undetermined part of y is actually $Q_{\mu-1}y \in N_{\mu-1}\cap S_{\mu-1}$.

Multiplication of (164) by projector functions discovers some further structure. In particular, multiplication by $\varPi_{\mu-1}$ yields

$$\varPi_{\mu-1}D^-(D\varPi_{\mu-1}x)' + \varPi_{\mu-1}y = \varPi_{\mu-1}G_\mu^-(q - B_{\mu-1}D^-D\varPi_{\mu-1}x),$$

hence we recognize an inherent explicit regular ODE with respect to $D\varPi_{\mu-1}x$, namely

$$(D\varPi_{\mu-1}x)' - (D\varPi_{\mu-1}D^-)'D\varPi_{\mu-1}x + D\varPi_{\mu-1}y + D\varPi_{\mu-1}G_\mu^- B_{\mu-1}D^-D\varPi_{\mu-1}x = D\varPi_{\mu-1}G_\mu^- q.$$

It is worth mentioning again, that, in contrast to regular DAEs, the properties of the flow of this ODE may depend on the choice of the admissible projector functions, as it is the case for example (154).
Multiplying (164) by $\varPi_{\mu-2}Q_{\mu-1}$ gives

$$\varPi_{\mu-2}Q_{\mu-1}x + \varPi_{\mu-2}Q_{\mu-1}y + \varPi_{\mu-2}Q_{\mu-1}G_\mu^- B_{\mu-1}D^-D\varPi_{\mu-1}x = \varPi_{\mu-2}Q_{\mu-1}G_\mu^- q.$$

Apart from the terms including y, these two formulas are the same as the corresponding ones in Section 6 on regular DAEs. However, the further equations that will be derived from (164) by multiplication with further projector functions are more difficult to survey. We restrict ourselves to several case studies.

Case 1: G_μ has full column rank.

This case can happen only if $k \geq m$, and $r_\mu = m$ holds true. Since G_μ is injective, due to Proposition 4.1, all intersections $(N_0 + \ldots + N_{i-1}) \cap N_i$, $i = 1, \ldots, \mu - 1$, are trivial, the components $\mathcal{U}_0, \ldots, \mathcal{U}_{\mu-2}$ vanish, and $\mathcal{V}_\ell$ simplifies to $\mathcal{V}_\ell = V_\ell$, $\ell = 0, \ldots, \mu - 2$. Moreover, $G_\mu y = 0$ implies $y = 0$.

The resulting special equation (164) reads

$$
\begin{aligned}
P_{\mu-1} D^-(D\Pi_{\mu-1}x)' + Q_{\mu-1}x \\
+ \sum_{\ell=0}^{\mu-2} \{ Q_\ell x - (I - \Pi_\ell) Q_{\ell+1} D^-(D\Pi_\ell Q_{\ell+1} x)' + V_\ell D\Pi_\ell x \} \\
+ G_\mu^- B_{\mu-1} D^- D\Pi_{\mu-1} x = G_\mu^- q.
\end{aligned} \tag{167}
$$

For $k = m$, that is, for regular DAEs with tractability index μ, this formula coincides in essence with formula (50) (several terms are arranged in a different way).

Applying the decoupling procedure from Section 6, we can prove (167) to represent a regular index μ DAE. Completed by an initial condition

$$
D(t_0)\Pi_{\mu-1}(t_0)x(t_0) = z_0 \in \operatorname{im} D(t_0)\Pi_{\mu-1}(t_0), \tag{168}
$$

this equation is uniquely solvable for x. That means, we have the option to consider the equation (167) to fully determine the solution x, and to treat equation (165) as an additional consistency condition.

Example 10.4 *Set $m = 2$, $k = 3$, $n = 1$, and write the system*

$$
\begin{aligned}
x_1' + x_2 &= q_1, \\
x_2 &= q_2, \\
x_2 &= q_3,
\end{aligned} \tag{169}
$$

as DAE (8) such that

$$
A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D = [1 \;\; 0], \quad G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
G_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mu = 1, \quad G_1^- = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
$$

G_1 has already maximal possible rank, $r_1 = 2$, and hence this DAE is tractable with index one. The consistency equation $\mathcal{W}_1(B\Pi_0 x - q) = 0$ means here $q_2 = q_3$. Equation (167) has the form

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1' + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} q_1 - q_2 \\ q_2 \end{bmatrix},
$$

which is a regular index-one DAE.

Example 10.5 *Set $m = 2$, $k = 3$, $n = 1$, and rewrite the system*

$$\begin{aligned} x_1' + x_2 &= q_1, \\ x_1 &= q_2, \\ x_2 &= q_3, \end{aligned} \tag{170}$$

as DAE (8). This leads to

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = [1\ 0], \quad G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mu = 1, \quad G_1^- = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

G_1 has maximal rank, $r_1 = 2$, this DAE is tractable with index one. Condition $\mathcal{W}_1(B_0 \Pi_0 x - q) = 0$ means now $x_1 = q_2$, and equation (167) specializes to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1' + \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} q_1 - q_3 \\ q_3 \end{bmatrix},$$

which is a regular index-one DAE.

Example 10.6 *Set $k = 5$, $m = 4$, $n = 4$, and put the DAE*

$$\begin{aligned} x_1' &= q_1, \\ x_2' + x_1 &= q_2, \\ x_3' + x_2 &= q_3, \\ x_4' + x_3 &= q_4, \\ x_4 &= q_5, \end{aligned} \tag{171}$$

into the form (8). This yields

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\mu = 0$. This DAE is interpreted as an explicit ODE for the components x_1, x_2, x_3, x_4 and the consistency condition $x_4 = q_5$.

Example 10.7 *The DAE*

$$\begin{aligned} x_2' + x_1 &= q_1, \\ x_3' + x_2 &= q_2, \\ x_3 &= q_3, \\ x_3' &= q_3', \end{aligned} \tag{172}$$

results from the index three system

$$\begin{aligned} x_2' + x_1 &= q_1, \\ x_3' + x_2 &= q_2, \\ x_3 &= q_3, \end{aligned} \tag{173}$$

by adding the differentiated version of the derivative-free equation. We may write (172) *in the form* (8) *with* $k = 4,\ m = 3,\ n = 2,$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ D^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compute

$$G_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ G_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Q_0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ G_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$r_0 = 2,\ r_1 = 2,\ r_2 = 3.$ *It results that* (172) *has tractability index two while* (173) *has tractability index three.*
System (172) *is overdetermined, and, in our view, the subsystem* $\mathcal{W}_2 Bx = \mathcal{W}_2 q$ *(cf.* (165)*), which means here in essence* $x_3 = q_3$*, is interpreted as a consistency condition. The main part* (167) *of the DAE reads*

$$\begin{aligned} x_2' + x_1 &= q_1, \\ x_2 &= q_2 - q_3', \\ x_3' &= q_3', \end{aligned}$$

and this is obviously a regular index two DAE.

The last example adresses an interesting general phenomenon: If one adds to a given DAE the differentiated version of a certain part of the derivative-free equations, then the tractability index reduces.
There are several possibilities to choose appropriate derivative-free equations to be differentiated. Here we concentrate on the part

$$\mathcal{W}_{\mu-1} Bx = \mathcal{W}_{\mu-1} q,$$

supposing the original DAE (8) to have tractability index $\mu \geq 2$.
Considering the inclusion $N_0 \subseteq S_1 \subseteq S_{\mu-1} = \ker \mathcal{W}_{\mu-1} B$ we can write this derivative-free part as

$$\mathcal{W}_{\mu-1} BD^- Dx = \mathcal{W}_{\mu-1} q,$$

and differentiation yields

$$\mathcal{W}_{\mu-1} BD^-(Dx)' + (\mathcal{W}_{\mu-1} BD^-)' Dx = (\mathcal{W}_{\mu-1} q)'. \tag{174}$$

The enlarged DAE (8), (174) is now

$$\underbrace{\begin{bmatrix} A \\ \mathcal{W}_{\mu-1}BD^- \end{bmatrix}}_{=:\tilde{A}}(Dx)' + \underbrace{\begin{bmatrix} B \\ (\mathcal{W}_{\mu-1}BD^-)'D \end{bmatrix}}_{=:\tilde{B}} x = \begin{bmatrix} q \\ (\mathcal{W}_{\mu-1}q)' \end{bmatrix}, \tag{175}$$

with $k+m =: \tilde{k}$ equations. The DAE (175) inherits the properly stated leading term from (8) because of $\ker \tilde{A} = \ker A$.
The next proposition says that the tractability index of (175) is less by one than that of (8).

Proposition 10.8 *If the DAE* (8) *has tractability index* μ*, and characteristic values* $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m,\ \mu \geq 2$*, then the DAE* (175) *has tractability index* $\tilde{\mu} = \mu - 1$*, and characteristic values* $\tilde{r}_i = r_i,\ i = 0, \ldots, \tilde{\mu} - 1,\ \tilde{r}_{\tilde{\mu}} = \tilde{r}_{\mu-1} = m$*.*

Proof: We have $N_0 \subseteq \ker \mathcal{W}_{\mu-1}B = S_{\mu-1}$,

$$\tilde{G}_0 = \tilde{A}D = \begin{bmatrix} AD \\ \mathcal{W}_{\mu-1}BD^-D \end{bmatrix} = \begin{bmatrix} G_0 \\ \mathcal{W}_{\mu-1}B \end{bmatrix},\ \tilde{r}_0 = r_0.$$

Set $\tilde{Q}_0 = Q_0$ and form $\tilde{G}_1 = \tilde{G}_0 + \tilde{B}\tilde{Q}_0 = \begin{bmatrix} G_1 \\ \mathcal{W}_{\mu-1}B \end{bmatrix}$.
If $\mu = 2$, then $\ker \tilde{G}_1 = \ker G_1 \cap \ker \mathcal{W}_1 B = N_1 \cap S_1 = \{0\}$. Then, $\tilde{r}_1 = m,\ \tilde{r}_0 < \tilde{r}_1$, and hence the new DAE (175) has tractability index one, and we are ready.
If $\mu \geq 3$ then $\ker \tilde{G}_1 = \ker G_1 \cap \ker \mathcal{W}_{\mu-1}B = N_1$, since $N_1 \subseteq S_2 \subseteq S_{\mu-1} = \ker \mathcal{W}_{\mu-1}B$. Moreover, $\tilde{r}_1 = r_1$.
Set $\tilde{Q}_1 = Q_1$ and form

$$\begin{aligned} \tilde{B}_1 &= \begin{bmatrix} B_1 \\ (\mathcal{W}_{\mu-1}BD^-)'D - \mathcal{W}_{\mu-1}BD^-(D\Pi_1D^-)'D \end{bmatrix} = \begin{bmatrix} B_1 \\ (\mathcal{W}_{\mu-1}BD^-)'D\Pi_1 \end{bmatrix}, \\ \tilde{G}_2 &= \begin{bmatrix} G_1 + B_1Q_1 \\ \mathcal{W}_{\mu-1}B \end{bmatrix} = \begin{bmatrix} G_2 \\ \mathcal{W}_{\mu-1}B \end{bmatrix},\ \tilde{N}_2 = N_2 \cap S_{\mu-1}. \end{aligned}$$

If $\mu = 3$, then $\tilde{N}_2 = N_2 \cap S_2 = \{0\}$, and $\tilde{r}_2 = m$, i.e. $\tilde{G}_2$ is injective, and the DAE (175) has tractability index two.
For $\mu > 3$, as long as $j \leq \mu - 2$, it results that

$$\begin{aligned} \tilde{G}_j &= \begin{bmatrix} G_j \\ \mathcal{W}_{\mu-1}B \end{bmatrix},\ \tilde{N}_j = N_j \cap S_{\mu-1} = N_j,\ \tilde{Q}_j = Q_j,\ \tilde{r}_j = r_j, \\ \tilde{B}_j &= \begin{bmatrix} B_j \\ (\mathcal{W}_{\mu-1}BD^-)'D\Pi_{j-1} - \mathcal{W}_{\mu-1}BD^-(D\Pi_jD^-)'D\Pi_{j-1} \end{bmatrix} = \begin{bmatrix} B_j \\ (\mathcal{W}_{\mu-1}BD^-)'D\Pi_j \end{bmatrix}. \end{aligned}$$

Finally,

$$\tilde{G}_{\mu-1} = \begin{bmatrix} G_{\mu-1} \\ \mathcal{W}_{\mu-1}B \end{bmatrix},\ \tilde{N}_{\mu-1} = N_{\mu-1} \cap S_{\mu-1} = \{0\},\ \tilde{r}_{\mu-1} = m,$$

that is, $\tilde{G}_{\mu-1}$ is injective, and the DAE (175) has tractability index $\tilde{\mu} = \mu - 1$. $\square$

We mention that $\tilde{\mathcal{W}}_{\tilde{\mu}} = \begin{bmatrix} \mathcal{W}_{\mu-1} & \\ & I - \mathcal{W}_{\mu-1} \end{bmatrix}$ is a projector function with $\ker \tilde{\mathcal{W}}_{\tilde{\mu}} = \operatorname{im} \tilde{G}_{\tilde{\mu}}$, and now the equation $\mathcal{W}_{\mu-1}Bx = \mathcal{W}_{\mu-1}q$ is interpreted as consistency condition, while its differentiated version is included into the main part (167), as it is the case in Example 10.7.

Case 2: The DAE is tractable with index one, and G_1 has a nontrivial nullspace.

The decomposed system (164), (165) has the form

$$\begin{aligned} D^-(Dx)' + Q_0x + y + G_1^- B_0 D^- Dx &= G_1^- q && (176)\\ \mathcal{W}_1 B_0 D^- Dx &= \mathcal{W}_1 q, && (177) \end{aligned}$$

with $G_1 y = 0$, i.e. $y = (I - G_0^- B_0 Q_0)Q_0 y$, $Q_0 y \in N_0 \cap S_0$. The inherent explicit ODE is here

$$(Dx)' - R'Dx + Dy + DG_1^- BD^- Dx = DG_1^- q, \tag{178}$$

and multiplication of (176) by Q_0 gives

$$Q_0 x + Q_0 y + Q_0 G_1^- B_0 D^- Dx = Q_0 G_1^- q. \tag{179}$$

For each arbitrarily fixed continuous $Q_0 y \in N_0 \cap S_0$, equation (176) represents a regular index-one DAE.

We consider (177) as a consistency condition. If $\operatorname{im} G_1 = \mathbb{R}^k$, $m \geq k$, are true, i.e. if G_1 has full row-rank, then this condition disappears.

Example 10.9 *Set $m = m_1 + m_2 + m_3$, $k = k_1 + k_2 + k_3$, $n = m_1$, $m_1 = k_1$, $m_2 = k_2$, $k_3 \geq 0$, $m_3 \geq 0$, and consider the DAE (8) with the coefficients*

$$A = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & B_{13} \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has the detailed form

$$\begin{aligned} x_1' + B_{13}x_3 &= q_1, \\ x_2 &= q_2, \\ 0 &= q_3. \end{aligned}$$

This DAE plays its role in the strangeness index framework (e.g. [KM06]). Derive

$$G_0 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad G_1 = \begin{bmatrix} I & 0 & B_{13} \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

and $r_0 = m_1$, $\quad r_1 = m_1 + m_2$ and $\operatorname{im} G_1 = \operatorname{im}[AD \;\; B] = \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \times \{0\}$. Therefore, G_1 has maximal possible rank, and hence the problem is tractable with index 1. The consistency condition (177) means simply $0 = q_3$, if $k_3 > 0$. It disappears for $k_3 = 0$.

Moreover, here we have $N_0 = \{z \in \mathbb{R}^m : z_1 = 0\}$, $S_0 = \{z \in \mathbb{R}^m : z_2 = 0\}$, $N_0 \cap S_0 = \{z \in \mathbb{R}^m : z_1 = 0,\, z_2 = 0\}$. $G_1 y = 0$ means $y_1 + B_{13}y_3 = 0$, $y_2 = 0$. The free component $Q_0 y \in N_0 \cap S_0$ is actually y_3 (if $m_3 > 0$), so that $y_1 = -B_{13}y_3$ follows.
It results that

$$G_1^- = \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix}, \quad G_1^- B_0 D^- = 0,$$

and the equation (176) reads in detail

$$\begin{aligned} x_1' - B_{13}y_3 &= q_1, \\ x_2 &= q_2, \\ x_3 + y_3 &= 0. \end{aligned}$$

For each given function y_3, this is obviously a regular index one DAE.

The characteristic values r_i as well as the tractability index are invariant under regular scalings and transformations of the unknown function (cf. Section 5). We derive a similar result on the structure of an index-one DAE via transformations.

Proposition 10.10 *Let $m > k$, and the DAE* (8) *be tractable with index one. Then there are nonsingular matrix functions $L \in \mathcal{C}(J, L(\mathbb{R}^k))$, $L^* = L^{-1}$, $K \in \mathcal{C}(J, L(\mathbb{R}^m))$, $K^* = K^{-1}$, such that the premultiplication by L and the transformation of the unknown function $x = K\bar{x}$, $\bar{x} = \begin{array}{l} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \begin{array}{l} \} \, r_1 \\ \} \, m - r_1 \end{array} \end{array}$, lead to the equivalent DAE*

$$\bar{A}_1(\bar{D}_1\bar{x}_1)' + \bar{B}_{11}\bar{x}_1 + \bar{B}_{12}\bar{x}_2 = \bar{q}_1, \tag{180}$$

$$\bar{B}_{21}\bar{x}_1 = \bar{q}_2, \tag{181}$$

with

$$LA = \begin{bmatrix} \bar{A}_1 \\ 0 \end{bmatrix}, \quad DK = \begin{bmatrix} \bar{D}_1 & 0 \end{bmatrix}, \quad LBK = \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & 0 \end{bmatrix}, \quad Lq = \begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix} \begin{array}{l} \} \, r_1 \\ \} \, k - r_1 \end{array},$$

and equation (180) *is a regular DAE with tractability index one with respect to $\bar{x}_1$. If $r_1 = k$, i.e. if G_1 has full row-rank, then the second equation* (181) *disappears. In general, it holds that* $\ker \bar{B}_{21} \supseteq \ker \bar{D}_1$.

Proof: We choose Q_0, $\mathcal{W}_0$ to be the orthogonal projectors onto N_0 and $\operatorname{im} G_0$, and consider the matrix function

$$\mathcal{G}_1 = G_0 + \mathcal{W}_0 B Q_0,$$

that has constant rank r_1. Compute L so that

$$L\mathcal{G}_1 = \begin{bmatrix} \check{\mathcal{G}}_1 \\ 0 \end{bmatrix} \begin{array}{l} \} \, r_1 \\ \} \, k - r_1 \end{array}, \quad \operatorname{rank} \check{\mathcal{G}}_1 = r_1.$$

Then we provide a K to obtain

$$\check{\mathcal{G}}_1 K = [\underbrace{S}_{r_1} \quad \underbrace{0}_{m-r_1}], \qquad S \text{ nonsingular.}$$

This yields

$$L(G_0 + \mathcal{W}_0 B Q_0)K = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \quad L(G_0 + \mathcal{W}_0 B Q_0)K \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0,$$

and further $G_0 K \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0$, $\mathcal{W}_0 B Q_0 K \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0$, $P_0 K \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0$, $DK \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0$.
In particular, $\bar{D} := DK = [\bar{D}_1 \; 0]$ must be true, and $\operatorname{im} \bar{D}_1 = \operatorname{im} D$. Denoting $\tilde{P}_0 := \bar{D}_1^+ \bar{D}_1$,

$\tilde{Q}_0 := I - \tilde{P}_0 \in \mathcal{C}(\mathcal{I}, L(\mathbb{R}^{r_1}))$ we find $\bar{Q}_0 = K^* Q_0 K = \begin{bmatrix} \tilde{Q}_0 & 0 \\ 0 & I \end{bmatrix}$ to be the orthogonal projector onto $\ker \bar{D} = K^* \ker D$.
Next we scale the DAE (8) by L and transform $x = K\bar{x}$. Because of $\operatorname{im} A \subseteq \operatorname{im} \mathcal{G}_1$, we must have

$$\bar{A} := LA = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \begin{matrix} \} \ r_1 \\ \} \ k - r_1 \end{matrix}, \quad \ker \bar{A} = \ker A = \ker A_1.$$

From $\operatorname{im} BQ_0 \subseteq \operatorname{im} G_1 = \operatorname{im} \mathcal{G}_1$ we derive, with $\bar{B} := LBK = \begin{bmatrix} \underbrace{\begin{matrix}\bar{B}_{11} \\ \bar{B}_{21}\end{matrix}}_{r_1} & \underbrace{\begin{matrix}\bar{B}_{12} \\ \bar{B}_{22}\end{matrix}}_{m-r_1} \end{bmatrix} \begin{matrix} \} \ r_1 \\ \} \ k - r_1 \end{matrix}$, that $\operatorname{im} \bar{B}\bar{Q}_0 \subseteq \operatorname{im} L\mathcal{G}_1$, hence $\bar{B}\bar{Q}_0$ has the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$, and $\bar{B}_{21}\tilde{Q}_0 = 0$, $\ker \bar{D}_1 \subseteq \ker \bar{B}_{21}$, $\bar{B}_{22} = 0$ must hold.
It remains to show that (180) has regular index one as a DAE for x_1 in $\mathbb{R}^{r_1}$. Obviously, this DAE for x_1 has a properly stated leading term, too. If we succeed showing $\bar{A}_1\bar{D}_1 + \tilde{\mathcal{W}}_0\bar{B}_{11}\tilde{Q}_0$ to be nonsingular, where $\tilde{\mathcal{W}}_0 := I - \bar{A}_1\bar{A}_1^+$, we are done. Notice that $\bar{\mathcal{W}}_0 := L\mathcal{W}_0 L^{-1}$ is the orthoprojector onto $\operatorname{im} \bar{G}_0^\perp = \operatorname{im} \bar{A}^\perp$. Because of $\bar{A} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$, we have $\bar{\mathcal{W}}_0 = \begin{bmatrix} \tilde{\mathcal{W}}_0 & 0 \\ 0 & I \end{bmatrix}$. Derive

$$\begin{aligned} \bar{A}_1\bar{D}_1 + \tilde{\mathcal{W}}_0\bar{B}_{11}\tilde{Q}_0 &= \begin{bmatrix} I & 0 \end{bmatrix} LADK \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} \bar{\mathcal{W}}_0 LBK\bar{Q}_0 \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \end{bmatrix} L(AD + \mathcal{W}_0 BQ_0)K \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = S, \end{aligned}$$

and S is nonsingular. □

Case 3: The DAE is tractable with index two, and G_2 has a nontrivial nullspace

The decomposed system (164), (165) is now

$$\begin{aligned} P_1 D^-(D\Pi_1 x)' + Q_1 x + Q_0 x - Q_0 Q_1 D^-(D\Pi_0 Q_1 x)' + \mathcal{V}_0 Dx + \mathcal{U}_0 (Dx)' & \\ + G_2^- B_1 D^- D\Pi_1 x + y = G_2^- q, & \qquad (182) \\ \mathcal{W}_2 B_1 D^- D\Pi_1 x = \mathcal{W}_2 q, & \qquad (183) \end{aligned}$$

with coefficients (cf. Proposition 4.1)

$$\begin{aligned} \mathcal{U}_0 &= -Q_0\{Q_0 + Q_1 Q_0 Q_1 P_0\}\Pi_0 D^- = -Q_0 Q_1 Q_0 Q_1 D^-, \\ \mathcal{V}_0 &= Q_0\{(P_0 + Q_1 Q_0)D^- R' - Q_1 D^-(D\Pi_1 D^-)'\}DD^- = -Q_0 Q_1 D^-(D\Pi_1 D^-)' DD^- \end{aligned}$$

and an arbitrary continuous function such that

$$G_2 y = 0. \qquad (184)$$

We multiply (182) by $D\Pi_1$, Q_1 and Q_0P_1, and obtain the system

$$(D\Pi_1x)' - (D\Pi_1D^-)'D\Pi_1x + D\Pi_1G_2^-B_1D^-D\Pi_1x + D\Pi_1y = D\Pi_1G_2^-q, \quad (185)$$

$$\begin{aligned} Q_1x + Q_1Q_0x - Q_1Q_0Q_1D^-(D\Pi_0Q_1x)' + Q_1\mathcal{V}_0Dx + Q_1\mathcal{U}_0(Dx)' \\ +Q_1G_2^-B_1D^-D\Pi_1x + Q_1y = Q_1G_2^-q, \quad (186) \end{aligned}$$

$$\begin{aligned} Q_0P_1Q_0x + Q_0P_1D^-(D\Pi_1x)' - Q_0P_1Q_0Q_1D^-(D\Pi_0Q_1x)' + Q_0P_1\mathcal{V}_0Dx \\ +Q_0P_1\mathcal{U}_0(Dx)' + Q_0P_1G_2^-B_1D^-D\Pi_1x + Q_0P_1y = Q_0P_1G_2^-q, \quad (187) \end{aligned}$$

which is a decomposed version of (182) due to $\Pi_0 + Q_0P_1 + Q_1 = I$, $\Pi_0 = D^-D\Pi_0$. Multiplying equation (186) by Π_0 and taking into account the property $\Pi_0Q_1Q_0 = 0$ we derive

$$\Pi_0Q_1x + \Pi_0Q_1G_2^-B_1D^-D\Pi_1x + \Pi_0Q_1y = \Pi_0Q_1G_2^-q. \quad (188)$$

Now it is evident that, for given y, and the initial condition

$$D(t_0)\Pi_1(t_0)x(t_0) = z_0 \in \operatorname{im} D(t_0)\Pi_1(t_0), \quad (189)$$

there is exactly one solution of the explicit ODE (185), that is, the solution component $\Pi_0x = D^-D\Pi_0x$ of the IVP for the DAE is uniquely determined. Having $D\Pi_1x$, we obtain the next component Π_0Q_1x from (188), and thus $Dx = D\Pi_1x + D\Pi_0Q_1x$. Then, formula (187) provides an expression for $Q_0P_1Q_0x$ in terms of the previous ones. Finally, multiplying (186) by Q_0 we find an expression $Q_0Q_1x + Q_0Q_1Q_0x = E$ with E depending on the already given terms y, $D\Pi_0Q_1x$, $D\Pi_1x$, Dx. In turn, this yields an expression for $Q_0Q_1Q_0x$, and then for $Q_0x = Q_0Q_1Q_0x + Q_0P_1Q_0$.
In summary, to each function y that satisfies condition (184), the system (185) – (187), completed by the initial condition (189), determines a unique solution $x = D^-D\Pi_1x + \Pi_0Q_1x + Q_0x$ of the DAE.
With regard of the discussion above (cf. (166)) the actual arbitrary part of y is $Q_1y \in N_1 \cap S_1$.
We mention that, for solvability, the component $D\Pi_0Q_1x$ must be continuously differentiable. Equation (188) shows the terms being responsible for that. For instance, if $\Pi_0Q_1G_2^-B_1D^-$ is a continuously differentiable matrix function, then the difference $D\Pi_0Q_1(G_2^-q - y)$ must be continuously differentiable.

Example 10.11 *Set $k = 3$, $m = 4$, $n = 2$, and consider the DAE (8) given by the coefficients*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This DAE reads in detail

$$\begin{aligned} x_1' + x_1 &= q_1, \\ x_2' + x_3 + x_4 &= q_2, \\ x_2 &= q_3. \end{aligned} \quad (190)$$

We provide the sequence

$$G_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad G_2^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{W}_2 = 0.$$

Thereby, the projector Q_1 satisfies the admissibility condition $X_1 \subset \ker Q_1$ with $X_1 := \{z \in \mathbb{R}^4 : z_1 = 0, z_2 = 0, z_3 = 0\}$ and $N_0 = (N_0 \cap N_1) \oplus X_1$. G_2 has maximal rank, $r_2 = k = 3$, thus the DAE is tractable with index two. The consistency condition (183) disappears. Compute further $\mathcal{V}_l = 0$ and $\mathcal{U}_l = 0$, so that the equation (182) simplifies to

$$P_1 D^-(D\Pi_1 x)' + Q_1 x + Q_0 x - Q_0 Q_1 D^-(D\Pi_0 Q_1 x)' + G_2^- B_1 D^- D\Pi_1 x + y = G_2^- q,$$

with

$$P_1 D^- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_0 Q_1 D^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad G_2^- B_1 D^- D\Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Taking into account that $G_2 y = 0$ is equivalent to $y_1 = 0,\ y_2 = 0,\ y_4 = -y_3,$, we find the equation (182) to be in detail:

$$\begin{aligned} x_1' + x_1 &= q_1, \\ x_2 &= q_3. \\ 2x_3 + y_3 &= q_2 - q_3, \\ x_4 - x_3 - x_2 + x_2' - y_3 &= 0. \end{aligned}$$

To each function y_3, this is a regular DAE with tractability index two. Its solutions are the solutions of the original DAE.

Linear Algebra – Basics

In this appendix we collect and complete well-known facts concerning projectors and subspaces of $\mathbb{R}^m$ (Section A), and generalized inverses (Section B). Section C provides material on matrix and projector valued functions with proofs, since these proofs are not easily available. In Section D we introduce $\mathcal{C}^k$-subspaces of $\mathbb{R}^m$ via $\mathcal{C}^k$-projector functions. We show $\mathcal{C}^k$-subspaces to be those which have local $\mathcal{C}^k$ basises.

A Projectors and subspaces

We collect some basic and useful properties of projectors and subspaces.

Definition A.1 (1) *A linear mapping $Q \in L(\mathbb{R}^m)$ is called a* projector, *if $Q^2 = Q$.*

(2) *A projector $Q \in L(\mathbb{R}^m)$ is called a projector* onto *$S \subseteq \mathbb{R}^m$ if $\operatorname{im} Q = S$.*

(3) *A projector $Q \in L(\mathbb{R}^m)$ is called a projector* along *$S \subseteq \mathbb{R}^m$ if $\ker Q = S$.*

(4) *A projector $Q \in L(\mathbb{R}^m)$ is called an* orthogonal *projector if $Q = Q^*$.*

Example: The m-dimensional matrix $Q = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{bmatrix}$ with arbitrary entries for $*$ becomes a projector onto the one-dimensional subspace spanned by the first column of Q along the $(m-1)$-dimensional subspace $\{v : v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, v_1 = 0\}$.

Lemma A.2 *Let P and $\bar{P}$ be projectors, and $Q := I - P$, $\bar{Q} := I - \bar{P}$ the complementary projectors . Then the following properties hold:*

(1) *$z \in \operatorname{im} Q \;\Leftrightarrow\; z = Qz$.*

(2) *If Q and $\bar{Q}$ project onto the same subspace S, then $\bar{Q} = Q\bar{Q}$ and $Q = \bar{Q}Q$ are valid.*

(3) *If P and $\bar{P}$ project along the same subspace S, then $\bar{P} = \bar{P}P$ and $P = P\bar{P}$ are true.*

(4) *Q projects onto S iff $P := I - Q$ projects along S.*

(5) *Each matrix of the form $I + PZQ$, with arbitrary matrix Z, is nonsingular and its inverse is $I - PZQ$.*

(6) *Each projector P is diagonalizable. Its eigenvalues are 0 and 1. The multiplicity of the eigenvalue 1 is $r = \operatorname{rank} P$.*

Proof:

1. $z = Qy \quad \to \quad Qz = Q^2y = Qy = z.$

2. $\bar{Q}z \in \operatorname{im}\bar{Q} = S = \operatorname{im}Q,$ also $\bar{Q}z = Q\bar{Q}z\ \forall z.$

3. $\bar{P}P = (I - \bar{Q})(I - Q) = I - \bar{Q} - Q + \bar{Q}Q = I - \bar{Q} = \bar{P}.$

4. $P^2 = P \quad \Leftrightarrow \quad (I - Q)^2 = I - Q \quad \Leftrightarrow \quad -Q + Q^2 = 0 \quad \Leftrightarrow \quad Q^2 = Q$ and $z \in \ker P \ \Leftrightarrow\ Pz = 0 \Leftrightarrow\ z = Qz \ \Leftrightarrow\ z \in \operatorname{im}Q.$

5. Multiplying $(I + PZQ)z = 0$ by $Q \Rightarrow Qz = 0$. Now with $(I + PZQ)z = 0$ follows $z = 0$.
$(I + PZQ)(I - PZQ) = I - PZQ + PZQ = I.$

6. Let $\bar{P}_1$ be a matrix of the r linearly independent columns of P and $\bar{Q}_2$ a matrix of the $m - r$ linearly independent columns of $I - P$. Then by construction $P\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix} = \begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}\begin{bmatrix}I & \\ & 0\end{bmatrix}$. Because of the nonsingularity of $\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}$ we have the structure $P = \begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}\begin{bmatrix}I & \\ & 0\end{bmatrix}\begin{bmatrix}\bar{P}_1 & \bar{Q}_2\end{bmatrix}^{-1}$. The columns of $\bar{P}_1$ resp. $\bar{Q}_2$ are the eigenvectors to the eigenvalues 1 resp. 0. □

Lemma A.3 *Let $A \in L(\mathbb{R}^n, \mathbb{R}^k)$, $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, $r := \operatorname{rank}(AD)$. Then the following two implications are valid:*

(1) $\ker A \cap \operatorname{im} D = 0,\ \operatorname{im}(AD) = \operatorname{im} A \Rightarrow \ker A \oplus \operatorname{im} D = \mathbb{R}^n.$

(2) $\ker A \oplus \operatorname{im} D = \mathbb{R}^n \Rightarrow$

- $\ker A \cap \operatorname{im} D = \{0\},$
- $\operatorname{im} AD = \operatorname{im} A,$
- $\ker AD = \ker D,$
- $\operatorname{rank} A = \operatorname{rank} D = r.$

Proof: (1) Because of $\operatorname{im}(AD) = \operatorname{im} A$, the matrix A has rank r and $\ker A$ has dimension $n - r$. Moreover, $\operatorname{rank} D \geq r$ must be true. The direct sum $\ker A \oplus \operatorname{im} D$ is well-defined, and it has dimension $n - r + \operatorname{rank} D \leq n$. This means that D has rank r. We are done with (1).

(2) The first relation is an inherent property of the direct sum. Let $R \in L(\mathbb{R}^n)$ denote the projector onto $\operatorname{im} D$ along $\ker A$. By means of suitable generalized inverses D^- and A^- of D and A we may write (Appendix B) $R = A^-A = DD^-$, $D = RD$, $A = AR$. This leads to

$$\begin{aligned} &\operatorname{im} AD \subseteq \operatorname{im} A = \operatorname{im} ADD^- \subseteq \operatorname{im} AD, \\ &\ker AD \subseteq \ker A^-AD = \ker D \subseteq \ker AD. \end{aligned}$$

The remaining rank property follows now from (1). □

Lemma A.4 *[GvL91, Ch. 12.4.2] Given are matrices G, Π, $\mathcal{N}$, $\mathcal{W}$ of suitable sizes such that*

$$\ker G = \operatorname{im} \mathcal{N},$$
$$\ker \Pi\mathcal{N} = \operatorname{im} \mathcal{W}.$$

Then it holds that

$$\ker G \cap \ker \Pi = \ker \mathcal{N}\mathcal{W}.$$

Proof: For $x \in \ker G \cap \ker \Pi$ we find $x = \mathcal{N}y, \Pi x = 0$, further $\Pi\mathcal{N}y = 0$, and hence $y = \mathcal{W}z$, $x = \mathcal{N}\mathcal{W}z \in \operatorname{im} \mathcal{N}\mathcal{W}$.
Conversely, each $x = \mathcal{N}\mathcal{W}z$ belongs obviously to $\ker G$, and $\Pi x = \Pi\mathcal{N}\mathcal{W}z = 0$. □

Lemma A.5 *$N, M \subseteq \mathbb{R}^m$ subspaces $\Rightarrow (N + M)^\perp = N^\perp \cap M^\perp$.*

Proof:

$$\begin{aligned}
(N + M)^\perp &= \{z \in \mathbb{R}^m : \forall w \in N + M : \langle z, w\rangle = 0\} \\
&= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N + w_M\rangle = 0\} \\
&= \{z \in \mathbb{R}^m : \forall w_N \in N, \forall w_M \in M : \langle z, w_N\rangle = 0,\ \langle z, w_M\rangle = 0\} \\
&= N^\perp \cap M^\perp.
\end{aligned}$$

□

Lemma A.6 (1) *Given two subspaces $N, X \subseteq \mathbb{R}^m$, $N \cap X = \{0\}$. Then $\dim N + \dim X \leq m$, and there is a projector $Q \in L(\mathbb{R}^m)$ such that $\operatorname{im} Q = N$, $\ker Q \supseteq X$.*

(2) *Given two subspaces $S, N \subseteq \mathbb{R}^m$. If the decomposition*

$$\mathbb{R}^m = S \oplus N$$

holds true, i.e. S and N are transversal , then there is a uniquely determined projector $P \in L(\mathbb{R}^m)$ such that $\operatorname{im} P = S$, $\ker P = N$.

(3) *An orthoprojector P projects onto $S := \operatorname{im} P$ along $S^\perp = \ker P$.*

(4) *Given the subspaces $K, N \subseteq \mathbb{R}^m$, $\widehat{N} := N \cap K$. If a further subspace $X \subseteq \mathbb{R}^m$ is a complement of $\widehat{N}$ in K, that means $K = \widehat{N} \oplus X$, then there is a projector $Q \in L(\mathbb{R}^m)$ onto N such that*

$$X \subseteq \ker Q. \tag{191}$$

Let d_K, d_N, u denote the dimensions of the subspaces $K, N, \widehat{N}$, respectively, then

$$d_K + d_N \leq m + u \tag{192}$$

holds.

(5) *If the subspace K in (4) is the nullspace of a certain projector $\Pi \in L(\mathbb{R}^m)$, that is $K = \ker \Pi = \operatorname{im}(I - \Pi)$, then*

$$\Pi Q(I-\Pi) = 0 \tag{193}$$

becomes true.

(6) *Given are the two projectors $\Pi, Q \in L(\mathbb{R}^m)$, further $P := I - Q$, $N := \operatorname{im} Q$, $K := \ker \Pi$. Then, supposed* (193) *is valid, the products ΠP, ΠQ, $P\Pi P$, $P(I-\Pi)$, $Q(I-\Pi)$ are projectors, too. The relation*

$$\ker \Pi P = \ker P\Pi P = N + K \tag{194}$$

holds true, and the subspace $X := \operatorname{im} P(I-\Pi)$ is the complement of $\widehat{N} := N \cap K$ in K, such that $K = \widehat{N} \oplus X$.
Moreover, the decomposition

$$\mathbb{R}^m = (N+K) \oplus \operatorname{im} P\Pi P = N \oplus \underbrace{X \oplus \operatorname{im} P\Pi P}_{\operatorname{im} P}$$

is valid.

(7) *If the projectors Π, Q in (6) are such that $\Pi^* = \Pi$, $(\Pi P)^* = \Pi P$, $(P(I-\Pi))^* = P(I-\Pi)$ and $Q\Pi P = 0$, then it follows that*

$$X = K \cap \widehat{N}^{\perp},\ \operatorname{im} P = X \oplus (N+K)^{\perp}.$$

Proof: (1): Let $x_1, \dots, x_r \in \mathbb{R}^m$ and $n_1, \dots, n_t \in \mathbb{R}^m$ be basises of X and N. Because of $X \cap N = \{0\}$ the matrix

$$F := [x_1 \dots x_r n_1 \dots n_t]$$

has full column rank and $r + t = \dim X + \dim N \leq m$. The matrix F^*F is invertible, and

$$Q := F \begin{array}{c} \begin{bmatrix} 0 & \\ & I \end{bmatrix} \\ r \quad t \end{array} (F^*F)^{-1}F^*$$

is a projector we looked for. Namely,

$$Q^2 = F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1}F^*F \begin{bmatrix} 0 & \\ & I \end{bmatrix} (F^*F)^{-1}F^* = Q, \quad \operatorname{im} Q = \operatorname{im} F \begin{bmatrix} 0 & \\ & I \end{bmatrix} = N,$$

and $z \in X$ implies that it has to have the structure $z = F \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \begin{matrix} \}r \\ \}t \end{matrix}$, which leads to $Qz = 0$.
(2): For transversal subspaces S and N we apply Assertion (1) with $t = m - r$, i.e. F is square. We have to show that P is unique. Supposed that there are two projectors P, $\bar{P}$ such that $\ker P = \ker \bar{P} = N$, $\operatorname{im} P = \operatorname{im} \bar{P} = S$, we immediately have $P = (\bar{P} + \bar{Q})P = \bar{P}P + \bar{Q}P = \bar{P}P = \bar{P}$.
(3): Let $S := \operatorname{im} P$ and $N := \ker P$. We choose a $v \in N$ and $y \in S$. Lemma A.2 (1) implies $y = Py$, therefore $\langle v, y\rangle = \langle v, Py\rangle = \langle P^*v, y\rangle$. With the symmetry of P we obtain

$\langle P^*v, y\rangle = \langle Pv, y\rangle = 0$, i.e. $N = S^\perp$.
(4): X has dimension $d_K - u$. Since the sum space $K + N = X \oplus N \subseteq \mathbb{R}^m$ may have at most dimension m, it results that $\dim(K + N) = \dim X + \dim N = d_K - u + d_N \leq m$, and assertion (1) provides Q.
(5): Take an arbitrary $z \in \operatorname{im}(I - \Pi) = K$ and decompose $z = z_{\widehat{N}} + z_X$. It follows that $\Pi Qz = \Pi Qz_{\widehat{N}} + \Pi \underbrace{Qz_X}_{=0} = \Pi z_{\widehat{N}} = 0$, and hence (193) is true.
(6): (193) means $\Pi Q = \Pi Q \Pi$ and hence

$$\begin{aligned}
&\Pi Q \Pi Q = \Pi QQ = \Pi Q,\\
&\Pi P \Pi P = \Pi(I - Q)\Pi P = \Pi P - \Pi \underbrace{Q \Pi P}_{=0} = \Pi P,
\end{aligned}$$

$$\begin{aligned}
(P\Pi P)^2 &= P\Pi P \Pi P = P\Pi P,\\
(P(I - \Pi))^2 &= P(I - \Pi)(I - Q)(I - \Pi) = P(I - \Pi) - P(I - \Pi)Q(I - \Pi)\\
&= P(I - \Pi) + P \underbrace{\Pi Q(I - \Pi)}_{=0},\\
(Q(I - \Pi))^2 &= Q(I - \Pi) - Q\Pi Q(I - \Pi) = Q(I - \Pi).
\end{aligned}$$

The representation $I - \Pi = Q(I - \Pi) + P(I - \Pi)$ corresponds to the decomposition $K = \widehat{N} \oplus X$.
Next we verify (194). The inclusion $\ker \Pi P \subseteq \ker P\Pi P$ is trivial. On the other side, $P\Pi Pz = 0$ implies $\Pi P\Pi Pz = 0$ and hence $\Pi Pz = 0$, and it follows $\ker \Pi P = \ker P\Pi P$. Now it is evident that $K + N \subseteq \ker \Pi P$. Finally, $\Pi Pz = 0$ implies $Pz \in K, z = Qz + Pz \in N + K$.
(7): From $Q\Pi P = 0$ and the symmetry of ΠP we know that $P\Pi P = \Pi P$, $\operatorname{im} P\Pi P = (N + K)^\perp$, $\operatorname{im} P = X \oplus (N + K)^\perp$. Next using Lemma A.5, compute $\widehat{N}^\perp = N^\perp + K^\perp$, and further

$$\begin{aligned}
K \cap \widehat{N}^\perp = K \cap (N^\perp + K^\perp) &= \{z \in \mathbb{R}^m : \Pi z = 0, z = z_{N^\perp} + z_{K^\perp}, z_{N^\perp} \in N^\perp, z_{K^\perp} \in K^\perp\}\\
&= \{z \in \mathbb{R}^m : z = (I - \Pi)z_{N^\perp}, z_{N^\perp} \in N^\perp\} = (I - \Pi)N^\perp\\
&= \operatorname{im}(I - \Pi)P^* = \operatorname{im}(P(I - \Pi))^* = \operatorname{im} P(I - \Pi) = X.
\end{aligned}$$

□

Lemma A.7 *Let $D \in L(\mathbb{R}^m, \mathbb{R}^n)$ be given, $M \subseteq \mathbb{R}^m$ be a subspace. $D^+ \in L(\mathbb{R}^n, \mathbb{R}^m)$ be the Moore-Penrose inverse of D. Then,*

(1) $\ker D^* = \operatorname{im} D^\perp$, $\operatorname{im} D = \ker D^{*\perp}$, $\ker D = \ker D^{+*}$, $\operatorname{im} D = \operatorname{im} D^{+*}$.

(2) $\ker D \subseteq M \Rightarrow (DM)^\perp = (\operatorname{im} D)^\perp \oplus D^{+*}M^\perp$.

(3) $\ker D \subseteq M \Rightarrow M^\perp = D^*(DM)^\perp$.

Proof: (1) The first two identities are shown in [BIG03] (Theorem 1, p.12).
If $z \in \ker D = \operatorname{im} I - D^+D$ with Lemma A.2(1) it is valid that $z = (I - D^+D)z$ or $D^+Dz = 0$. With (201) it holds $0 = D^+Dz = (D^+D)^*z = D^*D^{+*}z \Leftrightarrow D^{+*}z = 0$ because

of (198) for D^* and we have that $z \in \ker D^{+*}$. We prove $\operatorname{im} D = \operatorname{im} D^{+*}$ analogously.
(2) Let $T \in L(\mathbb{R}^m)$ be the orthoprojector onto M, i.e. $\operatorname{im} T = M$, $\ker T = M^\perp$, $T^* = T$.
$\Rightarrow DM = \operatorname{im} DT$,

$$\begin{aligned}(DM)^\perp &= (\operatorname{im} DT)^\perp = \ker (DT)^* = \ker TD^* = \{z \in \mathbb{R}^n : D^*z \in M^\perp\} \\ &= \underbrace{\ker D^*}_{=\operatorname{im} D^\perp} \oplus \{v \in \operatorname{im} D : D^*v \in M^\perp\}.\end{aligned}$$

It remains to show that

$$\{v \in \operatorname{im} D : D^*v \in M^\perp\} = D^{+*}M^\perp.$$

From $v \in \operatorname{im} D = \operatorname{im} DD^+$ we get with Lemma A.2(1) $v = DD^+v = (DD^+)^*v = D^{+*}D^*v$. Because of $D^*v \in M^\perp$ it holds $v \in D^{+*}M^\perp$. Conversely with Lemma A.2(4), $u \in D^{+*}M^\perp = \operatorname{im} D^{+*}(I-T)$ implies $u \in \operatorname{im} D^{+*} = \operatorname{im} D$, and $\exists w : u = D^{+*}(I-T)w$, $D^*u = D^*D^{+*}(I-T)w = D^+D(I-T)w$. Since $\operatorname{im}(I-T) = M^\perp \subseteq \ker D^\perp = \ker D^+D^\perp = \operatorname{im}(D^+D)^* = \operatorname{im} D^+D$, it holds that $D^+D(I-T) = I-T$, hence $D^*u = (I-T)w \in M^\perp$.

(3) This is a consequence of (2), because of

$$D^*(DM)^\perp = D^*[(\operatorname{im} D)^\perp \oplus D^{+*}M^\perp] = D^*D^{+*}M^\perp = D^+DM^\perp = M^\perp.$$

□

Lemma A.8 *([GM86], AppendixA, Theorem 13)*
Let $A, B \in L(\mathbb{R}^m)$, $\operatorname{rank} A = r < m$, $N := \ker A$, $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\}$. The following statements are equivalent:

(1) *Multiplication by a nonsingular $E \in L(\mathbb{R}^m)$ such that*

$$EA = \begin{bmatrix} \bar{A}_1 \\ 0 \end{bmatrix}, \quad EB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \operatorname{rank} \bar{A}_1 = r,$$

yields a nonsingular $\begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix}$.

(2) $N \cap S = \{0\}$.

(3) *$A + BQ$ is nonsingular for each projector Q onto N.*

(4) $N \oplus S = \mathbb{R}^m$.

(5) *The pair $\{A, B\}$ is regular with Kronecker index one.*

(6) *The pair $\{A, B + AW\}$ is regular with Kronecker index one for each arbitrary $W \in L(\mathbb{R}^m)$.*

Proof: (1) $\Rightarrow$ (2): With $\bar{N} := \ker \bar{A}_1 = \ker EA = \ker A = N$,

$$\bar{S} := \ker \bar{B}_2 = \{z \in \mathbb{R}^m : EBz \in \operatorname{im} EB\} = S,$$

we have

$$0 = \ker \begin{bmatrix} \bar{A}_1 \\ \bar{B}_2 \end{bmatrix} = \bar{N} \cap \bar{S} = N \cap S.$$

(2) $\Rightarrow$ (3): $(A+BQ)z = 0$ implies $BQz = -Az$, that is $Qz \in N \cap S$, thus $Qz = 0$, $Az = 0$, therefore $z = Qz = 0$.

(3) $\Rightarrow$ (4): Fix any projector $Q \in L(\mathbb{R}^m)$ onto N and introduce $Q_* := Q(A+BQ)^{-1}B$. We show Q_* to be a projector with $\operatorname{im} Q_* = N$, $\ker Q_* = S$ so that the assertion follows. Compute

$$Q_*Q = Q(A+BQ)^{-1}BQ = Q(A+BQ)^{-1}(A+BQ)Q = Q,$$

hence $Q_*^2 = Q_*$, $\operatorname{im} Q_* = N$. Further, $Q_*z = 0$ implies $(A+BQ)^{-1}Bz = (I-Q)(A+BQ)^{-1}Bz$, thus

$$Bz = (A+BQ)(I-Q)(A+BQ)^{-1}Bz = A(A+BQ)^{-1}Bz,$$

that is, $z \in S$. Conversely, $z \in S$ leads to $Bz = Aw$ and

$$Q_*z = Q(A+BQ)^{-1}Bz = Q(A+BQ)^{-1}Aw = Q(A+BQ)^{-1}(A+BQ)(I-Q)w = 0.$$

This proves the relation $\ker Q_* = S$.

(4) $\Rightarrow$ (5): Let Q_* denote the projector onto N along S, $P_* := I - Q_*$. Since $N \cap S = 0$ we know already that $G_* := A + BQ_*$ is nonsingular as well as the representation $Q_* = Q_*G_*^{-1}B$. It holds that

$$\begin{aligned} G_*^{-1}A &= G_*^{-1}(A+BQ_*)P_* = P_*, \\ G_*^{-1}B &= G_*^{-1}BQ_* + G_*^{-1}BP_* = G_*^{-1}(A+BQ_*)Q_* + G_*^{-1}BP_* = Q_* + G_*^{-1}BP_*. \end{aligned}$$

Consider the equation $(\lambda A + B)z = 0$, or the equivalent one $(\lambda G_*^{-1}A + G_*^{-1}B)z = 0$, i.e.

$$(\lambda P_* + G_*^{-1}BP_* + Q_*)z = 0. \tag{195}$$

Multiplying (195) by Q_* and taking into account that $Q_*G_*^{-1}BP_* = Q_*P_* = 0$ we find $Q_*z = 0$, $z = P_*z$. Now (195) writes

$$(\lambda I + G_*^{-1}B)z = 0.$$

If λ does not belong to the spectrum of the matrix $-G_*^{-1}B$, then it follows that $z = 0$. This means, $\lambda A + B$ is nonsingular except for a finite number of values λ, hence the pair $\{A, B\}$ is regular.

Transform $\{A, B\}$ into Weierstraß-Kronecker canonical form (cf. [LMT11b, Section 1]):

$$\bar{A} := EAF = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \quad \bar{B} := EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}, \quad J^\mu = 0,\ J^{\mu-1} \neq 0.$$

We derive further

$$\begin{aligned} &\bar{N} := \ker \bar{A} = F^{-1}\ker A, \quad \bar{S} := \{z \in \mathbb{R}^m : \bar{B}z \in \operatorname{im} \bar{A}\} = F^{-1}S, \\ &\bar{N} \cap \bar{S} = F^{-1}(N \cap S) = \{0\}, \text{ and} \\ &\bar{N} \cap \bar{S} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^m : z_1 = 0,\ Jz_2 = 0,\ z_2 \in \operatorname{im} J \right\}. \end{aligned}$$

Now it follows that $J = 0$ must be true since otherwise $\bar{N} \cap \bar{S}$ would be nontrivial.

(5) $\Rightarrow$ (1): This follows from $\bar{A} = EAF = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{B} = EBF = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$, $\bar{N} \cap \bar{S} = 0$ and $\bar{N} \cap \bar{S} = F^{-1}(N \cap S) = \{0\}$.

(6) $\Rightarrow$ (5) is trivial.

(2) $\Rightarrow$ (6): Set $\tilde{B} := B + AW, \tilde{S} := \{z \in \mathbb{R}^m : \tilde{B}z \in \operatorname{im} A\} = S$. Because of $\tilde{S} \cap N = S \cap N = \{0\}$, and the equivalence of assertion (2) and (5), which is proved already, the pair $\{A, \tilde{B}\}$ is regular with Kronecker index 1. $\square$

Lemma A.9 *Let $A, B \in L(\mathbb{R}^m)$ be given, A singular, $N := \ker A$, $S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\}$, and $N \oplus S = \mathbb{R}^m$. Then the projector Q onto N along S satisfies the relation*

$$Q = Q(A + BQ)^{-1}B. \tag{196}$$

Proof: First we notice that Q is uniquely determined. $A + BQ$ is nonsingular due to Lemma A.8. The arguments used in that lemma apply to show $Q(A+BQ)^{-1}B$ to be the projector onto N along S so that (196) becomes valid. $\square$

For any matrix $A \in L(\mathbb{R}^m)$ there exists an integer k such that

$$\begin{aligned} \mathbb{R}^m &= \operatorname{im} A^0 \supset \operatorname{im} A \supset \ldots \supset \operatorname{im} A^k = \operatorname{im} A^{k+1} = \ldots, \\ \{0\} &= \ker A^0 \subset \ker A \subset \ldots \subset \ker A^k = \ker A^{k+1} = \ldots, \end{aligned}$$

and $\operatorname{im} A^k \oplus \ker A^k = \mathbb{R}^m$. This integer $k \in \mathbb{N} \cup \{0\}$ is said to be the <u>index of A</u> , and we write $k = \operatorname{ind} A$.

Lemma A.10 *([GM86], Appendix A, Theorem 4)*
Let $A \in L(\mathbb{R}^m)$ be given, $k = \operatorname{ind} A$, $r = \operatorname{rank} A^k$, and let $s_1, \ldots, s_r \in \mathbb{R}^m$ and $s_{r+1}, \ldots, s_m \in \mathbb{R}^m$ be basises of $\operatorname{im} A^k$ and $\ker A^k$, respectively. Then, for $S = [s_1 \ldots s_m]$ the product $S^{-1}AS$ has the special structure

$$S^{-1}AS = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

where $M \in L(\mathbb{R}^r)$ is nonsingular and $N \in L(\mathbb{R}^{m-r})$ is nilpotent, $N^k = 0$, $N^{k-1} \neq 0$.

Proof: For $i \leq r$, it holds that $As_i \in A \operatorname{im} A^k = \operatorname{im} A^{k+1} = \operatorname{im} A^k$, therefore $As_i = \sum\limits_{j=1}^{r} s_j m_{ji}$. For $i \geq r+1$, it holds that $As_i \in \ker A^{k+1} = \ker A^k$, thus $As_i = \sum\limits_{j=r+1}^{m} s_j n_{ji}$. This yields the representations $A[s_1 \ldots s_r] = [s_1 \ldots s_r]M$ with $M = (m_{ij})_{i,j=1}^r$, and $A[s_{r+1} \ldots s_m] = [s_{r+1} \ldots s_m]N$, with $N = (n_{ij})_{i,j=r+1}^m$. The block M is nonsingular. Namely, for a $z \in \mathbb{R}^r$ with $Mz = 0$, we have $A[s_1 \ldots s_r]z = 0$, that is,

$$\sum_{j=1}^{r} z_j s_j \in \operatorname{im} A^k \cap \ker A \subseteq \operatorname{im} A^k \cap \ker A^k = \{0\},$$

which shows the matrix M to be nonsingular. It remains to verify the nilpotency of N. We have $AS = S \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$, hence $A^\ell S = S \begin{bmatrix} M^\ell & 0 \\ 0 & N^\ell \end{bmatrix}$. From $A^k s_i = 0,\ i \geq r+1$ it follows that $N^k = 0$ must be valid. It remains to prove the fact that $N^{k-1} \neq 0$. Since $\ker A^{k-1}$ is a proper subspace of $\ker A^k$ there is an index $i_* \geq r+1$ such that the basis element $s_{i_*} \in \ker A^k$ does not belong to $\ker A^{k-1}$. Then, $S \begin{bmatrix} M^{k-1} & 0 \\ 0 & N^{k-1} \end{bmatrix} e_{i_*} = A^{k-1} s_{i_*} \neq 0$, that is, $N^{k-1} \neq 0$. $\square$

B Generalized inverses

In [BIG03] we find a detailed collection of properties of generalized inverses for theory and application. We will here report the definitions and relations of generalized inverses we need for our considerations.

Definition B.1 *For a matrix $Z \in L(\mathbb{R}^n, \mathbb{R}^m)$, we call the matrix $Z^- \in L(\mathbb{R}^m, \mathbb{R}^n)$ a* *<u>reflexive generalized inverse</u>, if it fulfills*

$$\begin{aligned} ZZ^-Z &= Z \quad \textit{and} && (197) \\ Z^-ZZ^- &= Z^-. && (198) \end{aligned}$$

Z^- is called a $\{1,2\}$-inverse of Z in [BIG03].

The products $ZZ^- \in L(\mathbb{R}^m)$ and $Z^-Z \in L(\mathbb{R}^n)$ are projectors (cf. Appendix A). We have $(ZZ^-)^2 = ZZ^-ZZ^- = ZZ^-$ and $(Z^-Z)^2 = Z^-ZZ^-Z = Z^-Z$. We know that the rank of a product of matrices does not exceed the rank of any factor. Let Z has rank r_z. From (197) we obtain rank $r_z \leq$ rank r_{z^-} and from (198) the opposite, i.e. that both Z and Z^- and also the projectors ZZ^- and Z^-Z have the same rank.
Let $R \in L(\mathbb{R}^n)$ be any projector onto $\operatorname{im} Z$ and $P \in L(\mathbb{R}^m)$ any projector along $\ker Z$.

Lemma B.2 *With (197), (198) and the conditions*

$$\begin{aligned} Z^-Z &= P \quad \textit{and} && (199) \\ ZZ^- &= R && (200) \end{aligned}$$

the reflexive inverse Z^- is uniquely determined.

Proof: Let Y be a further matrix fulfilling (197), (198), (199) and (200).

$$\begin{aligned} Y &\overset{(198)}{=} YZY \overset{(197)}{=} YZZ^-ZY \overset{(200)}{=} YRZY \\ &\overset{(200)}{=} YR \overset{(200)}{=} YZZ^- \overset{(199)}{=} PZ^- \overset{(198)}{=} Z^-. \end{aligned}$$

□

If we choose for the projectors P and R the orthogonal ones the conditions (199) and (200) could be replaced by

$$\begin{aligned} Z^-Z &= (Z^-Z)^*, && (201) \\ ZZ^- &= (ZZ^-)^*. && (202) \end{aligned}$$

The resulting generalized inverse is called the Moore-Penrose-inverse and denoted by Z^+.

To represent the generalized reflexive inverse Z^- we want to use a decomposition of

$$Z = U \begin{bmatrix} S & \\ & 0 \end{bmatrix} V^{-1}$$

with nonsingular matrices U, V and S. Such a decomposition is e.g. available using an SVD or a Householder decomposition of Z.
A generalized reflexive inverse is given by

$$Z^- = V \begin{bmatrix} S^{-1} & M_2 \\ M_1 & M_1SM_2 \end{bmatrix} U^{-1} \qquad (203)$$

with M_1 and M_2 being matrices of free parameters that fulfill

$$P = Z^- Z = V \begin{bmatrix} I & 0 \\ M_1 S & 0 \end{bmatrix} V^{-1}$$

and

$$R = Z Z^- = U \begin{bmatrix} I & S M_2 \\ 0 & 0 \end{bmatrix} U^{-1}$$

(cf. also [Zie79]). There are two ways in looking at the parameter matrices M_1 and M_2. We can compute an arbitrary Z^- with fixed M_1 and M_2. Then also the projectors P and R are fixed by these parameter matrices. Or we provide the projectors P and R, then M_1 and M_2 are given and Z^- is fixed, too.

C Parameter dependent matrices and projectors

For any two continuosly differentiable matrix functions of appropriate size $F : \mathcal{I} \to L(\mathbb{R}^m.\mathbb{R}^k)$ and $G : \mathcal{I} \to L(\mathbb{R}^l.\mathbb{R}^m)$, $\mathcal{I} \subseteq \mathbb{R}$, an interval, the product $FG : \mathcal{I} \to L(\mathbb{R}^l.\mathbb{R}^k)$ is defined pointwise by $(FG)(t) := F(t)G(t),\ t \in \mathcal{I}$, and the product rule applies to the derivatives, i.e.

$$(FG)'(t) = F'(t)G(t) + F(t)G'(t).$$

In particular, this is valid for projector valued functions.
Let $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$ be a projector valued function and $Q = I - P$ the complementary one. The following three simply rules are useful in computations:

(1) $Q + P = I$, and hence $Q' = -P'$.

(2) $QP = PQ = 0$, and hence $Q'P = -QP',\ P'Q = -PQ'$.

(3) $PP'P = -PQ'P = PQP' = 0$ and, analogously, $QQ'Q = 0$.

Lemma C.1 (1) *If the matrix function $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$ is projector valued, that is, $P(t)^2 = P(t),\ t \in \mathcal{I}$, then it has constant rank r, and there are r linearly independent functions $\eta_1, \dots, \eta_r \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$ such that $\operatorname{im} P(t) = \operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\},\ t \in \mathcal{I}$.*

(2) *If a time-depending subspace $L(t) \subseteq \mathbb{R}^m,\ t \in \mathcal{I}$, with constant dimension r is spanned by functions $\eta_1, \dots, \eta_r \in \mathcal{C}^1(\mathcal{I}, \mathbb{R}^m)$, that means $L(t) = \operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\}$, $t \in \mathcal{I}$, then the orthoprojector function onto this subspace is continuously differentiable.*

(3) *Let the matrix function $A \in \mathcal{C}^k(\mathcal{I}, L(\mathbb{R}^m))$ have constant rank r. Then, there is a matrix function $M \in \mathcal{C}^k(\mathcal{I}, L(\mathbb{R}^m))$ being pointwise nonsingular such that $A(t)M(t) = [\underbrace{\tilde{A}(t)}_{r}\ 0], \operatorname{rank} \tilde{A}(t) = r$ for all $t \in \mathcal{I}$.*

Proof: (1) Denote $Q = I - P$, and let r be the maximal rank of $P(t)$ for $t \in \mathcal{I}$. We fix a value $\bar{t} \in \mathcal{I}$ such that $\operatorname{rank} P(\bar{t}) = r$. Let $\bar{\eta}_1, \dots, \bar{\eta}_r$ be a basis of $\operatorname{im} P(\bar{t})$.
For $i = 1, \dots, r$, the ordinary IVP

$$\eta'(t) = P'(t)\eta(t),\ t \in \mathcal{I}, \quad \eta(\bar{t}) = \bar{\eta}_i,$$

is uniquely solvable. The IVP solutions $\eta_1, \dots, \eta_r$ remain linearly independent on the entire interval $\mathcal{I}$ since they are so at $\bar{t}$.
Moreover, the function values of these functions remain in $\operatorname{im} P$, that is, $\eta_i(t) = P(t)\eta_i(t)$. Namely, multiplying the identity $\eta_i = P'\eta_i$ by Q gives $(Q\eta_i)' = -Q'Q\eta_i$, and because of $Q(\bar{t})\eta_i(\bar{t}) = Q(\bar{t})\bar{\eta}_i = 0$, the function $Q\eta_i$ must vanish identically.
It follows that $\operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\} \subseteq \operatorname{im} P(t)$ for all $t \in \mathcal{I}$, and $r \leq \operatorname{rank} P(t)$, and hence $r = \operatorname{rank} P(t)$ and $\operatorname{span}\{\eta_1(t), \dots, \eta_r(t)\} = \operatorname{im} P(t)$.
(2) The matrix function $\Gamma := [\eta_1 \quad \eta_r]$ the columns of which are the given functions $\eta_1, \dots, \eta_r$ is continuously differentiable and injective., and $\Gamma^*\Gamma$ is invertible. Then $P := \Gamma(\Gamma^*\Gamma)^{-1}\Gamma^*$ is continuously differentiable, The value $P(t)$ is an orthoprojector, further $\operatorname{im} P \subseteq \operatorname{im} \Gamma$ by construction, and $P\Gamma = \Gamma$, in consequence $\operatorname{im} P = \operatorname{im} \Gamma = L$.
(3) Proof see [Dol64]. □

For matrix functions depending on several variables we define products pointwise, too. More precisely, for $F : \Omega \to L(\mathbb{R}^m.\mathbb{R}^k)$ and $G : \Omega \to L(\mathbb{R}^l.\mathbb{R}^m)$, $\Omega \subseteq \mathbb{R}^p$, the product $FG : \Omega \to L(\mathbb{R}^l.\mathbb{R}^k)$ is defined pointwise by $(FG)(x) := F(x)G(x),\ x \in \Omega$.
We speak of a projector function $P : \Omega \to L(\mathbb{R}^l)$, if for all $x \in \Omega$, $P(x)^2 = P(x)$ holds true, and of an orthoprojector function, if, additionally, $P(x)^* = P(x)$. Saying that P is a projector function onto the subspace L we mean that P and L have a common definition domain, say Ω, and $\operatorname{im} P(x) = L(x),\ x \in \Omega$.

Lemma C.2 *Given is a matrix function* $A \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m, \mathbb{R}^n)),\ k \in \mathbb{N} \cup \{0\},\ \Omega \subseteq \mathbb{R}^p$ *open, that has constant rank* r.

(1) *Then the orthoprojector function onto* $\operatorname{im} A$ *is* k *times continuously differentiable.*

(2) *The orthoprojector function onto* $\ker A$ *is also* k *times continuously differentiable.*

Proof: (1) Let $\bar{x} \in \Omega$ be fixed, and $\bar{z}_1, \dots, \bar{z}_r$ be an orthonormal basis of $\operatorname{im} A(\bar{x})^\perp$. Denote $\bar{u}_i := A(\bar{x})\bar{z}_i,\ i = 1, \dots, r$. By construction, $\bar{u}_1, \dots, \bar{u}_r$ are linearly independent.
We form $u_i(x) := A(x)\bar{z}_i$ for $i = 1, \dots, r$, and then the matrix $U(x) := [u_1(x) \quad u_r(x)]$, $x \in \Omega$. The matrix $U(\bar{x})$ has full columnrank r. Therefore, there is a neighborhood $\mathcal{N}_{\bar{x}}$ of $\bar{x}$ such that $U(x)$ has full columnrank r on $\mathcal{N}_{\bar{x}}$. The Gram-Schmidt orthogonalization yields the factorization

$$U(x) = Q(x)R(x), \quad Q(x) \in L(\mathbb{R}^r, \mathbb{R}^n), \quad Q(x)^*Q(x) = I_r, \quad x \in \mathcal{N}_{\bar{x}},$$

with $R(x)$ being uppertriangular and nonsingular. It follows that $\operatorname{im} U(x) = \operatorname{im} Q(x)$ is true for $x \in \mathcal{N}_{\bar{x}}$.
Further, $U = A[\bar{z}_1 \quad \bar{z}_r]$ shows that U is k times continuously differentiable together with A. By construction, Q is as smooth as U. Finally, the matrix function $R_A := Q(Q^*Q)^{-1}Q^*$ is k times continuously differentiable, and it is an orthoprojector function, $\operatorname{im} R_A = \operatorname{im} Q = \operatorname{im} U = \operatorname{im} A$.
(2) This assertion is a consequence of (1). Considering the well-known relation $\ker A^\perp = \operatorname{im} A^*$ we apply (1) and find the orthoprojector function P_A onto $\ker A^\perp$ along $\ker A$ to be k times continuously differentiable, and $I - P_A$ has this property, too. □

Remark C.3 *By Lemma C.1 the orthogonal projector function* $P \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$*,* $\mathcal{I} \subseteq \mathbb{R}$ *an interval, generates globally on* $\mathcal{I}$ *defined basises* $\eta_1, \dots, \eta_r \in \mathcal{C}^1(\mathcal{I}, L(\mathbb{R}^m))$*,* $r = \operatorname{rank} P(t)$*,* $\operatorname{im} P(t) = \operatorname{im}[\eta_1(t), \dots, \eta_r(t)]$*,* $t \in \mathcal{I}$*.*
In the higher dimensional case, if $P \in \mathcal{C}^1(\Omega, L(\mathbb{R}^m))$*,* $\Omega \subseteq \mathbb{R}^p$ *open,* $p > 1$*, the situation is different. By Lemma D.2, item (8), there are local basises. However, in general, global basises do not neccessarily exist.*
For instance, the orthoprojector function onto the nullspace of the matrix function $M(x) = [x_1, x_2, x_3]$*,* $x \in \mathbb{R}^3 \setminus \{0\}$*, reads*

$$P(x) = \frac{1}{x_1^2 + x_2^2 + x_3^2} \begin{bmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2 + x_2^2 \end{bmatrix}.$$

This projector function is obviously continuously differentiable. On the other hand, the nullspace $\ker M(x) = \{z \in \mathbb{R}^3 : x_1z_1 + x_2z_2 + x_3z_3 = 0\}$ *allows only locally different descriptions by basises e.g.*

$$\ker M(x) = \operatorname{im} \begin{bmatrix} -\frac{x_2}{x_1} & -\frac{x_3}{x_1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{if } x_1 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & -\frac{x_3}{x_2} \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{if } x_1 = 0,\ x_2 \neq 0,$$

$$\ker M(x) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{if } x_1 = 0,\ x_2 = 0,\ x_3 \neq 0.$$

Proposition C.4 *Let, for a* $k \in \mathbb{N} \cup \{0\}$*, the matrix function* $D \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m, \mathbb{R}^n))$ *have constant rank on the open set* $\Omega \subseteq \mathbb{R}^p$*.*

(1) *Then the Moore-Penrose generalized inverse* D^+ *of* D *is as smooth as* D*.*

(2) *Let* $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^n))$ *be a projector function onto* $\operatorname{im} D$*, and* $P \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ *be a projector function such that* $\ker P = \ker D$*. Then the four conditions*

$$DD^-D = D, \quad D^-DD^- = D, \quad D^-D = P, \quad DD^- = R,$$

determine uniquely a function D^- *being pointwise a generalized inverse of* D*, and* D^- *is* k *times continuously differentiable.*

Proof: The first assertion is well-known, and can be found e.g. in [CM91].
The second assertion follows from the first one. We simply show the matrix function $D^- := PD^+R$ to be the required one. By Lemma B.2, the four conditions define pointwise a unique generalized inverse. Taking into account that $\operatorname{im} D = \operatorname{im} R = \operatorname{im} DD^+$ and $\ker D = \ker D^+D = \ker P$ we derive

$$\begin{aligned} D(PD^+R)D &= DD^+R = R, \\ (PD^+R)D(PD^+R) &= PD^+DD^+R = (PD^+R), \\ (PD^+R)D &= PD^+D = P, \\ D(PD^+R) &= DD^+R = R, \end{aligned}$$

so that the four conditions are fulfilled. Obviously, the product PD^+R inherits the smoothness of its factors. $\square$

What concerns the derivatives, the situation is more difficult, if several variables are involved. We use the symbols $F_x(x,t)$, $F_t(x,t)$ for the partial derivatives and partial Jacobian matrices of the function $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^k))$ with respect to $x \in \mathbb{R}^p$ and $t \in \mathbb{R}$, taken at the point $(x,t) \in \Omega \times \mathcal{I}$.
For the two functions $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^k))$ and $G \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^l, \mathbb{R}^m))$, the product $FG \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^l, \mathbb{R}^k))$ is defined pointwise. We have

$$(FG)_x(x,t)z = [F_x(x,t)z]G(x,t) + F(x,t)G_x(x,t)z \quad \text{for all } z \in \mathbb{R}^p.$$

Besides the partial derivatives we apply the *total derivative in jet variables.* For the function $F \in \mathcal{C}^1(\Omega \times \mathcal{I}, L(\mathbb{R}^m, \mathbb{R}^k))$, $\Omega \times \mathcal{I} \subseteq \mathbb{R}^p \times \mathbb{R}$, the function $F' \in \mathcal{C}(\Omega \times \mathcal{I} \times \mathbb{R}^p, L(\mathbb{R}^m, \mathbb{R}^k))$ defined by

$$F'(x,t,x^1) := F_x(x,t)x^1 + F_t(x,t), \quad x \in \Omega,\ t \in \mathcal{I},\ x^1 \in \mathbb{R}^p,$$

is named total derivative of F in jet variables. For the total derivative, the product rule

$$(FG)' = F'G + FG'$$

is easily checked to be valid.

Lemma C.5 *The total derivatives in jet variables P' and Q' of a continuously differentiable projector function P and its complementary one $Q = I - P$ satisfy the following relations:*

$$\begin{aligned} Q' &= -P', \\ Q'P &= -QP', \\ PP'P &= 0. \end{aligned}$$

Proof: The assertion follows from the identities $Q + P = I$ and $QP = 0$ by regarding the product rule. $\square$

Notice that, for each given function $x_* \in \mathcal{C}^1(\mathcal{I}_*, \mathbb{R}^p)$, $\mathcal{I}_* \subseteq \mathcal{I}$, with values in Ω, the resulting superposition $F(x_*(t),t)$ is continuously differentiable with respect to t on $\mathcal{I}_*$, and it possesses the derivative

$$(F(x_*(t),t))' := (F(x_*(.),.))'(t) = F'(x_*(t),t,x_*'(t)).$$

D Variable subspaces

Definition D.1 *Let $\Omega \subseteq \mathbb{R}^P$ be open and connected, $L(x) \subseteq \mathbb{R}^m$ be a subspace for each $x \in \Omega$. For $k \in \mathbb{N} \cup \{0\}$, L is said to be a $\mathcal{C}^k$-subspace on Ω, if there exists a projector function $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ which projects pointwise onto L, i.e. $R(x) = R(x)^2$,* $\operatorname{im} R(x) = L(x)$, *$x \in \Omega$. We write* $\operatorname{im} R = L$.

It should be mentioned at this place that the notion *smooth subspace* (smooth stands for $\mathcal{C}^1$) is applied in [GM86], Subsection 1.2.1, to subspaces depending on one real variable ($p = 1$) in the same way.

Lemma D.2 *Let $k \in \mathbb{N} \cup \{0\}$.*

(1) *A $\mathcal{C}^k$-subspace on an open connected Ω has constant dimension.*

(2) *The orthoprojector function onto a $\mathcal{C}^k$-subspace belongs to $\mathcal{C}^k$.*

(3) *If L is a $\mathcal{C}^k$-subspace, so is $L^\perp$.*

(4) *If L and N are $\mathcal{C}^k$-subspaces, and $L \cap N$ has constant dimension, then $L \cap N$ is a $\mathcal{C}^k$-subspace, too.*

(5) *If N and L are $\mathcal{C}^k$-subspaces, and $N \oplus L = \mathbb{R}^m$, then the projector onto N along L belongs to $\mathcal{C}^k$.*

(6) *If L and N are $\mathcal{C}^k$-subspaces, and $L \cap N$ has constant dimension, then there is a $\mathcal{C}^k$-subspace X such that $X \subseteq L$, and*

$$L = X \oplus (N \cap L),$$

as well as a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ with $\operatorname{im} R = N$, $\ker R \supseteq X$.

(7) *If L and N are $\mathcal{C}^k$-subspaces, and $N \cap L = 0$, then $L \oplus N$ is a $\mathcal{C}^k$-subspace, too.*

(8) *L is a $\mathcal{C}^k$-subspace on $\Omega \Leftrightarrow$ to each $\bar{x} \in \Omega$ there is a neighborhood $U_{\bar{x}} \subseteq \Omega$ and a local $\mathcal{C}^k$-basis $\eta_1, \ldots, \eta_{r(\bar{x})} \in \mathcal{C}^k(U_{\bar{x}}, \mathbb{R}^m)$ spanning L on $U_{\bar{x}}$, i.e.*

$$\operatorname{span}\{\eta_1(x), \ldots, \eta_{r(\bar{x})}(x)\} = L(x), \quad x \in U_{\bar{x}}.$$

Proof: (1) Let $x_0 \in \Omega$, let the columns of $\xi^0 := [\xi_1^0, \ldots, \xi_{r_{x_0}}^0]$ form a basis of $L(x_0)$, i.e. $L(x_0) = \operatorname{im} \xi^0$. $\xi(x) := R(x)\xi^0$ is a $\mathcal{C}^k$ matrix function, and since $\xi(x_0) = R(x_0)\xi^0 = \xi^0$ has full column rank r_{x_0}, there is a neighborhood $U_{x_0} \subset \Omega$ such that $\xi(x)$ has rank r_{x_0} for all $x \in U_{x_0}$. That means $\operatorname{im} \xi(x) \subseteq \operatorname{im} R(x)$,

$$\operatorname{rank} R(x) \geq \operatorname{rank} \xi(x) = r_{x_0}, \quad x \in U_{x_0}.$$

Denote by $r_{\min}$, $r_{\max}$ the minimal and maximal ranks of $R(x)$ on Ω, $0 \leq r_{\min} \leq r_{\max} \leq m$, and by $x_{\min}, x_{\max} \in \Omega$ points with $\operatorname{rank} R(x_{\min}) = r_{\min}$, $\operatorname{rank} R(x_{\max}) = r_{\max}$.
Since Ω is connected, there is a connecting curve of $x_{\min}$ and $x_{\max}$ belonging to Ω. We move on this curve from $x_{\max}$ to $x_{\min}$. If $r_{\min} < r_{\max}$, there must be a x_* on this curve with

$$r_* := \operatorname{rank} R(x_*) < r_{\max},$$

and in each arbitrary neighborhood of x_* there are points $\hat{x}$ with $\operatorname{rank} R(\hat{x}) = r_{\max}$.
At each $x \in \Omega$, as a projector, $R(x)$ has the only eigenvalues 1 and 0 (cf. Lemma A.2(6)). Hence, $R(x_*)$ has eigenvalue 1 with multiplicity r_*, and eigenvalue 0 with multiplicity $m - r_*$, $R(\hat{x})$ has eigenvalue 1 with multiplicity $r_{\max}$ and eigenvalue 0 with multiplicity $m - r_{\max}$.

Since eigenvalues depend continuously on the entries of a matrix, and the entries of $R(x)$ are $\mathcal{C}^k$-functions in x, the existence of x_* contradicts the continuity of eigenvalues. Therefore, $r_{\min} = r_{\max}$ must be valid.

(2) If L is a $\mathcal{C}^k$-subspace, by definition, there is a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ onto L, and the rank $R(x)$ is constant on Ω. By Lemma C.2, the orthoprojector function onto $\operatorname{im} R = L$ is k times continuously differentiable.

(3) If L is a $\mathcal{C}^k$-subspace, the orthoprojector R onto L belongs to $\mathcal{C}^k$. Then, $I - R$ is a $\mathcal{C}^k$-projector onto $\operatorname{im}(I-R) = L^\perp$.

(4) Suppose L, N are $\mathcal{C}^k$-subspaces in $\mathbb{R}^m$, and R_L, R_N corresponding projectors onto L and N. Then $F := \begin{bmatrix} I - R_L \\ I - R_N \end{bmatrix}$ is a $\mathcal{C}^k$-function, and $\ker F = L \cap N$. Since $L \cap N$ has constant dimension, F has constant rank, and therefore F^+ and F^+F are $\mathcal{C}^k$-functions. F^+F is the orthoprojector onto $\ker F$, thus $\ker F = L \cap N$ is a $\mathcal{C}^k$-subspace.

(5) Let N, L be $\mathcal{C}^k$-subspaces, $N \oplus L = \mathbb{R}^m$. For each arbitrary $x \in \Omega$, $R(x)$ is uniquely determined by $\operatorname{im} R(x) = L(x)$, $\ker R(x) = N(x)$, $R(x)^2 = R(x)$. We have to make sure that R belongs to $\mathcal{C}^k$. To each fixed $x_0 \in \Omega$ we consider basises $\xi_1^0, \dots, \xi_r^0$ of $L(x_0)$, and $\eta_1^0, \dots, \eta_{m-r}^0$ of $N(x_0)$, and consider

$$\xi(x) := R_L(x)\xi^0, \quad \eta(x) := R_N(x)\eta^0, \quad x \in \Omega,$$

where

$$\xi^0 = [\xi_1^0, \dots, \xi_r^0], \quad \eta^0 = [\eta_1^0, \dots, \eta_{m-r}^0],$$

and R_L, R_N are $\mathcal{C}^k$-projectors according to the $\mathcal{C}^k$-subspaces L and N. There is a neighborhood $U_{x_0} \subset \Omega$ of x_0, such that the columns of $\xi(x)$ and $\eta(x)$, for $x \in U_{x_0}$, are basises of $L(x)$ and $N(x)$, and the matrix $F(x) := [\xi(x), \eta(x)]$ is nonsingular for $x \in U_{x_0}$. Define, for $x \in U_{x_0}$,

$$\tilde{R}(x) := F(x) \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} F(x)^{-1},$$

such that

$$\tilde{R} \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m)), \quad \operatorname{im} \tilde{R}(x) = L(x), \quad \ker \tilde{R}(x) = N(x).$$

Since the projector corresponding to the decomposition $N(x) \oplus L(x) = \mathbb{R}^m$ is unique, we have $R(x) = \tilde{R}(x)$, $x \in U_{x_0}$, and hence R is $\mathcal{C}^k$ on U_{x_0}.

(6) Let L, N be $\mathcal{C}^k$-subspaces, $\dim(N \cap L) = \text{constant} =: u$. By (d), $N \cap L$ is a $\mathcal{C}^k$-subspace. We have $\mathbb{R}^m = (L \cap N) \oplus (L \cap N)^\perp$, $L = (L \cap N) \oplus (L \cap (L \cap N)^\perp)$, and $X := L \cap (L \cap N)^\perp$ is a $\mathcal{C}^k$-subspace, too. Further (cf. Lemma A.5), $(N+L)^\perp = N^\perp \cap L^\perp$ is also a $\mathcal{C}^k$-subspace. With $N + L = N \oplus X$ we find

$$\mathbb{R}^m = (N+L)^\perp \oplus (N+L) = (N+L)^\perp \oplus X \oplus N = S \oplus N, \quad S := (N+L)^\perp \oplus X.$$

Denote by $R^\perp$ and R_X the orthoprojectors onto the $\mathcal{C}^k$-subspaces $(N+L)^\perp$ and X. Due to $X \subseteq N + L$, $(N+L)^\perp \subseteq X^\perp$, hence $\operatorname{im} R_X \subseteq \ker R^\perp$, $\operatorname{im} R^\perp \subseteq \ker R_X$, it holds that $R_X R^\perp = 0$, $R^\perp R_X = 0$, hence $R_S := R^\perp + R_X$ is a projector and belongs to $\mathcal{C}^k$, $\operatorname{im} R_S = \operatorname{im} R^\perp + \operatorname{im} R_X = S$. This makes clear that S is also a $\mathcal{C}^k$-subspace.
Finally, due to $\mathbb{R}^m = S \oplus N$, there is a projector $R \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ with $\operatorname{im} R = N$, $\ker R = S \supset X$.

(7) By (6), due to $N \cap L = 0$, there are projectors $R_L, R_N \in \mathcal{C}^k(\Omega, L(\mathbb{R}^m))$ such that $\operatorname{im} R_L = L$, $N \subset \ker R_L$, $\operatorname{im} R_N = N$, $L \subset \ker R_N$, thus $R_L R_N = 0$, $R_N R_L = 0$, and $R := R_L + R_N$ is a $\mathcal{C}^k$-projector, too, and finally $\operatorname{im} R = \operatorname{im} R_L + \operatorname{im} R_N = L \oplus N$.

(8) If L is a $\mathcal{C}^k$-subspace then the orthogonal projector R on L along $L^\perp$ is $\mathcal{C}^k$. For each $x_0 \in \Omega$ and a basis $\xi_1^0, \ldots, \xi_r^0$ of $L(x_0)$, the columns of $\xi(x) := R(x)\xi^0$, $\xi = [\xi_1^0, \ldots, \xi_r^0]$, form a $\mathcal{C}^k$-basis of $L(x)$ locally on a neighborhood $U_{x_0} \subset \Omega$ of x_0.
Conversely, if there is a local $\mathcal{C}^k$-basis on the neighborhood $U_{\bar{x}}$ of $\bar{x}$, then one can show that the orthoprojector onto $L(x)$, $x \in U_{\bar{x}}$, can be represented by means of this basis. That means, L is $\mathcal{C}^k$ on $U_{\bar{x}}$. □

Corollary D.3 *Any projector function being continuous on an open connected set has constant rank there.*

Proof: The continuous projector function, say $P : \Omega \to L(\mathbb{R}^p)$, defines the $\mathcal{C}$-space $\operatorname{im} P$. Owing to Lemma D.2 item (1), $\operatorname{im} P$ has constant dimension, and hence P has constant rank. □

Technical Computations

E Proof of Lemma 3.7

Lemma 3.7 *If two projector function sequences $Q_0, \dots, Q_k$ and $\bar{Q}_0, \dots, \bar{Q}_k$ are both admissible, then the corresponding matrix functions and subspaces are related by the following properties:*

(a) $\ker \bar{\Pi}_j = \bar{N}_0 + \dots + \bar{N}_j = N_0 + \dots + N_j = \ker \Pi_j,\ j = 0, \dots, k,$

(b) $\bar{G}_j = G_j Z_j,$
$\bar{B}_j = B_j - G_j Z_j \bar{D}^-(D\bar{\Pi}_j \bar{D}^-)' D\Pi_j + G_j \sum_{l=0}^{j-1} Q_l \mathfrak{A}_{jl},\ j = 1, \dots, k,$
with nonsingular matrix functions $Z_0, \dots, Z_{k+1}$ given by
$Z_0 := I,\ Z_{i+1} := Y_{i+1} Z_i,\ i = 0, \dots, k,$

$$\begin{aligned} Y_1 &:= I + Q_0(\bar{Q}_0 - Q_0) = I + Q_0 \bar{Q}_0 P_0, \\ Y_{i+1} &:= I + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1} Q_i) + \sum_{l=0}^{i-1} Q_l \mathfrak{A}_{il} \bar{Q}_i,\ i = 1, \dots, k, \end{aligned}$$

and certain continuous coefficients $\mathfrak{A}_{il}$ that satisfy condition $\mathfrak{A}_{il} = \mathfrak{A}_{il}\bar{\Pi}_{i-1}$,

(c) $Z_i(\bar{N}_i \cap (\bar{N}_0 + \dots + \bar{N}_{i-1})) = N_i \cap (N_0 + \dots + N_{i-1}), \quad i = 1, \dots, k,$

(d) $\bar{G}_{k+1} = G_{k+1} Z_{k+1}, \quad \bar{N}_0 + \dots + \bar{N}_{k+1} = N_0 + \dots + N_{k+1},$
$Z_{k+1}(\bar{N}_{k+1} \cap (\bar{N}_0 + \dots + \bar{N}_k)) = N_{k+1} \cap (N_0 + \dots + N_k).$

Proof:
We have $G_0 = AD = \bar{G}_0, \quad B_0 = B = \bar{B}_0, \quad \ker P_0 = N_0 = \bar{N}_0 = \ker \bar{P}_0$, hence $P_0 = P_0 \bar{P}_0,\ \bar{P}_0 = \bar{P}_0 P_0$.
The generalized inverses D^- and $\bar{D}^-$ of D satisfy the properties $DD^- = D\bar{D}^- = R$, $D^- D = P_0,\ \bar{D}^- D = \bar{P}_0$, and therefore $\bar{D}^- = \bar{D}^- D \bar{D}^- = \bar{D}^- D D^- = \bar{P}_0 D^-,\ D^- = P_0 \bar{D}^-$.
Compare $G_1 = G_0 + B_0 Q_0$ and

$$\begin{aligned} \bar{G}_1 &= \bar{G}_0 + \bar{B}_0 \bar{Q}_0 = G_0 + B_0 \bar{Q}_0 = G_0 + B_0 Q_0 \bar{Q}_0 \\ &= (G_0 + B_0 Q_0)(P_0 + \bar{Q}_0) = G_1 Z_1, \end{aligned}$$

where $Z_1 := Y_1 := P_0 + \bar{Q}_0 = I + Q_0 \bar{Q}_0 P_0 = I + Q_0(\bar{Q}_0 - Q_0)$. Z_1 is invertible, it has the inverse $Z_1^{-1} = I - Q_0 \bar{Q}_0 P_0$.
The nullspaces N_1 and $\bar{N}_1$ are, due to $\bar{G}_1 = G_1 Z_1$, related by $\bar{N}_1 = Z_1^{-1} N_1 \subseteq N_0 + N_1$. This implies $\bar{N}_0 + \bar{N}_1 = N_0 + (Z_1^{-1} N_1) \subseteq N_0 + N_1$. From $N_1 = Z_1 \bar{N}_1 \subseteq N_0 + \bar{N}_1 = \bar{N}_0 + \bar{N}_1$, we obtain $\bar{N}_0 + \bar{N}_1 = N_0 + N_1$.

Since the projectors $\Pi_1 = P_0 P_1$ and $\bar{\Pi}_1 = \bar{P}_0 \bar{P}_1$ have the common nullspace $N_0 + N_1 = \bar{N}_0 + \bar{N}_1$, we may now derive

$$\begin{aligned} D\bar{P}_0\bar{P}_1\bar{D}^- &= D\bar{P}_0\bar{P}_1 P_0 P_1 \bar{P}_0 D^- = D\bar{P}_0\bar{P}_1 P_0 P_1 D^- = D\bar{P}_0\bar{P}_1\bar{D}^- D P_0 P_1 D^-, \\ DP_0 P_1 D^- &= DP_0 P_1 D^- D\bar{P}_0\bar{P}_1\bar{D}^-. \end{aligned}$$

Next we compute

$$\begin{aligned}
\bar{B}_1 &= \bar{B}_0\bar{P}_0 - \bar{G}_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'D\bar{P}_0 \\
&= B_0(P_0+Q_0)\bar{P}_0 - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-DP_0P_1D^-)'D \\
&= B_0P_0 + B_0Q_0\bar{P}_0 - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1 - G_1Z_1\bar{P}_0\bar{P}_1\bar{D}^-(DP_0P_1D^-)'D \\
&= B_1 + G_1D^-(DP_0P_1D^-)'D - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1 \\
&\quad\; -G_1Z_1\bar{P}_0\bar{P}_1D^-(DP_0P_1D^-)'D + B_0Q_0\bar{P}_0 \\
&= B_1 - G_1Z_1\bar{D}^-(D\bar{P}_0\bar{P}_1\bar{D}^-)'DP_0P_1 + \mathfrak{B}_1
\end{aligned}$$

with $\mathfrak{B}_1 := G_1Q_0\bar{P}_0 + G_1(I - Z_1\bar{\Pi}_1)D^-(D\Pi_1D^-)'D$.
The identity $0 = \bar{G}_1\bar{Q}_1 = G_1Z_1\bar{Q}_1 = G_1\bar{Q}_1 + G_1(Z_1 - I)\bar{Q}_1$ leads to $G_1\bar{Q}_1 = -G_1(Z_1 - I)\bar{Q}_1$ and further to

$$\begin{aligned}
&G_1(I - Z_1\bar{\Pi}_1) = G_1(I - \bar{\Pi}_1 - (Z_1 - I)\bar{\Pi}_1) = G_1(\bar{Q}_1 + \bar{Q}_0\bar{P}_1 - Q_0\bar{Q}_0P_0\bar{\Pi}_1) \\
&= G_1(-Q_0\bar{Q}_0P_0\bar{Q}_1 + \bar{Q}_0\bar{P}_1 - Q_0\bar{Q}_0P_0\bar{P}_1) = G_1(-Q_0\bar{Q}_0P_0 + \bar{Q}_0\bar{P}_1) \\
&= G_1(-Q_0\bar{Q}_0 + Q_0 + Q_0\bar{Q}_0\bar{P}_1) = G_1(-Q_0\bar{Q}_0\bar{Q}_1 + Q_0).
\end{aligned}$$

Inserting into the expression of $\mathfrak{B}_1$ yields
$\mathfrak{B}_1 = G_1Q_0\bar{P}_0 - G_1Q_0\bar{Q}_0\bar{Q}_1D^-(D\Pi_1D^-)'D = G_1Q_0\mathfrak{A}_{10}$ with

$$\mathfrak{A}_{10} := \bar{P}_0 - \bar{Q}_0\bar{Q}_1D^-(D\Pi_1D^-)'D$$

and $\mathfrak{A}_{10} = \mathfrak{A}_{10}\bar{P}_0$. In order to verify assertions (a) and (b) by induction, we assume the relations

$$\begin{aligned}
\bar{N}_0 + \cdots + \bar{N}_j &= N_0 + \cdots N_j, \\
\bar{G}_j &= G_jZ_j, \\
\bar{B}_j &= B_j - G_jZ_j\bar{D}^-(D\bar{\Pi}_j\bar{D}^-)'D\Pi_j + G_j\sum_{l=0}^{j-1} Q_l\mathfrak{A}_{jl}
\end{aligned} \tag{204}$$

to be valid for $j = 1, \ldots, i,\ i < k$, with nonsingular Z_i as described above.

By construction, Z_i is of the form $Z_j = Y_jZ_{j-1} = Y_jY_{j-1}\cdots Y_1$. By realizing the multiplication and rearranging the terms we find the expression

$$Z_j - I = \sum_{l=0}^{j-1} Q_l\mathcal{C}_{jl} \tag{205}$$

with continuous coefficients $\mathcal{C}_{jl}$.

It holds that $Y_1 - I = Q_0\bar{Q}_0P_0$ and

$$Y_j - I = (Y_j - I)\Pi_{j-2},\ j = 2, \ldots, i, \tag{206}$$

such that $(Y_j - I)(Z_{j-1} - I) = 0$ must be true. From this it follows that $Y_j(Z_{j-1} - I) = Z_{j-1} - I$, and $Z_j = Y_jZ_{j-1} = Y_j + Y_j(Z_{j-1} - I) = Y_j + Z_{j-1} - I = Y_j - I + Z_{j-1}$, i.e.,

$$\begin{aligned}
Z_j &= Y_j - I + \cdots + Y_1 - I + Z_0, \\
Z_j - I &= \sum_{l=1}^{j}(Y_l - I).
\end{aligned} \tag{207}$$

From (207) one can obtain special formulas for the coefficients $\mathcal{C}_{jl}$ in (205), but in our context there is no need for these special descriptions.

Now we compare $\bar{G}_{i+1}$ and G_{i+1}. We have

$$\bar{G}_{i+1} = \bar{G}_i + \bar{B}_i\bar{Q}_i = G_iZ_i + \bar{B}_i\bar{Q}_i.$$

Because of $\bar{B}_i = \bar{B}_i\bar{\Pi}_{i-1}$ we may write

$$\begin{aligned} \bar{B}_i\bar{Q}_i(Z_i - I) &= \bar{B}_i\bar{\Pi}_{i-1}\bar{Q}_i(Z_i - I) \\ &= \bar{B}_i\bar{\Pi}_{i-1}\bar{Q}_i\bar{\Pi}_{i-1}(Z_i - I) \end{aligned}$$

and using (205) and $Q_i = \bar{Q}_iQ_i$ we obtain

$$= 0,$$

i.e., $\bar{B}_i\bar{Q}_i = \bar{B}_i\bar{Q}_iZ_i$. This yields

$$\bar{G}_{i+1} = (G_i + \bar{B}_i\bar{Q}_i)Z_i.$$

Derive further

$$\bar{G}_{i+1}Z_i^{-1} = G_i + \bar{B}_i\bar{Q}_i = G_{i+1} + (\bar{B}_i\bar{Q}_i - B_iQ_i)$$

using (204) and $\bar{Q}_i = Q_i\bar{Q}_i$ we obtain

$$\begin{aligned} &= G_{i+1} + B_i(\bar{Q}_i - Q_i) + G_i\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1} + B_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + G_{i+1}\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1} + B_iQ_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) + G_{i+1}\sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_i \\ &= G_{i+1}Y_{i+1}, \end{aligned}$$

and $\bar{G}_{i+1} = G_{i+1}Y_{i+1}Z_i = G_{i+1}Z_{i+1}$, that is, $\bar{G}_{i+1}$ and G_{i+1} are related as demanded. Next we show the invertibility of Y_{i+1} and compute the inverse. Consider the linear equation $Y_{i+1}z = w$, i.e.,

$$z + Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)z + \sum_{l=0}^{i-1} Q_l\mathfrak{A}_{il}\bar{Q}_iz = w.$$

Because of (206) we immediately realize that

$$\Pi_iz = \Pi_iw, \quad z = w - (Y_{i+1} - I)\Pi_{i-1}z,$$

and

$$\Pi_{i-1}z + \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)z = \Pi_{i-1}w.$$

Taking into account that

$$\Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i) = \Pi_{i-1}Q_i\bar{Q}_i - \Pi_{i-1}Q_i = -\Pi_{i-1}Q_i\bar{P}_i = -\Pi_{i-1}Q_i\bar{\Pi}_{i-1}\bar{P}_i = -\Pi_{i-1}Q_i\bar{P}_i\Pi_i$$

we conclude

$$\Pi_{i-1}z = \Pi_{i-1}w - \Pi_{i-1}Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)w$$

and

$$\begin{array}{rcl} z &=& w - (Y_{i+1} - I)(I - Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i))w, \\ Y_{i+1}^{-1} &=& I - (Y_{i+1} - I)(I - Q_i(\bar{\Pi}_{i-1}\bar{Q}_i - \Pi_{i-1}Q_i)). \end{array}$$

The inverse $Z_{i+1}^{-1} = (Y_{i+1}\cdots Y_1)^{-1} = Y_1^{-1}\cdots Y_{i+1}^{-1}$ may be expressed as

$$Z_{i+1}^{-1} = I + \sum_{l=0}^{i} Q_l \mathfrak{E}_{i+1,l}$$

with certain continuous coefficients $\mathfrak{E}_{i+1,l}$.
We have

$$\begin{aligned} \bar{N}_{i+1} = Z_{i+1}^{-1}N_{i+1} &\subseteq N_0 + \cdots + N_{i+1}, \\ \bar{N}_0 + \cdots + \bar{N}_{i+1} = N_0 + \cdots + N_i + \bar{N}_{i+1} &\subseteq N_0 + \cdots + N_{i+1}, \\ N_0 + \cdots + N_{i+1} = N_0 + \cdots + N_i + (Z_{i+1}\bar{N}_{i+1}) &\subseteq N_0 + \cdots + N_i + \bar{N}_{i+1} = \bar{N}_0 + \cdots + \bar{N}_{i+1}, \end{aligned}$$

thus $\bar{N}_0 + \cdots + \bar{N}_{i+1} = N_0 + \cdots + N_{i+1}$.
It follows that

$$D\bar{\Pi}_{i+1}\bar{D}^- = D\bar{\Pi}_{i+1}\bar{D}^- D\Pi_{i+1}D^-.$$

Now we consider the terms $\bar{B}_{i+1}$ and B_{i+1}. We have

$$\begin{array}{rcl} \bar{B}_{i+1} &=& \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\bar{\Pi}_i \\ &=& \bar{B}_i\bar{P}_i - \bar{G}_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^- D\Pi_{i+1}D^-)'D\bar{\Pi}_i \\ &=& \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} - G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\Pi_{i+1}D^-)'D\bar{\Pi}_i \\ && \\ &=& \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\ && \quad -G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-\{(D\Pi_{i+1}D^-)'D\Pi_i - D\Pi_{i+1}D^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i\} \\ && \\ &=& \bar{B}_i\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\ && \quad -G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i. \end{array}$$

Taking into account the given result for $\bar{B}_i$ we obtain

$$\begin{aligned} \bar{B}_{i+1} = \{&B_i - G_iZ_i\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\}(P_i + Q_i)\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\ &- G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i \\ =\ &B_iP_i - G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ &+ B_iQ_i\bar{P}_i - G_iZ_i\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} \\ &- G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}D^-(D\Pi_{i+1}D^-)'D\Pi_i + G_{i+1}Z_{i+1}\bar{\Pi}_{i+1}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i, \end{aligned}$$

hence

$$\bar{B}_{i+1} = B_{i+1} - G_{i+1}Z_{i+1}\bar{D}^-(D\bar{\Pi}_{i+1}\bar{D}^-)'D\Pi_{i+1} + \mathfrak{B}_{i+1}$$

with

$$\begin{aligned}\mathfrak{B}_{i+1} \quad = \quad & B_iQ_i\bar{P}_i + G_i\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i + G_{i+1}(I - Z_{i+1}\bar{\Pi}_{i+1})D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ & -G_{i+1}(P_iZ_i - Z_{i+1}\bar{\Pi}_{i+1})\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.\end{aligned}$$

It remains to show that $\mathfrak{B}_{i+1}$ can be expressed as $G_{i+1}\sum\limits_{l=0}^{i}Q_l\mathfrak{A}_{i+1\,l}$. For this purpose we rewrite

$$\begin{aligned}\mathfrak{B}_{i+1} \quad = \quad & G_{i+1}Q_i\bar{P}_i + G_{i+1}\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i \\ & +G_{i+1}(I - \bar{\Pi}_{i+1} - (Z_{i+1} - I)\bar{\Pi}_{i+1})D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ & -G_{i+1}(Z_i - I - Q_iZ_i + I - \bar{\Pi}_{i+1} - (Z_{i+1} - I)\bar{\Pi}_{i+1})\bar{D}^-(D\bar{\Pi}_iD^-)'D\Pi_i.\end{aligned}$$

Take a closer look at the term $G_{i+1}(I - \bar{\Pi}_{i+1}) = G_{i+1}(\bar{Q}_{i+1} + (I - \bar{\Pi}_i)\bar{P}_{i+1})$.
By means of the identity $0 = \bar{G}_{i+1}\bar{Q}_{i+1} = G_{i+1}Z_{i+1}\bar{Q}_{i+1} = G_{i+1}\bar{Q}_{i+1} + G_{i+1}(Z_{i+1} - I)\bar{Q}_{i+1}$ we obtain the relation

$$G_{i+1}\bar{Q}_{i+1} = -G_{i+1}(Z_{i+1} - I)\bar{Q}_{i+1}$$

and hence

$$G_{i+1}(I - \bar{\Pi}_{i+1}) = G_{i+1}(-(Z_{i+1} - I)\bar{Q}_{i+1} + (I - \bar{\Pi}_i)\bar{P}_{i+1}).$$

This yields

$$\begin{aligned}\mathfrak{B}_{i+1} =& G_{i+1}Q_i\bar{P}_i + G_{i+1}\sum_{l=0}^{i-1}Q_l\mathfrak{A}_{il}\bar{P}_i \\ & + G_{i+1}\{-(Z_{i+1} - I)\bar{Q}_{i+1} + (I - \bar{\Pi}_i)\bar{P}_{i+1}) - (Z_{i+1} - I)\bar{\Pi}_{i+1}\}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ & - G_{i+1}\{Z_i - I - Q_iZ_i - (Z_{i+1} - I)\bar{Q}_{i+1} \\ & \qquad\qquad + (I - \bar{\Pi}_i)\bar{P}_{i+1} - (Z_{i+1} - I)\bar{\Pi}_{i+1}\}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.\end{aligned}$$

With

$$\begin{aligned}Z_{i+1} - I = \sum_{l=0}^{i}Q_l\mathfrak{C}_{i+1\,l},\ Z_i - I = \sum_{l=0}^{i-1}Q_l\mathfrak{C}_{il}, \\ I - \bar{\Pi}_i = (I - \Pi_i)(I - \bar{\Pi}_i) = Q_i + Q_{i-1}P_i + \cdots + Q_0P_1\cdots P_i)(I - \bar{\Pi}_i),\end{aligned}$$

by rearranging the terms we arrive at

$$\mathfrak{B}_{i+1} = G_{i+1}\sum_{l=0}^{i}Q_l\mathfrak{A}_{i+1\,l},$$

e.g. with

$$\begin{aligned}\mathfrak{A}_{i+1\,i} :=& \bar{P}_i + \{-\mathfrak{C}_{i+1\,i}(\bar{Q}_{i+1} + \bar{\Pi}_{i+1}) + (I - \bar{\Pi}_i)\bar{P}_{i+1}\}D^-(D\Pi_{i+1}D^-)'D\Pi_i \\ & - \{-Z_i - \mathfrak{C}_{i+1\,i}(\bar{Q}_{i+1} + \bar{\Pi}_{i+1}) + (I - \bar{\Pi}_i)\bar{P}_{i+1}\}\bar{D}^-(D\bar{\Pi}_i\bar{D}^-)'D\Pi_i.\end{aligned}$$

It is evident that all coefficients have the wanted property $\mathfrak{A}_{i+1\,l} = \mathfrak{A}_{i+1\,l}\bar{\Pi}_i$.
Finally, we are done with assertions (a), (b). At the same time, we have proved the first two relations in (d).
Assertion (c) is a consequence of (a), (b) and the special form (205) of the nonsingular matrix function Z_i. Namely, we have $Z_i(N_0 + \cdots + N_{i-1}) = N_0 + \cdots + N_{i-1}$, $Z_i\bar{N}_i = N_i$, thus

$$\begin{aligned}&Z_i(\bar{N}_i \cap (\bar{N}_0 + \cdots + \bar{N}_{i-1})) = (Z_i\bar{N}_i) \cap (Z_i(\bar{N}_0 + \cdots + N_{i-1})) = \\ &N_i \cap (Z_i(N_0 + \cdots + N_{i-1})) = N_i \cap (N_0 + \cdots + N_{i-1}).\end{aligned}$$

The same arguments apply for obtaining the third relation in (d). □

F Proof of Lemma 6.17

Lemma 6.17 *Let the DAE (44) with sufficiently smooth coefficients be regular with tractability index $\mu \geq 3$, and let $Q_0, \ldots, Q_{\mu-1}$ be admissible projector functions.*
Let $k \in \{1, \ldots, \mu-2\}$ be fixed, and let $\bar{Q}_k$ be an additional continuous projector function onto $N_k = \ker G_k$ such that $D\Pi_{k-1}\bar{Q}_kD^-$ is continuously differentiable and the inclusion $N_0 + \cdots + N_{k-1} \subseteq \ker \bar{Q}_k$ is valid. Then the following becomes true:

(1) *The projector function sequence*

$$\begin{aligned}&\bar{Q}_0 := Q_0, \ldots, \bar{Q}_{k-1} := Q_{k-1}, \\ &\bar{Q}_k, \\ &\bar{Q}_{k+1} := Z_{k+1}^{-1}Q_{k+1}Z_{k+1}, \ldots, \bar{Q}_{\mu-1} := Z_{\mu-1}^{-1}Q_{\mu-1}Z_{\mu-1},\end{aligned}$$

with the determined below continuous nonsingular matrix functions $Z_{k+1}, \ldots, Z_{\mu-1}$, is also admissible.

(2) *If, additionally, the projector functions $Q_0, \ldots, Q_{\mu-1}$ provide an advanced decoupling in the sense that the conditions (cf. Lemma 6.12)*

$$Q_{\mu-1*}\Pi_{\mu-1} = 0, \ \ldots, \ Q_{k+1*}\Pi_{\mu-1} = 0$$

are given, then also the relations

$$\bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = 0, \ \ldots, \ \bar{Q}_{k+1*}\bar{\Pi}_{\mu-1} = 0, \tag{208}$$

are valid, and further

$$\bar{Q}_{k*}\bar{\Pi}_{\mu-1} = (Q_{k*} - \bar{Q}_k)\Pi_{\mu-1}. \tag{209}$$

The matrix functions Z_i are consistent with those given in Lemma 3.7, however, for an easier reading we do not access this general lemma in the proof below. In the special case given here, Lemma 3.7 yields simply $Z_0 = I, Y_1 = Z_1 = I, \ldots, Y_k = Z_k = I$, and further

$$\begin{aligned}&Y_{k+1} = I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l\mathfrak{A}_{kl}\bar{Q}_k = (I + \sum_{l=0}^{k-1} Q_l\mathfrak{A}_{kl}Q_k)(I + Q_k(\bar{Q}_k - Q_k)), \\ &Z_{k+1} = Y_{k+1}, \\ &\quad Y_j = I + \sum_{l=0}^{j-2} Q_l\mathfrak{A}_{j-1l}Q_{j-1}, \quad Z_j = Y_jZ_{j-1}, \quad j = k+2, \ldots, \mu.\end{aligned}$$

Besides the general property $\ker \bar{\Pi}_j = \ker \Pi_j$, $j = 0, \ldots, \mu - 1$, which follows from Lemma 3.7, now it additionally holds that

$$\operatorname{im} \bar{Q}_k = \operatorname{im} Q_k, \quad \text{but} \quad \ker \bar{Q}_j = \ker Q_j, \quad j = k+1, \ldots, \mu - 1.$$

Proof of Lemma 6.17:
(1) Put $\bar{Q}_i = Q_i$ for $i = 0, \ldots, k-1$ such that $\bar{Q}_0, \ldots, \bar{Q}_k$ are admissible by the assumptions and the following relations are valid:

$$\begin{aligned}
\Pi_k &= \Pi_k \bar{\Pi}_k,\ \bar{\Pi}_k = \bar{\Pi}_k \Pi_k,\\
\bar{Q}_k P_k &= \bar{Q}_k \Pi_k,\\
Q_k \bar{P}_k &= Q_k(I - \bar{Q}_k) = Q_k - \bar{Q}_k = \bar{Q}_k Q_k - \bar{Q}_k = -\bar{Q}_k P_k,\\
\bar{\Pi}_k &= \Pi_{k-1}(P_k + Q_k)\bar{P}_k = \Pi_k + \Pi_{k-1} Q_k \bar{P}_k = (I - \Pi_{k-1}\bar{Q}_k)\Pi_k.
\end{aligned}$$

We verify the assertion level by level by induction. Set $\bar{G}_i = G_i, Z_i = I, \bar{B}_i = B_i$, for $i = 0, \ldots, k-1$, $\bar{G}_k = G_k$, $Z_k = I$, and derive

$$\begin{aligned}
\bar{B}_k &= B_{k-1}P_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_{k-1}\\
&= B_{k-1}P_{k-1} - G_k D^-\{D\bar{\Pi}_k D^-(D\Pi_k D^-)' + (D\bar{\Pi}_k D^-)' D\Pi_k D^-\} D\Pi_{k-1}\\
&= B_{k-1}P_{k-1} - G_k \bar{\Pi}_k D^-(D\Pi_k D^-)' D\Pi_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k\\
&= B_k + G_k(I - \bar{\Pi}_k) D^-(D\Pi_k D^-)' D\Pi_{k-1} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k\\
&= B_k + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} - G_k D^-(D\bar{\Pi}_k D^-)' D\Pi_k,
\end{aligned}$$

with regard of $G_k \bar{Q}_k = 0$ and $I - \bar{\Pi}_k = \bar{Q}_k + Q_{k-1}\bar{P}_k + \cdots + Q_0 P_1 \cdots P_{k-1}\bar{P}_k$ and with coefficients

$$\mathfrak{A}_{k,l} = Q_l P_{l+1} \cdots P_{k-1} \bar{P}_k D^-(D\bar{\Pi}_k D^-)' D\Pi_{k-1}.$$

Next we compute

$$\begin{aligned}
\bar{G}_{k+1} &= G_k + \bar{B}_k \bar{Q}_k = G_k + B_k \bar{Q}_k + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k\\
&= G_{k+1} + B_k(\bar{Q}_k - Q_k) + G_k \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k = G_{k+1} Z_{k+1},
\end{aligned}$$

$$Z_{k+1} = I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} \bar{Q}_k = (I + \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k)(I + Q_k(\bar{Q}_k - Q_k)),$$

$$Z_{k+1}^{-1} = (I - Q_k(\bar{Q}_k - Q_k))(I - \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k) = I - Q_k(\bar{Q}_k - Q_k) - \sum_{l=0}^{k-1} Q_l \mathfrak{A}_{k,l} Q_k.$$

Put $\bar{Q}_{k+1} = Z_{k+1}^{-1} Q_{k+1} Z_{k+1} = Z_{k+1}^{-1} Q_{k+1}$ such that

$$\bar{Q}_{k+1} P_{k+1} = 0,\ \bar{Q}_{k+1} = \bar{Q}_{k+1} \Pi_{k-1},\ \Pi_k \bar{Q}_{k+1} = \Pi_k Q_{k+1},$$

$\bar{\Pi}_k\bar{Q}_{k+1} = \bar{\Pi}_k\Pi_kQ_{k+1}$ is continuous and $D\bar{\Pi}_k\bar{Q}_{k+1}D^- = D\bar{\Pi}_kD^-D\Pi_kQ_{k+1}D^-$ is continuously differentiable, and hence $\bar{Q}_0, \ldots, \bar{Q}_k, \bar{Q}_{k+1}$ are admissible. It holds that

$$\Pi_{k+1} = \Pi_{k+1}\bar{\Pi}_{k+1}, \quad \bar{\Pi}_{k+1} = \bar{\Pi}_{k+1}\Pi_{k+1}, \quad \bar{\Pi}_{k+1} = (I - \Pi_{k-1}\bar{Q}_k)\Pi_{k+1}.$$

We obtain the expression

$$\bar{B}_{k+1} = B_{k+1} - \bar{G}_{k+1}D^-(D\bar{\Pi}_{k+1}D^-)'D\Pi_{k+1} + G_{k+1}\sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l},$$

with continuous coefficients $\mathfrak{A}_{k+1,l} = \mathfrak{A}_{k+1,l}\Pi_k = \mathfrak{A}_{k+1,l}\bar{\Pi}_k$, and then

$$\begin{aligned}\bar{G}_{k+2} &= \bar{G}_{k+1} + \bar{B}_{k+1}\bar{Q}_{k+1} = (G_{k+1} + \bar{B}_{k+1}Q_{k+1})Z_{k+1} = (G_{k+1} + B_{k+1}Q_{k+1} \\ &\quad + G_{k+1}\sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1} = G_{k+2}(I + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1} =: G_{k+2}Z_{k+2},\end{aligned}$$

with the nonsingular matrix function

$$Z_{k+2} = (I + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1})Z_{k+1} = I + Q_k(\bar{Q}_k - Q_k) + \sum_{l=0}^{k-1} Q_l\mathfrak{A}_{k,l}\bar{Q}_k + \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1}$$

such that

$$Z_{k+1}Z_{k+2}^{-1} = I - \sum_{l=0}^{k} Q_l\mathfrak{A}_{k+1,l}Q_{k+1}.$$

Letting $\bar{Q}_{k+2} = Z_{k+2}^{-1}Q_{k+2}Z_{k+2} = Z_{k+2}^{-1}Q_{k+2}$ we find

$$\begin{aligned} Q_{k+2}\bar{Q}_{k+2} = Q_{k+2}, \quad \bar{Q}_{k+2}Q_{k+2} = \bar{Q}_{k+2} \quad \bar{Q}_{k+2} = \bar{Q}_{k+2}\Pi_{k+1} = \bar{Q}_{k+2}\bar{\Pi}_{k+1}, \\ \bar{\Pi}_{k+1}\bar{Q}_{k+2} = \bar{\Pi}_{k+1}\Pi_{k+1}Q_{k+2}, \quad D\bar{\Pi}_{k+1}\bar{Q}_{k+2}D^- = D\bar{\Pi}_{k+1}D^-D\Pi_{k+1}Q_{k+2}D^-,\end{aligned}$$

so that $\bar{Q}_0, \ldots, \bar{Q}_{k+2}$ are known to be admissible.
Further, we apply induction. Let, for a certain $\kappa \geq k+2$, the projector functions $\bar{Q}_0, \ldots, \bar{Q}_\kappa$ be already shown to be admissible and, for $i = k+2, \ldots, \kappa$,

$$\begin{aligned} \bar{B}_{i-1} &= B_{i-1} - \bar{G}_{i-1}D^-(D\bar{\Pi}_{i-1}D^-)'D\Pi_{i-1} + G_{i-1}\sum_{l=0}^{i-2} Q_l\mathfrak{A}_{i-1,l}, \\ \mathfrak{A}_{i-1,l} &= \mathfrak{A}_{i-1,l}\Pi_{i-2}, \\ \bar{G}_i &= G_iZ_i, \quad Z_i = (I + \sum_{l=0}^{i-2} Q_l\mathfrak{A}_{i-1,l}Q_{i-1})Z_{i-1}, \\ \bar{Q}_i &= Z_i^{-1}Q_iZ_i = Z_i^{-1}Q_i, \quad \bar{\Pi}_i = (I - \Pi_{k-1}\bar{Q}_k)\Pi_i.\end{aligned}$$

Now we consider

$$\begin{aligned}\bar{B}_\kappa &= \bar{B}_{\kappa-1}\bar{P}_{\kappa-1} - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\bar{\Pi}_{\kappa-1} \\ &= \bar{B}_{\kappa-1}P_{\kappa-1} - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\Pi_\kappa - \bar{G}_\kappa\bar{\Pi}_\kappa D^-(D\Pi_\kappa D^-)'D\bar{\Pi}_{\kappa-1} \\ &= B_\kappa - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)'D\Pi_\kappa + G_\kappa\sum_{l=0}^{\kappa-2} Q_l\mathfrak{A}_{\kappa-1,l}P_{\kappa-1} + \mathfrak{C}_\kappa,\end{aligned}$$

with

$$
\begin{aligned}
\mathfrak{C}_\kappa &:= G_\kappa D^-(D\Pi_\kappa D^-)' D\Pi_{\kappa-1} - \bar{G}_\kappa \bar{\Pi}_\kappa D^-(D\Pi_\kappa D^-)' D\bar{\Pi}_{\kappa-1} - \bar{G}_{\kappa-1} D^-(D\bar{\Pi}_{\kappa-1} D^-)' D\Pi_{\kappa-1} \\
&= G_\kappa D^-(D\Pi_\kappa D^-)' D\Pi_{\kappa-1} - \bar{G}_\kappa \bar{\Pi}_\kappa D^-\{(D\Pi_\kappa D^-)' - D\Pi_\kappa D^-(D\bar{\Pi}_{\kappa-1} D^-)'\} D\Pi_{\kappa-1} \\
&\quad - \bar{G}_{\kappa-1} D^-(D\bar{\Pi}_{\kappa-1} D^-)' D\Pi_{\kappa-1} \\
&= G_\kappa (I - Z_\kappa \bar{\Pi}_\kappa) D^-(D\Pi_\kappa D^-)' D\Pi_{\kappa-1} - G_\kappa (P_{\kappa-1} Z_{\kappa-1} - Z_\kappa \bar{\Pi}_\kappa) D^-(D\bar{\Pi}_{\kappa-1} D^-)' D\Pi_{\kappa-1}.
\end{aligned}
$$

Regarding the relations $\Pi_\kappa Z_\kappa = \Pi_\kappa$ and $\Pi_\kappa Z_{\kappa-1} = \Pi_\kappa$ we observe that

$$
\Pi_\kappa (I - Z_\kappa \bar{\Pi}_\kappa) = 0, \quad \Pi_\kappa (P_{\kappa-1} Z_{\kappa-1} - Z_\kappa \bar{\Pi}_\kappa) = 0.
$$

The representation $I - \Pi_\kappa = Q_\kappa + Q_{\kappa-1} P_\kappa + \ldots + Q_0 P_1 \cdots P_\kappa$ admits of the expressions

$$
I - Z_\kappa \bar{\Pi}_\kappa = \sum_{l=0}^{\kappa} Q_l \mathfrak{E}_{\kappa,l}, \quad P_{\kappa-1} Z_{\kappa-1} - Z_\kappa \bar{\Pi}_\kappa = \sum_{l=0}^{\kappa} Q_l \mathfrak{F}_{\kappa,l}.
$$

Considering $G_\kappa Q_\kappa = 0$, this leads to the representations

$$
\mathfrak{C}_\kappa = \sum_{l=0}^{\kappa-1} Q_l \{\mathfrak{E}_{\kappa,l} D^-(D\Pi_\kappa D^-)' D\Pi_{\kappa-1} - \mathfrak{F}_{\kappa,l} D^-(D\bar{\Pi}_{\kappa-1} D^-)' D\Pi_{\kappa-1}\},
$$

and hence

$$
\bar{B}_\kappa = B_\kappa - \bar{G}_\kappa D^-(D\bar{\Pi}_\kappa D^-)' D\Pi_\kappa + G_\kappa \sum_{l=0}^{\kappa-1} Q_l \mathfrak{A}_{\kappa,l},
$$

with continuous coefficients

$$
\mathfrak{A}_{\kappa,l} = \mathfrak{A}_{\kappa,l} \Pi_{\kappa-1}, \ l = 0, \ldots, \kappa-1.
$$

It follows that

$$
\begin{aligned}
\bar{G}_{\kappa+1} &= \bar{G}_\kappa + \bar{B}_\kappa \bar{Q}_\kappa = G_\kappa Z_\kappa + \bar{B}_{\kappa+1} Z_\kappa^{-1} Q_\kappa Z_\kappa \\
&= \{G_\kappa + B_\kappa Q_\kappa + G_\kappa \sum_{l=0}^{\kappa-1} Q_l \mathfrak{A}_{\kappa,l} Q_\kappa\} Z_\kappa \\
&= G_{\kappa+1} \{I + \sum_{l=0}^{\kappa-1} Q_l \mathfrak{A}_{\kappa,l} Q_\kappa\} Z_\kappa =: G_{\kappa+1} Z_{\kappa+1}.
\end{aligned}
$$

Letting $\bar{Q}_{\kappa+1} = Z_{\kappa+1}^{-1} Q_{\kappa+1} Z_{\kappa+1} = Z_{\kappa+1}^{-1} Q_{\kappa+1}$ we find

$$
\begin{aligned}
\bar{Q}_{\kappa+1} = \bar{Q}_{\kappa+1} \Pi_\kappa = \bar{Q}_{\kappa+1} \Pi_\kappa \bar{\Pi}_\kappa = \bar{Q}_{\kappa+1} \bar{\Pi}_\kappa, \\
\bar{\Pi}_\kappa \bar{Q}_{\kappa+1} = \bar{\Pi}_\kappa \Pi_\kappa Q_{\kappa+1} \quad D\bar{\Pi}_\kappa \bar{Q}_{\kappa+1} D^- = D\bar{\Pi}_\kappa D^- D\Pi_\kappa Q_{\kappa+1} D^-,
\end{aligned}
$$

which shows the sequence $\bar{Q}_0, \ldots, \bar{Q}_{\kappa+1}$ to be admissible and all required relations to be valid. We are done with Assertion (1).

(2) Owing to Lemma 6.12, the functions

$$
\begin{aligned}
Q_{\mu-1\,*} &= Q_{\mu-1} G_\mu^{-1} B_{\mu-1}, \\
Q_{i\,*} &= Q_i P_{i+1} \cdots P_{\mu-1} G_\mu^{-1} \underbrace{\{B_i + G_i D^-(D\Pi_{\mu-1} D^-)' D\Pi_{i-1}\}}_{=: \mathfrak{B}_i}, \quad i = 1, \ldots, \mu-2,
\end{aligned}
$$

are continuous projector-valued functions such that

$$\operatorname{im} Q_{i*} = \operatorname{im} Q_i = \ker G_i, \quad Q_{i*} = Q_{i*}\Pi_{i-1}, \quad i = 1, \ldots, \mu-1.$$

Since $Q_0, \ldots, Q_{\mu-1}$ are admissible, for $j = 1, \ldots, \mu-2$, it holds that

$$\begin{aligned} Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_j &= Q_j P_{j+1} \cdots P_{\mu-1} P_{\mu-1} \cdots P_j = Q_j P_{j+1} \cdots P_{\mu-1} P_j \\ &= Q_j P_{j+1} \cdots P_{\mu-1} - Q_j = -Q_j (I - P_{j+1} \cdots P_{\mu-1}) \\ &= -Q_j \{ Q_{j+1} + P_{j+1} Q_{j+2} + \ldots + P_{j+1} \cdots P_{\mu-2} Q_{\mu-1} \}. \end{aligned} \tag{210}$$

Property (210) immediately implies

$$Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_j = Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_j \Pi_j, \tag{211}$$

$$Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_j \Pi_{\mu-1} = 0, \tag{212}$$

$$Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_i = Q_j P_{j+1} \cdots P_{\mu-1} G_\mu^{-1} G_j \quad \text{for} \quad i < j. \tag{213}$$

Analogous relations are valid also for the new sequence $\bar{Q}_0, \ldots, \bar{Q}_{\mu-1}$, and, additionally,

$$\bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \bar{G}_j = \bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} G_j, \tag{214}$$

$$\bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \bar{G}_j = \bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \bar{G}_j \Pi_j. \tag{215}$$

With regard of $\bar{Q}_l = \bar{Q}_l Q_l,\ Q_l = Q_l \bar{Q}_l$ for $l \geq k+1$, we have further

$$\bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \bar{G}_j \Pi_{\mu-1} = 0, \quad \text{for} \quad j \geq k. \tag{216}$$

Now, assume the projector function sequence $Q_0, \ldots, Q_{\mu-1}$ to provide an already advanced decoupling such that

$$Q_{\mu-1*}\Pi_{\mu-1} = 0, \ldots, Q_{k+1*}\Pi_{\mu-1} = 0.$$

Mind $k \leq \mu - 2$. Taking into account the relation $Q_{\mu-1} G_\mu^{-1} G_{\mu-1} = Q_{\mu-1} P_{\mu-1} = 0$, we immediately conclude

$$\begin{aligned} \bar{Q}_{\mu-1*}\bar{\Pi}_{\mu-1} = \bar{Q}_{\mu-1} \bar{G}_\mu^{-1} \bar{B}_{\mu-1} \bar{\Pi}_{\mu-1} &= \bar{Q}_{\mu-1} \underbrace{Q_{\mu-1} Z_\mu^{-1}}_{=Q_{\mu-1}} G_\mu^{-1} \bar{B}_{\mu-1} \underbrace{\Pi_{\mu-2} \bar{\Pi}_{\mu-1}}_{=\Pi_{\mu-1}} \\ &= \bar{Q}_{\mu-1} Q_{\mu-1} G_\mu^{-1} B_{\mu-1} \Pi_{\mu-1} = \bar{Q}_{\mu-1} Q_{\mu-1*} \Pi_{\mu-1} = 0. \end{aligned}$$

Next, for $k \leq i \leq \mu-2$, we investigate the terms

$$\begin{aligned} \bar{Q}_{i*}\bar{\Pi}_{\mu-1} &= \bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \bar{\mathfrak{B}}_i \bar{\Pi}_{\mu-1} \\ &= \bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \mathfrak{B}_i \bar{\Pi}_{\mu-1} + \mathfrak{D}_i, \end{aligned}$$

with $\mathfrak{D}_i := \bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \{\bar{\mathfrak{B}}_i - \mathfrak{B}_i\} \bar{\Pi}_{\mu-1}$. First we show that $\mathfrak{D}_i = 0$ thanks to (214)-(216). Namely, we have by definition

$$\begin{aligned} \mathfrak{D}_i = {} & \bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \{ \bar{B}_i + \bar{G}_i D^- (D\bar{\Pi}_{\mu-1} D^-)' D\bar{\Pi}_{i-1} - B_i \\ & - G_i D^- (D\Pi_{\mu-1} D^-)' D\Pi_{i-1} \} \bar{\Pi}_{\mu-1} \\ = {} & \bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\mu-1} \bar{G}_\mu^{-1} \{ -\bar{G}_i D^- (D\bar{\Pi}_i D^-)' D\Pi_i + G_i \sum_{l=0}^{i-1} Q_l \mathfrak{A}_{i,l} + \bar{G}_i D^- (D\bar{\Pi}_{\mu-1} D^-)' D\bar{\Pi}_{i-1} \\ & - G_i D^- (D\Pi_{\mu-1} D^-)' D\Pi_{i-1} \} \bar{\Pi}_{\mu-1}, \end{aligned}$$

yielding

$$\begin{aligned}\mathfrak{D}_i = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{&-(D\Pi_iD^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_iD^-)'D\Pi_i\\ &+(D\Pi_{\mu-1}D^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{\mu-1}D^-)'(D\Pi_{i-1}D^- - D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1})\\ &-(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}\\ = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{&(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_i+(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\\ &-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1}-(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{\mu-1}-(D\Pi_{\mu-1}D^-)'D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}.\end{aligned}$$

Due to $\Pi_i\bar{\Pi}_{\mu-1} = \Pi_{\mu-1}$ we arrive at

$$\begin{aligned}\mathfrak{D}_i &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-\{-(D\Pi_{\mu-1}D^-)'D\Pi_{k-1}\bar{Q}_kD^-D\Pi_{i-1}\}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_iD^-D\Pi_{\mu-1}D^-(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{i-1}\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\bar{G}_i\Pi_{\mu-1}D^-(D\Pi_{k-1}\bar{Q}_kD^-)'D\Pi_{i-1}\bar{\Pi}_{\mu-1} = 0,\end{aligned}$$

which proves the relation

$$\bar{Q}_{i*}\bar{\Pi}_{\mu-1} = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\mu-1}\bar{G}_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1} \tag{217}$$

for $k \le i \le \mu-2$. By means of the formula

$$Z_jZ_{j+1}^{-1} = I - \sum_{l=0}^{j-1} Q_l\mathfrak{A}_{j,l}Q_j$$

being available for $j = k+1,\dots,\mu-1$, we rearrange the terms in (217) as

$$\begin{aligned}\bar{Q}_{i*}\bar{\Pi}_{\mu-1} &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}Z_{i+1}Z_{i+2}^{-1}P_{i+2}\cdots Z_{\mu-1}^{-1}P_{\mu-1}Z_{\mu-1}Z_\mu^{-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}\\ &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}\\ &\quad+\sum_{j=i+1}^{\mu-2}\mathfrak{E}_{i,j}Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}+\mathfrak{E}_{i,\mu-1}Q_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1}.\end{aligned}$$

The very last term in this formula disappears because of

$$\begin{aligned}Q_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\bar{\Pi}_{\mu-1} &= Q_{\mu-1}G_\mu^{-1}B_i\bar{\Pi}_{\mu-1}\\ &= Q_{\mu-1}G_\mu^{-1}B_{\mu-1}\bar{\Pi}_{\mu-1} = Q_{\mu-1*}(I-\Pi_{k-1}Q_k)\Pi_{\mu-1} = Q_{\mu-1*}\Pi_{\mu-1} = 0.\end{aligned}$$

Next we prove to vanish also the involved sum. For this aims we consider the relation

$$(B_j - B_i)\Pi_{\mu-1} = -\sum_{l=i+1}^{j} G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1}, \quad \text{for} \quad j \ge i+1. \tag{218}$$

We first assume $i > k$ leading to $\mathfrak{B}_i\bar{\Pi}_{\mu-1} = \mathfrak{B}_i\Pi_{i-1}\Pi_{\mu-1} = \mathfrak{B}_i\Pi_{\mu-1}$ and further

$$\begin{aligned}&Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\ &= \underbrace{Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_j\Pi_{\mu-1}}_{=Q_{j*}\Pi_{\mu-1}=0}+Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}(\mathfrak{B}_i-\mathfrak{B}_j)\Pi_{\mu-1}\\ &= Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\{\sum_{l=i+1}^{j}G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1}+(G_j-G_i)(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\}.\end{aligned}$$

Applying once more the properties (211) and (213), we derive

$$
\begin{aligned}
&Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\
&\quad = Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}\{\sum_{l=i+1}^{j} G_lD^-(D\Pi_lD^-)'D\Pi_{\mu-1} + (G_j-G_i)(D\Pi_{\mu-1}D^-)'D\Pi_{\mu-1}\}\\
&\quad = Q_jP_{j+1}\cdots P_{\mu-1}G_\mu^{-1}G_j\Pi_jD^-\sum_{l=i+1}^{j}(D\Pi_lD^-)'D\Pi_{\mu-1} = 0.
\end{aligned}
$$

Now, for $i>k$, it results that

$$
\begin{aligned}
\bar{Q}_{i*}\Pi_{\mu-1} &= \bar{Q}_iZ_{i+1}^{-1}P_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1} = \bar{Q}_iQ_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_i\Pi_{\mu-1}\\
&= \bar{Q}_iQ_{i*}\Pi_{\mu-1} = 0,
\end{aligned}
$$

which verifies property (208). By the same means one obtains

$$
\begin{aligned}
\bar{Q}_{k*}\Pi_{\mu-1} &= \underbrace{\bar{Q}_kZ_{k+1}^{-1}}_{=Q_k}P_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_k\Pi_{\mu-1} = Q_kP_{k+1}\cdots P_{\mu-1}G_\mu^{-1}\mathfrak{B}_k\Pi_{\mu-1}\\
&= Q_{k*}\Pi_{\mu-1}.
\end{aligned}
$$

Finally, it remains to investigate the expression $\bar{Q}_{k*}\bar{\Pi}_{\mu-1}$. Since $\bar{Q}_{k*}$ also projects onto $\operatorname{im}\bar{Q}_k = \ker G_k$, it follows that $\bar{Q}_{k*}\bar{Q}_k = \bar{Q}_k$. This proves property (209), namely

$$
\begin{aligned}
\bar{Q}_{k*}\bar{\Pi}_{\mu-1} &= \bar{Q}_{k*}(I-\Pi_{k-1}\bar{Q}_k)\Pi_{\mu-1} = \bar{Q}_{k*}\Pi_{\mu-1} - \bar{Q}_{k*}\Pi_{k-1}\bar{Q}_k\Pi_{\mu-1}\\
&= Q_{k*}\Pi_{\mu-1} - \bar{Q}_k\Pi_{\mu-1} = (Q_{k*}-\bar{Q}_k)\Pi_{\mu-1}.
\end{aligned}
$$

□

G Admissible projectors for $Nx'+x=r$

In this part, admissible projectors are generated for the DAE (219) with a nilpotent matrix function N typical for the normal form in the framework of strangeness index (cf. [KM06]). Our admissible projectors are given explicitly by formulas (229) below, they have block upper triangular form in correspondence to the strict block upper triangular form of N.

Roughly speaking Lemma G.1 below is the technical key when proving that any DAE which has a well-defined regular strangeness index is at the same time regular in the tractability-index framework, and, in particular, the constant-rank requirements associated to the strangeness index are sufficient for the constant-rank conditions associated to the tractability index.

We deal with the special DAE

$$Nx'+x=r, \tag{219}$$

given by a matrix function $N\in C(\mathcal{I},L(\mathbb{R}^m))$, $\mathcal{I}\subseteq\mathbb{R}$ an interval, that has uniform on $\mathcal{I}$ strict block upper triangular structure

$$
N = \begin{array}{c} \begin{bmatrix} 0 & N_{12} & \dots & N_{1\mu} \\ & 0 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & 0 & N_{\mu-1\mu} \\ & & & 0 \end{bmatrix} \end{array}
\begin{array}{l} \}\ \ell_1 \\ \\ \\ \}\ \ell_{\mu-1} \\ \}\ \ell_\mu \end{array},
$$

$1 \leq \ell_1 \leq \ldots \leq \ell_\mu$, $\ell_1 + \ldots + \ell_\mu = m$, $\mu \geq 2$. The blocks N_{ii+1}, $i = 1, \ldots, \mu - 1$, are supposed to have full row rank each, i.e.

$$\operatorname{rank} N_{ii+1} = \ell_i, \quad i = 1, \ldots, \mu - 1. \tag{220}$$

This implies all powers of N to have constant rank, namely

$$\begin{aligned} \operatorname{rank} N &= \ell_1 + \ldots + \ell_{\mu-1}, \\ \operatorname{rank} N^k &= \ell_1 + \ldots + \ell_{\mu-k}, \quad k = 1, \ldots, \mu - 1, \\ \operatorname{rank} N^\mu &= 0. \end{aligned} \tag{221}$$

N is nilpotent with index μ, i.e. $N^{\mu-1} \neq 0$, $N^\mu = 0$. For $i = 1, \ldots, \mu - 1$, we introduce projectors $\mathcal{V}^{[1]}_{i+1i+1} \in C(\mathcal{I}, L(\mathbb{R}^{\ell_{i+1}}))$ onto the continuous subspace $\ker N_{i,i+1}$, and $\mathcal{U}^{[1]}_{i+1,i+1} := I_{\ell_{i+1}} - \mathcal{V}^{[1]}_{i+1i+1}$. $\mathcal{V}^{[1]}_{i+1i+1}$ and $\mathcal{U}^{[1]}_{i+1i+1}$ have constant rank $\ell_{i+1} - \ell_i$ and ℓ_i, respectively. Exploiting the structure of N we built a projector $\mathcal{V}^{[1]} \in C(\mathcal{I}, L(\mathbb{R}^m))$ onto the continuous subspace $\ker N$, which has a corresponding block upper triangular structure

$$\mathcal{V}^{[1]} = \begin{array}{l} \begin{bmatrix} I & & & & \\ & \mathcal{V}^{[1]}_{22} & * & \ldots & * \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & \mathcal{V}^{[1]}_{\mu\mu} \end{bmatrix} \end{array} \begin{array}{l} \} \ \ell_1 \\ \\ \\ \} \ \ell_{\mu-1} \\ \} \ \ell_\mu \end{array} . \tag{222}$$

The entries indicated by "$*$" are uniquely determined by the entries of N and generalized inverses N^-_{ii+1} with

$$N^-_{ii+1} N_{ii+1} = \mathcal{V}^{[1]}_{i+1i+1}, \quad N_{ii+1} N^-_{ii+1} = I_{\ell_i}, \quad i = 1, \ldots, \mu - 1.$$

In the following, we assume the nullspace $\ker N$ to be just a C^1 subspace, and the projector $\mathcal{V}^{[1]}$ to be continuously differentiable. Obviously, the property $N \in C^1(\mathcal{I}, L(\mathbb{R}^m))$ is sufficient for that but might be to generous. For this reason, we do not specify further smoothness conditions in terms of N but in terms of projectors and subspaces.
Taking use of $N = N\mathcal{U}^{[1]}$, $\mathcal{U}^{[1]} := I - \mathcal{V}^{[1]}$, we reformulate the DAE (219) as

$$N(\mathcal{U}^{[1]}x)' + (I - N\mathcal{U}^{[1]\prime})x = r. \tag{223}$$

The matrix function $N\mathcal{U}^{[1]\prime}$ is again strictly block upper triangular, and $I - N\mathcal{U}^{[1]\prime}$ is nonsingular, block upper triangular with identity diagonal blocks.

$$M_0 := (I - N\mathcal{U}^{[1]\prime})^{-1} N = \sum_{\ell=0}^{\mu-1} (N\mathcal{U}^{[1]\prime})^\ell N$$

has the same strict block upper triangular structure as N, the same nullspace, and entries $(M_0)_{ii+1} = N_{ii+1}$, $i = 1, \ldots, \mu - 1$. Scaling equation (223) by $(I - N\mathcal{U}^{[1]\prime})^{-1}$ yields

$$M_0(\mathcal{U}^{[1]}x)' + x = q, \tag{224}$$

where $q := (I - N\mathcal{U}^{[1]\prime})r$. By construction, the DAE (224) has a properly stated leading term (cf. Definition 2.1). Written as a general linear DAE

$$A(Dx)' + Bx = q$$

with $A = M_0$, $D = \mathcal{U}^{[1]}$, $B = I$, we have $\ker A = \ker M_0 = \ker N = \ker \mathcal{U}^{[1]}$, $\operatorname{im} D = \operatorname{im} \mathcal{U}^{[1]}$, $R = \mathcal{U}^{[1]}$.
Next we choose $D^- = \mathcal{U}^{[1]}$, and, correspondingly $P_0 = \mathcal{U}^{[1]}$, $Q_0 = \mathcal{V}^{[1]}$. With these projectors, $\Pi_0 = P_0$, and $G_0 = AD = M_0\mathcal{U}^{[1]} = M_0$, $B_0 = I$, we form a matrix function sequence and admissible projectors $Q_0, \dots, Q_\kappa$ for the DAE (224) as described in Section 3. In particular, we shall prove this DAE to be regular with tractability index μ.
The first matrix function (cf. Section 3) G_1 is

$$G_1 = M_0 + Q_0,$$

and $G_1 z = 0$, i.e. $(M_0 + Q_0)z = 0$, leads to $P_0 M_0 z = 0$, $Q_0 z = -Q_0 M_0 P_0 z$, $z = (I - Q_0 M_0)P_0 z$, $z \in \ker P_0 M$. Because of $P_0 M_0 = M_0^- M_0 M_0$, $M_0^2 = M_0 P_0 M_0$ the nullspaces of $P_0 M_0$ and M_0^2 coincide. The inclusion $\ker M_0 \subset \ker M_0^2 = \ker P_0 M_0$ allows for the decomposition $\ker M_0^2 = \ker M_0 \oplus P_0 \ker M_0^2$. If $\mathcal{V}^{[2]}$ denotes a projector onto $\ker M_0^2$, $\mathcal{U}^{[2]} := I - \mathcal{V}^{[2]}$, then it follows that

$$\begin{aligned}
\operatorname{im} \mathcal{V}^{[2]} &= \operatorname{im} \mathcal{V}^{[1]} \oplus \operatorname{im} \mathcal{U}^{[1]}\mathcal{V}^{[2]}, \\
\mathcal{V}^{[2]}\mathcal{V}^{[1]} &= \mathcal{V}^{[1]}, \quad (\mathcal{U}^{[1]}\mathcal{U}^{[2]})^2 = \mathcal{U}^{[1]}\mathcal{U}^{[2]}, \\
(\Pi_0 \mathcal{V}^{[2]})^2 &= \Pi_0 \mathcal{V}^{[2]}, \\
\operatorname{rank} \mathcal{U}^{[2]} &= \operatorname{rank} M_0^2 = \ell_1 + \ldots + \ell_{\mu-2}, \\
\operatorname{rank} \mathcal{V}^{[2]} &= \ell_{\mu-1} + \ell_\mu, \\
\operatorname{rank} \Pi_0 \mathcal{V}^{[2]} &= \operatorname{rank} \mathcal{V}^{[2]} - \operatorname{rank} \mathcal{V}^{[1]} = \ell_{\mu-1}.
\end{aligned}$$

The matrix function

$$Q_1 := (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]} \tag{225}$$

has the properties

$$Q_1 Q_0 = (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]}\mathcal{V}^{[1]} = (I - Q_0 M_0)\Pi_0 \mathcal{V}^{[1]} = (I - Q_0 M_0)\Pi_0 Q_0 = 0,$$

hence $Q_1 \cdot Q_1 = Q_1$, and

$$\begin{aligned}
G_1 Q_1 &= (M_0 + Q_0)(I - Q_0 M_0)\Pi_0 \mathcal{V}^{[2]} = (M_0 - Q_0 M_0 + Q_0)\Pi_0 \mathcal{V}^{[2]} \\
&= P_0 M_0 \Pi_0 \mathcal{V}^{[2]} = P_0 M_0 \mathcal{V}^{[2]} = 0.
\end{aligned}$$

It becomes clear, that Q_1 is actually the wanted projector onto $\ker G_1$, if $\operatorname{rank} Q_1 = m - \operatorname{rank} G_1$. $I - Q_0 M_0$ is nonsingular, and Q_1 has the same rank as $\Pi_0 \mathcal{V}^{[2]}$, that is, $\operatorname{rank} Q_1 = \ell_{\mu-1}$. Proposition 2.4(3) allows for an easy rank determination of the matrix function G_1. With

$$\mathcal{W}_0 := \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & I \end{bmatrix} \begin{matrix} \\ \\ \\ \} \; \ell_\mu \end{matrix}$$

we find $\operatorname{im} G_1 = \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 B_0 Q_0 = \operatorname{im} M_0 \oplus \operatorname{im} \mathcal{W}_0 Q_0$, thus $r_1 = r_0 + \operatorname{rank} \mathcal{V}_{\mu\mu}^{[1]} = m - \ell_\mu + \ell_\mu - \ell_{\mu-1} = m - \ell_{\mu-1}$. It comes out that Q_0, Q_1 are admissible, supposed $\pi_1 = \mathcal{U}^{[1]}\mathcal{U}^{[2]}$ is continuously differentiable.

Next, due to the structure of M_0^2, the projector $\mathcal{V}^{[2]}$ can be chosen to be block upper triangular,

$$\mathcal{V}^{[2]} = \begin{bmatrix} I & & & & \\ & I & & & \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}, \quad \mathcal{U}^{[2]} = I - \mathcal{V}^{[2]} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{bmatrix}.$$

The entries in the right lower corners shall play their role in rank calculations. They are

$$\mathcal{V}_{\mu\mu}^{[2]} = I - \mathcal{U}_{\mu\mu}^{[2]}, \quad \mathcal{U}_{\mu\mu}^{[2]} = (N_{\mu-2\mu-1}N_{\mu-1\mu})^{-}N_{\mu-2\mu-1}N_{\mu-1\mu}.$$

To realize this we just remember that the entry $(\mu-2,\mu)$ of M_0^2 is $[M_0^2]_{\mu-2\mu} = N_{\mu-2\mu-1}N_{\mu-1\mu}$. Both $N_{\mu-2\mu-1}$ and $N_{\mu-1\mu}$ have full row-rank $\ell_{\mu-2}$ resp. $\ell_{\mu-1}$. Therefore, the product $N_{\mu-2\mu-1}N_{\mu-1\mu}$ has full row-rank equal to $\ell_{\mu-2}$. From this it follows that

$$\operatorname{rank} \mathcal{V}_{\mu\mu}^{[2]} = \dim\ker N_{\mu-2\mu-1}N_{\mu-1\mu} = \ell_{\mu} - \ell_{\mu-2}.$$

Taking into account the inclusion

$$\operatorname{im} \mathcal{V}_{\mu\mu}^{[1]} = \ker N_{\mu-1\mu} \subseteq \ker N_{\mu-2\mu-1}N_{\mu-1\mu} = \operatorname{im} \mathcal{V}_{\mu\mu}^{[2]}$$

we find

$$\operatorname{rank} \mathcal{U}_{\mu\mu}^{[1]}\mathcal{V}_{\mu\mu}^{[2]} = \operatorname{rank} \mathcal{V}_{\mu\mu}^{[2]} - \operatorname{rank} \mathcal{V}_{\mu\mu}^{[1]} = \ell_{\mu-1} - \ell_{\mu-2}.$$

By Proposition 2.4(3), with the projector along $\operatorname{im} G_1$

$$\mathcal{W}_1 := \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \mathcal{U}_{\mu\mu}^{[1]} \end{bmatrix}, \quad \mathcal{W}_1 = \mathcal{W}_0\mathcal{U}^{[1]},$$

we compute (before knowing G_2 in detail)

$$\begin{aligned} &\operatorname{im} G_2 = \operatorname{im} G_1 \oplus \operatorname{im} \mathcal{W}_1 Q_1, \quad \mathcal{W}_1 Q_1 = \mathcal{W}_0\mathcal{U}^{[1]}\mathcal{V}^{[2]}, \\ &r_2 = r_1 + \operatorname{rank} \mathcal{W}_1 Q_1 = r_1 + \operatorname{rank} \mathcal{U}_{\mu\mu}^{[1]}\mathcal{V}_{\mu\mu}^{[2]} = m - \ell_{\mu-1} + \ell_{\mu-1} - \ell_{\mu-2} = m - \ell_{\mu-2}. \end{aligned}$$

We compute $G_2 = G_1 + (B_0\Pi_0 - G_1D^{-}(D\Pi_1D^{-})'D\Pi_0)Q_1$ (cf. Section 3) itself as

$$\begin{aligned} G_2 &= M_0 + Q_0 + \Pi_0 Q_1 - (M_0 + Q_0)P_0\Pi_1'\Pi_0 Q_1 \\ &= M_0 + Q_0 + \Pi_0 Q_1 - M_0 F_1 \Pi_0 Q_1, \end{aligned}$$

where $F_1 := P_0\Pi_1'\Pi_0 Q_1$ is block upper triangular as all its factors. It results that

$$G_2 = M_0 + Q_0 + (I - M_0F_1)P_0(I - \Pi_1),$$

G_2 is block upper triangular. Due to the nonsingularity of $I - M_0F_1$, as well as the simple property $(I - M_0F_1)Q_0 = Q_0$, we may use the description

$$G_2 = (I - M_0F_1)^{-1}\{M_1 + I - \Pi_1\},$$

where $M_1 := (I - M_0F_1)^{-1}M_0$ has again the strict block upper triangular structur of N, and entries $[M_1]_{ii+1} = N_{ii+1}$, $i = 1, \ldots, \mu - 1$. From the representation

$$\begin{aligned} \Pi_1 M_1 &= \Pi_1 P_0 M_1 = \Pi_1 P_0 (I + M_0F_1 + \ldots + (M_0F_0)^{\mu-1})M_0 \\ &= \Pi_1 (I + M_0F_1 + \ldots + (M_0F_1)^{\mu-1})P_0M_0 \end{aligned}$$

we know the inclusion $\ker \Pi_0 M_0 \subseteq \ker \Pi_1 M_1$ to be valid. Furthermore, we have $\ker M_0^2 M_1 = \ker \Pi_1 M_1$ because of the representations $\ker \mathcal{U}^{[2]} = \ker M_0^2 = \ker P_0M_0$, $\Pi_1 M_1 = P_0\mathcal{U}^{[2]}M_1 = P_0(M_0^2)^- M_0^2 M_1$, and $M_0^2M_1 = M_0^2\mathcal{U}^{[2]}M_1 = M_0^2P_0\mathcal{U}^{[2]}M_1 = M_0^2\Pi_1M_1$.

The next lemma shows that we may proceed further in this way to construct admissible projectors for the DAE (224). We shall use certain auxilary continuous matrix functions which are determined from level to level as

$$\begin{aligned} F_0 &:= 0, \\ F_i &:= F_{i-1} + \sum_{\ell=1}^{i} P_0 \Pi_\ell' \Pi_{i-1} Q_i = \sum_{j=1}^{i}\sum_{\ell=1}^{j} P_0 \Pi_\ell' \Pi_{j-1} Q_i, \quad i \geq 1, \qquad (226) \\ H_2 &:= H_1 := H_0 := 0, \\ H_i &:= H_{i-1} + \sum_{\ell=2}^{i-1} (I - H_{\ell-1}) P_0 (I - \Pi_{\ell-1}) \Pi_\ell' \Pi_{i-1} Q_i \\ &= \sum_{j=3}^{i}\sum_{\ell=2}^{j-1} (I - H_{\ell-1}) P_0 (I - \Pi_{\ell-1}) \Pi_\ell' \Pi_{j-1} Q_j, \quad i \geq 3. \qquad (227) \end{aligned}$$

This matrix functions inherit the block upper triangular structure. They disappear if the projectors $\Pi_1, \ldots, \Pi_i$ do not vary with time (what is given at least in the constant coefficient case).
It holds that $F_i = F_iP_0$, $H_i = H_iP_0$. The products F_iM_0 are strictly block upper triangular so that $I - M_0F_i$ is nonsingular, and

$$M_i := (I - M_0F_i)^{-1}M_0 \qquad (228)$$

has again strict block upper triangular structure. The entries $(j, j+1)$ of M_i coincide with that of N, i.e.

$$[M_i]_{jj+1} = N_{jj+1}. \qquad (229)$$

If the projectors $\Pi_0, \ldots, \Pi_i$ are constant, then we have simply $M_i = M_0 = N$.

Lemma G.1 *Let N be sufficiently smooth so that the continuous projectors Π_i arising below are even continuously differentiable. Let $k \in \mathbb{N}$, $k \leq \mu - 1$, and let $Q_0 := \mathcal{V}^{[1]}$ be given by* (222)*, and, for $i = 1, \ldots, k$,*

$$Q_i := \Big(I - \sum_{j=0}^{i-1} Q_j (I - H_{i-1})^{-1} M_{i-1}\Big) \Pi_{i-1} \mathcal{V}^{[i+1]}, \qquad (230)$$

$\mathcal{V}^{[i+1]} \in C(\mathcal{I}, L(\mathbb{R}^m))$ a block upper triangular projector onto $\ker M_0^2 M_1 \cdots M_{i-1}$, $\mathcal{U}^{[i+1]} := I - \mathcal{V}^{[i+1]}$. Then, the matrix functions $Q_0, \ldots, Q_k$ are admissible projectors for the DAE

(224) *on $\mathcal{I}$, and, for $i = 1, \ldots, k$, it holds that*

$$\Pi_{i-1}Q_i = \Pi_{i-1}\mathcal{V}^{[i+1]}, \quad \Pi_i = \mathcal{U}^{[1]} \cdots \mathcal{U}^{[i+1]}, \tag{231}$$
$$\ker \Pi_{i-1}M_{i-1} \subset \ker \Pi_i M_i, \tag{232}$$
$$\ker \Pi_i M_i = \ker M_0^2 M_1 \ldots M_i, \tag{233}$$
$$G_{i+1} = M_0 + Q_0 + (I - M_0F_i)(I - H_i)P_0(I - \Pi_i), \tag{234}$$
$$r_{i+1} = \operatorname{rank} G_{i+1} = m - \ell_{\mu-i-1}, \quad \operatorname{im} G_{i+1} = \operatorname{im} G_i \oplus \operatorname{im} \mathcal{W}_0\Pi_{i-1}Q_i,$$

and $I - H_i$ is nonsingular.

Before we turn to the proof of Lemma G.1 we realize that it provides admissible projectors $Q_0, \ldots, Q_{\mu-1}$ and characteristics $r_0 = m - \ell_\mu, \ldots, r_{\mu-1} = m - \ell_1 < m$. Because of the strict block upper triangular structure of $M_0, \ldots, M_{\mu-2}$, the product $M_0^2 M_1 \cdots M_{\mu-2}$ disappears (as N^μ does). This leads to $\mathcal{V}^{[\mu]} = I$, $\mathcal{U}^{[\mu]} = 0$, thus $\Pi_{\mu-1} = 0$, and

$$\begin{aligned} G_\mu &= M_0 + Q_0 + (I - M_0F_{\mu-1})(I - H_{\mu-1})P_0(I - \Pi_{\mu-1}) \\ &= M_0 + Q_0 + (I - M_0F_{\mu-1})(I - H_{\mu-1)}P_0 \\ &= (I - M_0F_{\mu-1})(I - H_{\mu-1})\{(I - H_{\mu-1})^{-1}M_{\mu-1} + I\}. \end{aligned}$$

The factors $I - M_0F_{\mu-1}$ and $I - H_{\mu-1}$ are already known to be nonsingular. $(I - H_{\mu-1})^{-1}M_{\mu-1}$ inherits the strict block upper triangular structure from $M_{\mu-1}$, but then $I + (I - H_{\mu-1})^{-1}M_{\mu-1}$ is nonsingular, and so is G_μ. By this we have proved an important consequence of Lemma G.1:

Proposition G.2 *Let N be sufficiently smooth to make the continuous projectors $\Pi_0, \ldots, \Pi_{\mu-2}$ even continuously differentiable. Then the DAE* (224) *is on $\mathcal{I}$ regular with tractability index μ and characteristic values*

$$r_i = m - \ell_{\mu-i}, \quad i = 0, \ldots, \mu - 1, \quad r_\mu = m.$$

It holds that $\Pi_{\mu-1} = 0$, and there is no inherent regular ODE within the DAE.

To prepare the proof of Lemma G.1 we give the following one.

Lemma G.3 *Let $\mathcal{V}_i \in L(\mathbb{R}^m)$ be idempotent, $\mathcal{U}_i := I - \mathcal{V}_i$, $L_i := \operatorname{im} \mathcal{V}_i$, $\nu_i := \operatorname{rank} \mathcal{V}_i$, $i = 1, \ldots, k$, and $L_i \subseteq L_{i+1}$, $i = 1, \ldots, k-1$.*
Then the products $\mathcal{U}_1\mathcal{V}_2, \ldots, \mathcal{U}_1 \cdots \mathcal{U}_{k-1}\mathcal{V}_k, \mathcal{U}_1\mathcal{U}_2, \ldots, \mathcal{U}_1 \cdots \mathcal{U}_k$ are projectors, too, and it holds that

$$\begin{aligned} &\mathcal{U}_1 \cdots \mathcal{U}_i \mathcal{V}_{i+1}\mathcal{V}_j = 0, \quad 1 \le j \le i, \quad i = 1, \ldots, k-1, \\ &\ker \mathcal{U}_1 \cdots \mathcal{U}_i = L_i, \quad i = 1, \ldots, k, \\ &L_k = L_1 \oplus \mathcal{U}_1 L_2 \oplus \ldots \oplus \mathcal{U}_1 \cdots \mathcal{U}_{k-1}L_k, \\ &\dim \mathcal{U}_1 \cdots \mathcal{U}_{k-1}L_k = \nu_k - \nu_{k-1}. \end{aligned} \tag{235}$$

Proof: The inclusions $L_1 \subseteq L_2 \subseteq \ldots \subseteq L_{i+1}$ lead to $\mathcal{V}_{i+1}\mathcal{V}_j = \mathcal{V}_j$, for $j = 1, \ldots, i$. Compute

$$\begin{aligned} \mathcal{U}_1\mathcal{V}_2\mathcal{U}_1\mathcal{V}_2 &= \mathcal{U}_1\mathcal{V}_2(I - \mathcal{V}_1)\mathcal{V}_2 = \mathcal{U}_1\mathcal{V}_2 - \mathcal{U}_1\mathcal{V}_1\mathcal{V}_2 = \mathcal{U}_1\mathcal{V}_2, \\ \mathcal{U}_1\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2 &= \mathcal{U}_1(I - \mathcal{V}_2)(I - \mathcal{V}_1)\mathcal{U}_2 = \mathcal{U}_1(I - \mathcal{V}_1 - \mathcal{V}_2 + \mathcal{V}_1)\mathcal{U}_2 = \mathcal{U}_1\mathcal{U}_2. \end{aligned}$$

$L_2 = \operatorname{im}\mathcal{V}_2 \subseteq \ker\mathcal{U}_1\mathcal{U}_2$ holds trivially. $z \in \ker\mathcal{U}_1\mathcal{U}_2$ means $(I-\mathcal{V}_1)(I-\mathcal{V}_2)z = 0$, hence $z = \mathcal{V}_1 z + \mathcal{V}_2 z - \mathcal{V}_1\mathcal{V}_2 z \in L_2$, so that $\ker\mathcal{U}_1\mathcal{U}_2 = L_2$ is true.
By induction, if $\mathcal{U}_1\cdots\mathcal{U}_{i-1}Q_i$, $\mathcal{U}_1\cdots\mathcal{U}_i$ are projectors, $\ker\mathcal{U}_1\cdots\mathcal{U}_i = L_i$, then these properties remain valid for $i+1$ instead of i. Namely,

$$\begin{aligned}
&\mathcal{U}_1\cdots\mathcal{U}_{i+1}\mathcal{U}_1\cdots\mathcal{U}_{i+1} = \mathcal{U}_1\cdots\mathcal{U}_i(I-\mathcal{V}_{i+1})\mathcal{U}_1\cdots\mathcal{U}_{i+1} = \mathcal{U}_1\cdots\mathcal{U}_i\mathcal{U}_1\cdots\mathcal{U}_{i+1} = \mathcal{U}_1\cdots\mathcal{U}_{i+1},\\
&\mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1}\mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1} = \mathcal{U}_1\cdots\mathcal{U}_i\mathcal{V}_{i+1},\\
&L_{i+1} = \ker\mathcal{U}_{i+1} \subseteq \ker\mathcal{U}_1\cdots\mathcal{U}_{i+1},
\end{aligned}$$

and $z \in \ker\mathcal{U}_1\cdots\mathcal{U}_{i+1}$ implies $\mathcal{U}_{i+1}z \in \operatorname{im}\mathcal{U}_1\cdots\mathcal{U}_i = L_i$, $z - \mathcal{V}_{i+1}z \in L_i$, hence $z \in L_i + L_{i+1} = L_{i+1}$. Now we can decompose

$$\begin{aligned}
L_2 &= L_1 \oplus \mathcal{U}_1 L_2,\\
L_3 &= L_1 \oplus \mathcal{U}_1 L_2 \oplus \mathcal{U}_1\mathcal{U}_2 L_3 = L_2 \oplus \mathcal{U}_1\mathcal{U}_2 L_3,\\
L_{i+1} &= \underbrace{L_1 \oplus \mathcal{U}_1 L_2 \oplus \ldots \oplus}_{=L_i}\mathcal{U}_1\cdots\mathcal{U}_i L_{i+1} = L_i \oplus \mathcal{U}_1\cdots\mathcal{U}_i L_{i+1},
\end{aligned}$$

and it results that $\dim\mathcal{U}_1\cdots\mathcal{U}_i L_{i+1} = v_{i+1} - v_i$, $i = 1,\ldots,k-1$. □

Proof of Lemma G.1: We apply induction. For $k=1$ the assertion is already proved, and the corresponding projector Q_1 is given by (225).
Let the assertion be true up to level k, and we are going to show its validity for level $k+1$. Stress one more that we deal with structured triangular matrices. We know already that $Q_0,\ldots,Q_k$ are admissible, and, in particular, it holds that $Q_iQ_j = 0$, for $0 \le j < i \le k$. A closer look to the auxilary matrix functions H_i (cf. (227)) shows that $H_iQ_1 = 0$, $H_iQ_2 = 0$, further $H_i\varPi_i = 0$, and

$$\varPi_{i-2}H_i = 0.$$

Namely, $\varPi_1 H_3 = \varPi_1 P_0(I-\varPi_1)\varPi_2'\varPi_1 Q_2 = 0$, and $\varPi_{j-3}H_{j-1} = 0$, for $j \le i$, implies $\varPi_{i-2}H_i = 0$ (due to $\varPi_{i-2}H_\ell = 0$, $\varPi_{i-2}P_0(I-\varPi_{\ell-1}) = 0$, $\ell = 1,\ldots,i-1$).
The functions $F_1,\ldots,F_k$ (cf. (226)) are well-defined, and they have the properties

$$(F_k - F_j)\varPi_k = 0, \quad (F_k - F_j)\varPi_j = F_k - F_j, \quad \text{for } j = 1,\ldots,k. \tag{236}$$

It follows that, for $j = 1,\ldots,k$,

$$(I - M_0F_k)^{-1}(I - M_0F_j) = I + (I - M_0F_k)^{-1}M_0(F_k - F_j)\varPi_j.$$

Next we verify the property

$$\varPi_{j-1}M_kQ_j = 0, \quad j = 0,\ldots,k. \tag{237}$$

From $G_jQ_j = 0$, $j = 0,\ldots,k$, we know

$$M_0Q_j + Q_0Q_j + (I - M_0F_{j-1})(I - H_{j-1})P_0(I - \varPi_{j-1})Q_j = 0. \tag{238}$$

Multiplication by $(I - M_0F_k)^{-1}$ leads to

$$M_kQ_j + Q_0Q_j + \{I + (I - M_0F_k)^{-1}M_0(F_k - F_{j-1})\varPi_{j-1}\}(I - H_{j-1})P_0(I - \varPi_{j-1})Q_j = 0,$$

and further, with account of $\Pi_{j-1}H_{j-1} = 0$, $\Pi_{j-1}P_0(I - \Pi_{j-1}) = 0$,

$$M_kQ_j + Q_0Q_j + (I - H_{j-1})P_0(I - \Pi_{j-1})Q_j = 0, \tag{239}$$

and hence $\Pi_{j-1}M_kQ_j = 0$, i.e. (237). Now it follows that $\Pi_kM_kQ_j = 0$, for $j = 0, \ldots, k$, hence

$$\Pi_kM_k = \Pi_kM_k\Pi_k, \tag{240}$$

a property that will appear to be very helpful.
Recall that we have already a nonsingular $I - H_k$, as well as

$$\begin{aligned} G_{k+1} &= M_0 + Q_0 + (I - M_0F_k)(I - H_k)P_0(I - \Pi_k) \\ &= (I - M_0F_k)(I - H_k)\{(I - H_k)^{-1}M_k + I - \Pi_k\}, \end{aligned} \tag{241}$$

and G_{k+1} has rank $r_{k+1} = m - \ell_{\mu-k-1}$. We have to show the matrix function

$$Q_{k+1} := \Big(I - \sum_{j=0}^{k} Q_j(I - H_k)^{-1}M_k\Big)\Pi_k\mathcal{V}^{[k+2]}$$

to be a suitable projector. We check first whether $G_{k+1}Q_{k+1} = 0$ is satisfied. Derive (cf. (241))

$$\begin{aligned} G_{k+1}Q_{k+1} &= (I - M_0F_k)\{M_k + (I - H_k)(I - \Pi_k)\}\Big(I - \sum_{j=0}^{k} Q_j(I - H_k)^{-1}M_k\Big)\Pi_k\mathcal{V}^{[k+2]} \\ &= (I - M_0F_k)\Big\{M_k - \sum_{j=1}^{k} M_kQ_j(I - H_k)^{-1}M_k \\ &\qquad -(I - H_k)\sum_{j=0}^{k} Q_i(I - H_k)^{-1}M_k\Big\}\Pi_k\mathcal{V}^{[k+2]} \\ &= (I - M_0F_k)\Big\{I - H_k - \sum_{j=1}^{k} M_kQ_j - (I - H_k)\sum_{j=0}^{k} Q_j\Big\}\cdot \\ &\qquad\qquad \cdot(I - H_k)^{-1}M_k\Pi_k\mathcal{V}^{[k+2]}. \end{aligned} \tag{242}$$

From (239) we obtain, for $j = 1, \ldots, k$,

$$\begin{aligned} M_kQ_j + (I - H_k)Q_j &= -Q_0Q_j - (I - H_{j-1})P_0(I - \Pi_{j-1})Q_j + (I - H_k)Q_j \\ &= P_0Q_j - H_kQ_j - (I - H_{j-1})(I - \Pi_{j-1})P_0Q_j \\ &= P_0Q_j - H_kQ_j - (I - H_{j-1})P_0Q_j + \Pi_{j-1}Q_j \\ &= -(H_k - H_{j-1})P_0Q_j + \Pi_{j-1}Q_j \end{aligned}$$

and, therefore,

$$\begin{aligned} \sum_{j=1}^{k}(M_kQ_j + (I - H_k)Q_j) &= \sum_{j=1}^{k}\Pi_{j-1}Q_j - \sum_{j=1}^{k}(H_k - H_{j-1})P_0Q_j \\ &= \sum_{j=1}^{k}\Pi_{j-1}Q_j - H_k. \end{aligned}$$

The last relation becomes true because of $(H_k - H_0)Q_1 = 0$, $(H_k - H_1)Q_2 = 0$, and the construction of H_i (cf. (227)),

$$\begin{aligned}
\sum_{j=1}^{k}(H_k - H_{j-1})P_0Q_j &= \sum_{j=3}^{k}(H_k - H_{j-1})P_0Q_j \\
&= \sum_{j=3}^{k}\Big[\sum_{\nu=j}^{k}\sum_{\ell=2}^{\nu-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{\nu-1}Q_\nu\Big]P_0Q_j \\
&= \sum_{j=3}^{k}\sum_{\ell=2}^{j-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{j-1}Q_j = H_k.
\end{aligned}$$

Together with (242) this yields

$$\begin{aligned}
G_{k+1}Q_{k+1} &= (I - M_0F_k)\Big\{I - H_k - \Big(\sum_{j=1}^{k}\mathit{\Pi}_{j-1}Q_j - H_k\Big) - Q_0\Big\}(I - H_k)^{-1}M_k\mathit{\Pi}_k\mathcal{V}^{[k+2]} \\
&= (I - M_0F_k)\Big\{I - Q_0 - \sum_{j=1}^{k}\mathit{\Pi}_{j-1}Q_j\Big\}(I - H_k)^{-1}M_k\mathit{\Pi}_k\mathcal{V}^{[k+2]} \\
&= (I - M_0F_k)\mathit{\Pi}_k(I - H_k)^{-1}M_k\mathit{\Pi}_k\mathcal{V}^{[k+2]}. \qquad (243)
\end{aligned}$$

For more specific information on $(I - H_k)^{-1}$ we consider the equation $(I - H_k)z = w$, i.e. (cf. (227))

$$(I - H_{k-1})z - \sum_{\ell=2}^{k-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{k-1}Q_kz = w. \qquad (244)$$

Because of $\mathit{\Pi}_{k-1}H_{k-1} = 0$, $\mathit{\Pi}_{k-1}H_{\ell-1} = 0$, $\mathit{\Pi}_{k-2}P_0(I - \mathit{\Pi}_{\ell-1}) = 0$, multiplication of (244) by $\mathit{\Pi}_{k-1}Q_k = \mathit{\Pi}_{k-1}Q_k\mathit{\Pi}_{k-1}$ yields $\mathit{\Pi}_{k-1}Q_kz = \mathit{\Pi}_{k-1}Q_kw$, such that

$$z = (I - H_{k-1})^{-1}\Big\{w + \sum_{\ell=2}^{k-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{k-1}Q_kw\Big\}$$

results, and further,

$$\begin{aligned}
(I - H_k)^{-1} &= (I - H_{k-1})^{-1}\Big(I - \sum_{\ell=2}^{k-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{k-1}Q_k\Big) \\
&= (I - H_3)^{-1}\Big(I + \sum_{\ell=2}^{3}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_3Q_4\Big)\cdots \\
&\qquad \cdots\Big(I + \sum_{\ell=2}^{k-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{k-1}Q_k\Big) \\
&= (I + P_0Q_1\mathit{\Pi}'_2\mathit{\Pi}_2Q_3)\cdots\Big(I + \sum_{\ell=2}^{k-1}(I - H_{\ell-1})P_0(I - \mathit{\Pi}_{\ell-1})\mathit{\Pi}'_\ell\mathit{\Pi}_{k-1}Q_k\Big).
\end{aligned}$$

This shows that $\Pi_k(I - H_k)^{-1} = \Pi_k$ holds true. On the other hand $F_k\Pi_k = 0$ is also given, what leads to

$$G_{k+1}Q_{k+1} = \Pi_k M_k \Pi_k \mathcal{V}^{[k+2]}.$$

With the help of (240), and taking into account that $\ker \Pi_k M_k = \ker M_0^2 M_1 \cdots M_k$, we arrive at

$$G_{k+1}Q_{k+1} = \Pi_k M_k \mathcal{V}^{[k+2]} = 0,$$

that is, the matrix function Q_{k+1} satisfies the condition $\operatorname{im} Q_{k+1} \subseteq \ker G_{k+1}$. The inclusions (cf. (232), (233))

$$\ker \Pi_{i-1} M_{i-1} = \ker M_0^2 M_1 \cdots M_{i-1} \subset \ker \Pi_i M_i = \ker M_0^2 M_1 \cdots M_i$$

are valid for $i = 1, \ldots, k$. This leads to

$$\operatorname{im} \mathcal{V}^{[1]} \subset \operatorname{im} \mathcal{V}^{[2]} \subset \ldots \subset \mathcal{V}^{[\kappa+2]}$$

what allows an application of Lemma G.3. We take use of the structural properties

$$\begin{aligned} \operatorname{rank} M_0^2 M_1 \cdots M_i &= \operatorname{rank} N^{i+2} = \ell_1 + \ldots + \ell_{\mu-i-2}, \\ \operatorname{rank} \mathcal{V}^{[i+2]} &= m - (\ell_1 + \ldots + \ell_{\mu-i-2}) = \ell_{\mu-i-1} + \ldots + \ell_\mu, \end{aligned}$$

so that Lemma G.3 yields

$$\operatorname{rank} \mathcal{U}^{[1]} \cdots \mathcal{U}^{[k+1]} \mathcal{V}^{[k+2]} = \operatorname{rank} \mathcal{V}^{[k+2]} - \operatorname{rank} \mathcal{V}^{[k+1]} = \ell_{\mu-k-1}.$$

Writing Q_{k+1} in the form

$$Q_{k+1} = \Big(I - \sum_{j=0}^{k} Q_j (I - H_k)^{-1} M_k \Pi_k\Big) \Pi_k \mathcal{V}^{[k+2]},$$

and realizing the first factor to be nonsingular, we conclude

$$\operatorname{rank} Q_{k+1} = \operatorname{rank} \Pi_k \mathcal{V}^{[k+2]} = \ell_{\mu-k-1} = m - \operatorname{rank} G_{k+1}.$$

Applying Lemma G.3 again we derive, for $j = 0, \ldots, k$,

$$\begin{aligned} Q_{k+1}Q_j &= \Big(I - \sum_{j=0}^{k} Q_j (I - H_k)^{-1} M_k\Big) \Pi_k \mathcal{V}^{[k+2]} Q_j, \\ \Pi_k \mathcal{V}^{[k+2]} Q_j &= \mathcal{U}^{[1]} \cdots \mathcal{U}^{[k+1]} \mathcal{V}^{[k+2]} Q_j \\ &= \mathcal{U}^{[1]} \cdots \mathcal{U}^{[k+1]} \mathcal{V}^{[k+2]} \mathcal{U}^{[1]} \cdots \mathcal{U}^{[j]} Q_j \\ &= \mathcal{U}^{[1]} \cdots \mathcal{U}^{[k+1]} \mathcal{V}^{[k+2]} \mathcal{U}^{[1]} \cdots \mathcal{U}^{[j]} \mathcal{V}^{[j+1]} = 0, \end{aligned}$$

such that $Q_{k+1}Q_j = 0$, $j = 0, \ldots, k$, and furthermore $Q_{k+1}Q_{k+1} = Q_{k+1}$. This completes the proof for Q_{k+1} to be a suitable projector function, and for $Q_0, \ldots, Q_k, Q_{k+1}$ to be admissible.

It remains to verify (232)–(234) for $i = k + 1$, to consider the rank of G_{k+2} as well as to show the nonsingularity of $I - H_{k+1}$.

First we consider the rank of G_{k+2}. Following Proposition 2.4(3) it holds that

$$\operatorname{im} G_{k+2} = \operatorname{im} G_{k+1} \oplus \operatorname{im} \mathcal{W}_{k+1} \Pi_k Q_{k+1},$$

with a projector $\mathcal{W}_{k+1}$ such that $\ker \mathcal{W}_{k+1} = \operatorname{im} G_{k+1}$. Because of

$$\begin{aligned}
\operatorname{im} G_{k+1} &= \operatorname{im} G_k \oplus \operatorname{im} \mathcal{W}_0 \Pi_{k-1} Q_k \\
&= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 Q_0 \oplus \ldots \oplus \operatorname{im} \mathcal{W}_0 \Pi_{k-1} Q_k \\
&= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 (Q_0 + \ldots + \Pi_{k-1} Q_k) \\
&= \operatorname{im} G_0 \oplus \operatorname{im} \mathcal{W}_0 (I - \Pi_k)
\end{aligned}$$

we may choose the projector

$$\mathcal{W}_{k+1} = \mathcal{W}_0 \Pi_k = \mathcal{W}_0 \Pi_k \mathcal{W}_0.$$

This leads to

$$\operatorname{im} G_{k+2} = \operatorname{im} G_{k+1} \oplus \operatorname{im} \mathcal{W}_0 \Pi_k Q_{k+1},$$

as well as to

$$\begin{aligned}
r_{k+2} &= r_{k+1} + \operatorname{rank} \mathcal{W}_0 \Pi_k Q_{k+1} = r_{k+1} + \operatorname{rank} [\Pi_k Q_{k+1}]_{\mu\mu} \\
&= r_{k+1} + \operatorname{rank} \mathcal{U}_{\mu\mu}^{[1]} \cdots \mathcal{U}_{\mu\mu}^{[k+1]} \mathcal{V}_{\mu\mu}^{[k+2]} = m - \ell_{\mu-k-1} + (\ell_{\mu-k-1} - \ell_{\mu-k-2}) \\
&= m - \ell_{\mu-k-2}.
\end{aligned}$$

Thereby, to show that $\operatorname{rank} \mathcal{U}_{\mu\mu}^{[1]} \cdots \mathcal{U}_{\mu\mu}^{[k+1]} \mathcal{V}_{\mu\mu}^{[k+2]} = \ell_{\mu-k-1} - \ell_{\mu-k-2}$ we recall that

$$\begin{aligned}
&\mathcal{V}_{\mu\mu}^{[1]} \text{ projects onto } \ker N_{\mu-1\mu}, \\
&\mathcal{V}_{\mu\mu}^{[2]} \text{ projects onto } \ker N_{\mu-2\mu-1} N_{\mu-1\mu}, \\
&\ldots \\
&\mathcal{V}_{\mu\mu}^{[k+1]} \text{ projects onto } \ker N_{\mu-k-1\mu-k} \cdots N_{\mu-1\mu}
\end{aligned}$$

and

$$\mathcal{V}_{\mu\mu}^{[k+2]} \text{ projects onto } \ker N_{\mu-k-2\mu-k-1} \cdots N_{\mu-1\mu},$$

and

$$\begin{aligned}
&\operatorname{im} \mathcal{V}_{\mu\mu}^{[1]} \subset \operatorname{im} \mathcal{V}_{\mu\mu}^{[2]} \subset \ldots \subset \operatorname{im} \mathcal{V}_{\mu\mu}^{[k+2]}, \\
&\operatorname{rank} \mathcal{V}_{\mu\mu}^{[i]} = \ell_\mu - \ell_{\mu-i}, \quad i = 1, \ldots, k+2.
\end{aligned}$$

Here, Lemma G.3 applies again, and it results that

$$\begin{aligned}
&\operatorname{rank} \mathcal{U}_{\mu\mu}^{[1]} \cdots \mathcal{U}_{\mu\mu}^{[k+1]} \mathcal{V}_{\mu\mu}^{[k+2]} = \operatorname{rank} \mathcal{V}_{\mu\mu}^{[k+2]} - \operatorname{rank} \mathcal{V}_{\mu\mu}^{[k+1]} \\
&= \ell_\mu - \ell_{\mu-k-2} - (\ell_\mu - \ell_{\mu-k-1}) = \ell_{\mu-k-1} - \ell_{\mu-k-2}.
\end{aligned}$$

So we are done with range and rank of G_{k+2}.
In the next step we provide G_{k+2} itself (cf. Section 3). Compute

$$\begin{aligned}
G_{k+2} &= G_{k+1} + \Pi_k Q_{k+1} - \sum_{j=1}^{k+1} G_j P_0 \Pi_j' \Pi_k Q_{k+1} \\
&= M_0 + Q_0 + (I - M_0 F_k)(I - H_k) P_0 (I - \Pi_k) + \Pi_k Q_{k+1} \\
&\quad - M_0 \Pi_1' \Pi_k Q_{k+1} - \sum_{j=2}^{k+1} \{ M_0 + (I - M_0 F_{j-1})(I - H_{j-1}) P_0 (I - \Pi_{j-1}) \} \Pi_j' \Pi_k Q_{k+1} \\
&= M_0 + Q_0 + (I - M_0 F_k) P_0 (I - \Pi_k) - (I - M_0 F_k) H_k + \Pi_k Q_{k+1} \\
&\quad - \sum_{j=1}^{k+1} M_0 \Pi_j' \Pi_k Q_{k+1} - \sum_{j=2}^{k+1} (I - M_0 F_{j-1})(I - H_{j-1}) P_0 (I - \Pi_{j-1}) \Pi_j' \Pi_k Q_{k+1}
\end{aligned}$$

and rearrange (cf. (226), (227)) certain terms to

$$(I - M_0F_k)P_0(I - \varPi_k) + \varPi_kQ_{k+1} - M_0\sum_{j=1}^{k+1} P_0\varPi_j'\varPi_kQ_{k+1} = (I - M_0F_{k+1})P_0(I - \varPi_{k+1})$$

and

$$\begin{aligned}
&(I - M_0F_k)H_k + \sum_{j=2}^{k}(I - M_0F_{j-1})(I - H_{j-1})P_0(I - \varPi_{j-1})\varPi_j'\varPi_kQ_{k+1} \\
&= (I - M_0F_k)\Big\{H_k + \sum_{j=2}^{k}(I - M_0F_k)^{-1}(I - M_0F_{j-1})(I - H_{j-1})\cdot \\
&\qquad\qquad\qquad \cdot P_0(I - \varPi_{j-1})\varPi_j'\varPi_kQ_{k+1}\Big\} \\
&= (I - M_0F_k)\Big\{H_k + \sum_{j=2}^{k}(I - H_{j-1})P_0(I - \varPi_{j-1})\varPi_j'\varPi_kQ_{k+1}\Big\} \\
&= (I - M_0F_k)H_{k+1} = (I - M_0F_{k+1})(I - M_0F_{k+1})^{-1}(I - M_0F_k)H_{k+1} \\
&= (I - M_0F_{k+1})H_{k+1} = (I - M_0F_{k+1})H_{k+1}P_0(I - \varPi_{k+1}),
\end{aligned}$$

what leads to

$$\begin{aligned}
G_{k+2} &= M_0 + Q_0 + (I - M_0F_{k+1})P_0(I - \varPi_{k+1}) - (I - M_0F_{k+1})H_{k+1}P_0(I - \varPi_{k+1}) \\
&= M_0 + Q_0 + (I - M_0F_{k+1})(I - H_{k+1})P_0(I - \varPi_{k+1}),
\end{aligned}$$

and we are done with G_{k+2} (cf. (234)).
Next, $I - H_{k+1}$ is nonsingular, since $(I - H_{k+1})z = 0$ implies $\varPi_kQ_{k+1}z = 0$, thus $(I - H_k)z = 0$, and, finally $z = 0$ due to the nonsingularity of $(I - H_k)$.
To complete the proof of Lemma G.1 we have to verify (232) and (233) for $i = k + 1$, supposed $\ker \varPi_{k-1}M_{k-1} \subseteq \ker \varPi_kM_k$, $\ker \varPi_kM_k = \ker M_0^2M_1\cdots M_k$ are valid. From $\varPi_kM_k = \varPi_kM_k\varPi_k$ (cf. (240)) and $\ker M_0^2M_1\cdots M_k = \ker \varPi_kM_k = \ker \mathcal{U}^{[k+2]}$ we obtain the relations

$$\begin{aligned}
\varPi_{k+1}M_{k+1} &= \varPi_k\mathcal{U}^{[k+2]}M_{k+1} = \varPi_k(M_0^2M_1\cdots M_k)^-M_0^2M_1\cdots M_kM_{k+1}, \\
M_0^2M_1\cdots M_{k+1} &= M_0^2M_1\cdots M_k\mathcal{U}^{[k+2]}M_{k+1} = M_0^2M_1\cdots M_k(\varPi_kM_k)^-\varPi_kM_k\mathcal{U}^{[k+2]}M_{k+1} \\
&= M_0^2M_1\cdots M_k(\varPi_kM_k)^-\varPi_kM_k\varPi_k\mathcal{U}^{[k+2]}M_{k+1} \\
&= M_0^2M_1\cdots M_k(\varPi_kM_k)^-\varPi_kM_k\varPi_{k+1}M_{k+1},
\end{aligned}$$

hence $\ker \varPi_{k+1}M_{k+1} = \ker M_0^2M_1\cdots M_{k+1}$ holds true. Additionally, from

$$\begin{aligned}
\varPi_{k+1}M_{k+1} &= \varPi_{k+1}(I - M_0F_{k+1})^{-1}(I - M_0F_k)M_k \\
&= \varPi_{k+1}[I + (I - M_0F_{k+1})^{-1}M_0(F_{k+1} - F_k)\varPi_k]M_k \\
&= \varPi_{k+1}[I + (I - M_0F_{k+1})^{-1}M_0(F_{k+1} - F_k)\varPi_k]\varPi_kM_k
\end{aligned}$$

we conclude the inclusion

$$\ker \varPi_kM_k \subseteq \ker \varPi_{k+1}M_{k+1}.$$

□

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