A posteriori error estimates of higher-order finite elements for frictional contact problems

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Abstract

In this paper, a posteriori estimates are derived for higher-order finite element methods and frictional contact problems. The discretization is based on a mixed approach where the geometrical and frictional constraints are captured by Lagrange multipliers. The use of higher-order polynomials leads to a certain non-conformity in the discretization which requires special attention in the error analysis. As a main result an error estimation is proposed which consists of the dual norm of a residual plus some computable remainder terms. The residual is estimated by well-known a posteriori error estimates for variational equations. The remainder terms represent typical sources resulting from the non-conforming mixed discretization. Numerical experiments confirm the applicability of the a posteriori estimates to adaptive mesh refinements.

Keywords: contact problems, friction, higher-order finite elements, error estimates
2000 MSC: 65N30, 65N15

1. Introduction

Frictional contact problems play an import role in many processes of mechanical engineering. Their modeling involves systems of partial differential equations with inequality conditions describing geometrical as well as frictional constraints. In the literature, a huge number of discretization schemes for the numerical solution of such contact problems is discussed. We refer to the monographs [1, 2, 3, 4] for an overview and to [5, 6, 7, 8] for some recent works on discretization and solution schemes for frictional contact problems .

Well-established approaches to solve static contact problems are given by mixed methods. They are usually derived from saddle point formulations, where the geometrical and frictional constraints are captured by Lagrange multipliers. A commonly used mixed method is proposed by Haslinger et al. in [9, 3]. It is widely studied for low-order finite elements and enhanced for many applications of frictional contact problems, cf. [10, 11, 12]. The main advantage of this approach is that the Lagrange multipliers can be interpreted as normal and tangential contact forces. Moreover, the constraints of the Lagrange multipliers are sign conditions and box constraints which are simpler than the contact conditions in non-mixed formulations.

The low-order finite element discretization is based on the usual (bi-/tri-)linear H^1-conforming ansatz functions for the displacement and piecewise constant functions for the Lagrange multipliers. It allows for a conforming discretization, where the discrete Lagrange multipliers fulfill the non-discrete constraints. This is not the case when the discetization is extended to higher-order finite elements, where piecewise polynomial and discontinuous functions are applied to discretize the Lagrange multipliers. Using polynomials of degree strictly greater than 1, one can only ensure the constraints of the Lagrange multipliers to hold in a finite set of discrete points, cf. [13, 14]. This results in a certain non-conformity of the discretization which has to be taken into account in the error analysis.

In this paper, a posteriori estimates for mixed finite element methods of higher-order are derived for frictional contact problems. In contrast to [15], we put special attention to the non-conformity error of the Lagrange multipliers.

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The basic idea in the derivation of the estimates is to consider a certain residual given by the discrete displacement and the discrete Lagrange multipliers as well as to carefully insert correction functions to encounter the non-conformity. As a main result, we state an error estimate consisting of the dual norm of the residual plus some computable remainder terms. It is easy to see that the dual norm of the residual can be estimated by the discretization error of an auxiliary problem that is simply a variational equation. Thus, well-known a posteriori error estimates for variational equations can be used. The remainder terms capture typical sources resulting from the non-conforming mixed discretization: the geometrical error, the violation of complementary conditions, errors with respect to the frictional constraints and, finally, errors resulting from the non-conformity of the Lagrange multipliers.

In fact, the use of higher-order schemes to discretize contact problems is not obvious due to the low-regularity nature of contact problems. However, using adaptivity, one may raise hope to recover optimal algebraic or even exponential convergence rates. In our numerical experiments, we apply the a posteriori estimates within an adaptive scheme which resolves the end points of the contact zone and the points where gliding switches to sticking. The use of adaptivity significantly improves the convergence of the higher-order scheme and makes it applicable to solve frictional contact problems.

The paper is organized as follows: In the Section 2, we introduce some notations concerning the usual Sobolev spaces and define the space of the displacements as well as the sets describing the constraints of the Lagrange multipliers. Frictional contact problems in linear elasticity can be modeled by Signorini's problem with Tresca friction, which can be seen as a simplification of Coulomb friction. Section 3 presents this model and its mixed variational formulation. The higher-order finite element discretization based on the mixed formulation is introduced in Section 4. The main results, the derivation of a posteriori error estimates, is described in Section 5. Numerical results confirming the applicability of the a posteriori estimates within adaptive schemes are presented in Section 6.

2. Notations

Let $\Omega \subset \mathbb{R}^k$, $k \in \{2,3\}$, be a domain with sufficiently smooth boundary $\Gamma := \partial\Omega$. Moreover, let $\Gamma_D \subset \Gamma$ be closed with positive measure and let $\Gamma_C \subset \Gamma\backslash\Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma\backslash\Gamma_D$. $L^2(\Omega)$, $H^k(\Omega)$ with $k \geq 1$ and $H^{1/2}(\Gamma_C)$ denote the usual Sobolev spaces and $H^1_D(\Omega) := \{v \in H^1(\Omega) \mid \gamma(v) = 0 \text{ on } \Gamma_D\}$ with the trace operator γ. The space $H^{-1/2}(\Gamma_C)$ is the topological dual space of $H^{1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$. Let $(\cdot,\cdot)_{0,\omega}$, $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$. Note, that the linear and bounded mapping $\gamma_C := \gamma_{|\Gamma_C} : H^1_D(\Omega) \to H^{1/2}(\Gamma_C)$ is surjective and continuous due to the assumptions on Γ_C, cf. [1, p.88]. For functions in $L^2(\Omega)$ or $L^2(\Gamma_C)$, the inequality symbols $\geq$ and $\leq$ are defined as "almost everywhere". For a function v, we define the positive part by $(v)_+ := \max\{v, 0\}$ and the cutoff functions $(\cdot)_\zeta$ by $(v)_\zeta := v$ if $|v| \leq \zeta$ and $(v)_\zeta := \zeta v/|v|$ otherwise. Here, ζ is a non-negative function and $|\cdot|$ the euclidian norm. We set $H^{1/2}_+(\Gamma_C) := \{v \in H^{1/2}(\Gamma_C) \mid v \geq 0\}$ and $L^2_{s,\tilde{s}}(\Gamma_C) := \{\mu \in (L^2(\Gamma_C))^{k-1} \mid |\mu| \leq \zeta(s,\tilde{s})\}$, where $\zeta(s,\tilde{s})$ is defined as $s/\tilde{s}$ on $\operatorname{supp}\tilde{s}$ and 0 on $\Gamma_C\backslash \operatorname{supp}\tilde{s}$ for $s \in L^2(\Gamma_C)$, $s \geq 0$ and $\tilde{s} \in \{1, s\}$. Furthermore, we define the dual cone $H^{-1/2}_+(\Gamma_C) := (H^{1/2}_+(\Gamma_C))' := \{\mu \in H^{-1/2}(\Gamma_C) \mid \forall w \in H^{1/2}_+(\Gamma_C) : \ \langle \mu, w\rangle \geq 0\}$ and set $\gamma_N := \gamma_{|\Gamma_N}$ with $\Gamma_N \subset \Gamma\backslash(\Gamma_D \cup \overline{\Gamma}_C)$.

For a displacement field $v \in (H^1_D(\Omega))^k$ we specify the linearized strain tensor as $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^\top)$ and the stress tensor as $\sigma(v)_{ij} := C_{ijkl}\varepsilon(v)_{kl}$ with $C_{ijkl} \in L^\infty(\Omega)$ and $C_{ijkl} = C_{jilk} = C_{klij}$ as well as $C_{ijkl}\tau_{ij}\tau_{kl} \geq \kappa\tau^2_{ij}$ for $\tau \in L^2(\Omega)^{k\times k}_{\text{sym}}$ and a $\kappa > 0$. Furthermore, n denotes the vector-valued function describing the outer unit normal vector with respect to Γ_C and t the $k\times(k-1)$-matrix-valued function containing the tangential vectors. We define $\sigma_{n,j} := \sigma_{ij}n_i$, $\sigma_{nn} := \sigma_{ij}n_i n_j$, and $\sigma_{nt,l} := \sigma_{ij}n_i t_{jl}$. Moreover, we set $\gamma_n(v) := \gamma_C(v_i)n_i$, $\gamma_t(v)_j := \gamma_C(v_i)t_{ij}$ and $\gamma_{N,i}(v) := \gamma_{|\Gamma_N}(v_i)$.

For the ease of the notation, we set

$$V := (H^1_D(\Omega))^k, \quad W_n := H^{-1/2}(\Gamma_C), \quad W_t := (L^2(\Gamma_C))^{k-1}, \quad \Lambda_n := H^{-1/2}_+(\Gamma_C), \quad \Lambda_t := L^2_{s,\tilde{s}}(\Gamma_C).$$

3. Signorini problem with Tresca friction

Frictional contact between a deformable elastic body and a rigid foundation is often modelled by the Signorini problem with Tresca friction, where a linear elastic material law is used to describe the deformation of an elastic body. We assume that the body is described by Ω and is clamped at the boundary part Γ_D. Furthermore, volume and surface forces given by the functions $f \in (L^2(\Omega))^k$ and $f_N \in (L^2(\Gamma_N))^k$ act on the body leading to its deformation.

To describe the geometrical contact, we assume that Γ_C is parameterized by a sufficiently smooth function $\phi : \mathbb{R}^{k-1} \to \mathbb{R}$ so that, without loss of generality, the geometrical constraints for a displacement v in the k-th component is given by $\phi(x) + v_k(x, \phi(x)) \leq \psi(x_1 + v_1(x, \phi(x)), \ldots, x_{k-1} + v_{k-1}(x, \phi(x)))$ with $x := (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$ and a sufficiently smooth function ψ describing the surface of the rigid foundation. Since this condition is non-linear in general, one usually applies the linearization $\gamma_n(v) \leq g$ with $g := (\psi(x) - \phi(x))(1 + (\nabla\phi(x))^\top \nabla\phi(x))^{-1/2}$, cf. [1, Ch.2].

Frictional contact conditions are introduced assuming that sliding does not occur if the magnitude of the tangential forces is below a critical value described by a function $s \in L^2(\Gamma_C)$ with $s \geq 0$. If the tangential forces reach this critical value, sliding is obtained in the direction of the tangential forces. Note that such Tresca friction can be extended to Coulomb friction setting s to the magnitude of the normal forces times a friction coefficient and integrating the problem into a fixed point scheme. With the linearized geometrical as well as the frictional contact conditions, the Signorini problem with Tresca friction is to find a displacement u such that

$$
\begin{gathered}
-\operatorname{div}\sigma(u) = f \text{ in } \Omega, \quad \sigma_n(u) = f_N \text{ on } \Gamma_N,\\
u_n - g \leq 0,\ \sigma_{nn}(u) \leq 0,\ \sigma_{nn}(u)(u_n - g) = 0 \text{ on } \Gamma_C,\\
|\sigma_{nt}(u)| \leq s \text{ with } \left\{ \begin{array}{ll} |\sigma_{nt}(u)| < s & \Rightarrow u_t = 0, \\ |\sigma_{nt}(u)| = s & \Rightarrow \exists \xi \geq 0 :\ u_t = -\xi\sigma_{nt}(u) \end{array} \right\} \text{ on } \Gamma_C.
\end{gathered}
$$

Here, $u_n := u_i n_i$ and $u_{t,j} := u_i t_{ij}$ on Γ_C. The function u is a solution if and only if $u \in K := \{v \in V \mid g - \gamma_n(v) \geq 0\}$ and the variational inequality

$$
(\sigma(u), \varepsilon(v - u))_0 + (s, |\gamma_t(v)| - |\gamma_t(u)|)_{0,\Gamma_C} \geq (f, v - u)_0 + (f_N, \gamma_N(v - u))_{0,\Gamma_N} \tag{1}
$$

is fulfilled for all $v \in K$, cf. [16, Sec. 5.4.5]. The inequality above is fulfilled if and only if u is a minimizer of the functional $J(v) := \frac{1}{2}(\sigma(v), \varepsilon(v))_0 - (f, v)_0 - (f_N, \gamma_N(v))_{0,\Gamma_N} + (s, |\gamma_t(v)|)_{0,\Gamma_C}$ in K. The functional J is strictly convex, continuous and coercive due to Cauchy's and Korn's inequalities. This implies the existence of a unique minimizer u. Given the Lagrange functional $\mathcal{L}(v, \mu_n, \mu_t) := J(v) + \langle \mu_n, \gamma_n(v) - g\rangle + (\mu_t, \tilde{s}\gamma_t(v))_{0,\Gamma_C}$ on $V \times \Lambda_n \times \Lambda_t$, the Hahn-Banach theorem and the fact that $(s, |\gamma_t(v)|)_{0,\Gamma_C} = \sup_{\mu_t \in \Lambda_t} (\mu_t, \tilde{s}\gamma_t(v))_{0,\Gamma_C}$ yield

$$
J(u) = \inf_{v \in V} \sup_{(\mu_n, \mu_t) \in \Lambda_n \times \Lambda_t} \mathcal{L}(v, \mu_n, \mu_t).
$$

Thus, u is a minimizer of J, whenever $(u, \lambda_n, \lambda_t) \in V \times \Lambda_n \times \Lambda_t$ is a saddle point of $\mathcal{L}$. The existence of a unique saddle point is guaranteed, since Λ_t is bounded and the inf-sup condition $\alpha\|\mu_n\|_{-1/2,\Gamma_C} \leq \sup_{v \in V, \|v\|_1 = 1} \langle \mu_n, \gamma_n(v)\rangle$ holds for a constant $\alpha > 0$ and all $\mu_n \in W_n$, cf. [1, 14]. In fact, it follows from the closed range theorem and the surjectivity of γ_n that the inf-sup condition is valid. Due to the stationary conditions, $(u, \lambda_n, \lambda_t) \in V \times \Lambda_n \times \Lambda_t$ is a saddle point of $\mathcal{L}$, if and only if it fulfills the mixed variational formulation

$$
\begin{aligned}
&(\sigma(u), \varepsilon(v))_0 = (f, v)_0 + (f_N, \gamma_N(v))_{0,\Gamma_N} - \langle \lambda_0, \gamma_n(v)\rangle - (\lambda_t, \tilde{s}\gamma_t(v))_{0,\Gamma_C},\\
&\langle \mu_n - \lambda_n, \gamma_n(u) - g\rangle + (\mu_t - \lambda_t, \tilde{s}\gamma_t(u))_{0,\Gamma_C} \leq 0
\end{aligned} \tag{2}
$$

for all $(v, \mu_n, \mu_t) \in V \times \Lambda_n \times \Lambda_t$.

Remark 3.1. The main advantage of the mixed approach is that the displacement and contact forces are given simultaneously. Under certain regularity assumptions, there holds $\lambda_n = -\sigma_{nn}(u)$ and $\tilde{s}\lambda_t = -\sigma_{nt}(u)$ so that the Lagrange multipliers can be interpreted as contact forces, cf. [9].

Remark 3.2. The choice of $\tilde{s} \in \{1, s\}$ indicates two equivalent mixed formulations. In the case $\tilde{s} = s$, the frictional function s is weakly included in the mixed formulation (2) and the set Λ_t is defined via constant box constraints. In the case $\tilde{s} = 1$, the pointwise box constraints are defined by the possibly non-constant function s which, then again, does not enter the weak formulation.

4. Higher-order finite element discretization

A higher-order finite element discretization based on quadrangles or hexahedrons is given as follows: Let $\mathcal{T}$ be a finite element mesh of Ω with mesh size h and let $\mathcal{E}$ be a finite element mesh of Γ_C with mesh size H. Furthermore, let $\Psi_T : [-1,1]^k \to T \in \mathcal{T}$ and $\Phi_T : [-1,1]^{k-1} \to T \in \mathcal{T}$ be bijective and sufficiently smooth transformations and let $p_T, q_E \in \mathbb{N}$ be degree distributions on $\mathcal{T}$ and $\mathcal{E}$, respectively. Using the polynomial (Serendipity) tensor product space $Q_{k,p}$ of order p on the reference element $[-1,1]^k$, we define

$$\begin{aligned} V_{hp} &:= \left\{ v \in V \mid \ \forall T \in \mathcal{T} : \ v_{|T} \circ \Psi_T \in (Q_{k,p_T})^k \right\}, \\ M_{Hq} &:= \left\{ \mu \in L^2(\Gamma_C) \mid \ \forall E \in \mathcal{E} : \ \mu_{|E} \circ \Phi_E \in Q_{k-1,q_E} \right\}. \end{aligned}$$

To identify adequate substitutions of Λ_n and Λ_t, we have to take into account that polynomials can not be easily ensured to be positive or bounded. Therefore, we enforce these properties in a finite set of discrete points only. For this purpose, let $\mathcal{M} \subset [-1,1]$ be a finite set of points and

$$\begin{aligned} \Lambda_{n,Hq} &:= \{\mu_{n,Hq} \in M_{Hq} \mid \forall E \in \mathcal{E} : \forall x \in \mathcal{M}^{k-1} : \ \mu_{n,Hq}(\Phi_E(x)) \geq 0\}, \\ \Lambda_{t,Hq} &:= \{\mu_{t,Hq} \in (M_{Hq})^{k-1} \mid \forall E \in \mathcal{E}, \forall x \in \mathcal{M}^{k-1} : \ |\mu_{t,Hq}(\Phi_E(x))| \leq (\zeta(s,\tilde{s}))(\Phi_E(x))\}. \end{aligned}$$

Note that the definition of $\Lambda_{n,Hq}$ and $\Lambda_{t,Hq}$ using discrete points leads to the non-conformity $\Lambda_{n,Hq} \not\subset \Lambda_n$ and $\Lambda_{t,Hq} \not\subset \Lambda_t$.

A discrete mixed formulation is to find $(u_{hp}, \lambda_{n,Hq}, \lambda_{t,Hq}) \in V_{hp} \times \Lambda_{n,Hq} \times \Lambda_{t,Hq}$ such that

$$\begin{aligned} &(\sigma(u_{hp}), \varepsilon(v_{hp}))_0 = (f, v_{hp})_0 + (f_N, \gamma_N(v_{hp}))_{0,\Gamma_N} - (\lambda_{n,Hq}, \gamma_n(v_{hp}))_{0,\Gamma_C} - (\mu_{t,Hq}, \tilde{s}\gamma_t(v_{hp}))_{0,\Gamma_C}, \\ &(\mu_{n,Hq} - \lambda_{n,Hq}, \gamma_n(u_{hp}) - g)_{0,\Gamma_C} + (\mu_{t,Hq} - \lambda_{t,Hq}, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \leq 0 \end{aligned} \tag{3}$$

for all $(v_{hp}, \mu_{n,Hq}, \mu_{t,Hq}) \in V_{hp} \times \Lambda_{n,Hq} \times \Lambda_{t,Hq}$. In order to ensure the stability of the discretization scheme, we have to verify the discrete inf-sup condition

$$\beta(\|\mu_{n,Hq}\|_{-1/2,\Gamma_C} + \|\mu_{t,Hq}\|_{-1/2,\Gamma_C}) \leq \sup_{v_{hp} \in V_{hp}, \|v_{hp}\|_1 = 1} (\mu_{n,Hq}, \gamma_n(v_{hp}))_{0,\Gamma_C} + (\mu_{t,Hq}, \tilde{s}\gamma_t(v_{hp}))_{0,\Gamma_C} \tag{4}$$

for all $(\mu_{n,Hq}, \mu_{t,Hq}) \in M_{Hq} \times (M_{Hq})^{k-1}$ and a constant $\beta > 0$ independent of the mesh sizes h and H as well as the polynomial degree distributions p and q.

Remark 4.1. The use of different mesh sizes and polynomial degrees with sufficiently small quotients h/H and q/p is the key to guarantee the discrete inf-sup condition (4). In our implementation, we usually ensure $h/H \leq 0.5$ and $q = p-1$, using hierarchical meshes with $\mathcal{E}$ being sufficiently coarser than $\mathcal{T}$. We refer to [17, 14], where this subject is explicitly outlined.

5. A posteriori error control

The basic idea for the estimation of $\|u - u_{hp}\|_1$ is to consider the residual $\mathrm{Res} \in V'$ defined by

$$\langle \mathrm{Res}, v\rangle := (f, v)_0 + (f_N, \gamma_N(v))_{0,\Gamma_N} - (\lambda_{n,Hq}, \gamma_n(v))_{0,\Gamma_C} - (\lambda_{t,Hq}, \gamma_t(v))_{0,\Gamma_C} - (\sigma(w), \varepsilon(v))_0.$$

We will show that an upper bound of $\|u - u_{hp}\|_1$ is given by the norm of the residual $\|\mathrm{Res}\|_{V'}$ plus some remainder terms. It is easy to see that $\|\mathrm{Res}\|_{V'}$ can be estimated by an arbitrary error estimation known from variational equations. Indeed, using Korn's inequality $\kappa\|v\|_1^2 \leq (\sigma(v), \varepsilon(v))_0$ and the continuity statement $(\sigma(v), \varepsilon(w))_0 \leq c\|v\|_1\|w\|_1$ for some positive constants κ and c and all $v, w \in V$, we easily find that

$$\kappa\|u^* - u_{hp}\|_1 \leq \|\mathrm{Res}\|_{V'} \leq c\|u^* - u_{hp}\|_1.$$

Here, $u^* \in V$ fulfills the variational equation

$$(\sigma(u^*), \varepsilon(v))_0 = (f, v)_0 + (f_N, \gamma_N(v))_{0,\Gamma_N} - \langle \lambda_{n,Hq}, \gamma_n(v)\rangle - (\lambda_{t,Hq}, \gamma_t(v))_{0,\Gamma_C}$$

for all $v \in V$. The unique existence of u^* is guaranteed by the Lax-Milgram lemma. Hence, estimating $\|\mathrm{Res}\|_{V'}$ implies the estimation of $\|u^* - u_{hp}\|_1$ and vice versa.

In the following, we assume $s \in L^\infty(\Gamma_C)$ and make use of the basic estimations

$$ab \leq \frac{1}{4\epsilon}a^2 + \epsilon b^2, \tag{5}$$

$$(a+b)^2 \leq 2a^2 + 2b^2 \tag{6}$$

for $a, b \in \mathbb{R}$ and $\epsilon > 0$. Furthermore, we use the continuity of γ_n and γ_t, i.e. $\|\gamma_t(v)\|_{1/2,\Gamma_C} \leq c_t\|v\|_1$ and $\|\gamma_n(v)\|_{1/2,\Gamma_C} \leq c_n\|v\|_1$ for some positive constants c_n and c_t and all $v \in V$.

Lemma 5.1. *There exists positive constants C_0 and C_1 such that*

$$\|u - u_{hp}\|_1^2 \leq C_0(\|\operatorname{Res}\|_{V'}^2 + \|\lambda_{n,Hq} - \mu_n\|_{-1/2,\Gamma_C}^2 + \|\lambda_{t,Hq} - \mu_t\|_{0,\Gamma_C}^2) \\ + C_1(\langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C})$$

for an arbitrary $(\mu_n, \mu_t) \in \Lambda_n \times \Lambda_t$.

Proof. There holds

$$\begin{aligned}
\kappa\|u - u_{hp}\|_1^2 &\leq (\sigma(u - u_{hp}), \varepsilon(u - u_{hp}))_0 \\
&= (\sigma(u), \varepsilon(u - u_{hp}))_0 - (f, u - u_{hp})_0 - (f_N, u - u_{hp})_{0,\Gamma_N} \\
&\quad + (\lambda_{n,Hq}, \gamma_n(u - u_{hp}))_{0,\Gamma_C} + (\lambda_{t,Hq}, \tilde{s}\gamma_t(u - u_{hp}))_{0,\Gamma_C} + \langle \operatorname{Res}, u - u_{hp}\rangle \\
&= \langle \lambda_{n,Hq} - \lambda_n, \gamma_n(u - u_{hp})\rangle + (\lambda_{t,Hq} - \lambda_t, \tilde{s}\gamma_t(u - u_{hp}))_{0,\Gamma_C} + \langle \operatorname{Res}, u - u_{hp}\rangle \\
&= \langle \lambda_{n,Hq} - \mu_n, \gamma_n(u - u_{hp})\rangle + (\lambda_{t,Hq} - \mu_t, \tilde{s}\gamma_t(u - u_{hp}))_{0,\Gamma_C} + \langle \mu_n - \lambda_n, \gamma_n(u) - g\rangle + (\mu_t - \lambda_t, \tilde{s}\gamma_t(u))_{0,\Gamma_C} \\
&\quad + \langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} + \langle \operatorname{Res}, u - u_{hp}\rangle \\
&\leq \max\{1, c_n, c_t\|\tilde{s}\|_{\infty,\Gamma_C}\}(\|\operatorname{Res}\|_{V'} + \|\lambda_{n,Hq} - \mu_n\|_{-1/2,\Gamma_C} + \|\lambda_{t,Hq} - \mu_t\|_{0,\Gamma_C})\|u - u_{hp}\|_1 \\
&\quad + \langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&\leq \frac{\max\{1, c_n, c_t\|\tilde{s}\|_{\infty,\Gamma_C}\}^2}{4\epsilon}(\|\operatorname{Res}\|_{(H_D^1(\Omega))'}^2 + \|\lambda_{n,Hq} - \mu_n\|_{-1/2,\Gamma_C}^2 + \|\lambda_{t,Hq} - \mu_t\|_{0,\Gamma_C}^2) + 3\epsilon\|u - u_{hp}\|_1^2 \\
&\quad + \langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}
\end{aligned}$$

where we use (5) in the last inequality. Subtraction of $3\epsilon\|u - u_{hp}\|_1^2$ and division by $\kappa - 3\epsilon$ yield the assertion with $0 < \epsilon < \kappa/3$, $C_0 := \max\{1, c_n, c_t\|\tilde{s}\|_{\infty,\Gamma_C}\}^2(4\epsilon(\kappa - 3\epsilon))^{-1}$ and $C_1 := (\kappa - 3\epsilon)^{-1}$. ☐

In order to obtain an a posteriori error estimation, we have to estimate the term $\langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}$. This is done in the proof of the following theorem.

Theorem 5.2. *Let*

$$\eta(\mu_n, \mu_t, z) := \|\operatorname{Res}\|_{V'}^2 + \|\lambda_{n,Hq} - \mu_n\|_{-1/2,\Gamma_C}^2 + \|\lambda_{t,Hq} - \mu_t\|_{0,\Gamma_C}^2 + \|z\|_{1/2,\Gamma_C}^2 \\ + |(\lambda_{n,Hq}, z)_{0,\Gamma_C}| + |\langle \mu_n, g - \gamma_n(u_{hp})\rangle| + (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}$$

for an arbitrary $(\mu_n, \mu_t) \in \Lambda_n \times \Lambda_t$ and $z \in Z := \{z \in H^{1/2}(\Gamma_C) \mid g - \gamma_n(u_{hp}) + z \in H_+^{1/2}(\Gamma_C)\}$. Then, there exists a constant $C > 0$ such that

$$\|u - u_{hp}\|_1^2 \leq C\eta(\mu_n, \mu_t, z).$$

Proof. There exists $\tilde{z} \in V \cap \ker \gamma_t$ with $\gamma_n(\tilde{z}) = z$ and $\|\tilde{z}\|_1 \leq \tilde{c}\|z\|_{1/2,\Gamma_C}$, cf. [18, Thm. 6.2.40]. Employing (5), we obtain from

$$
\begin{aligned}
&\langle \lambda_n - \mu_n, \gamma_n(u_{hp}) - g\rangle + (\lambda_t - \mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&= -\langle \lambda_n, g - \gamma_n(u_{hp}) + z\rangle + \langle \lambda_n, z\rangle + \langle \mu_n, g - \gamma_n(u_{hp})\rangle + (\lambda_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&\leq \langle \lambda_n, \gamma_n(\tilde{z})\rangle + \langle \mu_n, g - \gamma_n(u_{hp})\rangle + (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&= (f, \tilde{z})_0 + (f_N, \tilde{z})_{0,\Gamma_N} - (\sigma(u), \varepsilon(\tilde{z}))_0 + \langle \mu_n, g - \gamma_n(u_{hp})\rangle + (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&= (\sigma(u_{hp} - u), \varepsilon(\tilde{z}))_0 + (\lambda_{n,Hq}, \gamma_n(\tilde{z}))_{0,\Gamma_C} + \langle \mathrm{Res}, \tilde{z}\rangle + \langle \mu_n, g - \gamma_n(u_{hp})\rangle + (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&\leq c\|u - u_{hp}\|_1\|\tilde{z}\|_1 + (\lambda_{n,Hq}, z)_{0,\Gamma_C} + \|\mathrm{Res}\|_{V'}\|\tilde{z}\|_1 + \langle \mu_n, g - \gamma_n(u_{hp})\rangle + (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C} \\
&\leq \epsilon\|u - u_{hp}\|_1^2 + \tilde{c}\frac{c^2 + 2\epsilon}{4\epsilon}\|z\|_{1/2,\Gamma_C}^2 + |(\lambda_{n,Hq}, z)_{0,\Gamma_C}| + \frac{1}{2}\|\mathrm{Res}\|_{V'}^2 + |\langle \mu_n, g - \gamma_n(u_{hp})\rangle| \\
&\quad + (s, |\gamma_t(u_{hp}|)_{0,\Gamma_C} - (\mu_t, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}
\end{aligned}
$$

with $0 < \epsilon < 1/C_1$. From Theorem 5.1, we obtain that

$$\|u - u_{hp}\|_1^2 \leq C_1\epsilon\|u - u_{hp}\|_1^2 + \max\{C_0, C_1\}\max\{\tilde{c}(c^2 + 2\epsilon)(4\epsilon)^{-1}, 3/2, 1\}\eta(\mu_n, \mu_t, z).$$

Subtraction of $C_1\epsilon\|u - u_{hp}\|_1^2$ and division by $1 - C_1\epsilon$ complete the proof. □

In order to derive an a posteriori error estimation using Theorem 5.2, we have to specify suitable $z \in Z$, $\mu_n \in \Lambda_n$ and $\mu_t \in \Lambda_t$. In principle, we are free to choose each of them arbitrarily. However, it seems to be natural to employ functions which are close to $g - \gamma_n(u_{hp})$, $\lambda_{n,Hq}$ and $\lambda_{t,Hq}$. A practical choice is obviously given via the positive part $(\cdot)_+$ and the cutoff function $(\cdot)_\zeta$. Therewith, we obtain the following result.

Corollary 5.3. *There holds*

$$\|u - u_{hp}\|_1^2 \leq C\eta((\lambda_{n,Hq})_+, (\lambda_{t,Hq})_{\zeta(s,\tilde{s})}, (\gamma_n(u_{hp}) - g)_+). \tag{7}$$

Proof. Since $g - \gamma_n(u_{hp}) + (\gamma_n(u_{hp}) - g)_+ \geq 0$, we have $(\gamma_n(u_{hp}) - g)_+ \in Z$. Furthermore, there holds $(\lambda_{n,Hq})_+ \in \Lambda_n$ and $(\lambda_{t,Hq})_{\zeta(s,\tilde{s})} \in \Lambda_t$. □

The error contributions resulting from the insertion of $(\lambda_{n,Hq})_+$, $(\lambda_{t,Hq})_{\zeta(s,\tilde{s})}$ and $(\gamma_n(u_{hp}) - g)_+$ in η are interpretable as typical sources of discretization errors. The contributions $\|\lambda_{n,Hq} - (\lambda_{n,Hq})_+\|_{-1/2,\Gamma_C}$ and $\|\lambda_{t,Hq} - (\lambda_{t,Hq})_{\zeta(s,\tilde{s})}\|_{0,\Gamma_C}$ can be interpreted as measures for the non-conformity $\Lambda_{n,Hq} \not\subset \Lambda_n$ and $\Lambda_{t,Hq} \not\subset \Lambda_t$, respectively. The contribution $\|(\gamma_n(u_{hp}) - g)_+\|_{1/2,\Gamma_C}$ measures the error with respect to the geometrical constraint $\gamma_n(u) \leq g$, whereas $|(\lambda_{n,Hq}, (\gamma_n(u_{hp}) - g)_+)_{0,\Gamma_C}|$ and $|((\lambda_{n,Hq})_+, \gamma_n(u_{hp}) - g)_{0,\Gamma_C}|$ describe the error with respect to the complementary condition $(-\sigma_{nn}(u), g - \gamma_n(u))_{0,\Gamma_C} = (\lambda_n, g - \gamma_n(u))_{0,\Gamma_C} = 0$. Finally, $(s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - ((\lambda_{t,Hq})_{\zeta(s,\tilde{s})}, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}$ represents the violation of the frictional condition $(s, |\gamma_t(u)|)_{0,\Gamma_C} = (\lambda_t, \tilde{s}\gamma_t(u))_{0,\Gamma_C}$.

Remark 5.4. If $g \in H^1(\Gamma_C)$, we have $(\gamma_n(u_{hp}) - g)_+ \in H^1(\Gamma_C)$, cf. [2, Ch. I, Cor 2.1]. In this case, the term $\|(\gamma_n(u_{hp}) - g)_+\|_{1/2,\Gamma_C,n}$ can be further estimated by $\|(\gamma_n(u_{hp}) - g)_+\|_{0,\Gamma_C}\|(\gamma_n(u_{hp}) - g)_+\|_{1,\Gamma_C}$ up to a positive constant, cf [19, Ch. I.3.3]. In practice, $\|\lambda_{n,Hq} - (\lambda_{n,Hq})_+\|_{-1/2,\Gamma_C}$ can simply be estimated by $\|\lambda_{n,Hq} - (\lambda_{n,Hq})_+\|_{0,\Gamma_C}$.

To include the error of the Lagrange multipliers into the error estimation, we assume the inf-sup condition

$$\hat{\kappa}(\|\mu_n\|_{-1/2,\Gamma_C} + \|\mu_t\|_{-1/2,\Gamma_C}) \leq \sup_{v \in V, \|v\|_1 = 1} \langle \mu_n, \gamma_n(v)\rangle + (\mu_t, \tilde{s}\gamma_t(v))_{0,\Gamma_C} \tag{8}$$

for a constant $\hat{\kappa} > 0$ and all $(\mu_n, \mu_t) \in W_n \times W_t$. In the case $\tilde{s} = 1$, the inf-sup condition (8) directly results from the surjectivity of γ_n and γ_t.

Theorem 5.5. *Let the inf-sup condition* (8) *be fulfilled. Then, there exists a constant $\hat{C} > 0$ such that*

$$\|u - u_{hp}\|_1^2 + \|\lambda_n - \lambda_{n,Hq}\|_{-1/2,\Gamma_C}^2 + \|\lambda_t - \lambda_{t,Hq}\|_{-1/2,\Gamma_C}^2 \leq \hat{C}\eta(\mu_n, \mu_t, z)$$

for an arbitrary $(\mu_n, \mu_t) \in \Lambda_n \times \Lambda_t$ and $z \in Z$. In particular, there holds

$$\|u - u_{hp}\|_1^2 + \|\lambda_n - \lambda_{n,Hq}\|_{-1/2,\Gamma_C}^2 + \|\lambda_t - \lambda_{t,Hq}\|_{-1/2,\Gamma_C}^2 \leq \hat{C}\eta((\lambda_{n,Hq})_+, (\lambda_{t,Hq})_{\zeta(s,\tilde{s})}, (\gamma_n(u_{hp}) - g)_+).$$

Proof. We find

$$
\begin{aligned}
&\hat{\kappa}(\|\lambda_n - \lambda_{n,Hq}\|_{-1/2,\Gamma_C} + \|\lambda_t - \lambda_{t,Hq}\|_{-1/2,\Gamma_C}) \\
\leq &\sup_{v \in V, \|v\|_1 = 1} \langle \lambda_n - \lambda_{n,Hq}, \gamma_n(v)\rangle + (\lambda_t - \lambda_{t,Hq}, \tilde{s}\gamma_t(v))_{0,\Gamma_C} \\
= &\sup_{v \in V,\, \|v\|_V = 1} (f, v)_0 + (f_N, v)_{0,\Gamma_N} - (\sigma(u), \varepsilon(v))_0 - \langle \lambda_{n,Hq}, \gamma_n(v)\rangle - (\lambda_{t,Hq}, \tilde{s}\gamma_t(v))_{0,\Gamma_C} \\
= &\sup_{v \in V,\, \|v\|_V = 1} \langle \mathrm{Res}, v\rangle + (\sigma(u_{hp} - u), \varepsilon(v)) \leq \|\,\mathrm{Res}\,\|_{V'} + c\|u - u_{hp}\|_1 .
\end{aligned}
$$

Thus, we obtain from (6) and Theorem 5.1 that

$$
\begin{aligned}
\|u - u_{hp}\|_1^2 + \|\lambda_n - \lambda_{n,Hq}\|_{-1/2,\Gamma_C}^2 + \|\lambda_t - \lambda_{t,Hq}\|_{-1/2,\Gamma_C}^2 &\leq (1 + \frac{2c^2}{\hat{\kappa}^2})\|u - u_{hp}\|_1^2 + \frac{2}{\hat{\kappa}^2}\|\,\mathrm{Res}\,\|_{V'}^2 \\
&\leq \max\{C + \frac{2Cc^2}{\hat{\kappa}^2}, \frac{2}{\hat{\kappa}^2}\}\eta(u_{hp}, \lambda_{n,Hq}, \lambda_{t,Hq}, \mu_n, \mu_t, z).
\end{aligned}
$$

The second assertion follows by the same arguments as in Theorem 5.3. $\square$

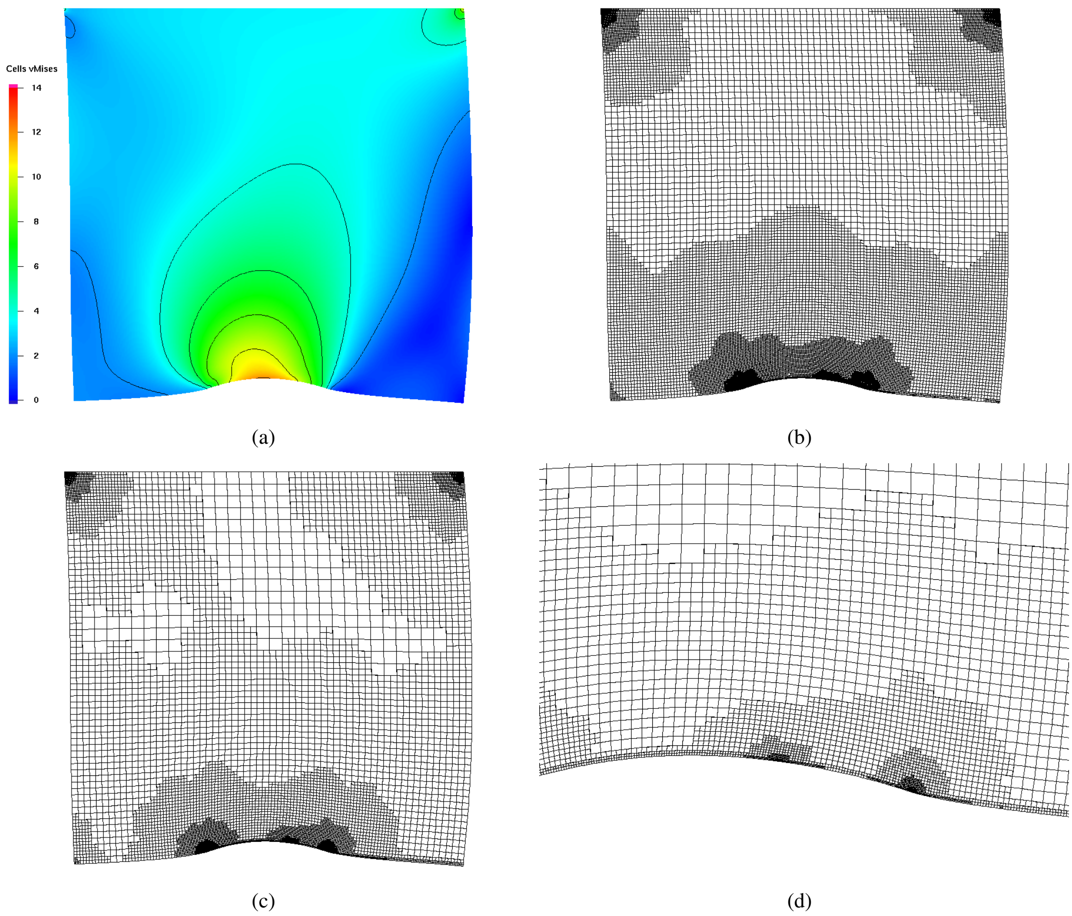

Figure 1: **(a)** Von Mises equivalent stress, **(b)** adaptive mesh for $p = 3$, **(c)** adaptive mesh for $p = 4$, **(d)** zoom to the contact zone ($p = 4$).

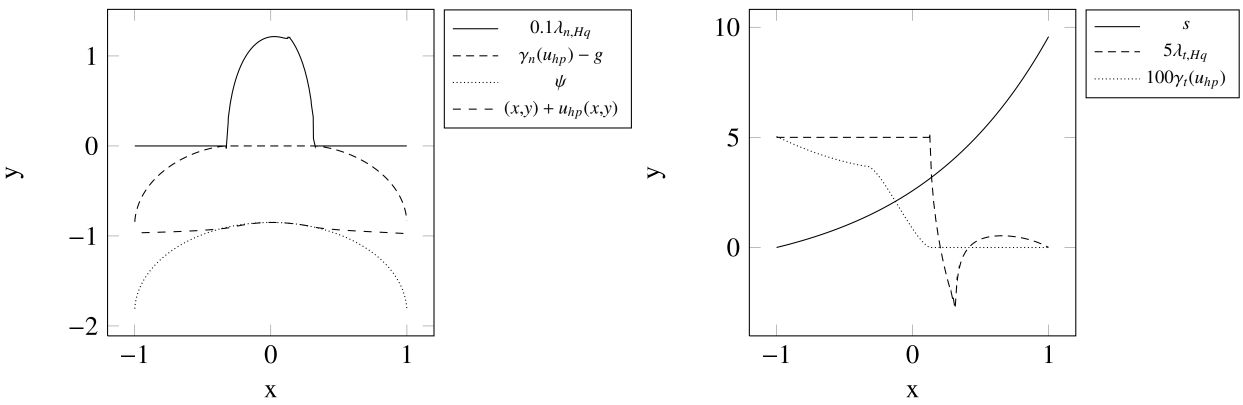

Figure 2: Displacement and Lagrange multipliers on Γ_C.

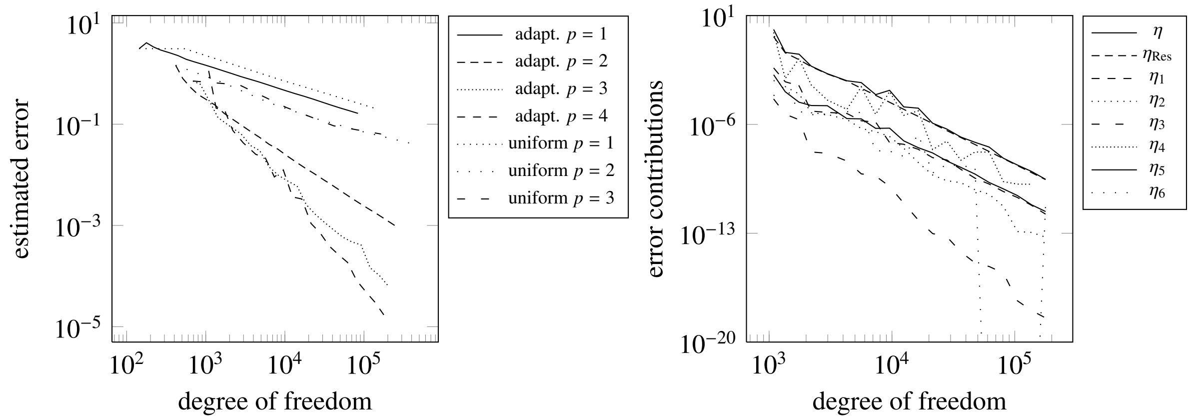

Figure 3: Estimated error and error contributions.

6. Numerical results

In this section, we study Signorini's problem with Tresca friction where $\Omega := (-1,1)^2$, $\Gamma_C := (-1,1) \times \{-1\}$, $\Gamma_D := [-1,1] \times \{1\}$, $\Gamma_N := \{-1\} \times (-1,1)$, $f := 0$, $f_N := -2$ and $s(x) = 1.5(\exp(x+1) - 1)$. The surface of the rigid foundation is described by $\psi(x) = (1-x^2)^{1/2} - 1.85$. We use Hooke's law for plane stress with Young's modulus $E := 70kN/mm^2$ and Poisson number $\nu := 0.33$. In Figure 1(a), the displacement and the von Mises stress $\sigma_v := (\sigma_{11} + \sigma_{22} - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2)^{1/2}$ are visualized for the resulting frictional contact problem. We see that the body gets in contact with the rigid foundation and is deformed in accordance with the surface of the foundation given by ψ. The surface load on Γ_N results in a gliding area on the left of Γ_C and an area of sticking on the right. In Figure 2, the discrete displacement u_{hp} as well as the Lagrange multipliers $\lambda_{n,Hq}$ and $\lambda_{t,Hq}$ are depicted. The discretization is given by $p = 2$, $q = 1$, $h = 7.8125 \cdot 10^{-3}$, $H = 2h$ on a uniform mesh. Here and in the following, $\mathcal{M}$ is the set of $q+1$ Chebyshev points and we choose $\tilde{s} = s$, see Remark 3.2. We observe in Figure 2 that the discrete displacement and the Lagrange multipliers reflect the geometrical contact conditions $u_n - g \leq 0$, $\lambda_n \geq 0$ and $\lambda_n(u_n - g) = 0$ as well as the frictional constraints $|\lambda_t| = |\sigma_{nt}(u)| \leq 1$, $|\lambda_t| < 1 \Rightarrow u_t = 0$ and $|\lambda_t| = 1 \Rightarrow u_t = \xi s \lambda_t$ with $\xi \geq 0$.

In the experiments, we study the applicability of the a posteriori error estimates as stated in the Theorems 5.3 and 5.5 within an h-adaptive scheme. To get a reliable error estimation, we have to estimate the dual norm of the residual

$\|\operatorname{Res}\|_{V'}$ and to compute the following error contributions,

$$\begin{aligned}
&\eta_1 := |(\lambda_{n,Hq}, (\gamma_n(u_{hp}) - g)_+)_{0,\Gamma_C}|, \quad \eta_2 := |((\lambda_{n,Hq})_+, g - \gamma_n(u_{hp}))_{0,\Gamma_C}|,\\
&\eta_3 := \|(\gamma_n(u_{hp}) - g)_+\|_{0,\Gamma_C} \|(\gamma_n(u_{hp}) - g)_+\|_{1,\Gamma_C}, \quad \eta_4 := \|\lambda_{n,Hq} - (\lambda_{n,Hq})_+\|^2_{0,\Gamma_C},\\
&\eta_5 := (s, |\gamma_t(u_{hp})|)_{0,\Gamma_C} - ((\lambda_{t,Hq})_{\zeta(s,\tilde{s})}, \tilde{s}\gamma_t(u_{hp}))_{0,\Gamma_C}, \quad \eta_6 := \|\lambda_{t,Hq} - (\lambda_{t,Hq})_{\zeta(s,\tilde{s})}\|^2_{0,\Gamma_C}.
\end{aligned}$$

To estimate $\|\operatorname{Res}\|_{V'}$ we use a standard residual error estimator η_{Res}, which is defined by $\eta_{\mathrm{Res}} := \sum_{T\in\mathcal{T}}(h_T^2 R_T^2 + \sum_{e\in\mathcal{E}_T} h_E R_E^2)$ for

$$R_T := \|f + \operatorname{div}\sigma(u_{hp})\|_{0,T}, \ T \in \mathcal{T},$$

$$R_E := \begin{cases} \frac{1}{2}\|[\sigma_n(u_{hp})]\|_{0,E}, & e \in \mathcal{E}^\circ,\\ \|\sigma_n(u_{hp}) - f_N\|_{0,E}, & E \in \mathcal{E}_N,\\ \|\sigma_{nn}(u_{hp}) + \lambda_{n,Hq}\|_{0,E} + \|\sigma_{nt}(u_{hp}) + \tilde{s}\lambda_{t,Hq}\|_{0,E}, & E \in \mathcal{E}_C, \end{cases}$$

where $\mathcal{E}_T$ is the set of edges of $T \in \mathcal{T}_h$, $\mathcal{E}^\circ$ contains the internal edges, $\mathcal{E}_N$ and $\mathcal{E}_C$ contain the edges on Γ_N and Γ_C. As usual, $[\cdot]_E$ denotes the jump across an edge $E \in \mathcal{E}^\circ$, whereas h_T and h_E denote the diameter of T and E, respectively. The overall error estimation is then given by

$$\eta := \eta_{\mathrm{Res}} + \sum_{i=1}^{6}\eta_i.$$

We localize the error contributions $\eta_1, \ldots, \eta_6$ by adding the local edge contributions to the element contributions which are associated to the adjacent mesh elements. Unfortunately, this localization is not possible for the contribution η_3. Since η_3 seems to be of higher-order (see below), we omit this contribution for the adaptive mesh refinement. Furthermore, we use a simple fixed fraction strategy where a fixed fraction (here 10%) of the mesh $\mathcal{T}$ is refined which is associated to the largest error contributions. Adaptive meshes are shown in Figure 1(b) and (c) for the polynomial degrees $p = 3$ and $p = 4$. In addition to the local refinements at both ends of the Dirichlet boundary, we find particularly strong local refinements at those points where the geometrical condition switches from contact to non-contact and the frictional condition from gliding to sticking, cf. Figure 1(d). These local refinements directly result from the regularity property of the Lagrange multipliers. Due to the switching from contact to non-contact and from gliding to sticking the Lagrange multipliers may still be continuous, but they are generally not of higher regularity. This also influences the local regularity of the displacement solution and, therefore, limits the use of non-adaptive higher-order finite element methods.

Thanks to the local refinements of the adaptive scheme, we are able to significantly improve the convergence of the proposed higher-order discretization and recover nearly optimal algebraic rates. This can be seen in Figure 3(left), where the estimated errors resulting from adaptive as well as uniform mesh refinements are compared for the polynomial degrees $p = 1, 2, 3, 4$.

Finally, we consider the non-conformity of the higher-order discretization. For this purpose, we study the error contributions η_{Res} and $\eta_1, \ldots, \eta_6$. Figure 3(right) shows the contributions for the adaptive mesh for $p = 4$. Except for the error contribution η_3, all contributions are nearly of the same order. However, the contributions $\eta_1, \eta_2, \eta_5, \eta_6$ are essentially smaller than η_{Res} and η_4 so that they do not significantly affect the overall estimation. In contrast, the contribution η_4 has an essential effect on the estimation. The error contribution η_3 seems to be of higher-order so that it may be omitted at least in the mesh adaptation.

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