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Three-loop universal structure constants in $\mathcal{N} = 4$ susy Yang-Mills theory

or

Very many harmonic sums

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Abstract

We present a conjecture for the normalisation of the twist two conformal partial waves in a double OPE limit of the four-point function of stress tensor multiplets in $\mathcal{N} = 4$ super Yang-Mills theory up to three loops. This contains information about the structure constants in the OPE.

Like the twist two anomalous dimensions our result is expressed as a linear combination of harmonic sums whose argument is the spin of the exchanged operators.

To arrive at the result we derive asymptotic expansions for the twist two part of two unknown three-loop integrals using the method of expansion by regions, complemented by some intuition gained on the example of the ladder integrals up to three loops.

1 Introduction

In $\mathcal{N}=4$ super Yang-Mills theory the loop corrections to the four-point function of stress-energy tensor multiplets take a factorised form: A superconformal invariant is multiplied by an x-space function expressed in terms of finite conformal integrals [1, 2]. The one- and two-loop corrections to the x-space part have been known for a while [3, 4, 5]. The three-loop contribution could possibly still be calculated using the methods of [4] based on superconformal invariance and $\mathcal{N}=2$ supergraphs, but such an endeavour would surely be rather cumbersome.

However, it is not necessary to start from off-shell Feynman graphs in order to construct the loop corrections to the four-point correlator. One can rather sort the set of candidate scalar l-loop conformal integrals into orbits under an enlarged permutation symmetry discovered in [2], and then fix the overall coefficient of each orbit either by appealing to the correlator/amplitude duality [6] or by independent criteria relating to the suppression of the highest logarithmic singularities in accordance with the expected singular behaviour of the correlator in a Euclidean coincidence limit or in a light-cone limit, c.f. [7]. In the latter paper the integrand of the four-point correlator was constructed up to six loops relying on these criteria.

In a Euclidean double OPE limit in which the positions of the four operators approach each other pairwise we can extract anomalous dimensions from a decomposition in terms of conformal partial waves [8]. Here the two pairs of operators have an operator product expansion expressed as an infinite series of other operators, and the four-point function is essentially reduced to an infinite sum over the two-point function of these "exchanged" operators. In this paper we analyse the exchange of twist two operators. Their anomalous dimensions are given by a universal function depending on the spin. More precisely, perturbative calculations up to three loops have shown that this function is a linear combination of harmonic sums [9, 10]. This result has been of fundamental importance for the construction of an integrable system describing the higher-loop anomalous dimensions [11] in the so-called $sl(2)$ or twist sector.

The double OPE limit of the four-point function equally contains information about the structure constants in the OPE or equivalently the three-point functions $\langle \mathcal{T}\,\mathcal{T}\,\mathcal{O}^{(s)}\rangle$ where $\mathcal{T}$ is the stress tensor multiplet and $\mathcal{O}^{(s)}$ is any (twist) operator occurring in the OPE[1]. We show in this article that up to three loops the structure constants of twist two operators are given by a universal function written in terms of harmonic sums. Recently there has been rising interest in an integrable systems explanation of structure constants [12]. We hope to foster this development with the formulae here presented.

At one and two loops the quantum corrections to the four-point correlator are encoded in the functions

$$
\begin{aligned}
F^{(1)} &= g(1,2,3,4)\,, && (1)\\
F^{(2)} &= \frac{1}{2}\,g(1,2,3,4)^2\,\left(x_{12}^2x_{34}^2 + x_{13}^2x_{24}^2 + x_{14}^2x_{23}^2\right) && (2)\\
&+\; 2\,\left(h(1,2;3,4) + h(1,3;2,4) + h(1,4;2,3)\right)\,.
\end{aligned}
$$

[1]This picture does not require an explicit definition of the quantum corrected operators.

At three-loop level we use the result from [2]

$$
\begin{aligned}
F^{(3)} \;=\;& 2\, g(1,2,3,4)\, \left(x_{12}^2 x_{34}^2\, h(1,2;3,4) + x_{13}^2 x_{24}^2\, h(1,3;2,4) + x_{14}^2 x_{23}^2\, h(1,4;2,3)\right) \qquad (3)\\
&+ 6\, l(1,2;3,4) + 6\, l(1,3;2,4) + 6\, l(1,4;2,3) + 4\, E(1;3,4;2) + 4\, E(1;2,4;3) + 4\, E(1;2,3;4)\\
&+ (x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)\, H(1,2;3,4) + (x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2)\, H(1,3;2,4) +\\
&+ (x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2)\, H(1,4;2,3)\,.
\end{aligned}
$$

This is the full answer; there are no non-planar corrections up to this order. The definition of the integrals is

$$
\begin{aligned}
g(1,2,3,4) &= \frac{1}{4\pi^2} \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,,\\
h(1,2;3,4) &= x_{34}^2\, \frac{1}{(4\pi^2)^2} \int \frac{d^4x_5\, d^4x_6}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2)}\,, \qquad (4)\\
l(1,2;3,4) &= x_{34}^4\, \frac{1}{(4\pi^2)^3} \int \frac{d^4x_5\, d^4x_6\, d^4x_7}{(x_{15}^2 x_{35}^2 x_{45}^2) x_{57}^2 (x_{37}^2 x_{47}^2) x_{67}^2 (x_{26}^2 x_{36}^2 x_{46}^2)}\,,\\
E(1,2;3,4) &= x_{23}^2 x_{24}^2\, \frac{1}{(4\pi^2)^3} \int \frac{d^4x_5\, d^4x_6\, d^4x_7\;\, x_{17}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{57}^2 (x_{27}^2 x_{37}^2 x_{47}^2) x_{67}^2 (x_{16}^2 x_{26}^2 x_{46}^2)}\,,\\
H(1,2;3,4) &= x_{34}^2\, \frac{1}{(4\pi^2)^3} \int \frac{d^4x_5\, d^4x_6\, d^4x_7\;\, x_{56}^2}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) x_{57}^2 (x_{37}^2 x_{47}^2) x_{67}^2 (x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2)}\,.
\end{aligned}
$$

In (3) we encounter two conformal non-ladder three-loop integrals E, H (for “easy” and “hard”) which are not explicitly known. Fortunately, one can extract the twist two trajectory directly from the integrals by the method of ”expansion by regions” [13]. To this end we move one point to infinity — which is always possible due to conformal invariance — and then study an expansion of the resulting three-loop integrals in terms of the remaining small distance of the OPE limit. The procedure collapses the scalar three-point integrals to two-point tensor integrals. We evaluate the genuine three-loop pieces by tensor reduction and an integration routine of the Mincer system [14]. The information so obtained is not quite sufficient for our purpose so that we have to supplement it by some intuition about the form of the results.

It is in principle possible to extend our analysis to higher twist but we would meet more and more difficulties in extracting enough information from the integrals using Mincer.

The article is structured as follows: In Section 1 we derive an asymptotic expansion of the box integrals in terms of harmonic sums with only positive indices. Section 2 discusses the method of expansion by regions and our results for the twist two part of the unknown integrals E, H. In Section 4 we discuss the conformal partial wave decomposition of the correlator.

2 Asymptotic expansion of the box-integrals

The massless L-loop boxes (equivalently the $(L+1)$-rung four-point ladder diagrams) have been evaluated in [15]. They are conformal integrals given by the appropriate weight factor times $\Phi^{(L)}(x,y)$ with the functions

$$
\begin{aligned}
\Phi^{(L)}(x,y) \;=\;& -\frac{1}{L!(L-1)!} \int_0^1 \frac{d\xi}{y\,\xi^2 + (1-x-y)\,\xi + x}\; * \qquad (5)\\
& \log^{L-1}(\xi)\, \left(\log\left(\frac{y}{x}\right) + \log(\xi)\right)^{L-1} \left(\log\left(\frac{y}{x}\right) + 2\,\log(\xi)\right)\,.
\end{aligned}
$$

The arguments are space time cross ratios. In a Euclidean coincidence limit $x_{12},\, x_{34}\, \to 0$

$$v\, =\, 1 - Y\, =\, \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\, \to 1\,, \qquad u\, =\, \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\, \to 0\,. \tag{6}$$

and similar with $x_3 \leftrightarrow x_4$. Due to the $x \leftrightarrow y$ reflection symmetry of $\Phi^{(L)}$ we may analyse all four such situations by considering the limit $x \to 1,\, y \to 0$ of (5). Since we will be interested in the y^0 part of the expansion we can conveniently put $y = 0$ in the first factor (the denominator under $d\xi$) of the integrand; the parameter integrals remain well-defined. The simplest expansions arise for $g(1,4,2,3)$, $h(1,4;2,3)$, $l(1,4;2,3)$ where we put $x = 1/v$, $y = u/v$. The ξ-integration leads to $Li_n(-Y)$ which is replaced by its Taylor series for small Y. To take the same limit on $h(1,3;2,4)$, $l(1,3;2,4)$ we choose $x = v$, $y = u$. The integration then yields $Li_n(-Y/(1-Y))$ which can also straightforwardly be expanded in Y.

The point exchange $x_3 \leftrightarrow x_4$ implies the transformation

$$u \,\to\, \frac{u}{1-Y}\,, \qquad Y \,\to\, -\frac{Y}{1-Y} \tag{7}$$

on the cross ratios. Combined with a division by $1 - Y$ due to the outer weight factor this exchanges the expansions of $h(1,4;2,3)$, $l(1,4;2,3)$ in the $x_{12}, x_{34} \,\to\, 0$ limit with those of $h(1,3;2,4)$, $l(1,3;2,4)$, respectively. The one-loop box g_{1234} is totally symmetric, and indeed the two choices for x, y yield the same series.

The third case, $g(1,2,3,4)$, $h(1,2;3,4)$, $l(1,2;3,4)$ corresponding to the choice

$$x \,=\, \frac{1}{u}\,, \qquad y \,=\, \frac{v}{u} \tag{8}$$

(or its reflection) is mapped into itself. It cannot be analysed by the same simple manipulation of the integrand in (5). For the fully symmetric one-loop box this poses no problem because the series expansion must fall upon what we had before. In order to analyse this limit of the higher box integrals we write the first factor of the integrand as

$$\frac{u}{\lambda}\left(\frac{d\xi}{\xi - \xi_+} - \frac{d\xi}{\xi - \xi_-}\right)\,, \quad \xi_\pm \,=\, \frac{2 - Y - u \pm \lambda}{2\,(1-Y)}\,, \qquad \lambda = \sqrt{-4\,u + (Y+u)^2}\,. \tag{9}$$

Using

$$\frac{1}{\xi_\pm} = 1 - x_\pm\,, \qquad x_\pm = \frac{1}{2}\,(Y + u \pm \lambda) \tag{10}$$

we find

$$\begin{aligned}
\frac{\lambda}{u}\,\Phi^{(2)}(Y,u) \;&=\; -6\,(Li_4(1-x_+) - Li_4(1-x_-)) + 3\,\log(1-Y)\,\left(Li_3(1-x_+) - Li_3(1-x_-)\right) \\
&\quad -\frac{1}{2}\,\log^2(1-Y)\,\left(Li_2(1-x_+) - Li_2(1-x_-)\right)\,,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\lambda}{u}\,\Phi^{(3)}(Y,u) \;&=\; -20\,(Li_6(1-x_+) - Li_6(1-x_-)) + 10\,\log(1-Y)\,\left(Li_5(1-x_+) - Li_5(1-x_-)\right) \\
&\quad -2\,\log^2(1-Y)\,\left(Li_4(1-x_+) - Li_4(1-x_-)\right) \\
&\quad +\frac{1}{6}\,\log^3(1-Y)\,\left(Li_3(1-x_+) - Li_3(1-x_-)\right)\,.
\end{aligned} \tag{12}$$

We seek an asymptotic expansion in a Euclidean regime where $Y, u > 0$. In the coincidence limit, Y^2 and u are of the same order in small quantities. Elementary trigonometry shows $-4\,u + Y^2 \leq 0$ to leading order, so that the root λ is purely imaginary. Hence $x_\pm$ are complex with $\Re(x_\pm) > 0$. After expanding $Li_n(1 - x_\pm)$ in $x_\pm$ in this regime we treat the variables Y and u as independent and further expand first in Y and then in u. The individual polylogarithm terms contribute Y^n/u^m at order $n >= 0$, with $m = [n/2] + 1/2$, $[n/2]\,, \ldots$ up to positive powers of u, but negative and half-integer powers of u cancel in the complete expressions for $\Phi^{(L)}$. We retain only $O(u^0)$, the twist two trajectory.

When the branch point is approached from the left the polylogarithms are described by the following asymptotic series:

$$
\begin{aligned}
Li_2(1-x) &= \zeta_2 + \sum_1^\infty \frac{x^n}{n} \left(- \left(\frac{1}{n} - \log(x) \right) I_0 + J_1 \right) , \\
Li_3(1-x) &= \zeta_3 + \sum_1^\infty \frac{x^n}{n} \left(-\zeta_2\, I_o - \left(\frac{1}{n} - \log(x) \right) I_1 + J_2 \right) , \\
Li_4(1-x) &= \zeta_4 + \sum_1^\infty \frac{x^n}{n} \left(-\zeta_3\, I_o - \zeta_2\, I_1 - \left(\frac{1}{n} - \log(x) \right) I_2 + J_3 \right) , \\
Li_5(1-x) &= \zeta_5 + \sum_1^\infty \frac{x^n}{n} \left(-\zeta_4\, I_o - \zeta_3\, I_1 - \zeta_2\, I_2 - \left(\frac{1}{n} - \log(x) \right) I_3 + J_4 \right) , \\
Li_6(1-x) &= \zeta_6 + \sum_1^\infty \frac{x^n}{n} \left(-\zeta_5\, I_o - \zeta_4\, I_1 - \zeta_3\, I_2 - \zeta_2\, I_3 - \left(\frac{1}{n} - \log(x) \right) I_4 + J_5 \right)
\end{aligned} \tag{13}
$$

with the functions

$$
\begin{aligned}
I_0 &= 1 , \\
I_1 &= -\,S_1 , \\
I_2 &= -\frac{1}{2}\,S_2 + \frac{1}{2}\,S_1^2 , \\
I_3 &= -\frac{1}{3}\,S_3 + \frac{1}{2}\,S_1 S_2 - \frac{1}{6}\,S_1^3 , \\
I_4 &= -\frac{1}{4}\,S_4 + \frac{1}{3}\,S_1 S_3 + \frac{1}{8}\,S_2^2 - \frac{1}{4}\,S_1^2 S_2 + \frac{1}{24}\,S_1^4
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
J_1 &= 0 , \\
J_2 &= S_2 , \\
J_3 &= S_3 - \,S_1 S_2 , \\
J_4 &= S_4 - \,S_1 S_3 - \frac{1}{2}\,S_2^2 + \frac{1}{2}\,S_1^2 S_2 , \\
J_5 &= S_5 - \,S_1 S_4 - \frac{5}{6}\,S_2 S_3 + \frac{1}{2}\,S_1^2 S_3 + \frac{1}{2}\,S_1 S_2^2 - \frac{1}{6}\,S_1^3 S_2 .
\end{aligned} \tag{15}
$$

In these formulae the range of the harmonic sums is from 1 to n, so they denote $S_1(n), S_2(n)$ etc.

The expansions are of pure transcendentality: A sum $\sum_n x^n/n^m = Li_m(x)$ is obviously assigned weight m. The more general expression $\sum_n x^n/n^m\, S_l(n)$ will be regarded as a weight $l+m$ object if S_l is any harmonic sum (or a product thereof) of total weight l. The redundant symbols I_o and J_1 were introduced only to emphasize the pretty iterative pattern, by which the coefficients of ζ-values in the asymptotic expansion of $Li_n(1-x)$ is known from the expansions of $Li_m(1-x)$ with $m<n$. The I_n, J_n functions that we display here have been matched on Mathematica output. Unfortunately, Mathematica runs into problems when the order of the expansion becomes too high. The iteration as well as the fact that I_n, J_n can apparently always be expressed as products of simple ζ-values were observed on the lower examples in the list, where the fits can contain only a very limited number of distinct structures. Once a fit has been established it is a trivial matter to continue the original series up to very high orders.

The asymptotic expansions for the entire box integrals (the complete $\Phi^{(L)}$ functions including rational pre-factors) are all similar to (13). We have begun by analysing the highest logarithms and/or ζ-value contributions to understand what type of object to fit to the series.

By the simple manipulation on the integrand of (5) skeched in the beginning of this section we found at $O(u^0)$:

$$x_{13}^4\, g(1,4,2,3) \;\rightarrow\; \sum_{n=1}^{\infty} \frac{Y^{n-1}}{n}\left[-\log(u)+\frac{2}{n}\right], \tag{16}$$

$$x_{13}^4\, h(1,4;2,3) \;\rightarrow\; \sum_{n=1}^{\infty} \frac{Y^{n-1}}{n}\left[\frac{1}{2}\log^2(u)\frac{1}{n}-\log(u)\frac{3}{n^2}+\frac{6}{n^3}\right], \tag{17}$$

$$x_{13}^4\, l(1,4;2,3) \;\rightarrow\; \sum_{n=1}^{\infty} \frac{Y^{n-1}}{n}\left[-\frac{1}{6}\log^3(u)\,\frac{1}{n^2}+\frac{1}{2}\log^2(u)\,\frac{4}{n^3}-\log(u)\,\frac{10}{n^4}+\frac{20}{n^5}\right]. \tag{18}$$

The limits of $h(1,3;2,4)$, $l(1,3;2,4)$ are of a more complicated form: The resulting series can be fitted on linear combinations of harmonic sums with exclusively positive indices:

$$\begin{aligned} &x_{13}^4\, h(1,3;2,4) \;\rightarrow \\ &\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n}\left[\frac{1}{2}\log^2(u)\, S_1-\log(u)\left(\frac{S_1}{n}+2\,S_2\right)+\left(\frac{S_1}{n^2}+\frac{2\,S_2}{n}-S_1S_2+2\,S_3+2\,S_{1,2}\right)\right], \end{aligned} \tag{19}$$

$$\begin{aligned} &x_{13}^4\, l(1,3;2,4) \;\rightarrow \\ &\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n}\left[-\frac{1}{6}\log^3(u)\left(\frac{S_1^2}{2}+\frac{S_2}{2}\right)+\frac{1}{2}\log(u)^2\left(\frac{S_1^2}{2\,n}+\frac{S_2}{2\,n}+S_1\,S_2+S_3+S_{1,2}\right)\right. \\ &\quad -\log(u)\left(\frac{S_1^2}{2\,n^2}+\frac{S_2}{2\,n^2}+\frac{S_1\,S_2}{n}+\frac{S_3}{n}+\frac{S_{1,2}}{n}+S_2^2+S_1\,S_3+2\,S_4+2\,S_{1,3}+S_{1,1,2}-S_{1,2,1}\right) \\ &\quad +\frac{S_1^2}{2\,n^3}+\frac{S_2}{2\,n^3}+\frac{S_1\,S_2}{n^2}+\frac{S_3}{n^2}+\frac{S_{1,2}}{n^2}+\frac{S_2^2}{n}+\frac{S_1\,S_3}{n}+\frac{2\,S_4}{n}+\frac{2\,S_{1,3}}{n}+\frac{S_{1,1,2}}{n}-\frac{S_{1,2,1}}{n} \\ &\quad \left.+2\,S_2\,S_3+S_1\,S_4+3\,S_5+3\,S_{1,4}+S_{2,3}+2\,S_{1,1,3}-2\,S_{1,3,1}+S_{2,1,2}-S_{2,2,1}-S_{1,1,2,1}+S_{1,2,1,1}\right]. \end{aligned} \tag{20}$$

In these equations — like anywhere in this article — all harmonic sums have argument n unless explicitly stated otherwise.

The coincidence limit on the remainig cases $h(1,2;3,4)$, $l(1,2;3,4)$ requires the more complicated procedure outlined above. We can match the results at $O(u^0)$ by the following expressions:

$$x_{13}^4\, h(1,2;3,4) \;\to\; \sum_{n=0}^{\infty} \frac{Y^n}{n+1} \left[6\,\zeta(3) + S_{1,2} - \, S_{2,1} \right], \tag{21}$$

$$\begin{aligned} x_{13}^4\, l(1,2;3,4) \;\to\; \sum_{n=0}^{\infty} \frac{Y^n}{n+1} \Big[& 20\,\zeta(5) + \zeta(3)\left(S_1^2 - S_2\right) \\ & - S_{2,3} + S_{3,2} + \, S_{1,1,3} - S_{3,1,1} - \, S_{1,2,2} + S_{2,2,1} - \, S_{1,1,2,1} + S_{1,2,1,1} \Big] . \end{aligned}$$

Although this is not manifest, these expansions — as well as the simple result for $g(1,4,2,3)$ — are mapped onto themselves under the point exchange $x_3 \leftrightarrow x_4$.

3 Limits of E and H by Asymptotic Expansion

The non-ladder integrals in the three-loop correction to the four-point function are not yet explicitly known. Fortunately, the method of "asymptotic expansion of Feynman integrals" [13] allows us to analyse Feynman diagrams in any limit; coincidence limits on finite Euclidean integrals are almost the defining examples. Like in the case of the box integrals we will obtain the leading terms of a power series in Y at u^0 and seek a fit on harmonic sums.

Recall the definition

$$E(1,3;2,4) \;=\; \int \frac{d^4x_5 d^4x_6 d^4x_7\, x_{23}^2 x_{34}^2 x_{17}^2}{x_{15}^2 x_{16}^2 x_{25}^2 x_{27}^2 x_{46}^2 x_{47}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{57}^2 x_{67}^2} \;=\; \frac{1}{x_{13}^2 x_{24}^2} \Phi^{(E)}\,(u,v) \tag{22}$$

where the second equality follows by conformal covariance. The integral representation is invariant under the exchange of points 2 and 4. As this exchanges u and v we conclude that $\Phi^{(E)}(u,v) \;=\; \Phi^{(E)}(v,u)$. Next, the exchange of points 1 and 3 has the same effect on the cross ratios, whereby we can conclude that $E(1,3;2,4) \;=\; E(3,1;2,4)$ despite of the apparent asymmetry of the integrand between x_1 and x_3. All in all we are left with three cases to analyse: $E(1,2;3,4)$ and $E(1,3;2,4)$, $E(1,4;2,3)$. In the limit $x_{12}, x_{34} \to 0$, the first case stays apart, while the other two are related by the exchange of points 3 and 4.

3.1 $E(1,4;2,3)$ and $E(1,3;2,4)$

Due to conformal covariance the integrals can be uniquely reconstructed from a limit where, say, point 4 is moved to infinity by replacing

$$x_{12}^2 x_{34}^2 \;\leftrightarrow\; x_{12}^2\,, \qquad x_{13}^2 x_{24}^2 \;\leftrightarrow\; x_{13}^2\,, \qquad x_{14}^2 x_{23}^2 \;\leftrightarrow\; x_{23}^2\,. \tag{23}$$

Note that in the limit

$$u \;\to\; \frac{x_{12}^2}{x_{13}^2}\,, \qquad v \;\to\; \frac{x_{23}^2}{x_{13}^2} \;=\; 1 - \frac{2(x_{12}.x_{13})}{x_{13}^2} + u\,. \tag{24}$$

We consider

$$E(1;2,3) \;=\; \lim_{x_4 \to \infty} x_4^2\, E(1,4;2,3) \;=\; \int \frac{d^4x_5 d^4x_6 d^4x_7\, x_{17}^2}{x_{15}^2 x_{16}^2 x_{25}^2 x_{27}^2 x_{36}^2 x_{37}^2 x_{57}^2 x_{67}^2} \tag{25}$$

Thanks to translation invariance we may further put $x_1 = 0$. Let us re-label $x_2 = p_1$, $x_3 = p_2, x_5 = k_1$, $x_6 = k_2$, $x_7 = k_3$. The integral becomes

$$E(p_1, p_2) = \int \frac{d^4k_1 d^4k_2 d^4k_3 \; k_3^2}{k_1^2 k_2^2 (k_1-k_3)^2 (k_2-k_3)^2 (k_1-p_1)^2 (k_3-p_1)^2 (k_2-p_2)^2 (k_3-p_2)^2} \tag{26}$$

and we are interested in the limit $p_1 \to 0$. We might simply try to expand the integrand in p_1 using

$$\frac{1}{(k_1-p_1)^2} = \frac{1}{k_1^2} \sum_{n_1=0}^{\infty} \left(\frac{2(k_1.p_1) - p_1^2}{k_1^2} \right)^{n_1}, \qquad \frac{1}{(k_3-p_1)^2} = \frac{1}{k_3^2} \sum_{n_2=0}^{\infty} \left(\frac{2(k_3.p_1) - p_1^2}{k_3^2} \right)^{n_2}. \tag{27}$$

These equations are only valid if $k_1^2, k_3^2 > p_1^2$, of course, and for example in calculations with orthogonal polynomials one would indeed subdivide the integration domains according to the validity of such expansions. Here we rather put the measure into $D = 4 - 2\,\epsilon$ dimensions in order to regularise the IR singularities $1/(k_1^2)^{m_1}$ and $1/(k_3^2)^{m_2}$ that arise by extending the integration domain to the origin. We have expanded the integrand according to the formal assigment $k_1, k_2, k_3 = O(p_2) >> p_1$. We call this the "top region". In the "bottom region" we declare $k_1, k_2, k_3 = O(p_1) << p_2$ and likewise employ the geometric series to expand the $k_2 - p_2$ and $k_3 - p_2$ propagators in k_2, k_3, respectively. Here we find UV poles arising from the part of the integration domain where the momenta are large. The method of "expansion by regions" consists of evaluating not only the top and bottom regions, but the sum of all eight possibilities arising from $k_i = O(p_1)$ or $O(p_2)$. All singularities cancel and the logarithms combine into powers of $\log u$. The expansion by regions is equivalent to the "expansion by subgraphs" which in turn has been proven by renormalisation theory to yield valid asymptotic expansions [13].

In massless theories, in some regions one encounters "no-scale" integrals $\int d^Dk/(k^2)^\alpha = 0$. In the case at hand we find non-vanishing contributions in the regions

$$\begin{aligned} &R_1 \,:\, k_1, k_2, k_3 \sim p_2\,, && R_2 \,:\, k_2, k_3 \sim p_2;\; k_1 \sim p_1\,, \\ &R_3 \,:\, k_2 \sim p_2;\; k_1, k_3 \sim p_1\,, && R_4 \,:\, k_1, k_2, k_3 \sim p_1\,. \end{aligned} \tag{28}$$

Here R_1, R_4 are the "top" and "bottom" problems mentioned in the last paragraph. Quite generally, the top problem is an l-loop integral with very high indices (exponents) on some propagators and therefore the hardest to solve. The bottom problem is often of the same topology but the indices are all as in the original integral. The mixed cases break into an m-loop integral depending on the small scale and an $(l-m)$-loop integral depending on the large scale.

The original integral $E(1; 2, 3)$ has dimension $[1/p^2]$. Let us consider the bottom problem R_4: To lowest order we put $k_2 - p_2, k_3 - p_2 \to p_2$. It follows that the corresponding three-loop integral must go as p_1^2. We conclude that the bottom problem does not contribute at order u^0. The same applies to R_3: To lowest order we have $1/(k_3 - p_2)^2 \to 1/p_2^2$. We find a one-loop integral that has dimension of $1/(p_2^2)^{1+\epsilon}$, so once again the two-loop integral depending on p_1 must produce at least one power of u.

The leading term of the top problem has p-dependence $1/(p_2^2)^{1+3\epsilon}$, that of R_2 is $1/((p_2^2)^{1+2\epsilon}(p_1^2)^\epsilon)$. Since there are only two contributing regions we should find that the leading poles from R_1, R_2 are equal and of opposite sign. Further, a pole ϵ^{-n} yields logarithms to the n-th power in the finite part of the ϵ expansion. The logarithms from the two regions can only combine into $\log(u)$, if the leading singularity is a simple pole.

We are finally in a position to consider the top region R_1 in detail. We can drop p_1^2 from (27) because we want to restrict to $O(u^0)$. We wish to calculate

$$E(p_1,p_2)^{\rm top}_{u^0} \; = \; \sum_{n_1,n_2=0}^{\infty} \int \frac{d^Dk_1 d^Dk_2 d^Dk_3 \; (2(k_1.p_1))^{n_1} (2(k_3.p_1))^{n_2}}{(k_1^2)^{2+n_1}(k_1-k_3)^2\, k_2^2\, (k_3^2)^{n_2}(k_2-k_3)^2(k_2-p_2)^2(k_3-p_2)^2} \,. \tag{29}$$

In terms of the classification of three-loop propagator type integrals in [14] this integral is of topology O_2, the two-loop master T_1 with a bubble insertion in an outer line. The k_1 integration (i.e. the momentum going around the bubble) can be executed using

$$\int \frac{d^Dk \; P_s(p)}{(k^2)^\alpha((k-p)^2)^\beta} \; = \; \frac{1}{(p^2)^{\alpha+\beta-D/2}} \sum_{i=0}^{[s/2]} G(\alpha,\beta,s,i) \left\{ \frac{(k^2)^i}{i!} \left(\frac{\Box_k}{4}\right)^i \, P_s(k) \right\}_{k=p} \tag{30}$$

where $P_s(k)$ is any polynomial of order s and $\Box_k \, = \, \partial^2/\partial k^\mu \partial k_\mu$. Last,

$$G(\alpha,\beta,s,i) \; = \; \frac{\Gamma(\alpha+\beta+i-D/2)\,\Gamma(D/2-\alpha+s-i)\,\Gamma(D/2-\beta+i)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(D-\alpha-\beta+s)} \,. \tag{31}$$

In our case the numerator polynomial is $k_1^{\mu_1}\ldots k_1^{\mu_{n_1}}$, all contracted onto p_1. In applying (30) to (29) we discard terms with $i > 0$ since they yield p_1^2. It follows

$$E(p_1,p_2)^{\rm top}_{u^0} \; = \; \sum_{n_1,n_2=0}^{\infty} G(2+n_1,1,n_1,0) \int \frac{d^Dk_2 d^Dk_3 \; (2(k_3.p_1))^{n_1+n_2}}{k_2^2\,(k_3^2)^{1+\epsilon+n_1+n_2}(k_2-k_3)^2(k_2-p_2)^2(k_3-p_2)^2} \,. \tag{32}$$

The region R_2 gives a very similar result: We expand $1/(k_3-p_1)^2$ in p_1 and $1/(k_3-k_1)^2$ in k_1. Powers of k_1^2/k_3^2 can be dropped because they vanish under the k_1 integral, which is a one-loop bubble with ingoing momentum p_1. It multiplies a T_1 topology depending on p_2. Using (30) for the bubble we find

$$E(p_1,p_2)^{R_2}_{u^0} \; = \; \frac{1}{(p_1^2)^\epsilon} \sum_{n_1,n_2=0}^{\infty} G(1,1,n_1,0) \int \frac{d^Dk_2 d^Dk_3 \; (2(k_3.p_1))^{n_1+n_2}}{k_2^2\,(k_3^2)^{1+n_1+n_2}(k_2-k_3)^2(k_2-p_2)^2(k_3-p_2)^2} \,. \tag{33}$$

From the last two equations it is already clear that the leading poles will cancel:

$$G(2+n_1,1,n_1,0)|_{\epsilon^{-1}} \; = \; \frac{1}{\epsilon}\,\frac{1}{n+1} \; = \; -\,G(1,1,n_1,0)|_{\epsilon^{-1}} \tag{34}$$

Recall that the first two arguments of the G function label the indices of the propagators in a one-loop bubble integral, reflecting the fact that $G(2+n_1,1,n_1,0)$ has a pole of IR origin while $G(1,1,n_1,0)$ contains a UV divergence. This is a very direct illustration of the cancellation of singularities due to extension of the integration domains to the whole of Minkowski space. The remaining two loop integrals in (32) and (33) are equal to leading order in ϵ. The traceless part of the numerator (i.e. no p_1^2) always leads to finite integrals so that the recombination of the logarithms can take place as we had anticipated.

Summing up, conformal invariance has permitted to reduce a four-point integral to a (generic) three-point one. The strategy of "expansion by regions" makes it possible to extract a power series in a small ingoing "momentum" $p_1 \, = \, x_{21}$ whereby the three-point integral collapses to a collection of two-point problems on the expense of introducing numerators with open indices.

Even if there are numerators, the "rule of the triangle" [14] can be used to solve the two-loop master topology T_1 with all integer exponents as in (33) or one non-integer exponent on an outer line as in (32). The triangle rule is an IBP identity[2] which reduces the exponents of one of

[2]integration by parts

the k_2, $k_2 - k_3$, $k_2 - p_2$ lines in favour of increasing those of k_3, $k_3 - p_2$. In the case at hand a single application of the triangle rule would be sufficient because the lines involving k_2 all have exponent one. Hence one of them will be cancelled in every term upon which the integral can be evaluated by iterated use of equation (30).

In practice we rather implement a different strategy: We will use tensor reduction in order to pull p_1 off the integrals and then apply the rule of the triangle to the resulting scalar integrals. In the top and bottom problems of the expansion by regions in other limits on E and H we encounter the three-loop FA topology which is implemented in the powerful "Mincer" system [14]. Since this programme only deals with scalar integrals we have to understand the tensor reduction at any rate.

Let an l-loop integral depending on a single outer scale p_2 have a numerator $N^{\mu_1 \ldots \mu_s}$ with s open indices and some denominator D. Upon integration

$$\int \frac{d^D k_1 \ldots d^D k_l \, N^{\mu_1 \ldots \mu_s}}{D} \; = \; P_0^{\mu_1 \ldots \mu_s}(p_2) \, I_0(p_2) + \ldots + P_{[s/2]}^{\mu_1 \ldots \mu_s}(p_2) \tag{35}$$

where

$$P_i^{\mu_1 \ldots \mu_s}(p_2) \; = \; \frac{(p_2^2)^i}{i!} \left(\frac{\Box_{p_2}}{4} \right)^i p_2^{\mu_1} \ldots p_2^{\mu_s} \, . \tag{36}$$

As in (30) the d'Alembertian replaces $p_2^{\mu_i} p_2^{\mu_j}$ by $\eta^{\mu_i \mu_j}$ in a totally symmetric fashion. Symbolic differentiation gives a fast and simple algorithmic realisation of the tensor decomposition because we need not explicitly symmetrise.

In order to determine the I_i in (35) we project the entire equation with $P_0(p_2), \ldots, P_{[s/2]}(p_2)$ obtaining

$$J_i \; = \; M_{ij} \, I_j \, , \qquad M_{ij} \; = \; \left\{ \frac{(k^2)^i}{i!} \left(\frac{\Box_k}{4} \right)^i \frac{(p_2^2)^j}{j!} \left(\frac{\Box_{p_2}}{4} \right)^j (k.p_2)^s \right\}_{k = p_2} \, , \tag{37}$$

$$J_i \; = \; \left\{ \frac{(k^2)^i}{i!} \left(\frac{\Box_k}{4} \right)^i \int \frac{d^D k_1 \ldots d^D k_l \;\; k^{\mu_1} \ldots k^{\mu_s} \, N_{\mu_1 \ldots \mu_s}}{D} \right\}_{k = p_2} \, . \tag{38}$$

The matrix M_{ij} depends on the regulator because $\partial_p . p \; = \; D \; = \; 4 - 2\,\epsilon$. However, the tensor reduction procedure is well-defined also in exactly four dimensions, so that M must have a power series expansion in ϵ. To construct the inverse up to and including ϵ^3 we only have to invert the lowest term and complete it by linear perturbation theory; in practice the whole procedure was still fast up to spin 80 in the most straightforward implementation. In addition, we can limit our scope to the construction of I_0 because the trace terms in $P_i(p_2)$, $i > 0$ obviously yield powers of u upon multiplication with p_1. Finally, we point out that all steps of the algorithm can be performed on the numerator N before calling any integration routine.

In our results we can finally replace p_1, p_2 by x_{21}, x_{31}, respectively, and then recover the dependence on the fourth point by identifying $2(x_{12}.x_{13})/x_{13}^2 \; = \; Y + O(u)$ and extending the overall denominator to $x_{13}^2 x_{24}^2$, c.f. (23) and (24). The limit of $E(1,3;2,4)$ can be obtained from this expansion by the transformation (7) corresponding to the exchange of points 3 and 4 and division by $(1 - Y)$ owing to the overall denominator. The asymptotic series so obtained have very much the same features as the limits of the box integrals. By a short Mathematica script for the evaluation of the T_1 integrals via the triangle rule we could easily generate the asymptotic expansion up to order Y^{31}, which is sufficient to determine all coefficients in fits on harmonic sums with only positive indices (possibly divided by powers of their arguments):

$$
\begin{aligned}
&x_{13}^4\, E(1,4;2,3) \;\rightarrow \qquad\qquad (39)\\
&\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n} \Bigg[-\log(u)\,\zeta(3)\Big(6\,S_1\Big) - \log(u)\left(\frac{S_1^2}{n^2} - \frac{S_1\,S_2}{n} - S_{1,3} + S_{3,1} + 2\,S_{1,1,2} - 2\,S_{2,1,1}\right)\\
&\quad + \zeta(3)\left(\frac{4\,S_1}{n} - 4\,S_1^2 + 12\,S_2\right)\\
&\quad + \frac{2\,S_1^2}{n^3} + \frac{2\,S_1\,S_2}{n^2} - \frac{2\,S_2^2}{n} - \frac{2\,S_1\,S_3}{n} - 4\,S_{1,4} + 4\,S_{4,1} + 4\,S_{1,1,3} - 4\,S_{3,1,1} + 4\,S_{1,2,2} - 4\,S_{2,2,1}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&x_{13}^4\, E(1,3;2,4) \;\rightarrow \qquad\qquad (40)\\
&\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n} \Bigg[-\log(u)\,\zeta(3)\left(\frac{6}{n}\right) - \log(u)\Bigg(\frac{2\,S_1}{n^3} - \frac{S_1^2}{2\,n^2} - \frac{S_2}{2\,n^2} - \frac{2\,S_3}{n} + \frac{S_{1,2}}{n} - \frac{S_2^2}{2} - S_1\,S_3\\
&\quad - \frac{3\,S_4}{2} + 3\,S_{1,3} - S_{1,2,1} + S_{2,1,1}\Bigg) + \zeta(3)\left(\frac{10}{n^2} + \frac{4\,S_1}{n} - 2\,S_2\right) + \frac{8\,S_1}{n^4} - \frac{5\,S_1^2}{2\,n^3} + \frac{3\,S_2}{2\,n^3}\\
&\quad + \frac{S_1\,S_2}{n^2} - \frac{3\,S_3}{n^2} - \frac{S_{1,2}}{n^2} - \frac{S_2^2}{n} + \frac{S_1\,S_3}{n} - \frac{8\,S_4}{n} + \frac{4\,S_{1,3}}{n} + \frac{S_{1,1,2}}{n} - \frac{3\,S_{1,2,1}}{n} + \frac{2\,S_{2,1,1}}{n}\\
&\quad - 5\,S_1\,S_4 - 4\,S_2\,S_3 - 9\,S_5 + 13\,S_{1,4} + 5\,S_{2,3} - 2\,S_{1,1,3} - 2\,S_{1,3,1} + 4\,S_{3,1,1}\\
&\quad + S_{2,1,2} - S_{2,2,1} + S_{1,1,2,1} - S_{1,2,1,1}\Bigg].
\end{aligned}
$$

3.2 $H(1,2;3,4)$

This is a single-log limit of the H integral. The derivation of an asymptotic expansion for the twist two trajectory is very similar to the single-log limit of the E integral, just that here the $x_3 \leftrightarrow x_4$ exchange maps the expansion to itself. In the limit $x_4 \to \infty$ we obtain the reduced integral

$$
H(1,2;3) \;=\; \lim_{x_4\to\infty} x_4^4\, H(1,2;3,4) \;=\; \int \frac{d^4x_5 d^4x_6 d^4x_7\, x_{56}^2}{x_{15}^2 x_{16}^2 x_{25}^2 x_{26}^2 x_{35}^2 x_{36}^2 x_{37}^2 x_{57}^2 x_{67}^2} \qquad (41)
$$

We put $x_5 = k_1$, $x_6 = k_2$, $x_7 = k_3$, $x_1 = 0$, $x_2 = p_1$, $x_3 = p_2$ as before and expand using the geometric series according to which "loop momenta" are supposed to be of order p_1 (small) or order p_2 (large). There are two contributing regions: The top problem $k_i \sim p_2$ and $k_1 \sim p_1$, $k_2, k_3 \sim p_2$ (and symmetrically $k_1 \leftrightarrow k_2$). The bottom problem is non-vanishing but only comes in at $O(u)$.

The top region yields an FA topology, with high indices for two propagators (p_4, p_5 w.r.t. the definitions of [14]). The second region yields a one-loop bubble times the two-loop master T_1 quite as R_2 in the evaluation of E in the last section. The Mincer system could derive the expansion of the top problem only up to and including $O(Y^{20})$. This may point to an installation problem, but more likely this means that the recursion for the evaluation of the T_1 master with a non-integer exponent on the central line (which is internally encountered upon applying the rule of the triangle to the FA topology) is not tabulated to sufficiently high orders. We obtain an asymptotic series containing $\log(u)\,\zeta(3)$, $\log(u)$, $\zeta(3)$ and purely rational terms. The coefficient of $\log(u)\,\zeta(3)$ must have transcendentally weight two, and from the numerators we could in fact easily recognise $Y^{n-1} S_1(n)/(n+1)$. Fits of this type are quickly found also for the non-ζ log term and the $\zeta(3)$ term without logarithm. However, the expansion up to spin 20 does not furnish enough information to fix all 32 constants in our ansatz $S_1(n)/(n+1)^5, \ldots, S_5(n)/(n+1), \ldots$ for the purely rational part (all sums were taken to have positive indices only).

To make progress we invoke invariance under $x_3 \leftrightarrow x_4$ acting on the expansion

$$\lim_{x_2 \to x_1} H(1,2;3) \; = \; \sum_{n=1}^{\infty} Y^{n-1} \Big(\log(u)\, a_n + b_n \Big) + O(u) \,. \tag{42}$$

On the cross ratios we have the aforementioned transformation (7). It remains to divide by $(1-Y)^2$, to re-expand in Y and to equate with the original series. The resulting system of equations allows to express one half of the a_n, b_n in terms of the others. We substitute the ansatz

$$a_n \; = \; c_{5,1} \frac{S_4(n)}{n+1} + \ldots + c_{2,1}\, \zeta(3) \frac{S_1(n)}{n+1} \,, \qquad b_n \; = \; c_{6,1} \frac{S_5(n)}{n+1} + \ldots + c_{3,1}\, \zeta(3) \frac{S_2(n)}{n+1} + \ldots \tag{43}$$

into the first, say, 60 such equations finding seven linear relations between the 15 constants $c_{5,i}$, one for three $a_{3,i}$ and fifteen equations on the set of 32 constants $c_{6,i}$. The actual power series derived from the integral is consistent with these conditions and now suffices to pin down the fit at $O(u^0)$:

$$x_{13}^4 \, H(1,2;3,4) \; \rightarrow \tag{44}$$

$$\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n+1} \Bigg[- \log(u)\, \zeta(3) \Big(24\, S_1 \Big) - \log(u) \left(-2\, S_2^2 + 4\, S_1\, S_3 + 2\, S_4 - 4\, S_{1,3} + 4\, S_{1,1,2} - 4\, S_{1,2,1} \right)$$
$$+ \zeta(3) \left(\frac{48\, S_1}{n+1} - 6\, S_1^2 + 6\, S_2 \right) - \frac{4\, S_2^2}{n+1} + \frac{8\, S_1\, S_3}{n+1} + \frac{4\, S_4}{n+1} - \frac{8\, S_{1,3}}{n+1} + \frac{8\, S_{1,1,2}}{n+1} - \frac{8\, S_{1,2,1}}{n+1}$$
$$+ 2\, S_2\, S_3 + 8\, S_1\, S_4 + 10\, S_5 - 8\, S_{1,4} - 12\, S_{2,3} + 10\, S_{1,1,3} - 8\, S_{1,3,1} - 2\, S_{3,1,1}$$
$$- 2\, S_{1,2,2} + 2\, S_{2,2,1} - 2\, S_{1,1,2,1} + 2\, S_{1,2,1,1} \Bigg] \tag{45}$$

3.3 $E(1,2;3,4)$

We send point 3 to infinity and identify $x_1 = p_1$, $x_2 = 0$, $x_4 = p_2$, $x_5 = k_1$, $x_6 = k_2$, $x_7 = k_3$. The integral to expand is

$$E(p_1, p_2) \; = \; p_2^2 \int \frac{d^D k_1 d^D k_2 d^D k_3 \; (k_3^2 - 2\, k_3.p_1 + p_1^2)}{k_1^2\, k_2^2\, k_3^2 (k_1-k_3)^2 (k_2-k_3)^2 (k_1-p_1)^2 (k_2-p_1)^2 (k_1-p_2)^2 (k_3-p_2)^2} \tag{46}$$

There are six contributing regions: The top and bottom problems, two regions with two integration momenta treated as large and one as small, and two for the opposite. One of the latter does not contribute at $O(u^0)$. The top problem is of topology O_2, the bottom one of topology FA and the three other regions give (nested) bubble integrals. For the top problem and the trivial regions we need not appeal to Mincer, for the bottom problem we used its inbuilt FA routine once again. Unfortunately the programme was not able to go beyond Y^{17}. To make matters worse, the $\log^3$ part of the asymptotic expansion is easy to guess and one sees that there are series of the type $y^{n-1}\, S_2(n)/n$ and $y^{n-1}\, S_2(n)/(n+1)$, whereby there are twice as many constants as before.

On the other hand, the expansion by regions must make sense also if the three terms of the numerator are treated separately: $E(p_1,p_2)$ could be a generic three-point integral, for which the strategy should be operational independently of the fact that it is a limit of a conformal four-point integral. Interestingly we find that the k_3^2 numerator is linked to the $S(n)/(n+1)$ type expansion while the other pieces cause the $S(n)/n$ part. Consequently, running the expansion by regions separately for the two parts of the numerator we can double the number of conditions available from the integral itself.

The integral must have point exchange symmetry $x_1 \to x_2$. This maps the $S(n)/n$ part of the fit to both pieces, but fortunately the $S(n)/(n+1)$ piece is sent onto itself. In this sector we could solve up to five constants. Next, in all the combinations of harmonic sums that we have encountered up to now the coefficients are integer or half integer. Using this knowledge one can simply play through all values for the remaining undetermined coefficients in a typical range and see where the dependent constants come out with denominator 1 or 2. Admittedly, this is a somewhat experimental approach, but there is very clearly only one reasonable solution:

$$
\begin{aligned}
&x_{13}^4\, E(1,2;3,4) \to \qquad (47)\\
&\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n} \Bigg[-\frac{1}{6} \log^3(u) \left(\frac{2\,S_1}{n} - S_1^2 - S_2 \right) \\
&\quad + \frac{1}{2} \log^2(u) \left(\frac{4\,S_1}{n^2} - \frac{3\,S_1^2}{2\,n} + \frac{3\,S_2}{2\,n} - S_1\,S_2 - 3\,S_{1,2} \right) \\
&\quad - \log(u) \Bigg(\frac{6\,S_1}{n^3} - \frac{2\,S_1^2}{n^2} + \frac{4\,S_2}{n^2} - \frac{3\,S_1\,S_2}{n} + \frac{S_3}{n} - \frac{2\,S_{1,2}}{n} - \frac{3\,S_2^2}{2} + \frac{5\,S_4}{2} \\
&\quad - 5\,S_{1,3} - S_{1,1,2} + 2S_{1,2,1} - S_{2,1,1} \Bigg) + 20\,\zeta(5) + \zeta(3) \left(-\frac{2\,S_1}{n} - S_1^2 - 5\,S_2 \right) \\
&\quad + \frac{8\,S_1}{n^4} - \frac{5\,S_1^2}{2\,n^3} + \frac{13\,S_2}{2\,n^3} - \frac{5\,S_1\,S_2}{n^2} + \frac{2\,S_3}{n^2} - \frac{S_{1,2}}{n^2} - \frac{2\,S_1\,S_3}{n} - \frac{5\,S_2^2}{2\,n} + \frac{3\,S_4}{2\,n} - \frac{5\,S_{1,3}}{n} \\
&\quad - \frac{2\,S_{1,1,2}}{n} + \frac{3\,S_{1,2,1}}{n} - \frac{S_{2,1,1}}{n} - S_2\,S_3 + 9\,S_5 - 4\,S_{1,4} - 4\,S_{2,3} \\
&\quad - S_{1,1,3} + 4\,S_{1,3,1} - 3\,S_{3,1,1} + S_{1,2,2} - 2\,S_{2,1,2} + S_{2,2,1} + S_{1,1,2,1} - S_{1,2,1,1} \Bigg] + \\
&\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n+1} \Bigg[-\frac{1}{6} \log^3 \left(S_1^2 + 3\,S_2 \right) + \frac{1}{2} \log^2(u) \left(\frac{2\,S_1^2}{n+1} + \frac{6\,S_2}{n+1} + 4\,S_3 + 4\,S_{1,2} \right) \\
&\quad - \log(u) \Bigg(\frac{4S_1^2}{(n+1)^2} + \frac{12\,S_2}{(n+1)^2} + \frac{8\,S_3}{n+1} + \frac{8\,S_{1,2}}{n+1} - 3\,S_1\,S_3 + S_2^2 + 2\,S_4 + 8\,S_{1,3} \\
&\quad + S_{1,1,2} - 2\,S_{1,2,1} + S_{2,1,1} \Bigg) + \zeta(3) \left(S_1^2 + 3\,S_2 \right) + \frac{8S_1^2}{(n+1)^3} + \frac{24\,S_2}{(n+1)^3} \\
&\quad + \frac{16\,S_3}{(n+1)^2} + \frac{16\,S_{1,2}}{(n+1)^2} - \frac{6\,S_1\,S_3}{n+1} + \frac{2\,S_2^2}{n+1} + \frac{4\,S_4}{n+1} + \frac{16\,S_{1,3}}{n+1} \\
&\quad + \frac{2\,S_{1,1,2}}{n+1} - \frac{4\,S_{1,2,1}}{n+1} + \frac{2\,S_{2,1,1}}{n+1} - 6\,S_1\,S_4 - S_2\,S_3 - 7\,S_5 + 10\,S_{1,4} + 4\,S_{2,3} \\
&\quad + S_{1,1,3} - 4\,S_{1,3,1} + 3\,S_{3,1,1} - S_{1,2,2} + 2\,S_{2,1,2} - S_{2,2,1} - S_{1,1,2,1} + S_{1,2,1,1} \Bigg]
\end{aligned}
$$

3.4 $H(1,4;2,3)$ and $H(1,3;2,4)$

The H integral enjoys the same type of flip symmetry as the ladder graphs and E. In order to analyse $H(1,4;2,3)$ we may thus start from $H(2,3;1,4)$ instead. We send point 4 to infinity and identify $x_1 = 0$, $x_2 = p_1$, $x_3 = p_2, x_5 = k_1$, $x_6 = k_2$, $x_7 = k_3$. In momentum space notation:

$$H(p_1,p_2) \;=\; \int \frac{d^D k_1 d^D k_2 d^D k_3 \; (k_1^2 - 2(k_1.k_2) + k_2^2)}{k_1^2 k_2^2 k_3^2 (k_1-k_3)^2 (k_2-k_3)^2 (k_1-p_1)^2 (k_1-p_2)^2 (k_2-p_1)^2 (k_2-p_2)^2} \tag{48}$$

If $k_1, k_2 \sim p_2$; $k_3 \sim p_1$ or $k_1, k_2 \sim p_1$; $k_3 \sim p_2$ we find no-scale integrals. The contributing regions are thus the top and bottom problems — both of topology FA — and the two regions $R_2: \; k_1, k_3 \sim p_2$; $k_2 \sim p_1$ (and its mirror image with $k_1 \leftrightarrow k_2$), and $R_3: \; k_1, k_3 \sim p_1$; $k_2 \sim p_2$ (and the same with $k_1 \leftrightarrow k_2$).

The leading $1/\epsilon^3$ pole must be universal to all regions for the recombination of the logarithms into powers of $\log(u)$ to happen. We can use the trivial cases R_2, R_3 to try and understand the structure of a fit. Once again, the two types of numerator terms $A = k_1^2 + k_2^2$ and $B = -2(k_1.k_2)$ lead to different structures: The A terms give a series like $Y^{n-1}\, S(n)/n^m$, the B term causes $Y^{n-1}\, S(n)/(n+1)^m$. The Mincer system is able to deal with the top problem up to and including Y^{19}, thus we obtain 20 equations on the coefficients in an ansatz for each sector. It turns out that in the logarithm terms and the $\zeta(3)$ bit of the A part only $Y^{n-1}\, S(n)/n^1$ occurs, see below. Imposing this for the rational terms at u^0 as well we can comfortably fix all coefficients.

With some hindsight ($S_{1,1,1,1}, S_{1,1,1,1,1}$ and high powers of S_1 do not occur) the ansatz for the B part can be limited to 28 coefficients, for which there are 20 equations, so what can be done? Exchanging $x_1 \leftrightarrow x_2$ is an active transformation mapping our case to $H(1,3;2,4)$. The usual transformation (7) on the cross ratios must be followed by dividing out $(1-Y)^2$ due to the higher weight of the integral. This leads to different behaviour under the map: We find a single series $Y^{n-1}\, S(n)/(n+1)^m$, where for the first time the $m = 0$ cases also occur. The resulting ansatz for the rational terms is fairly large because there are many independent harmonic sums (or products thereof) with positive indices adding up to total transcendentally weight six.

Hence a priori we cannot expect any constraint from point exchange. As a matter of fact the simultaneous existence of the two expansions does impose a few conditions: We do not find any condition on the coefficients for the B fit for the $\log^3(u)$ terms of $H(1,4;2,3) = H(2,3;1,4)$, but one at $\log^2(u)$, two at $\log(u)$ and three in the rational part. Hence we can restrict our ansatz for the rational terms of the B series to five unknown constants. Supplemented by the guess that all coefficients are integer multiples of eight it was not hard to play through the possibilities in a likely range; again we find one solution, which is presented below. This "diophantine" problem may seem a weak constraint, yet the difference in complexity is absolutely striking between the preferrred solution which we display and any other random try (where one puts integer or half-integer guesses for the independent parameters and inspects the values of the dependent parameters).

$$\begin{aligned}
& x_{13}^4 \, H(1,4;2,3) \to \\
& \sum_{n=1}^{\infty} \frac{Y^{n-1}}{n} \Big[-\frac{1}{6} \, \log^3(u) \; \left(2\, S_1^2\right) + \frac{1}{2} \, \log^2(u) \; \left(8\, S_1\, S_2\right) \\
& \quad - \log(u) \; \left(8\, S_2^2 + 16\, S_1 S_3 + 4\, S_4 - 8\, S_{1,3} + 8\, S_{1,1,2} - 8\, S_{2,1,1}\right) + \zeta(3) \; \left(-16\, S_1^2\right) \\
& \quad + 24\, S_2\, S_3 + 32\, S_1 S_4 + 16\, S_5 - 32\, S_{1,4} + 16\, S_{1,1,3} - 16\, S_{3,1,1} + 16\, S_{1,2,2} - 16\, S_{2,2,1} \Big] +
\end{aligned} \tag{49}$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{Y^{n-1}}{n+1}\Bigg[-\frac{1}{6}\,\log^3(u)\,\left(-\frac{4\,S_1}{n+1}-2\,S_1^2+4\,S_2\right)\\
&\quad+\frac{1}{2}\,\log^2(u)\left(-\frac{8\,S_1}{(n+1)^2}-\frac{4\,S_2}{n+1}-8\,S_1\,S_2+12\,S_3\right)\\
&\quad-\log(u)\left(\frac{8\,S_1^2}{(n+1)^2}-\frac{24\,S_2}{(n+1)^2}+\frac{8\,S_3}{n+1}-\frac{16\,S_{1,2}}{n+1}-8\,S_2^2-16\,S_1\,S_3+20\,S_4+8\,S_{1,3}\right.\\
&\quad\left.-8\,S_{1,1,2}+8\,S_{2,1,1}\right)+\zeta(3)\,\left(\frac{32\,S_1}{n+1}+16\,S_1^2-32\,S_2\right)\\
&\quad+\frac{64\,S_1}{(n+1)^4}+\frac{32\,S_1^2}{(n+1)^3}-\frac{64\,S_2}{(n+1)^3}+\frac{32\,S_1\,S_2}{(n+1)^2}-\frac{16\,S_3}{(n+1)^2}-\frac{64\,S_{1,2}}{(n+1)^2}-\frac{16\,S_2^2}{(n+1)}+\frac{24\,S_4}{n+1}\\
&\quad-\frac{32\,S_{1,3}}{n+1}-\frac{16\,S_{1,1,2}}{n+1}+\frac{16\,S_{2,1,1}}{n+1}-24\,S_2\,S_3-32\,S_1\,S_4+24\,S_5+32\,S_{1,4}-16\,S_{1,1,3}+32\,S_{3,1,1}\\
&\quad-16\,S_{1,2,2}-16\,S_{2,1,2}+16\,S_{2,2,1}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&x_{13}^4\,H(1,3;2,4)\to \qquad (50)\\
&\sum_{n=1}^{\infty} Y^{n-1}\Bigg[-\frac{1}{6}\,\log^3(u)\,\left(-\frac{4\,S_1}{(n+1)^2}+\frac{S_1^2}{n+1}+\frac{S_2}{n+1}-2\,S_1\,S_2+2\,S_3+2\,S_{1,2}\right)\\
&\quad+\frac{1}{2}\,\log^2(u)\,\left(-\frac{8\,S_1}{(n+1)^3}-\frac{4\,S_2}{(n+1)^2}+\frac{4\,S_1\,S_2}{n+1}-4\,S_1\,S_3+12\,S_4+2\,S_{1,2,1}-2\,S_{2,1,1}\right)\\
&\quad-\log(u)\left(-\frac{4\,S_1^2}{(n+1)^3}-\frac{12\,S_2}{(n+1)^3}+\frac{8\,S_1\,S_2}{(n+1)^2}-\frac{8\,S_3}{(n+1)^2}-\frac{8\,S_{1,2}}{(n+1)^2}+\frac{2\,S_2^2}{n+1}+\frac{14\,S_1\,S_3}{n+1}\right.\\
&\quad-\frac{12\,S_{1,3}}{n+1}+\frac{2\,S_{1,1,2}}{n+1}-\frac{2\,S_{2,1,1}}{n+1}+4\,S_2\,S_3-2\,S_1\,S_4+42\,S_5-14\,S_{1,4}-10\,S_{2,3}\\
&\quad\left.+4\,S_{1,1,3}+4\,S_{1,3,1}-8\,S_{3,1,1}-2\,S_{2,1,2}+2\,S_{2,2,1}-2\,S_{1,1,2,1}+2\,S_{1,2,1,1}\right)\\
&\quad+\zeta(3)\,\left(\frac{32\,S_1}{(n+1)^2}-\frac{8\,S_1^2}{n+1}-\frac{8\,S_2}{n+1}+16\,S_1\,S_2-16\,S_3-16\,S_{1,2}\right)\\
&\quad+\frac{64\,S_1}{(n+1)^5}-\frac{16\,S_1^2}{(n+1)^4}-\frac{16\,S_2}{(n+1)^4}+\frac{16\,S_1\,S_2}{(n+1)^3}-\frac{32\,S_3}{(n+1)^3}-\frac{32\,S_{1,2}}{(n+1)^3}+\frac{44\,S_1\,S_3}{(n+1)^2}-\frac{12\,S_4}{(n+1)^2}\\
&\quad-\frac{56\,S_{1,3}}{(n+1)^2}+\frac{4\,S_{1,1,2}}{(n+1)^2}-\frac{4\,S_{2,1,1}}{(n+1)^2}+\frac{16\,S_2\,S_3}{n+1}+\frac{44\,S_1\,S_4}{n+1}+\frac{20\,S_5}{n+1}-\frac{52\,S_{1,4}}{n+1}-\frac{28\,S_{2,3}}{n+1}\\
&\quad+\frac{8\,S_{1,1,3}}{n+1}-\frac{8\,S_{3,1,1}}{n+1}-\frac{4\,S_{2,1,2}}{n+1}+\frac{4\,S_{2,2,1}}{n+1}+\frac{2}{3}\,S_1^2\,S_2^2-\frac{4}{3}\,S_3^2+22\,S_2\,S_4+\frac{20}{3}\,S_1\,S_5\\
&\quad+108\,S_6-48\,S_{1,5}-56\,S_{2,4}+\frac{52}{3}\,S_{1,1,4}+\frac{10}{3}\,S_{1,4,1}-\frac{62}{3}\,S_{4,1,1}\\
&\quad+\frac{26}{3}\,S_{1,2,3}-\frac{4}{3}\,S_{1,3,2}+\frac{26}{3}\,S_{2,1,3}+\frac{38}{3}\,S_{2,3,1}-\frac{28}{3}\,S_{3,1,2}+\frac{14}{3}\,S_{3,2,1}\\
&\quad-4\,S_{1,1,1,3}-4\,S_{1,1,3,1}+8\,S_{1,3,1,1}-\frac{8}{3}\,S_{1,1,2,2}-\frac{2}{3}\,S_{1,2,1,2}-\frac{14}{3}\,S_{1,2,2,1}-\frac{2}{3}S_{2,1,1,2}\\
&\quad-\frac{26}{3}\,S_{2,1,2,1}+\frac{4}{3}\,S_{2,2,1,1}+2\,S_{1,1,1,2,1}-2\,S_{1,1,2,1,1}\Bigg]
\end{aligned}
$$

4 CPWA analysis

At tree level, the correlation function is given by products of free scalar propagators:

$$G_4^{(0)}(1,2,3,4) = \frac{(N_c^2-1)^2}{4\,(4\pi^2)^4}\left[\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{34}^2}{x_{34}^2}\right)^2 + \left(\frac{y_{13}^2}{x_{13}^2}\frac{y_{24}^2}{x_{24}^2}\right)^2 + \left(\frac{y_{41}^2}{x_{41}^2}\frac{y_{23}^2}{x_{23}^2}\right)^2\right] \tag{51}$$
$$+\frac{N_c^2-1}{(4\pi^2)^4}\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{41}^2}{x_{41}^2} + \frac{y_{12}^2}{x_{12}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{13}^2}{x_{13}^2} + \frac{y_{13}^2}{x_{13}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{41}^2}{x_{41}^2}\right),$$

The loop-corrections to G_4 take a factorised form [1, 2]:

$$G_4^{(l)}(1,2,3,4) = \frac{2\,(N_c^2-1)}{(4\pi^2)^4} \times R(1,2,3,4) \times F^{(l)}(x_i) \qquad (\text{for } l \geq 1) \tag{52}$$

Here $R(1,2,3,4)$ is a universal, $l-$independent rational function of the space-time, x_i, and harmonic, y_i, coordinates at the four external points $1,2,3,4$:

$$\begin{aligned} R(1,2,3,4) = {} & \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) \\ & + \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}(x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2) \\ & + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2}(x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2) \\ & + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} + \frac{y_{13}^4 y_{24}^4}{x_{13}^2 x_{24}^2} + \frac{y_{14}^4 y_{23}^4}{x_{14}^2 x_{23}^2}, \end{aligned} \tag{53}$$

while $F^{(\ell)}(x_i)$ are functions of x_i only, which are explicitly stated up to three loops in terms of the box integrals and E and H in formulae (2), (3) in the introduction.

In the OPE limit $x_2 \to x_1$, $x_4 \to x_3$ the weight 2 operators at points 1,2 and 3,4 respectively fuse into an expansion in terms of operators "exchanged" between the two halves of the four-point function. The exchanged operators carry twist (dilatation weight - spin), spin and $SU(4)$ quantum numbers. Since we are fusing two operators in the **20'** representation, the exchanged operators must carry one of the representations in the product $\mathbf{20'} \times \mathbf{20'} = \mathbf{1} + \mathbf{15} + \mathbf{20'} + \mathbf{84} + \mathbf{105} + \mathbf{175}$. The correlator as written in the last two formulae has six "channels" distinguished by the y variables pertaining to the internal symmetry group. It has been worked out in [16] which linear combination of these channels correspond to the exchange of operators in a given representation. In particular, if we label the channels according to

$$G_4 \;=\; y_{12}^4 y_{34}^4\, A_1 + y_{13}^4 y_{24}^4\, A_2 + y_{14}^4 y_{23}^4\, A_3 + y_{12}^2 y_{34}^2\, y_{13}^2 y_{24}^2\, A_4 + y_{12}^2 y_{34}^2\, y_{14}^2 y_{23}^2\, A_5 + y_{13}^2 y_{24}^2\, y_{14}^2 y_{23}^2\, A_6 \tag{54}$$

then the **20'** exchange corresponds to $A_2 + A_3 + \frac{5}{3}\,A_4 + \frac{5}{3}\,A_5 + \frac{1}{6}\,A_6$. Both at tree- and at loop-level we find that the leading power singularity is $1/(x_{12}^2 x_{34}^2)$ coming from A_4, A_5:

$$\begin{aligned} \lim_{x_{12},x_{34}\to 0} G_4^{\mathbf{20'}} \;=\; & \frac{5(N^2-1)}{3\,(4\pi^2)^4\, x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2} * \\ & \left(\left(\frac{1}{1-Y}+1\right) + 2\left(\frac{1}{1-Y} - 1 - Y\right)\sum_{l=1}^{\infty} \lim_{x_{12},x_{34}\to 0} a^l\, F^{(l)}(x_i)\right) + O(u) \end{aligned} \tag{55}$$

where a is the effective coupling. Powers of u can be discarded because they correspond to higher twist. On the other hand, the expansion in the Y variable is associated to the spin of the exchanged twist two operators.

In terms of the elementary field the operators we discuss are schematically realised as $\mathcal{O}^{(s)} = \mathrm{tr}(W D^{\{\mu_1} \ldots D^{\mu_s\}} W)$. The positioning of the trace-free symmetrised Yang-Mills covariantised derivatives $D^\mu = \partial^\mu + i\, g\, [A^\mu, \bullet]$ on the two W fields decides about the anomalous dimension of the operators. All these composites have twist two because there are two scalar elementary fields of dimension one carrying the derivatives. The spin 0 operator in this class is the protected primary $\mathcal{O}$. At spin one we only find $\partial^\mu \mathcal{O}$, a "conformal descendent" of $\mathcal{O}$. By tree-level orthogonalisation one sees that there is one new primary operator at every even spin, all other combinations of derivatives give descendents of operators with lower spin. The descendents do show in the OPE, but their occurrence is statically linked to that of the primary field. One can resum the contribution of the descendents into a "conformal partial wave" labelled by the primary operator, see [8, 16, 17] and references therein:

$$\mathrm{cpwa}(s) \, = \, N(s) \;\; u^{\gamma/2}\, Y^s \; {}_2F_1\Big(s + 1 + \gamma/2, s + 1 + \gamma/2, 2 + 2\, s + \gamma;\, Y\Big) \tag{56}$$

When expanding in the effective coupling a the cpwa furnish logarithms to be matched on those of the $F^{(l)}$ in (56). Our task is thus to solve

$$\left(\left(\frac{1}{1-Y} + 1\right) + 2\left(\frac{1}{1-Y} - 1 - Y\right) \sum_{l=1}^{\infty} \lim_{x_{12}, x_{34} \to 0} a^l\, F^{(l)}(x_i)\right) \, = \, \sum_{s=0}^{\infty} \mathrm{cpwa}(s) \tag{57}$$

for $\gamma(s), N(s)$. As it should, the normalisation of the cpwa turns out to be zero if the spin is odd. For even spin we find

$$\begin{aligned} N(s) \;\; &= \;\; 2 \left(\frac{\Gamma\left(s+1+\frac{\gamma}{2}\right)^2}{\Gamma\left(2\, s + 1 + \gamma\right)} - \frac{1}{4} \sum_{i=2}^{\infty} \zeta(i)\, b_i \right) \\ & \quad * \Big(1 + a\, c_{1,2} + a^2 \left(\zeta(3)\, c_{2,1} + c_{2,4}\right) + a^3 \Big(\zeta(5)\, c_{3,1} + \zeta(3)\, c_{3,3} + c_{3,6}\Big) + \ldots \Big) \end{aligned} \tag{58}$$

where

$$a \, = \, \frac{g^2\, N}{4\, \pi^2}\,, \qquad b_2 \, = \, -\gamma^2 + \Big(S_1(2\, s) - S_1(s)\Big)\, \gamma^3 + \ldots\,, \qquad b_3 \, = \, \gamma^3 + \ldots\,. \tag{59}$$

The b_i cancel the explicit dependence of the Γ ratio on ζ values; the first factor in (58) is meant to be purely rational. The anomalous dimension depends on the spin and has an expansion $\gamma \, = \, a\, \gamma_1 + a^2\, \gamma_2 + a^3\, \gamma_3 + \ldots$

$$\gamma_1 \;\; = \;\; 2\, S_1\,, \tag{60}$$

$$\gamma_2 \;\; = \;\; -\, S_{-3} - 2\, S_{-2}\, S_1 - 2\, S_1\, S_2 - S_3 + 2\, S_{-2,1}\,, \tag{61}$$

$$\begin{aligned} \gamma_3 \;\; = \;\; & 3\, S_{-5} + 8\, S_{-4}\, S_1 + S_{-2}^2\, S_1 + 6\, S_{-3}\, S_1^2 + S_{-3}\, S_2 + 4\, S_{-2}\, S_1\, S_2 + 2\, S_1\, S_2^2 \\ & + 2\, S_{-2}\, S_3 + 2\, S_1^2\, S_3 + S_2\, S_3 + 3\, S_1\, S_4 + S_5 - 6\, S_{-4,1} - 12\, S_1\, S_{-3,1} - 6\, S_{-3,2} \\ & - 4\, S_1^2\, S_{-2,1} - 2\, S_2\, S_{-2,1} - 10\, S_1\, S_{-2,2} - 6\, S_{-2,3} + 12\, S_{-3,1,1} + 16\, S_1\, S_{-2,1,1} \\ & + 12\, S_{-2,1,2} + 12\, S_{-2,2,1} - 24\, S_{-2,1,1,1}\,. \end{aligned} \tag{62}$$

In these formulae all harmonic sums depend on the argument s. Our result is in complete agreement with the literature [9, 10].

The tree-level normalisation is easily recognised to be the ratio $2\,(s!)^2/(2s)!$ which we take out of the entire normalisation factor. A fit of the one-loop normalisation on harmonic sums is then in fact possible, but only if in addition $S(2s)$ are taken into account. The result for the coefficients is compatible with promoting the the factorials at tree to the Γ functions in (58); this mimics the Y dependent part of the cpwa. At two and three loops the Γ functions correctly incorporate all $S(2s)$ terms. Below we state our results for the coefficients in the second line of (58). In these formulae all harmonic sums have argument s once again.

$$c_{1,2} = -S_2\,, \tag{63}$$

$$c_{2,1} = 3\,S_1\,, \tag{64}$$

$$c_{2,4} = \frac{5}{2}\,S_{-4} + S_{-2}^2 + 2\,S_{-3}\,S_1 + S_{-2}\,S_2 + S_2^2 + 2\,S_1\,S_3 + \frac{5}{2}\,S_4 - 2\,S_{-3,1} - S_{-2,2} - 2\,S_{1,3}\,, \tag{65}$$

$$c_{3,1} = -\frac{25}{2}\,S_1\,, \tag{66}$$

$$c_{3,3} = -3\,S_{-3} - 10\,S_{-2}\,S_1 + \frac{4}{3}\,S_1^3 - 6\,S_1\,S_2 - \frac{4}{3}\,S_3 + 6\,S_{-2,1}\,, \tag{67}$$

$$\begin{aligned} c_{3,6} = {} & -11\,S_{-6} + \frac{5}{2}\,S_{-3}^2 - 5\,S_{-4}\,S_{-2} - \frac{41}{2}\,S_{-5}\,S_1 - S_{-3}\,S_{-2}\,S_1 - 5\,S_{-4}\,S_1^2 - 2\,S_{-2}^2\,S_1^2 \\ & + \frac{4}{3}\,S_{-3}\,S_1^3 - \frac{13}{2}\,S_{-4}\,S_2 - \frac{3}{2}\,S_{-2}^2\,S_2 - 10\,S_{-3}\,S_1\,S_2 - 2\,S_{-2}\,S_2^2 - S_2^3 - \frac{16}{3}\,S_{-3}\,S_3 \\ & - 8\,S_{-2}\,S_1\,S_3 - 6\,S_1\,S_2\,S_3 - 3\,S_3^2 - 3\,S_{-2}\,S_4 + 9\,S_1^2\,S_4 - 4\,S_2\,S_4 + \frac{15}{2}\,S_1\,S_5 - \frac{13}{2}\,S_6 \\ & + 14\,S_{-5,1} + 11\,S_1\,S_{-4,1} + 9\,S_{-4,2} - 12\,S_1\,S_{-3,-2} + 10\,S_{-2}\,S_{-3,1} - 4\,S_1^2\,S_{-3,1} \\ & + 8\,S_2\,S_{-3,1} + 4\,S_1\,S_{-3,2} + 9\,S_{-3,3} - 10\,S_{-3}\,S_{-2,1} + 14\,S_{-2}\,S_1\,S_{-2,1} - \frac{8}{3}\,S_1^3\,S_{-2,1} \\ & + 4\,S_1\,S_2\,S_{-2,1} + \frac{20}{3}\,S_3\,S_{-2,1} + 10\,S_{-2,1}^2 + 10\,S_{-2}\,S_{-2,2} - 6\,S_1^2\,S_{-2,2} + 6\,S_2\,S_{-2,2} \\ & + 6\,S_1\,S_{-2,3} + 11\,S_{-2,4} - 6\,S_2\,S_{1,3} - 4\,S_1\,S_{1,4} - 4\,S_{1,5} + 4\,S_1\,S_{2,3} + 4\,S_{2,4} - 12\,S_{-4,1,1} \\ & + 8\,S_1\,S_{-3,1,1} - 2\,S_{-3,1,2} - 2\,S_{-3,2,1} - 24\,S_1\,S_{-2,-2,1} - 20\,S_{-2}\,S_{-2,1,1} + 16\,S_1^2\,S_{-2,1,1} \\ & - 8\,S_2\,S_{-2,1,1} + 16\,S_1\,S_{-2,1,2} - 6\,S_{-2,1,3} + 16\,S_1\,S_{-2,2,1} + 4\,S_{-2,2,2} - 6\,S_{-2,3,1} - 4\,S_1\,S_{1,1,3} \\ & - 8\,S_{1,1,4} + 8\,S_{1,3,2} - 8\,S_{-3,1,1,1} - 48\,S_1\,S_{-2,1,1,1} - 20\,S_{-2,1,1,2} - 20\,S_{-2,1,2,1} - 20\,S_{-2,2,1,1} \\ & + 16\,S_{1,1,1,3} + 64\,S_{-2,1,1,1,1} \end{aligned} \tag{68}$$

5 Conclusions

In a double concidence limit $x_2 \to x_1$, $x_4 \to x_3$, the four-point function of stress tensor multiplets $\mathcal{T}$ is reduced to an OPE (operator product expansion) $\mathcal{T}(x_1)\mathcal{T}(x_2) = \sum_s c(s,x)\,\mathcal{O}^{(s)}(x_1)$, and similarly at the other end, so that one obtains (the sum over) the square of the structure constants c with the two-point function of the exchanged operator $\mathcal{O}^{(s)}$. This is a scheme invariant combination from which one can read off the structure constants if the two-point function in the middle is assumed to be normalised to one.

Not only the primary operators but also their conformal descendents (x-derivatives) are exchanged. The descendents are usually put together with the primary fields in conformal blocks called "conformal partial waves" (cpwa). We have derived explicit results for the twist two operators in the **20'** representation of $SU(4)$: Their anomalous dimensions come out as linear combinations of harmonic sums in full agreement with the literature [9, 10]. What is more, also the constants multiplying the twist two cpwa are elaborated in terms of harmonic sums.

We have stopped short of predicting the structure constants from these results because the absolute normalisation of the cpwa is not known[3]. Our results for the constant terms naturally factors into two pieces. It is tempting to associate the first of these factors with the normalisation of the cpwa and the second with the structure constants.

The fact that the entire result is expressed in terms of harmonic sums is a clear hint at an integrable systems explanation, c.f. [11, 12]. This issue will be addressed in future work; we are confident that the normalisation question will be understood if such an interpretation is found.

The asymptotic expansion of the individual conformal integrals is given in terms of harmonic sums with positive indices only, and products thereof with negative powers of their argument. These results should help with the construction of an explicit expression for the unknown integrals in terms of special functions of the polylogarithm type. Interestingly, the cpwa decomposition leads to formulae in terms of harmonic sums only, but here the sums can have negative indices. S_{-1} does not occur, and in most of the higher sums only the outermost index can be negative (two exceptions).

In deriving the asymptotic series by expansion by regions we have met a number of structural properties, i.e. that given numerator terms lead to expansions that can be matched on the distinct structures $Y^{n-1}\, S(n)/n^m$, $Y^{n-1}\, S(n)/(n+1)^m$ with or without the $m = 0$ case. We have used conformal symmetry to make four-point integrals into three-point integrals. The latter are generic by inspection; one may wonder whether any three-point integral can be written as a spin expansion in terms of harmonic sums, or whether the examples here are somehow specific to the $\mathcal{N} = 4$ SYM theory.

Obviously, our work can be extended to the twist three, four, ... trajectories corresponding to powers of the second cross ratio u. We expect that the coefficients will pick up a second parameter; it remains open for the moment whether the rational number in front of each harmonic sum will simply start to depend on the twist, whether each trajectory is completely different, or indeed if the Euler-Zagier sums are not sufficient to express the complete expansion.

Last, on very many occasions — but not always — the coefficients of a set of harmonic sums related by index permutations add up to zero in our formulae. This hints at the existence of a special basis w.r.t. which the results would take a much simpler form.

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6 Appendix: The 20 channel in terms of harmonic sums

In the four-point correlation function at two and three loops the integrals come with very specific rational factors depending on Y. It is an interesting question how the expansions in terms of harmonic sums over powers of their arguments change by including the Y expansion of the rational factors, and which type of expansion will lead to a similar result if multiplied by such factors. In the **20'** channel we obtain a fit of the same type as for the asymptotic expansions of the individual integrals even if the factor $1/(1-Y)-1-Y$ from the R polynomial projected onto the **20'** representation is included. We display the result as an illustration of the universality of the basis of harmonic sums we were using, but also in the hope that the formulae may be useful in an attempt on deriving an explicit result for the correlation function in terms of harmonic polylogarithms and related functions. The tree-level $1/(1-Y)+1+O(u)$ is thus followed by the loop correction $2\sum_{l=1}^{\infty} a^l\, f^{(l)} + O(u)$ with the twist 2 contributions

$$f^{(1)} \;=\; \sum_{n=2}^{\infty} Y^n \left[\log(u)\,\left(-\frac{1}{2\,n}+\frac{1}{2}\,S_1\right)+\frac{1}{n^2}-S_2\right], \tag{69}$$

$$f^{(2)} \;=\; \sum_{n=2}^{\infty} Y^n \left[\log^2(u)\,C_{2;2}+\log(u)\,C_{2,3}+C_{2,4}+\zeta(3)\,D_{2,1}\right] \tag{70}$$

where

$$D_{2,1} \;=\; -\frac{3}{n}+3\,S_1\,, \tag{71}$$

$$C_{2,2} \;=\; -\frac{1}{2\,n^2}+\frac{1}{4}\,S_1^2+\frac{1}{4}\,S_2\,, \tag{72}$$

$$C_{2,3} \;=\; \frac{3}{n^3}-\frac{S_1}{2\,n^2}+\frac{S_2}{2\,n}-\frac{3}{2}\,S_1\,S_2-\frac{3}{2}\,S_3\,, \tag{73}$$

$$\begin{aligned} C_{2,4} \;=\;& -\frac{6}{n^4}+\frac{S_1}{n^3}+\frac{S_2}{2\,n^2}+\frac{S_1\,S_2}{n}-\frac{S_3}{2\,n}-\frac{2\,S_{1,2}}{n} \\ & -S_1\,S_{1,2}+\frac{1}{4}\,S_2^2+3\,S_1\,S_3+\frac{13}{4}\,S_4-\frac{5}{2}\,S_{1,3}+3\,S_{1,1,2} \end{aligned} \tag{74}$$

and

$$f^{(3)} \;=\; \sum_{n=2}^{\infty} Y^n \left[\log^3(u)\,C_{3;3}+\log^2(u)\,C_{3;4}+\log(u)\Big(C_{3;5}+\zeta(3)\,D_{3;2}\Big)+C_{3;6}+\zeta(3)\,D_{3;3}+\zeta(5)\,D_{3;1}\right] \tag{75}$$

with

$$D_{3;1} \;=\; \frac{25}{n}-\,25\,S_1\,, \tag{76}$$

$$D_{3;2} \;=\; -\,\frac{6}{n^2}+3\,S_1^2+3\,S_2\,, \tag{77}$$

$$D_{3;3} \;=\; \frac{14}{n^3}-\frac{10\,S_1}{n^2}-\frac{2\,S_1^2}{n}+\frac{2\,S_2}{n}+\frac{2}{3}\,S_1^3-8\,S_2\,S_1-\frac{14}{3}\,S_3+8\,S_{1,2}\,, \tag{78}$$

$$C_{3;3} = -\frac{1}{2\,n^3} + \frac{S_2}{6\,n} + \frac{1}{12}\,S_1^3 + \frac{1}{12}\,S_1\,S_2 + \frac{1}{6}\,S_{1,2}\,, \tag{79}$$

$$\begin{aligned} C_{3;4} = {} & \frac{6}{n^4} - \frac{S_1}{n^3} - \frac{S_1^2}{4\,n^2} - \frac{S_2}{4\,n^2} + \frac{S_1\,S_2}{2\,n} - \frac{S_3}{2\,n} - \frac{S_{1,2}}{2\,n} \\ & - S_1^2\,S_2 - S_1\,S_3 - \frac{3}{4}\,S_2^2 - \frac{3}{4}\,S_4 - \frac{1}{2}\,S_{1,3}\,, \end{aligned} \tag{80}$$

$$\begin{aligned} C_{3;5} = {} & -\frac{30}{n^5} + \frac{6\,S_1}{n^4} + \frac{S_1^2}{2\,n^3} + \frac{5\,S_2}{2\,n^3} + \frac{5\,S_1\,S_2}{2\,n^2} + \frac{5\,S_3}{2\,n^2} - \frac{2\,S_{1,2}}{n^2} - \frac{3\,S_1\,S_{1,2}}{2\,n} \\ & + \frac{3\,S_1^2\,S_2}{4\,n} + \frac{S_1\,S_3}{2\,n} - \frac{3\,S_2^2}{4\,n} + \frac{S_4}{n} - \frac{2\,S_{1,3}}{n} - \frac{1}{2}\,S_1^2\,S_{1,2} - S_1\,S_{1,3} - 2\,S_2\,S_{1,2} \\ & + \frac{13}{4}\,S_1^2\,S_3 + \frac{9}{4}\,S_1\,S_2^2 + \frac{29}{4}\,S_1\,S_4 + \frac{17}{4}\,S_2\,S_3 + 6\,S_5 - 5\,S_{1,4} + S_{2,3} - \frac{5}{2}\,S_{1,1,3} \\ & + S_{1,2,2} + 6\,S_{1,1,1,2}\,, \end{aligned} \tag{81}$$

$$\begin{aligned} C_{3;6} = {} & \frac{60}{n^6} - \frac{12\,S_1}{n^5} - \frac{6\,S_2}{n^4} - \frac{S_1^2}{n^4} - \frac{3\,S_1\,S_2}{n^3} - \frac{6\,S_3}{n^3} + \frac{4\,S_{1,2}}{n^3} - \frac{S_1\,S_{1,2}}{n^2} \\ & - \frac{3\,S_2^2}{2\,n^2} + \frac{S_1^2\,S_2}{2\,n^2} - \frac{6\,S_1\,S_3}{n^2} - \frac{8\,S_4}{n^2} + \frac{10\,S_{1,3}}{n^2} + \frac{6\,S_2\,S_{1,2}}{n} + \frac{2\,S_1\,S_{1,3}}{n} - \frac{5\,S_1\,S_2^2}{2\,n} \\ & - \frac{S_1^2\,S_3}{n} - \frac{13\,S_1\,S_4}{2\,n} - \frac{3\,S_5}{n} + \frac{16\,S_{1,4}}{n} + \frac{2\,S_{2,3}}{n} - \frac{3\,S_{1,2,2}}{n} - \frac{125}{6}\,S_6 + \frac{13}{12}\,S_2^3 \\ & + 3\,S_1\,S_2\,S_{1,2} + \frac{21}{2}\,S_2\,S_{1,3} + 2\,S_3\,S_{1,2} + \frac{1}{2}\,S_1^2\,S_{1,3} + 3\,S_1\,S_{1,4} - S_1\,S_{2,3} - \frac{11}{2}\,S_2\,S_{1,1,2} \\ & - \frac{1}{2}\,S_1^2\,S_{1,1,2} - S_1\,S_{1,1,3} - 3\,S_1\,S_{1,2,2} + 4\,S_1\,S_{1,1,1,2} - \frac{1}{4}\,S_1^2\,S_2^2 - 9\,S_1\,S_2\,S_3 - \frac{41}{4}\,S_2\,S_4 \\ & - \frac{7}{2}\,S_3^2 - \frac{17}{4}\,S_1^2\,S_4 - 19\,S_1\,S_5 + S_{1,2}^2 + 24\,S_{1,5} + 6\,S_{2,4} + 2\,S_{1,1,4} - S_{1,2,3} \\ & - 6\,S_{1,3,2} - 3\,S_{1,1,1,3} + S_{1,1,2,2} - 10\,S_{1,1,1,1,2}\,. \end{aligned} \tag{82}$$

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