Global Unique Solvability for a Quasi-Stationary Water Network Model

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Abstract This paper analyzes a water pipe network model. In contrast to works existing so far, the underlying model equation for the pipe flow is not stationary but quasi-stationary. In exchange, the model is kept simple in terms of considered control devices. The model gives rise to a differential-algebraic equation (DAE), for which an index analysis, a decoupling and a proof of global unique solvability is established. Two important concepts to analyze DAEs are the Tractability Index [9] and the Strangeness Index [8]. In this paper, we make use of the mixed Tractability-Strangeness Index (TSI), introduced in greater detail in [6, 7]. It allows for a topological decoupling of the model DAE.

1 Introduction

As the permanent availability of clean water is of high importance for human societies, water distribution systems have to be run reliably. The circulation of a fair amount of water at any time and a certain pressure at extraction points have to be assured. At the same time, energetic costs caused by pumps and other control devices in the system should be minimized. A good modeling and simulation of the dynamics in a water distribution system is indispensable for an efficient control of it.

Hydraulic dynamics in a water pipe can be modeled by means of the equation of motion and the continuity equation. Depending on the degree of dynamics occurring in the pipe, it is appropriate to use different model levels. This work focuses on a

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quasi-stationary model level, which is based on [11]. In this quasi-stationary model level, the system of equations arising in the network model is a differential-algebraic equation (DAE).

For an index analysis and a decoupling of the DAE, we make use of the topology of the network. This topological approach, which is presented in greater detail in [6, 7], additionally permits a considerable reduction of the nonlinear equations to solve. Furthermore, we prove that there is a unique solution of the model DAE over an arbitrary compact time interval.

The structure of this paper is as follows. In the section 2, we introduce the topological approach for the numerical analysis of DAEs. The network model and the arising DAE is established in section 3. Section 4 presents theoretical graph preparations, which will be used in section 5. The index analysis, the decoupling and the proof of the global unique solvability of the model DAE are presented in section 5.

Numerous works have already investigated water network analysis problems. We name only few of them here. [13] and [15] have introduced the Global Gradient Algorithm (GGA). The GGA is the hydraulic algorithm for EPANET, a public-domain water distribution system modeling software which has become the standard in water distribution network analysis. More recent works can be found in [4, 12, 14]. While [12] investigates topological index conditions, in [14] an extension of the Global Gradient Algorithm is developed and [4] deals with numerical convergence issues. These three papers are based on the network model of [1, 2], which consists of tanks, pumps, valves and a static pipe model. In contrast to this network model we only deal with pipes, but the pipe model is not static.

2 DAEs and the Tractability-Strangeness Index

As we will see in the next section, the presented network model gives rise to a DAE. Therefore this section provides definitions and theorems required for DAE analysis.

Definition 1 *(Semi-Linear DAE with Constant Leading Coefficients)*
Let $\mathscr{I} \subset \mathbb{R}$ be a compact interval and let $\mathscr{D} \subset \mathbb{R}^n$ be open and connected. A semi-linear differential-algebraic equation (DAE) with constant leading coefficients is an equation of the form

$$A(Dz(t))' + b(z(t),t) = 0 \tag{1}$$

with $t \in \mathscr{I}$, $A \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$ and $b \in C^1(\mathscr{D} \times \mathscr{I})$.

We want the matrices A and D to match each other, i.e. the product AD should not cut off information neither from A nor from D. Therefore, we define properly formulated DAEs.

Definition 2 *(Properly Formulated DAE)*
A DAE (1) *is properly formulated, if*

$$\mathit{ker}\, A \oplus \mathit{im}\, D = \mathbb{R}^m.$$

With the next definition we want to fix a splitting of $\mathbb{R}^n$ and $\mathbb{R}^m$ with respect to a matrix function $M \in C(\mathscr{D} \times \mathscr{I}, \mathbb{R}^{m \times n})$. Therefore define the direct difference. Let V be a vector space and W be a subspace of V. Then $U := V \ominus W$ is a subspace of V such that $U \oplus W = V$.

Definition 3 *(Basis Functions)*
Let $\mathscr{I} \subset \mathbb{R}$ be a compact interval and $\mathscr{D} \subset \mathbb{R}^n$ be open and connected. Let $M \in C(\mathscr{D} \times \mathscr{I}, \mathbb{R}^{m \times n})$ be a matrix function. Define integers n_x, n_y, m_v and $m_w \in \mathbb{N}$ such that

$$n_y = \dim(\mathit{ker}\, M(z,t)), \qquad n_x = \dim \mathbb{R}^n - n_y,$$
$$m_w = \dim(\mathit{ker}\, M^T(z,t)), \quad m_v = \dim \mathbb{R}^m - m_w.$$

Choose four matrix functions

$$P : \mathscr{D} \times \mathscr{I} \to \mathbb{R}^{n \times n_x}, \quad Q : \mathscr{D} \times \mathscr{I} \to \mathbb{R}^{n \times n_y},$$
$$V : \mathscr{D} \times \mathscr{I} \to \mathbb{R}^{m \times m_v}, \quad W : \mathscr{D} \times \mathscr{I} \to \mathbb{R}^{m \times m_w}$$

such that the columns of $P(z,t)$ are a basis of a space $\mathbb{R}^n \ominus \mathit{ker}\, M(z,t)$, the columns of $Q(z,t)$ are a basis of $\mathit{ker}\, M(z,t)$, the columns of $V(z,t)$ are a basis of a space $\mathbb{R}^m \ominus \mathit{ker}\, M^T(z,t)$ and the columns of $W(z,t)$ are a basis of $\mathit{ker}\, M^T(z,t)$ $\forall (z,t) \in \mathscr{D} \times I$.
We refer to P as the complementary kernel function of M, to Q as the kernel function of M, to V as the complementary transposed kernel function and to W as the transposed kernel function. The four matrix functions together are called the associated basis functions of M.

Remark 2.1
If M is a constant function, i.e. $M(z,t) \equiv M$, the associated basis functions are constant as well and we refer to them as the associated basis matrices. Furthermore it holds $n_x + n_y = n$, $m_v + m_w = m$ and $n_x = m_v$ in general.

For the analysis of DAEs, several index concepts can be used. Two important ones among them are the Tractability Index [9] and the Strangeness Index [8]. For our purposes, it is convenient to make use of a mixed index concept the index of which is called Tractability-Strangeness Index (TSI), see [6, 7]. In the following, we present the construction of the matrix chain of the TSI up to index 2. We denote the Jacobian of the non-linear function $b(z,t)$ in equation (1) by

$$B(z,t) := \frac{\partial}{\partial z} b(z,t)$$

and define

$$G_0 := AD.$$

Let $P,\ Q,\ V$ and W be the associated basis matrix functions of G_0, see Definition 3. Since G_0 is constant, $P,\ Q,\ V$ and W are also constant. We denote

$$G_1 := V^T ADP, \quad B^V_{x_1}(z,t) := V^T B(z,t)P, \quad B^V_{y_1}(z,t) := V^T B(z,t)Q,$$

$$B^W_{x_1}(z,t) := W^T B(z,t)P, \quad B^W_{y_1}(z,t) := W^T B(z,t)Q.$$

For the next step, let $Q_{y_1}(z,t)$ and $W_{y_1}(z,t)$ be the kernel basis function and the transposed kernel function of $B^W_{y_1}(z,t)$, let $Q_{x_1}(z,t)$ be the kernel basis function of $W^T_{y_1}(z,t)B^W_{x_1}(z,t)$ and let $W_{x_1}(z,t)$ be the transposed kernel function of $G_1 Q_{x_1}(z,t)$. We define

$$B^W_{y_2}(z,t) := W^T_{x_1}(z,t)B^V_{y_1}(z,t)Q_{y_1}(z,t)$$

and the characteristic values $r_0,\ r_1$ and r_2 of the DAE (1) such that

$$r_0 := \mathrm{rk}\, AD\ , r_1 := r_0 + \mathrm{rk}\, B^W_{y_1}(z,t),\ r_2 := r_1 + \mathrm{rk}\, B^W_{y_2}(z,t)$$

under the assumption that all defined associated basis functions are continuous and have constant rank. With the help of the matrix chain we define the TSI up to 2.

Definition 4 *(Tractability-Strangeness Index (TSI))*
Let the DAE (1) *be properly formulated and let $\mathscr{G} \subset \mathscr{D} \times \mathscr{I}$ be open and connected. Then the DAE* (1) *is*

(i) regular with TSI 0 *on $\mathscr{G}$, if $r_0 = n$,*
(ii) regular with TSI 1 *on $\mathscr{G}$, if $r_0 < r_1 = n$,*
(iii) regular with TSI 2 *on $\mathscr{G}$, if $r_1 < r_2 = n$*

We still have to show that the TSI is well defined, i.e. the index must be independent of the choice of the basis functions.

Theorem 2.2 *(Rank Independence)*
Let the DAE (1) *be properly formulated and let $\mathscr{G} \subset \mathscr{D} \times \mathscr{I}$ be open and connected. For a given $\mu \in \mathbb{N}$, let exist a basis function sequence associated to the DAE. Then the characteristic values $r_0,\ r_1,\ r_2$ and the TSI itself are independent of the choice of the involved basis functions.*

A proof is given in [7].

3 Quasi-Stationary Network Model

A water pipe network consists of several different elements, such as nodes, pipes, valves, pumps and tanks. In order to reduce the complexity, the model analyzed in the following takes into account just nodes and pipes. All other elements are not regarded here. The nodes are either pressure nodes, where the pressure is set externally and known, or demand nodes, where the demand, i.e. the extraction out of the system, is set externally and known. For the nodes, we establish the mass balance, which is analogous to Kirchhoff's first law in electrical networks, and obtain an algebraic system of equations containing information about the flows, i.e. the condition that the sum of the flows m_i through the pipes incident to node v must be equal to the demand q_s at the node:

$$\sum_{i \in I_{in}} m_i(t) - \sum_{i \in I_{out}} m_i(t) = q_s(t) \tag{2}$$

I_{in} is the set of pipes incident to v and directed towards v, I_{out} is the set of pipes incident to v and directed away from v. In I_{in}, a flow towards node v is considered positive, a flow away from the node is considered negative. In I_{out}, a flow towards node v is considered negative, a flow away from the node is considered positive.
For the pipes, we have the equations of motion to get information about the pressure loss between two nodes. We assign a direction to every pipe and a number to every node and every pipe in the network. Figure 3 shows an example network. The dots represent the nodes, while the lines between the nodes represent pipes. The arrowheads describe the direction of the pipes.

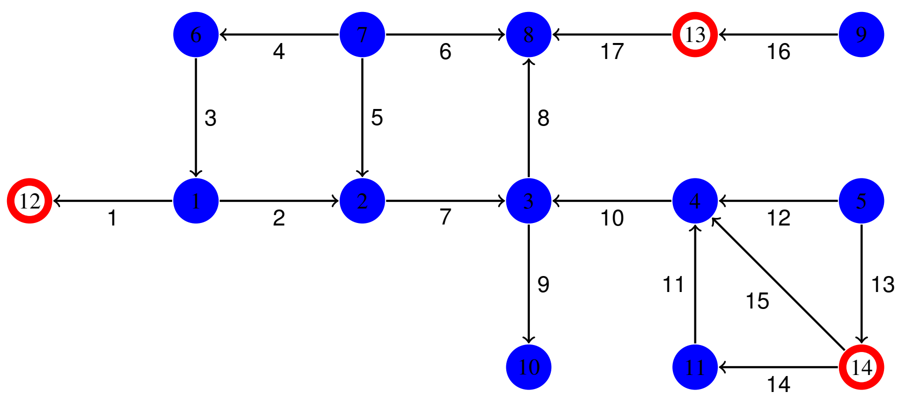

Fig. 1 Example network. The pressure nodes are red and unfilled and the demand nodes are blue and solid.

We assume the water network to be dominated by laminar flows, which allows us to consider the water motion as a one-dimensional flow along the length of the pipes. Furthermore we assume a network with significant time-dependent changes of flow, but without hydraulic shocks. These assumptions cause a pipe model called the *quasi-stationary model level*, which will be presented in the following
We consider an arbitrary water pipe of the network. Let $t \in [t_0, T] =: \mathscr{I}$ denote the time and $x \in [0, L] =: \Omega$ denote the space. We define the functions

$$m : \mathscr{I} \times \Omega \mapsto \mathbb{R}, \; p : \mathscr{I} \times \Omega \mapsto \mathbb{R},$$

with $m(t,x)$ the water flow and $p(t,x)$ the water pressure. The rigid water column model (see [11]) yields the continuity equation

$$\frac{\partial m}{\partial x}(t,x) = 0, \tag{3}$$

i.e. the water is assumed to be incompressible. Thus $m(t,x) = m(t)$. For the equation of motion, we obtain with [11] the PDE

$$\frac{dm}{dt}(t) + A\frac{\partial p}{\partial x}(t,x) + \rho A g sin\alpha + \frac{1}{\rho}\frac{\lambda}{2DA} m(t)\,|m(t)| = 0, \tag{4}$$

with ρ the water density, g the gravity constant and A, α, λ and D constant pipe characteristics. More precisely, for a specific pipe, A denotes the cross-sectional area, α the angle of elevation, λ the Darcy friction factor and D the diameter.
We see that $\frac{\partial p}{\partial x}(t,x)$ is constant with respect to x as the other terms in the equation do not depend on x. As a consequence, the difference quotient $\frac{p(t,x+h)-p(t,x)}{h}$ is analytically exact for any pair (x,h) such that $x,\ (x+h) \in \Omega$. Our aim is to find the pressure at the nodes, formally speaking $p(t,0)$ and $p(t,L)$. Therefore, we set $x=0$ and $h=L$ and obtain

$$\frac{\partial p}{\partial x}(t,0) = \frac{p(t,L)-p(t,0)}{L} \tag{5}$$

Inserting equation (5) into the pipe equation (4) and multiplying by $\frac{L}{A}$ yields the ODE

$$\frac{L}{A}m'(t) + p(t,L) - p(t,0) + L\rho g sin\alpha + \frac{L}{\rho}\frac{\lambda}{2DA^2} m(t)\,|m(t)| = 0,$$

with $m'(t) := \frac{dm}{dt}(t)$. For a compact notation, we define $S := \frac{L}{A}$, $H := -L\rho g sin\alpha$, $c := \frac{L}{\rho}\frac{\lambda}{2DA^2}$ and the function $g : \mathbb{R} \mapsto \mathbb{R},\ g(x) := cx\,|x|$ and we obtain the pipe equation

$$Sm'(t) + p(t,L) - p(t,0) + g(m(t)) = H. \tag{6}$$

At this point we can already see that the node equations (2) together with the pipe equations (6) yield a DAE for a complete network.

3.1 Representing a Water Pipe Network by a Graph

We want to establish the pipe equation (6) and the node equation (2) for the whole network. Therefore, it is convenient to use some common graph theoretical definitions and notation. We define a connected oriented graph $G = (N,E)$ (see [5]) representing the pipe network. The set of n_N nodes is denoted by $N(G) = N =$

$\{v_1, v_2, \ldots, v_{n_N}\}$ and the set of n_E edges is denoted by $E(G) = E = \{e_1, e_2, \ldots, e_{n_E}\}$. A particular edge e_i represents a particular pipe, referred to as the i-th pipe. As we consider oriented graphs, any edge is given an arbitrary but fixed direction. Flows against the edge direction are taken into account negatively, flows with the edge direction are taken into account positively. It is useful to split the node set into disjoint sets N_p and N_q such that $N_p \sqcup N_q = N$, here $\sqcup$ denote the disjoint intersection of two sets. N_p represents the set of pressure nodes and N_q represents the set of demand nodes. We order the nodes as follows: The demand nodes N_q are numbered $v_1, ..., v_{n_{N_q}}$ and the pressure nodes are numbered $v_{n_{N_q}+1}, ..., v_{n_N}$. The graph G can be identified with a (complete) incidence matrix $A^c = A^c(G) \in \mathbb{R}^{n_E \times n_N}$, defined as

$$(A^c)_{i,j} := \begin{cases} 1, \text{ if } v_j \text{ is the end node of } e_i \\ -1, \text{ if } v_j \text{ is the start node of } e_i \\ 0, \text{ else} \end{cases}$$

Based on A^c, we can define the reduced incidence matrix $A_r \in \mathbb{R}^{n_E \times n_{N_q}}$ and the complementary reduced incidence matrix $A_r^p \in \mathbb{R}^{n_E \times n_{N_p}}$ such that

$$\left(A_r \; A_r^p\right) := A^c.$$

The columns of A_r correspond to demand nodes and the columns of A_r^p correspond to pressure nodes.

3.2 Network Model DAE

Considering the whole pipe network, we can establish the pipe equation (6) for every pipe and the node equation (2) for every demand node in the network. That yields a system of $n := n_E + n_{N_q}$ equations.
For the purpose of a compact representation we modify the notation of the previous sections. Henceforth, $m(t) \in \mathbb{R}^{n_E}$, $p(t) \in \mathbb{R}^{n_{N_q}}$, $q_s(t) \in \mathbb{R}^{n_{N_q}}$ and $H \in \mathbb{R}^{n_E}$ denote vectors:

$$m(t) = \begin{pmatrix} \vdots \\ m_i(t) \\ \vdots \end{pmatrix}, \; p(t) = \begin{pmatrix} \vdots \\ p_i(t) \\ \vdots \end{pmatrix}, \; q_s(t) = \begin{pmatrix} \vdots \\ q_{s_i}(t) \\ \vdots \end{pmatrix}, \; H = \begin{pmatrix} \vdots \\ -\rho g L_i sin(\alpha_i) \\ \vdots \end{pmatrix},$$

where m_i denotes the flow through the i-th pipe represented by the edge e_i, p_i denotes the pressure at demand node v_i and q_{s_i} denotes the demand at demand node v_i. Furthermore, L_i denotes the length of the i-th pipe and α_i its angle of elevation.
We define the vector $p_s \in \mathbb{R}^{n_{N_p}}$ such that

$$p_s(t) = \begin{pmatrix} \vdots \\ p_{s_i}(t) \\ \vdots \end{pmatrix}$$

and p_{s_i} is the pressure at pressure node $v_{n_{N_q}+i}$. The function g is henceforth defined as follows:

$$g : \mathbb{R}^{n_E} \to \mathbb{R}^{n_E},\ g(m) := \begin{pmatrix} \vdots \\ c_i m_i |m_i| \\ \vdots \end{pmatrix}$$

where

$$c_i := \frac{L_i}{\rho} \frac{\lambda_i}{2D_i A_i^2} \tag{7}$$

and the subscript i points to pipe i. Finally, $S \in \mathbb{R}^{n_E \times n_E}$ is defined as a diagonal matrix such that $S_{ii} := \frac{L_i}{A_i}$. Putting the node equations and the pipe equations together, we obtain the n-dimensional system of network model equations

$$Sm'(t) + A_r p(t) + g(m(t)) = H - A_r^p p_s(t) \tag{8a}$$

$$A_r^T m(t) = q_s(t) \tag{8b}$$

Note that m and p are unknown and to be found, whereas everything else is known. The system is non-linear due to non-linearity of g. In the stationary model level, the derivative vanishes and we obtain

$$A_r p(t) + g(m(t)) = H - A_r^p p_s(t) \tag{9a}$$

$$A_r^T m(t) = q_s(t). \tag{9b}$$

An analysis of the algebraic equation (9) arising in the stationary model level, notably a proof of the global unique solvability of the equation, can be found in [10]. Defining $z(t) := \begin{pmatrix} m(t) \\ p(t) \end{pmatrix} \in \mathbb{R}^n$ and

$$\begin{aligned}
A &:= \begin{pmatrix} I_{n_E} \\ 0 \end{pmatrix} \in \mathbb{R}^{n \times n_E}, \\
D &:= \begin{pmatrix} S & 0 \end{pmatrix} \in \mathbb{R}^{n_E \times n}, \\
\tilde{B} &:= \begin{pmatrix} 0 & A_r \\ A_r^T & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}, \\
f(z(t),t) &:= \begin{pmatrix} g(m(t)) \\ 0 \end{pmatrix} - \begin{pmatrix} H - A_r^p p_s(t) \\ q_s(t) \end{pmatrix} \in \mathbb{R}^n, \\
b(z(t),t) &:= \tilde{B} z(t) + f(z(t),t) \in \mathbb{R}^n,
\end{aligned}$$

yields the following equation which is equivalent to (8):

$$A(Dz(t))' + b(z(t),t) = 0. \tag{10}$$

This is a semi-linear DAE with constant leading coefficients as in Definition 2. For the index analysis and the decoupling of this DAE, we need the associated basis functions of Definition 3. The network topology will help us to find such basis functions. The next section will provide the essential definitions and results on the network topology.

4 Analysis of the Network Topology

We use the graph theoretical terms *spanning tree, fundamental cycle, subgraph, path and forest*. For a definition of these terms see [5]. In the connected directed graph $G(N,E)$, we choose a spanning tree T. We call an edge $e \in E(G) \setminus E(T)$ a cycle edge and denote the set of n_C cycle edges by $E_{red}(T)$. By construction, it holds

$$E(T) \sqcup E_{red}(T) = E(G).$$

We denote the set of fundamental cycles of T by $C_T = \left\{C_1, \ldots, C_{n_{C_T}}\right\}$ and we assign an orientation to every fundamental cycle $C_i,\ i = 1, \ldots, n_{C_T}$. Next, we define Z as a subgraph of T with $n_{E(T)} - (n_{N_p} - 1)$ edges such that Z does not contain any paths between pressure nodes $v_i \in N_p$, called pressure paths. An edge $e \in E(T) \setminus E(Z)$ is called pressure path edge and the set of n_P pressure path edges is denoted by $E_{red}(Z)$. By construction, it holds

$$E(Z) \sqcup E_{red}(Z) = T.$$

Adding a pressure path edge $e \in E_{red}(Z)$ to Z induces a unique pressure path in Z. We call such an inducible pressure path a fundamental pressure path of Z and denote the set of fundamental paths by $P_Z = \left\{P_1, \ldots, P_{n_{P_Z}}\right\}$. We assign a direction to every fundamental pressure path $P_i,\ i = 1, \ldots, n_{P_Z}$, which yields a start node and an end node for each of them. As $E_{red}(T)$ only contains elements of $E(G) \setminus E(T)$ and $E_{red}(Z)$ only contains elements of $E(T)$, it holds $E_{red}(Z) \cap E_{red}(T) = \emptyset$. We define

$$E_{red}(T,Z) := E_{red} := E_{red}(T) \sqcup E_{red}(Z)$$

In the following, speaking about the sets E and N, we always refer to $E(G)$ and $N(G)$. For any tree T of a graph G it holds

$$n_{N(T)} - 1 = n_{E(T)} = n_E - n_{E_{red}(T)}. \tag{11}$$

Moreover, we know that for any subgraph Z of T defined as above, it holds

$$n_{E_{red}(Z)} = n_{N_p} - 1. \tag{12}$$

We can conclude that

$$n_{(E \setminus E_{red})} = n_E - n_{E_{red}(T)} - n_{E_{red}(Z)} = n_{N(T)} - 1 - (n_{N_p} - 1) = n_{N_q}. \tag{13}$$

For an easier definition of some matrices later on, we renumber the edges $E = (E \setminus E_{red}) \sqcup E_{red}(Z) \sqcup E_{red}(T))$ such that

$$E = \{\underbrace{e_1, \ldots, e_{n_{N_q}}}_{\in E \setminus E_{red}}, \underbrace{e_{n_{N_q}+1}, \ldots, e_{n_{N_q}+n_{P_Z}}}_{\in E_{red}(Z)}, \underbrace{e_{n_{N_q}+n_{P_Z}+1}, \ldots, e_n}_{\in E_{red}(T)}\}$$

Figure 2 shows the example network of Figure 1 with the renumbered edges. Note that the choice of cycle edges $E_{red}(T)$ and pressure path edges $E_{red}(Z)$ is not unique, but the number n_C of cycle edges and the number n_P of pressure path edges are invariant under the choice.

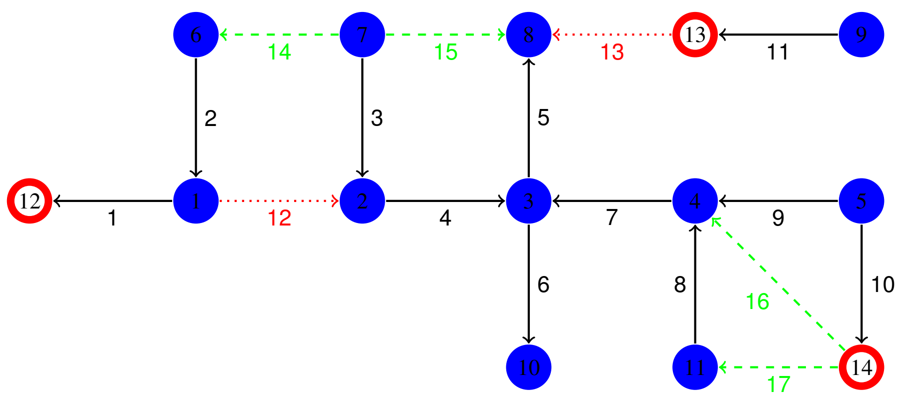

Fig. 2 Renumbered example network. The pressure nodes are red and unfilled and the demand nodes are blue and solid. The cycles edges are green and dashed while the pressure path edges are red and dotted.

Now we can define three important matrices. The matrix $A_t \in \mathbb{R}^{n_{N_q} \times n_E}$ is defined as

$$A_t := (I_{n_{N_q}} \; 0)$$

with $I_{n_{N_q}}$ the identity matrix in $\mathbb{R}^{n_{N_q} \times n_{N_q}}$. The cycle matrix $A_C \in \mathbb{R}^{n_{C_T} \times n_E}$ of a spanning tree T in G is defined as

$$(A_C)_{i,j} := \begin{cases} 1, & \text{if } e_j \text{ is directed as the orientation of cycle } C_i \\ -1, & \text{if } e_j \text{ is directed opposite to the orientation of cycle } C_i \\ 0, & \text{if } e_j \notin E(C_i) \end{cases}$$

The pressure path matrix $A_P \in \mathbb{R}^{n_{P_Z} \times n_E}$ of Z is defined as

$$(A_P)_{i,j} := \begin{cases} 1, \text{ if } e_j \text{ is directed as pressure path } P_i \\ -1, \text{ if } e_j \text{ is directed opposite to pressure path } P_i \\ 0, \text{ if } e_j \notin E(P_i) \end{cases}$$

We define

$$\begin{pmatrix} A_P \\ A_C \end{pmatrix} := A_{PC} \in \mathbb{R}^{(n_{E_{red}} \times n_E)}.$$

The proof of the next lemma can be found in [3].

Lemma 4.1
Let T be a tree with one pressure node. Then, the reduced incidence matrix of T is non-singular.

With the help of Lemma 4.1 we prove the following results.

Lemma 4.2
For the matrices A_t, A_r and A_{PC} hold:

(i) $R := \begin{pmatrix} A_t \\ A_{PC} \end{pmatrix} \in \mathbb{R}^{n_E \times n_E}$ *and* $A_t A_r \in \mathbb{R}^{n_{N_q} \times n_{N_q}}$ *are non-singular.*
(ii) $A_{PC} A_r = 0$.

PROOF:
(i) To show that R is non-singular, we show that it is of the form

$$\begin{array}{cc} & \begin{array}{ccc} E \setminus E_{red} & E_{red}(Z) & E_{red}(T) \end{array} \\ \begin{array}{c} A_t \\ A_P \\ A_C \end{array} & \begin{pmatrix} R_1 & 0 & 0 \\ * & R_2 & 0 \\ * & * & R_3 \end{pmatrix} = R \end{array}$$

with R_i non-singular and $*$ arbitrary matrices. A_t is of the form $(R_1 \ 0 \ 0)$ with $R_1 = I_{n_{N_q}}$ by construction. Coming to A_P, we firstly note that by construction every edge $e_i \in E_{red}(Z)$ is element of exactly one fundamental pressure path P_i, and two different edges $e_i, e_j \in E_{red}(Z)$ can not be element of the same fundamental pressure path. Thus, the columns of R_2 are pairwise independent positive or negative unit vectors and R_2 is non-singular. Furthermore, we remember that

$$E_{red}(T) \cap E(T) = \emptyset$$

by definition of $E_{red}(T)$. Thus, an edge $e_i \in E_{red}(T)$ can not be element of a fundamental pressure path P_i since the fundamental paths P_i are defined as subgraphs of T. The third constituting matrix block of A_P is hence a zero matrix. Analogously to R_2, the matrix R_3 is non-singular because every edge $e_i \in E_{red}(T)$ is element of

exactly one fundamental cycle C_i, and two different edges $e_i, e_j \in E_{red}(T)$ can not be element of the same fundamental cycle.
Now we show the regularity of $A_t A_r$. By construction of A_t, the matrix $A_t A^c \in \mathbb{R}^{n_{N_q} \times n_N}$ is an incidence matrix corresponding to the graph G_{red}, with

$$G_{red} := (N, E_{red})$$

G_{red} is a forest F with n_{N_p} trees by construction of E_{red}. Each of these trees contains exactly one pressure node. We renumber the edges and nodes of G_{red} such that the incidence matrix $\widetilde{A_t A}$ of the renumbered graph is of the form

$$\widetilde{A_t A} = \begin{pmatrix} *^1 & & \\ & \ddots & \\ & & *^k \end{pmatrix}$$

with $*^i$ the incidence matrices of the trees of the forest F. Then, the reduced incidence matrix of the renumbered graph is of the form

$$\widetilde{(A_t A)}_r = \begin{pmatrix} *_r^1 & & \\ & \ddots & \\ & & *_r^k \end{pmatrix},$$

with $*_r^i,\ i = 1, \ldots, k$ the reduced incidence matrices of the trees of the forest F. The matrices $*_r^i,\ i = 1, \ldots, k$ are non-singular with Lemma 4.1. Thus, $\widetilde{(A_t A)}_r$ is non-singular. Since renumbering of a graph involves only changes of rows and columns of the corresponding (reduced) incidence matrix, it does not have an impact singularity or non-singularity of the matrix. Thus $(A_t A)_r = A_t A_r$ is non-singular.
(ii) Firstly, we prove that $A_C A_r$ is a zero matrix. We consider an arbitrary entry

$$(A_C A_r)_{i,j} = \sum_{k=1}^{n_E} (A_C)_{i,k} (A_r)_{k,j}, \text{ with } i \in n_C,\ j \in n_{N_q}$$

and show that it is zero.
$(A_C)_{i,k}$ is non-zero if and only if $e_k \in E(C_i)$. Hence we only have to consider k such that $e_k \in E(C_i)$.
We consider the cases $v_j \notin N(C_i)$ and $v_j \in N(C_i)$ separately.

(a) If $v_j \notin N(C_i)$, then $(A_r)_{k,j} = 0$ because v_j can be neither the start node nor the end node of $e_k \in E(C_i)$.
(b) If $v_j \in N(C_i)$, there are exactly two $k \in C_i$, say k_1 and k_2, such that $(A_r)_{k_1,j}$ and $(A_r)_{k_2,j}$ are non-zero. e_{k_1} and e_{k_2} are the edges of C_i joining in v_j. If the edges e_{k_1} and e_{k_2} are directed the same way in C_i, $(A_C)_{i,k_1}$ and $(A_C)_{i,k_2}$ have the same algebraic sign and $(A_r)_{k_1,j}$ and $(A_r)_{k_2,j}$ have opposite algebraic signs. In this case,

$$(A_C A_r)_{i,j} = (A_C)_{i,k_1}(A_r)_{k_1,j} + (A_C)_{i,k_2}(A_r)_{k_2,j} = 0.$$

If e_{k_1} and e_{k_2} are directed opposite to each other, $(A_C)_{i,k_1}$ and $(A_C)_{i,k_2}$ have opposite algebraic signs and $(A_r)_{k_1,j}$ and $(A_C)_{k_2,j}$ have the same algebraic sign. Again,

$$(A_C A_r)_{i,j} = (A_C)_{i,k_1}(A_r)_{k_1,j} + (A_C)_{i,k_2}(A_r)_{k_2,j} = 0.$$

Finally, we show that $A_P A_r = 0$ is a zero matrix. As in the proof for $A_C A_r$, consider an arbitrary entry

$$(A_P A_r)_{i,j} = \sum_{k=1}^{n_E} (A_P)_{i,k}(A_r)_{k,j} = 0, \text{ with } i \in n_P,\ j \in n_{N_q}.$$

Substituting paths for cycles, we obtain $(A_P A_r)_{i,j} = 0$ for j such that v_j is neither the start node nor the end node of P_i analogously to the proof of $A_C A_r = 0$. But as both the start node and the end node considered here are pressure nodes, the corresponding columns are omitted in A_r and hence we do not have to take them into account. □

We close this section with the following corollary:

Corollary 4.3
The matrix A_r has full column rank.

PROOF:
By construction of A_t, $A_t A_r$ arises from A_r when the last $n_{E_{red}}$ rows are omitted, i.e. the columns of $A_t A_r$ are 'truncated' columns of A_r. $A_t A_r$ has full row rank since it is non-singular with Lemma 4.2 and thus the 'truncated' columns are linearly independent. Then, the 'non-truncated' columns of A_r must be linearly independent as well and $A_t A_r$ has full column rank. □

5 Index Analysis an Global Solvability

This section is split into three parts. First the index of the DAE (10) is analyzed, then we will provide a topologically motivated decoupling procedure and last we will prove the global unique solvability of the (10).
We undertake an index analysis of the DAE (10) using topological matrices for the matrix chain. Firstly, we check if the DAE (10) is properly formulated. The matrix $A = \begin{pmatrix} I_{n_E} & 0 \end{pmatrix}^T \in \mathbb{R}^{n \times n_E}$ has full column rank and thus $\ker A = \{0\}$. The matrix $D = \begin{pmatrix} S & 0 \end{pmatrix} \in \mathbb{R}^{n_E \times n}$ has n_E linearly independent rows since S is a positive definite diagonal matrix and thus $\operatorname{im} D = \mathbb{R}^{n_E}$. Consequently, $\ker A \oplus \operatorname{im} D = \mathbb{R}^{n_E}$ and the DAE (10) is properly formulated. For the index analysis, we need the Jacobian of b with respect to z. That is

$$B(z) := b_z(z) = \begin{pmatrix} \ddots & & & \\ & 2c_i\,|m_i| & & A_r \\ & & \ddots & \\ & A_r^T & & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Throughout this section, we might drop the argument t if that yields a better view.

Theorem 5.1 *(Index Analysis)*
Let the network be connected and let there be at least one pressure node and one demand node in the network. Then the resulting DAE (10) *has Tractability-Strangeness Index* 2*.*

PROOF:
We start with the matrix

$$G_0 = AD = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

$S \in \mathbb{R}^{n_E \times n_E}$ is a positive definite diagonal matrix. Thus,

$$r_0 = \text{rk } G_0 = n_E < n$$

and the TSI is at least 1. By symmetry of G_0, it holds $\ker G_0 = \ker G_0^T$ and we can choose

$$Q = W = \begin{pmatrix} 0 \\ I_{n_{N_q}} \end{pmatrix} \in \mathbb{R}^{n\times n_{N_q}},$$

as kernel matrix and transposed kernel matrix of G_0 and

$$P = V = \begin{pmatrix} I_{n_E} \\ 0 \end{pmatrix} \in \mathbb{R}^{n\times n_E}$$

as complementary kernel matrix and transposed complementary kernel matrix of G_0. For the first sequence of matrices, we obtain

$$\begin{aligned} G_1 &= V^T G_0 P = S, \\ B_{x_1}^V(z) &= V^T B(z) P = g_z(z), \\ B_{y_1}^V &= V^T B(z) Q = A_r, \\ B_{x_1}^W &= W^T B(z) P = A_r^T, \\ B_{y_1}^W &= W^T B(z) Q = 0_{n_{N_q}} \in \mathbb{R}^{n_{N_q}\times n_{N_q}} \end{aligned}$$

Since $\text{rk } B_{y_1}^W = 0$, we obtain

$$r_1 = r_0 + \operatorname{rk} B_{y_1}^W = n_E < n$$

and the TSI is at least 2. To check if it is exactly 2 or bigger, we have to determine $\operatorname{rk} B_{y_2}^W(z)$. Therefore, we need a kernel matrix Q_{y_1} and a transposed kernel matrix W_{y_1} of $B_{y_1}^W = 0_{n_{N_q}}$. We can choose

$$W_{y_1} = Q_{y_1} = I_{n_{N_q}}.$$

It follows

$$W_{y_1}^T B_{x_1}^W = A_r^T.$$

Now we need a kernel matrix Q_{x_1} of $W_{y_1}^T B_{x_1}^W = A_r^T$. With Lemma 4.2(ii), we can choose

$$Q_{x_1} = A_{PC}^T \in \mathbb{R}^{n_E \times n_{E_{red}}}.$$

Finally, W_{x_1} is a transposed kernel matrix of $G_1 Q_{x_1} = SA_{PC}^T$. Without determining W_{x_1} explicitly, we show that $W_{x_1}^T A_r$ is non-singular. Firstly, we note that

$$\begin{pmatrix} A_r^T \\ A_{PC} \end{pmatrix} \begin{pmatrix} SA_{PC}^T & A_r \end{pmatrix} = \begin{pmatrix} * & A_r^T A_r \\ A_{PC} SA_{PC}^T & 0 \end{pmatrix}$$

is a non-singular matrix since A_{PC} and A_r^T have full row rank with Lemma 4.2(i) and Corollary 4.3 and thus $A_{PC}SA_{PC}^T$ and $A_r^T A_r$ are positive definite and therewith non-singular. It follows that $\begin{pmatrix} A_r & A_{PC}^T \end{pmatrix}^T$ and $\begin{pmatrix} SA_{PC}^T & A_r \end{pmatrix}$ are both non-singular since they are quadratic. Without determining the transposed complementary kernel matrix V_{x_1} of $G_1 Q_{x_1} = SA_{PC}^T$, we know by construction that

$$\begin{pmatrix} V_{x_1}^T \\ W_{x_1}^T \end{pmatrix} \in \mathbb{R}^{n_E \times n_E}$$

is non-singular. Thus,

$$\begin{pmatrix} V_{x_1}^T \\ W_{x_1}^T \end{pmatrix} \begin{pmatrix} SA_{PC}^T & A_r \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & W_{x_1}^T A_r \end{pmatrix}$$

is non-singular as the product of non-singular matrices. That implies that $W_{x_1}^T A_r \in \mathbb{R}^{n_{N_q} \times n_{N_q}}$ is non-singular and

$$\operatorname{rk} W_{x_1}^T A_r = n_{N_q}.$$

By definition,

$$B_{y_2}^W(z) = W_{x_1}^T(z) B_{y_1}^V Q_{y_1}(z) = W_{x_1}^T(z) A_r$$

and thus $\operatorname{rk} B^W_{y_2} = n_{N_q}$. Consequently,

$$r_2 = r_1 + \operatorname{rk} B^W_{y_2} = r_1 + \operatorname{rk} A_r = n_E + n_{N_q} = n$$

and the TSI is 2. $\square$

Next we presents a decoupling of the model DAE (8). We consider the matrix $\begin{pmatrix} A_t^T & A_{PC}^T \end{pmatrix} \in \mathbb{R}^{n_E \times n_E}$, which consists of the columns of the complementary kernel matrix $A_t^T \in \mathbb{R}^{n_E \times n_{N_q}}$ and the kernel matrix $A_{PC}^T \in \mathbb{R}^{n_E \times n_{E_{red}}}$ of $A_r^T \in \mathbb{R}^{n_{N_q} \times n_E}$. $\begin{pmatrix} A_t^T & A_{PC}^T \end{pmatrix}$ is non-singular with Lemma 4.2(i). Thus we transform the coordinates:

$$m = \begin{pmatrix} A_t^T & A_{PC}^T \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = A_t^T m_1 + A_{PC}^T m_2 \tag{14}$$

with $m_1 \in \mathbb{R}^{n_{N_q}}$ and $m_2 \in \mathbb{R}^{n_{E_{red}}}$. Note that $m \neq \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ in general.

Lemma 5.2 *(Decoupling)*
Let the network be connected and let there be at least one pressure node and one demand node in the network. With the coordinate transformation (14) *the DAE* (10) *can be transformed into the following form:*

$$m_1 = (A_r^T A_t^T)^{-1} q_s \tag{15a}$$

$$m_2' = (A_{PC} S A_{PC}^T)^{-1} (A_{PC}(r - S A_t^T m_1' - g(A_t^T m_1 + A_{PC}^T m_2))) \tag{15b}$$

$$p = (A_t A_r)^{-1} (A_t (r - S(A_t^T m_1 + A_{PC}^T m_2)' - g(A_t^T m_1 + A_{PC}^T m_2))) \tag{15c}$$

PROOF:
Inserting (14) into equation (8b) yields

$$A_r^T m = A_r^T (A_t^T m_1 + A_{PC}^T m_2) = A_r^T A_t^T m_1 = q_s$$

as $A_r^T A_{PC}^T = 0$ with Lemma 4.2(ii).
$A_r^T A_t^T \in \mathbb{R}^{n_{N_q} \times n_{N_q}}$ is a non-singular matrix with Lemma 4.2(i) and thus we obtain m_1 by solving a system of linear equations. In the next step, we multiply $\begin{pmatrix} A_t \\ A_{PC} \end{pmatrix}$ from the left to equation (8a) and insert the transformed coordinates (14):

$$\begin{aligned}
& S m' + A_r p + g(m) = H - A_r^p p_s \\
\Leftrightarrow & \begin{cases} A_t S(A_t^T m_1 + A_{PC}^T m_2)' + A_t A_r p + A_t g(A_t^T m_1 + A_{PC}^T m_2) = A_t (H - A_r^p p_s) \\ A_{PC} S(A_t^T m_1 + A_{PC}^T m_2)' + A_{PC} A_r p + A_{PC} g(A_t^T m_1 + A_{PC}^T m_2) = A_{PC}(H - A_r^p p_s) \end{cases} \\
\Leftrightarrow & \begin{cases} (A_t A_r) p & = A_t (H - A_r^p p_s - S(A_t^T m_1 + A_{PC}^T m_2)' - g(A_t^T m_1 + A_{PC}^T m_2)) \\ (A_{PC} S A_{PC}^T) m_2' & = A_{PC}(H - A_r^p p_s - S A_t^T m_1' - g(A_t^T m_1 + A_{PC}^T m_2)) \end{cases}
\end{aligned}$$

For a convenient notation, we define $r := H - A_r^p p_s$. The matrix $A_t A_r \in \mathbb{R}^{n_{N_q} \times n_{N_q}}$ is non-singular with Lemma 4.2(i). Since S is a positive definite diagonal matrix and A_{PC} has full column rank with Lemma 4.2(i), $A_{PC} S A_{PC}^T \in \mathbb{R}^{n_{E_{red}} \times n_{E_{red}}}$ is non-singular, too. That means we can solve the DAE (10) by solving the equations in the following order:

$$
\begin{aligned}
m_1 &= (A_r^T A_t^T)^{-1} q_s \\
m_2' &= (A_{PC} S A_{PC}^T)^{-1} (A_{PC}(r - S A_t^T m_1' - g(A_t^T m_1 + A_{PC}^T m_2))) \\
p &= (A_t A_r)^{-1} (A_t (r - S(A_t^T m_1 + A_{PC}^T m_2)' - g(A_t^T m_1 + A_{PC}^T m_2)))
\end{aligned}
$$

□

$A_r^T A_t^T$ is the incidence matrix of a forest. If the graph is sorted in a convenient way, $A_r^T A_t^T$ is an upper triangular matrix with diagonal elements $a_{ii} = \pm 1$ and for exactly one non-diagonal element per row it holds $a_{ij} = \pm 1$. All other components are zero. Therefore, we obtain m_1 with relatively low computing costs in practice.
In our renumbered example network 4, $A_r^T A_t^T$ is the incidence matrix of the graph containing every node of the network and every edge which is neither a cycle edge nor a path edge, i.e. the graph contains exactly the edges drawn in black. The system of ODEs (15b) has dimension $n_{E_{red}}$, which equals the number of cycle edges (drawn in green) plus the number of path edges (drawn in red).
In the next part we prove the global unique solvability of equation (10).

Theorem 5.3 *(Unique Solvability)*
Let the network be connected and let there be at least one pressure node and one demand node in the network. Furthermore let, on a compact time interval $\mathscr{I} = [t_0, T]$, the demand input functions $q_s(t)$ be continuously differentiable and let the pressure input functions $p_s(t)$ be continuous. Then for every initial value of the flow components $m(t_0)$ which fulfills $A_r^T m(t_0) = 0$ the DAE (16) has a global unique solution $(m_\star(t), p_\star(t))$ on $\mathscr{I}$ with $m_\star(t)$ being continuously differentiable and $p_\star$ being continuous.

PROOF:
With Lemma 5.2 it is sufficient to show that there is a global unique solution of (15). With the assumption that the initial value $m(t_0)$ fulfills $A_r^T m(t_0) = 0$ and equation (15a) we obtain a global unique solution $m_1(t) \in C^1(\mathscr{I})$.
If there is a global unique solution m_2 for the non-linear ODE (15b), then we can insert m_1 and m_2 into equation (15c) and obtain the global unique solvability of p. Hence we have to show the global unique solvability of (15b). Notice here that the initial values of the pressures p are algebraically determined.
For a more compact notation, we define $a(t) := A_{PC}(r(t) - S A_t^T m_1'(t))$ and $s(t) := A_t^T m_1(t)$. Then, equation (15b) is equivalent to

$$
m_2'(t) = (A_{PC} S A_{PC}^T)^{-1} (a(t) - A_{PC} g(s(t) + A_{PC}^T m_2(t))). \tag{16}
$$

The right hand side function of the ODE (16),

$$h : \mathbb{R}^{n_{E_{red}}} \times \mathbb{R} \to \mathbb{R}^{n_{E_{red}}},\ (x,t) \mapsto (A_{PC}SA_{PC}^T)^{-1}(a(t) - A_{PC}g(A_{PC}^T x + s(t))),$$

is continuously differentiable as the composition of continuously differentiable functions. Hence, h is locally Lipschitz-continuous for all $t \in \mathscr{I}$ with respect to x. Firstly, we show that any possible solution x^* of (16) is bounded on $\mathscr{I}$ by means of an a-priori estimate. Before we can start our estimate, we need some preparation. We use the notation

$$\|x\|_{p,M} := \|Mx\|_p$$

with M a compatible matrix with full column rank. We define $S^{\frac{1}{2}}$ such that $S^{\frac{1}{2}}S^{\frac{1}{2}} = S$ with $S^{\frac{1}{2}}$ being positive definite and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|S^{\frac{1}{2}}A_{PC}^T x\right\|_2^2 &= \frac{1}{2}\frac{d}{dt}(S^{\frac{1}{2}}A_{PC}^T x)^T(S^{\frac{1}{2}}A_{PC}^T x)\\ &= (S^{\frac{1}{2}}A_{PC}^T x)^T(S^{\frac{1}{2}}A_{PC}^T x)' = x^T A_{PC}SA_{PC}^T x'\end{aligned}$$

Furthermore,

$$c_a := \max_i \begin{pmatrix} \vdots \\ \max_t |a_i(t)| \\ \vdots \end{pmatrix} \text{ and } c_s := \max_i \begin{pmatrix} \vdots \\ \max_t |c_i s_i(t)| \\ \vdots \end{pmatrix}, \tag{17}$$

with c_i the pipe characteristics defined in equation (7), exist as a and s are continuous functions on $\mathscr{I}$.
Now we are ready for the estimate. Remember that x, a and s are functions on $\mathscr{I}$. The following relations hold pointwise for all $t \in \mathscr{I}$.

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|S^{\frac{1}{2}}A_{PC}^T x\right\|_2^2\\ =&x^T A_{PC}SA_{PC}^T x'\\ =&x^T\left(a - A_{PC}g(A_{PC}^T x + s)\right)\\ =&x^T a - \sum_{i=1}^n \left(c_i(A_{PC}^T x)_i^2\left|(A_{PC}^T x + s)_i\right| + (A_{PC}^T x)_i c_i s_i\left|(A_{PC}^T x + s)_i\right|\right)\\ \leq&x^T a - \sum_{i=1}^n (A_{PC}^T x)_i c_i s_i\left|(A_{PC}^T x + s)_i\right|\\ \leq&\sum_{i=1}^n |x_i a_i| + \sum_{i=1}^n \left|(A_{PC}^T x)_i\right| |c_i s_i| \left(\left|(A_{PC}^T x)_i\right| + |s_i|\right)\\ \overset{(17)}{\leq}&c_a\sum_{i=1}^n |x_i| + c_s\sum_{i=1}^n\left((A_{PC}^T x)_i^2 + c_s\left|(A_{PC}^T x)_i\right|\right)\\ =&c_a\|x\|_1 + c_s\|x\|_{2,A_{PC}^T}^2 + c_s^2\|x\|_{1,A_{PC}^T}\end{aligned}$$

Equivalence of norms on $\mathbb{R}^n$ yields the existence of constants $\tilde{c}, \tilde{c}_s, \bar{c} > 0$ such that

$$\frac{d}{dt}\left\|S^{\frac{1}{2}}A_{PC}^T x\right\|_2^2 \leq 2\tilde{c}\,\|x\|_2 + 2\tilde{c}_s\,\|x\|_2^2 \leq 2\bar{c}(\|x\|_2 + \|x\|_2^2).$$

We define $I_>$ and $I_<$ such that

$$\begin{aligned} \|x\| &\geq 1 \text{ on } I_> \\ \|x\| &< 1 \text{ on } I_< \end{aligned}$$

and obtain by integrating

$$\begin{aligned}
&\left\|S^{\frac{1}{2}}A_{PC}^T x\right\|_2^2 \\
\leq &\bar{c}\int_{t_0}^{t}(\|x\|_2 + \|x\|_2^2)ds + \left\|S^{\frac{1}{2}}A_{PC}^T x(t_0)\right\|_2 \\
= &\bar{c}\int_{I_<}(\|x\|_2 + \|x\|_2^2)ds + \bar{c}\int_{I_>}(\|x\|_2 + \|x\|_2^2)ds + \left\|S^{\frac{1}{2}}A_{PC}^T x(t_0)\right\|_2 \\
\leq &\bar{c}\int_{I_<}2ds + \bar{c}\int_{I_>}(\|x\|_2 + \|x\|_2^2)ds + \left\|S^{\frac{1}{2}}A_{PC}^T x(t_0)\right\|_2 \\
\leq &2\bar{c}(T-t_0) + 2\bar{c}\int_{t_0}^{t}(\|x\|_2^2)ds + \left\|S^{\frac{1}{2}}A_{PC}^T x(t_0)\right\|_2 \\
= &2\bar{c}\int_{t_0}^{t}\|x\|_2^2\,ds + \bar{k},
\end{aligned}$$

with $\bar{k} := 2\bar{c}(T-t_0) + \left\|S^{\frac{1}{2}}A_{PC}^T x(t_0)\right\|_2$. Exploiting the equivalence of norms again yields

$$\|x\|_\infty^2 \leq \hat{c}\int_{t_0}^{t}\|x\|_\infty^2\,ds + \hat{k}.$$

At this point, we can see that the conditions for the Gronwall-inequality are met and we obtain the a-priori estimate for any possible solution x^* of (16)

$$\|x^*(t)\|_\infty^2 \leq \hat{k}e^{\hat{c}(t-t_0)}$$

and thus

$$\|x^*(t)\|_\infty \leq ke^{c(t-t_0)} \tag{18}$$

with $c = \frac{1}{2}\hat{c}$ and $k = \sqrt{\hat{k}}$. We define

$$R := \mathscr{I} \times [-ke^{c(T-t_0)}, ke^{c(T-t_0)}]^{n_{E_{red}}}.$$

The estimate (18) ensures that any possible solution $x^*(t)$ cannot cross the limits of R for $t \in \mathscr{I}$. Since the right hand side function is locally Lipschitz-continuous with respect to x for all $t \in \mathscr{I}$, the theorem of Picard-Lindelöf yields existence and uniqueness of a solution $x^* = m_2 \in C^1([t_0,t_1])$ with $t_0 < t_1 < T$. By means of the a-priori-estimate, this solution can be extended uniquely to the limits of the time interval $\mathscr{I}$, i.e. there is a solution $x^* = m_2 \in C^1(\mathscr{I})$. □

Remark 5.4
Notice that the initial values of the pressure variables can not be chosen freely.

6 Conclusions

In this paper, we have shown global unique solvability of a water network model with quasi-stationary models for the pipe equation. We have analyzed the resulting DAE with the Tractability-Strangeness Index, which allows for a topological decoupling. By means of this decoupling, we reduce the non-linear equations to solve substantially. i.e. the reduced system has the size of the of the quantity of cycles in the system plus the quantity of pressure nodes minus one. Furthermore, we have shown that the index of the model DAE is 2.
The considered model takes into account only nodes and pipes, whereas [12] has shown global unique solvability for a more complex model with respect to the considered control devices. But the model level of [12] is stationary with respect to the underlying pipe equation. The challenge arising naturally is to 'combine' the results of this paper and [12], i.e. to show global unique solvability for complex models in the quasi-stationary model level. Thereafter, an extension to the dynamic model level being able to model hydraulic shocks in a pipe appropriately would be desirable.

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