Numerical Analysis of Nonlinear PDAEs: A Coupled Systems Approach and its Application to Circuit Simulation

Lennart Jansen and Michael Matthes

Abstract Applications like electrical circuits including electromagnetic devices, semiconductor devices or thermal elements, and water or gas transportation networks give rise to a mix of partial differential equations and differential-algebraic equations. Such a mix is called a partial differential-algebraic equation (PDAE). In this paper we investigate a prototype for nonlinear coupled PDAE systems. The objectives are to prove the global existence and uniqueness of a solution, the convergence of Galerkin equations and a perturbation result for this prototype class. Regarding the applications we consider the simulation of electric circuits including thermal resistors. With a new decoupling technique we are able to reformulate the MNA equations up to index 2 such that we can apply the results of the prototype.

1 Introduction

Numerous mathematical models in science and engineering give rise to systems comprising partial differential equations (PDEs) and differential-algebraic equations (DAEs). These systems are called partial differential-algebraic equations (PDAEs) and occur frequently in application areas such as electric circuit simulation, flexible multibody systems, gas or water distribution network simulation or chemical engineering, see [8, 29, 32]. Research is mainly focused on the space-discretized system in literature and as in this paper the non-discretized system is investigated we will use the term abstract differential-algebraic equation (ADAE). As the index for DAEs the definition and determination of indexes for linear ADAEs has received attention

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in recent literature, see [8, 21, 20, 32, 23, 24]. ADAEs are also treated w.r.t. existence under the term degenerate differential equations mainly for the linear case, see [15], but also for certain classes of nonlinear ADAEs, see [14, 27]. However, the theoretical treatment of especially nonlinear ADAEs and their numerical treatment is still at an initial stage. In this paper we are interested in a systematic treatment of nonlinear ADAEs. We formulate a parabolic prototype of an ADAE which we are going to investigate with regard to the following guiding questions.

1. What conditions must be met for an ADAE to be solvable? When is the solution unique?
2. How does the solution change if the system is perturbed? Are there index criteria – similar to DAEs – which describe the perturbation behavior?
3. How should an ADAE be discretized? Is the solution of the semi-discretized system unique and does it converge to the exact solution?

While the first question is a core question in mathematics, the second one aims at suitable perturbation estimates. The second and the last question are closely linked. It is known that using different discretization schemes the semi-discretization of the ADAE may act like a deregularization (increasing the Perturbation Index) or regularization (decreasing the Perturbation Index), see [16]. So the determination of a perturbation estimate for the ADAE is important to predict the perturbation behavior to be expected from a corresponding discretized system. Furthermore it should be stressed that the convergence of solutions of the semi-discretized ADAE to a solution of the original ADAE is merely investigated in literature where numerical analysis is mainly concerned with the semi-discretized ADAE itself and its discretization in time.

The prototype to be presented is tailored for the treatment of so-called coupled systems. They arise especially in circuit simulation and have gained increasing significance in the last ten to fifteen years. Ever more increasing demands on high performance chips result in higher complexity, package densities and operating frequencies of integrated circuits. The well-established standard approach of describing the behavior of the circuit by the equations of the Modified Nodal Analysis (MNA), see [10, 12, 9], is not sufficient anymore for capturing all physical effects. Electromagnetic effects or heating effects, stemming from the surrounding circuitry, for example, or more accurate switching behavior of semiconductors cannot be neglected anymore when simulating correct physical behavior. So the MNA equations, yielding a DAE, are usually complemented by a suitable system of PDEs describing these additional effects or elements. The MNA equations and the PDE system interchange information via certain coupling terms, additional coupling equations, certain variables serving as input for boundary conditions of the PDE system or even more general parametric coupling. The resulting system is in general very complex and nonlinear, see e.g. [17, 3, 32, 4, 1, 11, 2, 28, 5, 22].

The paper is organized as follows. After presenting some preliminaries we present the prototype system and develop an existence and uniqueness result via the Galerkin approach. A perturbation result can also be easily derived. Then, in section 4 we apply the results for the prototype to the equations of the MNA coupled to the heat equation. To realize the application of the results we use a new decoupling approach for electric circuits.

2 Preliminaries

In this paper we consider real Banach spaces V with norm $\|\cdot\|_V$. Its dual space is denoted by V^* and it is a Banach space with its induced norm $\|\cdot\|_{V^*}$. Furthermore we write

$$\langle v^*, v\rangle_V := v^*(v) \quad \forall v \in V$$

for $v^* \in V^*$. Let $(v_n) \subseteq V$ be a sequence, then we say that v_n converges to $v \in V$ (short: $v_n \to v$) as $n \to \infty$, if it converges in the norm, i.e. if $\|v_n - v\|_V \to 0$ as $n \to \infty$. The sequence (v_n) converges weakly to v (short: $v_n \rightharpoonup v$) as $n \to \infty$ if

$$\langle v^*, v_n\rangle_V \to \langle v^*, v\rangle_V \quad \forall v^* \in V^* \quad \text{as } n \to \infty.$$

The real Banach space V is a Hilbert space if there is a scalar product on V which we denote by $(\cdot\,|\,\cdot)_V$. If $V = \mathbb{R}^n$, $n \in \mathbb{N}$, the Euclidean scalar product and its induced norm are denoted by

$$(x\,|\,y) := x^\top y, \quad \|x\| := \sqrt{x^\top x} \qquad \forall x, y \in \mathbb{R}^n.$$

A matrix $A \in \mathbb{R}^{m \times n}$ can be measured by the induced operator norm

$$\|A\|_* := \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Another important notion is Lipschitz continuity and strong monotonicity. Let $f : V \times M \to W$ be a map and $M \subseteq X$ be a subset of the Banach space X. Then f is Lipschitz continuous on V if there is $L > 0$ such that

$$\|f(v,z) - f(\bar{v},z)\|_W \leq L\,\|v - \bar{v}\|_V \quad \forall v, \bar{v} \in V, z \in M.$$

If $W = V^*$ we say that f is strongly monotone on V if there is $\mu > 0$ such that

$$\langle f(v,z) - f(\bar{v},z), v - \bar{v}\rangle_V \geq \mu\,\|v - \bar{v}\|_V^2 \quad \forall v, \bar{v} \in V, z \in M.$$

If $\mu = 0$ we call f monotone on V. Note here that L and μ do not depend on $z \in M$.

3 A parabolic prototype

In this section we discuss a parabolic prototype of a coupled system. Let $\mathcal{I} := [t_0, T]$ be an interval and $V \subseteq H \subseteq V^*$ be an evolution triple, i.e.

(i) V is a real, separable and reflexive Banach space with dual space V^*,
(ii) H is a real, separable Hilbert space,
(iii) V is dense in H and the embedding $V \subseteq H$ is continuous, i.e. there is a constant $c > 0$ such that

$$\|v\|_H \leq c\,\|v\|_V \quad \forall v \in V.$$

Consider the system

$$x'(t) + f(x(t), y(t), u(t), t) = 0, \quad t \in \mathcal{I}, \tag{1a}$$

$$g(x(t), y(t), t) = 0, \tag{1b}$$

$$u'(t) + \mathcal{B}u(t) + \mathcal{R}(u(t), x(t), y(t), t) = 0, \quad \text{in } V^*, \tag{1c}$$

$$x(t_0) = x_0, \quad u(t_0) = u_0 \tag{1d}$$

with functions $f : \mathbb{R}^{n_x+n_y} \times H \times \mathcal{I} \to \mathbb{R}^{n_x}$, $g : \mathbb{R}^{n_x+n_y} \times \mathcal{I} \to \mathbb{R}^{n_y}$ and operators $\mathcal{B} : V \to V^*$ and $\mathcal{R} : V \times \mathbb{R}^{n_x+n_y} \times \mathcal{I} \to V^*$. The unknowns are $x(t) \in \mathbb{R}^{n_x}$, $y(t) \in \mathbb{R}^{n_y}$ and $u(t) \in V$ for $t \in \mathcal{I}$. We also use the convention to write $z(t) = (x(t), y(t))$, $n_z = n_x + n_y$. We will also often omit the explicit time dependency of the variables. The initial values $x_0 \in \mathbb{R}^{n_x}$ and $u_0 \in H$ are given. Note that equations (1a), (1b) represent a semi-linear (finite dimensional) DAE whereas (1c) is an (infinite dimensional) evolution equation involving a generalized derivative where a solution u will be in the space

$$W_2^1 = W_2^1(\mathcal{I}; V, H) = \left\{ u \in L_2(\mathcal{I}, V) \,|\, u' \in L_2(\mathcal{I}, V^*) \right\}.$$

$L_2(\mathcal{I}, V)$ is the space of square integrable funtions on $\mathcal{I}$ with values in the Banach space V. The embedding $W_2^1 \subseteq C(\mathcal{I}, H)$ is continuous and all $u, v \in W_2^1$ and arbitrary s, t with $t_0 \leq s \leq t \leq T$ the following integration by parts formula holds:

$$(u(t)\,|\,v(t))_H - (u(s)\,|\,v(s))_H = \int_s^t \langle u'(\tau), v(\tau)\rangle_V + \langle v'(\tau), u(\tau)\rangle_V \,\mathrm{d}\tau$$

Here the values $u(t), v(t), u(s), v(s)$ are the values of the corresponding continuous functions $u, v : \mathcal{I} \to H$ from the embedding. For further background material on generalized derivatives on Banach spaces in the setting of evolution triples we refer to [34, 26].

The coupling of these two systems is realized by letting f depend on u and $\mathcal{R}$ depend on z. We will investigate system (1) regarding solvability, perturbation estimates and Galerkin convergence under appropriate assumptions. The unique solvability is obtained via a Galerkin approach. First we assemble the following assumptions.

Assumption 3.1
The following assumptions hold for system (1)*:*

(i) *Let* $\mathcal{I} := [t_0, T]$ *be an interval and* $V \subseteq H \subseteq V^*$ *be an evolution triple.*
(ii) *The initial values* $x_0 \in \mathbb{R}^{n_x}$, $u_0 \in H$ *are given.*
(iii) $f \in C(\mathbb{R}^{n_z} \times H \times \mathcal{I}, \mathbb{R}^{n_x})$ *is Lipschitz continuous w.r.t.* z *and* u.
(iv) $g \in C(\mathbb{R}^{n_z} \times \mathcal{I}, \mathbb{R}^{n_y})$ *is strongly monotone w.r.t.* $y \in \mathbb{R}^{n_y}$ *and Lipschitz continuous w.r.t.* x.
(v) $\mathcal{B} : V \to V^*$ *is linear, strongly monotone and bounded.*
(vi) $\mathcal{R} \in C(V \times \mathbb{R}^{n_z} \times \mathcal{I}, V^*)$ *is monotone w.r.t.* u*, i.e.*

$$\langle \mathcal{R}(u,z,t) - \mathcal{R}(\bar{u},z,t), u - \bar{u} \rangle_V \geq 0 \quad \forall u, \bar{u} \in V, z \in \mathbb{R}^{n_z}, t \in \mathcal{I}$$

and Lipschitz continuous w.r.t. z*. Furthermore there are* $c_{\mathcal{R},1}, c_{\mathcal{R},2} > 0$ *such that*

$$\|\mathcal{R}(u,0,t)\|_{V^*} \leq c_{\mathcal{R},1} \|u\|_V + c_{\mathcal{R},2} \quad \forall u \in V.$$

(vii) *Let* $\dim V = \infty$ *and* $\{v_1, v_2, \dots\}$ *be a basis of* V*. Set* $V_n := \{v_1, \dots, v_n\}$ *and let there be a sequence* $(u_{n0}) \subseteq V$ *with* $u_{n0} \in V_n$ *and* $u_{n0} \to u_0$ *in* H *as* $n \to \infty$.

Remark 3.2
We remark here that condition (iv) takes care of the global solvability of g*. Since* g *is continuous and strongly monotone equation* (1b) *is uniquely solvable, i.e. there is a solution function* $\psi_g \in C(\mathbb{R}^{n_x} \times \mathcal{I}, \mathbb{R}^{n_y})$ *such that* $y = \psi_g(x,t)$ *whenever* $g(x,y,t) = 0$ *for all* x, y, t*. Furthermore* ψ_g *is Lipschitz continuous w.r.t.* x *because* g *is.*

For proving unique solvability and convergence of the Galerkin solutions we proceed as follows. First, we show the uniqueness (Lemma 3.3) of a possible solution to (1). Then we prove a priori estimates for the Galerkin solutions (Lemma 3.4) and prove the unique solvability of the Galerkin equations (Lemma 3.5) which are given as follows:

$$x_n'(t) + f(x_n(t), y_n(t), u_n(t), t) = 0, \quad t \in \mathcal{I} \tag{2a}$$
$$g(x_n(t), y_n(t), t) = 0, \tag{2b}$$
$$\langle u_n'(t), v_i \rangle_V + \langle \mathcal{B} u_n(t), v_i \rangle_V + \langle \mathcal{R}(u_n(t), x_n(t), y_n(t), t), v_i \rangle_V = 0, \tag{2c}$$
$$x_n(t_0) = x_0, \quad u_n(t_0) = u_{n0}, \tag{2d}$$

for $i = 1, \dots, n$. The operator equation (2c) is formulated on the finite dimensional subspace $V_n \subseteq V$. So $u_n(t)$ is in V_n which also influences the finite dimensional variable z through the coupling. Hence $z_n = (x_n, y_n)$, too, depends on the Galerkin step n. Finally, we will be able to prove solvability and convergence of the Galerkin solutions (Theorem 3.6).

Lemma 3.3 (Uniqueness)
Let Assumption 3.1 be fulfilled. If $(z,u) \in C(\mathcal{I}, \mathbb{R}^{n_z} \times H)$ *with* $x \in C^1(\mathcal{I}, \mathbb{R}^{n_x})$ *and* $u \in W_2^1(\mathcal{I}; V, H)$ *is a solution to* (1) *then* (z,u) *is unique.*

PROOF:
Let (z,u), $(\bar{z},\bar{u})$ be two solutions to (1). We define $\Delta u := u-\bar{u}$, $\Delta z := z-\bar{z}$, $\Delta x := x-\bar{x}$ and $\Delta y := y-\bar{y}$. It is $\Delta u(t_0) = 0$ and since $u,\bar{u} \in W_2^1(\mathcal{I};V,H)$ we can apply the integration by parts formula for generalized derivatives, cf. e.g. [34, chapter 23.5], and obtain

$$\begin{aligned}
\frac{1}{2}\|\Delta u(t)\|_H^2 &= \frac{1}{2}\|\Delta u(t)\|_H^2 - \frac{1}{2}\|\Delta u(t_0)\|_H^2 \\
&= \int_{t_0}^t \langle \Delta u'(s), \Delta u(s)\rangle \mathrm{d}s \\
&= -\int_{t_0}^t \langle \mathcal{B}\Delta u(s), \Delta u(s)\rangle_V + \langle \mathcal{R}(u(s),z(s),s) - \mathcal{R}(\bar{u}(s),\bar{z}(s),s), \Delta u(s)\rangle_V \mathrm{d}s \\
&\leq -\mu \int_{t_0}^t \|\Delta u(s)\|_V^2 \,\mathrm{d}s - \int_{t_0}^t \langle \mathcal{R}(u(s),z(s),s) - \mathcal{R}(\bar{u}(s),z(s),s), \Delta u(s)\rangle_V \mathrm{d}s \\
&\quad - \int_{t_0}^t \langle \mathcal{R}(\bar{u}(s),z(s),s) - \mathcal{R}(\bar{u}(s),\bar{z}(s),s), \Delta u(s)\rangle_V \mathrm{d}s
\end{aligned}$$

In the last line we used the strong monotonicity of $\mathcal{B}$ with $\mu > 0$. Since $\mathcal{R}$ is monotone in u and Lipschitz continuous in z we have that

$$-\langle \mathcal{R}(u(s),z(s),s) - \mathcal{R}(\bar{u}(s),z(s),s), \Delta u(s)\rangle_V \leq 0$$

and

$$\begin{aligned}
\langle \mathcal{R}(\bar{u}(s),z(s),s) - \mathcal{R}(\bar{u}(s),\bar{z}(s),s), \Delta u(s)\rangle_V &\leq L_{\mathcal{R}} \|\Delta z(s)\| \, \|\Delta u(s)\|_V \\
&\leq \frac{\mu}{2}\|\Delta u(s)\|_V^2 + \frac{2L_{\mathcal{R}}^2}{\mu}\|\Delta z(s)\|^2
\end{aligned}$$

using the classical inequality that for all $\alpha > 0$ it holds that

$$2|xy| \leq \alpha^{-1}x^2 + \alpha y^2 \quad \forall x,y \in \mathbb{R}. \tag{3}$$

Hence

$$\|\Delta u(t)\|_H^2 + \mu \int_{t_0}^t \|\Delta u(s)\|_V^2 \,\mathrm{d}s \leq \frac{4L_{\mathcal{R}}^2}{\mu} \int_{t_0}^t \|\Delta z(s)\|^2 \,\mathrm{d}s$$

From this it can be concluded that

$$\|\Delta u(t)\|_H^2 \leq c_u \int_{t_0}^t \|\Delta x(s)\|^2 + \|\Delta y(s)\|^2 \,\mathrm{d}s \tag{4}$$

holds for a constant $c_u > 0$. For the algebraic part (1b) we have

$$y(t) = \psi_g(x(t),t), \qquad \bar{y}(t) = \psi_g(\bar{x}(t),t)$$

and using the Lipschitz continuity of ψ_g we observe that

$$\|\Delta y(t)\|^2 \le c_y \|\Delta x(t)\|^2 \tag{5}$$

for a constant $c_y > 0$. Using that

$$2\Delta x(t)^\top \Delta x'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \|\Delta x(t)\|^2$$

we see multiplying (1a) from the left by $\Delta x(t)^\top$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta x(t)\|^2 = \Delta x(t)^\top \left(f(\bar{z}(t),\bar{u}(t),t) - f(z(t),u(t),t)\right).$$

Integration over $[t_0,t]$ gives

$$\begin{aligned}\frac{1}{2}\|\Delta x(t)\|^2 = \frac{1}{2}\|\Delta x(t)\|^2 - \frac{1}{2}\|\Delta x(t_0)\|^2 \\ = \int_{t_0}^{t} \Delta x(s)^\top \left(f(\bar{z}(s),\bar{u}(s),s) - f(z(s),u(s),s)\right) \mathrm{d}s\end{aligned}$$

because $\Delta x(t_0) = 0$. With the Lipschitz continuity of f and the classical inequality (3) we estimate

$$\begin{aligned}&2\Delta x(s)^\top \left(f(\bar{z}(s),\bar{u}(s),s) - f(z(s),u(s),s)\right) \\ &\le 2L_f \|\Delta x(s)\| \left(\|\Delta z(s)\| + \|\Delta u(s)\|_H\right) \\ &\le c_x \left(\|\Delta x(s)\|^2 + \|\Delta y(s)\|^2 + \|\Delta u(s)\|_H^2\right)\end{aligned}$$

for scalars $c_x,\ L_f > 0$. Hence we get

$$\|\Delta x(t)\|^2 \le c_x \int_{t_0}^{t} \|\Delta x(s)\|^2 + \|\Delta y(s)\|^2 + \|\Delta u(s)\|_H^2 \,\mathrm{d}s \tag{6}$$

Now inserting (5) into (4), (6) and adding them gives

$$\|\Delta x(t)\|^2 + \|\Delta u(t)\|_H^2 \le c_{xu} \int_{t_0}^{t} \|\Delta x(s)\|^2 + \|\Delta u(s)\|_H^2 \,\mathrm{d}s$$

for a constant $c_{xu} > 0$. An application of the Gronwall Lemma reveals that

$$\|\Delta x(t)\|^2 + \|\Delta u(t)\|_H^2 = 0$$

and we deduce that $\Delta x = 0$, $\Delta u = 0$ and with (5) also $\Delta y = 0$. □

Lemma 3.4 (A priori estimates)
Let Assumption 3.1 be fulfilled. If $(z_n,u_n) \in C(\mathcal{I},\mathbb{R}^{n_z} \times V_n)$ *with* $x_n \in C^1(\mathcal{I},\mathbb{R}^{n_x})$ *and* $u_n \in W_2^1(\mathcal{I};V,H)$ *is a solution to the Galerkin equations* (2)*, then there is a* $C > 0$ *such that*

$$\max_{t\in\mathcal{I}} \|z_n(t)\| \leq C,$$
$$\max_{t\in\mathcal{I}} \|u_n(t)\|_H \leq C,$$
$$\|u_n\|_{L_2(\mathcal{I},V)} \leq C,$$
$$\|w_n\|_{L_2(\mathcal{I},V^*)} \leq C$$

where $w_n \in L_2(\mathcal{I},V^*)$ is defined by

$$\langle w_n(t),v\rangle_V := \langle \mathcal{B}u_n(t),v\rangle_V + \langle \mathcal{R}(u_n(t),z_n(t),t),v\rangle_V, \quad \forall v\in V,\ n\in\mathbb{N},\ t\in\mathcal{I}.$$

PROOF:
Let (z_n,u_n) be a solution of the Galerkin equations (2). Since $u_n \in W_2^1$ we can apply the integration by parts formula and get

$$\begin{aligned}\frac{1}{2}\|u_n(t)\|_H^2 - \frac{1}{2}\|u_{n0}\|_H^2 &= \int_{t_0}^t \langle u_n'(s),u_n(s)\rangle_V \,\mathrm{d}s\\ &\overset{(2c)}{=} -\int_{t_0}^t \langle \mathcal{B}u_n(s),u_n(s)\rangle_V + \langle \mathcal{R}(u_n(s),z_n(s),s),u_n(s)\rangle_V \,\mathrm{d}s\\ &\leq -\mu\int_{t_0}^t \|u_n(s)\|_V^2\,\mathrm{d}s - \int_{t_0}^t \langle \mathcal{R}(0,z_n(s),s),u_n(s)\rangle_V \,\mathrm{d}s\end{aligned}$$

using the strong monotonicity of $\mathcal{B}$ and the monotonicity of $\mathcal{R}$, i.e.

$$\begin{aligned}&-\langle \mathcal{R}(u_n(s),z_n(s),s),u_n(s)\rangle_V\\ &= -\langle \mathcal{R}(u_n(s),z_n(s),s)-\mathcal{R}(0,z_n(s),s),u_n(s)\rangle_V - \langle \mathcal{R}(0,z_n(s),s),u_n(s)\rangle_V\\ &\leq -\langle \mathcal{R}(0,z_n(s),s),u_n(s)\rangle_V.\end{aligned}$$

Furthermore we estimate

$$\begin{aligned}\langle \mathcal{R}(0,z_n(s),s),u_n(s)\rangle_V &= \langle \mathcal{R}(0,z_n(s),s)-\mathcal{R}(0,0,s),u_n(s)\rangle_V + \langle \mathcal{R}(0,0,s),u_n(s)\rangle_V\\ &\leq (L_{\mathcal{R}}\|z_n(s)\| + \|\mathcal{R}(0,0,s)\|_{V^*})\,\|u_n(s)\|_V\\ &\leq \frac{\mu}{2}\|u_n(s)\|_V^2 + \frac{2}{\mu}(L_{\mathcal{R}}\|z_n(s)\| + \|\mathcal{R}(0,0,s)\|_{V^*})^2\\ &\leq \frac{\mu}{2}\|u_n(s)\|_V^2 + \frac{4}{\mu}\left(L_{\mathcal{R}}^2\|z_n(s)\|^2 + \|\mathcal{R}(0,0,s)\|_{V^*}^2\right)\end{aligned}$$

for a constant $L_{\mathcal{R}} > 0$. Here we used the Lipschitz continuity of $\mathcal{R}$ w.r.t. z and the classical inequality (3). Hence we derive

$$\frac{1}{2}\|u_n(t)\|_H^2 + \frac{\mu}{2}\int_{t_0}^t \|u_n(s)\|_V^2\,\mathrm{d}s \leq \frac{1}{2}\|u_{n0}\|_H^2 + \frac{4}{\mu}\int_{t_0}^t L_{\mathcal{R}}^2\|z_n(s)\|^2 + \|\mathcal{R}(0,0,s)\|_{V^*}^2\,\mathrm{d}s.$$

Since $u_{n0} \to u_0$ in H as $n\to\infty$ the term $\|u_{n0}\|_H$ is bounded. The operator $\mathcal{R}$ is continuous and $\mathcal{I}$ compact, so $\|\mathcal{R}(0,0,s)\|_{V^*}$ is bounded as well. We get the estimates

$$\|u_n(t)\|_H^2 \leq c_{1,u,H} + c_{2,u,H} \int_{t_0}^{t} \|x_n(s)\|^2 + \|y_n(s)\|^2 \, \mathrm{d}s \tag{7}$$

and

$$\|u_n\|_{L_2(\mathcal{I},V)}^2 \leq c_{1,u,L_2} + c_{2,u,L_2} \|z_n\|_{L_2(\mathcal{I},\mathbb{R}^{n_z})}^2 \tag{8}$$

with constants $c_{1,u,H}$, $c_{2,u,H}$, c_{1,u,L_2}, $c_{2,u,L_2} > 0$. For the algebraic part (2b) we get

$$\|y_n(t)\| = \left\|\psi_g(x_n(t),t)\right\| \leq L_{\psi_g} \|x_n(t)\| + \left\|\psi_g(0,t)\right\|$$

for a $L_{\psi_g} > 0$ because of the Lipschitz continuity of ψ_g. ψ_g is continuous and $\mathcal{I}$ is compact, so $\left\|\psi_g(0,t)\right\|$ is bounded and we conclude that

$$\|y_n(t)\|^2 \leq c_{1,y} + c_{2,y} \|x_n(t)\|^2 \tag{9}$$

for constants $c_{1,y}$, $c_{2,y} > 0$. Using integration by parts and (2a) we observe

$$\begin{aligned}
\frac{1}{2}\|x_n(t)\|^2 - \frac{1}{2}\|x_0\|^2 &= \int_{t_0}^{t} x_n(s)^\top x_n'(s) \mathrm{d}s \\
&\leq \int_{t_0}^{t} \|x_n(s)\| \, \|f(z_n(s),u_n(s),s)\| \, \mathrm{d}s \\
&\leq \int_{t_0}^{t} \bar{c}_1 \|x_n(s)\| \left(\|z_n(s)\| + \|u_n(s)\|_H + \|f(0,0,s)\|\right) \mathrm{d}s \\
&\leq \bar{c} \int_{t_0}^{t} \|x_n(s)\|^2 + \|z_n(s)\|^2 + \|u_n(s)\|_H^2 + \|f(0,0,s)\|^2 \, \mathrm{d}s
\end{aligned}$$

for $\bar{c}_1, \bar{c} > 0$. Thus

$$\|x_n(t)\|^2 \leq c_{1,x} + c_{2,x} \int_{t_0}^{t} \left(\|x_n(s)\|^2 + \|y_n(s)\|^2 + \left\|u_n(s)^2\right\|_H \right) \mathrm{d}s \tag{10}$$

with constants $c_{1,x}$, $c_{2,x} > 0$ since f is continuous. Inserting (9) in (7), (10) and adding them gives

$$\|x_n(t)\|^2 + \|u_n(t)\|_H^2 \leq c_{1,xu} + c_{2,xu} \int_{t_0}^{t} \|x_n(s)\|^2 + \|u_n(s)\|_H^2 \, \mathrm{d}s$$

with constants $c_{1,xu}$, $c_{2,xu} > 0$. An application of the Gronwall Lemma reveals that there is a $\bar{C} > 0$ (independent of t and n) such that

$$\|x_n(t)\|^2 \leq \bar{C}, \qquad \|u_n(t)\|_H^2 \leq \bar{C}.$$

Applying this to (9) gives the desired bound on $\|y_n(t)\|$ and so $\|z_n(t)\|$ is bounded. Since z_n is uniformly bounded we also get the boundedness of $\|u_n\|_{L_2(\mathcal{I},V)}$ with (8). It still needs to be shown that $w_n \in L_2(\mathcal{I}, V^*)$ is bounded uniformly. We see

$$\begin{aligned}\langle w_n(t),v\rangle_V &= \langle \mathcal{B}u_n(t),v\rangle_V + \langle \mathcal{R}(u_n(t),z_n(t),t),v\rangle_V \\ &\le (c_{\mathcal{R}}\,\|u_n(t)\|_V + L_{\mathcal{R}}\,\|z_n(t)\| + \|\mathcal{R}(u_n(t),0,t)\|)\,\|v\|_V \\ &\le (c_{w,1}\,(\|u_n(t)\|_V + \|z_n(t)\|) + c_{w,2})\,\|v\|_V\end{aligned}$$

with constants $c_{w,1},\ c_{w,2} > 0$ because $\|\mathcal{R}(u_n(t),0,t)\| \le c_{\mathcal{R},1}\,\|u_n(t)\| + c_{\mathcal{R},2}$. Hence

$$\begin{aligned}\|w_n\|_{L_2(\mathcal{I},V^*)} &= \int_{\mathcal{I}} \|w_n(t)\|_{V^*}^2\,\mathrm{d}t \\ &\le c_{w,L_2,1}\int_{\mathcal{I}} \|u_n(t)\|_V^2 + \|z_n(t)\|^2\,\mathrm{d}t + c_{w,L_2,2}\end{aligned}$$

with $c_{w,L_2,1},\ c_{w,L_2,2} > 0$. Since $\|u_n\|_{L_2(\mathcal{I},V)}$ and $\max_{t\in\mathcal{I}}\|z_n(t)\|$ are bounded we obtain the desired result. □

Lemma 3.5 (Unique solvability of the Galerkin equations)
Let Assumption 3.1 be fulfilled. Then the Galerkin equations (2) *have a unique solution* $(z_n,u_n) \in C(\mathcal{I},\mathbb{R}^{n_z}\times V_n)$ *with* $x_n \in C^1(\mathcal{I},\mathbb{R}^{n_x})$ *and* $u_n \in W_2^1(\mathcal{I};V,H)$.

PROOF:
Inserting the algebraic constraint (2b), reformulated as

$$y_n(t) = \psi_g(x_n(t),t),$$

into (2a) and (2c) we set

$$\widetilde{f}(x_n(t),u_n(t),t) := f((x_n(t),\psi_g(x_n(t),t))^\top,u_n(t),t), \tag{11}$$

$$\widetilde{\mathcal{R}}(u_n(t),x_n(t),t) := \mathcal{R}(u_n(t),(x_n(t),\psi_g(x_n(t),t))^\top,t). \tag{12}$$

We have $\widetilde{f} \in C(\mathbb{R}^{n_x}\times V_n\times\mathcal{I},\mathbb{R}^{n_x})$ and $\widetilde{\mathcal{R}} \in C(V_n\times\mathbb{R}^{n_x}\times\mathcal{I},V^*)$ because f, $\mathcal{R}$ and ψ_g are continuous. Considering now (2c) we represent

$$u_n(t) = \sum_{j=1}^n \alpha_{nj}(t)v_j, \qquad u_{n0} = \sum_{j=1}^n \alpha_{nj}^0 v_j$$

with coefficients $\alpha_{nj}(t),\ \alpha_{nj}^0 \in \mathbb{R}$, $t\in\mathcal{I}$. With (12) we have for $i = 1\ldots,n$:

$$\sum_{j=1}^n \alpha_{nj}'(t)(v_j\,|\,v_i)_H + \sum_{j=1}^n \alpha_{nj}(t)\langle\mathcal{B}v_j,v_i\rangle_V + \langle\widetilde{\mathcal{R}}(\sum_{j=1}^n \alpha_{nj}(t)v_j,x_n(t),t),v_i\rangle_V = 0$$

$$\sum_{j=1}^n \alpha_{nj}(t_0)v_j = \sum_{j=1}^n \alpha_{nj}^0 v_j$$

Setting

$$\alpha_n(t) := \big(\alpha_{n1}(t)\ \ldots\ \alpha_{nn}(t)\big)^\top, \qquad \alpha_n^0 := \big(\alpha_{n1}^0(t)\ \ldots\ \alpha_{nn}^0(t)\big)^\top$$

we write in matrix notation

$$G\alpha_n'(t) + B\alpha_n(t) + r(\alpha_n(t), x_n(t), t) = 0$$

with

$$\begin{aligned} G &:= ((v_j \,|\, v_i)_H)_{i,j=1,\ldots,n}, \\ B &:= (\langle \mathcal{B} v_j, v_i \rangle_V)_{i,j=1,\ldots,n}, \\ r(\alpha_n(t), x_n(t), t) &:= (\langle \widetilde{\mathcal{R}}(\sum_{j=1}^{n} \alpha_{nj}(t) v_j, x_n(t), t), v_i \rangle_V)_{i=1,\ldots,n}. \end{aligned}$$

We have that $r \in C(\mathbb{R}^{n+n_x} \times \mathcal{I}, \mathbb{R}^n)$ because $\widetilde{\mathcal{R}}$ is continuous. Consider now the initial value problem

$$x_n'(t) = -\widetilde{f}(x_n(t), \sum_{j=1}^{n} \alpha_{nj}(t) v_j, t), \qquad x_n(t_0) = x_0 \tag{13a}$$

$$\alpha_n'(t) = -G^{-1}\left(B\alpha_n(t) + r(\alpha_n(t), x_n(t), t)\right), \qquad \alpha_n(t_0) = \alpha_n^0 \tag{13b}$$

which can be solved with the Peano Theorem in a neighborhood $J := [t_0, T_J) \subseteq \mathcal{I}$ of t_0. Let $(x_n^*, \alpha_n^*) \in C^1(J, \mathbb{R}^{n_x} \times \mathbb{R}^n)$ be this solution to (13). Then

$$\begin{aligned} u_n^*(t) &:= \sum_{j=1}^{n} \alpha_{jn}^*(t) v_j, \\ y_n^*(t) &:= \psi_g(x_n^*(t), t), \\ z_n^*(t) &:= \left(x_n^*(t) \; y_n^*(t)\right)^\top \end{aligned}$$

solves (2). We have $u_n^* \in W_2^1$ and the initial value condition is fulfilled because the v_j, $j = 1, \ldots, n$, are linearly independent. Due to Lemma 3.4 we have that $\|x_n^*(t)\|$ and $\|u_n^*(t)\|_V$ are uniformly bounded by a constant $C > 0$. We set

$$\|\alpha_n^*(t)\|_{H_n} := \left\| \sum_{j=1}^{n} \alpha_{nj}^*(t) v_j \right\|_H = \|u_n^*(t)\|_H$$

and have

$$\|\alpha_n^*(t)\| \le C_{H_n} \|\alpha_n^*(t)\|_{H_n} \le C_{H_n} C$$

because the norm $\|\cdot\|_{H_n}$ is equivalent to $\|\cdot\|$ on $\mathbb{R}^n$ ($C_{H_n} > 0$). This is due to the fact that the v_j are linearly independent. So the solution (x_n^*, α_n^*) can be extended to the end of the interval, cf. [35, p.800]. The uniqueness follows using the same arguments as in Lemma 3.3. □

Theorem 3.6 (Solvability and Galerkin convergence)
Let Assumption 3.1 be fulfilled. Then the original system (1) *has a unique solution*

$(z,u) \in C(\mathcal{I},\mathbb{R}^{n_z} \times H)$ with $x \in C^1(\mathcal{I},\mathbb{R}^{n_x})$ and $u \in W_2^1(\mathcal{I};V,H)$. Furthermore we have for the solution (z_n,u_n) of the Galerkin equations (2) *that*

$$\max_{t\in\mathcal{I}} \|z_n(t)-z(t)\| \to 0,$$
$$\max_{t\in\mathcal{I}} \|u_n(t)-u(t)\|_H \to 0,$$
$$\|u_n-u\|_{L_2(\mathcal{I},V)} \to 0$$

as $n \to \infty$.

PROOF:
The proof proceeds in several steps. We will first present the outline of the proof before proving the details. Therefore let (z_n,u_n) be the solution of the Galerkin equations (2) and w_n defined as in Lemma 3.4.
Step 1. We show that the sequence (x_n) is equicontinuous. Using then the uniform a priori estimates from Lemma 3.4 we apply the Arzelà-Ascoli Theorem and the Theorem of Eberlein and Šmuljan, cf. [34, Theorem 21.D]. The latter states that every bounded sequence in a reflexive Banach space has a weakly convergent subsequence. So there exists a subsequence $(x_{n'},u_{n'})$ and $x \in C(\mathcal{I},\mathbb{R}^{n_x})$, $u \in L_2(\mathcal{I},V)$, $w \in L_2(\mathcal{I},V^*)$ and $u_T \in H$ such that

$$x_{n'} \rightrightarrows x, \qquad u_{n'} \rightharpoonup u \text{ in } L_2(\mathcal{I},V),$$
$$u_{n'}(T) \rightharpoonup u_T \text{ in } H, \qquad w_{n'} \rightharpoonup w \text{ in } L_2(\mathcal{I},V^*)$$

as $n' \to \infty$. With $\rightrightarrows$ we denote uniform convergence. Furthermore we have $y_n(t) := \psi_g(x_n(t),t)$ and set $y(t) := \psi_g(x(t),t)$. We see due to the Lipschitz continuity of ψ_g that $y_{n'} \rightrightarrows y$ and so $z_{n'} \rightrightarrows z$.
Step 2. We show:
(2.I) The key equation

$$\phi(T)(u_T\,|\,v)_H - \phi(t_0)(u_0\,|\,v)_H = -\int_{\mathcal{I}} \langle w(t),v\rangle_V\, \phi(t)\mathrm{d}t + \int_{\mathcal{I}} (u(t)\,|\,v)_H \phi'(t)\mathrm{d}t \quad (14)$$

holds for all $v \in V,\ \phi \in C^\infty(\mathcal{I})$.
(2.II) The limits u, w and u_T satisfy

$$\langle u'(t),v\rangle_V + \langle w(t),v\rangle_V = 0 \quad \forall v \in V,$$
$$u(t_0) = u_0, \quad u(T) = u_T, \quad u \in W_2^1(\mathcal{I};V,H).$$

(2.III) For the given limit $z \in C(\mathcal{I},\mathbb{R}^{n_z})$ it is $w(t) = \mathcal{B}(u(t),z(t),t)$ for all $t \in \mathcal{I}$. So the limits u and z fulfill equation (1c).
Step 3. $u_{n'} \to u$ in $C(\mathcal{I},H)$ as $n' \to \infty$.
Step 4. The limits (z,u) satisfy the complete system (1) and $x \in C^1(\mathcal{I},\mathbb{R}^{n_x})$.
Step 5. The preceding argumentation was done for a subsequence n' of the original sequence n. The limits fulfill (1) and are unique because of Lemma 3.3. Now we apply the convergence principle form [33, Proposition 10.13 (1)] which states the

following: Given a Banach space W, a sequence (w_n) in W and $w \in W$ fixed. If every subsequence of (w_n) has, in turn, a subsequence which converges to w (in the norm $\|\cdot\|_W$) then the original sequence converges to w, i.e. $\|w_n - w\|_W \to 0$ as $n \to \infty$. The arguments above can be applied to any subsequence $\bar{n}$ with a corresponding subsequence $\bar{n}'$. Hence we have the convergence of the whole sequence $((z_n, u_n))$ in $C(\mathcal{I}, \mathbb{R}^{n_z} \times H)$. Additionally we have the weak convergence of the complete sequence (u_n) in $L_2(\mathcal{I}, V)$.
Step 6. It holds $u_n \to u$ in $L_2(\mathcal{I}, V)$ as $n \to \infty$. This completes the outline of the proof.
Ad (1). Because of the a priori estimates from Lemma 3.4 and the Lipschitz continuity of f there is a $D > 0$ (independent of n) such that

$$\max_{t \in \mathcal{I}} \|f(z_n(t), u_n(t), t)\| \leq D.$$

Let $\varepsilon > 0$ and $\delta(\varepsilon) := \frac{\varepsilon}{D}$. Then for $t, \bar{t} \in \mathcal{I}$ with $|t - \bar{t}| < \delta(\varepsilon)$ we see integrating (2a) over $[t, \bar{t}]$ that

$$\|x_n(\bar{t}) - x_n(t)\| \leq \int_t^{\bar{t}} \|f(z_n(s), u_n(s), s)\| \, \mathrm{d}s \leq D\,|t - \bar{t}| < \varepsilon.$$

Ad (2.I). We now write n instead of n'. Let $\phi \in C^\infty(\mathcal{I})$, $v \in V_k$, $k \in \mathbb{N}$ fixed, $n \geq k$. Since $u_n,\ \phi v \in W_2^1$ the integration by parts formula can be applied and we obtain

$$\begin{aligned}(u_n(T)\,|\,\phi(T)v)_H - (u_{n0}\,|\,\phi(t_0)v)_H &= \int_{\mathcal{I}} \langle u_n'(t), \phi(t)v\rangle_V + \langle \phi'(t)v, u_n(t)\rangle_V \,\mathrm{d}t \\ &\overset{(2c)}{=} \int_{\mathcal{I}} -\langle w_n(t), \phi(t)v\rangle_V + (u_n(t)\,|\,v)_H \phi'(t)\,\mathrm{d}t\end{aligned}$$

Since $u_n(T) \rightharpoonup u_T$ and $u_{n0} \to u_0$ in H we have

$$(u_n(T)\,|\,v)_H \to (u_T\,|\,v)_H, \qquad (u_{n0}\,|\,v)_H \to (u_0\,|\,v)_H$$

as $n \to \infty$. From $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ we deduce with the Hölder inequality that

$$\begin{aligned}\int_{\mathcal{I}} \langle w_n(t), \phi(t)v\rangle_V \,\mathrm{d}t &\to \int_{\mathcal{I}} \langle w(t), \phi(t)v\rangle_V \,\mathrm{d}t \\ \int_{\mathcal{I}} (v\,|\,u_n(t))_H \phi'(t)\,\mathrm{d}t &\to \int_{\mathcal{I}} (u(t)\,|\,v)_H \phi'(t)\,\mathrm{d}t\end{aligned}$$

as $n \to \infty$ because the embedding $V \subseteq H$ is continuous. So equation (14) is fulfilled for all $v \in \bigcup_{k \in \mathbb{N}} V_k$ which is dense in V. With a common density argument we verify (14) for all $v \in V$.
Ad (2.II). For $\phi \in C_0^\infty(\mathcal{I})$ and $v \in V$ we obtain

$$\int_{\mathcal{I}} \langle w(t), v\rangle_V \phi(t)\,\mathrm{d}t \overset{(14)}{=} \int_{\mathcal{I}} (u(t)\,|\,v)_H \phi'(t)\,\mathrm{d}t$$

and hence $u' \in L_2(\mathcal{I}, V^*)$ exists and $u' = -w$. So $u \in W_2^1$ and integration by parts with $\phi v \in W_2^1$ for $v \in V$ and $\phi \in C^\infty(\mathcal{I})$ reveals:

$$\begin{aligned}(u(T)\,|\,\phi(T)v)_H - (u(t_0)\,|\,\phi(t_0)v)_H &= \int_{\mathcal{I}} \langle u'(t), \phi(t)v\rangle_V + \langle \phi'(t)v, u(t)\rangle_V \,\mathrm{d}t \\ &= \int_{\mathcal{I}} -\langle w(t), \phi(t)v\rangle_V + (u(t)\,|\,v)_H \phi'(t)\,\mathrm{d}t \\ &\overset{(14)}{=} (u_T\,|\,\phi(T)v)_H - (u_0\,|\,\phi(t_0)v)_H\end{aligned}$$

Appropriate choices of ϕ and a density argument reveal that $u(T) = u_T$ and $u(t_0) = u_0$.
Ad (2.III). We set $X := L_2(\mathcal{I}, V)$, then $X^* = L_2(\mathcal{I}, V^*)$. For the limit $z \in C(\mathcal{I}, \mathbb{R}^{n_z})$ we define

$$\widetilde{\boldsymbol{\mathcal{B}}} : X \to X^*, \quad (\widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}))(t) := \boldsymbol{\mathcal{B}}\widetilde{u}(t) + \boldsymbol{\mathcal{R}}(\widetilde{u}(t), z(t), t), \quad \widetilde{u} \in X,\ t \in \mathcal{I}.$$

We also write $(\widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}))(t) = \widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}(t))$. As for the w_n in the proof of Lemma 3.4 it can be shown that $\widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}) \in X^*$ because z is bounded and $\widetilde{u} \in X$. We show:

(i) $\widetilde{\boldsymbol{\mathcal{B}}}$ is strongly monotone,
(ii) $\widetilde{\boldsymbol{\mathcal{B}}}$ is hemicontinuous,
(iii) $\widetilde{\boldsymbol{\mathcal{B}}}(u_n) \rightharpoonup w$ as $n \to \infty$ and
(iv) $\widetilde{\boldsymbol{\mathcal{B}}}(u) = w$.

The operator $\widetilde{\boldsymbol{\mathcal{B}}}$ is said to be hemicontinuous if the real function

$$t \mapsto \langle \boldsymbol{\mathcal{B}}(u + tv), w\rangle_X$$

is continuous on $[0,1]$ for all $u, v, w \in X$.
Ad (i). Let $\widetilde{u}_1, \widetilde{u}_2 \in X$. Then

$$\begin{aligned}\langle \widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}_1) - \widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}_2), \widetilde{u}_1 - \widetilde{u}_2\rangle_X &= \int_{\mathcal{I}} \langle \boldsymbol{\mathcal{R}}(\widetilde{u}_1(t), z(t), t) - \boldsymbol{\mathcal{R}}(\widetilde{u}_2(t), z(t), t), \widetilde{u}_1(t) - \widetilde{u}_2(t)\rangle_V \,\mathrm{d}t \\ &\quad + \int_{\mathcal{I}} \langle \boldsymbol{\mathcal{B}}(\widetilde{u}_1(t) - \widetilde{u}_2(t)), \widetilde{u}_1(t) - \widetilde{u}_2(t)\rangle_V \,\mathrm{d}t \\ &\geq \mu_{\boldsymbol{\mathcal{B}}} \|\widetilde{u}_1 - \widetilde{u}_2\|_X^2\end{aligned}$$

because $\boldsymbol{\mathcal{B}}$ is strongly monotone and $\boldsymbol{\mathcal{R}}$ is monotone w.r.t. u.
Ad (ii). We follow a standard argument here, cf. [35, chapter 30.3b. (IV)]. We first remark that

$$\langle (\widetilde{\boldsymbol{\mathcal{B}}}(\widetilde{u}))(t), v\rangle_V \leq \left(c_{\widetilde{\boldsymbol{\mathcal{B}}},1}\left(\|\widetilde{u}(t)\|_V + \|z(t)\|\right) + c_{\widetilde{\boldsymbol{\mathcal{B}}},2}\right)\|v\|_V$$

for all $\widetilde{u} \in X$, $v \in V$ and $t \in \mathcal{I}$. This follows as in the proof of Lemma 3.4 for the w_n.
Let now $\bar{u}, \bar{w}, \bar{v} \in X$, $t \in \mathcal{I}$ and $s_k \to s$ as $k \to \infty$ with $0 \leq s, s_k \leq 1$. We then have

$$\left|\langle\widetilde{\mathcal{B}}(\bar{u}(t)+s_k\bar{v}(t)),\bar{w}(t)\rangle_V\right| \leq \left(c_{\widetilde{\mathcal{B}},1}\left(\|\bar{u}(t)+s_k\bar{v}(t)\|_V+\|z(t)\|\right)+c_{\widetilde{\mathcal{B}},2}\right)\|\bar{w}(t)\|_V \leq q(t)$$

with

$$q(t) := c_{q,1}\left(\|\bar{u}(t)\|_V+\|\bar{v}(t)\|_V+\|z(t)\|+c_{q,2}\right)\|\bar{w}(t)\|_V$$

for constants $c_{q,1},\ c_{q,2} > 0$ because $s_k \leq 1$. Therefore the majorant function q is integrable because $\bar{u}$, $\bar{v}$, z and $\bar{w}$ are. Furthermore we have that

$$\langle\widetilde{\mathcal{B}}(\bar{u}(t)+s_k\bar{v}(t)),\bar{w}(t)\rangle_V \to \langle\widetilde{\mathcal{B}}(\bar{u}(t)+s\bar{v}(t)),\bar{w}(t)\rangle_V \quad \text{as } k\to\infty$$

because of the continuity of $\mathcal{B}$ and $\mathcal{R}$. From the principle of majorized convergence, cf. [35, p.1015], it follows that

$$\begin{aligned}\lim_{k\to\infty}\langle\widetilde{\mathcal{B}}(\bar{u}+s_k\bar{v}),\bar{w}\rangle_X &= \lim_{k\to\infty}\int_{\mathcal{I}}\langle\widetilde{\mathcal{B}}(\bar{u}(t)+s_k\bar{v}(t)),\bar{w}(t)\rangle_V\,\mathrm{d}t\\ &= \langle\widetilde{\mathcal{B}}(u+sv),w\rangle_X\end{aligned}$$

This shows the hemicontinuity of $\widetilde{\mathcal{B}}$.
Ad (iii). Let $h \in X^{**}$ and with the Hölder inequality and the Lipschitz continuity of $\mathcal{R}$ ($L_{\mathcal{R}} > 0$) it follows that

$$\begin{aligned}\langle h,\widetilde{\mathcal{B}}(u_n)-w\rangle_{X^*} &\leq \|h\|_{X^{**}}\left\|\widetilde{\mathcal{B}}(u_n)-w_n\right\|_{X^*}+|\langle h,w_n-w\rangle_{X^*}|\\ &\leq \|h\|_{X^{**}}\left(\int_{\mathcal{I}}\|\mathcal{R}(u_n(t),z(t),t)-\mathcal{R}(u_n(t),z_n(t),t)\|_{V^*}^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\quad +|\langle h,w_n-w\rangle_{X^*}|\\ &\leq \underbrace{\|h\|_{X^{**}}L_{\mathcal{R}}\left(\int_{\mathcal{I}}\|z(t)-z_n(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}}}_{\to 0 \text{ as } n\to\infty}+\underbrace{|\langle h,w_n-w\rangle_{X^*}|}_{\to 0 \text{ as } n\to\infty}\end{aligned}$$

because $z_n \rightrightarrows z$ and $w_n \rightharpoonup w$ in X^* as $n\to\infty$.
Ad (iv). We have $u_n \rightharpoonup u$ in X and $\widetilde{\mathcal{B}}(u_n) \rightharpoonup w$ in X^* as $n\to\infty$. Since $\widetilde{\mathcal{B}}$ is hemicontinuous and monotone it remains to show that

$$\overline{\lim_{n\to\infty}}\langle\widetilde{\mathcal{B}}(u_n),u_n\rangle_X \leq \langle w,u_n\rangle_X$$

and the fundamental monotonicity trick can be applied, cf. [35, p.474]. Then we can deduce that $\widetilde{\mathcal{B}}(u) = w$. Integration by parts and the Galerkin equations yield

$$\begin{aligned}
&\frac{1}{2}\|u_n(T)\|_H^2 - \frac{1}{2}\|u_n(t_0)\|_H^2\\
&= \int_{\mathcal{I}} \langle u_n'(t), u_n(t)\rangle_V \,\mathrm{d}t\\
&\overset{(2c)}{=} -\int_{\mathcal{I}} \langle w_n(t), u_n(t)\rangle_V \,\mathrm{d}t\\
&= -\int_{\mathcal{I}} \langle (\widetilde{\mathcal{B}}(u_n))(t), u_n(t)\rangle_V + \langle \mathcal{R}(u_n(t), z_n(t), t) - \mathcal{R}(u(t), z(t), t), u_n(t)\rangle_V \,\mathrm{d}t.
\end{aligned}$$

We have that $u_n(t_0) \to u_0$ in H and $u_n(T) \rightharpoonup u(T)$ in H and hence

$$|u(T)| \leq \varliminf_{n\to\infty} |u_n(T)|,$$

cf. [34, Proposition 21.23 (c)]. Furthermore the Hölder inequality gives

$$\begin{aligned}
&\int_{\mathcal{I}} \langle \mathcal{R}(u_n(t), z_n(t), t) - \mathcal{R}(u_n(t), z(t), t), u_n(t)\rangle_V \,\mathrm{d}t\\
&\leq L_{\mathcal{R}} \int_{\mathcal{I}} \|z_n(t) - z(t)\| \, \|u_n(t)\|_V \,\mathrm{d}t\\
&\leq \bar{c}\, \|z_n - z\|_\infty \|u_n\|_X \to 0 \text{ as } n \to \infty
\end{aligned}$$

for $\bar{c} > 0$ because $\|u_n\|_X$ is bounded and $z_n \rightrightarrows z$. We conclude:

$$\begin{aligned}
\varlimsup_{n\to\infty} \langle \widetilde{\mathcal{B}}(u_n), u_n\rangle_X &\leq \frac{1}{2}\|u(t_0)\|_H^2 - \frac{1}{2}\|u(T)\|_H^2\\
&= -\int_{\mathcal{I}} \langle u'(t), u(t)\rangle_V \,\mathrm{d}t\\
&= \int_{\mathcal{I}} \langle w(t), u(t)\rangle_V \,\mathrm{d}t = \langle w, u\rangle_X
\end{aligned}$$

Ad (3). We now show the convergence of u_n to u in $C(\mathcal{I}, H)$. Remember that with n we still denote a subsequence of the original sequence. In analogy to the proof of [34, Theorem 23.A] there is a sequence (p_n) of polynomials $p_n : \mathcal{I} \to V_n$ with

$$p_n \to u \text{ in } W_2^1 \quad \text{as } n \to \infty \tag{15}$$

because $\bigcup_n V_n \subseteq V$ dense. The embedding $W_2^1 \subseteq C(\mathcal{I}, H)$ is continuous, so we have

$$\max_{t\in\mathcal{I}} \|u(t) - p_n(t)\|_H \leq c_1 \|u - p_n\|_{W_2^1} \to 0 \quad \text{as } n \to \infty$$

with a $c_1 > 0$. So it suffices to show that

$$\max_{t\in\mathcal{I}} \|u_n(t) - p_n(t)\|_H \to 0 \quad \text{as } n \to \infty.$$

Clearly

$$\begin{aligned}\|u_n(t_0)-p_n(t_0)\|_H &\le \|u_n(t_0)-u(t_0)\|_H+\|u(t_0)-p_n(t_0)\|_H\\ &\le \|u_{n0}-u_0\|_H+\max_{t\in\mathcal{I}}\|u_n(t)-p_n(t)\|_H\to 0 \quad \text{as } n\to\infty\end{aligned}$$

because $u_{n0}\to u_0$ in H. We will now show that

$$\max_{t\in\mathcal{I}}\|u_n(t)-p_n(t)\|_H^2-\|u_n(t_0)-p_n(t_0)\|_H^2\to 0 \quad \text{as } n\to\infty \tag{16}$$

which then proves the convergence of u_n in $C(\mathcal{I},H)$. It is

$$\begin{aligned}&\langle u_n'(t)-u'(t),u_n(t)-p_n(t)\rangle_V\\ &\overset{(2c)}{=}-\langle w_n(t)+u'(t),u_n(t)-p_n(t)\rangle_V\\ &\overset{(1c)}{=}\langle w(t)-w_n(t),u_n(t)-p_n(t)\rangle_V\\ &=\langle(\widetilde{\mathcal{B}}(u))(t)-(\widetilde{\mathcal{B}}(u_n))(t)+(\widetilde{\mathcal{B}}(u_n))(t)-w_n(t),u_n(t)-p_n(t)\rangle_V\\ &=\underbrace{\langle(\widetilde{\mathcal{B}}(u))(t)-(\widetilde{\mathcal{B}}(u_n))(t),u_n(t)-u(t)\rangle_V}_{\le 0}+\underbrace{\langle(\widetilde{\mathcal{B}}(u))(t)-(\widetilde{\mathcal{B}}(u_n))(t),u(t)-p_n(t)\rangle_V}_{\le\left\|(\widetilde{\mathcal{B}}(u))(t)-(\widetilde{\mathcal{B}}(u_n))(t)\right\|_{V^*}\|u(t)-p_n(t)\|_V}\\ &\quad+\underbrace{\langle(\widetilde{\mathcal{B}}(u_n))(t)-w_n(t),u_n(t)-p_n(t)\rangle_V}_{\le L_{\mathcal{R}}\|z(t)-z_n(t)\|\|u_n(t)-p_n(t)\|_V}\\ &\le\left\|(\widetilde{\mathcal{B}}(u))(t)-(\widetilde{\mathcal{B}}(u_n))(t)\right\|_{V^*}\|u(t)-p_n(t)\|_V+L_{\mathcal{R}}\|z(t)-z_n(t)\|\,\|u_n(t)-p_n(t)\|_V\end{aligned}$$

with $L_{\mathcal{R}}>0$. Integration by parts gives

$$\begin{aligned}&\frac{1}{2}\|u_n(t)-p_n(t)\|_H^2-\frac{1}{2}\|u_n(t_0)-p_n(t_0)\|_H^2\\ &=\int_{t_0}^t\langle u_n'(s)-p_n'(s),u_n(s)-p_n(s)\rangle\mathrm{d}s\\ &=\int_{t_0}^t\langle u'(s)-p_n'(s),u_n(s)-p_n(s)\rangle+\langle u_n'(s)-u'(s),u_n(s)-p_n(s)\rangle\mathrm{d}s\\ &\le\|u-p_n\|_{W_2^1}\|u_n-p_n\|_X+\left\|(\widetilde{\mathcal{B}}(u))-(\widetilde{\mathcal{B}}(u_n))\right\|_{X^*}\|u-p_n\|_{W_2^1}\\ &\quad+L_{\mathcal{R}}\|z-z_n\|_\infty\|u_n-p_n\|_X\end{aligned}$$

Since $u_n\rightharpoonup u$ and $p_n\to u$ in W_2^1 as $n\to\infty$ the sequences (u_n) and (p_n) are bounded in X. Furthermore $\widetilde{\mathcal{B}}:X\to X^*$ is bounded and hence the sequence $(\widetilde{\mathcal{B}}(u_n))$ is bounded with the same reasoning as for w_n. This implies that the terms

$$\|u_n-p_n\|_X\,,\quad \left\|(\widetilde{\mathcal{B}}(u))-(\widetilde{\mathcal{B}}(u_n))\right\|_{X^*}$$

are bounded. Finally, we see that the right hand side tends to zero because $z_n\rightrightarrows z$ and (15) holds.

Ad (4). We have already seen that equation (1c) is fulfilled by the limit u given the limit z. Furthermore we can rewrite (2a) as follows

$$x_n(t) = x_{n0} - \int_{t_0}^{t} f(z_n(s), u_n(s), s)\mathrm{d}s.$$

It is $x_{n0} = x_0$ and letting $n \to \infty$ we observe

$$\begin{aligned}&\left\| x_{n0} - \int_{t_0}^{t} f(z_n(s), u_n(s), s)\mathrm{d}s - x_0 + \int_{t_0}^{t} f(z(s), u(s), s)\mathrm{d}s \right\| \\ &\leq c_f \int_{t_0}^{t} \|z_n(s) - z(s)\| + \|u_n(s) - u(s)\|_H \,\mathrm{d}s \\ &\leq c_f (t - t_0) \max_{s \in \mathcal{I}} (\|z_n(s) - z(s)\| + \|u_n(s) - u(s)\|_H) \to 0\end{aligned}$$

with $c_f > 0$ using the Lipschitz continuity of f and (3). Since $x_n \rightrightarrows x$ the limits z, u satisfy

$$x(t) = x_0 + \int_{t_0}^{t} f(z(s), u(s), s)\mathrm{d}s \quad \forall t \in \mathcal{I}.$$

Hence (1a) is fulfilled and $x \in C^1(\mathcal{I}, \mathbb{R}^{n_x})$ because f is continuous. By the definition of y the algebraic equation (1b) is automatically satisfied.
Ad (6). With integration by parts we get

$$\begin{aligned}\frac{1}{2}\|u(T) - u_n(T)\|_H^2 - \frac{1}{2}\|u(t_0) - u_n(t_0)\|_H^2 &= \int_{\mathcal{I}} \langle [u(t) - u_n(t)]', u(t) - u_n(t) \rangle_V \mathrm{d}t \\ &\overset{(1c)}{=} -\langle \widetilde{\mathcal{B}}(u), u - u_n \rangle_X - \langle u_n', u - u_n \rangle_X\end{aligned}$$

and

$$\begin{aligned}(u_n(T) \,|\, u(T))_H - (u_n(t_0) \,|\, u(t_0))_H &= \int_{\mathcal{I}} \langle u_n'(t), u(t) \rangle_V + \langle u'(t), u_n(t) \rangle_V \mathrm{d}t \\ &\overset{(1c)}{=} \langle u_n', u \rangle_X - \langle \widetilde{\mathcal{B}}(u), u_n \rangle_X.\end{aligned}$$

So we have

$$\frac{1}{2}\|u(T) - u_n(T)\|_H^2 = \frac{1}{2}\|u(t_0) - u_n(t_0)\|_H^2 - \langle \widetilde{\mathcal{B}}(u) + u_n', u - u_n \rangle_X \tag{17}$$

and

$$\langle u_n', u \rangle_X = \langle \widetilde{\mathcal{B}}(u), u_n \rangle_X + (u_n(T) \,|\, u(T))_H - (u_n(t_0) \,|\, u(t_0))_H. \tag{18}$$

For convenience we set

$$\bar{w}_n(t) := \mathcal{B}u(t) + \mathcal{R}(u(t), z_n(t), t), \quad \forall t \in \mathcal{I}.$$

With the strong monotonicity of $\mathcal{B}$ and the monotonicity of $\mathcal{R}$ we obtain

$$
\begin{aligned}
&\mu_{\mathcal{B}} \|u-u_n\|_X^2 \\
\leq\ & \int_{\mathcal{I}} \langle \mathcal{B}(u(t)-u_n(t)), u(t)-u_n(t)\rangle_V \,\mathrm{d}t \\
\leq\ & \int_{\mathcal{I}} \langle \mathcal{B}(u(t)-u_n(t)) + \mathcal{R}(u(t),z_n(t),t) - \mathcal{R}(u_n(t),z_n(t),t), u(t)-u_n(t)\rangle_V \,\mathrm{d}t \\
\leq\ & \langle \bar{w}_n - w_n, u-u_n\rangle_X + \frac{1}{2}\|u(T)-u_n(T)\|_H^2 \\
\overset{(17)}{=}\ & \langle \bar{w}_n - w_n - \widetilde{\mathcal{B}}(u) - u_n', u-u_n\rangle_X + \frac{1}{2}\|u(t_0)-u_n(t_0)\|_H^2 \\
\overset{(2c)}{=}\ & \langle \bar{w}_n - \widetilde{\mathcal{B}}(u), u-u_n\rangle_X - \langle w_n + u_n', u\rangle_X + \frac{1}{2}\|u(t_0)-u_n(t_0)\|_H^2 \\
\overset{(18)}{=}\ & \langle \bar{w}_n - \widetilde{\mathcal{B}}(u), u-u_n\rangle_X - \langle w_n, u\rangle_X - \langle \widetilde{\mathcal{B}}(u), u_n\rangle_X \\
& -(u_n(T)\,|\,u(T))_H + (u_n(t_0)\,|\,u(t_0))_H + \frac{1}{2}\|u(t_0)-u_n(t_0)\|_H^2
\end{aligned}
$$

With the Hölder inequality we see that

$$\langle \bar{w}_n - \widetilde{\mathcal{B}}(u), u-u_n\rangle_X \leq \bar{c}\,\|z_n - z\|_\infty \|u-u_n\|_X \to 0$$

with $\bar{c} > 0$ as $n \to \infty$ because $z_n \rightrightarrows z$ and (u_n) is bounded in X. Since $u_n \rightharpoonup u$ in X and $w_n \rightharpoonup w = \widetilde{\mathcal{B}}(u)$ in X^* we have

$$\langle w_n, u\rangle_X \to \langle \widetilde{\mathcal{B}}(u), u\rangle_X \quad \text{and} \quad \langle \widetilde{\mathcal{B}}(u), u_n\rangle_X \to \langle \widetilde{\mathcal{B}}(u), u\rangle_X \text{ as } n \to \infty.$$

Since $u_n(t_0) \to u(t_0)$ in H and $u_n(T) \rightharpoonup u(T)$ in H we have with the integration by parts formula that

$$
\begin{aligned}
\lim_{n\to\infty} \mu_{\mathcal{B}} \|u-u_n\|_X^2 &= -2\langle \widetilde{\mathcal{B}}(u), u\rangle_X - \|u(T)\|_H^2 + \|u(t_0)\|_H^2 \\
&= -2\langle \widetilde{\mathcal{B}}(u), u\rangle_X - 2\langle u', u\rangle_X \overset{(1c)}{=} 0.
\end{aligned}
$$

So $u_n \to u$ in $L_2(\mathcal{I}, V)$ as $n \to \infty$. □

Remark 3.7 (Algebraic part of (1)**)**
In the investigated system (1) *the algebraic part is given by the function g which does not depend on u. Allowing g to depend on u in the form*

$$g(z(t), u(t), t) = 0$$

instead of (1b) *makes the problem more complex. Apart from having to solve the algebraic part w.r.t. y with a continuous solution function that is Lipschitz continuous in x and u, it is not obvious how to achieve uniform convergence of y_n for the Galerkin sequence. This was crucial in the proof of Theorem 3.6 to obtain the convergence of u_n in $C(\mathcal{I}, H)$. If, however, $\mathcal{R}(u,z,t) = \mathcal{R}(u,x,t)$ only depends on x instead of z completely, then the convergence of y_n is not important to get the conver-*

gence of u_n in $C(\mathcal{I},H)$. We then conclude the uniform convergence by representing y_n with the solution function.

For obtaining a perturbation result for system (1) we study the perturbed system

$$x'(t) + f(z(t),u(t),t,\delta_x(t)) = 0, \quad t \in \mathcal{I}, \tag{19a}$$
$$g(z(t),t,\delta_y(t)) = 0, \tag{19b}$$
$$u'(t) + \mathcal{B}u(t) + \mathcal{R}(u(t),z(t),t) + \delta_u(t) = 0, \quad \text{in } V^*, \tag{19c}$$
$$x(t_0) = x_0^\delta, \quad u(t_0) = u_0^\delta \tag{19d}$$

for perturbations $\delta_x(t) \in \mathbb{R}^{n_x}$, $\delta_y(t) \in \mathbb{R}^{n_y}$ and $\delta_u(t) \in V^*$ and perturbed initial values $x_0^\delta \in \mathbb{R}^{n_x}$ and $u_0^\delta \in H$. We obtain the following result showing that the prototype system (1) has Perturbation Index 1.

Theorem 3.8 (Perturbation result)
Consider system (19) *together with the following assumptions:*

(i) Let $\mathcal{I} := [t_0,T]$ be an interval and $V \subseteq H \subseteq V^*$ be an evolution triple.
(ii) The initial values $x_0, x_0^\delta \in \mathbb{R}^{n_x}$, $u_0, u_0^\delta \in H$ are given.
(iii) $f \in C(\mathbb{R}^{n_z} \times H \times \mathcal{I} \times \mathbb{R}^{n_x}, \mathbb{R}^{n_x})$ is Lipschitz continuous w.r.t. z, u and δ_x.
(iv) $g \in C(\mathbb{R}^{n_z} \times \mathcal{I} \times \mathbb{R}^{n_y}, \mathbb{R}^{n_y})$ is uniquely solvable w.r.t. $y \in \mathbb{R}^{n_y}$, i.e. there is a solution function $\psi_g \in C(\mathbb{R}^{n_x} \times \mathcal{I} \times \mathbb{R}^{n_y}, \mathbb{R}^{n_y})$ such that $y = \psi_g(x,t,\delta_y(t))$ whenever $g(x,y,t,\delta_y(t)) = 0$ for all x,y,t,δ_y. Furthermore ψ_g is Lipschitz continuous w.r.t. x and δ_y.
(v) $\mathcal{B} : V \to V^*$ is linear, strongly monotone and bounded.
(vi) $\mathcal{R} \in C(V \times \mathbb{R}^{n_z} \times \mathcal{I}, V^*)$ is monotone w.r.t. u, i.e.

$$\langle \mathcal{R}(u,z,t) - \mathcal{R}(\bar{u},z,t), u - \bar{u} \rangle_V \geq 0 \quad \forall u, \bar{u}, v \in V, z \in \mathbb{R}^{n_z}, t \in \mathcal{I}$$

and Lipschitz continuous w.r.t. z. Furthermore there are constants $c_{\mathcal{R},1}$, $c_{\mathcal{R},2} >$ 0 such that

$$\|\mathcal{R}(u,0,t)\|_{V^*} \leq c_{\mathcal{R},1} \|u\|_V + c_{\mathcal{R},2} \quad \forall u \in V.$$

(vii) Let $\dim V = \infty$ and $\{v_1, v_2, \dots\}$ be a basis of V. Set $V_n := \operatorname{span}\{v_1, \dots, v_n\}$ and let there be two sequences $(u_{n0}), (u_{n0}^\delta) \subseteq V$ with $u_{n0}, u_{n0}^\delta \in V_n$ and

$$u_{n0} \to u_0 \text{ in } H, \quad u_{n0}^\delta \to u_0^\delta \text{ in } H \quad \text{as } n \to \infty.$$

(viii) $\delta_x \in C(\mathcal{I}, \mathbb{R}^{n_x})$, $\delta_y \in C(\mathcal{I}, \mathbb{R}^{n_y})$ and $\delta_u \in C(\mathcal{I}, V^*)$.

Then the perturbed system (19) *has a unique solution $(z^\delta, u^\delta) \in C(\mathcal{I}, \mathbb{R}^{n_z} \times H)$ with $z^\delta = (x^\delta, y^\delta)^\top$, $x^\delta \in C^1(\mathcal{I}, \mathbb{R}^{n_x})$ and $u^\delta \in W_2^1(\mathcal{I}; V, H)$. Let (z,u) be the solution for $(\delta_x, \delta_y, \delta_u) = 0$ with initial values $x(t_0) = x_0$ and $u(t_0) = u_0$. Then there is a $C > 0$ such that*

$$\left\|z-z^{\delta}\right\|_{\infty}+\max_{t\in\mathcal{I}}\left\|u(t)-u^{\delta}(t)\right\|_{H}+\left\|u-u^{\delta}\right\|_{L_2(\mathcal{I},V)}$$
$$\leq C\left(\left\|x_0-x_0^{\delta}\right\|+\left\|u_0-u_0^{\delta}\right\|_{H}+\|\delta_x\|_{\infty}+\left\|\delta_y\right\|_{\infty}+\max_{t\in\mathcal{I}}\|\delta_u(t)\|_{V^*}\right).$$

PROOF:
Solvability. For given perturbations δ_x, δ_y and δ_u we define

$$\begin{aligned} f^{\delta}(z(t),u(t),t) &:= f(z(t),u(t),t,\delta_x(t)),\\ g^{\delta}(z(t),t) &:= g(z(t),t,\delta_y(t)),\\ \mathcal{R}^{\delta}(u(t),z(t),t) &:= \mathcal{R}(u(t),z(t),t)+\delta_u(t) \end{aligned}$$

for all $t\in\mathcal{I}$. Then the functions f^{δ}, g^{δ} and the operator $\mathcal{R}^{\delta}$ inherit all the properties from the functions f, g and the operator $\mathcal{R}$ and this makes Theorem 3.6 applicable. Hence we get the desired unique solution (z^{δ},u^{δ}) for the initial values x_0^{δ} and $u_0^{\delta}\in H$. For $(\delta_x,\delta_y,\delta_u)=0$ and initial values x_0 and $u_0\in H$ we denote the solution by (z,u).
Perturbation estimate. Building the difference between the perturbed and the unperturbed operator equation gives for $t\in\mathcal{I}$:

$$(u^{\delta}-u)'(t)+\mathcal{B}(u^{\delta}(t)-u(t))+\mathcal{R}(u^{\delta}(t),z^{\delta}(t),t)-\mathcal{R}(u(t),z(t),t)+\delta_u(t)=0$$

Integration by parts yields

$$\begin{aligned} &\frac{1}{2}\left\|u^{\delta}(t)-u(t)\right\|_{H}^{2}-\frac{1}{2}\left\|u^{\delta}(t_0)-u(t_0)\right\|_{H}^{2}\\ =&\int_{t_0}^{t}\langle u^{\delta}(s)-u(s),u^{\delta}(s)-u(s)\rangle_V\,\mathrm{d}s\\ =&-\int_{t_0}^{t}\langle\mathcal{B}(u^{\delta}(s)-u(s)),u^{\delta}(s)-u(s)\rangle_V\,\mathrm{d}s\\ &-\int_{t_0}^{t}\langle\mathcal{R}(u^{\delta}(s),z^{\delta}(s),s)-\mathcal{R}(u(s),z(s),s)+\delta_u(s),u^{\delta}(s)-u(s)\rangle_V\,\mathrm{d}s \end{aligned}$$

With the strong monotonicity of $\mathcal{B}$ ($\mu_{\mathcal{B}}>0$) and the Lipschitz continuity of $\mathcal{R}$ ($L_{\mathcal{R}}>0$) we obtain

$$
\begin{aligned}
&\frac{1}{2}\left\|u^\delta(t)-u(t)\right\|_H^2+\mu_{\mathcal{B}}\left\|u^\delta-u\right\|_{L_2(\mathcal{I},V)}^2\\
\leq\ &\frac{1}{2}\left\|u^\delta(t_0)-u(t_0)\right\|_H^2+\int_{t_0}^t\underbrace{\langle\mathcal{R}(u(s),z(s),s)-\mathcal{R}(u^\delta(s),z(s),s),u^\delta(s)-u(s)\rangle_V}_{\leq 0}\,\mathrm{d}s\\
&+\int_{t_0}^t\langle\mathcal{R}(u^\delta(s),z(s),s)-\mathcal{R}(u^\delta(s),z^\delta(s),s)+\delta_u(s),u^\delta(s)-u(s)\rangle_V\,\mathrm{d}s\\
\leq\ &\frac{1}{2}\left\|u^\delta(t_0)-u(t_0)\right\|_H^2+\int_{t_0}^t\left(L_{\mathcal{R}}\left\|z(s)-z^\delta(s)\right\|+\|\delta_u(s)\|_{V^*}\right)\left\|u^\delta(s)-u(s)\right\|_V\mathrm{d}s\\
\leq\ &\frac{1}{2}\left\|u^\delta(t_0)-u(t_0)\right\|_H^2+\frac{\mu_{\mathcal{B}}}{2}\left\|u^\delta-u\right\|_{L_2(\mathcal{I},V)}^2\\
&+\frac{4}{\mu_{\mathcal{B}}}\int_{t_0}^t L_{\mathcal{R}}^2\left\|z(s)-z^\delta(s)\right\|^2+\|\delta_u(s)\|_{V^*}^2\,\mathrm{d}s.
\end{aligned}
$$

Hence there is a constant $c_1 > 0$ such that

$$
\left\|u^\delta-u\right\|_{L_2(\mathcal{I},V)}\leq c_1\max_{t\in\mathcal{I}}\left(\left\|z(t)-z^\delta(t)\right\|+\|\delta_u(t)\|_{V^*}\right). \tag{20}
$$

and

$$
\max_{t\in\mathcal{I}}\left\|u^\delta(t)-u(t)\right\|_H\leq c_1\max_{t\in\mathcal{I}}\left(\left\|z(t)-z^\delta(t)\right\|+\|\delta_u(t)\|_{V^*}\right). \tag{21}
$$

Furthermore we have from (19b) that

$$
y^\delta(t)=\psi_g(x^\delta(t),t,\delta_y(t))\quad\text{and}\quad y(t)=\psi_g(x(t),t,0).
$$

We obtain

$$
\begin{aligned}
\left\|y(t)-y^\delta(t)\right\|&\leq\left\|\psi_g(x(t),t,0)-\psi_g(x^\delta(t),t,0)\right\|+\left\|\psi_g(x^\delta(t),t,0)-\psi_g(x^\delta(t),t,\delta_y(t))\right\|\\
&\leq c_y\left(\left\|x(t)-x^\delta(t)\right\|+\|\delta_y(t)\|\right)
\end{aligned}
$$

for a constant $c_y > 0$. Integration of (19a) and the Lipschitz continuity of f gives

$$
\left\|x^\delta(t)-x(t)\right\|\leq\left\|x_0^\delta-x_0\right\|+c_x\int_{t_0}^t\left\|z^\delta(s)-z(s)\right\|+\left\|u^\delta(s)-u(s)\right\|_H+\|\delta_x(s)\|\mathrm{d}s \tag{22}
$$

Inserting into each other, using (21) and the Gronwall Lemma gives the desired result. □

4 Application of the parabolic prototype in circuit simulation

In this section we discuss a coupled system in circuit simulation where the prototype system (1) from the previous section can be applied. First we briefly summarize the well-known equations of the Modified Nodal Analysis. In MNA the circuit's topology is modeled by a network graph consisting of nodes and branches. The physical behavior is described by the Kirchhoff circuit laws and the characteristic equations for the basic elements, namely resistors, capacitors, inductors, current and voltage sources, see [10, 25]. We only consider elements with two contacts here, i.e. each element is represented by a branch of the network. The resulting system of equations results in a DAE, cf. [13, 30]. More precisely, the MNA equations, cf. [18, 9], are given for any time $t \in \mathcal{I} := [t_0, T]$:

$$A_C \frac{\mathrm{d}}{\mathrm{d}t}(C(t)A_C^\top e(t)) + A_R g_R(A_R^\top e(t), t) + A_L j_L(t) + A_V j_V(t) + A_I i_s(t) = 0 \quad \text{(23a)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(L(t) j_L(t)) - A_L^\top e(t) = 0 \quad \text{(23b)}$$

$$A_V^\top e(t) - v_s(t) = 0 \quad \text{(23c)}$$

The unknowns are the node potentials $e(t) \in \mathbb{R}^{n_e}$ neglecting the mass node, the currents through inductors $j_L(t) \in \mathbb{R}^{n_L}$ and the currents through voltage sources $j_V(t) \in \mathbb{R}^{n_V}$. The matrices $A_X \in \mathbb{R}^{n_e \times n_X}$ for $X \in \{R,C,L,V,I\}$ are the element-related (reduced) incidence matrices having entries from $\{-1,0,1\}$. The input functions i_s and v_s are related to the independent current and voltage sources respectively while the matrices $C(t)$, $L(t)$ and the functions g_R account for the physical relations for charge, flux and conductance.
We make the following assumption:

Assumption 4.1

(i) *$C(t) \in \mathbb{R}^{n_C \times n_C}$ and $L(t) \in \mathbb{R}^{n_L \times n_L}$ are positive definite matrices for all $t \in \mathcal{I}$ which are continuously differentiable in t.*
(ii) *$g_R \in C(\mathbb{R}^{n_R} \times \mathcal{I}, \mathbb{R}^{n_R})$ is strongly monotone and Lipschitz continuous w.r.t. the first argument.*
(iii) *The source terms are continuously differentiable, i.e. $i_s \in C^1(\mathcal{I}, \mathbb{R}^{n_I})$ and $v_s \in C^1(\mathcal{I}, \mathbb{R}^{n_V})$.*

In the classical formulation of the MNA equations (23) heating effects of certain circuit elements are not included. Nevertheless it is well known that resistors, for example, may depend significantly on their temperature. Due to miniaturization in chip design heating effects become ever more important. Accordingly, the influence on the circuit's behavior has to be simulated as well. In [4] a first coupled thermal-electric model was described which adds thermal effects to the circuit by means of an additional 1D heat equation. Furthermore, comprehensive information on various heating models for resistors and diodes is given. This approach has been extended to coupled systems involving semiconductors, cf. [7], and 2D/3D heat diffusion effects, cf. [11]. In [22] this is put into a weak formulation suitable to apply the parabolic

prototype. We only give here a short overview of the coupled model and refer for more details to [22, 11, 4].
Here some thermally active elements are added to the common circuit elements. The temperature distribution is described by a heat diffusion equation in the physical region Ω containing all K thermal elements. So thermal interaction between the elements is possible. The temperature of every element is influenced by an applied potential via a power term.
The thermally active elements are inserted into the electrical network by a corresponding incidence matrix

$$A_T \in \mathbb{R}^{n_e \times K}, \quad A_T = \big(A_{T,1}, \ldots, A_{T,K}\big)$$

which is defined by

$$(A_T)_{ij} := \begin{cases} 1, & \text{if the branch of the thermally active element } j \text{ leaves node } i, \\ -1, & \text{if the branch of the thermally active element } j \text{ enters node } i, \\ 0, & \text{else.} \end{cases}$$

In addition we describe all the branch currents of the thermally active elements by

$$j_T(t) = g_T(A_T^\top e(t), t, \theta(t))$$

with a function $g_T : \mathbb{R}^K \times \boldsymbol{\mathcal{I}} \times \mathbb{R}^K \to \mathbb{R}^K$. The vectors $\theta(t) \in \mathbb{R}^K$ represent the junction temperature collected for all elements. So the temperature has an influence on the conductance of the thermal element. Finally we consider the following system

$$A_C \frac{\mathrm{d}}{\mathrm{d}t}(C(t)A_C^\top e) + A_{\boldsymbol{\mathcal{R}}} g_{\boldsymbol{\mathcal{R}}}(A_{\boldsymbol{\mathcal{R}}}^\top e, t, \boldsymbol{\mathcal{K}}(u)) + A_L j_L + A_V j_V + A_I i_s(t) = 0 \tag{24a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(L(t) j_L) - A_L^\top e = 0 \tag{24b}$$

$$A_V^\top e - v_s(t) = 0 \tag{24c}$$

$$u' + \boldsymbol{\mathcal{B}} u + \boldsymbol{\mathcal{W}}(u, A_T^\top e) = 0 \tag{24d}$$

with

$$A_{\boldsymbol{\mathcal{R}}} = (A_R \quad A_T), \quad g_{\boldsymbol{\mathcal{R}}}(A_{\boldsymbol{\mathcal{R}}}^\top e(t), t, \boldsymbol{\mathcal{K}}(u(t))) = \begin{pmatrix} g_R(A_R^\top e(t), t) \\ g_T(A_T^\top e(t), t, \boldsymbol{\mathcal{K}}(u(t))) \end{pmatrix}.$$

The time dependency of the variables in equation (24) is dropped for a shorter notation. This system is already in variational formulation with the spaces $V := H_0^1(\Omega)$ and $H := L_2(\Omega)$. Hence $V^* = H^{-1}(\Omega)$ and $V \subseteq H \subseteq V^*$ forms an evolution triple. The infinite dimensional equation (24d) is formulated in V^* with the variable $u(t) : \Omega \to \mathbb{R}$ being the homogenized temperature. We let the operators $\boldsymbol{\mathcal{K}} : H \to \mathbb{R}^K$, $\boldsymbol{\mathcal{B}} : V \to V^*$ and $\boldsymbol{\mathcal{W}} : V \times \mathbb{R}^K \to V^*$ fulfill the following assumption:

Assumption 4.2
The operators $\mathcal{K} : H \to \mathbb{R}^K$, $\mathcal{B} : V \to V^*$ and $\mathcal{W} : V \times \mathbb{R}^K \to V^*$ and the function g_T satisfy

(i) $\mathcal{K}$ *is Lipschitz continuous,*
(ii) $\mathcal{B}$ *is linear, strongly monotone and bounded,*
(iii) $\mathcal{W} \in C(V \times \mathbb{R}^K, V^*)$ *is monotone on* V*, Lipschitz continuous on* $\mathbb{R}^K$ *and* $\mathcal{W}(u,0) = 0$ *for all* $u \in V$*,*
(iv) $g_T \in C(\mathbb{R}^K \times \mathcal{I} \times \mathbb{R}^K, \mathbb{R}^K)$ *is Lipschitz continuous w.r.t. to the first and third component.*

We remark here that although the conditions above are assumed for the operators and functions they can be matched by relevant underlying power and resistance terms, cf. [22]. System (24) has to be supplemented by appropriate initial conditions which we will discuss later. In the following we are going to apply the solvability result (Theorem 3.6) to system (24) and remark additional assumptions such that the perturbation result (Theorem 3.8) can also be applied to system (24). To apply Theorem 3.6 to the system (24) it has to be decoupled and therefore we need the following topological assumption:

Assumption 4.3
We assume that the nodes of every thermal element are connected by a path consisting only of capacitors and voltage sources.

We introduce a splitting method with the help of basis functions which will be used for the decoupling, cf. [19]. Let $M \in \mathbb{R}^{m\times n}$ be a matrix and let B_q be a basis of the kernel of M. Let B_p be an extension of B_q such that $\begin{pmatrix} B_\mathrm{q} & B_\mathrm{p} \end{pmatrix}$ is a basis of $\mathbb{R}^n$. Let $n_y := \dim(\ker M)$ and $n_x := n - n_y$ be the dimensions of the subspaces and let $\mathrm{q} \in \mathbb{R}^{n\times n_y}, \mathrm{p} \in \mathbb{R}^{n\times n_x}$ be matrices with the basis vectors of B_q and B_p as columns, respectively. Let $z \in \mathbb{R}^n$ then $\begin{pmatrix} \mathrm{p} & \mathrm{q} \end{pmatrix}$ induces a coordinate transformation such that

$$z = \begin{pmatrix} \mathrm{p} & \mathrm{q} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{p}x + \mathrm{q}y$$

with $x \in \mathbb{R}^{n_x}$ and $y \in \mathbb{R}^{n_y}$. Notice that

$$Mz = M(\mathrm{p}x + \mathrm{q}y) = M\mathrm{p}x + M\mathrm{q}y = M\mathrm{p}x$$

since the columns of q are elements of the kernel of M. Such a coordinate transformation as well as a multiplication by $\begin{pmatrix} \mathrm{p} & \mathrm{q} \end{pmatrix}^\top$ from the left is an equivalent transformation since we deal with a constant coordinate transformation and a multiplication with a constant non-singular matrix. We apply this basis splitting approach to the incidence matrices.
Let q_V be the basis function associated to the kernel of $A_V^\top$ and let p_V be its extension. Then we call

$$A_{\bar{V}X} := \mathrm{q}_V^\top A_X, \quad X \in \{C, \mathcal{R}, L, I\}$$

the V-reduced incidence matrix of the capacitors, thermal resistors, resistors, inductors or current sources, respectively. Further let q_C be the basis function associated to the kernel of $A_{\bar{V}C}^\top$ and let p_C be its extension. Analogously we call

$$A_{\bar{V}\bar{C}X} := \mathrm{q}_C^\top \mathrm{q}_V^\top A_X, \quad X \in \{R, L, I\}$$

the VC-reduced incidence matrix of the resistors, inductors or current sources, respectively. At last we obtain the basis function q_R associated to the kernel of $A_{\bar{V}\bar{C}R}^\top$ and its extension p_R and denote by

$$A_{\bar{V}\bar{C}\bar{R}X} := \mathrm{q}_R^\top \mathrm{q}_C^\top \mathrm{q}_V^\top A_X, \quad X \in \{L, I\}$$

the VCR-reduced incidence matrix of the inductors or current sources, respectively. Consider an arbitrary electric circuit. Remove all voltage sources and identify all nodes which were connected by voltage sources. We call this new circuit the V-reduced circuit. The V-reduced incidence matrices defined above are the incidence matrices of the V-reduced circuit, if we choose the basis function in a special topological way, cf. [19]. Analogously we can interpret the VC-reduced and the VCR-reduced incidence matrices. This topological interpretation is not necessary for the results in this paper, but it makes the following decoupling approach more demonstrative.

Notice that $\mathrm{q}_C^\top \mathrm{q}_V^\top A_T = 0$ due to Assumption 4.3. Successively we split the potential variable e into

$$\begin{aligned} e &= \mathrm{p}_V e_V + \mathrm{q}_V(\mathrm{p}_C e_C + \mathrm{q}_C(\mathrm{p}_R e_R + \mathrm{q}_R e_L)) \\ &= \mathrm{p}_V e_V + \mathrm{q}_V \mathrm{p}_C e_C + \mathrm{q}_V \mathrm{q}_C \mathrm{p}_R e_R + \mathrm{q}_V \mathrm{q}_C \mathrm{q}_R e_L \end{aligned}$$

with the help of the basis splitting approach. The equations of (24) will also be split successively in the order (24c),(24a),(24b) and (24d).

Equation (24c) provides

$$A_V^T e = v_s(t) \quad \Rightarrow \quad A_V^T \mathrm{p}_V e_V = v_s(t) \quad \Rightarrow \quad e_V = (A_V^T \mathrm{p}_V)^{-1} v_s(t) =: v_s^*(t)$$

and therefore e_V can be written as a known time depending function. Next we split equation (24a) by multiplying $\mathrm{p}_V^\top$, $\mathrm{p}_C^\top \mathrm{q}_V^\top$, $\mathrm{p}_R^\top \mathrm{q}_C^\top \mathrm{q}_V^\top$ and $\mathrm{q}_R^\top \mathrm{q}_C^\top \mathrm{q}_V^\top$ from the left and obtain an explicit description of the currents through the voltage sources

$$j_V = -(\mathrm{p}_V^\top A_V)^{-1} \mathrm{p}_V^\top (A_C(C(t) A_C^\top e)' + A_{\boldsymbol{\mathcal{R}}} g_{\boldsymbol{\mathcal{R}}}(A_{\boldsymbol{\mathcal{R}}}^\top e, t, \boldsymbol{\mathcal{K}}(u)) + A_L j_L + A_I i_s(t))$$

and a system which does not depend on these currents

$$\mathrm{p}_C^\top (A_{\bar{V}C}(C(t) A_C^\top e)' + A_{\bar{V}\boldsymbol{\mathcal{R}}} g_{\boldsymbol{\mathcal{R}}}(A_{\boldsymbol{\mathcal{R}}}^\top e, t, \boldsymbol{\mathcal{K}}(u)) + A_{\bar{V}L} j_L + A_{\bar{V}I} i_s(t)) = 0 \tag{25a}$$

$$\mathrm{p}_R^\top (A_{\bar{V}\bar{C}R} g_R(A_R^\top e, t) + A_{\bar{V}\bar{C}L} j_L + A_{\bar{V}\bar{C}I} i_s(t)) = 0 \tag{25b}$$

$$A_{\bar{V}\bar{C}\bar{R}L} j_L + A_{\bar{V}\bar{C}\bar{R}I} i_s(t) = 0. \tag{25c}$$

Remember at this point that $\mathrm{q}_C^\top \mathrm{q}_V^\top A_T = 0$ due to the topological assumption and notice that the thermal resistors do not appear in equation (25b).
Let q_{LI} be the associated basis functions of the kernel of $A_{\bar{V}\bar{C}\bar{R}L}$ and p_{LI} being its extension then we split the currents through the inductors into

$$j_L = \mathrm{p}_{LI} j_{LI} + \mathrm{q}_{LI} j_{L\bar{I}}.$$

Equation (25c) then provides a explicit formula for j_{LI} by

$$\begin{aligned}
&A_{\bar{V}\bar{C}\bar{R}L} j_L = -A_{\bar{V}\bar{C}\bar{R}I} i_s(t) \\
\Rightarrow &A_{\bar{V}\bar{C}\bar{R}L} \mathrm{p}_{LI} j_{LI} = -A_{\bar{V}\bar{C}\bar{R}I} i_s(t) \\
\Rightarrow &j_{LI} = -(A_{\bar{V}\bar{C}\bar{R}L} \mathrm{p}_{LI})^{-1} A_{\bar{V}\bar{C}\bar{R}I} i_s(t) =: i_s^*(t).
\end{aligned}$$

With the help of $v_s^*(t)$ and $i_s^*(t)$ we define the functions

$$\begin{aligned}
g_{\bar{V}\boldsymbol{\mathcal{R}}}(e_R, e_C, t, \boldsymbol{\mathcal{K}}(u)) &:= g_{\boldsymbol{\mathcal{R}}}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R + A_{\bar{V}\boldsymbol{\mathcal{R}}}^\top \mathrm{p}_C e_C + A_{\boldsymbol{\mathcal{R}}}^\top \mathrm{p}_V v_s^*(t), t, \boldsymbol{\mathcal{K}}(u)) \\
g_{\bar{V}\bar{C}R}(x, e_C, t) &:= g_R(x + A_{\bar{V}R}^\top \mathrm{p}_C e_C + A_R^\top \mathrm{p}_V v_s^*(t), t).
\end{aligned}$$

and insert the variable splitting of the potentials and the current of the inductors into (25a), (25b) and (24b) and obtain

$$\begin{aligned}
\mathrm{p}_C^\top A_{\bar{V}C}(C(t) A_{\bar{V}C}^\top \mathrm{p}_C e_C)' + \mathrm{p}_C^\top A_{\bar{V}\boldsymbol{\mathcal{R}}} g_{\bar{V}\boldsymbol{\mathcal{R}}}(e_R, e_C, t, \boldsymbol{\mathcal{K}}(u)) + \mathrm{p}_C^\top A_{\bar{V}L} \mathrm{q}_{LI} j_{L\bar{I}} + i_C(t) &= 0 \\
\mathrm{p}_R^\top A_{\bar{V}\bar{C}R} g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R, e_C, t) + \mathrm{p}_R^\top A_{\bar{V}\bar{C}L} \mathrm{q}_{LI} j_{L\bar{I}} + i_R(t) &= 0 \\
(L(t) \mathrm{q}_{LI} j_{L\bar{I}})' - A_{\bar{V}L}^\top \mathrm{p}_C e_C - A_{\bar{V}\bar{C}L}^\top \mathrm{p}_R e_R - A_{\bar{V}\bar{C}\bar{R}L}^\top e_L - A_L^\top \mathrm{p}_V v_s^*(t) + (L(t) \mathrm{p}_{LI} i_s^*(t))' &= 0
\end{aligned}$$

with

$$\begin{aligned}
i_C(t) &:= \mathrm{p}_C^\top A_{\bar{V}C}(C(t) A_C^\top \mathrm{p}_V v_s^*(t))' + \mathrm{p}_C^\top A_{\bar{V}I} i_s(t) + \mathrm{p}_C^\top A_{\bar{V}L} \mathrm{p}_{LI} i_s^*(t) \\
i_R(t) &:= \mathrm{p}_R^\top A_{\bar{V}\bar{C}I} i_s(t) + \mathrm{p}_R^\top A_{\bar{V}\bar{C}L} \mathrm{p}_{LI} i_s^*(t).
\end{aligned}$$

Next we split equation (24b) by multiplying $\mathrm{p}_{LI}^\top, \mathrm{q}_{LI}^\top$ from the left and obtain a reduced system in $e_C, e_R, j_{L\bar{I}}$:

$$\begin{aligned}
\mathrm{p}_C^\top A_{\bar{V}C}(C(t) A_{\bar{V}C}^\top \mathrm{p}_C e_C)' + \mathrm{p}_C^\top A_{\bar{V}\boldsymbol{\mathcal{R}}} g_{\bar{V}\boldsymbol{\mathcal{R}}}(e_R, e_C, t, \boldsymbol{\mathcal{K}}(u)) + \mathrm{p}_C^\top A_{\bar{V}L} \mathrm{q}_{LI} j_{L\bar{I}} + i_C(t) &= 0 \\
\mathrm{q}_{LI}^\top (L(t) \mathrm{q}_{LI} j_{L\bar{I}})' - \mathrm{q}_{LI}^\top A_{\bar{V}L}^\top \mathrm{p}_C e_C - \mathrm{q}_{LI}^\top A_{\bar{V}\bar{C}L}^\top \mathrm{p}_R e_R + v_L(t) &= 0 \\
\mathrm{p}_R^\top A_{\bar{V}\bar{C}R} g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R, e_C, t) + \mathrm{p}_R^\top A_{\bar{V}\bar{C}L} \mathrm{q}_{LI} j_{L\bar{I}} + i_R(t) &= 0
\end{aligned}$$

with $v_L(t) := -\mathrm{q}_{LI}^\top A_L^\top \mathrm{p}_V v_s^*(t) + \mathrm{q}_{LI}^\top (L(t) \mathrm{p}_{LI} i_s^*(t))'$ and an explicit presentation for

$$e_L = (\mathrm{p}_{LI}^\top A_{\bar{V}\bar{C}\bar{R}L}^\top)^{-1} \mathrm{p}_{LI}^\top ((L(t) j_L)' - A_{\bar{V}L}^\top \mathrm{p}_C e_C - A_{\bar{V}\bar{C}L}^\top \mathrm{p}_R e_R - A_L^\top \mathrm{p}_V v_s^*(t)).$$

We define the variables

$$x := \begin{pmatrix} e_C & j_{L\bar{I}} \end{pmatrix}^\top \in \mathbb{R}^{k_C + k_{L\bar{I}}}, \quad y := \begin{pmatrix} e_R & e_V \end{pmatrix}^\top \in \mathbb{R}^{k_R + k_V}$$

and the functions

$$g(x,y,t) := \begin{pmatrix} \mathrm{p}_R^\top A_{\bar V\bar CR} g_{\bar V\bar CR}(A_{\bar V\bar CR}^\top \mathrm{p}_R e_R, e_C, t) + \mathrm{p}_R^\top A_{\bar V\bar CL}\mathrm{q}_{LI} j_{L\bar I} + i_R(t) \\ e_V - (A_V^T \mathrm{p}_V)^{-1} v_s(t) \end{pmatrix},$$

$$f(x,y,u,t)$$
$$:= \begin{pmatrix} C_{\bar VC}(t)^{-1}(C'_{\bar VC}(t)e_C + \mathrm{p}_C^\top A_{\bar V\boldsymbol{\mathcal{R}}} g_{\bar V\boldsymbol{\mathcal{R}}}(e_R, e_C, t, \boldsymbol{\mathcal{K}}(u)) + \mathrm{p}_C^\top A_{\bar VL}\mathrm{q}_{LI} j_{L\bar I} + i_C(t)) \\ L_{L\bar I}(t)^{-1}(L'_{L\bar I}(t) j_{L\bar I} - \mathrm{q}_{LI}^\top A_{\bar VL}^\top \mathrm{p}_C e_C - \mathrm{q}_{LI}^\top A_{\bar V\bar CL}^\top \mathrm{p}_R e_R + v_L(t)) \end{pmatrix},$$

and

$$\boldsymbol{\mathcal{R}}(u,x,y,t) := \boldsymbol{\mathcal{W}}(u, A_T^\top \mathrm{p}_V e_V + A_{\bar VT}^\top \mathrm{p}_C e_C)$$

with $C_{\bar VC}(t) := \mathrm{p}_C^\top A_{\bar VC} C(t) A_{\bar VC}^\top \mathrm{p}_C$ and $L_{L\bar I}(t) := \mathrm{q}_{LI}^\top L(t)\mathrm{q}_{LI}$. Furthermore define

$$\zeta := \begin{pmatrix} j_V & e_L & j_{LI} \end{pmatrix}^\top \in \mathbb{R}^{n_V + k_L + k_{LI}}$$

and

$$h(x,y,u,t)$$
$$:= \begin{pmatrix} -(\mathrm{p}_V^\top A_V)^{-1}\mathrm{p}_V^\top (A_C(C(t)A_C^\top e)' + A_{\boldsymbol{\mathcal{R}}} g_{\boldsymbol{\mathcal{R}}}(A_{\boldsymbol{\mathcal{R}}}^\top e, t, \boldsymbol{\mathcal{K}}(u)) + A_L j_L + A_I i_s(t)) \\ (\mathrm{p}_{LI}^\top A_{\bar V\bar C\bar RL}^\top)^{-1}\mathrm{p}_{LI}^\top((L(t)j_L)' - A_{\bar VL}^\top \mathrm{p}_C e_C - A_{\bar V\bar CL}^\top \mathrm{p}_R e_R - A_L^\top \mathrm{p}_V e_V) \\ -(A_{\bar V\bar C\bar RL}\mathrm{p}_{LI})^{-1} A_{\bar V\bar C\bar RI} i_s(t) \end{pmatrix},$$

such that (24) can be written equivalently as

$$x' + f(x,y,u,t) = 0 \tag{26a}$$
$$g(x,y,t) = 0 \tag{26b}$$
$$u' + \boldsymbol{\mathcal{B}}u + \boldsymbol{\mathcal{R}}(u,x,y,t) = 0. \tag{26c}$$

and

$$\zeta = h(x',x,y,u,t) \tag{27}$$

With the help of this decoupling procedure we will show the unique solvability of the coupled system (24).

Theorem 4.4 (Unique solvability of (24)**)**
Let be $\boldsymbol{\mathcal{I}} := [t_0, T]$ and $V := H_0^1(\Omega)$, $H := L_2(\Omega)$ with Ω being defined as before. Let Assumptions 4.1, 4.2 and 4.3 be fulfilled. Furthermore we assume

(i) The initial values $u(t_0) = u_0 \in H$, $(e_C(t_0), j_{L\bar I}(t_0)) = (e_{C0}, j_{L\bar I0})$ are given.
(ii) Let $\{v_1, v_2, \dots\}$ be a basis of V, set $V_n := \{v_1, \dots, v_n\}$ and let there be a sequence $(u_{n0}) \subseteq V$ with $u_{n0} \in V_n$ and $u_{n0} \to u_0$ in H as $n \to \infty$.

with $e = \mathrm{p}_V e_V + \mathrm{q}_V \mathrm{p}_C e_C + \mathrm{q}_V \mathrm{q}_C \mathrm{p}_R e_R + \mathrm{q}_V \mathrm{q}_C \mathrm{q}_R e_L$ and $j_L = \mathrm{p}_{LI} j_{LI} + \mathrm{q}_{LI} j_{L\bar I}$. Then the system (24) has a unique solution $(e, j_L, j_V, u) \in C(\boldsymbol{\mathcal{I}}, \mathbb{R}^{n_e + n_L + n_V} \times H)$ with $(e_V, e_C, j_L) \in C^1(\boldsymbol{\mathcal{I}}, \mathbb{R}^{k_V + k_C + n_L})$ and $u \in W_2^1(\boldsymbol{\mathcal{I}}; V, H)$. Furthermore we have for the *solution* $(e_n,(j_L)_n,(j_V)_n,u_n)$ *of the corresponding Galerkin equations that*

$$\begin{aligned} \|e_n-e\|_\infty+\|(j_L)_n-j_L\|_\infty+\|(j_V)_n-j_V\|_\infty &\to 0 \\ \max_{t\in\mathcal{I}}\|u_n(t)-u(t)\|_H &\to 0 \\ \|u_n-u\|_{L_2(\mathcal{I},V)} &\to 0 \end{aligned}$$

as $n\to\infty$.

PROOF:
System (24) can be split into (26) and (27). In the following we will show the unique solvability of the system (26) with the help of Theorem 3.6 which will directly lead to the unique existence of the variable part ζ due to the explicit continuous description in equation (27). Together this provides the unique solvability of the original system (24). Analogously we obtain the convergence of the Galerkin equations, since we can describe the variable ζ by

$$\zeta=h(x',x,y,u,t)=h(-f(x,y,u,t),x,y,u,t)=:h_\zeta(x,y,u,t)$$

with $h_\zeta:\mathbb{R}^{k_C+k_{L\bar{I}}}\times\mathbb{R}^{k_R+k_V}\times H\to\mathbb{R}^{n_V+k_L+k_{LI}}$ continuous.
Therefore consider (26)

$$\begin{aligned} x'+f(x,y,u,t) &= 0 \\ g(x,y,t) &= 0 \\ u'+\mathcal{B}u+\mathcal{R}(u,x,y,t) &= 0. \end{aligned}$$

In order to apply Theorem 3.6 we have to check Assumption 3.1. Here (i), (ii), (v) and (vii) are obvious.
(iii) As a combination of continuous functions the function f is also continuous. For the Lipschitz continuity of f w.r.t. x,y and u we remark that the time dependent matrices $C_{\bar{V}C}(t)$ and $L_{L\bar{I}}(t)$ are continuously differentiable hence $C_{\bar{V}C}(t)^{-1}$, $L_{L\bar{I}}(t)^{-1}$, $C'_{\bar{V}C}(t)$ and $L'_{L\bar{I}}(t)$ are continuous and bounded since $\mathcal{I}$ is compact. Furthermore Assumption 4.1 provides the Lipschitz continuity of g_R w.r.t. to the first and third component while Assumption 4.1 provides the Lipschitz continuity of g_T w.r.t. to the first and third component hence f is Lipschitz continuous as a composition of Lipschitz continuous function.
(iv) g is Lipschitz continuous w.r.t. to x as a composition of Lipschitz continuous functions. To show that $g(x,y,t)$ is strongly monotone w.r.t. y we have to show that

$$r(e_R,e_C,j_{L\bar{I}},t):=\mathrm{p}_R^\top A_{\bar{V}\bar{C}R}g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top\mathrm{p}_Re_R,e_C,t)+\mathrm{p}_R^\top A_{\bar{V}\bar{C}L}\mathrm{q}_{LI}j_{L\bar{I}}+i_R(t)$$

is strongly monotone w.r.t. e_R, therefore let $e_R,\bar{e_R}\in\mathbb{R}^{k_R},(e_C,j_{L\bar{I}},t)\in\mathbb{R}^{k_C+k_{L\bar{I}}}\times\mathcal{I}$:

$$
\begin{aligned}
&(r(e_R,e_C,j_{L\bar{I}},t)-r(\bar{e_R},e_C,j_{L\bar{I}},t)\,|\,e_R-\bar{e_R})\\
=&(\mathrm{p}_R^\top A_{\bar{V}\bar{C}R}g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R,e_C,t)-\mathrm{p}_R^\top A_{\bar{V}\bar{C}R}g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R\bar{e_R},e_C,t)\,|\,e_R-\bar{e_R})\\
=&(g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R,e_C,t)-g_{\bar{V}\bar{C}R}(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R\bar{e_R},e_C,t)\,|\,A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R-A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R\bar{e_R})\\
=&(g_R(y+A_{\bar{V}R}^\top \mathrm{p}_C e_C+A_R^\top \mathrm{p}_V v_s^*(t),t)-g_R(\bar{y}+A_{\bar{V}R}^\top \mathrm{p}_C e_C+A_R^\top \mathrm{p}_V v_s^*(t),t)\,|\,y-\bar{y})\\
\geq&\mu_R\,\|y-\bar{y}\|^2\\
=&\mu_R\left\|A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R(e_R-\bar{e_R})\right\|^2\\
\geq&\mu_R\left\|(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R)^+\right\|_*^{-1}\|(e_R-\bar{e_R})\|^2\\
=&\mu_r\,\|(e_R-\bar{e_R})\|^2
\end{aligned}
$$

with $y:=A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R e_R$, $\bar{y}:=A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R\bar{e_R}$ and $\mu_r:=\mu_R\left\|(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R)^+\right\|_*^{-1}$. Here $(A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R)^+$ is the Moore-Penrose inverse of $A_{\bar{V}\bar{C}R}^\top \mathrm{p}_R$, cf. [6].

$$
\mathcal{R}(u,x,y,t):=\mathcal{W}(u,A_T^\top \mathrm{p}_V e_V+A_{\bar{V}T}^\top \mathrm{p}_C e_C)
$$

(vi) The continuity of $\mathcal{R}$ is obvious. $\mathcal{R}(u,x,y,t)$ is Lipschitz continuous w.r.t. x and y since $\mathcal{W}$ is Lipschitz continuous w.r.t. the second argument and $A_T^\top \mathrm{p}_V$ and $A_{\bar{V}T}^\top \mathrm{p}_C$ are constant matrices. The boundedness follows because

$$
\mathcal{R}(u(t),0,0,t)=\mathcal{W}(u(t),0)=0.
$$

So Theorem 3.6 is applicable and gives the desired result for the decoupled system and hence also for the system (24). □

Finally we remark that Theorem 3.8 can be applied to system (24) under two additional assumption. First we have to assume that the index 1 conditions for an electric circuit are fulfilled, cf. [31]. Secondly we change Assumption 4.3 to

Assumption 4.5
We assume that the nodes of every thermal element are connected by a path consisting only of capacitors.

Under these conditions we obtain an inequality (22) for the variables of the coupled system (24), cf. [22].

5 Outlook and Conclusion

In this paper we developed a parabolic ADAE prototype. For this prototype we showed unique solvability, a Perturbation Index estimate and the convergence of its Galerkin equations. With the help of this prototype we proved the unique solvability

and the Galerkin convergence of an Index 2 circuit including thermal resistors under the topological Assumption 4.3.
The Perturbation Index estimate was shown in [22] for electric circuits under the topological Index 1 conditions. Therefore the next step would be to also generalize this result to Index 2 circuits. At last we mention three long term objects: First the influence of the infinite variable u on the algebraic equations should be investigated, see Remark 1. Secondly more prototypes should be developed. In [22] a elliptic prototype can be found. In the introduction it was mentioned that electromagnetic devices, gas networks and water networks are important application fields for ADAEs and since these system are hyperbolic, a hyperbolic prototype should be developed. Thirdly the Galerkin convergence can be enhanced on the one hand by proving a convergence order and on the other hand by combing it with the convergence prove of a time discretization.

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