The complete one-loop dilatation operator of planar real β-deformed $\mathcal{N} = 4$ SYM theory

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Abstract. We determine the missing finite-size corrections to the asymptotic one-loop dilatation operator of the real β-deformed $\mathcal{N} = 4$ SYM theory for the gauge groups $U(N)$ and $SU(N)$ in the 't Hooft limit. In the $SU(N)$ case, the absence of the $U(1)$ field components leads to a new kind of finite-size effect, which we call prewrapping. We classify which states are potentially affected by it at generic loop orders and comment on the necessity to include it into the integrability-based description. As a further result, we identify classes of n-point correlation functions which at all loop orders in the planar theory are given by the values of their undeformed counterparts. Finally, we determine the superconformal multiplet structure and one-loop anomalous dimensions of all single-trace states with classical scaling dimension $\Delta_0 \leq 4.5$.

Contents

1 Introduction

1.1 General setup and conformality

The AdS/CFT correspondence [1–3] predicts dualities between certain string theories in anti-de Sitter (AdS) space and conformal field theories (CFTs). Its most prominent example relates type IIB string theory in $\text{AdS}_5 \times \text{S}^5$ with N units of five-form flux to the four-dimensional maximally ($\mathcal{N} = 4$) supersymmetric Yang-Mills (SYM) theory with gauge group $SU(N)$. It is most accessible in the 't Hooft (planar) limit [4], where $N \to \infty$ and the Yang-Mills coupling constant $g_{\text{YM}} \to 0$ such that the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ is kept fixed: the string theory becomes free, and in the gauge theory non-planar vacuum diagrams are suppressed.[1] Starting from this maximally (super)symmetric setup, further examples can be found e.g. by discrete orbifold projections [6, 7] or by applying deformations which depend on continuous parameters [8–11].

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[1]Non-planar non-vacuum diagrams may, however, become planar when connected to external states, and thus may contribute in the 't Hooft limit [5]. They give rise to so-called finite-size corrections, which are the main object of this work.

The prime example of such continuous deformations is the correspondence between the $\mathcal{N} = 1$ supersymmetric real β-deformed $\mathcal{N} = 4$ SYM theory (β-deformation) and the type II B string theory in the Lunin-Maldacena background. In $\mathcal{N} = 1$ superspace the gauge-theory action reads

$$S = \frac{1}{2g_{\mathrm{YM}}^2} \int \mathrm{d}x^4 \, \mathrm{d}\theta^2 \, \mathrm{tr} \left(W^\alpha W_\alpha \right) + \int \mathrm{d}x^4 \, \mathrm{d}\theta^4 \, \mathrm{tr} \left(\mathrm{e}^{-g_{\mathrm{YM}}V} \, \bar{\Phi}_i \, \mathrm{e}^{g_{\mathrm{YM}}V} \, \Phi^i \right)$$
$$+ i g_{\mathrm{YM}} \int \mathrm{d}x^4 \, \mathrm{d}\theta^2 \, \mathrm{tr} \left(\Phi^1 \Phi^2 \Phi^3 \, \mathrm{e}^{-i\frac{\beta}{2}} - \Phi^1 \Phi^3 \Phi^2 \, \mathrm{e}^{i\frac{\beta}{2}} \right) + \mathrm{h.c.} \,, \tag{1}$$

which reduces to the one of $\mathcal{N} = 4$ SYM theory if one sets the real parameter β to zero. The above deformation is a special case of the more general exactly marginal Leigh-Strassler deformations [12]. In [8] Lunin and Maldacena discussed the corresponding string theory background, which can be constructed by applying a TsT transformation, i.e. a T-duality, a shift (s) of an angular variable and another T-duality, to the S^5-factor in $\mathrm{AdS}_5 \times \mathrm{S}^5$. This breaks the isometry group $SO(6)$ of the S^5 to its $U(1)_{Q^1} \times U(1)_{Q^2} \times U(1)_R$ Cartan subgroup. In [8] it was also found that the β-deformation can be formulated by replacing all products in the undeformed superspace action by noncommutative $*$-products that introduce phase factors depending on the $U(1)_{Q^1} \times U(1)_{Q^2}$ Cartan charges of the respective fields.

All fields of the $\mathcal{N} = 4$ SYM theory transform in the adjoint representation of the gauge group. In the action, their representation matrices are contracted forming a single trace in colour space. Moreover, in the interactions each of these matrix products can be rephrased in terms of a commutator. Hence, if one considers gauge group $U(N)$, the $U(1)$ components of all fields decouple, and the $U(N)$ theory is essentially the same as the $SU(N)$ theory.

The β-deformation, however, does distinguish between the gauge groups $U(N)$ and $SU(N)$. While the $U(1)$ component of the vector superfield V still decouples, this is no longer the case for the matter superfields $\Phi^i, \bar{\Phi}^i$ of flavours $i = 1, 2, 3$. Beyond that, the $U(N)$ theory is not even conformally invariant. This can be seen e.g. in a component expansion of the action (1), where quantum corrections induce the running of a quartic double-trace coupling. While its tree-level value vanishes in the $U(N)$ theory, it is at its nonvanishing IR fix point value in the $SU(N)$ theory [13].[2] This value is found by integrating out the F-term auxiliary fields from the component expansion of (1), cf. [14]. Although coming with a prefactor of $\frac{1}{N}$, these double-trace terms do contribute in the 't Hooft limit. As we have argued in detail in [14], they were, however, neglected in the proof of conformal invariance in [15], which only considers planar single-trace couplings.

In a gauge theory without conformal symmetry, the (anomalous) scaling dimensions of gauge invariant composite operators depend on the β-functions of the couplings and thus are renormalisation-scheme dependent starting from the second loop order. In a CFT, however, the β-functions vanish and with them this

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[2]Note that the nonsupersymmetric three-parameter γ_i-deformation, which was proposed as candidate gauge theory of a generalisation of the Lunin-Maldacena AdS/CFT setup in [10], is not conformally invariant, neither for $U(N)$ nor $SU(N)$ – and not even in the 't Hooft limit. There, additional double-trace couplings are induced whose β-functions have no fixed points as functions of λ – see our work [14].

renormalisation-scheme dependence. The anomalous dimensions are then observables, and the AdS/CFT correspondence conjectures that they match the energies of respective string states in the gravitational theory.

1.2 The dilatation operator in $\mathcal{N} = 4$ SYM theory

The scaling dimensions can be measured as eigenvalues of the the generator of dilatations, known as dilatation operator. In the 't Hooft limit it admits a perturbative expansion in the small effective planar coupling g:

$$D = D_0 + g^2 D_2 + \mathcal{O}(g^3), \qquad g = \frac{\sqrt{\lambda}}{4\pi}, \tag{2}$$

where only the classical piece D_0 and the one-loop correction D_2 are shown. These operators, as well as higher-order corrections, can (in principle) be computed via Feynman diagrams, see [16, 17] for reviews. Moreover, it is sufficient to consider the action of D on the subset of gauge-invariant composite operators with a single colour-trace. On these single-trace operators, which can be represented as cyclic spin chains, D_2 acts as an integrable Hamiltonian. This integrability appears to persist also beyond the first loop order. The conjecture of all-loop integrability has led to enormous progress in checking and understanding the AdS/CFT correspondence, see the review collection [18] for a comprehensive list of references.

On the basis of cyclic spin-chain states of length L, the dilatation operator D can be written in terms of a (site-independent) density $\mathfrak{D}$ as

$$D_{2K} = \sum_{i=1}^{L} D_{i,i+1,\dots,i+K}, \qquad D_{i,i+1\dots,i+K} = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{i-1} \otimes \mathfrak{D}_{2K} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{L-i-K}, \tag{3}$$

where $i + L$ is identified with i. Here, D_{2K} denotes the contribution at order g^{2K} and, for the sake of simplicity, we have neglected length-changing contributions appearing at loop orders $K > 1$. Note that the density $\mathfrak{D}_{2K}$ has an interaction range $R = K+1$, i.e. a maximum of R neighbouring sites interact with each other. In particular, the one-loop density $\mathfrak{D}_2$ contains at most nearest-neighbour interactions. For the $\mathcal{N} = 4$ SYM theory $\mathfrak{D}_2^{\mathcal{N}=4}$ was found by Beisert in [19].

Since R increases by one with each additional loop order, an obvious problem occurs when R exceeds the length L of the state, which happens at loop-orders $K \geq L$. This necessitates finite-size corrections, i.e. length-dependent corrections to the asymptotic density. In the field-theory picture of $\mathcal{N} = 4$ SYM theory, such corrections occur in the form of wrapping diagrams and were analysed in detail in [5]. They originate from the fact that in the 't Hooft limit diagrams with external legs may contribute even if their N-power is naively subleading. The decision whether a diagram contributes can a priori only be made for diagrams in which all colour lines are closed, i.e. external lines have to be connected to external states (composite operators). In the notation of [5], a connected diagram with external legs and without external states is called planar if it contributes at leading order in the $\frac{1}{N}$-expansion when its external legs are planarly contracted with a single-trace vertex. Besides these diagrams, in the 't Hooft limit there may be contributions from non-planar connected

diagrams, which effectively are multi-trace interactions. Such an interaction enhances its N-power if one of its colour-trace factors is fully and planarly contracted with a colour-trace of the same length in an external state.

In the integrability-based description of $\mathcal{N} = 4$ SYM theory, wrapping effects are incorporated in terms of Lüscher corrections, Y-system and the thermodynamic Bethe ansatz (TBA). They all correct the result from the asymptotic Bethe ansatz and match the available field-theory data – see [17, 20–23] for reviews. More recently, these formalisms were further developed to the so-called finite non-linear integral equations (FiNLIE) [24] and the quantum algebraic curve (QAC) [25].

1.3 The dilatation operator in the β-deformation

The β-deformation is closely connected to the $\mathcal{N} = 4$ SYM theory, such that many of the methods and also some of the results can be adopted. In particular, it is claimed to be as integrable as its undeformed parent theory, see [26] for a review.

At the asymptotic level, the formulation of the β-deformation in terms of non-commutative Moyal-like $*$-products allows the adaptation of a particular theorem derived for spacetime noncommutative field theories in [27]: the contribution of a planar diagram in the deformed theory is given by its undeformed counterpart times a phase factor that is determined by the order and charges of the external fields alone. Beisert and Roiban used this theorem in [28] to propose a planar one-loop dilatation-operator density in the deformed theory. In addition, they deformed the asymptotic Bethe ansatz by introducing twisted boundary conditions.

Subsequently, also wrapping corrections were investigated. They can conveniently be studied for so-called single-impurity operators, which are single-trace operators containing $L - 1$ chiral scalar fields of one flavour and a single chiral scalar field of a different flavour. For $L \geq 3$ the respective field-theory results of [29] were reproduced in [30] for $\beta = \frac{1}{2}$ and in [31] and [32] for generic β.[3] They are insensitive to the choice of $U(N)$ or $SU(N)$ as gauge group.

For the $L = 2$ single-impurity operator, however, a sensitivity to that choice was observed already some years earlier in [34]: for $SU(N)$ its one-loop anomalous dimension vanishes identically whereas for $U(N)$ it is nonvanishing. In [11] it was noted that the latter result is reproduced by the one-loop dilatation operator as well as the asymptotic Bethe ansatz, both proposed in [28]. But up to now the results of [28] could not be modified to incorporate the conformally invariant $SU(N)$ theory, which is the CFT candidate for the AdS/CFT correspondence.[4]

The situation becomes worse at the level of wrapping corrections. According to field-theory calculations, the two-loop anomalous dimension of the $L = 2$ single-impurity state vanishes for gauge group $SU(N)$ [35]. The corresponding integrability-based result of [32], however, is logarithmically divergent,[5] which cannot even be

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[3]See [32, 33] for higher-order results.

[4]See [11] for some comments concerning the differences between $U(N)$ and $SU(N)$ gauge group in the deformed AdS/CFT correspondence.

[5]Such a divergence was encountered earlier in the expressions for the ground-state energy of the TBA [36]. In [37] it was found that the divergent ground-state energy vanishes in the undeformed theory when a regulating twist is introduced in the AdS$_5$ directions. This regularisation extends to the ground state of the supersymmetric deformations [38].

correct for gauge group $U(N)$.

In this paper we address the above problems from a field-theory perspective.

1.4 Organisation of our paper

This work is organised as follows.

In Section 2 we give a short review of the β-deformation and Filk's theorem in [27], which was used to derive the one-loop dilatation-operator density in [28]. We analyse in which cases this theorem can be adapted to the β-deformation and find that this adaptability is generically limited by the occurrence of finite-size effects. This implies that the proposal of [28] is a priori valid only asymptotically. For a certain class of operators, however, the theorem can be generalised to include these effects. We find that their n-point correlation functions are given by the values of their undeformed counterparts at all loop orders in the planar theory.

In Section 3 we identify a new type of finite-size effect, which captures the differences between the planar correlation functions of the $U(N)$ and $SU(N)$ β-deformation. In particular, this effect accounts for the aforementioned sensitivity of certain anomalous dimensions to the choice of the gauge group. As it starts to affect single-trace operators one loop order earlier than wrapping, we call it *prewrapping*. It is caused by the double-trace structure in the $SU(N)$ propagator and has no net effect in the undeformed theory. We classify which types of operators may be affected by prewrapping.

In Section 4 we determine the missing finite-size corrections to the asymptotic dilatation operator of [28]. For gauge group $SU(N)$ those are the aforementioned prewrapping corrections at $L = 2$, whereas for gauge group $U(N)$ they are the ordinary wrapping corrections at $L = 1$. We thus obtain the complete planar one-loop dilatation operator of the β-deformation.

In Section 5 we identify all $\mathcal{N} = 1$ superconformal primary states of the β-deformation with classical scaling dimension $\Delta_0 \leq 4.5$. We then apply the one-loop dilatation operator of Section 4 to these states and compute their anomalous dimensions for both gauge groups, stressing in particular the differences.

Our summary and outlook can be found in Section 6. There, we also comment on the implications our findings have on integrability. If the β-deformation is indeed as integrable as $\mathcal{N} = 4$ SYM theory, a consistent incorporation of prewrapping into the integrability-based descriptions must exist. In particular, this should cure the divergence encountered in [32].

Several appendices contain details of the calculations as well as a table of primary states and their anomalous dimensions supplementing the analysis of Section 5.

2 Filk's theorem in the β-deformation

In this section we review the $*$-product formulation of the β-deformation and a theorem that was first derived for spacetime noncommutative field theories by Filk in [27]. We analyse the subtleties which arise when it is adapted to the β-deformation and determine the limits of its applicability. In the asymptotic regime we review how it can be used to derive the dilatation operator of [28]. Beyond that, we show that

certain classes of n-point correlation functions are at all loop orders in the planar theory given by the values of their undeformed counterparts.

The β-deformation can be realised via a noncommutative Moyal-like $*$-product [8], which for two fields A and B it is defined as

$$A * B = A B \, e^{\frac{i}{2}(\mathbf{q}_A \times \mathbf{q}_B)} \, . \tag{4}$$

Here $\mathbf{q}_A = (q_A^1, q_A^2, q_A^3)$ and $\mathbf{q}_B = (q_B^1, q_B^2, q_B^3)$ are the charge vectors of the fields associated with the Cartan subgroup of the $SU(4)_R$ symmetry group of the undeformed theory, see Table 1. Their antisymmetric product is defined as

$$\mathbf{q}_A \times \mathbf{q}_B = -\beta \sum_{a,b,c=1}^{3} \epsilon_{abc} q_A^a q_B^b \, , \tag{5}$$

where ϵ is the three-dimensional antisymmetric tensor normalised to $\epsilon_{123} = 1$. In fact, the antisymmetric product (5) depends only on $Q^1 = q^1 - q^2$ and $Q^2 = q^2 - q^3$. It is insensitive to the $U(1)_R$ charge $r = \frac{2}{3}(q^1 + q^2 + q^3)$. Thus, the $*$-product is applicable also in the superfield formulation, leading to the action (1). This latter basis of the $\mathfrak{su}(4)$ Cartan charges was originally used in [8] and reappears in the representation theoretic considerations in Section 5 and Appendix B.

B	A_μ	ϕ^1	ϕ^2	ϕ^3	ψ_α^1	ψ_α^2	ψ_α^3	ψ_α^4	F^1	F^2	F^3
q_B^1	0	1	0	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	-1	-1
q_B^2	0	0	1	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	-1	0	-1
q_B^3	0	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	-1	-1	0
Q_B^1	0	1	-1	0	1	-1	0	0	1	-1	0
Q_B^2	0	0	1	-1	0	1	-1	0	0	1	-1
r_B	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$

Table 1: Cartan charges of the fields, including the F-term auxiliary fields F^i, in two different bases. The respective anti-fields carry the opposite charges.

The action of the β-deformation is obtained from the one of $\mathcal{N} = 4$ SYM theory by replacing all products by $*$-products and thus all commutators by $*$-commutators:

$$[A \overset{*}{,} B] = A * B - B * A \, . \tag{6}$$

For gauge group $SU(N)$, however, this has to be done in the superspace action (1), or in the component expansion before the auxiliary fields are integrated out. In this case a double-trace coupling is generated which is at its fix-point value, cf. [14]. Although formally suppressed by $\frac{1}{N}$, it is necessary for the conformal invariance also of the planar theory. Introducing $*$-products in the component field action without auxiliary fields misses this coupling at tree-level and induces its running at loop-level.

In the conventions of [14] the component action[6] reads

$$
\begin{aligned}
S = \int \mathrm{d}^4x \Big[\mathrm{tr} \Big(&-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - (\mathrm{D}^\mu \bar{\phi}_j)\,\mathrm{D}_\mu\,\phi^j + i\bar{\psi}_A^{\dot{\alpha}}\,\mathrm{D}_{\dot{\alpha}}{}^{\alpha}\psi_\alpha^A \\
&+ g_{\mathrm{YM}}\Big(\frac{i}{2}\epsilon_{ijk}\phi^i\{\psi^{\alpha j}\,\overset{*}{,}\,\psi_\alpha^k\} - \phi^j\{\bar{\psi}_4^{\dot{\alpha}}\,,\bar{\psi}_{\dot{\alpha}j}\} + \mathrm{h.c.}\Big) \\
&- \frac{g_{\mathrm{YM}}^2}{4}[\bar{\phi}_j\,,\phi^j][\bar{\phi}_k\,,\phi^k] + \frac{g_{\mathrm{YM}}^2}{2}[\bar{\phi}_j\,\overset{*}{,}\,\bar{\phi}_k][\phi^j\,\overset{*}{,}\,\phi^k]\Big) \\
&- \frac{s}{N}\frac{g_{\mathrm{YM}}^2}{2}\,\mathrm{tr}\big([\bar{\phi}_j\,\overset{*}{,}\,\bar{\phi}_k]\big)\,\mathrm{tr}\big([\phi^j\,\overset{*}{,}\,\phi^k]\big) \Big]\,,
\end{aligned}
\tag{7}
$$

with spacetime indices $\mu,\nu = 0,1,2,3$, spinor indices $\alpha = 1,2$, $\dot{\alpha} = \dot{1},\dot{2}$, flavour indices $i,j,k = 1,2,3$ and $A = 1,2,3,4$, as well as the gauge-group parameter

$$
s = \begin{cases} 0 & \text{for } U(N) \\ 1 & \text{for } SU(N) \end{cases}.
\tag{8}
$$

We have kept only those $*$-products that do introduce net deformations. In particular, since the gauge field A_μ and the gluino ψ^4 are uncharged under (Q^1, Q^2), all interactions in which they occur are undeformed.

The $*$-product of the β-deformation (4) is similar to the Moyal $*$-product used to formulate a specific type of spacetime noncommutative field theories.[7] This similarity allows to adapt a particular theorem for this class of noncommutative field theories derived by Filk in [27]: the deformed version of a planar Feynman diagram is equal to its undeformed counterpart times a phase factor which is determined by the order and charges of the external fields alone. A completely explicit formulation of the relation between the deformed and undeformed diagrams of colour-ordered amplitudes is given in [41]. If the fields entering such an amplitude are $A_1, A_2, \ldots, A_n$ in cyclic order, its phase factor is the one of $A_1 * A_2 * \cdots * A_n$. This relation can be depicted as

$$
\tag{9}
$$

where Φ denotes the phase factor of the $*$-product.

A direct application of relation (9) to the planar diagrams contributing to $\mathfrak{D}_2$ led to the following proposal for the one-loop dilatation-operator density of the deformed theory [28]:

$$
\begin{aligned}
(\mathfrak{D}_2^\beta)_{A_i A_j}^{A_k A_l} &= \Phi(A_k * A_l * A_j * A_i)(\mathfrak{D}_2^{\mathcal{N}=4})_{A_i A_j}^{A_k A_l} \\
&= \mathrm{e}^{\frac{i}{2}(\mathbf{q}_{A_k}\times\mathbf{q}_{A_l} - \mathbf{q}_{A_i}\times\mathbf{q}_{A_j})}(\mathfrak{D}_2^{\mathcal{N}=4})_{A_i A_j}^{A_k A_l}\,,
\end{aligned}
\tag{10}
$$

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[6]The double-trace term in the action was written explicitly in [39] but follows directly from the procedure mentioned much earlier in [34].

[7]See [40] for a review.

where A_i, A_j denote the incoming and A_k, A_l the outgoing fields. They are taken from the alphabet

$$\mathcal{A} = \{ \mathrm{D}^k \, \phi^i, \mathrm{D}^k \, \bar{\phi}^i, \mathrm{D}^k \, \psi_\alpha^A, \mathrm{D}^k \, \bar{\psi}_{\dot\alpha}^A, \mathrm{D}^k \, \mathcal{F}_{\alpha\beta}, \mathrm{D}^k \, \bar{\mathcal{F}}_{\dot\alpha\dot\beta} \} \,, \tag{11}$$

where the field strength $F_{\mu\nu}$ has been split into its self-dual and anti-self-dual part $\mathcal{F}$ and $\bar{\mathcal{F}}$, respectively, and all Minkowski indices are translated to spinor indices using the relations in Appendix C. The abbreviation $\mathrm{D}^k \, \bar{\psi}_{\dot\alpha}^2$ stands for expressions with arbitrarily many covariant derivatives acting on $\bar{\psi}_{\dot\alpha}^2$ and totally symmetrised in both kinds of spinor indices.

Although the dilatation-operator density (10) seems to follow immediately from relation (9), one has to be very careful when adapting Filk's theorem to the β-deformation. The β-deformation is not a spacetime noncommutative field theory, and also its Moyal-like $*$-product is distinctly different from the one in [27]. In spacetime noncommutativity the phase as well as the ordering principle of fields entering an interaction depend on the spacetime coordinates or, after Fourier transformation, the momenta. In the $*$-product of the β-deformation the phase depends on the (Q^1, Q^2) charges of the fields, while the ordering principle is colour arrangement. Colour and (Q^1, Q^2) charge are, however, independent of each other. Traces, i.e. colour-neutral object, may be (Q^1, Q^2)-charged, as e.g. the trace factors in the double-trace term in (7). In this case, the prescription of replacing all products by $*$-products becomes generically ill-defined since it is in conflict with the cyclic invariance of the trace.[8] This can exemplarily be seen for $\mathrm{tr}[\phi^i \phi^j] = \mathrm{tr}[\phi^j \phi^i]$, whereas in the $*$-deformed case $\mathrm{tr}[\phi^i * \phi^j] \neq \mathrm{tr}[\phi^j * \phi^i]$ for $i \neq j$. The fact that colour and (Q^1, Q^2) charge are not connected also manifests itself in a problem concerning external states and the notion of planarity. The commonly repeated statement that 'only planar diagrams contribute in the planar / 't Hooft limit' is true only for diagrams in which all external legs have been connected to external states (composite operators). Before the connection to external states, some of the contributing diagrams are non-planar. These are the diagrams giving rise to the finite-size corrections [5]. As external states are not explicitly covered in [27], a priori all diagrams have to be considered without external states, leaving a subdiagram of the elementary interactions. In particular, the resulting subdiagrams of finite-size corrections are non-planar and hence are not captured by (9).

We can, however, extend the applicability of (9) beyond the subdiagrams of the elementary interactions. In theories with spacetime noncommutativity, external states can immediately be incorporated. They can be added to the action as local interactions at which momentum is conserved. Hence, as far as the deformation is concerned, they are on equal footing with the elementary vertices. In particular, momentum conservation implies that the deformation via the Moyal $*$-product is well defined since the resulting phase factor is invariant under a cyclic relabelling of the external momenta. Hence, in spacetime-noncommutative theories there is no need for treating

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[8]The $*$-deformation of the double-trace term in the component action (7) is, however, unambiguous because it arises from the $*$-deformation of charge-neutral single-trace terms in the $\mathcal{N} = 1$ superspace action (1) or in its off-shell component expansion (before the auxiliary fields F^i are integrated out). Accordingly, working in superspace or off-shell component space, all vertices are of flavour-neutral single-trace type.

any external state separately from the elementary vertices when applying Filk's theorem. The above considerations can be adapted in parts to the β-deformation. The similarity between the β-deformation and spacetime noncommutativity holds only if colour and (Q^1, Q^2) neutrality coincide, i.e. the fields within a single colour trace must have vanishing (Q^1, Q^2) net charge. This concerns external states consisting of one or several colour traces each of which is uncharged under the $U(1)_{Q^1} \times U(1)_{Q^2}$ flavour symmetry. Relation (9) then applies to the diagrams including such states, where on the left hand side the elementary vertices as well as the external states are $*$-deformed.

In the β-deformation, the extension of relation (9) to diagrams containing external states with (Q^1, Q^2)-charge-neutral trace factors has severe consequences. Since it is not necessary to remove such external states, diagrams which are planar only due to the presence of these states also obey (9), even if their subdiagrams built from the elementary interactions are non-planar. In particular, any gauge-invariant correlation function of such external states is independent of the deformation. This follows immediately from evaluating (9) with no ($n = 0$) external legs. It implies that e.g. the anomalous scaling dimensions and the structure constants for external states (composite operators) of this class are at all loops independent of the deformation parameter β and are directly given by their $\mathcal{N} = 4$ SYM counterparts.

One example in the subclass of (Q^1, Q^2)-charge-neutral single-trace operators is the Konishi operator $\sum_{i=1}^{3} \operatorname{tr}[\phi^i \bar{\phi}^i]$, whose anomalous dimension in the undeformed theory has recently been determined up to eight loops in the framework of integrability [25].[9] By the above argument, this result is valid also in the β-deformation. Another example is the family of chiral primary operators with length $L = 3k$ and $(q^1, q^2, q^3) = (k, k, k)$:

$$\mathcal{O}_k = \operatorname{tr}[(\phi^1)^k (\phi^2)^k (\phi^3)^k] + \text{all permutations} \,. \tag{12}$$

In contrast to the Konishi operator, the operators $\mathcal{O}_k$ themselves are altered by the $*$-deformation.[10] In [42] the three-point functions $\langle \mathcal{O}_k \mathcal{O}_{k'} \mathcal{O}_{k''} \rangle$ of the deformed operators were investigated at one-loop order in the planar gauge theory and at strong coupling in the Lunin-Maldacena background. In these regimes they were found to be independent of β. From the above argument it is clear that these three-point functions as well as all their (higher) n-point functions are independent of β at all loop orders.[11]

If one removes the external single-trace operators in a planar two-point diagram, one either obtains a planar single-trace diagram as in (9) or a nonplanar double-trace diagram. The latter type of diagram generates the finite-size effects. The applicability of [27] to external single-trace operators with vanishing (Q^1, Q^2) charge leads to the

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[9]The anomalous dimension was obtained via one of the $\mathcal{N} = 4$ descendents of the Konishi operator which is not an $\mathcal{N} = 1$ descendent. While in the β-deformation the anomalous dimension of this descendent is altered, the $\mathcal{N} = 4$ SYM theory result remains valid for the Konishi primary.

[10]Cf. the alternative deformation prescription in [34], which differs only by an overall factor.

[11]See [11] for a generalisation of earlier arguments given in [43,44] for rational β that the anomalous dimensions of the operators (12) vanish as for their undeformed counterparts in $\mathcal{N} = 4$ SYM theory.

following relation for double-trace interactions that contribute in the planar limit:[12]

$$
\underbrace{\raisebox{-1em}{β-diagram}}_{} = \underbrace{\raisebox{-1em}{$\mathcal{N}=4$-diagram}}_{} \times \underbrace{\Phi\left(A_1 * \ldots * A_i\right)\Phi\left(A_{i+1} * \ldots * A_n\right)}_{\Phi\left(A_1 * \ldots * A_i * A_{i+1} * \ldots * A_n\right)},
$$

$$(13)$$

where $\delta\mathbf{Q} = (\delta Q^1, \delta Q^2)$ denotes the charge flow between the separate traces. As in (9) the grey-shaded regions represent arbitrary planar interactions.

Regardless of the above generalisations concerning (Q^1, Q^2)-neutral single-trace operators, finite-size corrections still set the limit of applicability of relation (9) for (Q^1, Q^2)-charged ones. These corrections can, however, not contribute if the maximal interaction range R of the connected subdiagrams at a given loop-order K is strictly smaller than the length L of the single-trace operator. In particular, (9) can be applied to the K-loop asymptotic dilatation-operator density in (3), where asymptotic now means $L > R = K + 1$.[13]

In the special case of the one-loop dilatation operator our previous discussion guarantees that (10) is correct asymptotically, i.e. for $L \geq 3$, but it does not include finite-size effects, which may – in principle – contribute at $L \leq 2$. In Section 4 we calculate the corresponding finite-size corrections. For gauge group $U(N)$ these are the conventional wrapping corrections at $L = 1$. For gauge group $SU(N)$ a new type of finite-size effect occurs at $L = 2$, which we discuss in the next section.

3 A new type of finite-size effect

In this section we describe a new type of finite-size effect which appears in particular in the β-deformation with gauge group $SU(N)$, while the undeformed $\mathcal{N} = 4$ SYM theory is insensitive to it. Moreover, we classify which operators are potentially affected.

As explained before, finite-size corrections arise in the 't Hooft limit from certain multi-trace colour structures whose apparently suppressed N-powers are enhanced to the leading order when attached to external states. In a theory with $U(N)$ gauge group and only single-trace vertices, such as the $\mathcal{N} = 4$ SYM theory, these multi-trace structures can only be generated by the wrapping effect, i.e. by interactions that

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[12]This relation was also obtained in [39] for the dominant contribution to certain multi-trace amplitudes at large but finite N.

[13]Note that the K-loop asymptotic dilatation-operator density $\mathfrak{D}_{2K}$ incorporates interaction subdiagrams of ranges $R' \leq K + 1$. In the cases where $R' < K + 1$ it acts as identity on the remaining $K + 1 - R'$ spin-chain sites, which corresponds to bare propagators in the two-point function. The relation (13) applied for $i = 2$, together with the fact that $\Phi(A * \bar{A}) = 1$, guarantees that the phase factor obtained from the interaction subdiagram is exactly the one obtained from applying (9) to $\mathfrak{D}_{2K}$. Disconnected interaction subdiagrams are also captured by relation (13). They do, however, not contribute to the dilatation operator, since they do not have overall UV divergencies.

wrap at least once around an external state. This is only possible if the loop-order K of the interaction equals or exceeds the length L of the composite single-trace operator, i.e. $K \geq L$. In the following we will argue that for $SU(N)$ gauge group an additional source for multi-trace structures and hence finite-size corrections exists: the propagator.

We denote by $(\mathrm{T}^a)^i{}_j$ the generators of the gauge group with fundamental indices $i, j = 1, \ldots, N$ and adjoint index $a = s, \ldots, N^2 - 1$. Recall that $s = 0$ for $U(N)$ and $s = 1$ for $SU(N)$, as defined in (8). The generators are normalised such that

$$\mathrm{tr}(\mathrm{T}^a \, \mathrm{T}^b) = \delta^{ab} \, . \tag{14}$$

They obey the following important identity:

$$\sum_{a=s}^{N^2-1} (\mathrm{T}^a)^i{}_j (\mathrm{T}^a)^k{}_l = \delta^i_l \delta^k_j - \frac{s}{N} \delta^i_j \delta^k_l \, , \tag{15}$$

which occurs as the colour part of the propagator of adjoint fields. In double-line notation it can be depicted as

$$\delta^i_l \delta^k_j - \frac{s}{N} \delta^i_j \delta^k_l = \begin{array}{c} i \qquad\qquad l \\ \longrightarrow \\ j \qquad\qquad k \\ \longleftarrow \end{array} - \frac{s}{N} \begin{array}{c} i \\ j \end{array} \begin{array}{c} l \\ k \end{array} \, . \tag{16}$$

Let us investigate in which cases the $\frac{1}{N}$ prefactor of the second term in (16) is enhanced in N such that it contributes at the same leading order as the first term. In a generic diagram contributing to the n-point function of gauge-invariant composite operators, the colour lines in (16) are closed by other propagators, vertices and operators. This allows two distinct connections of the indices i, j, k, l in (16): either i is connected with l and k with j or i with j and k with l. In the first case, we obtain

$$\underbrace{\begin{array}{c} i \qquad\qquad l \\ \longrightarrow \\ j \qquad\qquad k \end{array}}_{N^2} - \frac{s}{N} \underbrace{\begin{array}{c} i \\ j \end{array} \begin{array}{c} l \\ k \end{array}}_{N^1} \propto \left(1 - \frac{s}{N^2}\right) N^2 \, . \tag{17}$$

The second term is suppressed by $\frac{1}{N^2}$ since besides the $\frac{1}{N}$ prefactor it has one index loop less than the first term. In the second case, however, we have

$$\underbrace{\begin{array}{c} i \qquad\qquad l \\ \longrightarrow \\ j \qquad\qquad k \end{array}}_{N^1} - \frac{s}{N} \underbrace{\begin{array}{c} i \\ j \end{array} \begin{array}{c} l \\ k \end{array}}_{N^2} \propto (1 - s) \, N \, . \tag{18}$$

Here, the second term is of the same order in N as the first term, since its prefactor $\frac{1}{N}$ is compensated by a factor N from an additional index loop compared to the first term.

These results can be easily interpreted: the second term in (16) contributes in leading order precisely if only the $U(1)$ component propagates. This can only happen if by cutting a single propagator the diagram of the gauge-invariant n-point function decomposes into two parts, i.e. if it is one-particle reducible. For gauge group

$SU(N)$, i.e. for $s = 1$, such diagrams do not contribute, as the second (double-trace) contribution in (16) cancels the first (single-trace) contribution. For the particular case of two-point functions, the affected diagrams are of s-channel[14] type and have the generic form

$$-\frac{s}{N} \qquad \propto (1-s)N^{2L-1}\,, \qquad (19)$$

where the two external states of length L are depicted in light grey and the area in dark grey stands for possible additional planar interactions. The interactions have to reduce the L fields of each of the two operators into a single field. Since the reduction of two fields into a single field comes with one factor of the effective planar coupling g,[15] the diagrams of type (19) are at least of order $\mathcal{O}(g^{2L-2})$. This is one loop order lower than the leading wrapping order and hence we call this new finite-size effect *prewrapping*.[16] The consequence of prewrapping is the vanishing of all s-channel diagrams in the $SU(N)$ case ($s = 1$) in contrast to the $U(N)$ case ($s = 0$), cf. (19).

Note that the quartic scalar interactions from the action (7) also fit into the above analysis. In fact, the superspace action contains only cubic interactions between chiral superfields. In the component expansion including auxiliary fields this is still the case. The quartic scalar vertices only appear when the auxiliary fields are integrated out. In particular, the elimination of the F^i auxiliary fields generates the double-trace term in the action (7) via (15). This term is just another description of the prewrapping effect caused by the propagator of the F^i auxiliary fields. It is thus perfectly acceptable and for a homogeneous description of prewrapping also advisable to conduct the analysis in one of the two former formulations.

Since the undeformed theory is insensitive to the difference between the gauge groups $U(N)$ and $SU(N)$, prewrapping must not have a net effect there. This means the contributions to both colour structures in (19) must vanish separately. The occurring cancellation among the contributing diagrams with different flavour structure can be most easily seen at the example of $L = 2$ states at one loop. In this case, the cyclicity of the trace symmetrises the operators with respect to their flavour degrees of freedom, while the commutator interaction reducing these two fields to the s-channel propagator is antisymmetric.[17] The same argument goes through for longer states if we consider the last remaining pair of fields in the aforementioned reduction

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[14]We trust that the reader will not confuse the Mandelstam variable s with the gauge group parameter s previously defined in (8).

[15]This consideration holds for all v-valent vertices in the actions (1) and (7) since they are of order $\mathcal{O}(g^{v-2})$.

[16]For the sake of simplicity, we are neglecting length-changing interactions in the main text. These are, however, also affected and easy to incorporate. For a two-point function connecting an operator of length L with one of length L', the critical order simply becomes $g^{L+L'-2}$.

[17]For two fermions, the state is of course antisymmetric and the interaction is symmetric.

to a single field. In the deformed theory, whose $*$-commutator interactions are not antisymmetric, this cancellation ceases to happen.

Based on the above consideration, we can classify for which operators prewrapping can occur. In prewrapping diagrams, the whole composite operator is reduced to a single field, which hence must carry the complete $\mathfrak{su}(4)$ Cartan charge of that operator. In particular, a field (or auxiliary field) with such charges must exist, cf. Table 1. In addition, its (Q^1, Q^2) charge has to be nonvanishing; fields with vanishing (Q^1, Q^2) charge have undeformed interactions, which leads to an automatic cancellation as in $\mathcal{N} = 4$ SYM theory.

Let us apply these criteria to the closed subsectors of the theory, which for the undeformed theory were classified in [19]. It becomes an easy combinatorial exercise to give all operators that are potentially affected. For the compact closed subsectors the results are displayed in Table 2. It shows that all respective candidates for prewrapping are obtained by acting on the $L = 2$ single-impurity state $\mathrm{tr}[\phi^2\phi^3]$ with charge conjugation and/or a $\mathbb{Z}_3$ symmetry. The latter symmetry cyclically relabels the scalars ϕ^i and fermions ψ^i, as well as their conjugates, leaving the action invariant.[18] Recall that the sensitivity of this state to the choice of the gauge group was already observed in [34], and we can now identify this phenomenon as a manifestation of prewrapping.

Subsector	Fields	Prewrapping candidates
$SU(2)$	ϕ^1, ϕ^2	$\mathrm{tr}[\phi^1\phi^2]$
$SU(2)$	$\phi^1, \bar{\phi}^2$	none
$U(1\|1)$	ϕ^1, ψ_1^4	none
$U(1\|1)$	ϕ^1, ψ_1^1	none
$U(1\|1)$	$\bar{\phi}^1, \psi_1^2$	none
$U(1\|1)$	$\bar{\phi}^1, \psi_1^3$	none
$U(1\|2)$	ϕ^1, ϕ^2, ψ_1^4	$\mathrm{tr}[\phi^1\phi^2]$
$U(1\|2)$	$\bar{\phi}^2, \bar{\phi}^3, \psi_1^1$	$\mathrm{tr}[\bar{\phi}^2\bar{\phi}^3]$
$U(1\|2)$	$\phi^1, \bar{\phi}^2, \psi_1^1$	none
$U(1\|2)$	$\phi^1, \bar{\phi}^3, \psi_1^1$	none
$U(1\|3)$	$\phi^1, \phi^2, \phi^3, \psi_1^4$	$\mathrm{tr}[\phi^1\phi^2] + \mathbb{Z}_3$
$U(1\|3)$	$\phi^1, \bar{\phi}^2, \bar{\phi}^3, \psi_1^1$	$\mathrm{tr}[\bar{\phi}^2\bar{\phi}^3]$
$SU(2\|3)$	$\phi^1, \phi^2, \phi^3, \psi_1^4, \psi_2^4$	$\mathrm{tr}[\phi^1\phi^2] + \mathbb{Z}_3$
$SU(2\|3)$	$\phi^1, \bar{\phi}^2, \bar{\phi}^3, \psi_1^1, \psi_2^1$	$\mathrm{tr}[\bar{\phi}^2\bar{\phi}^3]$

Table 2: Candidates for prewrapping in the compact closed subsectors. We have omitted subsectors that are related to the above via the $\mathbb{Z}_3$ symmetry or charge conjugation.

The analogous analysis for the noncompact closed subsectors is equally straightforward and we only want to stress one noteworthy feature. In closed subsectors that restrict the flavour content, no combination of $\mathfrak{su}(4)$ charged fields $\{\phi^i, \bar{\phi}^i, \psi_\alpha^A, \bar{\psi}_{\dot{\alpha}}^A\}$ exists whose total $\mathfrak{su}(4)$ charge vanishes.[19] As a consequence, the aforementioned

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[18]This can most easily be seen in the superspace formulation (1).

[19]Note that our criteria are insensitive to the difference between the closed subsectors with unrestricted flavour content, namely $U(1, 1|4)$ and $U(1, 2|4)$, and the full theory. The latter is treated below.

criteria can only be fulfilled by operators containing a finite number of these fields. They may, however, contain an arbitrary number of field strengths and covariant derivatives if those are admitted in the sector.

In the full theory, large families of operators exist that are candidates for prewrapping, e.g.

$$\mathrm{tr}\left[\phi^2\phi^3(\phi^1\bar{\phi}^1)^i(\phi^2\bar{\phi}^2)^j(\phi^3\bar{\phi}^3)^k(\psi^1\bar{\psi}^1)^l(\psi^2\bar{\psi}^2)^m(\psi^3\bar{\psi}^3)^n(\psi^4\bar{\psi}^4)^o\mathcal{F}^p\bar{\mathcal{F}}^q\right]\,,\qquad(20)$$

where $i,j,k,l,m,n,o,p,q \in \mathbb{N}_0$ and we have suppressed covariant derivatives, which can additionally appear on all of the fields, and spinor indices for simplicity.

4 The complete one-loop dilatation operator

In this section we determine the missing finite-size corrections and obtain the complete one-loop dilatation operator for the planar β-deformation – first for gauge group $SU(N)$ and then for gauge group $U(N)$.

4.1 Gauge group $SU(N)$

The dilatation operator is most directly extracted from the UV divergences of correlation functions involving either one or two composite operators. According to the previous discussion, it is hence affected by prewrapping if the gauge group is $SU(N)$. In particular, this means that the asymptotic one-loop result (10) can be extended to the full one-loop result in the β-deformation with $SU(N)$ gauge group by removing non-vanishing s-channel contributions for states with length $L = 2$. In the undeformed $\mathcal{N} = 4$ SYM theory this is not necessary, since cancellations between different s-channel diagrams result in a vanishing net contribution, as already discussed in the previous section.

In the following we consider the operator $\mathcal{O} = \mathrm{tr}[\psi_\alpha^1 \phi^2]$ as an example and evaluate the result of the asymptotic dilatation operator (10) acting on it. We explicitly show how the corresponding s-channel diagrams cancel in the undeformed theory, while in the β-deformation a net contribution persists. This is the correct s-channel contribution in the $U(N)$ theory, but it has to be removed in the $SU(N)$ theory. After the example, we show how with a simple prescription this can efficiently be done for all one-loop states.

The operator $\mathcal{O} = \mathrm{tr}[\psi_\alpha^1 \phi^2]$ maps to the spin-chain state $|\mathcal{O}\rangle$, and we act on it with the one-loop dilatation operator D_2. On this state D_2 is the sum of two insertions $D_2 = D_{12} + D_{21}$, where D_{ij} denotes the dilatation-operator density $\mathfrak{D}_2$ acting on the two legs of $|\mathcal{O}\rangle$ in the specific order ij, cf. (3). Hence,

$$D_2|\mathcal{O}\rangle = D_{12}|\mathcal{O}\rangle + D_{21}|\mathcal{O}\rangle \qquad (21)$$

is a linear combination of certain $L = 2$ states.

In the undeformed theory it can be calculated using the so-called harmonic action [19] for $\mathfrak{D}_2^{\mathcal{N}=4}$, see Appendix C for details.[20] We focus on the matrix element

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[20]Note the factor 2 difference in our conventions in comparison to [19].

$\langle \mathcal{O}|D_2|\mathcal{O}\rangle$. It is the sum of the following four contributions:

$$\langle\psi^1,\phi^2|\,\mathfrak{D}_2^{\mathcal{N}=4}|\psi^1,\phi^2\rangle = +3\,,$$
$$\langle\phi^2,\psi^1|\,\mathfrak{D}_2^{\mathcal{N}=4}|\psi^1,\phi^2\rangle = -1\,,$$
$$\langle\phi^2,\psi^1|\,\mathfrak{D}_2^{\mathcal{N}=4}|\phi^2,\psi^1\rangle = +3\,,$$
$$\langle\psi^1,\phi^2|\,\mathfrak{D}_2^{\mathcal{N}=4}|\phi^2,\psi^1\rangle = -1\,,$$

$$(22)$$

where $|\psi^1,\phi^2\rangle$ denotes an ordered two-site state with ψ_α^1 at site one and ϕ^2 at site two. These contributions can be understood in terms of Feynman diagrams. The first two lines are given by

$$(23)$$

where scalars are depicted by solid lines, fermions by dashed lines, gauge fields by wiggly lines, the 'blob' represents one-loop self-energy insertions and the composite operators are depicted by bold horizontal lines. Underneath the diagrams we have displayed the respective individual contributions to the harmonic action, which were calculated from the black parts of the diagrams[21] via the Feynman rules given in detail in [14]. The extension of the bold horizontal lines beyond the points where the elementary field lines originate indicates that we have only kept those terms that would also contribute when connected to an arbitrarily long operator. These are the single-trace terms with the correct colour order. The third and forth line of (22) are given by the reflections of the above diagrams with respect to the vertical axis. The reflected diagrams give the same contributions as the unreflected ones. The four contributions from the s-channel diagram do indeed cancel each other such that $\mathfrak{D}_2^{\mathcal{N}=4}$ yields the correct result for the $\mathcal{N}=4$ SYM theory.

In the deformed theory the corresponding cancellation between the different contributions from the deformed *asymptotic* dilatation operator density does not occur. The four lines in (22) acquire the phases 1, $e^{i\beta}$, 1 and $e^{-i\beta}$, respectively, as follows from both the relation in (10) and the explicit Feynman diagram calculation. The net contribution from the s-channel diagrams is nonvanishing and given by

$$1 - e^{i\beta} + 1 - e^{-i\beta} = 4\sin^2\tfrac{\beta}{2}\,. \qquad (24)$$

---

[21]Note that the black parts as well as the labelling directly correspond to the diagrams of operator renormalisation, which is the most direct way to obtain the action of $\mathfrak{D}_2$ as operator on the spin chain, cf. [17]. The respective diagrams can be obtained from those of the two-point function, which are depicted by the grey completion, by amputating the outgoing operators and propagators.

In the $SU(N)$ theory, this contribution has to vanish because of prewrapping.

A priori this discrepancy requires the computation and subtraction of all deformed one-loop s-channel Feynman diagrams involving $L = 2$ operators. Fortunately, this is not necessary. In the remainder of this subsection we argue that a surprising short-cut is available.[22] It relies on the relatively small number and simple structure of Feynman diagrams at one-loop level and is proven in three steps. First, we show that for certain vertices, which correspond to specific combinations of external states, the automatic cancellations between different s-channel contributions take place as in the undeformed theory. Second, we identify the combinations of external states which contain spurious s-channel contributions. These contributions can be removed by setting the deformation parameter β to zero, which restores the cancellations from the undeformed theory. Third, we show that this procedure does not alter any non-s-channel contributions with these external states. It can hence be applied at the level of $\mathfrak{D}_2$.

Recall that in the β-deformation only interactions between matter-type superfields $\{\Phi^i, \bar{\Phi}^i\}$ – or their respective components $\{\phi^i, \psi_\alpha^i, F^i, \bar{\phi}^i, \bar{\psi}_{\dot\alpha}^i, \bar{F}^i\}$ – are deformed. Interactions involving at least one vector superfield V – or its gauge-type on-shell components in Wess-Zumino gauge $\{A_\mu, \psi_\alpha^4, \bar{\psi}_{\dot\alpha}^4\}$ – are undeformed. The contributions from s-channel diagrams in which both vertices are undeformed cancel as in the undeformed theory. Moreover, s-channel diagrams involving one deformed and one undeformed vertex automatically have a vanishing contribution also in the deformed theory, as the combination of symmetric state and commutator-type vertex in either initial or final state suffices for a cancellation. Hence, a nonvanishing net contribution can only come from s-channel diagrams in which both vertices are deformed, implying that all fields are of matter type. As an immediate consequence, the dilatation operator obtained from (10) gives the correct result if at least one of the external legs is of gauge-type.

We have depicted all s-channel diagrams with only matter-type fields in the first row of Table 3. They describe interactions of two incoming matter fields $\{\phi^i, \psi_\alpha^i\}$ or anti-matter fields $\{\bar{\phi}^i, \bar{\psi}_{\dot\alpha}^i\}$ which become two outgoing matter or anti-matter fields, respectively.[23] We can remove their contributions to the dilatation operator by setting the deformation parameter β to zero whenever these combinations of external fields occur. This restores the cancellations of the undeformed theory.

We want to be able to apply this procedure to the dilatation-operator density $\mathfrak{D}_2$ instead of only to individual diagrams. Hence, we have to justify that non-s-channel diagrams with the above configurations of four external matter-type fields either do not exist or are not affected. The latter is the case if the diagrams are independent of β, i.e. undeformed. Therefore, we have to analyse only deformed diagrams with matter-type external fields. They necessarily contain also internal fields of only matter type and are a priori of s-channel, t-channel or self-energy type. We have depicted the respective t-channel diagrams in the second row of Table 3. They do not exist for those combinations of external fields for which s-channel diagrams occur and thus are

---

[22]Note that this short-cut works in the supersymmetric β-deformation but fails to work in the nonsupersymmetric γ_i-deformation.

[23]Note that in the picture of the two-point function these diagrams are connecting two matter fields $\{\phi^i, \psi_\alpha^i\}$ of an $L = 2$ operator $\mathcal{O}$ with two *anti-matter* fields $\{\bar{\phi}^i, \bar{\psi}_{\dot\alpha}^i\}$ of a second $L = 2$ operator $\bar{\mathcal{O}}'$, or, respectively, anti-matter fields in the former to matter fields in the latter.

	in components				$\mathcal{N}=1$
s-channel + vertical & horizontal reflections + twists	$\psi^i \quad \psi^j$ / $\psi^i \quad \psi^j$	$\phi^i \quad \psi^j$ / $\phi^i \quad \psi^j$	$\phi^i \quad \phi^j$ / $\phi^i \quad \phi^j$	$=$ $\phi^i \quad \phi^j$ / $\phi^i \quad \phi^j$	$\Phi^i \quad \Phi^j$ / $\Phi^i \quad \Phi^j$
t-channel + vertical & horizontal reflections	$\bar\psi^j \quad \psi^i$ / $\psi^i \quad \bar\psi^j$	$\bar\psi^j \quad \phi^i$ / $\phi^i \quad \bar\psi^j$	$\bar\phi^j \quad \phi^i$ / $\phi^i \quad \bar\phi^j$	$=$ $\bar\phi^j \quad \phi^i$ / $\phi^i \quad \bar\phi^j$	$\bar\Phi^j \quad \Phi^i$ / $\Phi^i \quad \bar\Phi^j$

Table 3: Asymptotic range $R = 2$ diagrams with two deformed vertices. Scalars and chiral superfields are depicted by solid lines, fermions by dashed lines, F auxiliary fields by dotted lines, gauge fields by wiggly lines and the composite operators are depicted by bold horizontal lines. Twist signifies the vertical reflection of only the upper half of a diagram, respectively. Scalars are treated on the same footing as the matter fermions, as the quartic vertices can be rewritten as cubic vertices with 'propagating' auxiliary fields. Covariant derivatives are suppressed in the notation.

not altered by our procedure. It remains to be shown that the contributions from the self-energy-type diagrams are not affected either. As their subdiagrams of elementary interactions have range $R = 1$ and are connected to an operator of length $L = 2$, we can apply (9) with $n = 2$, which immediately[24] shows that their contributions are independent of β.

As the vertices only depend on the flavours of the fields involved, the same analysis is true for covariant derivatives acting on these fields. In particular it holds for those combinations forming the alphabet $\mathcal{A}$ given in (11). For the translation of the above considerations we define the following two subsets of $\mathcal{A}$:

$$
\begin{aligned}
\mathcal{A}_{\mathrm{matter}} &= \{\mathrm{D}^k\,\phi^1, \mathrm{D}^k\,\phi^2, \mathrm{D}^k\,\phi^3, \mathrm{D}^k\,\psi_\alpha^1, \mathrm{D}^k\,\psi_\alpha^2, \mathrm{D}^k\,\psi_\alpha^3\}, \\
\bar{\mathcal{A}}_{\mathrm{matter}} &= \{\mathrm{D}^k\,\bar\phi^1, \mathrm{D}^k\,\bar\phi^2, \mathrm{D}^k\,\bar\phi^3, \mathrm{D}^k\,\bar\psi_{\dot\alpha}^1, \mathrm{D}^k\,\bar\psi_{\dot\alpha}^2, \mathrm{D}^k\,\bar\psi_{\dot\alpha}^3\}.
\end{aligned}
\tag{25}
$$

According to the above discussion, the complete one-loop dilatation operator of the planar β-deformation with gauge group $SU(N)$ is given by the following density:

$$
(\mathfrak{D}_2^\beta)_{A_i A_j}^{A_k A_l} = \mathrm{e}^{\frac{i}{2}(\mathbf{q}_{A_k} \times \mathbf{q}_{A_l} - \mathbf{q}_{A_i} \times \mathbf{q}_{A_j})} \Bigg|_{\substack{\beta=0 \text{ if } L=2 \text{ and} \\ (A_i, A_j, A_k, A_l \in \mathcal{A}_{\mathrm{matter}} \text{ or} \\ A_i, A_j, A_k, A_l \in \bar{\mathcal{A}}_{\mathrm{matter}})}} (\mathfrak{D}_2^{\mathcal{N}=4})_{A_i A_j}^{A_k A_l},
\tag{26}
$$

where the rule for the implementation of prewrapping introduces an explicit dependence on the operator length L, as expected for finite-size effects.

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[24]Recall that the $*$-product of a fields with its conjugate reduces to the ordinary product.

4.2 Gauge group $U(N)$

For gauge group $U(N)$, the asymptotic result (10) is valid for $L \geq 2$. However, the $L = 1$ states, corresponding to the $U(1)$ modes of the fields, acquire anomalous dimensions. For the matter-type fields the respective eigenvalues E of the one-loop dilatation operator – which are the anomalous dimensions divided by g^2 – are

$$E_{\text{tr}\,\phi^i} = E_{\text{tr}\,\bar{\phi}^i} = E_{\text{tr}\,\psi^i_\alpha} = E_{\text{tr}\,\bar{\psi}^i_{\dot{\alpha}}} = 4\sin^2\frac{\beta}{2}\,. \tag{27}$$

The result for the scalar fields can be directly obtained from the self-energy diagrams given in the appendix of [14]. Supersymmetry demands that the respective result for the fermions is the same, and we have confirmed this by an explicit calculation using the Feynman rules described in [14]. The $U(1)$ components of the gluino and gauge field still decouple in the β-deformation and hence the following anomalous dimensions vanish:

$$E_{\text{tr}\,\mathcal{F}_{\alpha\beta}} = E_{\text{tr}\,\mathcal{F}_{\dot{\alpha}\dot{\beta}}} = E_{\text{tr}\,\psi^4_\alpha} = E_{\text{tr}\,\bar{\psi}^4_{\dot{\alpha}}} = 0\,. \tag{28}$$

5 The spectrum

In this section we employ the one-loop dilatation operator to compute the anomalous dimensions of all single-trace operators with classical scaling dimension $\Delta_0 \leq 4.5$. The results are structured according to primary states and for gauge group $SU(N)$ are shown in Table 4, as well as Table 6 in Appendix A. An analogous table for $\mathcal{N} = 4$ SYM theory can be found in [19]. Table 5 enlists the primary states of the $U(N)$ theory that do either not exist in the $SU(N)$ theory or that differ in their anomalous dimensions. In the following we introduce the notation and derive the results presented in the tables. We conclude with a discussion of the differences with respect to the gauge-group choice and derive an all-loop result.

The β-deformation breaks the $SU(4)_R$ R-symmetry group of $\mathcal{N} = 4$ SYM theory to $U(1)_{Q^1} \times U(1)_{Q^2} \times U(1)_R$. The corresponding conserved charges span the Cartan subalgebra of $\mathfrak{su}(4)_R$, see Section 2. Furthermore, the action is invariant under a $\mathbb{Z}_3$ symmetry that cyclically rotates the three matter superfields into each other and leaves the gauge superfield invariant. This is supplemented by exchanging two matter superfields and simultaneously replacing β by $2\pi - \beta$. As the one-loop spectrum is invariant under this transformation of β, it is invariant under the resulting larger $\mathbb{S}_3$ symmetry. Finally, the one-loop spectrum is invariant under charge conjugation, which exchanges j and $\bar{j}$ and sends the $\mathfrak{su}(4)_R$ Cartan charges to their negatives.

We label primary states by the classical scaling dimension Δ_0, the $\mathfrak{su}(2) \times \overline{\mathfrak{su}}(2)$ spin $[j,\bar{j}]$, the $\mathfrak{su}(4)_R$ Cartan charges[25] (q^1, q^2, q^3) and the length L, which is only preserved at one-loop order.[26] If several primary states of the $\mathcal{N} = 1$ superconformal

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[25]See Table 1 for the translation between q^1, q^2, q^3 and Q^1, Q^2, r.

[26]For a comparison with the $\mathcal{N} = 4$ SYM theory in [19], one should keep in mind that our spins $j, \bar{j}$ are half integers whereas the spins s_1, s_2 of [19] are integers, our E is twice the one of [19] and the charges q^1, q^2, q^3 are related to the $\mathfrak{su}(4)$ Dynkin labels q_1, p, q_2 as

$$q^1 = \frac{1}{2}(q_1 - q_2)\,, \quad q^2 = -\frac{1}{2}(q_1 + q_2)\,, \quad q^3 = -\frac{1}{2}(q_1 + 2p + q_2)\,. \tag{29}$$

group are related via the $\mathbb{S}_3$ symmetry, we only give the one with the highest q^3 charge. Moreover, if two primary states are related by charge conjugation, we only give the one with the higher $\mathfrak{su}(2)$ spin j and subordinately the highest q^3 charge.

We have determined the $\mathcal{N} = 1$ multiplet content of the β-deformation in analogy to the $\mathcal{N} = 4$ SYM theory case [45, 46] by applying the Eratostenes super-sieve algorithm [45] to the refined partition function. The necessary $\mathcal{N} = 1$ characters can e.g. be found in [47, 48]. A more detailed description of the procedure is presented in Appendix B. We have computed the anomalous dimensions by acting with the dilatation operator on the space of states with specified quantum numbers, subsequently diagonalising the resulting mixing matrix; see Appendix C for an explicit expression of the harmonic action. Finally, we have assigned the anomalous dimensions to the multiplets.

In the Tables 4, 5 and 6 the anomalous dimensions divided by g^2 are given as the solutions of polynomial equations of the form

$$E^n = \sum_{k=0}^{n-1} a_k E^k = a_{n-1} E^{n-1} + \cdots + a_1 E + a_0 \,, \tag{30}$$

which are abbreviated as $\{a_{n-1}, \ldots, a_1, a_0\}$. Moreover, we capture the dependence on the deformation parameter β in terms of $s_n = \sin^2 \frac{n\beta}{2}$. For example, $\{12, -32s_2\}$ stands for the two solutions of the quadratic equation $E^2 = 12E - 32 \sin^2 \beta$, namely $E = 6 \pm 2\sqrt{9 - 8 \sin^2 \beta}$. In particular, $\{a_0\}$ means $E = a_0$.

Certain representations are only irreducible in the interacting theory, where they have an anomalous dimension. In the free theory, where their highest weight states are at the unitarity threshold, these representations are reducible and split into several irreducible ones, see Appendix B.1 for details. In Tables 4, 5 and 6 we have marked the corresponding highest-weight states of the free theory that have to be dropped in the interacting theory with a $\star$. This happens e.g. for the Konishi multiplet whose highest-weight state is contained Table 4 as the third state in the second line with $E = 12$. In the interacting theory, the second state in the 13$^{\text{th}}$ line of Table 4 is a descendant of this state. Note that representations with vanishing one-loop anomalous dimensions – as e.g. the first and second in state in the second line of Table 4 – may in general still move away from the unitary threshold at higher loop orders. For the latter two states this is, however, not the case. In the undeformed theory they are descendants of the $L = 2$ protected state and thus have vanishing all-loop anomalous dimensions. As they have vanishing $U(1)_{Q^1} \times U(1)_{Q^2}$ charge, the argument from Section 2 can be applied and their anomalous dimensions stay zero at all loop orders in the β-deformation. Analogous considerations hold for all tabled multiplets at the unitary threshold with vanishing one-loop anomalous dimensions.

For $\Delta_0 \leq 4.5$, and presumably also for higher Δ_0, only one multiplet is affected by prewrapping at one-loop level, namely $\text{tr}[\phi^2\phi^3]$ (and its images under $\mathbb{S}_3$ and charge conjugation). Naively, one might expect a whole tower of affected multiplets, built from $n \geq 0$ covariant derivatives distributed on $\text{tr}[\phi^2\phi^3]$ in a similar fashion as the $SL(2)$ sector is built on $\text{tr}[\phi^3\phi^3]$. For the conformal primary states[27] corresponding

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[27]Similar to the situation in $\mathcal{N} = 4$ SYM theory only the primary state with $n = 0$ is a highest-weight state under the full symmetry group. For the $n \geq 1$ states the true highest-weight state is a fermionic state with classical scaling dimension smaller by $\frac{1}{2}$.

Δ_0	$[j,\bar{j}]_{(q^1,q^2,q^3)}$	L	E
2	$[0,0]_{(-1,0,1)}$	2	$\{8s_1\}$
2	$[0,0]_{(0,0,0)}$	2	$\{0\}, \{0\}, \{12\}$
2	$[0,0]_{(0,0,2)}$	2	$\{0\}$
2	$[0,0]_{(0,1,1)}$	2	$\{0\}$
$\frac{5}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	2	$\{12,-32s_1\}$
$\frac{5}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	2	$\{0\}$
3	$[0,0]_{(-1,0,2)}$	3	$\{8s_1\}$
3	$[0,0]_{(-1,1,1)}$	3	$\{12,-32s_2\}$
3	$[0,0]_{(0,0,1)}$	3	$\{12,-32s_1\}^\star, \{20,-96,128s_1\}$
3	$[0,0]_{(0,0,3)}$	3	$\{0\}$
3	$[0,0]_{(0,1,2)}$	3	$\{8s_1\}^\star$
3	$[0,0]_{(1,1,1)}$	2	$\{0\}$
3	$[0,0]_{(1,1,1)}$	3	$\{0\}, \{12\}^\star$
3	$[\frac{1}{2},\frac{1}{2}]_{(-1,0,1)}$	2	$\{4(3-s_1)\}$
3	$[\frac{1}{2},\frac{1}{2}]_{(0,0,0)}$	2	$\{0\}, \{12\}, \{12\}, \{12\}$
3	$[1,0]_{(0,0,1)}$	2	$\{12\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},-\frac{1}{2},\frac{3}{2})}$	3	$\{4(3-2s_1)\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{20,-96,128s_2)\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	3	$\{8\}, \{8\}, \{12\}, \{12\}, \{12\}^\star, \{12,-32s_1\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	3	$\{4(3-s_1)\}^\star, \{20,-32(3+s_1),32(8s_1+s_2)\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{0\}, \{0\}, \{8\}, \{8\}, \{8\}, \{12\}, \{12\}, \{12\}^\star, \{12\}^\star, \{12\}^\star$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{5}{2})}$	3	$\{0\}$
$\frac{7}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{3}{2},\frac{3}{2})}$	3	$\{12,-32s_1\}^\star$
$\frac{7}{2}$	$[1,\frac{1}{2}]_{(-\frac{1}{2},-\frac{1}{2},\frac{3}{2})}$	2	$\{12\}$
$\frac{7}{2}$	$[1,\frac{1}{2}]_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	2	$\{12\}, \{12\}$
$\frac{7}{2}$	$[\frac{3}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	2	$\{12\}$

Table 4: Anomalous dimensions of all primary states with classical scaling dimension $\Delta_0 < 4$ for gauge group $SU(N)$ in the notation introduced after (30). The dependence on the deformation parameter is encoded in $s_n = \sin^2 \frac{n\beta}{2}$. Highest-weight states of the free theory that join lower-lying multiplets in the interacting theory are marked with a $\star$. Each state has to be supplemented by its images under the $\mathbb{S}_3$ symmetry and charge conjugation.

Δ_0	$[j,\bar{j}]_{(q^1,q^2,q^3)}$	L	$E_{U(N)}$	$E_{SU(N)}$
1	$[0,0]_{(0,0,1)}$	1	$\{4s_1\}$	$-$
$\frac{3}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	1	$\{0\}$	$-$
2	$[0,0]_{(0,1,1)}$	2	$\{8s_1\}$	$\{0\}$

Table 5: Comparison of the anomalous dimensions of all primary states with classical scaling dimension $\Delta_0 \leq 4.5$ differing for gauge groups $U(N)$ and $SU(N)$. The notation is introduced after (30). Each state has to be supplemented by its images under the $\mathbb{S}_3$ symmetry and charge conjugation.

to $n \geq 1$, however, contributions from s-channel diagrams vanish due to cancellations in the spacetime part that are independent of the deformation, see Appendix D for details.

At two-loop order the two-point function of the state $\mathrm{tr}[\phi^2\phi^3]$ was investigated in [35]. While the two-point function receives finite $\frac{1}{N}$ corrections, its anomalous dimension remains zero for gauge group $SU(N)$.[28] We now show that this state is even protected at all orders in planar perturbation theory. In the $\mathcal{N} = 1$ super-space formulation, the superfield $\mathrm{tr}[\Phi^2\Phi^3]$ contains $\mathrm{tr}[\phi^2\phi^3]$ as its lowest component in the θ-expansion. Its anomalous dimension can be extracted from the overall UV divergence of the correlation function $\langle \Phi^2(x)\Phi^3(y)\,\mathrm{tr}[\Phi^2\Phi^3](0)\rangle$. According to the finiteness-conditions [49], an $\mathcal{N} = 1$ superspace Feynman diagram of range $R = 2$ that contributes to this correlation function can only have an overall UV divergence if at least one of its vertices is not part of a loop. This condition is only fulfilled by the diagrams of s-channel type, shown at one-loop in the upper right corner of Table 3. At higher loops arbitrary interactions supplement the lower (black) half of this diagram. For gauge group $SU(N)$, these diagrams vanish by the prewrapping effect as discussed before. Moreover, the remaining $R = 1$ diagrams are self-energy corrections of the elementary superfields, which are finite. The above correlation function is therefore finite and the anomalous dimension vanishes.

6 Conclusion and outlook

In this paper we have analysed finite-size corrections in the real β-deformed $\mathcal{N} = 4$ SYM theory in the 't Hooft limit. We have constructed the complete one-loop dilatation operator by incorporating these corrections into the proposal of [28]. The latter (asymptotic) result was obtained by applying Filk's theorem from spacetime noncommutative field theory to the undeformed dilatation-operator density, i.e. to the combination of respective planar single-trace Feynman diagrams.

We have analysed in detail the limits and implications of Filk's theorem when applied in the β-deformation. We have found that generic external states have to be removed from a diagram before the theorem is applicable. If the diagram is associated with a finite-size effect, it becomes non-planar after this truncation. Hence, finite-size corrections in general invalidate the results relying on Filk's theorem. External multi-trace states in which all traces are neutral under the (Q^1, Q^2) global charge, however, need not be removed from the planar $\mathcal{N} = 4$ SYM diagrams, and Filk's theorem can be applied to the entire diagrams undergoing the deformation. This implies that all n-point correlation functions of the resulting deformed states are identical to their counterparts in the undeformed $\mathcal{N} = 4$ SYM theory at any loop order. In particular, the anomalous dimension of the Konishi primary operator $\sum_{i=1}^{3} \mathrm{tr}[\phi^i\bar{\phi}^i]$ is undeformed.

From the findings summarised in the above paragraph it follows that the one-loop dilatation operator of [28] is valid only up to finite-size corrections. In the non-conformal $U(N)$ theory, it has to be supplemented by the anomalous dimensions of the $L = 1$ states, which are affected by the well-known finite-size effect of wrapping. In the conformal $SU(N)$ theory, we have identified a new type of finite-size effect,

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[28]Note that this is incorrectly summarised in [11].

which has to be taken into account for certain $L = 2$ states. It is caused by the $SU(N)$ propagators of the adjoint fields and starts to affect states of length L at loop order $K = L - 1$. Since this is one loop order lower than the critical wrapping order $K = L$, we call it *prewrapping*. We have identified criteria for states which may be affected by it. In all compact closed subsectors, prewrapping at all loop orders affects only the state $\text{tr}[\phi^2\phi^3]$ (and its five images under the $\mathbb{Z}_3$ symmetry and charge conjugation). In the full theory, prewrapping candidates with generic lengths $L \geq 3$ exist for sufficiently high loop orders.

At one loop, we have found that the prewrapping-effect can be incorporated into the dilatation operator of the deformed theory without explicitly calculating Feynman diagrams, simply by removing the deformation whenever the external states match certain criteria. This procedure strongly relies on the small number and simple structure of one-loop diagrams, and we doubt that it can be extended to higher loops.

We have employed our result to determine the one-loop spectrum of the theory with classical scaling dimension $\Delta_0 \leq 4.5$. At this loop order, only the superconformal multiplet with highest-weight state $\text{tr}[\phi^2\phi^3]$ (and its five images under the $\mathbb{Z}_3$ symmetry and charge conjugation) are affected by prewrapping. The absence of prewrapping for other one-loop candidate multiplets can be traced back to cancellations among Feynman diagrams. Yet, it would be desirable to understand if there is a deeper principle behind it.

While we have found and analysed prewrapping in the weakly coupled gauge theory, it remains an open problem to understand this effect in the strongly coupled dual string theory. As we have argued, the prewrapping-affected state $\text{tr}[\phi^2\phi^3]$ is protected at any order in perturbation theory. Hence, it should be a supergravity mode and the calculation suggested in [11] should yield a vanishing correction to its mass at strong coupling. It would be desirable to check this explicitly and to understand how the deformation increases the energy of e.g. the state dual to $\text{tr}[\phi^2\phi^3\phi^3]$, which is not protected by prewrapping. We hope that this might help to understand the subtleties related to the choice of $U(N)$ or $SU(N)$ as gauge group on the string theory side. This concerns in particular the role of the $U(1)$ mode.

Last but not least, the prewrapping effect has important consequences for the integrability-based descriptions of the β-deformation. The existence of prewrapping means that finite-size effects start one loop order earlier than in the undeformed theory. The asymptotic Bethe equations are reliable only up to this lower loop order and need to be supplemented already before the finite-size wrapping effect is incorporated. A very interesting possibility is that a correct inclusion of prewrapping, i.e. the removal of $U(1)$ modes, could cure the divergencies at $L = 2$ encountered in the TBA and Y-system equations mentioned in the introduction. In fact, our result allows for a patch-work solution to the spectral problem in all compact closed subsectors: the anomalous dimension of the states $\text{tr}[\phi^i\phi^j]$ are zero at all loop orders and the anomalous dimensions of all remaining ones can be computed with the current approach to integrability. It remains an important challenge to reproduce these results from a homogeneous, purely integrability-based approach. Conclusive tests of possible modifications to incorporate prewrapping into the integrability-based description do, however, require to work in non-compact subsectors or the complete theory, where $L \geq 3$ prewrapping candidates exist. The outcome of such tests will

show whether the β-deformation with gauge group $SU(N)$ is indeed as integrable as its undeformed parent theory.[29] Clearly, it is important to collect more, higher-loop field-theory results to guide and test these modifications of integrability.

Acknowledgements

A   Table of anomalous dimensions

In this appendix we provide the anomalous dimensions of all primary states with classical scaling dimensions $\Delta_0 = 4, \frac{9}{2}$. The following table should be understood as continuation of Table 6 in Section 5.

Δ_0	$[j,\bar{\jmath}]_{(q^1,q^2,q^3)}$	L	E
4	$[0,0]_{(-2,0,2)}$	4	$\{12, -32s_2\}$
4	$[0,0]_{(-1,-1,2)}$	4	$\{20, -96, 128s_3\}$
4	$[0,0]_{(-1,0,1)}$	4	$\{8\}, \{8\}, \{8\}, \{8\}, \{12, -32s_1\},$
			$\{56, -1232, 64(207 + 4s_1 + s_2), -128(540 + 59s_1 + 18s_2 + s_3),$
			$1024(135 + 68s_1 + 25s_2 + 4s_3), -1024(190s_1 + 88s_2 + 26s_3 + s_4)\}$
4	$[0,0]_{(-1,0,3)}$	4	$\{8s_1\}$
4	$[0,0]_{(-1,1,2)}$	4	$\{4(3 - 2s_1)\}^\star, \{8(1 + s_1), -16(2s_1 + s_2)\}$
4	$[0,0]_{(0,0,0)}$	4	$\{0\}, \{0\}, \{0\}, \{8\}, \{8\}, \{8\}, \{8\}, \{8\}, \{8\}, \{12\}, \{12\}, \{12\}, \{12\},$
			$\{12\}, \{12\}, \{12\}, \{12\}, \{20, -80\}, \{20, -80\}, \{26, -128\}$
4	$[0,0]_{(0,0,2)}$	3	$\{12\}$
4	$[0,0]_{(0,0,2)}$	4	$\{20, -96, 128s_2\}^\star, \{28, -240, 640, -512s_2\}$
4	$[0,0]_{(0,0,4)}$	4	$\{0\}$
4	$[0,0]_{(0,1,1)}$	3	$\{20, -96, 128s_1\}$
4	$[0,0]_{(0,1,1)}$	4	$\{8\}, \{8\}, \{8\}^\star, \{8\}^\star, \{12\}^\star, \{12\}^\star, \{12, -32s_1\}^\star,$
			$\{32, -320, 64(15 + 4s_1), -1536s_1\}$
4	$[0,0]_{(0,1,3)}$	4	$\{8s_1\}^\star$
4	$[0,0]_{(0,2,2)}$	4	$\{12, -32s_2\}^\star$

Table 6 – continued on next page

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[29]A correct incorporation of prewrapping would also have important consequences for the possible application of integrability beyond conformality in the non-conformal γ_i-deformation. See [14] for a discussion.

Δ_0	$[j,\bar{j}]_{(q^1,q^2,q^3)}$	L	E
4	$[0,0]_{(1,1,2)}$	3	$\{0\}$
4	$[0,0]_{(1,1,2)}$	4	$\{20,-96,128s_1\}^\star$
4	$[\frac{1}{2},\frac{1}{2}]_{(-1,0,2)}$	3	$\{4(5-s_1),-8(12-4s_1-s_2)\}$
4	$[\frac{1}{2},\frac{1}{2}]_{(-1,1,1)}$	3	$\{8\},\{8\},\{12\},\{12\},\{12\}^\star$
4	$[\frac{1}{2},\frac{1}{2}]_{(0,0,1)}$	3	$\{8\},\{8\},\{12\},\{12\}^\star,\{12\}^\star,\{20,-96,128s_1\},$
			$\{35,-396,12(120-s_1)\},\{35,-396,12(120-s_1)\}$
4	$[1,0]_{(-1,-1,2)}$	3	$\{12\}$
4	$[1,0]_{(-1,0,1)}$	3	$\{32,-16(21-s_1),16(72-8s_1-s_2)\}$
4	$[1,0]_{(0,0,0)}$	3	$\{8\},\{8\},\{8\},\{12\},\{12\},\{12\},\{12\}^\star,\{18\}$
4	$[1,0]_{(0,0,2)}$	3	$\{8\},\{12\}^\star$
4	$[1,0]_{(0,1,1)}$	3	$\{8\},\{8\},\{12\},\{12\}^\star,\{12\}^\star$
4	$[1,0]_{(1,1,2)}$	3	$\{12\}^\star$
4	$[1,1]_{(-1,0,1)}$	2	$\{\frac{4}{3}(9+2s_1)\}$
4	$[1,1]_{(0,0,0)}$	2	$\{12\},\{12\},\{12),\{\frac{50}{3}\}$
4	$[\frac{3}{2},\frac{1}{2}]_{(0,0,1)}$	2	$\{12\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{5}{2},-\frac{1}{2},\frac{3}{2})}$	4	$\{8\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{5}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{12\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{5}{2},\frac{1}{2},\frac{1}{2})}$	4	$\{28,-240,640,-512s_3\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},-\frac{3}{2},\frac{3}{2})}$	4	$\{8\},\{8\},\{12\}^\star$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},-\frac{1}{2},\frac{1}{2})}$	3	$\{32,-28(12-s_1),8(144-46s_1+s_2)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},-\frac{1}{2},\frac{1}{2})}$	4	$\{8\},\{32,-16(21-s_1),16(72-8s_1-s_2)\}^\star,$
			$\{106,-4985-32s_1,4(34219+744s_1+16s_2),$
			$\quad-16(151779+7540s_1+320s_2),$
			$\quad64(454285+43714s_1+2759s_2+2s_3),$
			$\quad-64(3715060+639132s_1+53612s_2+176s_3+s_4),$
			$\quad1024(1280225+381886s_1+40154s_2+357s_3+8s_4),$
			$\quad-1024(4553400+2387478s_1+304021s_2+5682s_3+242s_4),$
			$\quad4096(2358000+2353306s_1+355064s_2+11892s_3+755s_4+2s_5),$
			$\quad-4096(2160000+5310120s_1+935079s_2+50086s_3+4234s_4,$
			$\qquad\quad+50s_5+s_6),$
			$\quad32768(653112s_1+132666s_2+10408s_3+1105s_4+32s_5+2s_6)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},-\frac{1}{2},\frac{5}{2})}$	4	$\{20-8s_1,-16(6-4s_1-s_2)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{3}{2},\frac{1}{2},\frac{3}{2})}$	4	$\{80,-4(692+5s_1-4s_2),8(6760+177s_1-109s_2+s_3),$
			$\quad-16(40768+2630s_1-1187s_2+42s_3),$
			$\quad32(155360+21264s_1-6464s_2+648s_3+33s_4),$
			$\quad-64(365184+100981s_1-17319s_2+4983s_3+581s_4+8s_5),$
			$\quad128(483840+281360s_1-14562s_2+20629s_3+3800s_4+131s_5-4s_6),$
			$\quad-512(138240+212579s_1+12194s_2+22021s_3+5472s_4+344s_5$
			$\qquad\quad-12s_6),$
			$\quad512(268176s_1+42376s_2+38000s_3+11729s_4+1156s_5-20s_6-4s_7)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}$	3	$\{0\},\{8\},\{8\},\{8\},\{12\},\{12\},\{12\},\{15\},\{15\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}$	4	$\{0\},\{8\},\{8\},\{8\},\{8\},\{8\},\{8\},\{8\}^\star,\{8\}^\star,\{8\}^\star,\{12\},\{12\},$
			$\{12\}^\star,\{12\}^\star,\{12\}^\star,\{12\}^\star,\{15\},\{15\},\{15\},\{15\},\{18\}^\star,$
			$\{20,-80\},\{20,-80\},\{20,-80\}$

Table 6 – continued on next page

Δ_0	$[j,\bar{\jmath}]_{(q^1,q^2,q^3)}$	L	E
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},-\frac{1}{2},\frac{3}{2})}$	4	$\{8\}$, $\{8\}$, $\{12\}$, $\{20, -96, 128s_2\}$,
			$\{35, -396, 12(120-s_2)\}$, $\{35, -396, 12(120-s_2)\}$,
			$\{40, -576, 3520, -64(120-s_2)\}$, $\{40, -576, 3520, -64(120-s_2)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	4	$\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{12\}$, $\{12\}$, $\{12\}$, $\{12\}$, $\{12\}$, $\{12\}$,
			$\{38, -456, 64(27-s_1)\}$, $\{32, -320, 64(15+4s_1), -1536s_1\}$,
			$\{58, -16(80-s_1), 13376 - 640s_1, -128(515-48s_1), 4096(30+s_1),$
			$\quad -8192(12s_1+s_2)\}$,
			$\{78, -2561, 45808, -256(1880-s_1), 2965440 - 9216s_1,$
			$\quad -15360(645-7s_1), 1024(13500-396s_1+s_2)\}$,
			$\{78, -2561, 45808, -256(1880-s_1), 2965440 - 9216s_1,$
			$\quad -15360(645-7s_1), 1024(13500-396s_1+s_2)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},\frac{1}{2},\frac{5}{2})}$	4	$\{4(5-s_1), -8(12-4s_1-s_2)\}^\star$,
			$\{28, -48(5+s_1), 640(1+s_1), -64(31s_1+2s_2+s_3)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(-\frac{1}{2},\frac{3}{2},\frac{3}{2})}$	4	$\{8\}$, $\{8\}$, $\{8\}^\star$, $\{8\}^\star$, $\{12\}^\star$, $\{12\}^\star$,
			$\{32, -320, 64(15+4s_2), -1536s_2\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	3	$\{8\}$, $\{12\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	4	$\{8\}^\star$, $\{8\}^\star$, $\{12\}^\star$, $\{20, -96, 128s_1\}^\star$, $\{35, -396, 12(120-s_1)\}^\star$,
			$\{35, -396, 12(120-s_1)\}^\star$, $\{28, -240, 640, -512s_1\}$,
			$\{40, -576, 3520, -64(120-s_1)\}$, $\{40, -576, 3520, -64(120-s_1)\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{7}{2})}$	4	$\{0\}$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(\frac{1}{2},\frac{3}{2},\frac{5}{2})}$	4	$\{20, -32(3+s_1), 32(8s_1+s_2)\}^\star$
$\frac{9}{2}$	$[\frac{1}{2},0]_{(\frac{3}{2},\frac{3}{2},\frac{3}{2})}$	4	$\{0\}$, $\{8\}^\star$, $\{8\}^\star$, $\{8\}^\star$, $\{12\}^\star$, $\{12\}^\star$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(-\frac{3}{2},-\frac{1}{2},\frac{3}{2})}$	3	$\{4(5+2s_1), -4(24+25s_1-2s_2)\}$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(-\frac{3}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{35, -396, 12(120-s_2)\}$, $\{35, -396, 12(120-s_2)\}$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})}$	3	$\{8\}$, $\{8\}$, $\{12\}$, $\{12\}$, $\{12\}^\star$, $\{\frac{86}{3}, -\frac{8}{3}(75+2s_1)\}$,
			$\{38, -465, 24(75+2s_1)\}$, $\{38, -465, 24(75+2s_1)\}$,
			$\{38, -456, 64(27-s_1)\}$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(-\frac{1}{2},-\frac{1}{2},\frac{5}{2})}$	3	$\{8\}$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(-\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	3	$\{\frac{8s_1}{3}+12\}^\star$,
			$\{66, -3(595+4s_1), 4(6329+128s_1+4s_2),$
			$\quad -12(16552+667s_1+36s_2), \quad 48(17040+1131s_1+78s_2+s_3),$
			$\quad -32(43200+4216s_1+324s_2+16s_3-s_4)\}$
$\frac{9}{2}$	$[1,\frac{1}{2}]_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{8\}$, $\{12\}$, $\{12\}$, $\{12\}$, $\{12\}^\star$, $\{12\}^\star$, $\{12\}^\star$,
			$\{15\}$, $\{15\}$, $\{15\}$, $\{15\}$, $\{15\}$, $\{15\}$, $\{\frac{50}{3}\}^\star$
$\frac{9}{2}$	$[\frac{3}{2},0]_{(-\frac{1}{2},-\frac{1}{2},\frac{3}{2})}$	3	$\{12\}^\star$
$\frac{9}{2}$	$[\frac{3}{2},0]_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	3	$\{38, -456, 64(27-s_1)\}$
$\frac{9}{2}$	$[\frac{3}{2},0]_{(\frac{1}{2},\frac{1}{2},\frac{3}{2})}$	3	$\{8\}$, $\{12\}$
$\frac{9}{2}$	$[\frac{3}{2},0]_{(\frac{3}{2},\frac{3}{2},\frac{3}{2})}$	3	$\{12\}^\star$
$\frac{9}{2}$	$[\frac{3}{2},1]_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2})}$	2	$\{\frac{86}{3}, -\frac{8}{3}(75+2s_1)\}$

Table 6: Anomalous dimensions of all primary states with classical scaling dimension $\Delta_0 = 4, \frac{9}{2}$ for gauge group $SU(N)$ in the notation introduced after (30). The dependence on the deformation parameter is encoded in $s_n = \sin^2 \frac{n\beta}{2}$. Highest-weight states of the free theory that join lower-lying multiplets in the interacting theory are marked with a $\star$. Each state has to be supplemented by its images under the $\mathbb{S}_3$ symmetry and charge conjugation.

B Representation content of the β-deformation

In this appendix we describe how to determine the representation content of the β-deformation. In Subsection B.1 we review some facts about the unitary representations of the $\mathcal{N} = 1$ superconformal algebra and give the explicit formulae for their characters. In Subsection B.2 we show how to adapt them to the β-deformation. In Subsection B.3 we describe how to use the refined partition function and the characters to determine the representation content via the so-called Eratostenes super-sieve algorithm.

B.1 Representations and characters of $\mathfrak{su}(2,2|1)$

In this subsection we summarise some facts about the unitary representations of the $\mathcal{N} = 1$ superconformal algebra, which were first classified in [50]. We stick to the notation of [47], which uses the same as the work [46] for the undeformed $\mathcal{N} = 4$ SYM theory. We only give the final results necessary to understand our calculations and refer the reader to the literature for their derivations as well as the underlying theory.

The $\mathcal{N} = 1$ superconformal algebra $\mathfrak{su}(2,2|1)$ is generated by the Lorentz transformations $M_{\mu\nu}$, the translations P_μ, the dilatations D, the special conformal transformations K_μ, the supersymmetry transformations $Q_\alpha, \bar{Q}_{\dot\alpha}$, the superconformal transformations $S_\alpha, \bar{S}_{\dot\alpha}$ and the $U(1)_R$-symmetry generator R. In the spinor basis, the Lorentz generators $M_{\mu\nu}$ can be written in terms of the $\mathfrak{su}(2) \times \overline{\mathfrak{su}}(2)$ generators J_+, J_-, J_3 and $\bar{J}_+, \bar{J}_-, \bar{J}_3$. Using the Pauli matrices σ^μ, the translations and special conformal transformations can be rephrased as $P_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} P_\mu$, $K_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} K_\mu$.

A highest-weight (superconformal primary) state $|\Delta, r, j, \bar{j}\rangle_{\text{hw}}$ of $\mathfrak{su}(2,2|1)$ is specified by the requirements

$$
\begin{aligned}
(K_{\alpha\dot\alpha}, S_\alpha, \bar{S}_{\dot\alpha}, J_+, \bar{J}_+)|\Delta, r, j, \bar{j}\rangle_{\text{hw}} &= 0\,, \\
(D, R, J_3, \bar{J}_3)|\Delta, r, j, \bar{j}\rangle_{\text{hw}} &= (\Delta, r, j, \bar{j})|\Delta, r, j, \bar{j}\rangle_{\text{hw}}\,.
\end{aligned}
\tag{31}
$$

All other states in the multiplet can be obtained by acting on this state with the lowering operators $P_{\alpha\dot\alpha}$, Q_α, $\bar{Q}_{\dot\alpha}$, J_- and $\bar{J}_-$.

The character of a representation can be thought of as a (refined) partition function, counting the number of states in the multiplet with a specified set of quantum numbers. The highest-weight state $|\Delta, r, j, \bar{j}\rangle_{\text{hw}}$ is represented by the monomial $d^{2\Delta}\, u^r\, x^{2j}\, \bar{x}^{2\bar{j}}$. The action of the lowering operators increases or decreases the quantum numbers according to $P_{\alpha\dot\alpha} \sim d^2\, x^{\pm 1}\, \bar{x}^{\mp 1}$, $Q_\alpha \sim d\, u^{-1}\, x^{\pm 1}$, $\bar{Q}_{\dot\alpha} \sim d\, u\, \bar{x}^{\mp 1}$, where $\alpha = 1, 2$ correspond to x, x^{-1} and $\dot\alpha = \dot 1, \dot 2$ to $\bar{x}^{-1}, \bar{x}$. We denote the corresponding character by $\chi^{t,\bar{t}}_{(\Delta, r, j, \bar{j})}(d, u, x, \bar{x})$. Here $t, \bar{t}$ are the fractions of the respective supercharges $Q_\alpha, \bar{Q}_{\dot\alpha}$ that cannot be used to generate new states within the multiplet. They are connected to possible constraints that will be specified later.

For a generic long representation we have $t = \bar{t} = 0$ and the character is

$$
\chi^{0,0}_{(\Delta, r, j, \bar{j})}(d, u, x, \bar{x}) = d^{2\Delta}\, u^r\, \chi_{2j+1}(x)\, \chi_{2\bar{j}+1}(\bar{x})\, \mathcal{P}(d, x, \bar{x})\, \mathcal{Q}(d\, u^{-1}, x)\, \mathcal{Q}(d\, u, \bar{x})\,, \tag{32}
$$

where

$$
\mathcal{P}(d, x, \bar{x}) = \prod_{\epsilon, \eta = \pm 1} \frac{1}{(1 - d^2\, x^\epsilon\, \bar{x}^\eta)}\,, \qquad \mathcal{Q}(d, x) = \prod_{\epsilon = \pm 1} (1 + d\, x^\epsilon)\,, \tag{33}
$$

and

$$\chi_n(x) = \frac{x^n - x^{-n}}{x - x^{-1}} \tag{34}$$

is the character of the usual n-dimensional representation of $\mathfrak{su}(2)$. Note that $\mathcal{P}(d, x, \bar{x})$, $\mathcal{Q}(d\, u^{-1}, x)$ and $\mathcal{Q}(d\, u, \bar{x})$ account for the bosonic and fermionic – but otherwise unconstrained – action of $P_{\alpha\dot\alpha}$, Q_α and $\bar{Q}_{\dot\alpha}$, respectively, on the highest-weight state.

For unitary representations two different kinds of constraints may occur for Q_α and $\bar{Q}_{\dot\alpha}$, respectively. They also lead to constraints on the representation labels $\Delta, r, j, \bar{\jmath}$. The first kind is called shortening conditions and reads

$$
\begin{aligned}
\bar{t} = 1: \qquad \Delta &= +\frac{3}{2}r\,, \qquad \bar{Q}_{\dot\alpha}|\Delta, r, j, 0\rangle_{\text{hw}} = 0\,, \\
t = 1: \qquad \Delta &= -\frac{3}{2}r\,, \qquad Q_\alpha|\Delta, r, 0, \bar{\jmath}\rangle_{\text{hw}} = 0\,,
\end{aligned}
\tag{35}
$$

i.e. $\bar{Q}_{\dot\alpha}$ and Q_α, respectively, act as zero. In the corresponding characters the respective factors of $\mathcal{Q}(d\, u, \bar{x})$ and $\mathcal{Q}(d\, u^{-1}, x)$ are absent:

$$
\begin{aligned}
\chi^{0,1}_{(+\frac{3}{2}r, r, j, 0)}(d, u, x, \bar{x}) &= d^{+3r}\, u^r\, \chi_{2j+1}(x)\, \mathcal{P}(d, x, \bar{x})\, \mathcal{Q}(d\, u^{-1}, x)\,, \quad r \geq +\frac{2}{3}(j+1)\,, \\
\chi^{1,0}_{(-\frac{3}{2}r, r, 0, \bar{\jmath})}(d, u, x, \bar{x}) &= d^{-3r}\, u^r\, \chi_{2\bar{\jmath}+1}(\bar{x})\, \mathcal{P}(d, x, \bar{x})\, \mathcal{Q}(d\, u, \bar{x})\,, \qquad r \leq -\frac{2}{3}(\bar{\jmath}+1)\,.
\end{aligned}
\tag{36}
$$

The second kind is called semi-shortening conditions and reads

$$
\begin{aligned}
\bar{t} = \frac{1}{2}: \quad \Delta &= +\frac{3}{2}r + 2\bar{\jmath} + 2\,, \qquad \Big(\bar{Q}_1 + \frac{1}{2\bar{\jmath}}\bar{Q}_2\bar{J}_-\Big)|\Delta, r, j, \bar{\jmath}\rangle_{\text{hw}} = 0 \ \text{ for } \bar{\jmath} > 0\,, \\
& \hspace{5.5cm} \bar{Q}_1|\Delta, r, j, 0\rangle_{\text{hw}} = 0\,, \\
t = \frac{1}{2}: \quad \Delta &= -\frac{3}{2}r + 2j + 2\,, \qquad \Big(Q_2 - \frac{1}{2j}Q_1 J_-\Big)|\Delta, r, j, \bar{\jmath}\rangle_{\text{hw}} = 0 \ \text{ for } j > 0\,, \\
& \hspace{5.5cm} Q_2|\Delta, r, 0, \bar{\jmath}\rangle_{\text{hw}} = 0\,,
\end{aligned}
\tag{37}
$$

i.e. the action of $\bar{Q}_1$, or respectively Q_2, yields a state that can also be obtained via the other lowering operators. Accordingly, the monomials corresponding to these states must only appear once in the character, and the additionally occurring monomials capturing the action of $\bar{Q}_1$, or respectively Q_2, have to be removed:

$$
\begin{aligned}
\chi^{0,\frac{1}{2}}_{(+\frac{3}{2}r+2\bar{\jmath}+2, r, j, \bar{\jmath})}(d, u, x, \bar{x}) &= d^{+3r+4\bar{\jmath}+4}\, u^r\, \chi_{2j+1}(x)\big(\chi_{2\bar{\jmath}+1}(\bar{x}) + d\, u\, \chi_{2\bar{\jmath}+2}(\bar{x})\big) \\
& \hspace{2cm} \mathcal{P}(d, x, \bar{x})\, \mathcal{Q}(d\, u^{-1}, x)\,, \qquad r \geq \frac{2}{3}(j - \bar{\jmath})\,, \\
\chi^{\frac{1}{2},0}_{(-\frac{3}{2}r+2j+2, r, j, \bar{\jmath})}(d, u, x, \bar{x}) &= d^{-3r+4j+4}\, u^r\, \big(\chi_{2j+1}(x) + d\, u^{-1}\, \chi_{2j+2}(x)\big)\chi_{2\bar{\jmath}+1}(\bar{x}) \\
& \hspace{2cm} \mathcal{P}(d, x, \bar{x})\, \mathcal{Q}(d\, u, \bar{x})\,, \qquad r \leq \frac{2}{3}(j - \bar{\jmath})\,.
\end{aligned}
\tag{38}
$$

So far, (semi-)shortening conditions have only been applied for either Q_α or $\bar{Q}_{\dot\alpha}$. If both t and $\bar{t}$ are nonzero, the algebra relation $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$ requires that the contributions from the respective $P_{\alpha\dot\alpha}$ are also removed. In particular, the contribution

from $P_{2\mathrm{i}}$ has to be eliminated for $t = \bar{t} = \frac{1}{2}$. In this case we have

$$
\begin{aligned}
\chi^{\frac{1}{2},\frac{1}{2}}_{(j+\bar{\jmath}+2,\frac{2}{3}(j-\bar{\jmath}),j,\bar{\jmath})}(d,u,x,\bar{x}) = u^{\frac{2}{3}(j-\bar{\jmath})}\big(&\mathcal{D}_{j,\bar{\jmath}}(d,x,\bar{x}) + u^{-1}\,\mathcal{D}_{j+\frac{1}{2},\bar{\jmath}}(d,x,\bar{x}) \\
&+ u\,\mathcal{D}_{j,\bar{\jmath}+\frac{1}{2}}(d,x,\bar{x}) + \mathcal{D}_{j+\frac{1}{2},\bar{\jmath}+\frac{1}{2}}(d,x,\bar{x})\big)\,,
\end{aligned}
\tag{39}
$$

where

$$
\mathcal{D}_{j,\bar{\jmath}}(d,x,\bar{x}) = d^{2(j+\bar{\jmath}+2)}\big(\chi_{2j+1}(x)\,\chi_{2\bar{\jmath}+1}(\bar{x}) - d^2\chi_{2j}(x)\,\chi_{2\bar{\jmath}}(\bar{x})\big)\mathcal{P}(d,x,\bar{x})\,.
\tag{40}
$$

The remaining cases are

$$
\begin{aligned}
\chi^{\frac{1}{2},1}_{(j+1,+\frac{2}{3}(j+1),j,0)}(d,u,x,\bar{x}) &= u^{+\frac{2}{3}(j+1)}\big(\mathcal{E}_j(d,x,\bar{x}) + u^{-1}\,\mathcal{E}_{j+\frac{1}{2}}(d,x,\bar{x})\big)\,, \\
\chi^{1,\frac{1}{2}}_{(\bar{\jmath}+1,-\frac{2}{3}(\bar{\jmath}+1),0,\bar{\jmath})}(d,u,x,\bar{x}) &= u^{-\frac{2}{3}(\bar{\jmath}+1)}\big(\bar{\mathcal{E}}_{\bar{\jmath}}(d,x,\bar{x}) + u\,\bar{\mathcal{E}}_{\bar{\jmath}+\frac{1}{2}}(d,x,\bar{x})\big)\,,
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
\mathcal{E}_j(d,x,\bar{x}) &= d^{2j+2}\big(\chi_{2j+1}(x) - d^2\chi_{2j}(x)\,\chi_2(\bar{x}) + d^4\chi_{2j-1}(x)\big)\mathcal{P}(d,x,\bar{x})\,, \\
\bar{\mathcal{E}}_{\bar{\jmath}}(d,x,\bar{x}) &= d^{2\bar{\jmath}+2}\big(\chi_{2\bar{\jmath}+1}(\bar{x}) - d^2\chi_2(x)\,\chi_{2\bar{\jmath}}(\bar{x}) + d^4\chi_{2\bar{\jmath}-1}(\bar{x})\big)\mathcal{P}(d,x,\bar{x})\,.
\end{aligned}
\tag{42}
$$

Unitarity requires that $\Delta \geq \max(2 + 2\bar{\jmath} + \frac{3}{2}r, 2 + 2j - \frac{3}{2}r)$, unless one of the shortening conditions in (35) is fulfilled in which case $\Delta = \pm\frac{3}{2}r$. At this so-called unitary threshold otherwise irreducible representations become reducible, which is also reflected in the characters. The resulting reductions of the characters that alter the value of $\bar{t}$ are given by

$$
\begin{aligned}
\chi^{0,0}_{(+\frac{3}{2}r+2\bar{\jmath}+2,r,j,\bar{\jmath})}(d,u,x,\bar{x}) &= \chi^{0,\frac{1}{2}}_{(+\frac{3}{2}r+2\bar{\jmath}+2,r,j,\bar{\jmath})}(d,u,x,\bar{x}) \\
&\quad + \chi^{0,\frac{1}{2}}_{(+\frac{3}{2}r+2\bar{\jmath}+\frac{5}{2},r+1,j,\bar{\jmath}-\frac{1}{2})}(d,u,x,\bar{x}) && \text{for } \bar{\jmath} > 0\,, \\
\chi^{0,0}_{(+\frac{3}{2}r+2,r,j,0)}(d,u,x,\bar{x}) &= \chi^{0,\frac{1}{2}}_{(+\frac{3}{2}r+2,r,j,0)}(d,u,x,\bar{x}) + \chi^{0,1}_{(+\frac{3}{2}r+3,r+2,j,0)}(d,u,x,\bar{x})\,, \\
\chi^{\frac{1}{2},0}_{(j+\bar{\jmath}+2,\frac{2}{3}(j-\bar{\jmath}),j,\bar{\jmath})}(d,u,x,\bar{x}) &= \chi^{\frac{1}{2},\frac{1}{2}}_{(j+\bar{\jmath}+2,\frac{2}{3}(j-\bar{\jmath}),j,\bar{\jmath})}(d,u,x,\bar{x}) \\
&\quad + \chi^{0,\frac{1}{2}}_{(j+\bar{\jmath}+\frac{5}{2},\frac{2}{3}(j-\bar{\jmath})+1,j,\bar{\jmath}-\frac{1}{2})}(d,u,x,\bar{x}) && \text{for } \bar{\jmath} > 0\,, \\
\chi^{\frac{1}{2},0}_{(j+2,\frac{2}{3}j,j,0)}(d,u,x,\bar{x}) &= \chi^{\frac{1}{2},\frac{1}{2}}_{(j+2,\frac{2}{3}j,j,0)}(d,u,x,\bar{x}) + \chi^{0,1}_{(j+3,\frac{2}{3}j+2,j,0)}(d,u,x,\bar{x})\,,
\end{aligned}
\tag{43}
$$

and they can be translated directly to the respective reductions of the representations. The analogous relations that alter t have been omitted and can be obtained by replacing

$$
\chi^{\tilde{t},\tilde{\bar{t}}}_{(\tilde{\Delta}(r,j,\bar{\jmath}),\tilde{r}(r,j,\bar{\jmath}),\tilde{\jmath}(j,\bar{\jmath}),\tilde{\bar{\jmath}}(j,\bar{\jmath}))} \rightarrow \chi^{\tilde{\bar{t}},\tilde{t}}_{(\tilde{\Delta}(-r,\bar{\jmath},j),-\tilde{r}(-r,\bar{\jmath},j),\tilde{\bar{\jmath}}(\bar{\jmath},j),\tilde{\jmath}(\bar{\jmath},j))}\,,
\tag{44}
$$

where the quantities with tildes stand for the abstract functions of $r, j, \bar{\jmath}$ that are specified in χ. If a multiplet acquires an anomalous dimension, it moves away from the unitary threshold[30] and the pairs of representations on the right hand sides of (43) join again.

---

[30]Recall that $\Delta = \Delta_0 + \gamma$, where Δ_0 denotes the classical scaling dimension and $\gamma = g^2 E + \mathcal{O}(g^3)$ the anomalous one.

B.2 Adaption to the β-deformation

The results reviewed above can be easily applied to the β-deformation. We have identified Q_α with the $\mathcal{N}=4$ supercharge $Q_\alpha^{A=4}$ and the $U(1)_R$-symmetry charge r with the combination of $\mathfrak{su}(4)$ Cartan charges specified in Table 1. Furthermore, the generators of the superconformal algebra $\mathfrak{su}(2,2|1)$ are supplemented by the generators Q^1 and Q^2 of the global $U(1)_{Q^1} \times U(1)_{Q^2}$ symmetry. In the characters, they are represented by the fugacities v and w. As all irreducible representations of abelian groups are one-dimensional, the above character formulae simply have to be supplemented by $v^{Q^1} w^{Q^2}$.

B.3 The super-sieve algorithm

Using the above characters, the $\mathcal{N}=1$ representation content of the free β-deformation can be determined directly from the refined partition function in analogy to the $\mathcal{N}=4$ SYM theory case [45], see also [46].[31]

The refined partition function of single-trace operators in the β-deformation can be obtained via Polya theory, see e.g. [46]. It is given by[32]

$$Z(d,x,\bar{x},u,v,w) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \ln\left[1 - z((-1)^{k+1}d^k, x^k, \bar{x}^k, u^k, v^k, w^k)\right]$$
$$- s\, z(d,x,\bar{x},u,v,w), \tag{45}$$

where $\varphi(k)$ is the Euler totient function giving the number of positive integers less than or equal to k that are relative prime to k. The single-site partition function can be found in analogy to [46], and it reads

$$\begin{aligned}
z(d,x,\bar{x},u,v,w) = {} & (u^{\frac{2}{3}}v^1 + u^{\frac{2}{3}}v^{-1}w^1 + u^{\frac{2}{3}}w^{-1})\mathcal{E}_0(d,x,\bar{x}) \\
& + (u^{-\frac{2}{3}}v^{-1} + u^{-\frac{2}{3}}v^1w^{-1} + u^{-\frac{2}{3}}w^1)\bar{\mathcal{E}}_0(d,x,\bar{x}) \\
& + (u^{-\frac{1}{3}}v^1 + u^{-\frac{1}{3}}v^{-1}w^1 + u^{-\frac{1}{3}}w^{-1} + u^1)\mathcal{E}_{\frac{1}{2}}(d,x,\bar{x}) \\
& + (u^{\frac{1}{3}}v^{-1} + u^{\frac{1}{3}}v^1w^{-1} + u^{\frac{1}{3}}w^1 + u^{-1})\bar{\mathcal{E}}_{\frac{1}{2}}(d,x,\bar{x}) \\
& + \mathcal{E}_1(d,x,\bar{x}) + \bar{\mathcal{E}}_1(d,x,\bar{x}),
\end{aligned} \tag{46}$$

where $\mathcal{E}$ and $\bar{\mathcal{E}}$ were defined in (42). The first line in (46) accounts for the scalars, the second one for the anti-scalars, the third for the fermions, the fourth for the anti-fermions and the last one for the self-dual and anti-self-dual component of the field strength, cf. Table 1.

Using the refined partition function and the characters, the representation content can be determined via the Eratostenes super-sieve algorithm proposed in [45]. Starting with the refined partition function (45), one identifies as a highest-weight state the state corresponding to the monomial with the smallest exponent of d and subordinately largest exponent of x, $\bar{x}$. One then determines the character of the

---

[31]Note that the free β-deformation is identical to the free $\mathcal{N}=4$ SYM theory.

[32]The sign factor in front of d^k takes care of the fact that a state with integer or half-integer classical scaling dimension obeys the Bose-Einstein or Fermi-Dirac statistic, respectively. Note that we organise signs slightly different than the authors of [46].

representation containing it as highest-weight state and subtracts this character from the refined partition function. The result of this subtraction serves as input of the next iteration. In this way, the refined partition function can be uniquely expressed as the sum of the characters of the representations in the (free) theory.

For the $\mathcal{N} = 4$ SYM theory the $\mathcal{N} = 4$ representation content was determined for $\Delta_0 \leq 4$ in [45]. Hence, one can alternatively use the super-sieve algorithm to decompose the $\mathcal{N} = 4$ characters of these representations into $\mathcal{N} = 1$ characters.[33] The results of both methods agree.

C The harmonic action

In this appendix we present a completely explicit expression of the harmonic action in a form suitable for an implementation e.g. in `Mathematica`. Although apparently different it is nevertheless equivalent to the one given in [19].

The single-site states of the spin chain of $\mathcal{N} = 4$ SYM theory are taken from the alphabet given in (11). The occurring covariant derivatives have been translated from Minkowski indices to spinor indices using the Pauli matrices $(\sigma^\mu)_{\alpha\dot\alpha}$: $D_{\alpha\dot\alpha} = D_\mu(\sigma^\mu)_{\alpha\dot\alpha}$. Using the antisymmetric products of Pauli matrices $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$, the field strength is translated to the spinor basis and split into its self-dual and anti-self-dual part as $\mathcal{F}_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu}$ and $\bar{\mathcal{F}}_{\dot\alpha\dot\beta} = (\bar\sigma^{\mu\nu})_{\dot\alpha\dot\beta} F_{\mu\nu}$, respectively.

In terms of the bosonic $\mathfrak{su}(2)$ and $\overline{\mathfrak{su}}(2)$ oscillators $\mathbf{a}^\dagger_\alpha$, $\alpha = 1, 2$, and $\mathbf{b}^\dagger_{\dot\alpha}$, $\dot\alpha = \dot1, \dot2$, as well as the fermionic $\mathfrak{su}(4)$ oscillators $\mathbf{c}^\dagger_A$, $A = 1, 2, 3, 4$, the fields of the alphabet can be written as

$$
\begin{aligned}
D^k \mathcal{F} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k+2}(\mathbf{b}^\dagger)^k && |\,0\,\rangle\,, \\
D^k \psi^A &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^k && \mathbf{c}^{\dagger A}|\,0\,\rangle\,, \\
D^k \varphi^{AB} &\;\widehat{=}\; (\mathbf{a}^\dagger)^k && (\mathbf{b}^\dagger)^k && \mathbf{c}^{\dagger A}\mathbf{c}^{\dagger B}|\,0\,\rangle\,, \\
D^k \bar\psi^{ABC} &\;\widehat{=}\; (\mathbf{a}^\dagger)^k && (\mathbf{b}^\dagger)^{k+1}\mathbf{c}^{\dagger A}\mathbf{c}^{\dagger B}\mathbf{c}^{\dagger C}|\,0\,\rangle\,, \\
D^k \bar{\mathcal{F}} &\;\widehat{=}\; (\mathbf{a}^\dagger)^k && (\mathbf{b}^\dagger)^{k+2}\mathbf{c}^{\dagger 1}\mathbf{c}^{\dagger 2}\mathbf{c}^{\dagger 3}\mathbf{c}^{\dagger 4}|\,0\,\rangle\,,
\end{aligned}
\tag{48}
$$

where $\phi^i \propto \varphi^{i4}$, $\bar\phi_i \propto \epsilon_{ABi4}\varphi^{AB}$, with antisymmetric φ and $\bar\psi^{ABC} = \epsilon_{ABCD}\bar\psi^D$.[34] We denote the numbers of $\mathbf{a}^\dagger_1$, $\mathbf{a}^\dagger_2$, $\mathbf{b}^\dagger_{\dot1}$, $\mathbf{b}^\dagger_{\dot2}$, $\mathbf{c}^\dagger_1$, $\mathbf{c}^\dagger_2$, $\mathbf{c}^\dagger_3$, $\mathbf{c}^\dagger_4$ oscillators at spin-chain site i by $a^1_{(i)}, a^2_{(i)}, b^{\dot1}_{(i)}, b^{\dot2}_{(i)}, c^1_{(i)}, c^2_{(i)}, c^3_{(i)}, c^4_{(i)}$. These are connected to the spins j and $\bar j$ as $j = \frac{1}{2}(a^1 \text{-} a^2)$ and $\bar j = \frac{1}{2}(b^{\dot2} \text{-} b^{\dot1})$.

For two initial and final single-site states defined by canonically ordered oscillators

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[33]Note that the $\mathfrak{su}(4)$ Cartan charges used in [46] have to be translated to the basis $U(1)_{Q^1} \times U(1)_{Q^2} \times U(1)_R$, which leads to the following replacements in the character formulae of [45]:

$$
u_1 \to u^{-\frac{1}{3}} v^1\,, u_2 \to u^{-\frac{1}{3}} v^{-1} w^1\,, u_3 \to u^{-\frac{1}{3}} w^{-1}\,, u_4 \to u^1\,.
\tag{47}
$$

[34]The precise constants of proportionality are of no importance here, as they only lead to a change of basis. The corresponding similarity transformation leaves the spectrum of the dilatation operator invariant.

with the occupation numbers

$$A_{(1)} = (a^1_{(1)}, a^2_{(1)}, b^{\dot{1}}_{(1)}, b^{\dot{2}}_{(1)}, c^1_{(1)}, c^2_{(1)}, c^3_{(1)}, c^4_{(1)}) \, ,$$
$$A_{(2)} = (a^1_{(2)}, a^2_{(2)}, b^{\dot{1}}_{(2)}, b^{\dot{2}}_{(2)}, c^1_{(2)}, c^2_{(2)}, c^3_{(2)}, c^4_{(2)}) \, ,$$

(49)

and

$$A_{(3)} = (a^1_{(3)}, a^2_{(3)}, b^{\dot{1}}_{(3)}, b^{\dot{2}}_{(3)}, c^1_{(3)}, c^2_{(3)}, c^3_{(3)}, c^4_{(3)}) \, ,$$
$$A_{(4)} = (a^1_{(4)}, a^2_{(4)}, b^{\dot{1}}_{(4)}, b^{\dot{2}}_{(4)}, c^1_{(4)}, c^2_{(4)}, c^3_{(4)}, c^4_{(4)}) \, ,$$

(50)

respectively, such that $A_{(1)} + A_{(2)} = A_{(3)} + A_{(4)}$, the concrete expression of the harmonic action reads

$$\langle A_{(3)}, A_{(4)} | \mathfrak{D}_2^{\mathcal{N}=4} | A_{(1)}, A_{(2)} \rangle$$

$$= \prod_{\alpha=1}^{2} \left(\sum_{a^\alpha=\max(a^\alpha_{(3)}-a^\alpha_{(2)},0)}^{\min(a^\alpha_{(1)},a^\alpha_{(3)})} \binom{a^\alpha_{(1)}}{a^\alpha} \binom{a^\alpha_{(2)}}{a^\alpha_{(3)} - a^\alpha} \right)$$

$$\prod_{\dot{\alpha}=\dot{1}}^{\dot{2}} \left(\sum_{b^{\dot{\alpha}}=\max(b^{\dot{\alpha}}_{(3)}-b^{\dot{\alpha}}_{(2)},0)}^{\min(b^{\dot{\alpha}}_{(1)},b^{\dot{\alpha}}_{(3)})} \binom{b^{\dot{\alpha}}_{(1)}}{b^{\dot{\alpha}}} \binom{b^{\dot{\alpha}}_{(2)}}{b^{\dot{\alpha}}_{(3)} - b^{\dot{\alpha}}} \right)$$

$$\prod_{a=1}^{4} \left(\sum_{c^a=\max(c^a_{(3)}-c^a_{(2)},0)}^{\min(c^a_{(1)},c^a_{(3)})} \binom{c^a_{(1)}}{c^a} \binom{c^a_{(2)}}{c^a_{(3)} - c^a} \right)$$

$$c\Big[\sum_{i=1}^{2} \big(\sum_{\beta=1}^{2} a^\beta_{(i)} + \sum_{\dot{\beta}=\dot{1}}^{\dot{2}} b^{\dot{\beta}}_{(i)} + \sum_{B=1}^{4} c^B_{(i)} \big),$$

$$\sum_{\beta=1}^{2}(a^\beta_{(1)} - a^\beta) + \sum_{\dot{\beta}=\dot{1}}^{\dot{2}}(b^{\dot{\beta}}_{(1)} - b^{\dot{\beta}}) + \sum_{B=1}^{4}(c^B_{(1)} - c^B),$$

$$\sum_{\beta=1}^{2}(a^\beta_{(3)} - a^\beta) + \sum_{\dot{\beta}=\dot{1}}^{\dot{2}}(b^{\dot{\beta}}_{(3)} - b^{\dot{\beta}}) + \sum_{B=1}^{4}(c^B_{(3)} - c^B) \Big]$$

$$(-1)^{(c^1_{(1)}-c^1+c^2_{(1)}-c^2+c^3_{(1)}-c^3+c^4_{(1)}-c^4)(c^1_{(3)}-c^1+c^2_{(3)}-c^2+c^3_{(3)}-c^3+c^4_{(3)}-c^4)}$$

$$(-1)^{(c^2+c^3+c^4)(c^1_{(1)}+c^1_{(3)})}$$

$$(-1)^{(c^1_{(2)}-c^1_{(3)}+c^1+c^3+c^4)(c^2_{(1)}+c^2_{(3)})}$$

$$(-1)^{(c^1_{(2)}-c^1_{(3)}+c^1+c^2_{(2)}-c^2_{(3)}+c^2+c^4)(c^3_{(1)}+c^3_{(3)})}$$

$$(-1)^{(c^1_{(2)}-c^1_{(3)}+c^1+c^2_{(2)}-c^2_{(3)}+c^2+c^3_{(2)}-c^3_{(3)}+c^3)(c^4_{(1)}+c^4_{(3)})} \, .$$

(51)

The coefficients $c[n, n_{12}, n_{21}]$ are given in terms of the the harmonic numbers $h(k) = \sum_{i=1}^{k} \frac{1}{i}$ and the Euler gamma function Γ as

$$c[n, n_{12}, n_{21}] = \begin{cases} 2h(\frac{1}{2}n) & \text{if } n_{12} = n_{21} = 0 \, , \\ 2(-1)^{1+n_{12}n_{21}} \frac{\Gamma(\frac{1}{2}(n_{12}+n_{21}))\Gamma(1+\frac{1}{2}(n-n_{12}-n_{21}))}{\Gamma(1+\frac{1}{2}n)} & \text{else.} \end{cases}$$

(52)

D A cancellation mechanism

In this appendix we argue that in the tower of conformal primary states built from n covariant derivatives $D_{\alpha\dot\alpha}$ distributed on $\mathrm{tr}[\phi^2\phi^3]$ only the lowest ($n=0$) state is affected by prewrapping as the contributions from s-channel diagrams to all elements with $n \geq 1$ vanish due to cancellations in the spacetime part.

A conformal primary state is, by definition, annihilated by all raising operators. In the oscillator picture of [19] these are

$$J_+ = \mathbf{a}_1^\dagger \mathbf{a}^2 \,, \quad \bar{J}_+ = \mathbf{b}_{\dot{2}}^\dagger \mathbf{b}^{\dot{1}} \quad \text{and} \quad K^{\alpha\dot\alpha} = \mathbf{a}^\alpha \mathbf{b}^{\dot\alpha} \,. \tag{53}$$

It is easy to see that the operators

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!^2(n-k)!^2} \, \mathrm{tr}[(\mathrm{D}_{1\dot{2}})^k \phi^2 (\mathrm{D}_{1\dot{2}})^{n-k}\phi^3] \tag{54}$$

form indeed a tower of conformal[35] primaries if we identify $\mathrm{D}_{\alpha\dot\alpha} = \mathbf{a}_\alpha^\dagger \mathbf{b}_{\dot\alpha}^\dagger$ and recall that ϕ^2, ϕ^3 are built from only $\mathbf{c}^\dagger$-oscillators, cf. Appendix C.

In dimensional regularisation with $D = 4 - 2\varepsilon$, the relevant one-loop tensor integrals are

$$\frac{p_{(\alpha_1\dot\alpha_1} \cdots p_{\alpha_n\dot\alpha_n)}}{p^{2(2-\frac{D}{2})}} G_{(n)}(1,1) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_{(\alpha_1\dot\alpha_1} \cdots l_{\alpha_n\dot\alpha_n)}}{l^2(p-l)^2} \,, \tag{55}$$

where

$$G_{(n)}(1,1) = \frac{1}{(4\pi)^2\varepsilon} \frac{1}{n+1} + \mathcal{O}(\varepsilon^0) \,, \tag{56}$$

and the parenthesis denote total symmetrisation in both kinds of spinor indices [51]. Hence, the divergent part of the integrals found from the s-channel Feynman diagrams involving the individual operators in the sum (54) are given by

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(p_{1\dot{2}} - l_{1\dot{2}})^k (l_{1\dot{2}})^{n-k}}{l^2(p-l)^2} \sim \frac{1}{(4\pi)^2\varepsilon} \sum_{m=0}^{k} \frac{(-1)^m}{n-k+m+1} \binom{k}{m} = \frac{1}{(4\pi)^2\varepsilon} \frac{k!(n-k)!}{(n+1)!} \,. \tag{57}$$

The divergence of the one-loop s-channel diagrams involving the operators (54) is obtained by replacing the trace factor in (54) with the result of (57). This yields zero unless $n=0$.

References

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[2] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys. Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

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[35]Note that the corresponding *super*conformal primaries have classical scaling dimensions $n + \frac{3}{2}$ if $n \geq 1$.

[3] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[4] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," *Nucl.Phys.* **B72** (1974) 461.

[5] C. Sieg and A. Torrielli, "Wrapping interactions and the genus expansion of the 2-point function of composite operators," *Nucl.Phys.* **B723** (2005) 3–32, arXiv:hep-th/0505071 [hep-th].

[6] S. Kachru and E. Silverstein, "4-D conformal theories and strings on orbifolds," *Phys.Rev.Lett.* **80** (1998) 4855–4858, arXiv:hep-th/9802183 [hep-th].

[7] A. E. Lawrence, N. Nekrasov, and C. Vafa, "On conformal field theories in four-dimensions," *Nucl.Phys.* **B533** (1998) 199–209, arXiv:hep-th/9803015 [hep-th].

[8] O. Lunin and J. M. Maldacena, "Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals," *JHEP* **0505** (2005) 033, arXiv:hep-th/0502086 [hep-th].

[9] S. Frolov, R. Roiban, and A. A. Tseytlin, "Gauge-string duality for superconformal deformations of $\mathcal{N} = 4$ super Yang-Mills theory," *JHEP* **0507** (2005) 045, arXiv:hep-th/0503192 [hep-th].

[10] S. Frolov, "Lax pair for strings in Lunin-Maldacena background," *JHEP* **0505** (2005) 069, arXiv:hep-th/0503201 [hep-th].

[11] S. Frolov, R. Roiban, and A. A. Tseytlin, "Gauge-string duality for (non)supersymmetric deformations of $\mathcal{N} = 4$ super Yang-Mills theory," *Nucl.Phys.* **B731** (2005) 1–44, arXiv:hep-th/0507021 [hep-th].

[12] R. G. Leigh and M. J. Strassler, "Exactly marginal operators and duality in four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory," *Nucl.Phys.* **B447** (1995) 95–136, arXiv:hep-th/9503121 [hep-th].

[13] T. J. Hollowood and S. P. Kumar, "An $\mathcal{N} = 1$ duality cascade from a deformation of $\mathcal{N} = 4$ SUSY Yang-Mills theory," *JHEP* **0412** (2004) 034, arXiv:hep-th/0407029 [hep-th].

[14] J. Fokken, C. Sieg, and M. Wilhelm, "Non-conformality of γ_i-deformed $\mathcal{N} = 4$ SYM theory," arXiv:1308.4420 [hep-th].

[15] S. Ananth, S. Kovacs, and H. Shimada, "Proof of all-order finiteness for planar β-deformed Yang-Mills," *JHEP* **0701** (2007) 046, arXiv:hep-th/0609149 [hep-th].

[16] J. A. Minahan, "Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in $\mathcal{N} = 4$ Super Yang-Mills," *Lett.Math.Phys.* **99** (2012) 33–58, arXiv:1012.3983 [hep-th].

[17] C. Sieg, "Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory," *Lett.Math.Phys.* **99** (2012) 59–84, `arXiv:1012.3984 [hep-th]`.

[18] N. Beisert *et al.*, "Review of AdS/CFT Integrability: An Overview," *Lett.Math.Phys.* **99** (2012) 3–32, `arXiv:1012.3982 [hep-th]`.

[19] N. Beisert, "The complete one loop dilatation operator of $\mathcal{N} = 4$ superYang-Mills theory," *Nucl.Phys.* **B676** (2004) 3–42, `arXiv:hep-th/0307015 [hep-th]`.

[20] R. A. Janik, "Review of AdS/CFT Integrability, Chapter III.5: Lüscher Corrections," *Lett.Math.Phys.* **99** (2012) 277–297, `arXiv:1012.3994 [hep-th]`.

[21] N. Gromov and V. Kazakov, "Review of AdS/CFT Integrability, Chapter III.7: Hirota Dynamics for Quantum Integrability," *Lett.Math.Phys.* **99** (2012) 321–347, `arXiv:1012.3996 [hep-th]`.

[22] Z. Bajnok, "Review of AdS/CFT Integrability, Chapter III.6: Thermodynamic Bethe Ansatz," *Lett.Math.Phys.* **99** (2012) 299–320, `arXiv:1012.3995 [hep-th]`.

[23] C. Ahn and R. I. Nepomechie, "Review of AdS/CFT Integrability, Chapter III.2: Exact World-Sheet S-Matrix," *Lett.Math.Phys.* **99** (2012) 209–229, `arXiv:1012.3991 [hep-th]`.

[24] N. Gromov and A. Sever, "Analytic Solution of Bremsstrahlung TBA," *JHEP* **1211** (2012) 075, `arXiv:1207.5489 [hep-th]`.

[25] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Quantum spectral curve for AdS_5/CFT_4," `arXiv:1305.1939 [hep-th]`.

[26] K. Zoubos, "Review of AdS/CFT Integrability, Chapter IV.2: Deformations, Orbifolds and Open Boundaries," *Lett.Math.Phys.* **99** (2012) 375–400, `arXiv:1012.3998 [hep-th]`.

[27] T. Filk, "Divergencies in a field theory on quantum space," *Phys.Lett.* **B376** (1996) 53–58.

[28] N. Beisert and R. Roiban, "Beauty and the twist: The Bethe ansatz for twisted $\mathcal{N} = 4$ SYM," *JHEP* **0508** (2005) 039, `arXiv:hep-th/0505187 [hep-th]`.

[29] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, "Single impurity operators at critical wrapping order in the β-deformed $\mathcal{N} = 4$ SYM," *JHEP* **0908** (2009) 034, `arXiv:0811.4594 [hep-th]`.

[30] J. Gunnesson, "Wrapping in maximally supersymmetric and marginally deformed $\mathcal{N} = 4$ Yang-Mills," *JHEP* **0904** (2009) 130, `arXiv:0902.1427 [hep-th]`.

[31] N. Gromov and F. Levkovich-Maslyuk, "Y-system and β-deformed $\mathcal{N} = 4$ Super-Yang-Mills," *J.Phys.* **A44** (2011) 015402, `arXiv:1006.5438 [hep-th]`.

[32] G. Arutyunov, M. de Leeuw, and S. J. van Tongeren, "Twisting the Mirror TBA," *JHEP* **1102** (2011) 025, `arXiv:1009.4118 [hep-th]`.

[33] Z. Bajnok and O. el Deeb, "6-loop anomalous dimension of a single impurity operator from AdS/CFT and multiple zeta values," *JHEP* **1101** (2011) 054, `arXiv:1010.5606 [hep-th]`.

[34] D. Z. Freedman and U. Gursoy, "Comments on the β-deformed $\mathcal{N} = 4$ SYM theory," *JHEP* **0511** (2005) 042, `arXiv:hep-th/0506128 [hep-th]`.

[35] S. Penati, A. Santambrogio, and D. Zanon, "Two-point correlators in the β-deformed $\mathcal{N} = 4$ SYM at the next-to-leading order," *JHEP* **0510** (2005) 023, `arXiv:hep-th/0506150 [hep-th]`.

[36] S. Frolov and R. Suzuki, "Temperature quantization from the TBA equations," *Phys.Lett.* **B679** (2009) 60–64, `arXiv:0906.0499 [hep-th]`.

[37] M. de Leeuw and S. J. van Tongeren, "The spectral problem for strings on twisted $\mathrm{AdS}_5 \times \mathrm{S}^5$," *Nucl.Phys.* **B860** (2012) 339–376, `arXiv:1201.1451 [hep-th]`.

[38] S. Frolov. private communication.

[39] Q. Jin and R. Roiban, "On the non-planar β-deformed $\mathcal{N} = 4$ super-Yang-Mills theory," *J.Phys.* **A45** (2012) 295401, `arXiv:1201.5012 [hep-th]`.

[40] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys.Rept.* **378** (2003) 207–299, `arXiv:hep-th/0109162 [hep-th]`.

[41] V. V. Khoze, "Amplitudes in the β-deformed conformal Yang-Mills," *JHEP* **0602** (2006) 040, `arXiv:hep-th/0512194 [hep-th]`.

[42] J. R. David and A. Sadhukhan, "Structure constants of β deformed super Yang-Mills," `arXiv:1307.3909 [hep-th]`.

[43] D. Berenstein, V. Jejjala, and R. G. Leigh, "Marginal and relevant deformations of $\mathcal{N} = 4$ field theories and noncommutative moduli spaces of vacua," *Nucl.Phys.* **B589** (2000) 196–248, `arXiv:hep-th/0005087 [hep-th]`.

[44] D. Berenstein and R. G. Leigh, "Discrete torsion, AdS/CFT and duality," *JHEP* **0001** (2000) 038, `arXiv:hep-th/0001055 [hep-th]`.

[45] M. Bianchi, J. F. Morales, and H. Samtleben, "On stringy $\mathrm{AdS}_5 \times \mathrm{S}^5$ and higher spin holography," *JHEP* **0307** (2003) 062, `arXiv:hep-th/0305052 [hep-th]`.

[46] M. Bianchi, F. Dolan, P. Heslop, and H. Osborn, "$\mathcal{N} = 4$ superconformal characters and partition functions," *Nucl.Phys.* **B767** (2007) 163–226, `arXiv:hep-th/0609179 [hep-th]`.

[47] F. Dolan and H. Osborn, "Applications of the Superconformal Index for Protected Operators and q-Hypergeometric Identities to $\mathcal{N} = 1$ Dual Theories," *Nucl.Phys.* **B818** (2009) 137–178, `arXiv:0801.4947 [hep-th]`.

[48] V. Dobrev, "Explicit character formulae for positive energy unitary irreducible representations of D = 4 conformal supersymmetry," *J.Phys.* **A46** (2013) 405202, `arXiv:1208.6250 [hep-th]`.

[49] C. Sieg, "Superspace computation of the three-loop dilatation operator of $\mathcal{N} = 4$ SYM theory," *Phys.Rev.* **D84** (2011) 045014, `arXiv:1008.3351 [hep-th]`.

[50] M. Flato and C. Fronsdal, "Representations of Conformal Supersymmetry," *Lett.Math.Phys.* **8** (1984) 159.

[51] K. Chetyrkin, A. Kataev, and F. Tkachov, "New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique," *Nucl.Phys.* **B174** (1980) 345–377.