Dynamic Lattice Supersymmetry
in $\mathfrak{gl}(n|m)$ Spin Chains

DAVID MEIDINGER, VLADIMIR MITEV

Institut für Mathematik und Institut für Physik,
Humboldt-Universität zu Berlin
IRIS Haus, Zum Großen Windkanal 6, 12489 Berlin, Germany

david.meidinger@physik.hu-berlin.de
mitev@math.hu-berlin.de

Abstract

Supersymmetry operators that change the length of the spin chain have appeared in numerous contexts, ranging from the AdS/CFT correspondence to statistical mechanics models. In this article, we present, via an analysis of the Bethe equations, all homogeneous, rational and trigonometric, integrable $\mathfrak{gl}(n|m)$ spin chains for which length-changing supersymmetry can be present. Furthermore, we write down the supercharges explicitly for the simplest new models, namely the $\mathfrak{sl}(n|1)$ spin chains with the $(n-1)$-fold antisymmetric tensor product of the fundamental representation at each site and check their compatibility with integrability.

Contents

1 Introduction

Since the seminal work of [1], integrable spin chains have become increasingly important in the field of gauge theory and, in particular, have been instrumental for the proposed solution of the spectral problem of $\mathcal{N} = 4$ super Yang-Mills, see [2] and reference therein. Nevertheless, considerable conceptual and technical difficulties remain and the subject is far from closed. The spin chains arising from the perturbative treatment of supersymmetric gauge theories have properties that seem quite unusual from the point of view of the "classical" models like the Heisenberg chain, such as long-range interactions that, starting at higher loop order, wrap around the spin chain, as well as length-changing or "dynamic" symmetry generators. It is this latter property that will be the focal point of this article.

So far, the issue of length-changing symmetry generators has been approached in essentially two ways. First, one can attempt to construct the symmetry algebra perturbatively as in [3–5]. The second option involves considering the chain in the infinite length limit. In the setup of the coordinate Bethe Ansatz, the length-changing operators then simply acquire a momentum dependence and act with a non-trivial braiding which allows for an exact solution for the relevant S-matrix, [6–8]. This asymptotic solution [9] can then be used as an input of the thermodynamic Bethe Ansatz which provides a way to compute the finite size spectrum, see for instance [10–12]. Unfortunately, this method does not provide an insight into the way the length-changing symmetry is realized at finite length.

Length-changing symmetries are not unique to gauge theories and their related spin chains. Similar structures have also been discovered in statistical mechanics and condensed matter theory. Investigating lattice fermion models with exclusion rules that are constructed to possess supersymmetry [13–15], it was found that these models map to integrable spin chains for which the supersymmetry generators become length-changing. These results were later

generalized to elliptic and higher spin models [16, 17], in a fashion that made some of the underlying general structure more transparent. A closer look shows that these models are similar to the ones arising in some subsectors[1] of the gauge theories at one-loop, where the manifest spin chain symmetry algebra $\mathfrak{g}$ commutes with the $\mathfrak{gl}(1|1)$ algebra formed out of the Hamiltonian H, the two length-changing supercharges $\mathfrak{Q}$, $\mathfrak{Q}^\dagger$ and the length-measuring operator. A question thus presents itself: what are all the integrable models that posses such a structure? The reasons for investigating this problem are twofold. First, since all supercommutators of the charges close into $\mathfrak{g} \oplus \mathfrak{gl}(1|1)$, these models have an exact symmetry algebra and not one that is only known perturbatively up to some order in a coupling constant. Thus, they constitute nice toy models for an analysis of length-changing symmetries in spin chains. Second, the relation between integrability and the length-changing symmetry remains obscure. They can occur independently from each other and it is unclear what deeper meaning their simultaneous appearance in a given model has. A comprehensive analysis of all integrable models with length-changing supersymmetry could shed some light on the connection. To find promising candidates, we analyze the Bethe equations carefully and provide a set of necessary conditions for the presence of length-changing supersymmetry such that its algebra has the form $\mathfrak{g} \oplus \mathfrak{gl}(1|1)$.

The present article is structured as follows. We first introduce the reader in section 2 to the basic elements of length-changing or "dynamic" supersymmetry in spin chains and review the main elements of [16, 17], adapted to graded representations. In section 3, we analyze the Bethe equations for the rational and trigonometric $\mathfrak{gl}(n|m)$ spin chains and present the cases in which we can expect dynamic supersymmetry to be present. The simplest novel candidates, involving integrable $\mathfrak{sl}(n|1)$ spin chains, are then considered separately and the corresponding supercharges written explicitly in section 4. Finally, we devote section 5 to discussions and conclusions.

2   Dynamic supersymmetry

In this section, we want to present the main elements of dynamic supersymmetry in spin chains. Mainly, we shall review here the results of [16, 17], adapted to the language of superspaces. We refer the readers to the original publications for proofs. We start with a super vector space V and define the spin chain Hilbert space as

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_L, \qquad \mathcal{H}_L := V \otimes V \otimes \cdots \otimes V. \tag{1}$$

We have the shift operator $\mathbb{S}_L : \mathcal{H}_L \to \mathcal{H}_L$ acting in the usual way on homogeneous elements

$$\mathbb{S}_L v_1 \otimes \cdots \otimes v_L = (-1)^{|v_L| \sum_{i=1}^{L-1} |v_i|} v_L \otimes v_1 \otimes \cdots \otimes v_{L-1}, \tag{2}$$

---

[1]Examples include the non-compact fermionic $\mathfrak{sl}(2)$ [4], the $\mathfrak{psu}(1, 1|2)$ subsectors of $\mathcal{N} = 4$ SYM [18] and the $\mathfrak{osp}(4|2)$ subsector of ABJM [19].

where $|v| = 0$ for even and 1 for odd elements. We shall make use of the grading generator G whose action on homogeneous elements is $Gv = (-1)^{|v|}v$. Note that we shall consistently use the usual tensor product and not the graded one and take care of the signs by inserting the grading operator G explicitly.

Generalizing [17], we allow the length-changing symmetry operator $\mathfrak{Q} = \sum_{L=1}^{\infty} \mathfrak{Q}_L$ with $\mathfrak{Q}_L : \mathcal{H}_L \to \mathcal{H}_{L+1}$, to be either even or odd with respect to the grading of V. To treat both cases jointly, we set

$$\omega := (-1)^{|\mathfrak{Q}|+1}, \tag{3}$$

and use it to define the projection operators Π_L as

$$\Pi_L := \frac{1}{L} \sum_{j=0}^{L-1} \omega^{j(L+1)} \mathbb{S}_L^j. \tag{4}$$

The length-changing operator acts non-trivially only on the subsector $\bigoplus_{L=1}^{\infty} \Pi_L \mathcal{H}_L$ of the Hilbert space, made out of the subspaces that have shift eigenvalues ω^{L+1}. Thus, if $\mathfrak{Q}$ is bosonic, it acts on the subspaces that are alternate-cyclic, i.e. with shift eigenvalue $(-1)^{L+1}$, while if it is fermionic it acts non-trivially only on the cyclic ones. While changing the grading of V sends one case to the other, we shall consider both here for completeness.

The full length-changing operator $\mathfrak{Q}$ is made out of a sum of local ones. Given such a local length-changing operator $\mathfrak{q} : V \to V \otimes V$ with $|\mathfrak{q}| = |\mathfrak{Q}|$ that satisfies the nilpotency condition

$$\left((\mathfrak{q} \otimes \mathbb{1}) + \omega (G^{|\mathfrak{q}|} \otimes \mathfrak{q}) \right) \mathfrak{q}v = \chi \otimes v - v \otimes \chi, \qquad \forall v \in V \tag{5}$$

for some fixed $\chi \in V \otimes V$, we define the operators $\mathfrak{q}_i$ as follows:

$$\mathfrak{q}_{i+1} = \omega \mathbb{S}_{L+1} \mathfrak{q}_i \mathbb{S}_L^{-1}, \qquad \mathfrak{q}_0 := \omega \mathbb{S}_{L+1}^{-1} \mathfrak{q}_1 \mathbb{S}_L = \omega^L \mathbb{S}_{L+1} \mathfrak{q}_L, \tag{6}$$

where $\mathfrak{q}_1$ is just $\mathfrak{q} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. The global length-changing supersymmetry generator $\mathfrak{Q}_L : \mathcal{H}_L \to \mathcal{H}_{L+1}$ can then be expressed in three equivalent ways

$$\mathfrak{Q}_L := \sqrt{\frac{L+1}{L}} \Pi_{L+1} \sum_{i=1}^{L} \mathfrak{q}_i = \sqrt{\frac{L}{L+1}} \sum_{i=0}^{L} \mathfrak{q}_i \Pi_L = \sqrt{L(L+1)} \Pi_{L+1} \mathfrak{q}_k \Pi_L, \tag{7}$$

where $k \in \{1, \ldots, L\}$ is arbitrary. Thanks to the local condition (5), one can show that $\mathfrak{Q}_L$ is "nilpotent" in the sense that

$$\mathfrak{Q}_{L+1} \mathfrak{Q}_L = 0. \tag{8}$$

In most cases of interest, the space V is a representation of some superalgebra $\mathfrak{g}$. One can show via a straightforward computation that $\mathfrak{Q}_L$ commutes with the action of $\mathfrak{g}$ on $\mathcal{H}$, i.e. that

$$\Delta^{(L)}(X)\mathfrak{Q}_L - (-1)^{|\mathfrak{Q}||X|} \mathfrak{Q}_L \Delta^{(L-1)}(X) = 0, \qquad \forall X \in \mathfrak{g} \tag{9}$$

if $\mathfrak{q}$ commutes with $\mathfrak{g}$ up to a gauge transformation that vanishes on the restricted Hilbert space, i.e. we must have

$$\left(\mathfrak{q}X - (-1)^{|X||\mathfrak{q}|} \Delta(X) \mathfrak{q} \right)v = m_X \otimes v - \omega(-1)^{|v||m_X|} v \otimes m_X, \qquad \forall v \in V, \tag{10}$$

3

where m_X is a X-dependent element of V with $|m_X| = |X| + |\mathfrak{q}|$ and we have used the standard superalgebra coproduct $\Delta(X) = \Delta^{(1)}(X) = X \otimes \mathbb{1} + \mathrm{G}^{|X|} \otimes X$. Thus, the right hand side of (10) is graded symmetric for $\mathfrak{q}$ even and graded antisymmetric for $\mathfrak{q}$ odd.

In order to obtain the Hamiltonian, we also need a length-lowering operator. In all cases that we shall consider here, this operator is given by the superadjoint of $\mathfrak{Q}$. This means that we introduce a scalar product on V, extend it to $V \otimes V$ and define the superadjoint of a general operator $\mathcal{O} : V \to V \otimes V$ as

$$\left\langle \mathcal{O}^\dagger u, v \right\rangle_V := (-1)^{|\mathcal{O}||u|} \left\langle u, \mathcal{O}v \right\rangle_{V \otimes V}, \qquad \forall v \in V,\ u \in V \otimes V. \tag{11}$$

In this way, we obtain $\mathfrak{q}^\dagger : V \otimes V \to V$ and by conjugating (7) an expression for $\mathfrak{Q}_L^\dagger$, which is also nilpotent, $\mathfrak{Q}_{L-1}^\dagger \mathfrak{Q}_L^\dagger = 0$. We can then define a Hamiltonian density as

$$\mathrm{h}(\mathfrak{q}) = \omega\left(\mathfrak{q}^\dagger \otimes \mathbb{1}\right)\left(\mathrm{G}^{|\mathfrak{q}|} \otimes \mathfrak{q}\right) + \omega\left(\mathrm{G}^{|\mathfrak{q}|} \otimes \mathfrak{q}^\dagger\right)\left(\mathfrak{q} \otimes \mathbb{1}\right) + \mathfrak{q}\mathfrak{q}^\dagger + \frac{1}{2}\left(\mathfrak{q}^\dagger \mathfrak{q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{q}^\dagger \mathfrak{q}\right) \tag{12}$$

which gives us a nearest neighbor Hamiltonian operator for the whole chain

$$\mathrm{H}_L = \sum_{i=1}^{L} \mathrm{h}(\mathfrak{q})_{i,i+1}. \tag{13}$$

In the sectors of appropriate cyclicity, this Hamiltonian operator can be expressed as

$$\Pi_L \mathrm{H}_L \Pi_L = \mathfrak{Q}_L^\dagger \mathfrak{Q}_L + \mathfrak{Q}_{L-1} \mathfrak{Q}_{L-1}^\dagger, \tag{14}$$

which is the reason behind the existence of degeneracies at different lengths, since it implies that

$$\mathrm{H}_{L+1} \mathfrak{Q}_L = \mathfrak{Q}_L \mathrm{H}_L, \qquad \mathrm{H}_L \mathfrak{Q}_L^\dagger = \mathfrak{Q}_L^\dagger \mathrm{H}_{L+1}. \tag{15}$$

If we denote by Pr_L the projectors $\mathcal{H} \to \mathcal{H}_L$ and define

$$\mathfrak{H} := \sum_{L=1}^{\infty} \mathrm{H}_L \mathrm{Pr}_L, \quad \mathfrak{Q} := \sum_{L=1}^{\infty} \mathfrak{Q}_L \mathrm{Pr}_L, \qquad \mathfrak{L} := \sum_{L=1}^{\infty} L \mathrm{Pr}_L, \tag{16}$$

then we see that $\mathrm{span}\{\mathfrak{H}, \mathfrak{Q}, \mathfrak{Q}^\dagger, \mathfrak{L}\}$ forms a $\mathfrak{gl}(1|1)$ algebra that commutes with the action of $\mathfrak{g}$ on $\mathcal{H}$, so that the full symmetry of the model is $\mathfrak{g} \oplus \mathfrak{gl}(1|1)$.

Of course, for an arbitrary local nilpotent charge $\mathfrak{q}$, the Hamiltonian defined in (13) will not be integrable. In order to check the integrability of H_L, we need to find an appropriate R matrix, i.e. an operator $\mathrm{R}(u) : V \otimes V \to V \otimes V$ that reduces to the permutation operator $\mathbb{P}$ at some regularity point that we take to be $u = 0$ and that obeys the Yang-Baxter equation:

$$\mathrm{R}_{12}(u)\mathrm{R}_{13}(u+v)\mathrm{R}_{23}(v) = \mathrm{R}_{23}(v)\mathrm{R}_{13}(u+v)\mathrm{R}_{12}(u) \qquad \forall u, v \in \mathbb{C}. \tag{17}$$

Then, using an auxiliary space at site zero of the spin chain, one constructs a monodromy matrix and a transfer matrix as

$$\mathrm{M}_L(u) := \mathrm{R}_{0L}(u) \cdots \mathrm{R}_{01}(u), \qquad \mathrm{T}_L(u) := \mathrm{str}_0 \mathrm{M}_L(u). \tag{18}$$

The transfer matrix reduces to the shift operator at the regularity point $u = 0$, i.e. $\mathrm{T}_L(0) = \mathbb{S}_L$. The nearest neighbor Hamiltonian operator is then uniquely defined, up to some arbitrary complex length-depending constants[2] α_L and β_L, as the logarithmic derivative of the transfer matrix at $u = 0$ i.e.

$$\mathrm{H}_L = \alpha_L \frac{d}{du} \log \mathrm{T}_L(u)|_{u=0} + \beta_L \mathbb{1}. \tag{19}$$

If an R matrix exists such that (19) holds, we can write the Hamiltonian density of (13), up to two arbitrary complex constants, as the logarithmic derivative of the R matrix itself. Thus, in order to make sure that the Hamiltonian density of (12) leads to an integrable problem, we need to find an R matrix satisfying the Yang-Baxter equation (17) such that its logarithmic derivative at the regularity point gives $\mathrm{h}(\mathfrak{q})$.

We consider here the reverse problem, i.e. we look for integrable Hamiltonians for which a charge $\mathfrak{q}$ exists such that (14) holds. If such a charge exists for an integrable spin chains, then we expect that its effects should manifest themselves at the level of the Bethe equations in the form of degeneracies of energy for different lengths. Such degeneracies have been observed in numerous examples in AdS/CFT and condensed matter theory and the next section is devoted to classifying them in the context of $\mathfrak{gl}(n|m)$ integrable spin chains.

3   Analysis of the Bethe equations

In this section, we shall classify all periodic and homogeneous rational and trigonometric $\mathfrak{gl}(n|m)$ spin chains for which dynamic supersymmetry in the sense of section 2 is expected to be present by analyzing their Bethe equations. We begin by writing down the equations for arbitrary representations of $\mathfrak{gl}(n|m)$ and analyze extensively some low rank examples that already contain all the subtleties of the problem. Subsection 3.3 then presents the general result.

3.1   Bethe equations for $\mathfrak{gl}(n|m)$ spin chains

The initial ingredient is an R matrix, which in our case has either a $\mathcal{Y}(\mathfrak{gl}(n|m))$ or a $U_q(\widehat{\mathfrak{gl}}(n|m))$ symmetry, corresponding to rational or trigonometric $\mathfrak{gl}(n|m)$ spin chains respectively. This R matrix differs from the one of section 2, since it intertwines between the fundamental representation of the corresponding quantum group and the representation V from which the spin chain is built. One constructs the monodromy M and transfer matrix T as in (18), with the fundamental representation placed in the auxiliary space and diagonalizes the transfer matrix T using the Nested Bethe Ansatz [20]. One can of course not obtain the Hamiltonian operator directly in this way, since the T thus constructed does not reduce to the shift operator for any value of the spectral parameter, i.e. there is no regularity point. One can however construct the eigenvalues for the transfer matrices stemming from different auxiliary spaces

---

[2]These constants can be absorbed in the normalization of R.

from the fundamental one via fusion, so that degeneracies of the eigenvalues of T imply degeneracies in the eigenvalues of the Hamilton operator.

The Bethe equations we will investigate can be found in [20–24] and in a compact notation read

$$\frac{\Lambda_{k+1}(u_j^{(k)} + \frac{c_k}{2})}{\Lambda_k(u_j^{(k)} + \frac{c_k}{2})} = \gamma_k \frac{\mathcal{Q}_{k-1}(u_j^{(k)} - \frac{(-1)^{|k|}}{2})}{\mathcal{Q}_{k-1}(u_j^{(k)} + \frac{(-1)^{|k|}}{2})} \frac{\mathcal{Q}_k(u_j^{(k)} + (-1)^{|k|})}{\mathcal{Q}_k(u_j^{(k)} - (-1)^{|k+1|})} \frac{Q_{k+1}(u_j^{(k)} - \frac{(-1)^{|k+1|}}{2})}{\mathcal{Q}_{k+1}(u_j^{(k)} + \frac{(-1)^{|k+1|}}{2})}, \quad (20)$$

for $j = 1, \cdots, M_k$ and $k = 1, \cdots, n + m - 1$. We remind that the superscript (k) refers to the level of the nesting and the subscript j labels the distinct root of a single level, running from 1 to the total number M_k. We use the abbreviations

$$\gamma_k := (-1)^{|k|+|k+1|+1}, \qquad c_k := \sum_{i=1}^{k} (-1)^{|i|} \quad (21)$$

and choose our conventions such that the constant factors in the arguments of the $\mathcal{Q}$ functions are real. A given set of Bethe roots obeying these equations uniquely determine an eigenvector of the transfer matrix with eigenvalue

$$\Lambda(u) = \frac{1}{N^L} \sum_{k=1}^{n+m} (-1)^{|k|} \Lambda_k(u) \frac{\mathcal{Q}_{k-1}(u - \frac{c_k}{2} - \frac{(-1)^{|k|}}{2})}{\mathcal{Q}_{k-1}(u - \frac{c_k}{2} + \frac{(-1)^{|k|}}{2})} \frac{\mathcal{Q}_k(u - \frac{c_k}{2} + (-1)^{|k|})}{\mathcal{Q}_k(u - \frac{c_k}{2})}, \quad (22)$$

where we have introduced a normalizing factor N. We will parametrize the q-deformation, depending on the notational convenience, by either q, η or ζ, which are related by $q = e^\eta$ and $\eta = \frac{2\pi i}{\zeta}$. Defining the function

$$\mathfrak{s}(x) := \sinh(\eta x), \quad (23)$$

the $\mathcal{Q}$-functions and the weights are given by

$$\mathcal{Q}_i(u) = \prod_{j=1}^{M_i} (u - u_j^{(i)}), \qquad \Lambda_i(u) = \big(u - (-1)^{|i|} \lambda_i\big)^L \qquad \text{for } \mathcal{Y}(\mathfrak{gl}(n|m)),$$

$$\mathcal{Q}_i(u) = \prod_{j=1}^{M_i} \frac{\mathfrak{s}(u - u_j^{(i)})}{\mathfrak{s}(1)}, \qquad \Lambda_i(u) = \Big(\frac{\mathfrak{s}(u - (-1)^{|i|} \lambda_i)}{\mathfrak{s}(1)}\Big)^L \qquad \text{for } U_q(\widehat{\mathfrak{gl}}(n|m)). \quad (24)$$

Here, λ_i are the components of the $\mathfrak{gl}(n|m)$ highest weight of the representation V, i.e. they are the eigenvalues of the generators E_{ii} on the highest weight state of V. Finally, we introduce the normalized Dynkin labels

$$a_i := (-1)^{|i|} \lambda_i - (-1)^{|i+1|} \lambda_{i+1}, \quad (25)$$

which will be very useful for the subsequent discussion.

3.2 Low rank examples

Dynamic supersymmetry reveals itself by certain degeneracies in the Bethe equations. Before we present the general case, it is useful to first familiarize oneself with the general structure of these degeneracies by considering some simple low rank case, namely $\mathfrak{gl}(2)$, $\mathfrak{gl}(3)$ and $\mathfrak{gl}(2|1)$.

Models based on $\mathfrak{gl}(2)$

The Bethe equations[3] (20) for $\mathfrak{gl}(2)$ can be written as

$$\left(\frac{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_2)}{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_1)}\right)^L = -\prod_{i=1}^{M} \frac{\mathfrak{s}(u_j - u_i + 1)}{\mathfrak{s}(u_j - u_i - 1)}, \quad j = 1, \cdots, M. \tag{26}$$

Now imagine we have a set of M Bethe roots satisfying these equations for length L. We shall now try to obtain a solution for length $L - 1$, using the same set of roots plus an additional one u_*. Doing so, we assume that u_* is different[4] from the other Bethe roots, for otherwise, we would be constructing a zero state. The resulting $M + 1$ Bethe equations then read

$$\left(\frac{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_2)}{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_1)}\right)^L \frac{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_1)}{\mathfrak{s}(u_j + \frac{1}{2} - \lambda_2)} = -\prod_{i=1}^{M} \frac{\mathfrak{s}(u_j - u_i + 1)}{\mathfrak{s}(u_j - u_i - 1)} \frac{\mathfrak{s}(u_j - u_* + 1)}{\mathfrak{s}(u_j - u_* - 1)}, \tag{27}$$

for $j = 1, \cdots, M$ with one additional equation for u_*

$$\left(\frac{\mathfrak{s}(u_* + \frac{1}{2} - \lambda_2)}{\mathfrak{s}(u_* + \frac{1}{2} - \lambda_1)}\right)^{L-1} = \prod_{i=1}^{M} \frac{\mathfrak{s}(u_* - u_i + 1)}{\mathfrak{s}(u_* - u_i - 1)}. \tag{28}$$

We focus on the first M equations and use our knowledge that $u_1, \cdots, u_M$ satisfy the Bethe equations of length L to reduce them to two equations

$$\mathfrak{s}(u + \tfrac{1}{2} - \lambda_1) = \mathfrak{s}(u - u_* + 1), \qquad \mathfrak{s}(u + \tfrac{1}{2} - \lambda_2) = \mathfrak{s}(u - u_* - 1), \tag{29}$$

which should hold for any $u \in \mathbb{C}$, since we allow the Bethe roots u_j to be essentially arbitrary. Using the period ζ of the function $\mathfrak{s}$, we have

$$\frac{1}{2} - \lambda_1 = -u_* + 1 + m_1\zeta, \qquad \frac{1}{2} - \lambda_2 = -u_* - 1 + m_2\zeta, \tag{30}$$

where here and in the following all m_i are integers. Taking the sum and the difference of the last two equations we arrive at

$$m_3\zeta = a_1 + 2, \qquad u_* = \frac{1}{2}(\lambda_1 + \lambda_2 - 1 + m_4\zeta), \tag{31}$$

where the normalized Dynkin label was introduced in (25). Finally, the Bethe equation for the new root u_* simply reduces to the condition

$$\prod_{i=1}^{M} \frac{\mathfrak{s}(u_i + \frac{1}{2} - \lambda_2)}{\mathfrak{s}(u_i + \frac{1}{2} - \lambda_1)} = (-1)^{L-1} \tag{32}$$

which forces the states to lie in the alternate cyclic sectors, just as required for a bosonic length-changing operator, see (4). Moreover, since no two Bethe roots can be equal, a second

---

[3]Due to our normalizations, we can consider the trigonometric and rational case together, since no divergences appear for $\eta \to 0$.

[4]As noted in [25, 26], physical solutions with repeated roots appear in $\mathfrak{gl}(2)$ spin chains with $a_1 \geq 2$. The treatment of such cases requires the introduction of a regulator and exceeds the scope of this work.

insertion of u_* is not allowed. This is the manifestation in the Bethe equations of the nilpotency of the length-changing operators.

We will now show that the resulting doublets share the same transfer matrix eigenvalue, if the transfer matrix is properly normalized. For the case at hand the eigenvalue of the transfer matrix (22) is just

$$\Lambda(u) = \left(\frac{\mathfrak{s}(u - \lambda_1)}{\mathfrak{s}(1)N} \right)^L \prod_{i=1}^M \frac{\mathfrak{s}(u - u_i + \frac{1}{2})}{\mathfrak{s}(u - u_i - \frac{1}{2})} + \left(\frac{\mathfrak{s}(u - \lambda_2)}{\mathfrak{s}(1)N} \right)^L \prod_{i=1}^M \frac{\mathfrak{s}(u - u_i - \frac{3}{2})}{\mathfrak{s}(u - u_i - \frac{1}{2})}. \tag{33}$$

Going to $L-1$ and adding the extra root we find the eigenvalue

$$\Lambda_*(u) = \left(\frac{\mathfrak{s}(u - \lambda_1)}{\mathfrak{s}(1)N} \right)^L \prod_{i=1}^M \frac{\mathfrak{s}(u - u_i + \frac{1}{2})}{\mathfrak{s}(u - u_i - \frac{1}{2})} \left[\frac{\mathfrak{s}(u - u_* + \frac{1}{2})}{\mathfrak{s}(u - u_* - \frac{1}{2})} \frac{\mathfrak{s}(1)N}{\mathfrak{s}(u - \lambda_1)} \right]$$

$$+ \left(\frac{\mathfrak{s}(u - \lambda_2)}{\mathfrak{s}(1)N} \right)^L \prod_{i=1}^M \frac{\mathfrak{s}(u - u_i - \frac{3}{2})}{\mathfrak{s}(u - u_i - \frac{1}{2})} \left[\frac{\mathfrak{s}(u - u_* - \frac{3}{2})}{\mathfrak{s}(u - u_* - \frac{1}{2})} \frac{\mathfrak{s}(1)N}{\mathfrak{s}(u - \lambda_2)} \right]. \tag{34}$$

By (30), the normalization $N = \mathfrak{s}(1)^{-1}\mathfrak{s}(u - \lambda_2 + 1)$ makes the terms inside the square brackets of (34) equal to 1.

To recapitulate, we see that if the Dynkin label and the q deformation are related by $a_1 + 2 = m\zeta$, $m \in \mathbb{Z}$, then, given a solution to the Bethe equations at length L, we can construct a solution at length $L-1$ that shares the same eigenvalue of the transfer matrix by adding an extra root u_*. This procedure has two limitations. First, we cannot add u_* if it is already contained in the set of Bethe roots, in which case the solution that we started with must have descended from a solution at length $L+1$ that does not contain u_*. Thus, one can expect[5] that the symmetry is nilpotent and that the two states are in a doublet. Second, we are also prevented from adding u_* if it would bring us beyond the allowed number of roots, meaning that there are states that are singlets under the symmetry.

Furthermore, we need to be careful when solving the condition $a_1 + 2 = m\zeta$, because the points $q = \pm 1$ are singular for the algebra $U_q(\widehat{\mathfrak{gl}}(n|m))$, even though the Bethe equations are well-defined. Since $q = e^{2\pi i/\zeta}$, we need to exclude $\zeta = 1/m'$ and $\zeta = 2/m'$, $m' \in \mathbb{Z} \setminus \{0\}$. Thus, for the q-deformed spin chains, we exclude the Dynkin labels that obey $a_1 + 2 = \pm 1, \pm 2$. In the rational limit, the Bethe equations do not contain periodic functions, but the same conditions for the presence of supersymmetry hold as before with the restriction that all the integers m_i are set to zero, i.e. the representations for which supersymmetry is present in the rational limit are also those for which it is present for any value of the q-deformation. Therefore, we find only one possible representation for which length-changing supersymmetry is present in the rational limit, namely the non-compact one with $a_1 = -2$. The supercharges in this case were constructed in [4].

Summarizing, for the trigonometric models, we find the following representations for which length-changing supersymmetry is expected to be present

---

[5]Due to the possible presence of repeated roots, we cannot prove that the symmetry implied by the degeneracy should be nilpotent. Since the $\mathfrak{Q}$ charges that we want to construct are nilpotent by definition, the appearance of degeneracies in the Bethe equations is a necessary but not a sufficient condition.

Spin $a_1/2$	0	$1/2$	1	$3/2$	2	$5/2$	3	$7/2$	$\cdots$
$L \to L - 1$	—	$q^3 = 1$	$q^4 = 1$	$q^5 = 1$	$q^6 = 1$	$q^7 = 1$	$q^8 = 1$	$q^9 = 1$	$\cdots$
$L \to L + 1$	—	—	all q	—	—	$q^3 = 1$	$q^4 = 1$	$q^5 = 1$	$\cdots$

Table 1: Values of the deformation parameter q for which higher spin XXZ models exhibit dynamic supersymmetry.

- $a_1 + 2$ is a generic complex number, corresponding to an infinite dimensional representation of $\mathfrak{gl}(2)$ with $q = e^{\frac{2\pi i m}{a_1 + 2}}$, $m \in \mathbb{Z} \backslash \{0\}$.

- $a_1 + 2$ is an integer, which now implies that q has to be some root of unity satisfying $q^{a_1 + 2} = 1$. This includes all finite dimensional representations, except the trivial one, since $q^2 \neq 1$.

So far, we have considered transformations decreasing the length while adding a root. A peculiarity of the rank one case is the existence of transformations sending $L \to L + 1$. Since the derivation of the corresponding constraint equations is similar, we will just state the results: The constraints are

$$m_1 \zeta = a_1 - 2, \qquad u_* = \frac{1}{2}(\lambda_1 + \lambda_2 - 1 + m_2 \zeta). \tag{35}$$

Furthermore, the normalization of the transfer matrix (33) which makes the two solutions of the Bethe equations have the same eigenvalue is

$$N = \frac{\mathfrak{s}(u - \lambda_2 - 2)\, \mathfrak{s}(u - \lambda_2)}{\mathfrak{s}(1)\mathfrak{s}(u - \lambda_2 - 1)}. \tag{36}$$

In particular the $a_1 = 2$, or spin 1, representation exhibits dynamic supersymmetry in the rational limit and its supercharges can be found in [17]. We also find solutions in the q-deformed case starting from $a_1 = 5$ with $q^{a_1 - 2} = 1$. A summary of all these values can be found in table 1.

Models based on $\mathfrak{gl}(3)$

We now turn to the first algebra with rank greater than one, for which the Bethe equations are nested. For $\mathfrak{gl}(3)$ they read

$$\left(\frac{\mathfrak{s}(u_j^{(1)} + \frac{1}{2} - \lambda_2)}{\mathfrak{s}(u_j^{(1)} + \frac{1}{2} - \lambda_1)} \right)^L = -\prod_{i=1}^{M_1} \frac{\mathfrak{s}(u_j^{(1)} - u_i^{(1)} + 1)}{\mathfrak{s}(u_j^{(1)} - u_i^{(1)} - 1)} \prod_{i=1}^{M_2} \frac{\mathfrak{s}(u_j^{(1)} - u_i^{(2)} - \frac{1}{2})}{\mathfrak{s}(u_j^{(1)} - u_i^{(2)} + \frac{1}{2})}, \quad j = 1, \cdots, M_1$$

$$\left(\frac{\mathfrak{s}(u_j^{(2)} + 1 - \lambda_3)}{\mathfrak{s}(u_j^{(2)} + 1 - \lambda_2)} \right)^L = -\prod_{i=1}^{M_1} \frac{\mathfrak{s}(u_j^{(2)} - u_i^{(1)} - \frac{1}{2})}{\mathfrak{s}(u_j^{(2)} - u_i^{(1)} + \frac{1}{2})} \prod_{i=1}^{M_2} \frac{\mathfrak{s}(u_j^{(2)} - u_i^{(2)} + 1)}{\mathfrak{s}(u_j^{(2)} - u_i^{(2)} - 1)}, \quad j = 1, \cdots, M_2 \tag{37}$$

Our strategy remains the same, namely, we change the length and insert a new root u_*, which can now be of type 1 or 2. Let us denote the level at which the new root is inserted by ℓ. As one

can straightforwardly check, the equations one has to solve to find dynamic supersymmetry with $L \to L+1$ have no solution, so we will consider only $L \to L-1$ and discuss the two possibilities, $\ell = 1$ and $\ell = 2$, jointly.

Along the lines of the discussion of $\mathfrak{gl}(2)$, we have to solve the following system for the λ's and u_*, for arbitrary $u_j^{(k)}$'s:

$$
\begin{aligned}
&\underline{\ell = 1} &, \qquad &\underline{\ell = 2} \\
&\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_1) = \mathfrak{s}(u_j^{(1)} - u_* + 1), \qquad &\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_1) = \mathfrak{s}(u_j^{(1)} - u_* - \tfrac{1}{2}), \\
&\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_2) = \mathfrak{s}(u_j^{(1)} - u_* - 1), \qquad &\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_2) = \mathfrak{s}(u_j^{(1)} - u_* + \tfrac{1}{2}), \\
&\mathfrak{s}(u_j^{(2)} + 1 - \lambda_2) = \mathfrak{s}(u_j^{(2)} - u_* - \tfrac{1}{2}), \qquad &\mathfrak{s}(u_j^{(2)} + 1 - \lambda_2) = \mathfrak{s}(u_j^{(2)} - u_* + 1), \\
&\mathfrak{s}(u_j^{(2)} + 1 - \lambda_3) = \mathfrak{s}(u_j^{(2)} - u_* + \tfrac{1}{2}), \qquad &\mathfrak{s}(u_j^{(2)} + 1 - \lambda_3) = \mathfrak{s}(u_j^{(2)} - u_* - 1), \qquad (38)
\end{aligned}
$$

which can be rewritten in a form that separates the constraints on the weights from the required value of the extra root,

$$
\begin{aligned}
\underline{\ell = 1} \quad &a_1 + 2 = m_1 \zeta, \quad a_2 - 1 = m_2 \zeta, \quad u_* = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3 - \tfrac{3}{2} + m_3 \zeta), \\
\underline{\ell = 2} \quad &a_1 - 1 = m_1 \zeta, \quad a_2 + 2 = m_2 \zeta, \quad u_* = \tfrac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3 - 1 + m_3 \zeta). \qquad (39)
\end{aligned}
$$

At this point it is easy to see why the corresponding equations for $L \to L+1$ have no solutions. Note that in going from (38) to (39), the number of equations has been reduced. This was caused by the equality of the two equations involving λ_2. If we had chosen to send $L \to L+1$, this would not have been the case. Even worse, the two equations would have been $\lambda_2 - u_* = -\tfrac{1}{2} + m_1 \zeta$ and $\lambda_2 - u_* = \tfrac{1}{2} + m_1 \zeta$, which together imply that $q = 1$, a value that we need to exclude.

As in the rank one case, the right hand side of the Bethe equations for u_* becomes the shift eigenvalue of the original state if (38) is fulfilled. Since we are allowed to perform global shifts in the spectral parameter, the Bethe equations only depend on the Dynkin labels. This can be used to show that the shift eigenvalue of the original state has to be $(-1)^{L-1}$. Thus, the length-changing supersymmetry is still present only for the alternate-cyclic states, just as in the rank one case. Finally, the normalization of the transfer matrix that is necessary to make supersymmetry doublets have the same eigenvalue is

$$
N = \Lambda_3(u) \text{ for } \ell = 1, \qquad N = \Lambda_1(u) \text{ for } \ell = 2. \qquad (40)
$$

We can now discuss the different solutions to (39).

- If the Dynkin labels are generic complex numbers, there will be no value of ζ for which the first two equations of (39) can hold simultaneously.

- If the λ_i's are integers, we set $\zeta = \frac{A}{m}$ and recast (39) into the form

$$
\underline{\ell = 1} \quad \frac{A}{m} = \frac{a_1 + 2}{m_5} = \frac{a_2 - 1}{m_6}, \qquad \underline{\ell = 2} \quad \frac{A}{m} = \frac{a_1 - 1}{m_5} = \frac{a_2 + 2}{m_6}. \qquad (41)
$$

Here, A and m are in $\mathbb{Z}$. Stated in this form, one can see that the possible values of ζ are obtained if A is set to be the greatest common divisor of the two numerators:

$$
\begin{aligned}
\underline{\ell = 1} \qquad & A \equiv \gcd(a_1 + 2, a_2 - 1) \neq 1, 2\,, \\
\underline{\ell = 2} \qquad & A \equiv \gcd(a_1 - 1, a_2 + 2) \neq 1, 2\,,
\end{aligned}
\tag{42}
$$

where we imposed the constraint $q^2 \neq 1$. Thus, the possible values of q are given by the formula $q^A = 1$. There are many representations satisfying these constraints. As an example, one can take the two series $\lambda_1 \geq 2$, $\lambda_2 = 1$, and its dual $\lambda_1 \geq 2$, $\lambda_2 = \lambda_1 - 1$, with q given in each case by $q^{\lambda_1 + 1} = 1$. It should be stressed that the constraints have no solution for the fundamental representation.

- The rational solutions to (39), obtained by setting the m_i to zero, are infinite dimensional and have $\mathfrak{sl}(3)$ Dynkin labels $[a_1, a_2] = [-2, 1]$ for $\ell = 1$ and $[1, -2]$ for $\ell = 2$.

Models based on $\mathfrak{gl}(2|1)$

It is well known that Lie superalgebras admit different choices of Cartan matrices that lead to different gradations. In our experience, considering different gradations does not lead to novel models, so we will restrict the discussion to the distinguished gradation of $\mathfrak{gl}(2|1)$ with even indices 1 and 2 and an odd index 3. In this gradation, the Bethe equations read

$$
\begin{aligned}
\left(\frac{\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_2)}{\mathfrak{s}(u_j^{(1)} + \tfrac{1}{2} - \lambda_1)} \right)^L &= -\prod_{i=1}^{M_1} \frac{\mathfrak{s}(u_j^{(1)} - u_i^{(1)} + 1)}{\mathfrak{s}(u_j^{(1)} - u_i^{(1)} - 1)} \prod_{i=1}^{M_2} \frac{\mathfrak{s}(u_j^{(1)} - u_i^{(2)} - \tfrac{1}{2})}{\mathfrak{s}(u_j^{(1)} - u_i^{(2)} + \tfrac{1}{2})}, \quad j = 1, \cdots, M_1 \\
\left(\frac{\mathfrak{s}(u_j^{(2)} + 1 + \lambda_3)}{\mathfrak{s}(u_j^{(2)} + 1 - \lambda_2)} \right)^L &= \prod_{i=1}^{M_1} \frac{\mathfrak{s}(u_j^{(2)} - u_i^{(1)} - \tfrac{1}{2})}{\mathfrak{s}(u_j^{(2)} - u_i^{(1)} + \tfrac{1}{2})}, \quad j = 1, \cdots, M_2.
\end{aligned}
\tag{43}
$$

Insertion of the extra root at the "bosonic" level ($\ell = 1$) works very much as before. Again there is only the possibility to go from L to $L - 1$ and the constraint equations read

$$
a_1 + 2 = m_1 \zeta, \qquad a_2 - 1 = m_2 \zeta, \qquad u_* = \frac{1}{3}(\lambda_1 + \lambda_2 - \lambda_3 - \frac{3}{2} + m_3 \zeta).
\tag{44}
$$

In addition we find that the transfer matrix normalization is given by $N = \Lambda_3(u)$ and that the supersymmetry maps between states with shift eigenvalues $(-1)^{L-1}$.

If we want to insert the extra root at the "fermionic" level $\ell = 2$, there is a new subtlety. Due to the lack of self-scattering of magnons of type 2, the left hand side of the Bethe equations for this level has to become constant, meaning that it is not allowed to carry momentum. This gives us a first constraint:

$$
a_2 = m_1 \zeta.
\tag{45}
$$

The other constraints, arising from actual cancellations between momentum and scattering factors read

$$
a_1 \mp 1 = m_2 \zeta, \qquad u_* = \frac{1}{2}(\lambda_1 + \lambda_2 - 1 + m_3 \zeta),
\tag{46}
$$

for $L \to L \mp 1$. Thus, there are two important differences compared to the cases discussed so far. First, because there is only one cancellation involving each λ, both directions of the change of length are possible. Second, the Bethe equation for the extra root tells us that the shift eigenvalue of the original state is one, i.e. we are restricted to the cyclic sectors. The necessary normalization of the transfer matrix is $N = \Lambda_1(u)$. As for $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ it is easy to list the representations and values of the deformation parameter that satisfy the constraints (45) and (46). Restricting to representations with integer weights, one finds

- two rational solutions with Dynkin labels $[\pm 1, 0]$ for $L \to L \mp 1$. In section 4, we shall present the supercharges for the model with the fundamental representation explicitly.

- q-deformed solutions that have to satisfy $A = \gcd(a_1 \mp 1, a_2) \neq 1, 2$ for $L \to L \mp 1$. The deformation is then determined by $q^A = 1$.

3.3   Classification for arbitrary rank

After having gained some experience by considering low rank examples, we are now ready to tackle the general case. The aim of this section is to extend the classification of integrable models with dynamic supersymmetry to all rational and q-deformed models based on $\mathfrak{gl}(n|m)$ with arbitrary gradation and any representation.

The results obtained so far suggest a very nice strategy to achieve this: we will start by considering the slightly easier case of rational models, for which all cancellations have to hold directly, without exploiting any periodicities. This will yield a set of constraints $\{X_j(\lambda) = 0\}$. We can then immediately deduce the results for q-deformed chains: In principle any representation works, but we have to require[6] that

$$A = \gcd(\{X_j\}) \neq 1, 2. \tag{47}$$

If this is satisfied, q is forced to be an A'th root of unity, $q^A = 1$.

Before discussing the actual equations that need to be solved it is useful to examine some general requirements that have to be met.

General remarks

For higher rank algebras, the cancellations can only work if the level ℓ, at which the extra Bethe root is inserted, scatters with all momentum carrying levels and only with those. If it does not self-scatter, the level ℓ is also not allowed to carry momentum. For the Bethe equations under consideration, this implies that only the insertion level and its nearest neighboring ones are allowed to carry momentum. Also, for $\mathfrak{gl}(n|m)$, self-scattering is absent precisely if the level corresponds to a fermionic node in the Dynkin diagram of $\mathfrak{sl}(n|m)$, i.e. if $|\ell| \neq |\ell+1|$. Moreover, the momentum factors on the left hand side of level ℓ vanish exactly when $a_\ell = 0$.

---

[6]Note that in this scheme we are not allowed to rescale the constraints $X_j = 0 \to \mathrm{const} \times X_j = 0$.

Thus, without doing any calculations, we can restrict the Dynkin labels of representations with dynamic supersymmetry to be of the form

$$[a_1, \ldots, a_{\ell-1}, a_\ell, a_{\ell+1}, \ldots, a_{n+m-1}] = \begin{cases} [0, \ldots, 0, \#, \#, \#, 0, \ldots, 0] & \text{for } |\ell| = |\ell+1| \\ [0, \ldots, 0, \#, 0, \#, 0, \ldots, 0] & \text{for } |\ell| \neq |\ell+1| \end{cases}. \quad (48)$$

We can now proceed to determine the actual values of the non-zero Dynkin labels separately for the two cases.

Insertion at a bosonic level

To fix $a_{\ell-1}$, a_ℓ and $a_{\ell+1}$, we have to check all cancellations with the extra root, of which there are six, see (20). Just as in the $\mathfrak{gl}(3)$, we have to discard the possibility to change the length $L \to L+1$. For $L \to L-1$, the constraints in the rational case read

$$a_{\ell-1} = (-1)^{|\ell|}, \qquad a_\ell = -\big((-1)^{|\ell|} + (-1)^{|\ell+1|}\big), \qquad a_{\ell+1} = (-1)^{|\ell+1|}, \quad (49)$$

and determine the value of the additional Bethe root

$$u_* = \frac{1}{4}\left(\sum_{j=\ell-1}^{\ell+2} (-1)^{|j|}\lambda_j - c_\ell - c_{\ell+1} \right). \quad (50)$$

Now we can apply the argument given around equation (47). If we q-deform the Bethe equations, any Dynkin labels work, as long as they satisfy

$$A = \gcd\big[a_1, \ldots, a_{\ell-1}-(-1)^{|\ell|}, a_\ell+\big((-1)^{|\ell|}+(-1)^{|\ell+1|}\big), a_{\ell+1}-(-1)^{|\ell+1|}, \ldots, a_{n+m-1}\big] \neq 1, 2. \quad (51)$$

Then the valid deformation parameters are given by $q^A = 1$ and the extra Bethe root is given by $u_* + \frac{m}{4}\zeta$, $m \in \mathbb{Z}$, where u_* is the one of (50).

Regarding the momentum sectors the supersymmetry requires, it is an easy exercise to check that they are given by shift eigenvalues $(-1)^{L-1}$, as in the $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ cases discussed above, i.e. alternate cyclic. It is also not difficult to show that the transfer matrix eigenvalues (22) remain invariant under the addition of the extra root if they are normalized by $N = \Lambda_{\ell-1}(u)$. One merely has to apply the equalities in (49), modulo periodicity in the q-deformed case, to the eigenvalue with the extra root.

Insertion at a fermionic level

If the insertion level corresponds to a fermionic node in the Dynkin diagram we only need to determine $a_{\ell\pm1}$. Since $a_\ell = 0$, we see that there are no multiple cancellations involving λ_ℓ or $\lambda_{\ell+1}$ – thus both $L \to L-1$ as well as $L \to L+1$ are possible. One easily determines the constraints

$$a_{\ell-1} = \pm(-1)^{|\ell|}, \qquad a_{\ell+1} = \pm(-1)^{|\ell+1|}, \quad (52)$$

for $L \to L \mp 1$; the extra Bethe root is again given by (50) and is the same whether we send $L \to L-1$ or $L \to L+1$.

For the q-deformed case, we obtain that for $L \to L \mp 1$, the weights need to satisfy the constraints

$$A = \gcd\big[a_1, \ldots, a_{\ell-2}, a_{\ell-1} \mp (-1)^{|\ell|}, a_\ell, a_{\ell+1} \mp (-1)^{|\ell+1|}, a_{\ell+2}, \ldots, a_{n+m-1}\big] \neq 1, 2 \quad (53)$$

with $q^A = 1$, with the same u_* as in the bosonic case. Furthermore, invariance of the transfer matrix eigenvalues is possible and requires a normalization $N = \Lambda_{\ell-1}(u)$.

Special cases

For concreteness, we will now state the results for the most important cases, the bosonic algebras $\mathfrak{gl}(n)$ and for $\mathfrak{gl}(n|m)$ in the distinguished gradation explicitly.

For the bosonic Lie algebras $\mathfrak{gl}(n)$, we can only decrease the length while inserting the extra root. The constraints for insertion at level ℓ then read

$$[a_1, \ldots, a_{\ell-2}, a_{\ell-1}, a_\ell, a_{\ell+1}, a_{\ell+2}, \ldots, a_{n-1}] = [0, \ldots, 0, 1, -2, 1, 0, \ldots, 0]. \quad (54)$$

For the q-deformed case, simply use the formula (47). One should remember that the $\mathfrak{gl}(2)$ case is special and allows transformations $L \to L + 1$, see section 3.2. Note in particular that all representations for which dynamic supersymmetry works for the rational models are infinite dimensional, except the spin 1 representation of $\mathfrak{gl}(2)$, which is such a special case with $L \to L + 1$.

The distinguished gradation of $\mathfrak{gl}(n|m)$ is

$$|1| = \cdots = |n| = 0 \qquad \text{and} \qquad |n+1| = \cdots |n+m| = 1. \quad (55)$$

The Dynkin diagram related to this choice of grading has a single fermionic node at position n. Thus all bosonic insertion levels are covered by the discussion of $\mathfrak{gl}(n)$ above[7] and we just have to look at the special insertion $\ell = n$, for which the constraints read

$$[a_1, \ldots, a_{\ell-2}, a_{\ell-1}, a_\ell, a_{\ell+1}, a_{\ell+2}, \ldots, a_{n+m-1}] = [0, \ldots, 0, \mp 1, 0, \pm 1, 0, \ldots, 0] \text{ for } L \to L \pm 1. \quad (56)$$

Again one can apply (47) to obtain representations for which the supersymmetry is present in q-deformed models.

In our conventions, see [27], a representation of $\mathfrak{sl}(n|m)$ is finite dimensional if all a_j are integers, except for a_n which can be any complex number, and if $a_j \geq 0$ for $j = 1, \ldots, n-1$ and $a_j \leq 0$ for $j = n+1, \ldots, n+m-1$. Thus we find that representations which feature dynamic supersymmetry in their rational models with $L \to L+1$ are infinite dimensional. On the other hand, the $L \to L - 1$ cases are all finite dimensional. The nicest examples are the algebras $\mathfrak{sl}(n|1)$, for which the fermionic node is at the boundary of the Dynkin diagram. The representation for these Dynkin labels corresponds to a Young diagram consisting of $n-1$ antisymmetrized boxes. In particular, the $\mathfrak{sl}(2|1)$ spin chain features dynamic supersymmetry

---

[7]If one keeps in mind to replace λ_j for $j \geq n+1$ by $-\lambda_j$ – but this only affects the constraint for level n.

for the fundamental representation, as we have already seen in (3.2). In the next section we will construct the supercharges for these models.

As a final remark, we observe that, if we add the extra root at level ℓ while changing the length as $L \to L \pm 1$, the representations for which dynamic supersymmetry is expected to be present in the rational limit are exactly those whose highest weight is equal to $\pm \alpha_\ell$, where the α_i are the simple roots. Remember that increasing the length is not always possible. This observation may shed some light on the generalization of this procedure to other algebras.

4 The length-changing operators for $\mathfrak{sl}(n|1)$

In this section, we wish to construct explicitly the length-changing operators for one class of models, namely for the rational $\mathfrak{sl}(n|1)$ spin chains having the $(n-1)$-fold antisymmetric tensor product of the fundamental representation at each site. We begin by reviewing the $\mathfrak{sl}(2|1)$ case for which the solution was already given in [17] and which we generalize to the q-deformed case. From this special case, it is then straightforward to construct the local length-changing operators for all $\mathfrak{sl}(n|1)$ cases and compare the resulting Hamiltonian density with the one derived from integrability.

The methods presented in this section, namely obtaining the R matrices by fusion and comparing the resulting Hamiltonian density with the one obtained from the $\mathfrak{q}$ supercharges derived by requiring certain invariance conditions, can be expected to be generalizable to all $\mathfrak{gl}(n|m)$ spin chain examples found in section 3. We hope to return to this general problem in a future work.

4.1 The toy example

The simplest case involves the rational spin chain constructed using the fundamental representation of $\mathfrak{sl}(2|1)$ at each site. This spin chain is related to the t-J model and in [17] the local length-changing operator was given as

$$\mathfrak{q}e_i = \delta_{i3}\big(e_2 \otimes e_1 - e_1 \otimes e_2\big), \tag{57}$$

where e_i are the basis vectors of the fundamental representation, e_3 being fermionic. The only non-vanishing entries of the superadjoint are then $\mathfrak{q}^\dagger e_2 \otimes e_1 = -\mathfrak{q}^\dagger e_1 \otimes e_2 = e_3$. Plugging this in the formula for the Hamiltonian density (12) immediately leads to

$$\mathrm{h}(\mathfrak{q}) = \mathbb{1} - \mathbb{P}, \tag{58}$$

which can be written as $\mathbb{1} - \chi \partial_u \log(\mathrm{R}(u))|_u = 0$ using the normalized R matrix $\mathrm{R}(u) = \frac{1}{\chi}(u + \chi \mathbb{P})$. In the above χ is an arbitrary complex number that we keep for convenience. Thus we see that dynamic supersymmetry is present in this integrable spin chain.

Before moving on, we remark that the supercharge (57) can be easily generalized to the $\mathfrak{U}_q(\mathfrak{sl}(2|1))$ spin chain:

$$\mathfrak{q}e_i = \delta_{i3}\big(e_2 \otimes e_1 - qe_1 \otimes e_2\big), \tag{59}$$

for arbitrary non-vanishing q.

4.2 Representation theory

We now wish to summarize the main aspect of the representation theory of $\mathfrak{sl}(n|1)$ that we are going to make use of in the following.

Let V_F be the fundamental representation of $\mathfrak{g} := \mathfrak{sl}(n|1)$. Let $(e_i)_{i=1}^{n+1}$ be a basis of V_F, where the $\{e_i\}_{i=1}^{n}$ are bosonic and e_{n+1} is fermionic. The set of covariant $\mathfrak{g}$ representations can be classified by Young diagrams that lie in the "fat hook". We shall denote a Young diagram that has k_1 rows of length l_1, k_2 rows of length l_2 and so on, with $l_i > l_{i+1}$, by $l_1^{k_1} l_2^{k_2} \cdots$. We shall consider in particular the $n-1$ antisymmetric tensor product of the fundamental representation given by the Young diagram 1^{n-1} and we shall call this representation V in accordance with section 2.

In order to give explicit expressions in this representation, we need a good basis. The standard one is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_{n-1}} := \frac{1}{r!} \sum_{\sigma \in S_{n-1}} (-1)^{|\sigma|} \mathbb{P}_\sigma \left(e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \right). \tag{60}$$

We recall that the permutation $\mathbb{P}_\sigma$ will introduce additional signs when acting on fermions. For example, if $n = 4$, then e_5 is fermionic and $e_5 \wedge e_5 \wedge e_5 = e_5 \otimes e_5 \otimes e_5$. It is very useful to introduce another basis, that in a sense is obtained via a particle-hole transformation, as follows

$$|k_1, \ldots, k_r\rangle := \underbrace{e_1 \wedge \cdots \wedge \hat{e}_{k_1} \wedge \cdots \wedge \hat{e}_{k_2} \wedge \cdots \wedge \hat{e}_{k_r} \wedge \cdots \wedge e_n}_{n-r \text{ vectors}} \wedge \underbrace{e_{n+1} \wedge e_{n+1} \wedge \cdots \wedge e_{n+1}}_{r-1 \text{ vectors}},$$
$$\tag{61}$$

where $1 \le k_1 < k_2 \cdots < k_r \le n$ and $\hat{\ }$ indicates omission. It is obvious that there are exactly $\binom{n}{r}$ elements of the type $|k_1, \ldots, k_r\rangle$ and that $r = 1, \ldots, n$, which tells us that the dimension of V is equal to $2^n - 1$. It is furthermore clear that $|k_1, \ldots, k_r\rangle$ has the fermionic degree $r - 1$ mod 2 . The simple root raising and lowering operators act as

$$\begin{aligned}
E_{k_i,k_i+1} |k_1, \ldots, k_r\rangle &= \left(1 - \delta_{k_i+1,k_{i+1}}\right) |k_1, \ldots, \hat{k}_i, k_i + 1 \ldots, k_r\rangle, \\
E_{n,n+1} |k_1, \ldots, k_r\rangle &= \delta_{k_r,n}(r-1) |k_1, \ldots, k_{r-1}\rangle, \\
E_{k_i,k_i-1} |k_1, \ldots, k_r\rangle &= \left(1 - \delta_{k_i-1,k_{i-1}}\right) |k_1, \ldots, k_i - 1, \hat{k}_i, \ldots, k_r\rangle, \\
E_{n+1,n} |k_1, \ldots, k_r\rangle &= \left(1 - \delta_{k_r,n}\right) |k_1, \ldots, k_r, n\rangle.
\end{aligned} \tag{62}$$

In order to write the R matrix and the Hamiltonian density, we shall need the tensor product decomposition

$$V \otimes V = 1^{n-1} \otimes 1^{n-1} \cong \bigoplus_{i=0}^{n-1} 2^{n-1-i} 1^{2i}. \tag{63}$$

We now want to find the corresponding highest weight states. Using the basis of (61), we claim that the highest weight vectors v_i for the representation $2^{n-1-i} 1^{2i}$ are given by $v_0 := |n\rangle \otimes |n\rangle$

and by

$$v_i := \sum_{m=1}^{i} (-1)^{i(m-1)} \binom{i-1}{m-1} \sum_{\sigma \in S_{m,i+1-m}} (-1)^{|\sigma|} |n-i-1+\sigma(1), \ldots, n-i-1+\sigma(m)\rangle$$
$$\otimes |n-i-1+\sigma(m+1), \ldots, n-i-1+\sigma(i+1)\rangle, \quad (64)$$

for $i = 1, \ldots, n-1$. In the above, we have made use of the following group quotient

$$S_{i,j} := S_{i+j}/(S_i \times S_j) \quad (65)$$

in order to have the permutations keep the proper ordering. Note that in (64) $\sigma(j)$ is taken to be an element of $\{1, \ldots, i+1\}$ for all j.

We finish this subsection by considering the quadratic Casimir. We recall that given a basis T_a of a Lie superalgebra with a scalar product $\kappa_{ab} = (T_a, T_b)$, the quadratic Casimir $\mathfrak{C}_2$ is given by $\mathfrak{C}_2 = \kappa^{ab} T_a T_b$. A straightforward computation leads to the following general formula for the value of the properly normalized $\mathfrak{g}$ quadratic Casimir in a general representation of highest weight $\lambda = \sum_{i=1}^{n+1} \lambda_i \epsilon_i$:

$$\mathfrak{C}_2 = -\frac{1}{2} \Big[\sum_{k=1}^{n} \Big(\lambda_k - \frac{1}{n-1} \sum_{l=1}^{n+1} \lambda_l \Big) \Big(\lambda_k + n - 2k - \frac{1}{n-1} \sum_{l=1}^{n+1} \lambda_l \Big)$$
$$- \Big(\lambda_{n+1} + \frac{1}{n-1} \sum_{l=1}^{n+1} \lambda_l \Big) \Big(\lambda_{n+1} + n + \frac{1}{n-1} \sum_{l=1}^{n+1} \lambda_l \Big) \Big]. \quad (66)$$

Thus, in the representation $2^{n-1-i}1^{2i}$, the quadratic Casimir gives the value $i(i+1)$. If we define the operator $\mathbb{J}$ via $\mathfrak{C}_2 = \mathbb{J}(\mathbb{J}+1)$, then we see that on the tensor product $1^{n-1} \otimes 1^{n-1}$, this new operator can be written as

$$\mathbb{J} = \sum_{i=0}^{n-1} i \mathrm{P}^{2^{n-1-i}1^{2i}}. \quad (67)$$

4.3 The R matrix and the Hamiltonian

In this subsection, we shall construct the R matrix of the integrable $\mathfrak{g}$ spin chain with the representation 1^{n-1} at each site using the fusion procedure. We let χ be a complex parameter and define the fundamental basic R matrix that intertwines between two copies of the fundamental representation of $\mathfrak{g}$ to be

$$\mathrm{R}(u) := \frac{u}{\chi} + \mathbb{P}, \quad (68)$$

where $\mathbb{P}$ is the graded permutation operator. We have the relations

$$\mathrm{R}(0) = \mathbb{P}, \qquad \mathrm{R}(\chi) = 2\mathrm{P}^{2^1}, \qquad \mathrm{R}(-\chi) = -2\mathrm{P}^{1^2}, \quad (69)$$

as well as the Yang-Baxter equation

$$\mathrm{R}_{12}(u)\mathrm{R}_{13}(u+v)\mathrm{R}_{23}(v) = \mathrm{R}_{23}(v)\mathrm{R}_{13}(u+v)\mathrm{R}_{12}(u). \quad (70)$$

17

Here, P^{2^1} is the projector on the graded symmetric tensor product, while P^{1^2} projects on the antisymmetric tensor product. It is possible to generalize (69) and to write the projector on the $n-1$ fold antisymmetric tensor product of the fundamental representation using the basic R matrix as

$$P^{1^{n-1}} = (-1)^{\frac{(n-1)(n-2)}{2}} \Big(\prod_{k=1}^{n-1} k! \Big)^{-1} \prod_{i<j=1}^{n-1} R_{ij}\big(\chi(i-j)\big). \tag{71}$$

Using the fusion procedure[8], we can write the non-normalized R matrix that intertwines between two 1^{n-1} representations as

$$
\begin{aligned}
\tilde{R}^{(n)}(u) \equiv \tilde{R}^{1^{n-1},1^{n-1}}(u) = &\Big[\prod_{i<j=1}^{n-1} R_{ij}\big(\chi(i-j)\big) \Big] \Big[\prod_{i<j=n}^{2(n-1)} R_{ij}\big(\chi(i-j)\big) \Big] \\
&\times R_{1,2n-2}\big(u+(n-2)\chi\big) \cdots R_{1,n}\big(u\big) \\
&\times R_{2,2n-2}\big(u+(n-3)\chi\big) \cdots R_{2,n}\big(u-\chi\big) \cdots \\
&\times R_{n-1,2n-2}\big(u\big) \cdots R_{n-1,n}\big(u-(n-2)\chi\big) \\
&\times \Big[\prod_{i<j=1}^{n-1} R_{ij}\big(\chi(i-j)\big) \Big] \Big[\prod_{i<j=n}^{2n-2} R_{ij}\big(\chi(i-j)\big) \Big]. \tag{72}
\end{aligned}
$$

It is straightforward, if tedious, to see using (70) that the above satisfies the Yang-Baxter equations. Normalizing the expression properly and decomposing it using the projectors of (63), we obtain

$$
\begin{aligned}
R^{(n)}(u) &= \frac{1}{(n-1)!\chi^{n-1}} \sum_{i=0}^{n-1} \prod_{l=1}^{i}(u-l\chi) \prod_{l=i+1}^{n-1}(u+l\chi) P^{2^{n-1-i}1^{2i}}. \\
&= \frac{\Gamma(n+\frac{u}{\chi})}{\Gamma(n)\Gamma(1-\frac{u}{\chi})}(-1)^{\mathbb{J}}\frac{\Gamma(1+\mathbb{J}-\frac{u}{\chi})}{\Gamma(1+\mathbb{J}+\frac{u}{\chi})}, \tag{73}
\end{aligned}
$$

where we have used (67). The last expression agrees exactly with formula[9] (1) of [30] for $\chi = -1$, up to a multiplicative normalization.

At the special value $u = 0$, the R matrix reduces to the graded permutation and thanks to equation (19), we find that the general integrable Hamiltonian density is

$$h = \sum_{i=0}^{n-1} \left[\frac{\alpha}{\chi} \Big(\sum_{m=i+1}^{n-1} \frac{1}{m} - \sum_{m=1}^{i} \frac{1}{m} \Big) + \beta \right] P^{2^{n-1-i}1^{2i}}, \tag{74}$$

for two arbitrary complex number α and β. For reasons that shall become apparent in the next section, we wish that h should give 0 when acting on 2^{n-1} and 2 when acting on $2^{n-2}1^2$. The first condition then implies that $\beta = -\frac{\alpha}{\chi}\sum_{m=1}^{n-1}\frac{1}{m}$, while the second gives $\alpha = -\chi$. Thus, we arrive at the final expression for the Hamiltonian density

$$h = 2\sum_{i=1}^{n-1} h(i) P^{2^{n-1-i}1^{2i}}, \tag{75}$$

---

[8]See [28] for a review.
[9]This R matrix was first derived in [29] for the $\mathfrak{gl}(2)$ case.

where $h(i) := \sum_{m=1}^{i} \frac{1}{m}$ are the harmonic numbers.

4.4 The local length-changing operator

We are now looking for an operator $\mathfrak{q} : V \to V \otimes V$ such that the associated Hamiltonian density (12) agrees with the one (75) derived from the R matrix. In order to preserve the $\mathfrak{g}$ symmetry, we require that $\mathfrak{q}$ commutes with the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For $\mathfrak{sl}(n|1)$, the charge of the vector e_{n+1} is equal to the sum of the charges of all the bosonic vectors $(e_i)_{i=1}^{n}$. Therefore, when going from one to two sites, we need always to remove one e_{n+1} and add all the others in some appropriate order. In the hole basis (61), this means that $\mathfrak{q}$ takes the vector $|k_1, \ldots, k_r\rangle$ and maps it to a linear combination of $|k_{\sigma(1)}, \ldots, k_{\sigma(m)}\rangle \otimes |k_{\sigma(m+1)}, \ldots, k_{\sigma(r)}\rangle$, where σ is some arbitrary permutation and m can run from 1 to $r-1$. Essentially, it just distributes the holes on two sites. In particular, this implies that $\mathfrak{q}(k) = 0$. We can now try to find the action of $\mathfrak{q}$ on the highest weight state that has two holes. Invariance under $\mathfrak{h}$ implies

$$\mathfrak{q} \, |n-1, n\rangle = \gamma_1 \, |n-1\rangle \otimes |n\rangle + \gamma_2 \, |n\rangle \otimes |n-1\rangle \tag{76}$$

for arbitrary complex γ_i. We shall at this point impose two additional requirements that are motivated from the study of the $n = 2$ system, namely that $\mathfrak{q}$ commutes with all bosonic generators of $\mathfrak{g}$ and that its anticommutator with the simple negative fermionic root operator is given by

$$\Big(\mathfrak{q} E_{n+1,n} + \Delta(E_{n+1,n})\mathfrak{q}\Big)v = v \otimes |n\rangle - |n\rangle \otimes v, \qquad \forall v \in V. \tag{77}$$

The first requirement sets $\gamma_2 = -\gamma_1$ and we normalize γ_1 to one. By using the condition (77) and by requiring that $\mathfrak{q}$ commutes with the other simple generators, we guess the following expression[10] for the action of $\mathfrak{q}$:

$$\mathfrak{q} \, |k_1, \ldots, k_r\rangle = \sum_{m=1}^{r-1} (-1)^{rm} \sum_{\sigma \in S_{m,r-m}} (-1)^{|\sigma|} |k_{\sigma(1)}, \ldots, k_{\sigma(m)}\rangle \otimes |k_{\sigma(m+1)}, \ldots, k_{\sigma(r)}\rangle. \tag{78}$$

We remind that due to (65), the $k_{\sigma(i)}$ are all arranged as in (61), i.e. such that $k_{\sigma(i)} < k_{\sigma(i+1)}$ for $i = 1, \ldots m-1$ and for $i = m+1, \ldots, r-1$.

It is trivial to check, using (62), that $\mathfrak{q}$ commutes with all bosonic generators. Furthermore, an explicit computation confirms that (77) does indeed hold and that furthermore $\mathfrak{q}$ anticommutes with the fermionic raising generators. Thus, we can safely say that $\mathfrak{Q}$ commutes in the graded sense with the full action of $\mathfrak{g}$. Furthermore, we have checked that $\mathfrak{q}$ obeys equation (5) with $\chi = 0$, thus guaranteeing that the supercharge is nilpotent.

We now wish to define the local length-lowering operator by considering the adjoint of $\mathfrak{q}$. For this purpose, we introduce the scalar product on V_F by $\langle e_i, e_j \rangle = \delta_{ij}$. This together with

---

[10]While (78) seems superficially similar to the expression (64) for the highest weight vectors, observe that the right-hand side of (78) is always graded antisymmetric, while the v_i are symmetric for i even and antisymmetric for i odd.

(11) implies that $E_{ij}^\dagger = (-1)^{|i|(|j|+1)} E_{ji}$. In order for this to be compatible with the coproduct, we require that the scalar product is generalized to tensor products of V_F as

$$\langle e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_r}\rangle = (-1)^{\sum_{k=0}^{r-1}|i_{r-k}|\sum_{l=1}^{r-1-k}|j_l|}\delta_{i_1 j_1}\cdots\delta_{i_r j_r}. \qquad (79)$$

For the orthogonal basis (61) of V, we get the following expressions for the square of the norm:

$$\langle |k_1, \ldots, k_r\rangle, |k_1, \ldots, k_r\rangle\rangle = (-1)^{\frac{(r-2)(r-1)}{2}}\frac{(r-1)!}{(n-1)!}. \qquad (80)$$

It follows after a direct computation using (11) that

$$\mathfrak{q}^\dagger |k_1, \ldots, k_m\rangle \otimes |k_{m+1}, \ldots, k_r\rangle =$$
$$= (-1)^{rm}\frac{(m-1)!(r-m-1)!}{(r-1)!(n-1)!}(-1)^{|\sigma|}|k_{\sigma(1)}, \ldots, k_{\sigma(r)}\rangle. \qquad (81)$$

where σ is a permutation that brings the k_j in the correct order.

We now have all the ingredients to write the Hamiltonian density for the length-changing operators. Since $\mathfrak{q}$ is fermionic for our grading, we use (12) with $\omega = 1$, normalized to:

$$\mathrm{h}(\mathfrak{q}) = (n-1)!\Big[\big(\mathfrak{q}^\dagger \otimes \mathbb{1}\big)(\mathrm{G} \otimes \mathfrak{q}) + (\mathrm{G} \otimes \mathfrak{q}^\dagger)\big(\mathfrak{q} \otimes \mathbb{1}\big) + \mathfrak{q}\mathfrak{q}^\dagger + \frac{1}{2}\big(\mathfrak{q}^\dagger\mathfrak{q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathfrak{q}^\dagger\mathfrak{q}\big)\Big]. \qquad (82)$$

In order to compare with (75), we need to compute the action of $\mathrm{h}(\mathfrak{q})$ on the highest weight vectors v_i of (64). The case of v_0 is simple since $\mathfrak{q}|k\rangle = 0$ and $\mathfrak{q}^\dagger|n\rangle \otimes |n\rangle = 0$, so that

$$\mathrm{h}(\mathfrak{q})v_0 = 0. \qquad (83)$$

For the highest weight vector v_1 we find

$$\mathrm{h}(\mathfrak{q})v_1 = \mathfrak{q}\mathfrak{q}^\dagger v_1 = 2\mathfrak{q}|n-1, n\rangle = 2v_1. \qquad (84)$$

For v_2, we first see that $\mathfrak{q}^\dagger v_2 = 0$ and after some algebra arrive at $\mathrm{h}(\mathfrak{q})v_2 = 3v_2$ as required. The action on v_3 is already quite complicated, but a careful computation gives an eigenvalue of $\frac{11}{3} = 2h(3)$ in full agreement with (75). With the help of a computer algebra program, we were able to check that (82) agrees with (75) when acting on the highest weight vectors v_i with $i \leq 8$, so that we are confident that the relation holds in general.

5 Conclusion

In this article we examined the Bethe equations of the rational and trigonometric $\mathfrak{gl}(n|m)$ spin chains for degeneracies and could classify the cases for which dynamic, i.e. length-changing, supersymmetry in the sense of section 2 could be expected to be present. We then took the simplest new example for which the Bethe equations hinted at the existence of such a supersymmetry, namely the $\mathfrak{sl}(n|1)$ spin chains made out of copies of the $(n-1)$-fold antisymmetric tensor product of the fundamental, and constructed the supercharges explicitly,

20

checking in the process that they indeed reproduce the integrable Hamiltonian. It would be interesting to find a general way of constructing the supercharges that is valid for all candidate representations.

The procedure of analyzing the Bethe equations and constructing the supercharges can be generalized in several ways. The most promising one would be to allow for different representations to be present at each site of the chain and to introduce inhomogeneities in the transfer matrix. Of course, for the concept of dynamic supersymmetry to still make sense, some notion of cyclicity has to remain, which means that the most promising approach is to restrict oneself to alternating spin chains. For those cases, the generalization of section 3 is indeed straightforward but so far, we have not been able to write the supercharges explicitly for any new model[11]. Often, the main obstacle is related to the fact that the cyclicity condition gets replaced by a more stringent condition which requires that the sectors of interest be the combined eigenspaces of eigenvalue one of the transfer matrices[12] T_L and $\bar{T}_L$ at some special values of the spectral parameter. Unfortunately, these operators at those special values of the spectral parameter are often non-unitary and non-diagonalizable, which makes the precise description of the supercharges quite complicated.

Another important avenue of future research is the precise nature of the relationship between integrability and dynamic supersymmetry. We expect from the analysis of the Bethe equations that the transfer matrix commutes with the supercharges and, for the specific examples for which the supercharges have been constructed, this has been checked at level of the operators for small spin chain lengths. So far, there is no proof that this should hold for all lengths and it is to be hoped that elucidating this issue would also give one a way of finding a set of conditions for the local supercharges $\mathfrak{q}$ so that the resulting Hamiltonian is integrable.

Finally, gauge theory suggests that we should also generalize to spin chains that are long-ranged and for which the Hamiltonian itself is length-changing. For this purpose the works of [5] and [31, 32] might provide some clue. Of course, if might also be possible to find a non-perturbative way to freeze the length changing via some complicated basis transformation generalizing [13, 14, 17]. We hope to be able to answer at least some of these questions in future works.

Acknowledgments

---

[11]See [19] for an example appearing in the context of the ABJM duality.

[12]We recall that for alternating spin chains one has to construct two transfer matrices, depending on the kind of representation that is inserted in the auxiliary space and that the true transfer matrix which reduces to the shift operator is given by the product of the two.

References

[1] J. Minahan and K. Zarembo, *The Bethe ansatz for N=4 superYang-Mills*, *JHEP* **0303** (2003) 013, [`hep-th/0212208`].

[2] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, *Review of AdS/CFT Integrability: An Overview*, *Lett.Math.Phys.* **99** (2012) 3–32, [`arXiv:1012.3982`].

[3] N. Beisert, *The su(2|3) dynamic spin chain*, *Nucl.Phys.* **B682** (2004) 487–520, [`hep-th/0310252`].

[4] N. Beisert, *The Dilatation operator of N=4 super Yang-Mills theory and integrability*, *Phys.Rept.* **405** (2004) 1–202, [`hep-th/0407277`].

[5] B. I. Zwiebel, *Iterative Structure of the N=4 SYM Spin Chain*, *JHEP* **0807** (2008) 114, [`arXiv:0806.1786`].

[6] N. Beisert, *The SU(2|2) dynamic S-matrix*, *Adv.Theor.Math.Phys.* **12** (2008) 945–979, [`hep-th/0511082`].

[7] N. Beisert, *The Analytic Bethe Ansatz for a Chain with Centrally Extended su(2|2) Symmetry*, *J.Stat.Mech.* **0701** (2007) P01017, [`nlin/0610017`].

[8] G. Arutyunov, S. Frolov, and M. Zamaklar, *The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring*, *JHEP* **0704** (2007) 002, [`hep-th/0612229`].

[9] N. Beisert and M. Staudacher, *Long-range psu(2, 2|4) Bethe Ansätze for gauge theory and strings*, *Nucl.Phys.* **B727** (2005) 1–62, [`hep-th/0504190`].

[10] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states*, *Lett.Math.Phys.* **91** (2010) 265–287, [`arXiv:0902.4458`].

[11] N. Gromov, V. Kazakov, and P. Vieira, *Exact Spectrum of Planar $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling*, *Phys.Rev.Lett.* **104** (2010) 211601, [`arXiv:0906.4240`].

[12] G. Arutyunov, S. Frolov, and R. Suzuki, *Five-loop Konishi from the Mirror TBA*, *JHEP* **1004** (2010) 069, [`arXiv:1002.1711`].

[13] P. Fendley, K. Schoutens, and J. de Boer, *Lattice models with N=2 supersymmetry*, *Phys.Rev.Lett.* **90** (2003) 120402, [`hep-th/0210161`].

[14] P. Fendley, K. Schoutens, and B. Nienhuis, *Lattice fermion models with supersymmetry*, *J.Phys.* **A36** (2003) 12399–12424, [`cond-mat/0307338`].

[15] X. Yang and P. Fendley, *Non-local space-time supersymmetry on the lattice*, *J. Phys. A* **37** (2004) 8937, [`cond-mat/0404682`].

[16] C. Hagendorf and P. Fendley, *The Eight-vertex model and lattice supersymmetry*, *J.Statist.Phys.* **146** (2012) 1122–1155, [`arXiv:1109.4090`].

[17] C. Hagendorf, *Spin chains with dynamical lattice supersymmetry*, *J. Stat. Phys.* **150** (2013) 609–657, [`arXiv:1207.0357`].

[18] N. Beisert and B. I. Zwiebel, *On Symmetry Enhancement in the psu(1,1|2) Sector of N=4 SYM*, *JHEP* **0710** (2007) 031, [`arXiv:0707.1031`].

[19] B. I. Zwiebel, *Two-loop Integrability of Planar N=6 Superconformal Chern-Simons Theory*, *J.Phys.* **A42** (2009) 495402, [`arXiv:0901.0411`].

[20] S. Belliard and E. Ragoucy, *Nested Bethe ansatz for 'all' closed spin chains*, *J.Phys.* **A41** (2008) 295202, [`arXiv:0804.2822`].

[21] R. Frassek, T. Lukowski, C. Meneghelli, and M. Staudacher, *Baxter Operators and Hamiltonians for 'nearly all' Integrable Closed gl(n) Spin Chains*, `arXiv:1112.3600`.

[22] E. Ragoucy and G. Satta, *Analytical Bethe Ansatz for closed and open gl(M|N) super-spin chains in arbitrary representations and for any Dynkin diagrams*, *JHEP* **0709** (2007) 001, [`arXiv:0706.3327`].

[23] A. Arnaudon, N. Crampe, A. Doikou, L. Frappat, and E. Ragoucy, *Analytical Bethe Ansatz for closed and open gl(n)-spin chains in any representation*, *J.Stat.Mech.* **0502** (2005) P02007, [`math-ph/0411021`].

[24] D. Arnaudon, N. Crampe, A. Doikou, and L. Frappat, *Spectrum and Bethe ansatz equations for the $U_q(gl(N))$ closed and open spin chains in any representation*, `math-ph/0512037`.

[25] L. Avdeev and A. Vladimirov, *On Exceptional Solutions of the Bethe Ansatz equations*, *Theor.Math.Phys.* **69** (1987) 1071.

[26] W. Hao, R. I. Nepomechie, and A. J. Sommese, *Singular solutions, repeated roots and completeness for higher-spin chains*, `arXiv:1312.2982`.

[27] L. Frappat, P. Sorba, and A. Sciarrino, *Dictionary on Lie superalgebras*, `hep-th/9607161`.

[28] A. Zabrodin, *Discrete Hirota's equation in quantum integrable models*, `hep-th/9610039`.

[29] P. Kulish, N. Y. Reshetikhin, and E. Sklyanin, *Yang-Baxter Equation and Representation Theory. 1.*, *Lett.Math.Phys.* **5** (1981) 393–403.

[30] L. Ferro, T. Lukowski, C. Meneghelli, J. Plefka, and M. Staudacher, *Spectral Parameters for Scattering Amplitudes in N=4 Super Yang-Mills Theory*, `arXiv:1308.3494`.

[31] T. Bargheer, N. Beisert, and F. Loebbert, *Boosting Nearest-Neighbour to Long-Range Integrable Spin Chains*, *J.Stat.Mech.* **0811** (2008) L11001, [`arXiv:0807.5081`].

[32] T. Bargheer, N. Beisert, and F. Loebbert, *Long-Range Deformations for Integrable Spin Chains*, *J.Phys.* **A42** (2009) 285205, [`arXiv:0902.0956`].