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Towards non-commutative deformations of relativistic wave equations in 2+1 dimensions

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Abstract

We consider the deformation of the Poincaré group in 2+1 dimensions into the quantum double of the Lorentz group and construct Lorentz-covariant momentum-space formulations of the irreducible representations describing massive particles with spin 0, 1/2 and 1 in the deformed theory. We discuss ways of obtaining non-commutative versions of relativistic wave equations like the Klein-Gordon, Dirac and Proca equations in 2+1 dimensions by applying a suitably defined Fourier transform, and point out the relation between non-commutative Dirac equations and the exponentiated Dirac operator considered by Atiyah and Moore.

1 Introduction

It is well-known that the important linear wave equations of relativistic physics can be obtained by Fourier-transforming the irreducible representations of the Poincaré group. The Klein-Gordon, Dirac and Proca equations, for example, are Fourier-transforms of momentum-space constraints for, respectively, spin 0, 1/2 and 1 in Wigner's classification of irreducible Poincaré representations in terms of mass and spin [1, 2].

In this paper we discuss this picture for the case of 2+1 dimensional Minkowski space, and then consider a deformation of it, where the Poincaré symmetry is deformed into a non-cocommutative quantum group, namely the quantum double of the Lorentz group in 2+1 dimensions, or Lorentz double for short [3, 4, 5]. The deformation involves a parameter of dimension inverse mass, and deforms flat momentum space of ordinary special relativity into anti-de Sitter space; in 2+1 dimensions, this happens to be isometric to the identity component of the Lorentz group.

The Lorentz double plays an important role in (2+1)-dimensional quantum gravity [3, 5, 6, 7, 8]. In that context, the deformation parameter is related to Newton's constant. We will not discuss the gravitational interpretation much in this paper and refer to the review [9] for details and references. We further note that deformations of momentum space to a curved manifold are currently much studied in the context of 'relative locality' [10]. We expect our results to be relevant in that context, too.

In the deformed theory, position coordinates, which are translation generators in momentum space, no longer commute. Instead, they satisfy the Lie algebra of the Lorentz group in 2+1 dimensions and act on the Lorentz group-valued momenta by infinitesimal multiplication (see [11] and [9] for a review and further references). One therefore expects that Fourier-transforming the irreducible representations of the quantum double, whose elements are functions on momentum spaces, will lead to covariant wave equations on non-commutative spacetime.

In order to carry out this programme, we first need to write the unitary irreducible representations (UIR's) of the usual (2+1)-Poinaré group in a form that allows one to obtain covariant wave equations via Fourier-transform. This is discussed in Section 2. Even though the wave equations we obtain are the standard Klein-Gordon, Dirac and Proca equation in 2+1 dimensions, our method for obtaining them appears to be new.

Our discussion of the Poincaré UIR's follows a similar treatment of the Euclidean situation in [12], which is our main reference. However, the Lorentzian situation is considerably more involved than the Euclidean case. A full classification the UIR's of the Poincaré group in 2+1 dimensions was first given by Binegar in [13], where he also discusses the possibility – and difficulties – of writing the UIR's in terms of fields on Minkowski space obeying covariant wave equations. A complete analysis of relativistic wave equations in 2+1 dimensions is given in [14] from the point of view of generalised regular representations. Our approach of adapting the Euclidean discussion of [12] gives a less general treatment of the Poincaré UIR's, but maintains the link via Fourier transform between momentum space and position space. This is essential for our treatment of the Lorentz double.

In Section 3, we repeat the covariantisation procedure for the irreducible representations of the Lorentz double, still following and adapting the treatment of the Euclidean situation in [12]. In our final Section 4, we sketch how a Fourier transform adapted to quantum groups [15] can be used to yield non-commutative wave equations. The 'waves' in this case are elements of the Lorentz group. We also indicate how our treatment is linked to alternative approaches based on $\star$-products on $\mathbb{R}^3$ [16, 17, 18, 19, 20, 21, 23], and point out an interesting connection with the exponentiated Dirac operator proposed by Atiyah and Moore in [22].

2 Relativistic wave equations in 2+1 dimensions

2.1 Conventions and notation

We denote 2+1 dimensional Minkowski space by $\mathbb{R}^{2,1}$ and use the 'mostly minus' convention for the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1)$. We write elements of $\mathbb{R}^{2,1}$ as $x, y, \ldots$ with $x = (x^0, x^1, x^2)$ and

$$\eta(x, y) = \eta_{ab} x^a y^b = x^0 y^0 - x^1 y^1 - x^2 y^2. \tag{2.1}$$

Latin indices range over $0, 1, 2$ and summation over repeated indices is implied.

The group of linear transformations of $\mathbb{R}^{2,1}$ that leave η invariant is the Lorentz group $L_3 = O(2, 1)$. It has four connected components. We are mainly interested in the identity component – the subgroup of proper orthochronous Lorentz transformations, denoted $L_3^{+\uparrow}$.

The group of affine transformations that leave the Minkowski distance $\eta(x - y, x - y)$ invariant is the semidirect product $L_3 \ltimes \mathbb{R}^3$ of the Lorentz group with the abelian group of translations. We call it the extended Poincaré group. Its identity component is the Poincaré group, which we denote as

$$P_3 = L_3^{+\uparrow} \ltimes \mathbb{R}^3. \tag{2.2}$$

For the semidirect product we use the conventions of [12], which allow for an easy extension to the quantum group deformation in the next section but are different from those mostly used in the physics literature. In our conventions, the product of $(\Lambda_1, a_1), (\Lambda_2, a_2) \in P_3$ is given by

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_2 a_1 + a_2). \tag{2.3}$$

One advantage of this convention is that the ordering of the elements can be interpreted as a factorisation: $(\Lambda, a) = (\Lambda, 0)(\mathrm{I}, a)$, where I is the identity in $O(2, 1)$.

The action of $(\Lambda, a) \in P_3$ on the Minkowski space is then the right action

$$(\Lambda, a) : x \mapsto x \triangleleft (\Lambda, a) = \Lambda x + a. \tag{2.4}$$

For a full classification of possible excitations in 2+1 dimensional relativistic physics, including the anyonic ones, one needs to study the projective UIR's of P_3. These are given by the ordinary UIR's of the universal covering group $\widetilde{\tilde{P}_3}$ of P_3, which are studied in detail in [24]. Wave equations for anyonic wave functions with infinitely many components are investigated in [25]. In this paper, we work with the double cover $SL(2, \mathbb{R})$ of $L_3^{+\uparrow}$ and hence the double cover $\tilde{P}_3 = SL(2, \mathbb{R}) \ltimes \mathbb{R}^3$ of the Poincaré group. The main reason for this is the convenience of working with 2×2 matrices, and an easier link with the existing literature on the Lorentz double, which mostly uses a formulation based on $SL(2, \mathbb{R})$. Note also that, in 3+1, the double cover of the Poincaré group is the universal cover.

It turns out to be natural and convenient to interpret the translation group $\mathbb{R}^3$ as the vector space $sl(2, \mathbb{R})^*$ dual to $sl(2, \mathbb{R})$. Then $\tilde{P}_3 = SL(2, \mathbb{R}) \ltimes sl(2, \mathbb{R})^*$, where $SL(2, \mathbb{R})$ acts on $sl(2,\mathbb{R})^*$ via the coadjoint action. The right-action of $(g,a) \in \tilde{P}_3$ on Minkowski space $sl(2,\mathbb{R})^*$ is then given by

$$(g,a) : sl(2,\mathbb{R})^* \ni x \mapsto x \triangleleft (g,a) = \mathrm{Ad}_g^* x + a. \tag{2.5}$$

This action preserves the Minkowski metric η on $sl(2,\mathbb{R})^*$.[1]

The Lie algebra $p_3 = sl(2,\mathbb{R}) \ltimes sl(2,\mathbb{R})^*$ is six dimensional, with translation generators P_0, P_1 and P_2, rotation generator J_0 and boost generators J_1 and J_2. They satisfy the commutation relations:

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P^c, \qquad [P_a, P_b] = 0, \tag{2.6}$$

where indices are raised via the inverse Minkowski metric η^{ab} and ϵ_{abc} is the totally antisymmetric tensor in three dimensions normalised such that $\epsilon_{012} = 1$. We are using conventions where the structure constants in the Lie algebra are real. This has the advantage that we can exponentiate to obtain group elements without needing to insert the imaginary unit i. However, it has the drawback that eigenvalues of generators are typically imaginary. Our conventions differ from those in [12] in this respect.

The vector spaces $sl(2,\mathbb{R})$ and $sl(2,\mathbb{R})^*$, which make up p_3, are in duality, and the natural pairing between them is invariant and non-degenerate. This pairing plays an important role in the Chern-Simons formulation of 2+1 gravity [9, 26, 27], where it is normalised via Newton's constant in 2+1 dimensions:

$$\langle J_a, P_b \rangle = \frac{1}{8\pi G}\,\eta_{ab}, \qquad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0. \tag{2.7}$$

2.2 Irreducible unitary representations of $\tilde{P}_3$

The UIR's of $\tilde{P}_3$ are classified in terms of $SL(2,\mathbb{R})$ orbits in $(sl(2,\mathbb{R})^*)^*$ together with UIR's of associated stabiliser groups [2]. Since $(sl(2,\mathbb{R})^*)^* = sl(2,\mathbb{R})$, these orbits are nothing but adjoint orbits of $SL(2,\mathbb{R})$. The following is a convenient basis of $sl(2,\mathbb{R})$, whose detailed properties are summarised in Appendix A:

$$t^0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

However, we need to be careful about normalisation. The normalisation of $\{t^a\}_{a=0,1,2}$ is fixed by the commutation relations (A.3). The normalisation of the basis $\{P^{*a}\}_{a=0,1,2}$, which is dual to the basis $\{P_a\}_{a=0,1,2}$ used in (2.6), may be different. Therefore, we should allow

$$P^{*a} = \lambda t^a, \quad a = 0,1,2, \tag{2.9}$$

where λ is an arbitrary constant of dimension inverse mass. In Section 3 we use the invariant pairing (2.7) to identify $sl(2,\mathbb{R})^*$ with $sl(2,\mathbb{R})$ (and thus P^{*a} with J^a). The commutation relations (2.6) then fix $\lambda = 8\pi G$.

[1] We can think of η as being induced by the Killing form on the dual $(sl(2,\mathbb{R})^*)^*$, but this is not essential in the following.

We denote elements of momentum space $sl(2,\mathbb{R})$ as p, which we expand as

$$p = p_a P^{*a} = \lambda p_a t^a. \tag{2.10}$$

The adjoint action of $SL(2,\mathbb{R})$ on $sl(2,\mathbb{R})$ leaves invariant the inner product

$$-\frac{2}{\lambda^2}\mathrm{tr}(pq) = p_a q^a. \tag{2.11}$$

In the following, we take p^2 to mean $p_a p^a$, not the square of the matrix p.

The orbits of the $SL(2,\mathbb{R})$ adjoint action on $p \in sl(2,\mathbb{R})$ are labelled by the value of the invariant inner product p^2. The different cases are naturally distinguished by the spacelike (S), timelike (T) or lightlike (L) nature of the elements p on a given orbit.

T: There are two distinct possibilities corresponding to the different possible signs of a real parameter $m \neq 0$. Starting from the timelike representative element $\hat{p} = \lambda m t^0$, the orbits

$$O_m^{T\pm} = \{v\lambda m t^0 v^{-1} \mid v \in SL(2,\mathbb{R})\} = \{\lambda p_a t^a \in sl(2,\mathbb{R}) \mid p^2 = m^2, \pm m > 0\} \tag{2.12}$$

are the 'forward' and 'backward' sheets of the two-sheeted mass hyperboloid. The associated stabilisers are

$$N^T = \{\exp(\phi t^0) \mid \phi \in [0, 4\pi)\} \simeq U(1). \tag{2.13}$$

The UIR's are labelled by $s \in \frac{1}{2}\mathbb{Z}$; the half-integer values arising because of the range of ϕ for elements of the form $e^{\phi t^0} \in SL(2,\mathbb{R})$.

The parameters $|m|$ and s can be interpreted as the mass and the spin of a particle. We allow m to be either positive or negative, corresponding to the cases of a particle or antiparticle. Further note that, in contrast to the 3+1 dimensional case, the spin s can also be either positive or negative. In fact, spin in 2+1 dimensions violates parity P and time-reversal T unless two species with opposite spin are included in a theory [29].

S: Picking the spacelike representative element $\hat{p} = \lambda\mu t^1$, the resulting orbits

$$O_\mu^S = \{v\lambda\mu t^1 v^{-1} \mid v \in SL(2,\mathbb{R})\} = \{\lambda p_a t^a \in sl(2,\mathbb{R}) \mid p^2 = -\mu^2 < 0\} \tag{2.14}$$

are single-sheeted hyperboloids. The real parameter μ is strictly positive. The associated stabilisers are

$$N^S = \{\pm\exp(\vartheta t^1) \mid \vartheta \in \mathbb{R}\} \simeq \mathbb{R} \times \mathbb{Z}_2 \tag{2.15}$$

and their UIR's are labelled by pairs (s, ϵ), with $s \in \mathbb{R}$, $\epsilon = \pm 1$. Empirically, particles with spacelike momenta – so-called tachyons – do not exist in the physical 3+1 dimensions.

L: There are again two possibilities corresponding to the different possible signs of p_0. Picking the lightlike representative elements $\hat{p} = \pm E^+ = \pm(t^0 + t^2)$, we obtain the 'forward' and 'backward' light cones as orbits:

$$O^{L\pm} = \{\pm v E^+ v^{-1} \mid v \in SL(2,\mathbb{R})\} = \{\lambda p_a t^a \in sl(2,\mathbb{R}) \mid p^2 = 0, \pm p_0 > 0\}. \tag{2.16}$$

The stabiliser group in both cases is

$$N^L = \{\pm \exp(zE^+) \mid z \in \mathbb{R}\,\} \simeq \mathbb{R} \times \mathbb{Z}_2. \tag{2.17}$$

Its UIR's are again labelled by pairs (s, ϵ), with $s \in \mathbb{R}$, $\epsilon = \pm 1$.

V : The 'vacuum' orbit $\{0\}$ consists solely of the origin and the associated stabiliser is the whole group $SL(2,\mathbb{R})$. The irreducible representations of $SL(2,\mathbb{R})$ can, for instance, be found in [30].

There are two standard ways of writing down the UIR's of semidirect product groups like $\tilde{P}_3$, both using the orbits and stabiliser UIR's listed above. One uses sections of bundles over the homogeneous space $SL(2,\mathbb{R})/N$, where N denotes one of the stabiliser groups. The group action on such section involves multipliers or cocycles, see [2] for details. The other uses functions on $SL(2,\mathbb{R})$ satisfying an equivariance condition. This is the formulation we use here, referring the reader to [2, 5] for a translation between the two approaches.

For a given UIR of $\tilde{P}_3$ labelled by an orbit O with representative element $\hat{p}$, stabiliser group N and UIR ς of N, on a vector space V, the carrier space is

$$V_{O,\varsigma} = \{\, \psi : SL(2,\mathbb{R}) \to V \mid \psi(vn) = \varsigma^{-1}(n)\psi(v), \forall n \in N, \forall v \in SL(2,\mathbb{R})\,\}. \tag{2.18}$$

We also have to impose some kind of integrability condition. An element $(g,a) \in \tilde{P}_3$ acts on $\psi \in V_{P,\varsigma}$ via

$$\pi_{O,\varsigma}((g,a))\psi(v) = \exp(ia(\mathrm{Ad}_{g^{-1}v}(\hat{p})))\psi(g^{-1}v). \tag{2.19}$$

As we will subsequently focus on the case of timelike momenta, we give the carrier space for this case explicitly:

$$V_{ms} = \{\, \psi : SL(2,\mathbb{R}) \to \mathbb{C} \mid \psi(ve^{\alpha t^0}) = e^{-is\alpha}\psi(v), \forall(\alpha, v) \in [0, 4\pi) \times SL(2,\mathbb{R})\,\}. \tag{2.20}$$

The integrability condition is

$$\int_{SL(2,\mathbb{R})/N^T} |\psi|^2(w)\, d\nu(w) < \infty. \tag{2.21}$$

Here, $d\nu$ is the invariant measure on the homogeneous space $SL(2,\mathbb{R})/N^T$ (note that that $|\psi|^2$ only depends on $w \in SL(2,\mathbb{R})/N^T$).

An element $(g,a) \in \tilde{P}_3$ acts on $\psi \in V_{ms}$ via

$$\pi_{ms}((g,a))\psi(v) = \exp(ia(\mathrm{Ad}_{g^{-1}v}(\lambda m t^0)))\psi(g^{-1}v). \tag{2.22}$$

If we introduce the notation

$$p = \lambda m v t^0 v^{-1} \tag{2.23}$$

for an orbit element, this further simplifies to

$$\pi_{ms}((g,a))\psi(v) = e^{ia(\mathrm{Ad}_{g^{-1}}(p))}\psi(g^{-1}v). \tag{2.24}$$

2.3 Covariant momentum constraints

In a field theory, we are usually looking for wave functions that are defined on momentum or position space and which transform covariantly under the action of the Poincaré group [2, 13]. In our conventions, the required transformation behaviour reads

$$\pi((g,a))\phi(p) = e^{ia(\mathrm{Ad}_{g^{-1}}(p))}\rho(g)\phi(g^{-1}p), \tag{2.25}$$

where ρ is a (preferably finite-dimensional) representation of the full group $SL(2,\mathbb{R})$.

To obtain a covariant description, we employ the technique of [12]. In geometric terms, the approach taken there can be described as follows. The formulation (2.18) defines elements of the carrier space of an UIR as functions on the group obeying an equivariance condition. Replacing $SL(2,\mathbb{R})$ with a general Lie group G and considering a general stabiliser subgroup N, this is nothing but the equivariant formulation of sections of vector bundles over G/N. For $G = SU(2)$ and $H = U(1)$, these are the standard Hermitian line bundles over S^2.

The trick used in [12] is to view them as subbundles of the trivial bundle $S^2 \times \mathbb{C}^n$, where $\mathbb{C}^n$ is the standard n-dimensional UIR of $SU(2)$. In that way, sections become ordinary functions $S^2 \to \mathbb{C}^n$ obeying a linear constraint. In this construction, the unitarity of the $SU(2)$ action on $\mathbb{C}^n$ is essential for obtaining Hermitian line bundles. By thinking of S^2 as embedded in Euclidean (momentum) 3-space, one arrives at functions $\mathbb{R}^3 \to \mathbb{C}^n$ obeying a linear constraint. Applying an ordinary Fourier transform then produces functions on Euclidean (position) 3-space obeying a linear differential equation.

We would like to treat the Lorentzian situation analogously. However, the standard n-dimensional irreducible representations of $SL(2,\mathbb{R})$, reviewed in Appendix A, are not unitary, and therefore the procedure of [12] cannot be used to obtain all UIR's of $\tilde{P}_3$. We shall now show that it can be implemented for the UIR's (2.20) labelled by orbits containing timelike momenta. In that case the stabiliser group is the $U(1)$ subgroup of $SL(2,\mathbb{R})$ generated by t^0.

For a given ψ in (2.20), we define the maps

$$\tilde{\phi}^{\pm} : O_m^{T\pm} \to \mathbb{C}^{2|s|+1} \tag{2.26}$$

via

$$\tilde{\phi}^{\pm}(p) = \psi(v)\rho^{|s|}(v)||s|, s\rangle, \tag{2.27}$$

where p is related to v via (2.23), and the states $||s|, k\rangle$ form the basis (A.9) of the finite-dimensional $sl(2,\mathbb{R})$ irreducible representations in which t^0 is diagonal. Clearly

$$\rho^{|s|}(ve^{\alpha t^0})||s|, s\rangle = \rho^{|s|}(v)\rho^{|s|}(e^{\alpha t^0})||s|, s\rangle = \rho^{|s|}(v)e^{i\alpha s}||s|, s\rangle. \tag{2.28}$$

This cancels the phase picked up by ψ under the right-multiplication by $e^{\alpha t^0}$. Hence, $\tilde{\phi}^{\pm}$ only depends on $p \in O_m^{T\pm}$, even though both $\rho^{|s|}(v)$ and ψ depend on v.

We now see why this procedure is generally not feasible for UIR's (2.18) labelled by orbits containing spacelike or lightlike momenta, where the stabiliser groups are generated by spacelike and lightlike generators in $sl(2,\mathbb{R})$. Under the right-multiplication by $e^{\alpha t^1}$ resp. $e^{\alpha t^+}$, the elements of (2.18) pick up a phase that cannot be compensated using one of the finite-dimensional irreducible representations of $SL(2,\mathbb{R})$, as $\rho^{|s|}(t^1)$ has real eigenvalues and $\rho^{|s|}(t^+)$ has zero as the sole eigenvalue.

Similar restrictions were found in [13] for the existence of a finite-dimensional covariant description. More general covariant descriptions are given in [14]. However, these are not obtained directly from the standard UIR's of the Poincaré group. Instead, they are constructed using generalised regular representations.

The maps $\tilde{\phi}^{\pm}$ defined in (2.27) satisfy the constraint

$$\left(i\rho^{|s|}(t^a)p_a + ms\right)\tilde{\phi}^{\pm}(p) = 0, \tag{2.29}$$

as can be seen by writing (2.23) as $p_a t^a = vmt_0v^{-1}$:

$$\begin{aligned}\rho^{|s|}(t_a)p_a\tilde{\phi}^{\pm}(p) &= \rho^{|s|}(vmt^0v^{-1})\rho^{|s|}(v)\psi(v)||s|,s\rangle \\ &= \psi(v)\rho^{|s|}(v)mis||s|,s\rangle \\ &= ims\tilde{\phi}^{\pm}(p),\end{aligned}$$

as required. The equation (2.29) later becomes one of our wave equations and we refer to it as the spin constraint.

Following the method of [12], we now consider extensions of the function $\tilde{\phi}$, defined on the Lie algebra $sl(2,\mathbb{R})$. This will enable us to employ a standard Fourier transform for switching from momentum to position space. We embed the timelike orbits $O_m^{T\pm}$ into the Lie algebra $sl(2,\mathbb{R})$ and define

$$W_{ms} = \{\,\tilde{\phi} : sl(2,\mathbb{R}) \to \mathbb{C}^{2|s|+1} \mid (i\rho^{|s|}(t^a)p_a + ms)\tilde{\phi}(p) = 0,\, (p^2 - m^2)\tilde{\phi}(p) = 0\,\}, \tag{2.30}$$

which are representations of $\tilde{P}_3$. We call the condition

$$(p^2 - m^2)\tilde{\phi}(p) = 0 \tag{2.31}$$

the mass constraint; we will see that it is implied by the spin constraint for the cases $s = \pm\frac{1}{2}, \pm 1$.

To obtain irreducible representations of $\tilde{P}_3$, we still need to impose that $\tilde{\phi}$ has support only on either O_m^{T+} or O_m^{T-}, i.e.

$$\Theta(\mp p_0)\tilde{\phi} = 0, \tag{2.32}$$

where Θ is the Heaviside step function. We call this condition the sign constraint. We remark that though W_{ms} are reducible representations of $\tilde{P}_3$, they are irreducible representations of a suitable double cover of the extended Poincaré group, which includes time reversal (mapping O_m^{T+} to O_m^{T-} and vice versa).

The action of an element $(g,a) \in \tilde{P}_3$ on $\tilde{\phi} \in W_{ms}$ is

$$\left(\pi_{ms}((g,a))\tilde{\phi}\right)(p) = e^{ia(\mathrm{Ad}_{g^{-1}}p)}\rho^{|s|}(g)\tilde{\phi}(\mathrm{Ad}_{g^{-1}}p). \tag{2.33}$$

It commutes with the constraints (2.29) and (2.31), as required.

Before we can claim that this is an UIR, we need to define the inner product with respect to which the representations are unitary. We will do this below for spin 1/2 and spin 1. For a general discussion of the construction of the required invariant scalar product, see [2].

In the case $s = \frac{1}{2}$, the spin constraint (2.29) becomes the Dirac equation in momentum space

$$\left(it^a p_a + \frac{1}{2}m\right)\tilde{\phi}(p) = 0. \tag{2.34}$$

Applying $(it^a p_a - \frac{1}{2}m)$ to this and using (A.4), we see that (2.34) implies the mass constraint (2.31) but not the sign constraint (2.32). However, $\tilde{\phi}$ can be decomposed into positive and negative frequency parts $\tilde{\phi}^+$ and $\tilde{\phi}^-$ using a Foldy-Wouthuysen transformation; see [13] for details. This is completely analogous to the situation in 3+1 dimensions.

To see that (2.34) is indeed the Dirac equation in momentum space, we note that in 2+1 dimensions, Clifford generators (gamma matrices) satisfying

$$\{\gamma^a, \gamma^b\} = \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\,\mathrm{id}. \tag{2.35}$$

can be obtained from the $sl(2,\mathbb{R})$ generators (2.8) via

$$\gamma^a = 2it^a. \tag{2.36}$$

Thus we can write (2.34) as

$$(\gamma^a p_a + m)\tilde{\phi}(p) = 0. \tag{2.37}$$

The invariant scalar product on the space $W_{m,s=\frac{1}{2}}$ with constraints is

$$\left(\tilde{\phi}_1, \tilde{\phi}_2\right) = \int\limits_{O_m^{T+}\cup O_m^{T-}} \tilde{\phi}_1^\dagger\gamma^0\tilde{\phi}_2\,\frac{dp_1 dp_2}{|p_0|}. \tag{2.38}$$

The volume element is the standard Lorentz-invariant volume element on the mass shell. The Lorentz invariance of $\tilde{\phi}^\dagger\gamma^0\tilde{\phi}_2$ follows from the KAN or Iwasawa decomposition of an element $g \in SL(2,\mathbb{R})$ into $g = kv$, where k is a rotation (generated by t^0 and commuting with γ_0) and v is of the form

$$v = \begin{pmatrix} r & x \\ 0 & \frac{1}{r} \end{pmatrix},\ r > 0, x \in \mathbb{R}. \tag{2.39}$$

It satisfies $v^t\gamma^0 v = \gamma^0$.

For $s = 1$, $\tilde{\phi} = \tilde{\phi}_a t^a$ takes values in the adjoint representation of $sl(2,\mathbb{R})$. The constraint (2.29) then gives the Proca equations in momentum space

$$(ip_a \mathrm{ad}(t^a) + m)\tilde{\phi}(p) = 0, \tag{2.40}$$

or

$$[p_a t^a, \tilde{\phi}(p)] = im\tilde{\phi}(p). \tag{2.41}$$

Taking the Minkowski product (2.11) with $p_d t^d$ gives

$$p^a \tilde{\phi}_a(p) = 0. \tag{2.42}$$

The previous two equations together with the identity

$$[\xi, [\eta, \zeta]] = (\xi_a \zeta^a)\eta - (\xi_a \eta^a)\zeta, \quad \xi, \eta, \zeta \in sl(2, \mathbb{R}),\ \xi = \xi_a t^a \text{ etc.} \tag{2.43}$$

give the mass constraint (2.31). Like for spin 1/2, the equation (2.29) implies the mass constraint (2.31) but not the sign constraint (2.32).

The invariant scalar product on the space $W_{m,s=1}$ is

$$\left(\tilde{\phi}_1, \tilde{\phi}_2\right) = -\int_{O_m^{T+} \cup O_m^{T-}} \tilde{\phi}^*_{1a} \tilde{\phi}_2{}^a \, \frac{dp_1 dp_2}{|p_0|}. \tag{2.44}$$

This is manifestly Lorentz invariant, but it may not be obvious that (2.44) is indeed positive definite. This can be seen as follows: due to (2.42) $\tilde{\phi}$ is spacelike, and η is negative definite when restricted to spacelike vectors.

The wave equations for the cases $s = -\frac{1}{2}$ and $s = -1$ can be obtained from (2.37) and (2.41) by changing the sign in front of m.

2.4 Fourier transform to position space

The momentum-space form of the UIR's of $\tilde{P}_3$ in the previous sections were designed to be amenable to a standard Fourier transform. Defining

$$\phi(x) = \frac{1}{\sqrt{(2\pi)^3}} \int e^{ix(p)} \tilde{\phi}(p)\, d^3p, \tag{2.45}$$

the spin constraint (2.29) turns into the first order differential equation

$$(\rho^{|s|}(t^a)\partial_a + ms)\phi(x) = 0. \tag{2.46}$$

The mass constraint (2.31) becomes the Klein Gordon equation

$$(\Box + m^2)\phi = 0. \tag{2.47}$$

These are the general wave equations for massive particles with spin $s \in \frac{1}{2}\mathbb{Z}$ in 2+1 dimensions. An element $(g, a) \in \tilde{P}_3$ acts on the wave function ϕ via

$$\left(\pi_{ms}((g,a))\phi\right)(x) = \rho^{|s|}(g)\phi(\mathrm{Ad}^*_g x + a). \tag{2.48}$$

Specialising to spin 1/2, we obtain from (2.37) the Dirac equation in position space:

$$(i\gamma^a \partial_a - m)\phi = 0. \tag{2.49}$$

The invariant scalar product (2.38) can also be Fourier transformed to obtain the usual form [28]:

$$(\phi_1, \phi_2) = \int \phi_1^\dagger \phi_2 \, dx^1 dx^2. \tag{2.50}$$

For spin 1, the condition (2.41) becomes the Proca equation

$$\partial_a [t^a, \phi] = -m\phi, \tag{2.51}$$

and the constraint (2.42) becomes

$$\partial^a \phi_a = 0. \tag{2.52}$$

3 Deforming momentum space

3.1 Introduction and the quantum double $\mathcal{D}(SL(2,\mathbb{R}))$

We now repeat the analysis in the previous section for the case of the quantum double $\mathcal{D}(SL(2,\mathbb{R}))$ of $SL(2,\mathbb{R})$, or Lorentz double for short. Before summarising the defining properties of the quantum double of a Lie group, we make a few qualitative remarks which highlight the relation between the Lorentz double and the Poincaré group, following [5, 6].

The action (2.24) of a Poincaré group element on an element of one of its UIR's shows that pure translations act by a multiplication with a special function on the (linear) momentum space $sl(2,\mathbb{R})$, namely the plane wave $\psi_a(p) = e^{ia(p)}$. In the Lorentz double, this is deformed and generalised: the momentum space is exponentiated and extended to become the whole group manifold $SL(2,\mathbb{R})$. The space of functions on momentum spaces is generalised to a suitably well-behaved class, for example the class of continuous functions [4]. This deforms the translation part of the Poincaré group into something dual to the rotation/boost part: translations are functions on $SL(2,\mathbb{R})$ and rotations/boost are elements of $SL(2,\mathbb{R})$. By allowing linear combinations we obtain a Hopf algebra, consisting of two subalgebras which are in duality.

Generally, the quantum double of a Lie group is an example of a quantum double, which in turn is a special class of quantum groups [15, 33]. It can be defined in various ways. Here we use the form given in [3, 4] for locally compact Lie groups. As a vector space, the quantum double $D(G)$ of a Lie group G is the space of continuous, complex-valued functions $C(G \times G)$. Morally, one should think of this as the tensor product $C(G) \otimes C(G)$, with the first factor being the group algebra and the second factor being the function algebra on G. The product in the first factor is by convolution and the product in the second factor is pointwise, but twisted by the action of the first argument. The identity cannot be written as an element of $C(G \times G)$. Strictly speaking it should be added as a separate element, but it is convenient to formally express it as a delta-function.

In the conventions of [12] (which differ from those in [3, 4]), the product $\bullet$, coproduct δ, unit, co-unit, antipode and $*$-structure are as given below. The quantum double is quasitriangular [33], and the expression for the R-matrix can be found in [3, 4]. We do not require it here. In the equations below, all integrals over the group are with respect to the Haar measure on the group.

$$
\begin{aligned}
(F_1 \bullet F_2)(g,u) &:= \int_G F_1(z, zuz^{-1})\, F_2(z^{-1}g, u)\, dz, \\
1(g,u) &:= \delta_e(g), \\
(\Delta F)(g_1,u_1;g_2,u_2) &:= F(g_1, u_1u_2)\, \delta_{g_1}(g_2), \\
\epsilon(F) &:= \int_G F(g,e)\, dg, \\
(SF)(g,u) &:= F(g^{-1}, g^{-1}u^{-1}g), \\
F^*(g,u) &:= \overline{F(g^{-1}, g^{-1}ug)}.
\end{aligned}
\tag{3.1}
$$

3.2 Coordinates for $SL(2,\mathbb{R})$

We use two different kinds of coordinates for $SL(2,\mathbb{R})$. On the one hand, we use the exponential map to coordinatise $SL(2,\mathbb{R})$ in terms of its Lie algebra. The exponential map $\exp : sl(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is bijective for a sufficiently small neighbourhood of 0 resp. id, this is not the case globally. In fact, it is neither injective nor surjective as we shall see in our discussion of conjugacy classes below. As before, we write elements of $sl(2,\mathbb{R})$ as $p = \lambda p_a t^a$. Using the fact that $(\lambda p_a t^a)^2 = -\frac{\lambda^2}{4} p^2\, \mathrm{id}$, one finds:

$$
\exp(\lambda p_a t^a) = \begin{cases} \cos(\lambda\sqrt{p^2}/2)\,\mathrm{id} + \frac{p_a}{\sqrt{p^2}/2}\sin(\lambda\sqrt{p^2}/2) t^a, & \text{if } p^2 > 0, \\ \mathrm{id} + \lambda p_a t^a, & \text{if } p^2 = 0, \\ \cosh(\lambda\sqrt{-p^2}/2)\,\mathrm{id} + \frac{p_a}{\sqrt{-p^2}/2}\sinh(\lambda\sqrt{-p^2}/2) t^a, & \text{if } p^2 < 0. \end{cases}
\tag{3.2}
$$

On the other hand, we can realise $SL(2,\mathbb{R})$ as a submanifold of $\mathbb{R}^4$ and use Cartesian coordinates on $\mathbb{R}^4$. Defining

$$
u = \mathcal{P}_3\,\mathrm{id} + \lambda \mathcal{P}_a t^a = \begin{pmatrix} \mathcal{P}_3 + \frac{1}{2}\lambda\mathcal{P}_1 & \frac{1}{2}\lambda\mathcal{P}_0 + \frac{1}{2}\lambda\mathcal{P}_2 \\ -\frac{1}{2}\lambda\mathcal{P}_0 + \frac{1}{2}\lambda\mathcal{P}_2 & \mathcal{P}_3 - \frac{1}{2}\lambda\mathcal{P}_1 \end{pmatrix},
\tag{3.3}
$$

where Latin indices still take values $0, 1, 2$, the condition $u \in SL(2,\mathbb{R})$ is equivalent to

$$
\det u = \mathcal{P}_3^2 + \frac{\lambda^2}{4}\mathcal{P}^a\mathcal{P}_a = 1.
\tag{3.4}
$$

We regard $\mathcal{P}_a$ as the independent coordinates with $\mathcal{P}_3 = \pm\sqrt{1 - \frac{\lambda^2}{4}\mathcal{P}^a\mathcal{P}_a}$ valid for a patch with $\mathcal{P}^a\mathcal{P}_a < \frac{4}{\lambda^2}$. In the following, we refer to the subsets of $SL(2,\mathbb{R})$ with $\mathcal{P}_3 \gtrless 0$ as upper and lower half.

Comparing (3.2) and (3.3), we can easily write down a relation between the two coordinate systems on the intersections of their respective patches. The case $p^2 > 0$ is particularly important for us. Here one has

$$
\mathcal{P}_3 = \cos(\lambda\sqrt{p^2}/2), \qquad \mathcal{P}_a = p_a \frac{\sin(\lambda\sqrt{p^2}/2)}{\lambda\sqrt{p^2}/2}.
\tag{3.5}
$$

Taking the limit $\lambda \to 0$ corresponds to the flattening out of momentum space $SL(2,\mathbb{R}) = AdS_3$. It finally rips apart in the hyperplane of $\mathcal{P}_3 = 0$, producing not one but two copies of flat Minkowski momentum space situated at $\mathcal{P}_3 = \pm 1$. They would be identified if we had worked with $L_3^{+\uparrow}$ instead of $SL(2,\mathbb{R})$. If, on the other hand, we had worked with the universal covering group $\widetilde{SL(2,\mathbb{R})}$, we would have found a countable set of copies. For a discussion of $L_3^{+\uparrow}$ as momentum space in 2+1 dimensional gravity and 2+1 dimensional non-commutative scalar field theories, see [20].

This property of momentum space is an important consequence of the transition to the double cover or universal cover of P_3, compounding the more widely known manifestation via the spin of massive particles, which takes integer values in the case of P_3, half-integer values in the case of $\tilde{P}_3$ and real values in the case of the universal cover $\widetilde{\tilde{P}_3}$ (see our discussion in Section 2.1).

3.3 Irreducible unitary representations of $\mathcal{D}(SL(2,\mathbb{R}))$

The Lorentz double $\mathcal{D}(SL(2,\mathbb{R}))$ is a special example of a transformation group algebra, and its UIR's can best be understood in that general context. As shown in [4], they are labelled by conjugacy classes in $SL(2,\mathbb{R})$ and UIR's of the associated centraliser or stabiliser groups. As emphasised in [5, 6], this should be seen as deformation of the picture for the semi-direct product group $\tilde{P}_3$. In both cases the UIR's are labelled by $SL(2,\mathbb{R})$ orbits in momentum space and UIR's of associated stabilisers. The difference is that momentum space is linear for $\tilde{P}_3$ and curved for $\mathcal{D}(SL(2,\mathbb{R}))$.

The conjugacy classes of $SL(2,\mathbb{R})$ and their associated stabilisers have been classified in [4] and we list them here in a notation adapted to our needs. From the defining property of $SL(2,\mathbb{R}) = \{\, g \in GL(2,\mathbb{R}) \mid \det(g) = 1 \,\}$ it follows that the (generalised) eigenvalues λ_1, λ_2 of a given element multiply to one. They are thus either complex conjugate to each other or both real. The set of conjugacy classes can be organised according to the different possible eigenvalues. With important exceptions, the conjugacy classes can be obtained from the adjoint orbits in the Lie algebra $sl(2,\mathbb{R})$ by exponentiation. We have chosen a labelling of the conjugacy classes which mimicks the conventions we used for the adjoint orbits in the Lie algebra: we use the superscripts T, S and L for 'timelike', 'spacelike' and 'lightlike' to denote conjugacy classes whose elements can be obtained via exponentiated timelike, spacelike or lightlike elements of $sl(2,\mathbb{R})$. Our list also includes the stabiliser group of a representative element in each of the conjugacy classes.

L: For $\lambda_1 = e^{i\frac{\theta}{2}}$, $\lambda_2 = e^{-i\frac{\theta}{2}}$ $(0 < \theta < 2\pi)$, there are two disjoint conjugacy classes, with representative elements $\hat{h} = \exp(\pm\theta t^0)$ which are exponentials of timelike $sl(2,\mathbb{R})$ elements:

$$C^{T\pm}(\theta) = \{v \exp(\pm\theta t^0) v^{-1} \mid v \in SL(2,\mathbb{R}),\, \theta \in (0, 2\pi)\}. \tag{3.6}$$

The stabiliser group is

$$N^T = \{\exp(\phi t^0) \mid \phi \in [0, 4\pi)\} \simeq U(1), \tag{3.7}$$

with UIR's labelled by $s \in \frac{1}{2}\mathbb{Z}$.

S: For $\lambda_1 = e^{\frac{r}{2}}$, $\lambda_2 = e^{-\frac{r}{2}}$ ($r \in \mathbb{R}_+$), there is one conjugacy class, with representative element $\hat{h} = \exp(2rt^1)$ obtained by exponentiating a spacelike Lie algebra element:

$$C^S(r) = \{v \exp(rt^1) v^{-1} \mid v \in SL(2,\mathbb{R})\}. \tag{3.8}$$

It has stabiliser group

$$N^S = \{\pm \exp(\vartheta t^1) \mid \vartheta \in \mathbb{R}\} \simeq \mathbb{R} \times \mathbb{Z}_2, \tag{3.9}$$

with UIR's labelled by pairs (b, ϵ), with $b \in \mathbb{R}, \epsilon = \pm 1$.

-S: For $\lambda_1 = -e^{\frac{r}{2}}$, $\lambda_2 = -e^{-\frac{r}{2}}$ ($r \in \mathbb{R}_+$), there is likewise one conjugacy class: $C^{-S}(r)$, whose elements are obtained from those of $C^S(r)$ by multiplication with -id; they cannot be written as the exponential of a Lie algebra element. The stabiliser group is again N^S.

L$\pm$, **V**: For $\lambda_1 = \lambda_2 = 1$, we distinguish three conjugacy classes C^V, C^{L+} and C^{L-}. The 'vacuum' conjugacy class $C^V = \{\text{id}\}$ has stabiliser $SL(2,\mathbb{R})$, whose UIR's are discussed in [30]. The lightlike conjugacy classes have representative elements $\hat{h} = \exp(E_\pm)$, which are the exponentials of the lightlike elements $E_\pm$:

$$C^{L\pm} = \{v \exp(E_\pm) v^{-1} \mid v \in SL(2,\mathbb{R})\}. \tag{3.10}$$

The stabiliser group in both cases is

$$N^L = \{\pm \exp(zE_+) \mid z \in \mathbb{R}\} \simeq \mathbb{R} \times \mathbb{Z}_2, \tag{3.11}$$

with UIR's labelled by pairs (b, ϵ), with $b \in \mathbb{R}, \epsilon = \pm 1$.

-L$\pm$, **-V**: For $\lambda_1 = \lambda_2 = -1$, we distinguish three conjugacy classes, which are obtained by multiplying C^V, C^{L+} and C^{L-} by $-$id. They have the same stabiliser groups as C^V, C^{L+} and C^{L-}. Elements of C^{L+} and C^{L-} cannot be obtained by exponentiation.

The carrier spaces of the irreducible representations of $\mathcal{D}(SL(2,\mathbb{R}))$, discussed in [4], are again given in terms of functions on $SL(2,\mathbb{R})$ satisfying an equivariance condition. The equivariance condition only depends on the stabiliser group of a given conjugacy class, but not directly on the conjugacy class. Since the same stabiliser groups arise for orbits in $sl(2,\mathbb{R})$ as for conjugacy classes in $SL(2,\mathbb{R})$, the general form of the carrier spaces (2.18) of UIR's of $\tilde{P}_3$ is unchanged when replacing $\tilde{P}_3$ by $\mathcal{D}(SL(2,\mathbb{R}))$. However, the action of the elements of $\mathcal{D}(SL(2,\mathbb{R}))$ is different, and does depend on the conjugacy class labelling the representation.

Since we are only able to give covariant forms of momentum constraints in the case of massive particles, i.e. timelike momenta, we restrict ourselves to the corresponding irreducible representations of $\mathcal{D}(SL(2,\mathbb{R}))$. In the conventions of [12], an element $F \in \mathcal{D}(SL(2,\mathbb{R}))$ acts on $\psi \in V_{ms}$ as

$$(\Pi_{ms}(F)\psi)(v) = \int\limits_{SL(2,\mathbb{R})} F(z, z^{-1} v e^{m\lambda t^0} v^{-1} z) \psi(z^{-1} v) \, dz. \tag{3.12}$$

In the next section, we adapt the covariantisation procedure of Section 2.3 to this representation.

3.4 Deformed covariant constraints

As in Section 2.3, we begin by trading the equivariant function $\psi \in V_{ms}$ for a map

$$\tilde{\phi}^{\pm} : C^{L\pm}(\lambda m) \to \mathbb{C}^{2|s|+1} \tag{3.13}$$

via

$$\tilde{\phi}^{\pm}(u) = \psi(v)\rho^{|s|}(v)||s|, s\rangle, \tag{3.14}$$

where the states $||s|, k\rangle$ are again elements of the basis (A.9) and $u = ve^{m\lambda t^0}v^{-1} \in C^{L\pm}(\lambda m)$.

These functions satisfy the analogue of the spin constraint (2.29),

$$(\rho^{|s|}(u) - e^{i\lambda ms})\tilde{\phi}^{\pm}(u) = 0. \tag{3.15}$$

This can be shown by a short calculation which is entirely analogous to that following (2.29). Note that this is a rather natural condition: the value of the function $\tilde{\phi}^{\pm}$ at u lies in the eigenspace for eigenvalue $e^{i\lambda ms}$ of $\rho^{|s|}(u)$.

We now embed the conjugacy classes $C^{S\pm}(\lambda m)$ into the group $SL(2,\mathbb{R})$. They are characterised by

$$\mathcal{P}_3 = \cos\left(\frac{\lambda m}{2}\right), \quad \pm\mathcal{P}_0 > 0. \tag{3.16}$$

As before, we refer to the first of these equations as the mass constraint and to the second as the sign constraint. In terms of u, the mass constraint is

$$\left(\frac{1}{2}\mathrm{tr}(u) - \cos\left(\frac{\lambda m}{2}\right)\right)\tilde{\phi}(u) = 0. \tag{3.17}$$

We thus define the carrier spaces

$$\tilde{W}_{ms} = \{\tilde{\phi} : SL(2,\mathbb{R}) \to \mathbb{C}^{2|s|+1} \mid (\rho^{|s|}(u) - e^{im\lambda s})\tilde{\phi}(u) = 0, \left(\frac{1}{2}\mathrm{tr}(u) - \cos\left(\frac{\lambda m}{2}\right)\right)\tilde{\phi}(u) = 0\}, \tag{3.18}$$

and, as in the undeformed case, we will find that the mass constraint is actually implied by the spin constraint for spin 1/2 and spin 1. An element $F \in \mathcal{D}(SL(2,\mathbb{R}))$ acts on $\tilde{\phi} \in \tilde{W}_{ms}$ according to

$$(\Pi_{ms}(F)\tilde{\phi})(u) = \int_{SL(2,\mathbb{R})} F(z, z^{-1}uz)\rho^{|s|}(z)\tilde{\phi}(z^{-1}uz)\, dz. \tag{3.19}$$

For spinless particles, the covariant description involves a function $\tilde{\phi} : SL(2,\mathbb{R}) \to \mathbb{C}$. The spin constraint is empty, and we only have the mass constraint (3.17). Writing it in terms of $\mathcal{P}_3$ as in (3.16) and applying (3.4), we arrive at

$$\mathcal{P}_a\mathcal{P}^a\tilde{\phi} = \left(\frac{\sin(m\lambda/2)}{\lambda/2}\right)^2\tilde{\phi}. \tag{3.20}$$

This is our deformed Klein-Gordon equation in momentum space.

In the case $s = \frac{1}{2}$, we have functions $\tilde{\phi} : SL(2,\mathbb{R}) \to \mathbb{C}^2$ and the constraint (3.15) becomes simply

$$u\tilde{\phi}(u) = e^{\frac{i}{2}\lambda m}\tilde{\phi}(u). \tag{3.21}$$

Inserting $u = \mathcal{P}_3 \,\mathrm{id} + \lambda\mathcal{P}_a t^a$, this is equivalent to

$$\lambda\mathcal{P}_a t^a\tilde{\phi}(u) = (e^{\frac{i}{2}\lambda m} - \mathcal{P}_3)\tilde{\phi}(u). \tag{3.22}$$

However, since the vector (P_0, P_1, P_2) (like (p_0, p_1, p_2)) is timelike in the case under consideration, the Lie algebra element $\mathcal{P}_a t^a$ is conjugate to a rotation and has imaginary eigenvalues. Expanding $e^{\frac{i}{2}\lambda m} = \cos(\lambda m/2) + i\sin(\lambda m/2)$, the real part of (3.22) is the promised mass constraint $\mathcal{P}_3\tilde{\phi} = \cos(\lambda m/2)\tilde{\phi}$, while the imaginary part is

$$\left(i\mathcal{P}_a t^a + \frac{1}{2}\frac{\sin(\lambda m/2)}{\lambda/2}\right)\tilde{\phi}(u) = 0. \tag{3.23}$$

This is our deformed Dirac equation in momentum space. Using (2.36) to write it in terms of γ-matrices, we find

$$\left(\mathcal{P}_a\gamma^a + \frac{\sin(\lambda m/2)}{\lambda/2}\right)\tilde{\phi}(u) = 0. \tag{3.24}$$

Applying $(\mathcal{P}_a\gamma^a - \frac{\sin(\lambda m/2)}{\lambda/2})$ to (3.24) gives $\mathcal{P}_a\mathcal{P}^a\tilde{\phi} = \frac{\sin^2(\lambda m/2)}{\lambda^2/4}\tilde{\phi}$, which is equivalent to the squared version of the mass constraint. Note that the information whether the wave function has support on the upper or lower half of $SL(2,\mathbb{R})$ is thus no longer contained in the wave equation.

For $s = 1$ we again work with the adjoint representation of $SL(2,\mathbb{R})$ and think of $\tilde{\phi}$ as a map $\tilde{\phi} : SL(2,\mathbb{R}) \to sl(2,\mathbb{R})$, so we can expand $\tilde{\phi} = \tilde{\phi}_a t^a$. Hence, the constraint (3.15) becomes

$$u\tilde{\phi}(u)u^{-1} = e^{i\lambda m}\tilde{\phi}(u). \tag{3.25}$$

Expanding again $u = \mathcal{P}_3 \,\mathrm{id} + \lambda\mathcal{P}_a t^a$, and using the 'quaternionic' multiplication rule (A.2) of the generators t^a, we deduce

$$\lambda\mathcal{P}_3[\mathcal{P}_a t^a, \tilde{\phi}] - \frac{\lambda^2}{2}(\mathcal{P}_a\mathcal{P}^a)\tilde{\phi} + \frac{\lambda^2}{2}(\mathcal{P}^a\tilde{\phi}_a)(\mathcal{P}_b t^b) = (e^{i\lambda m} - 1)\tilde{\phi}, \tag{3.26}$$

where the evaluation at u is understood everywhere. Taking the Minkowski product (2.11) with $\mathcal{P}_b t^b$ and using that$(e^{i\lambda m} - 1) \neq 0$, we conclude that

$$\mathcal{P}^a\tilde{\phi}_a = 0. \tag{3.27}$$

Inserting this in (3.26) and applying (3.4) yields

$$\lambda\mathcal{P}_3[\mathcal{P}_a t^a, \tilde{\phi}] = (e^{i\lambda m} + 1 - 2\mathcal{P}_3^2)\tilde{\phi}. \tag{3.28}$$

Again we can argue from the representation theory of $sl(2,\mathbb{R})$ reviewed in Appendix A that the eigenvalues of $[\mathcal{P}_a t^a, \cdot]$ are imaginary. With $\mathcal{P}_3$ real and non-vanishing, we deduce that

$$(\cos(\lambda m) + 1 - 2\mathcal{P}_3^2)\tilde{\phi} = 0, \tag{3.29}$$

which is the squared mass constraint

$$\left(\mathcal{P}_3^2 - \cos^2 \frac{\lambda m}{2}\right) \tilde{\phi} = 0. \tag{3.30}$$

Inserting $\mathcal{P}_3 = \pm \cos \frac{\lambda m}{2}$ into (3.28), we finally arrive at

$$\pm i[\mathcal{P}_a t^a, \tilde{\phi}] = \frac{\sin(\lambda m/2)}{\lambda/2}\tilde{\phi}. \tag{3.31}$$

This is the deformed Proca equation in momentum space.

The wave equations in momentum space for the cases $s = -\frac{1}{2}$ and $s = -1$ can again be obtained by changing the sign in front of m in (3.24) and (3.31).

4 Towards non-commutative wave equations

The ordinary Fourier transform, as used in Section 2.4, takes the abelian algebra of functions on a vector space (in our case, momentum space) to the abelian algebra of functions on its dual (in our case, position space). It establishes the link between the UIR's of the Poincaré group and the fundamental wave equations of free, relativistic quantum theory.

In this paper we have taken the first steps towards establishing an analogous link for the deformation of Poincaré symmetry in 2+1 dimensions into the quantum double of the (double cover of the) Lorentz group, or Lorentz double. We succeeded in writing some of the irreducible representations of the Lorentz double in terms of $\mathbb{C}^n$-valued functions on the deformed momentum space $SL(2,\mathbb{R})$ obeying Lorentz-covariant constraints. The details are trickier than in the Euclidean situation considered in [12], and many open problems remain. This includes the extension to massless particles and to particles with spins other than 0,1/2 and 1. In particular, it would be interesting to consider anyonic excitations in this context. They arise naturally in the context of 2+1 gravity, where the spin is quantised in units which depend on the mass [5], and can only be understood in terms of the universal cover of $SL(2,\mathbb{R})$.

In this final section, we sketch different ways of implementing the Fourier transform in the deformed setting. The approach followed in the Euclidean context in [5] is based on a generalisation of Fourier transforms to Hopf algebras, where it takes elements of a given Hopf algebra to elements of its dual Hopf algebra [15]. In the deformed theory studied here, where momentum space is $SL(2,\mathbb{R})$, the 'algebra of momenta' is the algebra $C^\infty(SL(2,\mathbb{R}))$ of smooth functions on $SL(2,\mathbb{R})$, with pointwise multiplication. This is a commutative but not co-commutative algebra. The dual 'position algebra' can be taken to be a suitable class of functions on $SL(2,\mathbb{R})$ with multiplication given by convolution (i.e. a suitable version of the group algebra) or the universal enveloping algebra $U(sl(2,\mathbb{R}))$, with generators $\hat{x}^a = i\lambda t^a$ satisfying the $sl(2,\mathbb{R})$ commutation relations

$$[\hat{x}^a, \hat{x}^b] = i\lambda\epsilon^{abc}\hat{x}_c. \tag{4.1}$$

Arguments that this non-commutative 'spin-spacetime' arises in (2+1)-dimensional quantum gravity date back to the papers [31] and [11], and also more recently arose in [16]. They are naturally accommodated in the frame work of the Lorentz double, for which, in the terminology of [32], $U(sl(2,\mathbb{R}))$ is the 'Schrödinger representation'.

In [32], the authors consider the Euclidean situation $U(su(2))$, and go on to develop a bicovariant calculus on $U(su(2))$ as well as the notion of a quantum group Fourier transform. This was used in [12] to derive non-commutative linear differential equations in the Euclidean setting.

The Lorentzian case has received less attention, but most of the results of [32] are purely algebraic and carry over from the Euclidean to the Lorentzian setting without difficulty. The key result of [32] is the formula for the partial derivatives of plane waves

$$\psi(p;\hat{x}) = \exp(-ip_a\hat{x}^a) = \exp(\lambda p_a t^a) \in SL(2,\mathbb{R}), \tag{4.2}$$

where $\hat{x} = (\hat{x}^0, \hat{x}^1, \hat{x^2})$. In our coordinates (3.3), we have $\psi(p;\hat{x}) = \mathcal{P}_3 + \lambda\mathcal{P}_a t^a$ and the four-dimensional calculus in [32] would give

$$\partial_0\psi(p;\hat{x}) = \frac{1}{\lambda}(\mathcal{P}_3 - 1), \quad \partial_a\psi(p;\hat{x}) = i\mathcal{P}_a, \quad a = 0,1,2. \tag{4.3}$$

To obtain the analogue of wave equations in $U(sl(2,\mathbb{R}))$, we expand

$$\phi(\hat{x}) = \int_{SL(2,\mathbb{R})_\pm} d^3p J(p)\,\psi(p;\hat{x})\,\tilde{\phi}\left(e^{\lambda p_a t^a}\right), \tag{4.4}$$

where $SL(2,\mathbb{R})_\pm$ is the 'inside of the lightcone', or the union

$$SL(2,\mathbb{R})_\pm = \bigcup_{\lambda m\in(0,2\pi)} C^{T+}(\lambda m) \cup \bigcup_{\lambda m\in(-2\pi,0)} C^{T-}(\lambda m), \tag{4.5}$$

and $d^3p\,J(p)$ is the Haar measure on $SL(2,\mathbb{R})$ written in terms of the coordinatisation via the exponential map (which covers precisely $SL(2,\mathbb{R})_\pm$). The expression (4.4) is formal and adapted from similar expressions for the Euclidean version used in [18]. Even in that context, it has not been defined in a mathematically rigorous fashion.

Assuming the validity of the Fourier transform (4.4), non-commutative wave equations can now easily be read off from our momentum constraints in Section 3.4. The constraint (3.20) implies the non-commutative Klein Gordon equation

$$\left(\partial_a\partial^a + \left(\frac{\sin(m\lambda/2)}{\lambda/2}\right)^2\right)\phi = 0. \tag{4.6}$$

The deformed spin 1/2 constraint (3.24) takes the from of a non-commutative Dirac equation

$$\left(i\partial_a\gamma^a - \frac{\sin(\lambda m/2)}{\lambda/2}\right)\phi = 0, \tag{4.7}$$

and the deformed Proca constraint (3.31) turns into the non-commutative Proca equation

$$\pm\partial_a[t^a,\phi] = -\frac{\sin(\lambda m/2)}{\lambda/2}\phi, \tag{4.8}$$

which implies $\partial^a\phi_a = 0$.

A different approach to the Fourier transform maps functions on a Lie group G to functions on the the dual of the Lie algebra $\mathfrak{g}^*$, equipped with a $\star$-product. This is studied in different guises in [17, 18, 21, 34, 35] for the case of G being the rotation or Lorentz group in three dimension. In the case of $SL(2,\mathbb{R})$ and with our notation, the plane waves used for the Fourier transform are

$$\psi_\star(u;x) = \exp(i\mathcal{P}^a x_a), \tag{4.9}$$

where $u \in SL(2,\mathbb{R})$ is again parametrised as in (3.3), and $x \in sl(2,\mathbb{R})^*$ as in Section 2. They are ordinary $\mathbb{C}$-valued functions on $SL(2,\mathbb{R})$. The plane waves are multiplied with a $\star$-product

$$\psi_\star(u_{(1)};x) \star \psi_\star(u_{(2)};x) = \psi_\star(u_{(1)}u_{(2)};x), \tag{4.10}$$

which implies

$$\exp(i\mathcal{P}^a_{(1)}x_a) \star \exp(i\mathcal{P}^a_{(2)}x_a) = \exp(i\mathcal{P}^a_{1\oplus 2}x_a), \tag{4.11}$$

where $P^a_{1\oplus 2}$ is defined via

$$u_{(1)}u_{(2)} = \mathcal{P}^3_{1\oplus 2} + \lambda\mathcal{P}^a_{1\oplus 2}t_a. \tag{4.12}$$

The Fourier transform with such plane wave is the conventional integral

$$\phi_\star(x) = \int_{SL(2,\mathbb{R})_\pm} du\, \psi_\star(u;x)\, \tilde{\phi}(u). \tag{4.13}$$

The momentum space constraints (3.20), (3.23) and (3.31) on $\tilde{\phi}$ imply formally the same equation as (4.6), (4.7) and (4.8) for $\phi_\star(x)$, but with ∂_a now denoting the *usual* partial derivative $\partial/\partial x^a$.

There is a third way of carrying out a Fourier transform, which uses yet another form of plane waves on the group $SL(2,\mathbb{R})$. These waves fit into the general framework discussed in [23], and apply to group elements which are in the image of the exponential map. For $u = \exp(\lambda p_a t^a)$ and $x \in sl(2,\mathbb{R})^{**} \simeq \mathbb{R}^3$, we define the $\mathbb{C}$-valued function

$$\psi_{\star\star}(p;x) = \exp(ix^a p_a). \tag{4.14}$$

The Fourier transform now reads

$$\phi_{\star\star}(x) = \int_{SL(2,\mathbb{R})_\pm} d^3p J(p)\, \psi_{\star\star}(p;x)\tilde{\phi}\left(e^{\lambda p_a t^a}\right). \tag{4.15}$$

The momentum constraints (3.20), (3.23) and (3.31) on $\tilde{\phi}$ imply equations for $\phi_{\star\star}(x)$ involving *exponentiated* differential operators. For spin 1/2, for example, the left hand side of (3.21) produces

$$e^{-\frac{\lambda}{2}\gamma^a\partial_a}\phi_{\star\star}(x). \tag{4.16}$$

The exponentiated Dirac operator appearing here was considered in a very different context by Atiyah and Moore in [22]. The authors considered difference-differential versions of several fundamental equations of physics, including the Dirac equation, allowing for advanced and retarded as well as advanced-retarded versions. For spin 1/2, this involves in an essential way the exponential of the Dirac operator. Their work stresses the relation between exponentiated differential operators and difference equations, and explores the consequences of using such equations in fundamental physics. With few exceptions [36], this point of view has not received much attention in the context of generalised Fourier transforms and quantum groups.

We have seen three quite different approaches to translating the deformed momentum constraint derived in this paper into equations in spacetime. The first involves the language of quantum groups, the second a $\star$-product on spacetime and the third exponentials of differential operators, thus leading to difference equations. The relation between the first two approaches has been investigated in the literature, see e.g. [18]. The third point may add interesting additional insights.

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Appendix

A Basis and finite-dimensional representations of $sl(2,\mathbb{R})$

In the main text, we use the the basis $\{t^a,\ a=0,1,2\}$ of $sl(2,\mathbb{R})$ with

$$t^0=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad t^1=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad t^2=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{A.1}$$

The basis elements satisfy

$$t^at^b=-\frac{1}{4}\eta^{ab}\mathrm{id}+\frac{1}{2}\epsilon^{abc}t_c, \tag{A.2}$$

where id denotes the 2×2 identity matrix. As a result we have the commutation relations

$$[t^a,t^b]=\epsilon^{abc}t_c \tag{A.3}$$

and the anticommutation relations

$$\{t^a,t^b\}=t^at^b+t^bt^a=-\frac{1}{2}\eta^{ab}\mathrm{id}. \tag{A.4}$$

Finally, we note the orthogonality relations

$$-2\mathrm{tr}(t^a t^b) = \eta^{ab}. \tag{A.5}$$

The representation theory of $sl(2,\mathbb{R})$ is best studied in terms of raising and lowering operators

$$H = t^1, \qquad E_+ = t^2 + t^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = t^2 - t^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{A.6}$$

with commutation relations

$$[H, E_+] = E_+, \qquad [H, E_-] = -E_-, \qquad [E_+, E_-] = 2H. \tag{A.7}$$

It is well-known [1] that the finite-dimensional representations of the Lie algebra $sl(2,\mathbb{R})$ are parametrised by $j \in \frac{1}{2}(\mathbb{N} \cup 0)$. For each value of j there is a unique irreducible representation ρ^j on $V_j \simeq \mathbb{C}^{2j+1}$. The standard basis $\{w_j, w_{j-1}, \ldots w_{1-j}, w_{-j}\}$ of V_j is such that

$$\rho^j(H)w_k = kw_k, \qquad \rho^j(E_-)w_k = (j+k)w_{k-1}, \qquad \rho^j(E_+)w_k = (j-k)w_{k+1} \tag{A.8}$$

These representations are not unitary. Only $\rho^j(t^0)$ has imaginary eigenvalues and exponentiates to a unitary matrix. In the main text we work with an eigenbasis

$$\{|j,j\rangle, |j,j-1\rangle, \ldots |j,1-j\rangle, |j,-j\rangle\} \tag{A.9}$$

of $\rho^j(t^0)$ satisfying $\rho^j(t^0)|j,k\rangle = ik|j,k\rangle$.

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