Twist-2 at five loops: Wrapping corrections without wrapping computations

V. N. Velizhanin[*]

Institut für Mathematik und Institut für Physik
Humboldt-Universität zu Berlin
IRIS Adlershof, Zum Großen Windkanal 6
12489 Berlin, Germany
E-mail: velizh@physik.hu-berlin.de

ABSTRACT: Using known all-loop results from the BFKL and generalized double-logarithmic equations and large spin limit we have computed the five-loop anomalous dimension of twist-2 operators without consideration of any wrapping effects. One part of the anomalous dimension was calculated in a usual way with the help of Asymptotic Bethe Ansatz. The rest part, related with the wrapping effects, was reconstructed from known constraints with the help of methods from the numbers theory.

---

[*]On leave from Theoretical Physics Division, Petersburg Nuclear Physics Institute, St.Petersburg Russia

Contents

1. Introduction

During the investigation of AdS/CFT-correspondence [1, 2, 3] it was found that the anomalous dimension of BMN-operators [4] in $\mathcal{N} = 4$ SYM theory in the leading order of perturbative theory can be calculated with the help of integrability [5].[1] Generalization to the higher orders together with the investigations of the integrable structures from the superstring theory side, started in ref. [10], allowed to formulate all-loop asymptotic Bethe equations [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. The next step in the studying of integrability became the investigation of a wrapping effects for the operators with finite length [29, 30]. Initially, for the computation of the wrapping corrections it was suggested to use a results from the string theory side, where the wrapping corrections correspond to the effects of finite volume in which a string theory is considered (a generalization of Lüscher formulaes [31, 32]). Expanding the obtained exact result for the modification of the energy of string state under perturbative theory it is possible to calculate the wrapping corrections for the anomalous dimension of the corresponding finite size operators [33, 34, 35, 36]. Firstly, this method was applied for the simplest operator in $\mathcal{N} = 4$ SYM theory, i.e. for Konishi operator in the forth order of perturbative theory, where finite size effect appear from the first time [33]. The obtained result was confirmed

---

[1]Earlier, the similar integrability was opened in quantum chromodynamics in the Regge limit [6, 7, 8] and for some of operators [9].

by direct four-loop diagrammatic calculations from the field theory side [37, 38, 39, 40]. Then, this method was generalized to the twist-2 operators with an arbitrary Lorentz spin (arbitrary number of covariant derivatives in operator) in the same order [34]. The obtained result was found in a full agreement with the all-loop predictions, coming from the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [41, 42, 43] and from the results of the direct perturbative calculations of the leading transcendentality contribution [40], which serves as an important test of the correctness of the used methods. Then, Lüscher corrections were expanded to the next order for the calculation of the anomalous dimension of the Konishi operator [35] in the five loops and for the anomalous dimension of the twist-2 operators with the arbitrary Lorentz spin in the same order [36]. The last result was again in the full agreement with the predictions, coming from the BFKL equation [2]

In the same time the investigation of the more general integrable system were performed, which can be applied for the computations of the anomalous dimension of any operators in any order of perturbative theory along the approach, proposed by A. Zamolodchikov [46, 47]. In the result of such investigations the set of TBA equations and the Y-system were formulated [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. Both methods was tested with reproduction of the known four- and five-loop results and allow to obtain a lot of new results not only in $\mathcal{N} = 4$ SYM theory, but in other integrable models. However farther test of the developed methods based mainly on self-consistency and it will be nice to have some independent crosscheck. One of such important test can come from the analysis of the analytical properties of the full anomalous dimension of twist-2 operators, for which a lot of information are exist. All this predictions based on an analogical results, obtained earlier from the studying of the properties of twist-2 operators in QCD, where they give the leading contribution through the application of an operator product expansion to the deep-inelastic processes, actively studied experimentally. Initially, for these purpose were used all-loop predictions, coming from the BFKL and double-logarithmic equations [58, 59, 60, 61, 62, 63]. But some times ago we have found, that in $\mathcal{N} = 4$ SYM theory there is a rather simple generalization of double-logarithmic equation [64], which give a lot of constraints on analytical structure of anomalous dimension of twist-2 operators in any-loop orders. The presence a lot of information about analytical properties of twist-2 anomalous dimension led to the idea to use these information not for the testing of an obtained results, but directly for the computations of the general form of the anomalous dimension of twist-2 operator for the arbitrary Lorentz spin, i.e. for the reconstruction of the wrapping corrections. From the earlier obtained results in QCD and in $\mathcal{N} = 4$ SYM theory it is well known, that the anomalous dimension of twist-2 operators are expressed through the nested harmonic sums, defined as (see [65]):

$$S_a(M) = \sum_{j=1}^{M} \frac{(\text{sgn}(a))^j}{j^{|a|}}, \qquad S_{a_1,\ldots,a_n}(M) = \sum_{j=1}^{M} \frac{(\text{sgn}(a_1))^j}{j^{|a_1|}} S_{a_2,\ldots,a_n}(j). \qquad (1.1)$$

---

[2]Obtained formulaes for the leading and next-to-leading wrapping corrections were also applied for the calculations of the five- and six-loop anomalous dimension of the twist-3 operator [44, 45], for which wrapping effect appear in one order more with compare to twist-2 operator.

In other words the nested harmonic sums form the finite basis for the anomalous dimension of the twist-2 operators in each order of perturbative theory. According to the principle of maximal transcendentality [66] at ℓ order of the perturbation theory the anomalous dimension of twist-2 operators is expressed through the nested harmonic sums of the order $(2\ell - 1)$, or through the products of zeta functions and harmonic sums for which the sum of the arguments of the zeta functions and the orders of the harmonic sums is equal to $(2\ell - 1)$. For the fixing the coefficient in the ansatz, obtained from the corresponding basis, it is necessary to use all information, which we have in this order of perturbative theory. For the reconstruction of the general expression for the anomalous dimension of the twist-2 operators, coming from the ABA one can used results for the definite values of M. For the full anomalous dimension besides analytical properties we can used a generalized Gribov-Lipatov relation and looking for the reciprocity-respecting function, instead of the anomalous dimension, which is related to each other through:

$$\gamma(M) = \mathcal{P}\left(M + \frac{1}{2}\gamma(M)\right). \tag{1.2}$$

This allow considerably reduce the number of harmonic sums in basis, because these sums (or combinations of sums) should respect this symmetry. We have found, that more suitable move from the nested harmonic sums to the binomial harmonic sums, which are definite as (see [65]):

$$\mathbb{S}_{i_1,\ldots,i_k}(N) = (-1)^N \sum_{j=1}^{N} (-1)^j \binom{N}{j}\binom{N+j}{j} S_{i_1,\ldots,i_k}(j), \tag{1.3}$$

where $S_{i_1,\ldots,i_k}$ are the nested harmonic sums (1.1) and all indices a_i are always positive. Note again, that the number of binomial harmonic sums are much less, than the usual nested harmonic sums.

In the fourth order of perturbative theory an available information is enough and the coefficients in ansatz fix uniquely from the all know information (see [64]). Unfortunately, the number of sums in ansatz grows with the order of perturbative theory faster, than the number of constraints, coming from the know all-loop results, so, the numbers of unknown coefficients in ansatz are more than the number of available equations. However, in such case we can used the method, which we already used for the reconstruction of the six-loop anomalous dimension of the twist-3 operators in $\mathcal{N} = 4$ SYM theory and for the anomalous dimension of the non-singlet transverse twist-2 operators in QCD in third order of perturbative theory. This method based on the fact, that (seeking, desired) sought-for coefficients in ansatz should be integers numbers and, more over, some of them (large enough) should be zero. With this conditions we obtain the system of linear Diophantine equations, which can be solved with the different method from the numbers theory. In our previous papers we have used LLL-algorithm [67] to find a vector, which can solve an initial system. But we have found also other method, which will applied for the calculations of five-loop anomalous dimension of the twist-2 operators, presented in this paper.

The paper organized in the following way. In the Chapter 1 we give result for the five-loop anomalous dimension of the twist-2 operator, coming from the ABA, which was

obtained earlier in the paper of [36]. In the Chapter 2 we will obtained all expression (results), which will used then for the reconstruction of full anomalous dimension. The Chapter 3 devoted to the reconstruction of the five-loop anomalous dimension from the constraints, obtained in Chapter 2. We will applied two methods and will give simple example, which shows how the methods are works. In Conclusion we summarize our results and the overall methods.

2. Five-loop anomalous dimension from Bethe Ansatz

We start with the part of the anomalous dimension, which can be calculated with the help of asymptotic Bethe ansatz [20, 28]. The result at five-loop order together with the detailed description of computations can be found in ref. [36]. Here we present only final result in terms of the reciprocity-respecting functions (1.2) and the binomial harmonic sums (1.3).

Substituting the perturbative expansion for the anomalous dimension in eq. (1.2), one finds

$$\mathcal{P}^{\mathrm{ABA}}(M) = \sum_{l=1}^{\infty} g^{2l} \, \mathcal{P}^{\mathrm{ABA}}_{2l}(M) \,. \tag{2.1}$$

From eqs. (1.2) and (2.1) one can find, that at five-loop order the reciprocity function $\mathcal{P}_{10}$ is related with the anomalous dimension γ_{10} (see Appendix B of ref. [44]):

$$\mathcal{P}_{10}(M) = \mathcal{P}_{10} = \mathcal{P}_{10}^{\mathrm{rational}} + \mathcal{P}_{10}^{\zeta_3}\zeta_3 + \mathcal{P}_{10}^{\zeta_5}\zeta_5 \,, \tag{2.2}$$

$$\mathcal{P}_{10}^{\mathrm{rational}} = \gamma_{10}^{\mathrm{rational}} - \frac{1}{4}\left(\gamma_4\gamma_6 + \gamma_2\gamma_8^{\mathrm{rational}}\right)' + \frac{1}{32}\left(\gamma_2\gamma_4^2 + \gamma_2^2\gamma_6\right)''$$
$$- \frac{1}{384}\left(\gamma_2^3\gamma_4\right)''' + \frac{1}{30720}\left(\gamma_2^5\right)'''' \,, \tag{2.3}$$

$$\mathcal{P}_{10}^{\zeta_3} = \gamma_{10}^{\zeta_3} - \frac{1}{4}\left(\gamma_2\gamma_8^{\zeta_3}\right)' \,, \tag{2.4}$$

$$\mathcal{P}_{10}^{\zeta_5} = \gamma_{10}^{\zeta_5} \,, \tag{2.5}$$

where each prime marks derivative over M. The result for $\mathcal{P}_{10}^{\mathrm{rational}}(M)$ is equal to [36]:

$$\frac{\mathcal{P}_{10}^{\mathrm{rational}}}{128} = -5\,\mathbb{S}_{2,2,5} - \mathbb{S}_{2,6,1} + 19\,\mathbb{S}_{3,1,5} - 20\,\mathbb{S}_{3,2,4} + 21\,\mathbb{S}_{4,1,4} - 24\,\mathbb{S}_{4,2,3} + 25\,\mathbb{S}_{5,1,3}$$
$$-18\,\mathbb{S}_{5,2,2} + 7\,\mathbb{S}_{6,1,2} - 4\,\mathbb{S}_{6,2,1} - 2\,\mathbb{S}_{1,1,2,5} + 2\,\mathbb{S}_{1,1,6,1} - 2\,\mathbb{S}_{1,2,1,5} - \mathbb{S}_{1,2,2,4}$$
$$+\mathbb{S}_{1,2,3,3} + \mathbb{S}_{1,2,4,2} - 6\,\mathbb{S}_{1,2,5,1} + 23\,\mathbb{S}_{1,3,1,4} - 24\,\mathbb{S}_{1,3,2,3} - \mathbb{S}_{1,3,4,1} + 23\,\mathbb{S}_{1,4,1,3}$$
$$-20\,\mathbb{S}_{1,4,2,2} - \mathbb{S}_{1,4,3,1} + 13\,\mathbb{S}_{1,5,1,2} - 12\,\mathbb{S}_{1,5,2,1} + 6\,\mathbb{S}_{1,6,1,1} - 2\,\mathbb{S}_{2,1,1,5} + 5\,\mathbb{S}_{2,1,2,4}$$
$$+\mathbb{S}_{2,1,3,3} + \mathbb{S}_{2,1,4,2} - 5\,\mathbb{S}_{2,1,5,1} - 16\,\mathbb{S}_{2,2,1,4} + 17\,\mathbb{S}_{2,2,2,3} - 2\,\mathbb{S}_{2,2,3,2} + 14\,\mathbb{S}_{2,2,4,1}$$
$$-29\,\mathbb{S}_{2,3,1,3} + 25\,\mathbb{S}_{2,3,2,2} + 4\,\mathbb{S}_{2,3,3,1} - 19\,\mathbb{S}_{2,4,1,2} + 20\,\mathbb{S}_{2,4,2,1} - 12\,\mathbb{S}_{2,5,1,1} + 20\,\mathbb{S}_{3,1,1,4}$$
$$-22\,\mathbb{S}_{3,1,2,3} - 8\,\mathbb{S}_{3,1,3,2} + 6\,\mathbb{S}_{3,1,4,1} - 26\,\mathbb{S}_{3,2,1,3} + 36\,\mathbb{S}_{3,2,2,2} - 5\,\mathbb{S}_{3,2,3,1} - 6\,\mathbb{S}_{3,3,1,2}$$
$$+5\,\mathbb{S}_{3,3,2,1} - 2\,\mathbb{S}_{3,4,1,1} + 22\,\mathbb{S}_{4,1,1,3} - 24\,\mathbb{S}_{4,1,2,2} + 6\,\mathbb{S}_{4,1,3,1} - 18\,\mathbb{S}_{4,2,1,2} + 18\,\mathbb{S}_{4,2,2,1}$$
$$-2\,\mathbb{S}_{4,3,1,1} + 14\,\mathbb{S}_{5,1,1,2} - 10\,\mathbb{S}_{5,1,2,1} - 14\,\mathbb{S}_{5,2,1,1} + 8\,\mathbb{S}_{6,1,1,1} + 4\,\mathbb{S}_{1,1,1,1,5} - 6\,\mathbb{S}_{1,1,1,2,4}$$
$$-2\,\mathbb{S}_{1,1,1,3,3} - 2\,\mathbb{S}_{1,1,1,4,2} + 6\,\mathbb{S}_{1,1,1,5,1} - 4\,\mathbb{S}_{1,1,2,1,4} + 4\,\mathbb{S}_{1,1,2,2,3} + 5\,\mathbb{S}_{1,1,2,3,2}$$
$$-13\,\mathbb{S}_{1,1,2,4,1} + 24\,\mathbb{S}_{1,1,3,1,3} - 20\,\mathbb{S}_{1,1,3,2,2} - 5\,\mathbb{S}_{1,1,3,3,1} + 16\,\mathbb{S}_{1,1,4,1,2} - 19\,\mathbb{S}_{1,1,4,2,1}$$
$$+12\,\mathbb{S}_{1,1,5,1,1} - 4\,\mathbb{S}_{1,2,1,1,4} + 7\,\mathbb{S}_{1,2,1,2,3} + 4\,\mathbb{S}_{1,2,1,3,2} - 8\,\mathbb{S}_{1,2,1,4,1} - 19\,\mathbb{S}_{1,2,2,1,3}$$
$$+9\,\mathbb{S}_{1,2,2,2,2} + 24\,\mathbb{S}_{1,2,2,3,1} - 22\,\mathbb{S}_{1,2,3,1,2} + 31\,\mathbb{S}_{1,2,3,2,1} - 22\,\mathbb{S}_{1,2,4,1,1} + 22\,\mathbb{S}_{1,3,1,1,3}$$
$$-24\,\mathbb{S}_{1,3,1,2,2} + 6\,\mathbb{S}_{1,3,1,3,1} - 20\,\mathbb{S}_{1,3,2,1,2} + 23\,\mathbb{S}_{1,3,2,2,1} - 6\,\mathbb{S}_{1,3,3,1,1} + 16\,\mathbb{S}_{1,4,1,1,2}$$
$$-11\,\mathbb{S}_{1,4,1,2,1} - 22\,\mathbb{S}_{1,4,2,1,1} + 16\,\mathbb{S}_{1,5,1,1,1} - 4\,\mathbb{S}_{2,1,1,1,4} + 7\,\mathbb{S}_{2,1,1,2,3} + 4\,\mathbb{S}_{2,1,1,3,2}$$
$$-8\,\mathbb{S}_{2,1,1,4,1} + 3\,\mathbb{S}_{2,1,2,1,3} - 11\,\mathbb{S}_{2,1,2,2,2} + 14\,\mathbb{S}_{2,1,2,3,1} - 9\,\mathbb{S}_{2,1,3,1,2} + 16\,\mathbb{S}_{2,1,3,2,1}$$
$$-12\,\mathbb{S}_{2,1,4,1,1} - 16\,\mathbb{S}_{2,2,1,1,3} + 13\,\mathbb{S}_{2,2,1,2,2} + 5\,\mathbb{S}_{2,2,1,3,1} + 28\,\mathbb{S}_{2,2,2,1,2} - 67\,\mathbb{S}_{2,2,2,2,1}$$
$$+32\,\mathbb{S}_{2,2,3,1,1} - 22\,\mathbb{S}_{2,3,1,1,2} + 15\,\mathbb{S}_{2,3,1,2,1} + 31\,\mathbb{S}_{2,3,2,1,1} - 23\,\mathbb{S}_{2,4,1,1,1} + 20\,\mathbb{S}_{3,1,1,1,3}$$
$$-22\,\mathbb{S}_{3,1,1,2,2} + 6\,\mathbb{S}_{3,1,1,3,1} - 8\,\mathbb{S}_{3,1,2,1,2} + 2\,\mathbb{S}_{3,1,2,2,1} + 6\,\mathbb{S}_{3,1,3,1,1} - 20\,\mathbb{S}_{3,2,1,1,2}$$
$$+15\,\mathbb{S}_{3,2,1,2,1} + 25\,\mathbb{S}_{3,2,2,1,1} - 7\,\mathbb{S}_{3,3,1,1,1} + 16\,\mathbb{S}_{4,1,1,1,2} - 11\,\mathbb{S}_{4,1,1,2,1} - 11\,\mathbb{S}_{4,1,2,1,1}$$
$$-23\,\mathbb{S}_{4,2,1,1,1} + 16\,\mathbb{S}_{5,1,1,1,1} - 4\,\mathbb{S}_{1,1,1,1,2,3} - 4\,\mathbb{S}_{1,1,1,1,3,2} + 8\,\mathbb{S}_{1,1,1,1,4,1} + 4\,\mathbb{S}_{1,1,1,2,2,2}$$
$$-13\,\mathbb{S}_{1,1,1,2,3,1} + 18\,\mathbb{S}_{1,1,1,3,1,2} - 25\,\mathbb{S}_{1,1,1,3,2,1} + 16\,\mathbb{S}_{1,1,1,4,1,1} + 3\,\mathbb{S}_{1,1,2,1,2,2}$$
$$-5\,\mathbb{S}_{1,1,2,1,3,1} - 20\,\mathbb{S}_{1,1,2,2,1,2} + 53\,\mathbb{S}_{1,1,2,2,2,1} - 33\,\mathbb{S}_{1,1,2,3,1,1} + 18\,\mathbb{S}_{1,1,3,1,1,2}$$
$$-11\,\mathbb{S}_{1,1,3,1,2,1} - 25\,\mathbb{S}_{1,1,3,2,1,1} + 20\,\mathbb{S}_{1,1,4,1,1,1} + 3\,\mathbb{S}_{1,2,1,1,2,2} - 5\,\mathbb{S}_{1,2,1,1,3,1}$$
$$+14\,\mathbb{S}_{1,2,1,2,2,1} - 17\,\mathbb{S}_{1,2,1,3,1,1} - 18\,\mathbb{S}_{1,2,2,1,1,2} + 11\,\mathbb{S}_{1,2,2,1,2,1} + 52\,\mathbb{S}_{1,2,2,2,1,1}$$
$$-27\,\mathbb{S}_{1,2,3,1,1,1} + 16\,\mathbb{S}_{1,3,1,1,1,2} - 11\,\mathbb{S}_{1,3,1,1,2,1} - 11\,\mathbb{S}_{1,3,1,2,1,1} - 27\,\mathbb{S}_{1,3,2,1,1,1}$$
$$+20\,\mathbb{S}_{1,4,1,1,1,1} + 3\,\mathbb{S}_{2,1,1,1,2,2} - 5\,\mathbb{S}_{2,1,1,1,3,1} + 14\,\mathbb{S}_{2,1,1,2,2,1} - 17\,\mathbb{S}_{2,1,1,3,1,1}$$
$$+16\,\mathbb{S}_{2,1,2,2,1,1} - 11\,\mathbb{S}_{2,1,3,1,1,1} - 16\,\mathbb{S}_{2,2,1,1,1,2} + 11\,\mathbb{S}_{2,2,1,1,2,1} + 11\,\mathbb{S}_{2,2,1,2,1,1}$$
$$+45\,\mathbb{S}_{2,2,2,1,1,1} - 27\,\mathbb{S}_{2,3,1,1,1,1} + 16\,\mathbb{S}_{3,1,1,1,1,2} - 11\,\mathbb{S}_{3,1,1,1,2,1} - 11\,\mathbb{S}_{3,1,1,2,1,1}$$
$$-11\,\mathbb{S}_{3,1,2,1,1,1} - 27\,\mathbb{S}_{3,2,1,1,1,1} + 20\,\mathbb{S}_{4,1,1,1,1,1} - 16\,\mathbb{S}_{1,1,1,1,2,2,1} + 24\,\mathbb{S}_{1,1,1,1,3,1,1}$$
$$-28\,\mathbb{S}_{1,1,1,2,2,1,1} + 20\,\mathbb{S}_{1,1,1,3,1,1,1} - 20\,\mathbb{S}_{1,1,2,2,1,1,1} + 20\,\mathbb{S}_{1,1,3,1,1,1,1}$$
$$-20\,\mathbb{S}_{1,2,2,1,1,1,1} + 20\,\mathbb{S}_{1,3,1,1,1,1,1} - 20\,\mathbb{S}_{2,2,1,1,1,1,1} + 20\,\mathbb{S}_{3,1,1,1,1,1,1}\,. \tag{2.6}$$

Table 1: The five-loop function $\mathcal{P}_{10}^{\mathrm{rational}}(M)$.

The functions $\mathcal{P}_{10}^{\zeta_3}$ and $\mathcal{P}_{10}^{\zeta_5}$ look like [68, 36]

$$\frac{\mathcal{P}_{10}^{\zeta_3}}{256} = 3\mathbb{S}_{1,5} - 4\mathbb{S}_{2,4} - \mathbb{S}_{3,3} - \mathbb{S}_{4,2} + 3\mathbb{S}_{5,1} + 2\mathbb{S}_{1,1,4} - 4\mathbb{S}_{1,2,3} - 2\mathbb{S}_{1,3,2} + 2\mathbb{S}_{1,4,1}$$
$$-\mathbb{S}_{2,1,3} + 5\mathbb{S}_{2,2,2} - 2\mathbb{S}_{2,3,1} - 2\mathbb{S}_{3,2,1} + 2\mathbb{S}_{4,1,1} - 3\mathbb{S}_{1,1,2,2} + \mathbb{S}_{1,1,3,1} - \mathbb{S}_{1,2,1,2}$$
$$+2\mathbb{S}_{1,3,1,1} - \mathbb{S}_{2,1,1,2} + \mathbb{S}_{2,1,2,1} + \mathbb{S}_{3,1,1,1} \,, \tag{2.7}$$

$$\frac{\mathcal{P}_{10}^{\zeta_5}}{640} = \mathbb{S}_1(\mathbb{S}_{2,1} - \mathbb{S}_3) \,. \tag{2.8}$$

3. Weak-coupling constraints

In this section we will write down the known weak-coupling constraints on the five-loop anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory. We will use three classes of constraints, which are provided by the BFKL equation, by the generalized double-logarithmic equation and by the large M limit.

3.1 BFKL equation

The relation between the anomalous dimension of twist-2 operators and the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [41, 42, 43] and its next-to-leading logarithm approximation (NLLA) generalisation [69, 70] emerges upon analytic continuation of the function $\gamma(g, M)$ to complex values of M. This is straightforward in the one-loop case since

$$\gamma_2(M) = 8\,g^2\,S_1(M) = 8\,g^2\,(\Psi(M+1) - \Psi(1)) \,, \tag{3.1}$$

where $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the digamma function. At any loop order one expects singularities at all negative integer values of M. The first in this series of singular points,

$$M = -1 + \omega \,, \tag{3.2}$$

corresponds to the BFKL pomeron. In the above formula ω should be considered infinitesimally small. The BFKL equation predicts that, if expanded in g, the ℓ-loop anomalous dimension $\gamma_{2\ell}(\omega)$ exhibits poles in ω. Moreover, the residues and the order of the poles can be derived directly from the BFKL equation. Up to the next-to-leading logarithm approximation (NLLA) BFKL equation for twist-2 operators in $\mathcal{N} = 4$ SYM theory in the dimensional reduction scheme can be written as follows [69, 70]

$$\frac{\omega}{-4\,g^2} = \chi(\gamma) - g^2\,\delta(\gamma) \,, \tag{3.3}$$

where

$$\chi(\gamma) = \Psi\left(-\frac{\gamma}{2}\right) + \Psi\left(1 + \frac{\gamma}{2}\right) - 2\,\Psi(1) \,, \tag{3.4}$$

$$\delta(\gamma) = 4\,\chi''(\gamma) + 6\,\zeta(3) + 2\,\zeta(2)\,\chi(\gamma) + 4\,\chi(\gamma)\,\chi'(\gamma)$$

$$-\frac{\pi^3}{\sin\frac{\pi\gamma}{2}} - 4\,\Phi\left(-\frac{\gamma}{2}\right) - 4\,\Phi\left(1 + \frac{\gamma}{2}\right) \,. \tag{3.5}$$

The function $\Phi(\gamma)$ is given by

$$\Phi(\gamma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+\gamma)^2} \left[\Psi\left(k+\gamma+1\right) - \Psi(1) \right].$$ (3.6)

Using the perturbative expansion of anomalous dimension one easily determines the leading singularity structure

$$\gamma = \left(2 + 0\,\omega + \mathcal{O}(\omega^2)\right) \left(\frac{-4\,g^2}{\omega}\right) - \left(0 + 0\,\omega + \mathcal{O}(\omega^2)\right) \left(\frac{-4\,g^2}{\omega}\right)^2$$
$$+ \left(0 + \zeta_3\,\omega + \mathcal{O}(\omega^2)\right) \left(\frac{-4\,g^2}{\omega}\right)^3 - \left(4\,\zeta_3 + \frac{5}{4}\,\zeta_4\,\omega + \mathcal{O}(\omega^2)\right) \left(\frac{-4\,g^2}{\omega}\right)^4$$
$$- \left(0 + \left(2\,\zeta_2\,\zeta_3 + 16\,\zeta_5\right)\omega + \mathcal{O}(\omega^2)\right) \left(\frac{-4g^2}{\omega}\right)^5 \pm \dots.$$ (3.7)

3.2 Generalized double-logarithmic equation

Another class of constraints on the anomalous dimension of twist-2 operators is provided by the double-logarithmic asymptotic of the scattering amplitudes. The double-logarithmic asymptotic of the scattering amplitudes were investigated in QED and QCD in the papers [58, 59, 60] and [61, 62, 63] (see also `arXiv` version of ref. [66]). It corresponds to summing the leading terms $(\alpha \ln^2 s)^k$ in all orders of perturbation theory. In combination with a Mellin transformation, the double-logarithmic asymptotic allow to predict the singular part of anomalous dimensions near the point $M = -2 + \omega$. For our purpose and in our notation double-logarithmic equation has the following form

$$\gamma\left(2\,\omega + \gamma\right) = -16\,g^2\,.$$ (3.8)

The solution of this equation gives prediction for the highest pole (g^{2k}/ω^{2k-1}) in all orders of perturbative theory:

$$\gamma = -\omega + \omega\sqrt{1 - \frac{16g^2}{\omega^2}} = 2\,\frac{(-4\,g^2)}{\omega} - 2\,\frac{(-4\,g^2)^2}{\omega^3} + 4\,\frac{(-4\,g^2)^3}{\omega^5} - 10\,\frac{(-4\,g^2)^4}{\omega^7}$$
$$+ 28\,\frac{(-4\,g^2)^5}{\omega^9} - 84\,\frac{(-4\,g^2)^6}{\omega^{11}} + \dots.$$ (3.9)

The investigation of the analytical properties of the anomalous dimension of the twist-2 operators in $\mathcal{N} = 4$ SYM theory led to the suggestion about a simple generalization of the double-logarithmic equation [64].[3] The main idea was that in eq. (3.8) the corrections to the leading order equation will modified only the right-hand side and that such modification admit besides expansion over coupling constant g^2 only appearance of a regular terms over ω (and, possible, γ). Substitute the results for the analytical continuation of the anomalous dimension of twist-2 operators near $M = -2 + \omega$ into eq. (3.8) we indeed find the following form of the generalized double-logarithmic equation [64]

$$\gamma\left(2\,\omega + \gamma\right) = \sum_{k=1}\sum_{m=0} \mathfrak{C}_m^k\,\omega^m\,g^{2k}\,,$$ (3.10)

---

[3]For the first time, a such generalization was suggested by Lev N. Lipatov and Andrei Onishchenko at 2004, but was not published. Then, it was improved by Lev N. Lipatov in ref. [30].

where coefficients are listed in eq. (A.2) of Appendix of ref. [64].

This result has very remarkable consequence. If we solve equation (3.10), we have found

$$\gamma_{\mathrm{DL}}(\omega) = -\omega + \sqrt{\omega^2 + \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathfrak{C}_m^k \, \omega^m \, g^{2k}} \, . \tag{3.11}$$

Perturbatively expanding this solution we can predict in all orders of perturbative theory all poles up to $(g^2/\omega^2)^k \omega^{2\ell}$, if we know ℓ-loop anomalous dimension (or first ℓ orders in right hand side of eq. (3.10)).

3.3 Large M limit

Third class of constraints for the anomalous dimensions of composite operators in $\mathcal{N} = 4$ SYM theory comes from the consideration of large M limit. This limit is controlled by ABA and the corresponding predictions can be obtained at any order of perturbative theory. The main statement, interesting for us, that a wrapping corrections can not modified leading $\ln M$ behavior of large M limit.

4. Five loops from constraints

In this section we will describe in details the method of the reconstruction of the anomalous dimension of twist-2 operators from the above constraints and special numerical algorithms. Really, using generalized Gribov-Lipatov reciprocity we will reconstruct reciprocity-respecting function $\mathcal{P}^{\mathrm{wrap}}$, rather γ^{wrap} itself, because in this case the basis will consist from the binomial harmonic sums (1.3), the number of which is considerable less then the number of the usual nested harmonic sums (1.1). At five loops $\gamma_{10}^{\mathrm{wrap}}$ and $\mathcal{P}_{10}^{\mathrm{wrap}}$ are related as

$$\gamma_{10}^{\mathrm{wrap}} = \frac{1}{2} \mathcal{P}_8^{\mathrm{wrap}} \dot{\gamma}_2 + \frac{1}{2} \dot{\mathcal{P}}_8^{\mathrm{wrap}} \gamma_2 + \mathcal{P}_{10}^{\mathrm{wrap}} \, , \tag{4.1}$$

where

$$\mathcal{P}_8^{\mathrm{wrap}} = \mathcal{P}_2^2 \mathcal{T}_8 \, , \tag{4.2}$$

$$\mathcal{T}_8 = \Big(-5\,\zeta_5 + 2\,\mathbb{S}_2\,\zeta_3 + (\mathbb{S}_{2,1,2} - \mathbb{S}_{3,1,1}) \Big) \, , \tag{4.3}$$

$$\gamma_2 = \mathcal{P}_2 = 4\,\mathbb{S}_1 \, . \tag{4.4}$$

To find $\mathcal{P}_{10}^{\mathrm{wrap}}$ first of all we should write the full basis from the binomial harmonic sums, which will form the suitable ansatz. At five loops we can write the following general expression for the reciprocity-respecting function $\mathcal{P}_{10}^{\mathrm{wrap}}$:

$$\mathcal{P}_{10}^{\mathrm{wrap}} = \mathbb{S}_1^2 \, \mathcal{T}_{10} + c_1 \, \mathbb{S}_1 \big(\mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_{2,1} - \mathbb{S}_3 \big) \mathcal{T}_8 + c_2 \, \mathbb{S}_1^4 \, \mathcal{T}_8 \, . \tag{4.5}$$

The appearance of the structure with coefficients c_1 and c_2 are expected from the suggestion, that the more general integrable system applicable for the computations of the wrapping corrections represents as the set of (at least) two spin-chains, for which ABA are known (note, that $\mathbb{S}_1 \mathbb{S}_2 - \mathbb{S}_{2,1} - \mathbb{S}_3 = \mathcal{P}_4/8$). During the computations of the wrapping corrections at five loops in ref. [36] we assumed in advance the appearance of such terms,

what allow to find a final result much faster. Using the maximal transcendentality principle one can find, that at five loops the basis for $\mathcal{T}_{10}$ will contain $1 + 1 + 2 + 2^3 + 2^6 = 76$ binomial harmonic sums (multiplied by $\mathbb{S}_1^2$):

$$\Big\{ \zeta_7, \zeta_3^2\, \mathbb{S}_1, \zeta_5\, \mathbb{S}_2, \underline{\zeta_5\, \mathbb{S}_1^2}, \zeta_3\, \mathbb{S}_4, \zeta_3\, \mathbb{S}_{3,1}, \zeta_3\, \mathbb{S}_{2,2}, \zeta_3\, \mathbb{S}_{2,1,1}, \zeta_3\, \mathbb{S}_1\, \mathbb{S}_3, \zeta_3\, \mathbb{S}_1\, \mathbb{S}_{2,1}, \underline{\zeta_3\, \mathbb{S}_1^2\, \mathbb{S}_2}, \underline{\zeta_3\, \mathbb{S}_1^4},$$

$$\mathbb{S}_7, \mathbb{S}_{6,1}, \mathbb{S}_{5,2}, \mathbb{S}_{4,3}, \mathbb{S}_{3,4}, \mathbb{S}_{2,5}, \mathbb{S}_{5,1,1}, \mathbb{S}_{4,2,1}, \mathbb{S}_{4,1,2}, \mathbb{S}_{3,3,1}, \mathbb{S}_{3,2,2}, \mathbb{S}_{3,1,3}, \mathbb{S}_{2,4,1}, \mathbb{S}_{2,3,2}, \mathbb{S}_{2,2,3}, \mathbb{S}_{2,1,4},$$

$$\mathbb{S}_{4,1,1,1}, \mathbb{S}_{3,2,1,1}, \mathbb{S}_{3,1,2,1}, \mathbb{S}_{3,1,1,2}, \mathbb{S}_{2,3,1,1}, \mathbb{S}_{2,2,2,1}, \mathbb{S}_{2,2,1,2}, \mathbb{S}_{2,1,3,1}, \mathbb{S}_{2,1,2,2}, \mathbb{S}_{2,1,1,3},$$

$$\mathbb{S}_{3,1,1,1,1}, \mathbb{S}_{2,2,1,1,1}, \mathbb{S}_{2,1,2,1,1}, \mathbb{S}_{2,1,1,2,1}, \mathbb{S}_{2,1,1,1,2}, \mathbb{S}_{2,1,1,1,1,1},$$

$$\mathbb{S}_1\, \mathbb{S}_6, \mathbb{S}_1\, \mathbb{S}_{5,1}, \mathbb{S}_1\, \mathbb{S}_{4,2}, \mathbb{S}_1\, \mathbb{S}_{3,3}, \mathbb{S}_1\, \mathbb{S}_{2,4}, \mathbb{S}_1\, \mathbb{S}_{4,1,1}, \mathbb{S}_1\, \mathbb{S}_{3,2,1}, \mathbb{S}_1\, \mathbb{S}_{3,1,2}, \mathbb{S}_1\, \mathbb{S}_{2,3,1}, \mathbb{S}_1\, \mathbb{S}_{2,2,2}, \mathbb{S}_1\, \mathbb{S}_{2,1,3},$$

$$\mathbb{S}_1\, \mathbb{S}_{3,1,1,1}, \mathbb{S}_1\, \mathbb{S}_{2,2,1,1}, \mathbb{S}_1\, \mathbb{S}_{2,1,2,1}, \mathbb{S}_1\, \mathbb{S}_{2,1,1,2}, \mathbb{S}_1\, \mathbb{S}_{2,1,1,1,1},$$

$$\underline{\mathbb{S}_1^2\, \mathbb{S}_5}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{4,1}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{3,2}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{2,3}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{3,1,1}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{2,2,1}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{2,1,2}}, \underline{\mathbb{S}_1^2\, \mathbb{S}_{2,1,1,1}},$$

$$\underline{\mathbb{S}_1^3\, \mathbb{S}_4}, \underline{\mathbb{S}_1^3\, \mathbb{S}_{3,1}}, \underline{\mathbb{S}_1^3\, \mathbb{S}_{2,2}}, \underline{\mathbb{S}_1^3\, \mathbb{S}_{2,1,1}}, \underline{\mathbb{S}_1^4\, \mathbb{S}_3}, \underline{\mathbb{S}_1^4\, \mathbb{S}_{2,1}}, \underline{\mathbb{S}_1^5\, \mathbb{S}_2}, \underline{\mathbb{S}_1^7}\Big\}, \tag{4.6}$$

where we extract explicitly $\mathbb{S}_1$ from the binomial harmonic sums starting with unity $\mathbb{S}_{1,a,b,\ldots} \to \mathbb{S}_1\, \mathbb{S}_{a,b,\ldots}$. Really we will not use for our computations the underlining sums restricted only with the terms, which are not more then the first power of $\mathbb{S}_1$.

At the next step we take ansatz for $\mathcal{P}_{10}^{\mathrm{wrap}}$ from eqs. (4.5)-(4.6) multiplied by c_i and compute for them:

1) analytical continuation at $M = -2 + \omega$;

2) analytical continuation at $M = -1 + \omega$;

3) large M limit.

All above can be compute with the help of HARMPOL [71] and SUMMER [65] packages for FORM [72].

After that we form a set of equations, taking an expressions for ansatz as a left-hand side and the predictions from the Chapter 3 as a right-hand side. Thus, we obtain the system of linear equations on the coefficients under the following quantities:

1) from generalized double-logarithmic equation (3.10) at $M = -2 + \omega$:

$$\Big\{ \frac{1}{\omega^9}, \frac{1}{\omega^8}, \frac{1}{\omega^7}, \frac{\zeta_2}{\omega^7}, \frac{1}{\omega^6}, \frac{\zeta_2}{\omega^6}, \frac{\zeta_3}{\omega^6}, \frac{1}{\omega^5}, \frac{\zeta_2}{\omega^5}, \frac{\zeta_3}{\omega^5}, \frac{\zeta_4}{\omega^5}, \frac{1}{\omega^4}, \frac{\zeta_2}{\omega^4}, \frac{\zeta_3}{\omega^4}, \frac{\zeta_4}{\omega^4}, \frac{\zeta_5}{\omega^4}, \frac{\zeta_2\zeta_3}{\omega^4}, \frac{1}{\omega^3}, \frac{\zeta_2}{\omega^3}, \frac{\zeta_3}{\omega^3},$$

$$\frac{\zeta_4}{\omega^3}, \frac{\zeta_5}{\omega^3}, \frac{\zeta_2\zeta_3}{\omega^3}, \frac{\zeta_6}{\omega^3}, \frac{\zeta_3^2}{\omega^3}, \frac{1}{\omega^2}, \frac{\zeta_2}{\omega^2}, \frac{\zeta_3}{\omega^2}, \frac{\zeta_4}{\omega^2}, \frac{\zeta_5}{\omega^2}, \frac{\zeta_2\zeta_3}{\omega^2}, \frac{\zeta_6}{\omega^2}, \frac{\zeta_3^2}{\omega^2}, \frac{\zeta_7}{\omega^2}, \frac{\zeta_2\zeta_5}{\omega^2}, \frac{\zeta_3\zeta_4}{\omega^2}\Big\} \tag{4.7}$$

2) from BFKL equation (3.7) at $M = -1 + \omega$:

$$\Big\{ \frac{1}{\omega^9}, \frac{\zeta_2}{\omega^7}, \frac{\zeta_2}{\omega^6}, \frac{\zeta_4}{\omega^5}, \frac{\zeta_5}{\omega^4}, \frac{\zeta_2\zeta_3}{\omega^4}\Big\} \tag{4.8}$$

3) from large M limit:

$$S_\infty^2 \Big\{ \zeta_7, \zeta_2\zeta_5, \zeta_4\zeta_3, S_\infty\zeta_6, S_\infty\zeta_3^2, S_\infty^2\zeta_5, S_\infty^2\zeta_2\zeta_3\Big\}, \qquad S_\infty = S_1(\infty) \tag{4.9}$$

So, we have the system from 49 equations with 59 unknowns. However, some of them are linear depended. Remove such equations, we are left with 40 equations on 59 unknowns.

As was pointed out in Introduction the obtained equations have a rather special form - they give rise the system of linear Diophantine equations, because we are believe, that the coefficients in ansatz, which we are looking for, are the integer numbers. To solve this problem we will address to two methods from the numbers theory: LLL-algorithm and Linear Programming.

4.1 LLL-algorithm

The Lenstra-Lenstra-Lovász (LLL) lattice basis reduction algorithm is a polynomial time lattice reduction algorithm invented by Arjen Lenstra, Hendrik Lenstra and László Lovász in 1982 [67]. The original applications were to give polynomial time algorithms for factorizing polynomials with rational coefficients into irreducible polynomials, for finding simultaneous rational approximations to real numbers, and for solving the integer linear programming problem in fixed dimensions.

Let's give step-by-step description how to apply the LLL-algorithm to the solution of our problem on the toy example. We will use `LatticeReduce` function of `MATHEMATICA`, which realize LLL-algorithm.[4]

- We start from the system of equations:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0$$
$$\frac{15}{32}x_1 + \frac{27}{32}x_2 + \frac{33}{32}x_3 + \frac{39}{32}x_4 + \frac{21}{32}x_5 = \frac{9}{16} \tag{4.10}$$

- take matrix from the coefficients of above system of equations and right-hand side term as last columns

- divide each row of above matrix to the greatest common divisor (`GCD` function):

$$\mathsf{SE} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 5 & 9 & 11 & 13 & 7 & -6 \end{pmatrix}$$

- multiply SE to some huge integer number, for example 8^8

- create unity matrix $\mathbb{I}$ with rank equal to the length of row in SE

- append transpose SE to the right side of the unity matrix $\mathbb{I}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 8^8 & 5 \times 8^8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 8^8 & 9 \times 8^8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 8^8 & 11 \times 8^8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 8^8 & 13 \times 8^8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 8^8 & 7 \times 8^8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -6 \times 8^8 \end{pmatrix}$$

---

[4]See `Application` on http://reference.wolfram.com/mathematica/ref/LatticeReduce.html

- apply `LatticeReduce` to this matrix

As result we obtain the following matrix:

$$
\mathrm{RSE} = \begin{pmatrix}
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -8^8 & -8^8 \\
-1 & 0 & 0 & 0 & 0 & -1 & -8^8 & 8^8
\end{pmatrix}
\tag{4.11}
$$

We interesting only with rows, which are non zero at position 6 (number of variables plus one) and with the rest numbers at right equals zeroes. Only row 2 satisfies these criteria. Indeed, this is the solution, which we were looking for. This example is related with the rational part of the wrapping corrections for the four-loop anomalous dimension of twist-2 operators, which was calculated in ref. [34]. That is, we try to obtain the expression

$$
\Delta_8^{\mathrm{wrap,\, rational}}(M) = S_{-2,1} - \frac{S_{-3}}{2} - S_{-2}S_1 - S_1 S_2 - \frac{S_3}{2}
\tag{4.12}
$$

from the values at $M = 1$ and at $M = 2$ for the following basis with the binomial harmonic sums

$$
\{\mathbb{S}_{3,2}, \mathbb{S}_{2,3}, \mathbb{S}_{2,2,1}, \mathbb{S}_{2,1,2}, \mathbb{S}_{3,1,1}\} \; .
\tag{4.13}
$$

Obtained results is looks like:

$$
\Delta_8^{\mathrm{wrap,\, rational}}(M) = -\mathbb{S}_{2,1,2} + \mathbb{S}_{3,1,1} \; .
\tag{4.14}
$$

Note one important future of LLL-algorithm (and all other similar algorithm): a desired numbers should be small, at least most of them, because algorithm is looking for the vectors with minimal Euclidean norm. Therefore, for the reconstruction of the five-loop anomalous dimension of twist-2 operators we should eliminate some possible large numbers from the result, which we are looking for. From the four-loop anomalous dimension of twist-2 operators [34] and the five-loop anomalous dimension of twist-3 operators [44] we can find such large numbers as the coefficients in the front of terms in the basis with zeta-numbers ζ_i. To avoid possible difficulties we can eliminate the variables from the system of equations, which should be a large numbers as we believe. However we should keep as much equations as possible because each equation preserves information that its solution should be an integer numbers, i.e. if we exclude some of variables we can obtain after solution that these variables will not integers. First of all we should eliminate the variable related with $\mathbb{S}_1^2 \zeta_7$ term from the basis (4.6) as the most dangerous. Moreover, because we do not know overall normalization of the result, we can try a different variants for the normalization, for example in the form 2^{-i}, which will a prefactor of the right hand side of our equations (or we can rescale the binomial harmonic sums (1.3)). For the $i = 9$ we obtain the solution with rather small sum of absolute values and with a lot of zeros

$$
\{\, 4, 1, -6, -40, -4, 8, -8, -4, 0, 12, 0, 0, 0, 0, 0, 0, 2, 0, -2, 4, 4, -10, 0, 0, 4, -2, 0, 4, 4, -4, 4,
$$
$$
-4, -4, 4, -4, 0, 2, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, -2, 2, 0, 0, 0, 0, 0, 0, 0, -16, 0, \ldots \}\,, \tag{4.15}
$$

where first two numbers corresponds c_1 and c_2 and the rest numbers are listed according to the basis (4.6) with missing $\mathbb{S}_1^2 \zeta_7$ term, which can be obtained from one of the initial equations (its equals 105). Indeed, this solution (multiplied by common factor $2^{9-4} = 32$) gives the result, which is the same as was obtained earlier in ref. [36] with the calculation of the finite size effects.

In this way we get the result for the wrapping effect part of the five-loop anomalous dimension of twist-2 operators without any calculations at all, using only the properties of harmonic sums and the available information, obtained from the already known results from the low orders of perturbative theory.

4.2 Linear Programming

We have found also other method, which can be applied for the solution of the system of linear Diophantine equations. As we pointed out the LLL-algorithm seeking the vectors with the minimal Euclidean norms. However, it seems, that the more correct criteria is a minimization of the sum of absolute values of coefficients instead of the sums of their squares. In this case our problem is reduced to Integer Linear Problem, which is particular case of Linear Programming or Linear Optimization. Formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. We have a lot of constraints (equations) and want to find the best solution, which satisfied some global condition, in our case - the minimum of the sum of the absolute values. `MATHEMATICA` has a functions, which are related with an optimization. Most simple is `Minimize`. Application of `Minimize` is rather straightforward and we will not described it here.

In general, `Minimize` function tries to find the global minimum in the multidimensional space (equal to the number of variables) under the required conditions and then move to the nearest integer solution. In our case we will have 58-dimensional space and such task can not be solved with `MATHEMATICA` in a reasonable time. However we have found some interesting future of the solution for the five-loop anomalous dimension of twist-2 operators. In principal, we can use some additional information about anomalous dimension. Namely, we used the results for the one impurity state for the twist-2 operators in the β-deformed $\mathcal{N} = 4$ SYM theory, which are known even up to six-loop order [34, 35, 57]. This result corresponds to the value of the anomalous dimension of the twist-2 operators in $\mathcal{N} = 4$ SYM theory at $M = 1$. Exclude the variables related with $\mathbb{S}_1^2 \zeta_7$ and $\mathbb{S}_1^2 \zeta_5 \mathbb{S}_2$ terms from the basis (4.6) solving the most simple equations with these variables we obtain the system in which, as we believe, all unknowns should be less then $2^4 = 16$ (we also rescaled all binomial harmonic sums in basis (4.6) by factor $2^5 = 32$). Run `NMinimize` under this conditions we have found, that the obtained *numerical* solution is much close to the solution, which we should obtain. In other words the global minimum of the multidimensional problem lies near the exact solution!

Because the Linear Optimization is very actual at the present time there are a lot of specials programm, which realize this procedure in a rapid way with the usage of special algorithm and which can be run under modern computer clusters with parallelization. Usually Linear Optimization works with a positive integer quantities and an available

programs are restricted only to positive integer numbers. But because we do not know an exact sign of the coefficients in ansatz we should symmetrize the corresponding variables, doubling the number of unknowns. It would be very interesting find such a symmetry, which can fix these signs.

5. Conclusion and discussion

In this paper we reconstruct the five-loop anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory using a known constraints from the BFKL equation, generalized double-logarithmic equations and from the large spin limit with the help of numerical methods and without any special computations.

In principal described methods can be used for the solution of a similar problems, where desired results is looking in a known basis and some information about the properties of this results are available. In spite of simplicity the LLL-algorithm gives a desired results in a more simple way. As an example of its application we have reconstructed the next-to-leading order (NLO) wrapping correction to the anomalous dimension of the twist-2 operators in the β-deformed $\mathcal{N} = 4$ SYM theory from the results of ref. [73], which can be find in Appendix A. Note, however, that LLL-algorithm can not find a desired solution if the rank the system of equations is much less, then the numbers of variables. Linear Programming is more exact method, but more time-consuming. We used its for the reconstruction of the non-planar contribution to the four-loop anomalous dimension of the twist-2 operators in $\mathcal{N} = 4$ SYM theory from our results [74, 75] and some additional constraints for which the usage of LLL-algorithm is beyond its applicability [5]. Probably, we should combine these two methods using an output of LLL-algorithm as an input (first approximation or initial solution) of linear programming methods.

It seems, that at six loops an available information is not enough, because the basis from the binomial harmonic sums growth faster, then the number of constraints. There are some possibilities, which can help to solve this problem. First of all we can used some information, which is available for low values of M: six-loop anomalous dimension for one-impurity state in β-deformed $\mathcal{N} = 4$ SYM theory [57], six-loop anomalous dimension for Konishi [76, 77, 78] and other, but this information will based on the methods, which we do not want to use. Another possibility is try to find an additional symmetry, which can reduce the basis, either extend the number of constraints, for example, from the studying of large M limit or an expansion near $M = 0$ [79].

Acknowledgments

---

[5]We have found some reasonable solutions, but we are going to check their with the forthcoming direct perturbative calculations.

A. NLO wrapping corrections in the β-deformed $\mathcal{N} = 4$ SYM theory

In this Appendix we give a general expression for the next-to-leading order (NLO) wrapping correction to the anomalous dimension of the twist-2 operators in the β-deformed $\mathcal{N} = 4$ SYM theory from the results of ref. [73] reconstructed with the help of LLL-algorithm. The wrapping corrections in the leading order for twist-2 operators can be written as (see eq. (4.5) in ref. [73]):

$$E_{\text{LO}}^{\text{wrap}}(M) = 4g^6 \sin^2(2\pi\beta)\frac{S_1(M)S_{-2}(M)}{M(M+1)} \tag{A.1}$$

The results for the wrapping corrections in the next-to leading order can be find in the table from ref. [73] for the first ten even values of M. To reconstruct a general expression we write down the following basis:

$$\left\{ S_1\frac{S_1S_{-2}}{M}, S_1\frac{S_1S_{-2}}{M+1}, S_1S_2S_{-2}, S_1S_1S_{-3}, \mathbb{S}_5, \mathbb{S}_{4,1}, \mathbb{S}_{3,2}, \mathbb{S}_{2,3}, \mathbb{S}_{3,1,1}, \mathbb{S}_{2,2,1}, \mathbb{S}_{2,1,2}, \mathbb{S}_{2,1,1,1}, \right.$$
$$\left. \mathbb{S}_1\mathbb{S}_4, \mathbb{S}_1\mathbb{S}_{3,1}, \mathbb{S}_1\mathbb{S}_{2,2}, \mathbb{S}_1\mathbb{S}_{2,1,1}, \mathbb{S}_1^2\mathbb{S}_3, \mathbb{S}_1^2\mathbb{S}_{2,1} \right\}\frac{1}{M(M+1)}, \tag{A.2}$$

where first four terms come from the reciprocity $(\gamma_0 E_{\text{LO}}^{\text{wrap}})'$. We need only four first values from the table of ref. [73] to obtain with the help of `LatticeReduce` functions the following vector:

$$\{16, 0, 16, 16, 0, -1, -1, 0, -2, 1, -1, 0, -2, 4, -2, 0, 1, 0, 4, 0, 0, 0, 0\} \tag{A.3}$$

which give the desired result:

$$E_{\text{NLO}}^{\text{wrap}}(M) = g^8 \sin^2(2\pi\beta)\frac{1}{4\,M(M+1)}\left[16\left(S_1\frac{S_1S_{-2}}{M} + S_1S_2S_{-2} + S_1S_1S_{-3}\right)\right.$$
$$\left. -\mathbb{S}_{4,1} - \mathbb{S}_{3,2} - 2\,\mathbb{S}_{3,1,1} + \mathbb{S}_{2,2,1} - \mathbb{S}_{2,1,2} - 2\,\mathbb{S}_1\mathbb{S}_4 + 4\,\mathbb{S}_1\mathbb{S}_{3,1} - 2\,\mathbb{S}_1\mathbb{S}_{2,2} + \mathbb{S}_1^2\mathbb{S}_3\right]. \tag{A.4}$$

One can easely check with other values from the table of ref. [73] that this is indeed the correct answer.

References

[1] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[4] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, *Strings in flat space and pp waves from N = 4 super Yang Mills*, *JHEP* **04** (2002) 013, [hep-th/0202021].

[5] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, *JHEP* **03** (2003) 013, [hep-th/0212208].

[6] L. N. Lipatov, *High-energy asymptotics of multicolor QCD and exactly solvable lattice models*, hep-th/9311037.

[7] L. N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models*, *JETP Lett.* **59** (1994) 596–599. [Pisma Zh.Eksp.Teor.Fiz.59:571-574,1994].

[8] L. D. Faddeev and G. P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett.* **B342** (1995) 311–322, [hep-th/9404173].

[9] V. M. Braun, S. E. Derkachov, and A. N. Manashov, *Integrability of three-particle evolution equations in QCD*, *Phys. Rev. Lett.* **81** (1998) 2020–2023, [hep-ph/9805225].

[10] I. Bena, J. Polchinski, and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D69** (2004) 046002, [hep-th/0305116].

[11] N. Beisert, C. Kristjansen, and M. Staudacher, *The dilatation operator of N = 4 super Yang-Mills theory*, *Nucl. Phys.* **B664** (2003) 131–184, [hep-th/0303060].

[12] N. Beisert and M. Staudacher, *The N=4 SYM Integrable Super Spin Chain*, *Nucl. Phys.* **B670** (2003) 439–463, [hep-th/0307042].

[13] N. Beisert, *The Complete One-Loop Dilatation Operator of N=4 Super Yang-Mills Theory*, *Nucl. Phys.* **B676** (2004) 3–42, [hep-th/0307015].

[14] N. Beisert, *The su(2—3) dynamic spin chain*, *Nucl. Phys.* **B682** (2004) 487–520, [hep-th/0310252].

[15] D. Serban and M. Staudacher, *Planar N = 4 gauge theory and the Inozemtsev long range spin chain*, *JHEP* **06** (2004) 001, [hep-th/0401057].

[16] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024, [hep-th/0402207].

[17] N. Beisert, V. Dippel, and M. Staudacher, *A novel long range spin chain and planar N = 4 super Yang- Mills*, *JHEP* **07** (2004) 075, [hep-th/0405001].

[18] G. Arutyunov, S. Frolov, and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016, [hep-th/0406256].

[19] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054, [hep-th/0412188].

[20] N. Beisert and M. Staudacher, *Long-range PSU(2, 2|4) Bethe ansaetze for gauge theory and strings*, Nucl. Phys. **B727** (2005) 1–62, [`hep-th/0504190`].

[21] N. Beisert, *The su(2|2) dynamic S-matrix*, Adv. Theor. Math. Phys. **12** (2008) 945–979, [`hep-th/0511082`].

[22] R. A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, Phys. Rev. **D73** (2006) 086006, [`hep-th/0603038`].

[23] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, JHEP **07** (2006) 004, [`hep-th/0603204`].

[24] G. Arutyunov and S. Frolov, *On $AdS_5 \times S^5$ string S-matrix*, Phys. Lett. **B639** (2006) 378–382, [`hep-th/0604043`].

[25] N. Beisert, R. Hernandez, and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, JHEP **11** (2006) 070, [`hep-th/0609044`].

[26] B. Eden and M. Staudacher, *Integrability and transcendentality*, J. Stat. Mech. **0611** (2006) P11014, [`hep-th/0603157`].

[27] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, *The Four-Loop Planar Amplitude and Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory*, Phys. Rev. **D75** (2007) 085010, [`hep-th/0610248`].

[28] N. Beisert, B. Eden, and M. Staudacher, *Transcendentality and crossing*, J. Stat. Mech. **0701** (2007) P01021, [`hep-th/0610251`].

[29] S. Schafer-Nameki, M. Zamaklar, and K. Zarembo, *How Accurate is the Quantum String Bethe Ansatz?*, JHEP **12** (2006) 020, [`hep-th/0610250`].

[30] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher, and V. N. Velizhanin, *Dressing and Wrapping*, J. Stat. Mech. **0710** (2007) P10003, [`arXiv:0704.3586`].

[31] M. Luscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States*, Commun. Math. Phys. **104** (1986) 177.

[32] M. Luscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States*, Commun. Math. Phys. **105** (1986) 153–188.

[33] Z. Bajnok and R. A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, Nucl. Phys. **B807** (2009) 625–650, [`arXiv:0807.0399`].

[34] Z. Bajnok, R. A. Janik, and T. Lukowski, *Four loop twist two, BFKL, wrapping and strings*, Nucl. Phys. **B816** (2009) 376–398, [`arXiv:0811.4448`].

[35] Z. Bajnok, A. Hegedus, R. A. Janik, and T. Lukowski, *Five loop Konishi from AdS/CFT*, Nucl. Phys. **B827** (2010) 426–456, [`arXiv:0906.4062`].

[36] T. Lukowski, A. Rej, and V. N. Velizhanin, *Five-Loop Anomalous Dimension of Twist-Two Operators*, Nucl. Phys. **B831** (2010) 105–132, [`arXiv:0912.1624`].

[37] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, *Wrapping at four loops in N=4 SYM*, Phys. Lett. **B666** (2008) 100–105, [`arXiv:0712.3522`].

[38] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, *Anomalous dimension with wrapping at four loops in N=4 SYM*, Nucl. Phys. **B805** (2008) 231–266, [`arXiv:0806.2095`].

[39] V. N. Velizhanin, *The four-loop anomalous dimension of the Konishi operator in N=4 supersymmetric Yang-Mills theory*, JETP Lett. **89** (2009) 6–9, [arXiv:0808.3832].

[40] V. N. Velizhanin, *Leading transcedentality contributions to the four-loop universal anomalous dimension in N=4 SYM*, Phys. Lett. **B676** (2009) 112–115, [arXiv:0811.0607].

[41] L. N. Lipatov, *Reggeization of the Vector Meson and the Vacuum Singularity in Nonabelian Gauge Theories*, Sov. J. Nucl. Phys. **23** (1976) 338–345. [Yad.Fiz.23:642-656,1976].

[42] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *The Pomeranchuk Singularity in Nonabelian Gauge Theories*, Sov. Phys. JETP **45** (1977) 199–204. [Zh.Eksp.Teor.Fiz.72:377-389,1977].

[43] I. I. Balitsky and L. N. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, Sov. J. Nucl. Phys. **28** (1978) 822–829. [Yad.Fiz.28:1597-1611,1978].

[44] M. Beccaria, V. Forini, T. Lukowski, and S. Zieme, *Twist-three at five loops, Bethe Ansatz and wrapping*, JHEP **03** (2009) 129, [arXiv:0901.4864].

[45] V. N. Velizhanin, *Six-Loop Anomalous Dimension of Twist-Three Operators in N=4 SYM*, JHEP **11** (2010) 129, [arXiv:1003.4717].

[46] A. B. Zamolodchikov, *THERMODYNAMIC BETHE ANSATZ IN RELATIVISTIC MODELS. SCALING THREE STATE POTTS AND LEE-YANG MODELS*, Nucl. Phys. **B342** (1990) 695–720.

[47] A. B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, Phys. Lett. **B253** (1991) 391–394.

[48] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory*, Phys. Rev. Lett. **103** (2009) 131601, [arXiv:0901.3753].

[49] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states*, Lett. Math. Phys. **91** (2010) 265–287, [arXiv:0902.4458].

[50] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang- Mills Theory: Konishi Dimension at Any Coupling*, Phys. Rev. Lett. **104** (2010) 211601, [arXiv:0906.4240].

[51] D. Bombardelli, D. Fioravanti, and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal*, J. Phys. **A42** (2009) 375401, [arXiv:0902.3930].

[52] G. Arutyunov, S. Frolov, and R. Suzuki, *Exploring the mirror TBA*, JHEP **05** (2010) 031, [arXiv:0911.2224].

[53] G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model*, JHEP **05** (2009) 068, [arXiv:0903.0141].

[54] G. Arutyunov, S. Frolov, and R. Suzuki, *Five-loop Konishi from the Mirror TBA*, JHEP **04** (2010) 069, [arXiv:1002.1711].

[55] J. Balog and A. Hegedus, *5-loop Konishi from linearized TBA and the XXX magnet*, JHEP **06** (2010) 080, [arXiv:1002.4142].

[56] J. Balog and A. Hegedus, *The Bajnok-Janik formula and wrapping corrections*, JHEP **09** (2010) 107, [arXiv:1003.4303].

[57] Z. Bajnok and O. el Deeb, *6-loop anomalous dimension of a single impurity operator from AdS/CFT and multiple zeta values*, *JHEP* **01** (2011) 054, [arXiv:1010.5606].

[58] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov, and G. V. Frolov, *Doubly logarithmic asymptotic behavior in quantum electrodynamics*, *Sov. J. Nucl. Phys.* **6** (1968) 95. [Yad.Fiz. 6:129,1967].

[59] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov, and G. V. Frolov, *Backward electron - positron scattering at high- energies*, *Sov. J. Nucl. Phys.* **6** (1968) 262. [Yad.Fiz.6:361,1967].

[60] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov, and G. V. Frolov, *Double logarithmic asymptotics of quantum electrodynamics*, *Phys. Lett.* **22** (1966) 671–673.

[61] R. Kirschner and L. N. Lipatov, *DOUBLE LOGARITHMIC ASYMPTOTICS OF QUARK SCATTERING AMPLITUDES WITH FLAVOR EXCHANGE*, *Phys. Rev.* **D26** (1982) 1202–1205.

[62] R. Kirschner and L. n. Lipatov, *DOUBLY LOGARITHMIC ASYMPTOTIC OF THE QUARK SCATTERING AMPLITUDE WITH NONVACUUM EXCHANGE IN THE T CHANNEL*, *Sov. Phys. JETP* **56** (1982) 266–273. [Zh.Eksp.Teor.Fiz.83:488-501,1982].

[63] R. Kirschner and L. n. Lipatov, *Double Logarithmic Asymptotics and Regge Singularities of Quark Amplitudes with Flavor Exchange*, *Nucl. Phys.* **B213** (1983) 122–148.

[64] V. N. Velizhanin, *Double-logs, Gribov-Lipatov reciprocity and wrapping*, *JHEP* **08** (2011) 092, [arXiv:1104.4100].

[65] J. A. M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys.* **A14** (1999) 2037–2076, [hep-ph/9806280].

[66] A. V. Kotikov and L. N. Lipatov, *DGLAP and BFKL evolution equations in the N=4 supersymmetric gauge theory*, *Nucl. Phys.* **B661** (2003) 19–61, [hep-ph/0208220]. [Erratum-ibid.B685:405-407,2004].

[67] A. K. Lenstra, H. W. Lenstra, and L. Lovasz, *Factoring polynomials with rational coefficients*, *Math. Ann.* **261** (1982) 515–534.

[68] M. Beccaria, A. V. Belitsky, A. V. Kotikov, and S. Zieme, *Analytic solution of the multiloop Baxter equation*, *Nucl. Phys.* **B827** (2010) 565–606, [arXiv:0908.0520].

[69] V. S. Fadin and L. N. Lipatov, *BFKL pomeron in the next-to-leading approximation*, *Phys. Lett.* **B429** (1998) 127–134, [hep-ph/9802290].

[70] A. V. Kotikov and L. N. Lipatov, *NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories*, *Nucl. Phys.* **B582** (2000) 19–43, [hep-ph/0004008].

[71] E. Remiddi and J. A. M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys.* **A15** (2000) 725–754, [hep-ph/9905237].

[72] J. A. M. Vermaseren, *New features of FORM*, math-ph/0010025.

[73] M. de Leeuw and T. Lukowski, *Twist operators in N=4 beta-deformed theory*, *JHEP* **04** (2011) 084, [arXiv:1012.3725].

[74] V. N. Velizhanin, *The non-planar contribution to the four-loop universal anomalous dimension in N=4 Supersymmetric Yang-Mills theory*, *JETP Lett.* **89** (2009) 593–596, [arXiv:0902.4646].

[75] V. N. Velizhanin, *The non-planar contribution to the four-loop anomalous dimension of twist-2 operators: first moments in N=4 SYM and non-singlet QCD*, Nucl. Phys. **B846** (2011) 137–144, [arXiv:1008.2752].

[76] Z. Bajnok and R. A. Janik, *Six and seven loop Konishi from Luscher corrections*, JHEP **1211** (2012) 002, [arXiv:1209.0791].

[77] S. Leurent, D. Serban, and D. Volin, *Six-loop Konishi anomalous dimension from the Y-system*, Phys.Rev.Lett. **109** (2012) 241601, [arXiv:1209.0749].

[78] S. Leurent and D. Volin, *Multiple zeta functions and double wrapping in planar N=4 SYM*, arXiv:1302.1135.

[79] B. Basso, *An exact slope for AdS/CFT*, arXiv:1109.3154.