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Four-loop corrections with two closed fermion loops to fermion self energies and the lepton anomalous magnetic moment

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Abstract

We compute the eighth-order fermionic corrections involving two and three closed massless fermion loops to the anomalous magnetic moment of the muon. The required four-loop on-shell integrals are classified and explicit analytical results for the master integrals are presented. As further applications we compute the corresponding four-loop QCD corrections to the mass and wave function renormalization constants for a massive quark in the on-shell scheme.

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1 Introduction

In the last about ten years several groups have been active in computing four-loop corrections to various physical quantities. Among them are the order α_s^4 corrections to the R ratio and the Higgs decay into bottom quarks [1–3], four-loop corrections to moments of the photon polarization function [4–8] which lead to precise results for the charm and bottom quark masses (see, e.g., Ref. [9]), and the free energy density of QCD at high temperatures [10]. The integrals involved in such calculations are either four-loop massless two-point functions or four-loop vacuum integrals with one non-vanishing mass scale. In this paper we take the first steps towards the systematic study of a further class of four-loop single-scale integrals, the so-called on-shell integrals where in the loop massless and massive propagators may be present and the only external momentum is on the mass shell.

On-shell integrals enter a variety of physical quantities, where the anomalous magnetic moments and on-shell counterterms are prominent examples. The first systematic study of two-loop on-shell integrals needed for the evaluation of the on-shell mass and wave function renormalization constants ($Z_m^{\rm OS}$ and $Z_2^{\rm OS}$) for a heavy quark in QCD has been performed in Refs. [11, 12]. Already a few years later, in 1996 the analytical three-loop corrections to the lepton anomalous magnetic moment a_l became available [13]. This result has been checked in Refs. [14,15]. In Refs. [14,16] the three-loop on-shell integrals have been applied to QCD, namely the evaluation of $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$. The calculation of Ref. [14] has confirmed the numerical result of [17, 18] which has been available before. Both $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ have also been computed in Ref. [15]. Further application of three-loop on-shell integrals are discussed in Refs. [19, 20]. There is no systematic study of four-loop on-shell integrals available in the literature. Nevertheless, some four-loop results to the anomalous magnetic moment of the muon, a_μ, have been computed analytically, in particular contributions from closed electron loops. E.g., the contribution where the photon propagator of the one-loop diagram (see Fig. 1) is dressed by higher order corrections has been considered in several papers [21–27]. Four-loop corrections where one of the two photon propagators of the two-loop diagram is dressed by higher orders has been considered in Ref. [28, 29]. Contributions where both photon propagators get one-loop electron insertions are still missing. This gap will be closed in the present work. Let us mention that all four- and even five-loop results for a_l are available in the literature in numerical form [27, 30–33] (see also the review articles [34, 35]).

In this paper we take the first step towards the analytical calculation of four-loop on-shell integrals by considering the subclass with two or three closed massless fermion loops, which are marked by a factor n_l. Thus we are concerned with four-loop terms proportional to n_l^3 and n_l^2 which we consider for three physical quantities: the anomalous magnetic moment of the muon, a_μ, the on-shell mass renormalization constant, $Z_m^{\rm OS}$, and the on-shell wave function renormalization constant, $Z_2^{\rm OS}$, for a massive quark. For the latter QCD corrections to the quark two-point functions are computed whereas for the former muon-photon vertex diagrams have to be considered. Some sample Feynman diagrams are

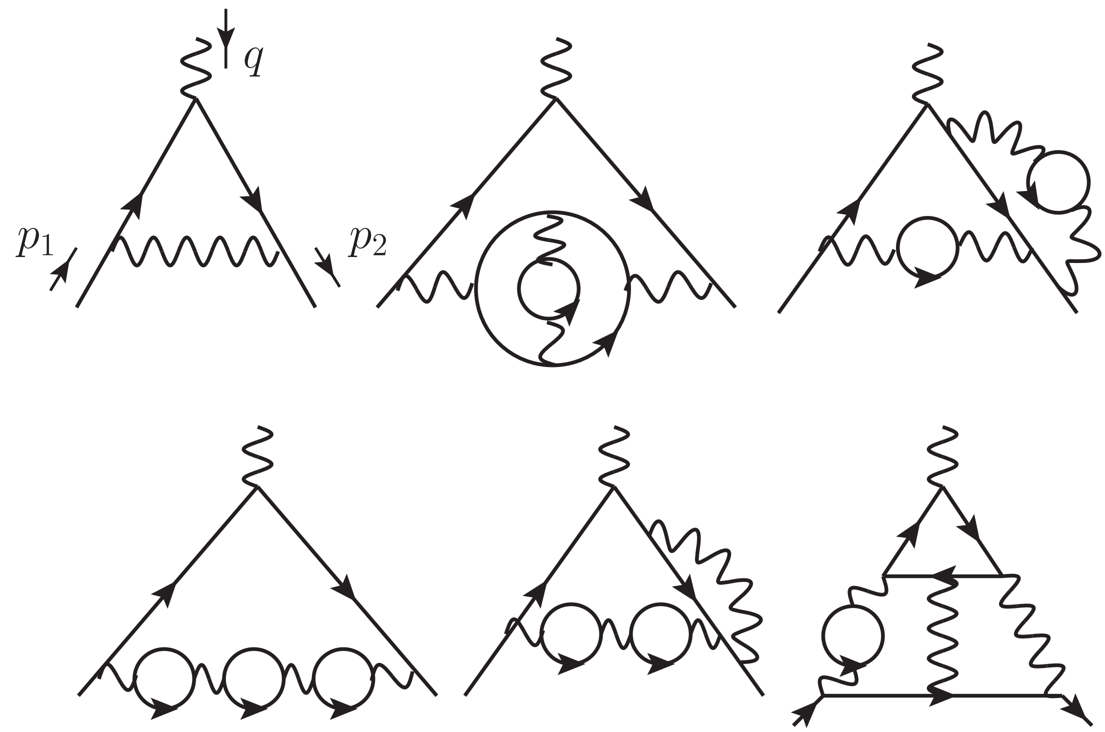

Figure 1: Sample Feyman diagrams for the photon-muon vertex contributing to a_μ. Wavy and straight lines represent photons and fermions, respectively. In this paper we consider the contribution where at least two of the closed loops correspond to massless fermions. The last diagram in the second line is a representative of the so-called "light-by-light" contribution.

given in Figs. 1 and 2. The precise definition of these quantities is provided in Sections 3 and 4.

The outline of the paper is as follows: in the next section we provide details of the four-loop on-shell integrals needed for our calculation. In particular, we identify all master integrals and provide analytical results in Appendix A. The renormalization constants $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ are discussed in Section 3 and Section 4 is devoted to the anomalous magnetic moment of the muon. We discuss the relation between the $\overline{\rm MS}$ and on-shell fine structure constant and provide analytical results for a_μ. Finally, we conclude in Section 5. Appendix B contains the analytic results for the relation between the fine structure constant defined in the $\overline{\rm MS}$ and on-shell scheme.

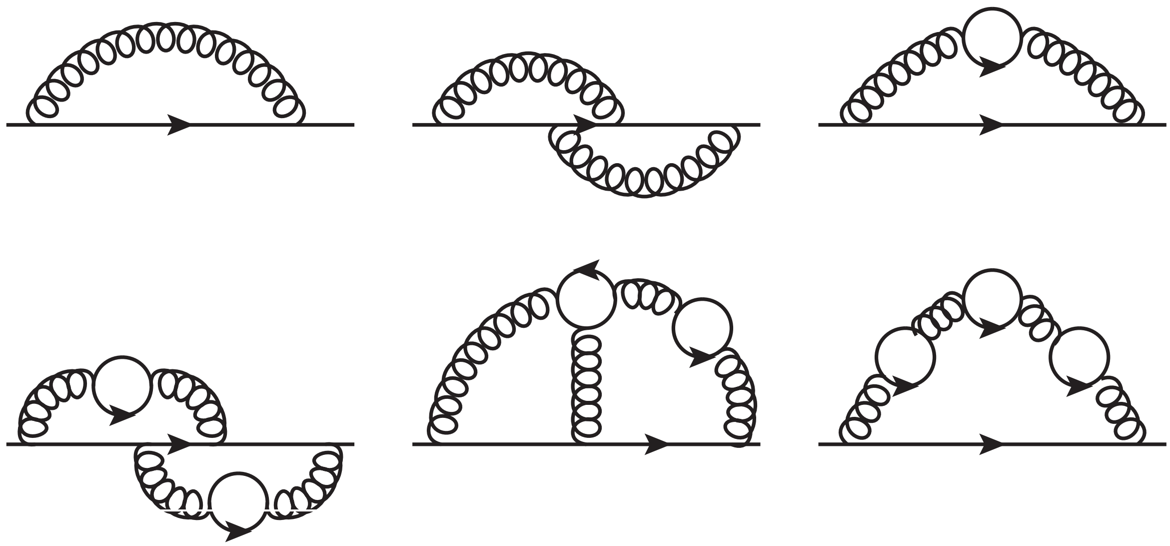

Figure 2: Sample Feynman diagrams for the QCD corrections to the fermion propagator contributing to $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$. Curly and straight lines represent gluons and fermions, respectively. In this paper we consider the contribution where at least two of the closed loops correspond to massless fermions.

2 Four-loop on-shell integrals

In this Section we present the setup used for the calculation and discuss the families of four-loop on-shell integrals needed for the n_l^2 and n_l^3 corrections for $Z_2^{\rm OS}$, $Z_m^{\rm OS}$ and a_μ. Since all three cases reduce to the calculation of corrections to the fermion propagator we consider in this Section the corresponding two-point function.

After the generation of the diagrams with `QGRAF` [36] we use `q2e` [37, 38] to translate the output into a `FORM` [39] readable form. In a next step `exp` [37, 38] is applied to map the momenta to one of five families. During the evaluation of the `FORM` code we apply projectors and take traces to end up with integrals which only contain scalar products in the numerator and quadratic denominators.

In the next step we have to reduce all occurring integrals to a minimal set of master integrals. This is done using two different programs in order to have a cross check for the calculation. On the one hand we use `crusher` [40] and on the other hand the `C++` version of `FIRE`.[1] Both programs implements Laporta's algorithm [42] for the solution of integration-by-parts identities [43]. We find complete agreement for the expressions where the physical quantities are expressed in terms of master integrals.

Let us mention that we have performed our calculations for general gauge parameter which drops out once the four-loop results for $Z_2^{\rm OS}$, $Z_m^{\rm OS}$ and a_μ are expressed in terms of

[1] The `Mathematica` version of `FIRE` is publicly available [41].

master integrals.[2]

Altogether we end up with 13 master integrals. Seven of them (shown in Fig. 3) are products of one- and two-loop integrals whereas the remaining six integrals (cf. Fig. 4) request a dedicated investigation. We calculate them using the Dimensional Recurrence and Analyticity (DRA) method introduced in [44]. In order to fix the position and order of the poles of the integrals, we use FIESTA [45, 46]. The remaining constants are fixed using the Mellin-Barnes technique [47–51]. In order to express the results in terms of the conventional multiple zeta values we apply the PSLQ algorithm [52] on high-precision numerical results (with several hundreds of decimal digits).[3]

The analytic results for the integrals in Fig. 4 are listed in Appendix A. Results in terms of Gamma functions for the integrals in Fig. 3 are easily obtained recursively using the formulae from the Appendix of Ref. [49]. For convenience also these results are given in Appendix A.

All results have been cross-checked numerically with the help of `FIESTA` [46] where an accuracy of at least four digits has been achieved.

3 Fermionic n_l^2 and n_l^3 contributions to $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$

Both $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ are obtained from the fermion two-point functions $\Sigma(q)$ which can be cast in the form

$$\Sigma(q, m_q) \;=\; m_q\,\Sigma_1(q^2, m_q) + (\not{q} - m_q)\,\Sigma_2(q^2, m_q)\,. \tag{1}$$

Here m_q represents a generic quark mass whereas bare, on-shell and $\overline{\rm MS}$ quark masses are denoted by m_q^0, M_q and $\bar{m}_q$.

The derivation of ready-to-use formulae for $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ is discussed at length in Refs. [14, 15]. Thus, let us for convenience only repeat the final formulae which are applied in our calculations. They read

$$Z_m^{\rm OS} \;=\; 1 + \Sigma_1(M_q^2, M_q)\,, \tag{2}$$

$$\left(Z_2^{\rm OS}\right)^{-1} \;=\; 1 + 2M_q^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)\,. \tag{3}$$

The expressions on the right-hand side are computed by introducing the momentum Q

[2] Note that $Z_m^{\rm OS}$ and a_μ have to be independent of the QCD gauge parameter ξ whereas we expect that the n_l^1 and n_l-independent terms of $Z_2^{\rm OS}$ do depend on ξ.

[3] Let us mention that the numerical evaluation of the factorizable four-loop master integrals for a_l which reduce to the evaluation of the corresponding three-loop master integrals in higher orders of ϵ was undertaken in Ref. [53] as a warm-up before a future full four-loop calculation. This was done with the method of [42] based on difference equations. The achieved accuracy of several dozen of decimal digits was not enough for using PSLQ.

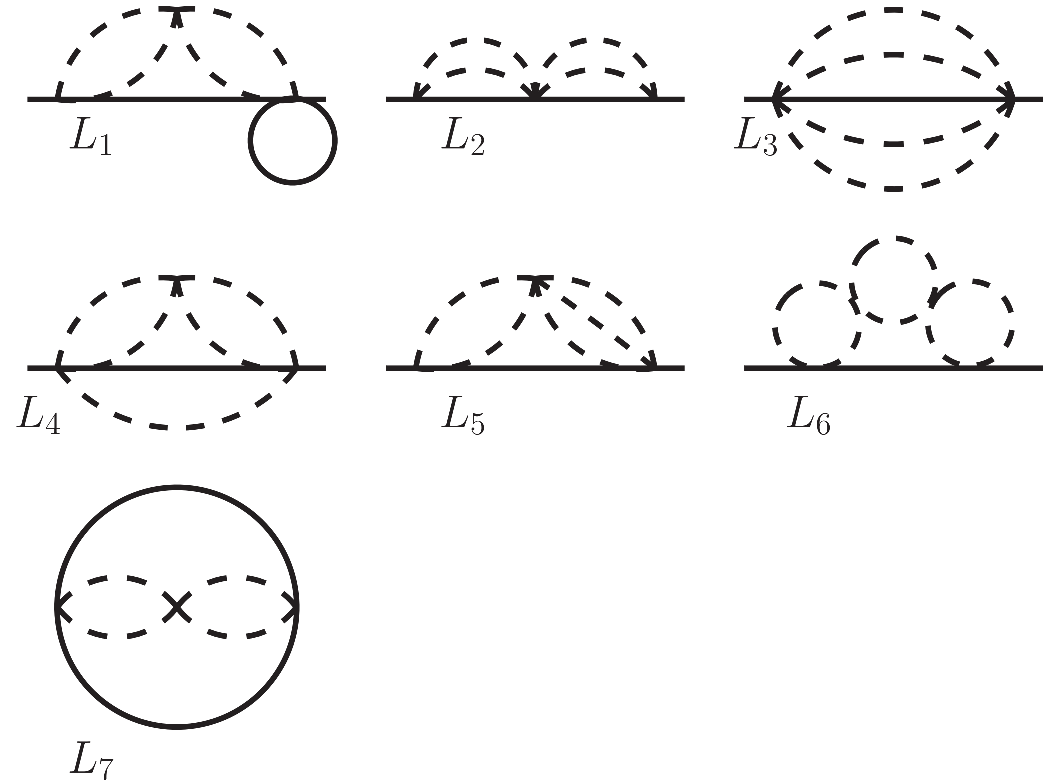

Figure 3: Master integrals for the n_l^2 and n_l^3 contribution which are easily obtained by applying one- and two-loop formulae, see e.g., Ref. [49]. Solid lines carry the mass M and dashed lines are massless. For L_1 to L_6 we have $q^2 = M^2$ where q is the external momentum; L_7 is a vacuum integral.

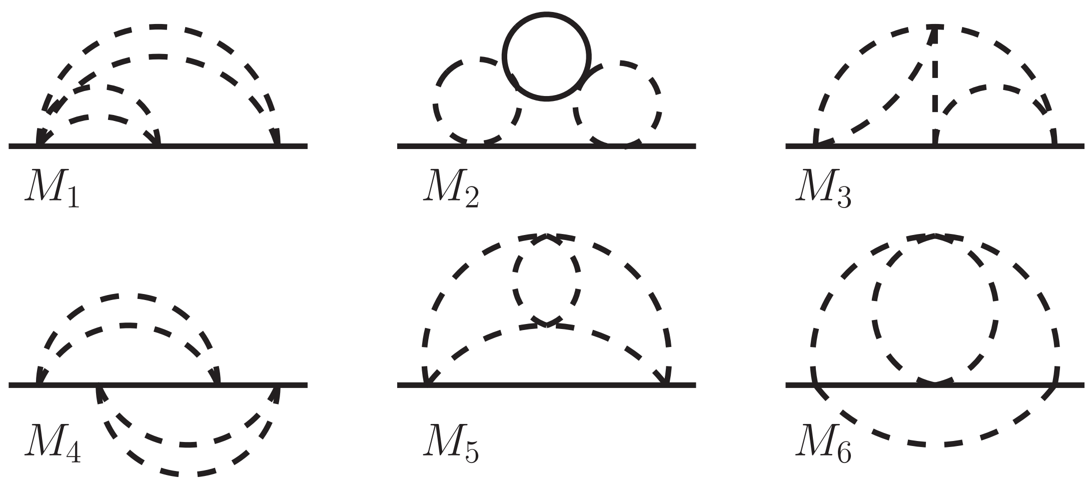

Figure 4: Non-trivial master integrals contributing to the n_l^2 contribution. The same notation as in Fig. 3 has been used.

with $Q^2 = M_q^2$ via $q = Q(1+t)$ which leads to the equation

$$\begin{aligned}
\mathrm{Tr}\left\{\frac{\not{Q} + M_q}{4M_q^2}\Sigma(q, M_q)\right\} &= \Sigma_1(q^2, M_q) + t\Sigma_2(q^2, M_q) \\
&= \Sigma_1(M_q^2, M_q) + \left(2M_q^2 \frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)\right) t \\
&\quad + \mathcal{O}(t^2) \,. \qquad (4)
\end{aligned}$$

Hence, to obtain Z_m^{OS} one only needs to calculate Σ_1 for $q^2 = M_q^2$. To calculate Z_2^{OS}, one has to compute the first derivative of the self-energy diagrams. Note that the renormalization of the quark mass is taken into account iteratively by explicitly calculating the corresponding counterterm diagrams.

We write the perturbative expansion for Z_m^{OS} in terms of the renormalized strong coupling as (γ_E is the Euler-Mascheroni number)

$$\begin{aligned}
Z_m^{\mathrm{OS}} &= 1 + \frac{\alpha_s(\mu)}{\pi}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}\delta Z_m^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}\delta Z_m^{(2)} \\
&\quad + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\epsilon}\delta Z_m^{(3)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^4\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-4\epsilon}\delta Z_m^{(4)} + \mathcal{O}\left(\alpha_s^5\right) \,. \qquad (5)
\end{aligned}$$

This allows us to take the ratio between the on-shell and $\overline{\mathrm{MS}}$ [54–56] mass renormalization constant which is given by

$$\begin{aligned}
z_m^{\mathrm{OS}}(\mu) &= \frac{\bar{m}_q(\mu)}{M_q} = \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}} \\
&= 1 + \frac{\alpha_s(\mu)}{\pi}\delta z_m^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\delta z_m^{(2)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3\delta z_m^{(3)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^4\delta z_m^{(4)} \\
&\quad + \mathcal{O}\left(\alpha_s^5\right) \qquad (6)
\end{aligned}$$

The coefficients $\delta z_m^{(i)}$ are by construction finite.

In the case of Z_2^{OS} we choose the bare coupling as expansion parameter which in many applications turns out to be convenient. Furthermore, the dependence on μ/M_q can be written in factorized form which leads to shorter expressions. Thus we have

$$\begin{aligned}
Z_2^{\mathrm{OS}} &= 1 + \frac{\alpha_s^0}{\pi}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}\delta Z_2^{(1)} + \left(\frac{\alpha_s^0}{\pi}\right)^2\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}\delta Z_2^{(2)} \\
&\quad + \left(\frac{\alpha_s^0}{\pi}\right)^3\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\epsilon}\delta Z_2^{(3)} + \left(\frac{\alpha_s^0}{\pi}\right)^4\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-4\epsilon}\delta Z_2^{(4)} + \mathcal{O}\left(\left(\alpha_s^0\right)^5\right) \,, \qquad (7)
\end{aligned}$$

where each term $\delta Z_2^{(n)}$ contains a factor $(\mu^2/M_q^2)^{n\epsilon}$.

We refrain from repeating the one-, two- and three-loop results for Z_m^{OS} and Z_2^{OS} since analytical expressions for general colour coefficients are available in the literature [14–16].

We split the four-loop coefficient according to the number of closed massless fermion loops and write ($i \in \{m, 2\}$)

$$\delta Z_i^{(4)} \;=\; \delta Z_i^{(40)} + \delta Z_i^{(41)} n_l + \delta Z_i^{(42)} n_l^2 + \delta Z_i^{(43)} n_l^3 \,. \tag{8}$$

with an analog notation for $\delta z_m^{(4)}$.

In the following we present analytical results for $\delta z_m^{(42)}$, $\delta z_m^{(43)}$, $\delta Z_2^{(42)}$ and $\delta Z_2^{(43)}$ which read

$$\begin{aligned}\delta z_m^{(43)} \;=\;& C_F T^3 \left(\frac{l_M^4}{144} + \frac{13 l_M^3}{216} + \left(\frac{89}{432} + \frac{\pi^2}{36} \right) l_M^2 + l_M \left(\frac{\zeta_3}{3} + \frac{1301}{3888} + \frac{13\pi^2}{108} \right) \right. \\ & \left. + \frac{317\zeta_3}{432} + \frac{71\pi^4}{4320} + \frac{89\pi^2}{648} + \frac{42979}{186624} \right) , \end{aligned} \tag{9}$$

$$\begin{aligned}\delta z_m^{(42)} \;=\;& C_F n_h T^3 \left(\frac{l_M^4}{48} + \frac{13 l_M^3}{72} + \frac{125 l_M^2}{144} + \frac{2489 l_M}{1296} + \frac{5\zeta_3}{144} - \frac{19\pi^4}{480} + \frac{\pi^2}{6} + \frac{128515}{62208} \right) \\ & + C_A C_F T^2 \left(-\frac{11 l_M^4}{192} - \frac{91 l_M^3}{144} + l_M^2 \left(-\frac{1}{12}\pi^2 a_1 - \frac{\zeta_3}{4} - \frac{\pi^2}{8} - \frac{6539}{2304} \right) \right. \\ & + l_M \left(\frac{a_1^4}{18} + \frac{1}{9}\pi^2 a_1^2 - \frac{11}{18}\pi^2 a_1 + \frac{4a_4}{3} - \frac{37\zeta_3}{16} - \frac{\pi^4}{216} - \frac{29\pi^2}{36} - \frac{15953}{2592} \right) \\ & - \frac{1}{45} a_1^5 + \frac{11 a_1^4}{54} - \frac{2}{27}\pi^2 a_1^3 + \frac{11}{27}\pi^2 a_1^2 - \frac{31\pi^4 a_1}{1080} - \frac{103}{108}\pi^2 a_1 + \frac{44 a_4}{9} + \frac{8 a_5}{3} \\ & \left. - \frac{41\zeta_5}{24} - \frac{13\pi^2\zeta_3}{48} - \frac{3245\zeta_3}{576} - \frac{4723\pi^4}{51840} - \frac{527\pi^2}{384} - \frac{2708353}{497664} \right) \\ & + C_F^2 T^2 \left(+ \frac{11 l_M^4}{384} + \frac{97 l_M^3}{576} + l_M^2 \left(\frac{1}{6}\pi^2 a_1 + \frac{\zeta_3}{8} - \frac{5\pi^2}{96} - \frac{157}{2304} \right) \right. \\ & + l_M \left(-\frac{1}{9} a_1^4 - \frac{2}{9}\pi^2 a_1^2 + \frac{11}{9}\pi^2 a_1 - \frac{8 a_4}{3} - \frac{11\zeta_3}{8} + \frac{11\pi^4}{216} - \frac{21\pi^2}{32} - \frac{50131}{20736} \right) \\ & + \frac{2 a_1^5}{45} - \frac{11 a_1^4}{27} + \frac{4}{27}\pi^2 a_1^3 - \frac{22}{27}\pi^2 a_1^2 + \frac{31}{540}\pi^4 a_1 + \frac{103}{54}\pi^2 a_1 - \frac{88 a_4}{9} - \frac{16 a_5}{3} \\ & \left. + \frac{305\zeta_5}{48} + \frac{3\pi^2\zeta_3}{8} - \frac{2839\zeta_3}{576} + \frac{3683\pi^4}{51840} - \frac{5309\pi^2}{3456} - \frac{2396921}{497664} \right) , \end{aligned} \tag{10}$$

$$\begin{aligned}\delta Z_2^{(43)} \;=\;& C_F T^3 \left(\frac{\mu^2}{M_q^2} \right)^{4\epsilon} \left(\frac{1}{144\epsilon^4} + \frac{65}{864\epsilon^3} + \frac{\frac{89}{192} + \frac{13\pi^2}{432}}{\epsilon^2} + \frac{\frac{151\zeta_3}{216} + \frac{73669}{31104} + \frac{845\pi^2}{2592}}{\epsilon} \right. \\ & \left. + \frac{9815\zeta_3}{1296} + \frac{589\pi^4}{4320} + \frac{1157\pi^2}{576} + \frac{2106347}{186624} \right) , \end{aligned} \tag{11}$$

$$\delta Z_2^{(42)} \;=\; \left(\frac{\mu^2}{M_q^2} \right)^{4\epsilon} \left[C_F n_h T^3 \left(\frac{1}{36\epsilon^4} + \frac{187}{864\epsilon^3} + \frac{\frac{10957}{5184} - \frac{5\pi^2}{108}}{\epsilon^2} \right. \right.$$

$$
\begin{aligned}
&+\frac{\frac{2}{3}\pi^2 a_1-\frac{71\zeta_3}{54}-\frac{1013\pi^2}{2592}+\frac{349615}{31104}}{\epsilon}-\frac{10}{9}a_1^4-\frac{20}{9}\pi^2 a_1^2+\frac{127}{18}\pi^2 a_1\\
&-\frac{80a_4}{3}-\frac{21719\zeta_3}{1296}-\frac{\pi^4}{360}-\frac{14027\pi^2}{15552}+\frac{13135057}{186624}\Bigg)\\
&+C_A C_F T^2\Bigg(-\frac{11}{192\epsilon^4}-\frac{761}{1152\epsilon^3}+\frac{-\frac{1}{6}\pi^2 a_1+\frac{\zeta_3}{16}-\frac{13\pi^2}{192}-\frac{64433}{13824}}{\epsilon^2}\\
&+\frac{\frac{5a_1^4}{18}+\frac{5}{9}\pi^2 a_1^2-\frac{163}{72}\pi^2 a_1+\frac{20a_4}{3}+\frac{37\zeta_3}{288}-\frac{647\pi^4}{8640}-\frac{1627\pi^2}{1152}-\frac{18287}{768}}{\epsilon}\\
&-\frac{5}{9}a_1^5+\frac{815a_1^4}{216}-\frac{50}{27}\pi^2 a_1^3+\frac{815}{108}\pi^2 a_1^2+\frac{1}{18}\pi^4 a_1-\frac{281}{18}\pi^2 a_1\\
&+\frac{815a_4}{9}+\frac{200a_5}{3}-\frac{2079\zeta_5}{32}-\frac{209\pi^2\zeta_3}{144}-\frac{27977\zeta_3}{1728}-\frac{7411\pi^4}{5760}\\
&-\frac{436741\pi^2}{41472}-\frac{60973393}{497664}\Bigg)\\
&+C_F^2 T^2\Bigg(\frac{11}{384\epsilon^4}+\frac{47}{192\epsilon^3}+\frac{\frac{1}{3}\pi^2 a_1-\frac{5\zeta_3}{16}-\frac{241\pi^2}{1152}+\frac{2363}{1536}}{\epsilon^2}\\
&+\frac{-\frac{5}{9}a_1^4-\frac{10}{9}\pi^2 a_1^2+\frac{163}{36}\pi^2 a_1-\frac{40a_4}{3}-\frac{773\zeta_3}{72}+\frac{383\pi^4}{1728}-\frac{1181\pi^2}{576}+\frac{2893}{2304}}{\epsilon}\\
&+\frac{10a_1^5}{9}-\frac{815a_1^4}{108}+\frac{100}{27}\pi^2 a_1^3-\frac{815}{54}\pi^2 a_1^2-\frac{1}{9}\pi^4 a_1+\frac{281}{9}\pi^2 a_1\\
&-\frac{1630a_4}{9}-\frac{400a_5}{3}+\frac{7145\zeta_5}{48}+\frac{187\pi^2\zeta_3}{48}-\frac{50209\zeta_3}{576}+\frac{8413\pi^4}{6480}\\
&-\frac{75089\pi^2}{4608}-\frac{261181}{55296}\Bigg)\Bigg]\,, \qquad (12)
\end{aligned}
$$

where $l_M = \ln \mu^2/M_q^2$, ζ_n is Riemann's zeta function, $a_1 = \ln 2$ and $a_n = \mathrm{Li}_n(1/2)$ $(n \geq 1)$. In the case of QCD the colour factors take the values $C_A = 3, C_F = 4/3$ and $T = 1/2$. In Eqs. (10) and (12) the contributions from closed heavy quark loops are marked by $n_h = 1$ which has been introduced for illustration.

In order to get an impression of the numerical size of the newly calculated terms we evaluate $z_m^{\rm OS}$ for $\mu = M_q$. After inserting the numerical values for the colour factors we obtain ($A_s \equiv \alpha_s(M_q)/\pi$)

$$
\begin{aligned}
z_m^{\rm OS} \;=\;& 1 - A_s 1.333 + A_s^2\left(-14.229 - 0.104\,n_h + 1.041\,n_l\right)\\
&+ A_s^3\left(-197.816 - 0.827\,n_h - 0.064\,n_h^2 + 26.946\,n_l - 0.022\,n_h n_l - 0.653\,n_l^2\right)\\
&+ A_s^4\left(-43.465\,n_l^2 - 0.017\,n_h n_l^2 + 0.678\,n_l^3 + \ldots\right) + \mathcal{O}\left(A_s^5\right)\,, \qquad (13)
\end{aligned}
$$

where the ellipses indicate n_l independent contributions and terms proportional to n_l which have not been computed. One observes that the n_l^2 contribution at two loops and the n_l^3 contribution at three loops are quite small. This is in contrast to the linear n_l

terms which can become quite sizeable. E.g., setting $n_l = 5$, which corresponds to the case of the top quark, we obtain (for $n_h = 1$)

$$
\begin{aligned}
z_m^{\rm OS} &= 1 - A_s 1.333 + A_s^2 \left(-14.332 + 5.207_{n_l}\right) \\
&\quad + A_s^3 \left(-198.707 + 134.619_{n_l} - 16.317_{n_l^2}\right) \\
&\quad + A_s^4 \left(-1087.060_{n_l^2} + 84.768_{n_l^3} + \ldots\right) + \mathcal{O}\left(A_s^5\right) . \qquad (14)
\end{aligned}
$$

At two-loop order the n_l contribution is only a factor of three smaller than the n_l-independent term, however, with an opposite sign. At three loops the linear-n_l term has almost the same order of magnitude than the constant contribution but again a different sign. It is remarkable that for $n_l = 5$ the coefficient of the four-loop n_l^2 term is more than a factor of five larger than the n_l-independent term at order α_s^3.

Let us finally compare our results with the approximate expressions obtained in Ref. [57] in the large-β_0 approximation. In Ref. [57] one finds for the quantity $M_q/\bar{m}_q(\bar{m}_q)$ the result ($a_s \equiv \alpha_s(\bar{m}_q)/\pi$)

$$
\begin{aligned}
\left.\frac{M_q}{\bar{m}_q(\bar{m}_q)}\right|_{\text{large}-\beta_0} &= 1 + a_s 1.333 + a_s^2 \left(17.186 - 1.041 n_l\right) \\
&\quad + a_s^3 \left(177.695 - 21.539 n_l + 0.653 n_l^2\right) \\
&\quad + a_s^4 \left(3046.294 - 553.872 n_l + 33.568 n_l^2 - 0.678 n_l^3\right) , \qquad (15)
\end{aligned}
$$

where for the renormalization scale $\mu = \bar{m}_q$ has been chosen. The coefficients of Eq. (15) should be compared with our findings which read

$$
\begin{aligned}
\frac{M_q}{\bar{m}_q(\bar{m}_q)} &= 1 + a_s 1.333 + a_s^2 \left(13.443 - 1.041 n_l\right) \\
&\quad + a_s^3 \left(190.595 - 26.655 n_l + 0.653 n_l^2\right) \\
&\quad + a_s^4 \left(c_0 + c_1 n_l + 43.396 n_l^2 - 0.678 n_l^3\right) , \qquad (16)
\end{aligned}
$$

where c_0 and c_1 are not yet known. By construction one finds agreement for the coefficient of n_l^3 since it has been used as input in Ref. [57]. As far as the n_l^2 term is concerned the exact coefficient is predicted with an accuracy of about 30%.

4 Fermionic n_l^2 and n_l^3 contributions to a_μ

It is convenient to introduce the form factors F_1 and F_2 of the photon-lepton vertex as

$$
\Gamma^\mu(q,p) = F_1(q^2)\gamma^\mu + i\frac{F_2(q^2)}{2M_l}\sigma_{\mu\nu}q^\nu , \qquad (17)
$$

where q is the incoming momentum in the photon line and M_l is the lepton mass. The anomalous magnetic moment is given by

$$a_l = \left(\frac{g-2}{2}\right)_l = F_2(0)\,. \tag{18}$$

In Eq. (17) also the momentum $p = (p_1 + p_2)/2$ has been introduced where $p_1^2 = p_2^2 = M_l^2$ are the momenta flowing through the external fermion lines (see Fig. 1 for the directions of the momenta).

The evaluation of a_l requires that $\Gamma^\mu(q,p)$ is computed in the limit $q \to 0$. Due to the factor q^ν in front of F_2 in Eq. (17) one has to perform an expansion of $\Gamma^\mu(q,p)$ up to linear terms in q which can be written as

$$\Gamma^\mu(q,p) = X^\mu(p) + q_\nu Y^{\mu\nu}(p) + \mathcal{O}\left(q^2\right)\,, \tag{19}$$

with $p^2 = M_l^2$. F_2 is conveniently obtained after the application of a projector given by (see, e.g., Ref. [58])

$$\begin{aligned} a_l = & \frac{1}{2M_l^2(D-1)(D-2)}\mathrm{Tr}\Bigg[\frac{D-2}{2}\left(M_l^2\gamma_\mu - Dp_\mu\not{p} - (D-1)M_l p_\mu\right)X^\mu \\ & + \frac{M_l}{4}\left(\not{p}+M_l\right)\left[\gamma_\nu,\gamma_\mu\right]\left(\not{p}+M_l\right)Y^{\mu\nu}\Bigg]\,, \end{aligned} \tag{20}$$

and thus a_l is reduced to the evaluation of on-shell two-point functions as described in Section 2.

We define the loop expansion of a_l in analogy to Eq. (5) (with α_s replaced by the fine structure constant) and introduce the same splitting according to the number of massless lepton loops as in Eq. (8).

The Feynman diagrams contributing to a_l respectively the coefficients of α^n and n_l^k can be subdivided to two classes: (i) the one where the external photon couples to the lepton at hand and (ii) the one where it couples to a lepton present in a closed loop. Sample diagrams are given in Fig. 1. In the following we refer to the diagrams of class (ii) as "light-by-light" contribution in analogy to the corresponding hadronic part.

In this paper four-loop corrections contributing to class (i) are evaluated which contain two or three closed massless fermion loops. They are used in order to compute electron loop contributions to a_μ neglecting terms of order M_e/M_μ.

For the diagrams in class (i) we can proceed as follows: In a first step we renormalize the fine structure constant in the $\overline{\mathrm{MS}}$ scheme, $\bar{\alpha}(\mu)$. The corresponding renormalization constant is easily obtained from the one for α_s after specifying the colour factors to QED. The $\overline{\mathrm{MS}}$ scheme has the advantage that the electron mass can be set to zero (which is not the case for the diagrams in class (ii)). After renormalizing the muon mass in the on-shell scheme we obtain a finite expression for a_μ which shows an explicit dependence on $\ln(\mu^2/M_\mu^2)$.

In a next step we replace $\bar{\alpha}(\mu)$ by its on-shell counterpart using the corresponding relation up to three loops. It can best be calculated by considering the photon two point function

$$(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \;=\; i\int \mathrm{d}x\, e^{iqx}\langle 0|j^\mu(x)j^\nu(x)|0\rangle \tag{21}$$

and employing the on-shell renormalization condition $\Pi(q^2 = 0) = 0$. The form of the renormalization condition reduces the problem to the calculation of two-scale vacuum integrals at three loops. Note, that for the renormalization of the fermion masses in the on-shell scheme the dependence on both masses has to be taken into account. In the limit $M_e \ll M_\mu$ we obtain (see also Refs. [25, 27, 59])

$$\begin{aligned}\frac{\bar{\alpha}(\mu)}{\alpha} \;=\;& 1 + \frac{\alpha}{3\pi}\left(L_\mu + L_e\right) + \left(\frac{\alpha}{\pi}\right)^2\left[\frac{15}{8} + \frac{L_\mu + L_e}{4} + \frac{(L_\mu + L_e)^2}{9}\right]\\ &+ \left(\frac{\alpha}{\pi}\right)^3\left(\frac{L_e^3}{27} + \frac{L_\mu L_e^2}{9} + \frac{5L_e^2}{24} + \frac{79L_e}{144} - \frac{695}{648} + \frac{\pi^2}{9} + \frac{7\zeta_3}{64} + \ldots\right) + \mathcal{O}(\alpha_s^4)\end{aligned} \tag{22}$$

with $L_\mu = \ln(\mu^2/M_\mu^2)$ and $L_e = \ln(\mu^2/M_e^2)$. The ellipses in the coefficient of $(\alpha/\pi)^3$ indicate terms which we left out since they are irrelevant for the n_l^2 contribution discussed in this paper. The complete result containing the exact dependence on M_e/M_μ is presented in Appendix B. Note that the result in Eq. (22) can be obtained from the one provided in Ref. [27] where the relation between $\bar{\alpha}(\mu)$ and α is given for one massive lepton.

Also in the case of a_l we refrain from listing the lower-order results which can be found in the literature [13, 32–35]. Rather we concentrate on the new correction terms at four loops. Adopting the notation from Eq. (8) we obtain the following results for the n_l^3 contribution

$$\begin{aligned}a_\mu^{(43)} \;=\;& \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324} + \frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832}\\ \approx\;& 7.196\,66\,,\end{aligned} \tag{23}$$

where $L_{\mu e} = \ln(M_\mu^2/M_e^2)$. The approximate results have been obtained with the help of [60] $M_\mu/M_e = 206.7682843(52)$. The result in Eq. (23) agrees with the one in Ref. [28, 29].

In the case of the n_l^2 contribution we split $a_\mu^{(42)}$ into two parts. The first one ($a_\mu^{(42)a}$) corresponds to the diagrams containing two closed fermion loops and the second one ($a_\mu^{(42)b}$) originates from diagrams with three closed fermion loops where one of them is a muon and two are electron loops. Thus, we have

$$a_\mu^{(42)} \;=\; a_\mu^{(42)a} + a_\mu^{(42)b}\,,$$

with

$$
\begin{aligned}
a_\mu^{(42)a} &= L_{\mu e}^2\left[\pi^2\left(\frac{5}{36}-\frac{a_1}{6}\right)+\frac{\zeta_3}{4}-\frac{13}{24}\right]+L_{\mu e}\left[-\frac{a_1^4}{9}+\pi^2\left(-\frac{2a_1^2}{9}+\frac{5a_1}{3}-\frac{79}{54}\right)\right.\\
&\quad\left.-\frac{8a_4}{3}-3\zeta_3+\frac{11\pi^4}{216}+\frac{23}{6}\right]-\frac{2a_1^5}{45}+\frac{5a_1^4}{9}+\pi^2\left(-\frac{4a_1^3}{27}+\frac{10a_1^2}{9}\right.\\
&\quad\left.-\frac{235a_1}{54}-\frac{\zeta_3}{8}+\frac{595}{162}\right)+\pi^4\left(-\frac{31a_1}{540}-\frac{403}{3240}\right)+\frac{40a_4}{3}+\frac{16a_5}{3}-\frac{37\zeta_5}{6}\\
&\quad+\frac{11167\zeta_3}{1152}-\frac{6833}{864}\\
&\approx -3.624\,27\,, \qquad (24)\\
a_\mu^{(42)b} &= \left(\frac{119}{108}-\frac{\pi^2}{9}\right)L_{\mu e}^2+\left(\frac{\pi^2}{27}-\frac{61}{162}\right)L_{\mu e}-\frac{4\pi^4}{45}+\frac{13\pi^2}{27}+\frac{7627}{1944}\\
&\approx 0.494\,05\,. \qquad (25)
\end{aligned}
$$

$a_\mu^{(42)b}$ agrees with Ref. [28,29]. Analytical results for $a_\mu^{(42)a}$ are not present in the literature since corrections originating from diagrams as the third one in the first row of Fig. 1 have not been considered yet. However, we can perform a numerical comparison with the results from Refs. [30, 33][4] which reads

$$
a_\mu^{(42)a}\Big|_{\mathrm{num}} = -3.642\,04(1\,12)\,, \qquad (26)
$$

There is a good agreement with the analytic result in Eq. (24). The deviation can be explained by corrections of order $M_e/M_\mu \approx 0.005$ or $(M_e/M_\mu)^2 \ln^3 M_\mu/M_e \approx 0.004$ [28,29] which are absent in our analytic expressions.

5 Conclusions

In this paper the first steps towards the evaluation of four-loop on-shell integrals have been undertaken. As an application within QCD we have computed the contributions involving two massless quark loops to the on-shell renormalization constants Z_2^{OS} and Z_m^{OS}. As an application in QED we have considered the contribution from four-loop diagrams involving two or three closed electron loops to the anomalous magnetic moment of the muon excluding, however, the light-by-light contribution.

We describe in some detail the techniques and the programs which have been used for the calculation. We are confident that they are generic enough to be applied to the n_l^1 and non-fermionic contribution. The only bottleneck might be the analytic evaluation of the master integrals so that maybe numerical methods have to be applied.

[4] In in Ref. [33] the contributions from closed electron and muon loops are always added whereas in our result at least two closed electron loops are present. We are deeply grateful to the authors of Ref. [33] for providing us the results for the contributions containing only electron loops Eq. (26).

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Appendix A: Analytic results for the master integrals

In this appendix we provide the analytic results of all master integrals where we assume an integration measure $\mathrm{d}^D k/(i\pi)^{D/2}$ with $D = 4 - 2\epsilon$. Furthermore we write scalar propagators of particles with mass M in the form $1/(-k^2 + M^2)$. For convenience we set $M = 1$ in the final result since the dependence on M can easily be restored.

The analytic results for the integrals in Fig. 3 read

$$
\begin{aligned}
L_1 &= \frac{\Gamma\left(5-\frac{3D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)^2\Gamma\left(\frac{D}{2}-1\right)^4\Gamma(3D-9)}{\Gamma(D-2)^2\Gamma(2D-5)}\,,\\
L_2 &= \frac{\Gamma(3-D)^2\Gamma\left(2-\frac{D}{2}\right)^2\Gamma\left(\frac{D}{2}-1\right)^4\Gamma(2D-5)^2}{\Gamma(D-2)^2\Gamma\left(\frac{3D}{2}-3\right)^2}\,, && (27)\\
L_3 &= \frac{\Gamma(5-2D)\Gamma\left(4-\frac{3D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)^4\Gamma(4D-9)}{\Gamma(2D-4)\Gamma\left(\frac{5D}{2}-5\right)}\,, && (28)\\
L_4 &= \frac{\Gamma(6-2D)\Gamma\left(5-\frac{3D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)^2\Gamma\left(\frac{D}{2}-1\right)^5\Gamma\left(\frac{3D}{2}-4\right)\Gamma(4D-11)}{\Gamma(4-D)\Gamma(D-2)^2\Gamma(2D-5)\Gamma\left(\frac{5D}{2}-6\right)}\,, && (29)\\
L_5 &= \frac{\Gamma(6-2D)\Gamma(3-D)\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)^5\Gamma(4D-11)}{\Gamma(D-2)\Gamma\left(\frac{3D}{2}-3\right)\Gamma\left(\frac{5D}{2}-6\right)}\,, && (30)\\
L_6 &= \frac{\Gamma(7-2D)\Gamma\left(2-\frac{D}{2}\right)^3\Gamma\left(\frac{D}{2}-1\right)^6\Gamma(4D-13)}{\Gamma(D-2)^3\Gamma\left(\frac{5D}{2}-7\right)}\,, && (31)\\
L_7 &= \frac{\Gamma(6-2D)\Gamma\left(5-\frac{3D}{2}\right)^2\Gamma\left(2-\frac{D}{2}\right)^2\Gamma\left(\frac{D}{2}-1\right)^4\Gamma\left(\frac{3D}{2}-4\right)}{\Gamma(10-3D)\Gamma(D-2)^2\Gamma\left(\frac{D}{2}\right)}\,. && (32)
\end{aligned}
$$

The analytic results for the integrals in Fig. 4 read

$$
\begin{aligned}
e^{4\epsilon\gamma_E}M_1 = {} & \frac{5}{24\epsilon^4} + \frac{25}{24\epsilon^3} + \left(\frac{205}{96} + \frac{17\pi^2}{72}\right)\epsilon^{-2} + \left(-\frac{323}{96} + \frac{85\pi^2}{72} + \frac{79\zeta_3}{18}\right)\epsilon^{-1} + \left(-\frac{55241}{1152}\right.\\
&\left. + \frac{409\pi^2}{288} + \frac{395\zeta_3}{18} + \frac{\pi^4}{8}\right)\epsilon^0 - \left(\frac{733351}{3456} + \frac{4199\pi^2}{288} - \frac{1943\zeta_3}{72} - \frac{5\pi^4}{8} - \frac{499\pi^2\zeta_3}{54}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{407\zeta_5}{6}\bigg)\epsilon^1 - \bigg(\frac{14346449}{41472} + \frac{383045\pi^2}{3456} + \frac{19057\zeta_3}{72} + \frac{437\pi^4}{96} - \frac{2495\pi^2\zeta_3}{54} - \frac{2035\zeta_5}{6} \\
&+ \frac{2027\pi^6}{11340} - \frac{2285\zeta_3^2}{27}\bigg)\epsilon^2 - \bigg(-\frac{391938053}{124416} + \frac{3517963\pi^2}{10368} + \frac{1751323\zeta_3}{864} + \frac{93347\pi^4}{1440} \\
&- \frac{443\pi^2\zeta_3}{216} + \frac{21905\zeta_5}{24} + \frac{2027\pi^6}{2268} - \frac{11425\zeta_3^2}{27} - \frac{2477\pi^4\zeta_3}{90} - \frac{9223\pi^2\zeta_5}{90} + \frac{11681\zeta_7}{42}\bigg)\epsilon^3 \\
&+ O\left(\epsilon^4\right) , \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
e^{4\epsilon\gamma_E} M_2 = &-\frac{5}{12\epsilon^4} - \frac{23}{8\epsilon^3} - \left(\frac{433}{48} + \frac{29\pi^2}{36}\right)\epsilon^{-2} - \left(-\frac{297}{32} + \frac{191\pi^2}{24} + \frac{275\zeta_3}{18}\right)\epsilon^{-1} - \bigg(-\frac{22765}{64} \\
&+ \frac{7177\pi^2}{144} - 24\pi^2 a_1 + \frac{2273\zeta_3}{12} + \frac{125\pi^4}{72}\bigg)\epsilon^0 - \bigg(-\frac{411105}{128} + \frac{8085\pi^2}{32} - 324\pi^2 a_1 \\
&+ \frac{105463\zeta_3}{72} + \frac{2747\pi^4}{240} + 80\pi^2 a_1^2 + 40 a_1^4 + \frac{1595\pi^2\zeta_3}{54} + \frac{3223\zeta_5}{6} + 960 a_4\bigg)\epsilon^1 \\
&- \bigg(-\frac{16944559}{768} + \frac{216731\pi^2}{192} - 2706\pi^2 a_1 + \frac{146091\zeta_3}{16} + \frac{43757\pi^4}{1440} + 1080\pi^2 a_1^2 \\
&+ 540 a_1^4 + \frac{16\pi^4 a_1}{3} - \frac{800}{3}\pi^2 a_1^3 - 80 a_1^5 + \frac{9785\pi^2\zeta_3}{36} - \frac{52351\zeta_5}{20} + \frac{112339\pi^6}{22680} + \frac{15125\zeta_3^2}{54} \\
&+ 12960 a_4 + 9600 a_5\bigg)\epsilon^2 - \bigg(-\frac{68697721}{512} + \frac{589805\pi^2}{128} - 18165\pi^2 a_1 + \frac{4851365\zeta_3}{96} \\
&- \frac{94853\pi^4}{960} + 9020\pi^2 a_1^2 + 4510 a_1^4 + 72\pi^4 a_1 - 3600\pi^2 a_1^3 - 1080 a_1^5 + \frac{336415\pi^2\zeta_3}{216} \\
&- \frac{8849321\zeta_5}{120} + 32640 s_6 - \frac{351599\pi^6}{15120} - 104\pi^4 a_1^2 + 744\pi^2 a_1^4 + \frac{400 a_1^6}{3} + 944\pi^2 a_1 \zeta_3 \\
&- \frac{375097\zeta_3^2}{36} + \frac{6875\pi^4\zeta_3}{108} + \frac{93467\pi^2\zeta_5}{90} + \frac{652775\zeta_7}{42} + 108240 a_4 + 129600 a_5 + 96000 a_6 \\
&+ 1856\pi^2 a_4\bigg)\epsilon^3 + O\left(\epsilon^4\right) , \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
e^{4\epsilon\gamma_E} M_3 = &-\frac{1}{6\epsilon^4} - \frac{7}{6\epsilon^3} - \left(\frac{10}{3} + \frac{13\pi^2}{18}\right)\epsilon^{-2} - \left(-\frac{61}{6} + \frac{73\pi^2}{18} + \frac{118\zeta_3}{9}\right)\epsilon^{-1} - \bigg(-\frac{851}{4} + \frac{83\pi^2}{18} \\
&+ \frac{637\zeta_3}{9} + \frac{37\pi^4}{10}\bigg)\epsilon^0 - \bigg(-\frac{14861}{8} - \frac{3467\pi^2}{36} + \frac{1003\zeta_3}{18} + \frac{1121\pi^4}{60} + \frac{1894\pi^2\zeta_3}{27} \\
&+ \frac{16018\zeta_5}{15}\bigg)\epsilon^1 - \bigg(-\frac{613975}{48} - \frac{25981\pi^2}{24} - \frac{68293\zeta_3}{36} + \frac{83\pi^4}{24} + \frac{9559\pi^2\zeta_3}{27} + \frac{79891\zeta_5}{15} \\
&+ \frac{59501\pi^6}{2835} + \frac{17704\zeta_3^2}{27}\bigg)\epsilon^2 - \bigg(-\frac{7539347}{96} - \frac{382349\pi^2}{48} - \frac{482627\zeta_3}{24} - \frac{426659\pi^4}{720} \\
&+ \frac{3757\pi^2\zeta_3}{54} + \frac{2525\zeta_5}{6} + \frac{43201\pi^6}{420} + \frac{88585\zeta_3^2}{27} + \frac{17204\pi^4\zeta_3}{45} + \frac{206434\pi^2\zeta_5}{45}
\end{aligned}
$$

$$
+\frac{1267243\zeta_7}{21}\Big)\epsilon^3+O\left(\epsilon^4\right)\,, \tag{35}
$$

$$
\begin{aligned}
e^{4\epsilon\gamma_E}M_4 = &-\frac{1}{3\epsilon^4}-\frac{5}{2\epsilon^3}-\left(\frac{55}{6}+\frac{4\pi^2}{9}\right)\epsilon^{-2}-\left(3+\frac{19\pi^2}{3}+\frac{56\zeta_3}{9}\right)\epsilon^{-1}-\Big(-250+\frac{1925\pi^2}{36}\\
&-32\pi^2a_1+\frac{464\zeta_3}{3}+\frac{19\pi^4}{45}\Big)\epsilon^0-\Big(-\frac{5091}{2}+\frac{2811\pi^2}{8}-432\pi^2a_1+\frac{14797\zeta_3}{9}+\frac{17\pi^4}{90}\\
&+\frac{320}{3}\pi^2a_1^2+\frac{160a_1^4}{3}+\frac{332\pi^2\zeta_3}{27}+\frac{1556\zeta_5}{15}+1280a_4\Big)\epsilon^1-\Big(-\frac{55049}{3}+\frac{95693\pi^2}{48}\\
&-3608\pi^2a_1+\frac{24831\zeta_3}{2}-\frac{1084\pi^4}{45}+1440\pi^2a_1^2+720a_1^4+\frac{160\pi^4a_1}{9}-\frac{3200}{9}\pi^2a_1^3\\
&-\frac{320a_1^5}{3}+\frac{1046\pi^2\zeta_3}{9}-8246\zeta_5+\frac{772\pi^6}{2835}+\frac{2648\zeta_3^2}{27}+17280a_4+12800a_5\Big)\epsilon^2\\
&-\Big(-\frac{458141}{4}+\frac{329467\pi^2}{32}-24220\pi^2a_1+\frac{938425\zeta_3}{12}-\frac{9979\pi^4}{36}+\frac{36080}{3}\pi^2a_1^2\\
&+\frac{18040a_1^4}{3}+240\pi^4a_1-4800\pi^2a_1^3-1440a_1^5+\frac{19930\pi^2\zeta_3}{27}-\frac{353044\zeta_5}{3}+44800s_6\\
&-\frac{76904\pi^6}{945}-176\pi^4a_1^2+976\pi^2a_1^4+\frac{1600a_1^6}{9}+\frac{4160}{3}\pi^2a_1\zeta_3-\frac{155668\zeta_3^2}{9}+\frac{4304\pi^4\zeta_3}{135}\\
&+\frac{7304\pi^2\zeta_5}{45}+\frac{4616\zeta_7}{21}+144320a_4+172800a_5+128000a_6+\frac{6272\pi^2a_4}{3}\Big)\epsilon^3\\
&+O\left(\epsilon^4\right)\,,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
e^{4\epsilon\gamma_E}M_5 = &-\frac{1}{12\epsilon^4}-\frac{13}{24\epsilon^3}-\left(\frac{15}{16}+\frac{13\pi^2}{36}\right)\epsilon^{-2}-\left(-\frac{1135}{96}+\frac{169\pi^2}{72}+\frac{86\zeta_3}{9}\right)\epsilon^{-1}-\Big(-\frac{28699}{192}\\
&+\frac{65\pi^2}{16}+\frac{559\zeta_3}{9}+\frac{149\pi^4}{90}\Big)\epsilon^0-\Big(-\frac{144429}{128}-\frac{14755\pi^2}{288}+\frac{227\zeta_3}{2}+\frac{1937\pi^4}{180}\\
&+\frac{1118\pi^2\zeta_3}{27}+\frac{7604\zeta_5}{15}\Big)\epsilon^1-\Big(-\frac{5327075}{768}-\frac{373087\pi^2}{576}-\frac{45889\zeta_3}{36}+\frac{749\pi^4}{40}\\
&+\frac{7267\pi^2\zeta_3}{27}+\frac{49426\zeta_5}{15}+\frac{14053\pi^6}{1620}+\frac{14063\zeta_3^2}{27}\Big)\epsilon^2-\Big(-\frac{58275695}{1536}-\frac{625859\pi^2}{128}\\
&-\frac{1192693\zeta_3}{72}-\frac{168143\pi^4}{720}+\frac{2951\pi^2\zeta_3}{6}+5805\zeta_5+\frac{182689\pi^6}{3240}+\frac{182819\zeta_3^2}{54}\\
&+\frac{51013\pi^4\zeta_3}{270}+\frac{98852\pi^2\zeta_5}{45}+\frac{1021711\zeta_7}{42}\Big)\epsilon^3+O\left(\epsilon^4\right)\,,
\end{aligned} \tag{37}
$$

$$
e^{4\epsilon\gamma_E}M_6 = -\frac{1}{12\epsilon^4}-\frac{17}{24\epsilon^3}-\left(\frac{149}{48}+\frac{13\pi^2}{36}\right)\epsilon^{-2}-\left(\frac{433}{96}+\frac{149\pi^2}{72}+\frac{23\zeta_3}{9}\right)\epsilon^{-1}-\Big(-\frac{3817}{64}
$$

$$+\frac{521\pi^2}{144}+\frac{173\zeta_3}{9}+\frac{341\pi^4}{180}\Big)\epsilon^0-\Big(-\frac{97165}{128}-\frac{9419\pi^2}{288}+\frac{1367\zeta_3}{18}+\frac{1667\pi^4}{180}$$
$$+\frac{659\pi^2\zeta_3}{27}+\frac{5939\zeta_5}{15}\Big)\epsilon^1-\Big(-\frac{4640963}{768}-\frac{80461\pi^2}{192}+\frac{1717\zeta_3}{36}+\frac{833\pi^4}{360}+\frac{3527\pi^2\zeta_3}{27}$$
$$+\frac{30599\zeta_5}{15}+\frac{57791\pi^6}{5670}+\frac{734\zeta_3^2}{27}\Big)\epsilon^2-\Big(-\frac{61900879}{1536}-\frac{410283\pi^2}{128}-\frac{55357\zeta_3}{24}$$
$$-\frac{193957\pi^4}{720}+\frac{8489\pi^2\zeta_3}{54}+\frac{45833\zeta_5}{30}+\frac{1106911\pi^6}{22680}+\frac{14197\zeta_3^2}{54}+\frac{21637\pi^4\zeta_3}{135}$$
$$+\frac{75407\pi^2\zeta_5}{45}+\frac{522017\zeta_7}{21}\Big)\epsilon^3+O\left(\epsilon^4\right)\,, \tag{38}$$

with $s_6=\sum_{m=1}^{\infty}\sum_{k=1}^{m}(-1)^{m+k}/(m^5k)=0.98744\ldots$..

Appendix B: Relation between $\bar{\alpha}(\mu)$ and α

In this Appendix we present the result for the relation between the fine structure constant defined in the $\overline{\rm MS}$ and on-shell renormalization scheme involving two massive leptons with masses m_1 and m_2. We label contributions from leptons with mass m_1 and m_2 by n_h and n_l, respectively. Our result reads

$$\begin{aligned}
\frac{\bar{\alpha}(\mu)}{\alpha}-1 &= \frac{1}{3}l_2n_l\frac{\alpha}{\pi}+\left\{\frac{1}{9}l_1l_2n_hn_l+\frac{1}{9}{l_2}^2{n_l}^2+\left(\frac{l_2}{4}+\frac{15}{16}\right)n_l\right\}\left(\frac{\alpha}{\pi}\right)^2\\
&+\left\{\frac{1}{27}{l_2}^3{n_l}^3+\frac{1}{9}l_1{l_2}^2n_h{n_l}^2+{n_l}^2\left(\frac{5{l_2}^2}{24}+\frac{79l_2}{144}+\frac{7\zeta_3}{64}+\frac{\pi^2}{9}-\frac{695}{648}\right)\right.\\
&+n_l\left(-\frac{1}{3}\pi^2a_1-\frac{l_2}{32}+\frac{\zeta_3}{192}+\frac{5\pi^2}{24}+\frac{77}{576}\right)\\
&+n_hn_l\left[\frac{79l_1x^2}{384}-\frac{79l_2\left(3x^2-8\right)}{1152}+\frac{5l_1l_2}{24}+\frac{1}{384}\left(-128x^4-15x^2-71\right)\ln^2(x)\right.\\
&+\frac{1}{3}\left(-x^4+x^3+x-1\right)\mathrm{Li}_2(1-x)+\frac{1}{3}\left(x^4+x^3+x+1\right)\mathrm{Li}_2(-x)\\
&+\frac{(5x^6+3x^4+3x^2+5)}{256x^3}\Big((\mathrm{Li}_2(1-x)+\mathrm{Li}_2(-x))\ln(x)-2\mathrm{Li}_3(1-x)-\mathrm{Li}_3(-x)\Big)\\
&+\frac{1}{3}\left(x^4+x^3+x+1\right)\ln(x)\ln(x+1)+\frac{(5x^6+3x^4+3x^2+5)\ln^2(x)\ln(x+1)}{512x^3}\\
&\left.\left.+\frac{405x^3\zeta_3+1152\pi^2\left(x^3+x\right)-5994x^2+243x\zeta_3-5126}{10368}\right]\right\}\left(\frac{\alpha}{\pi}\right)^3\\
&+\left\{n_h\leftrightarrow n_l,m_1\leftrightarrow m_2,x\leftrightarrow\frac{1}{x}\right\},
\end{aligned}\tag{39}$$

with $x=m_1/m_2$, $l_k=\ln(\mu^2/m_k^2)$ and $a_1=\ln 2$.

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