HU-EP-14/11
AEI-2014-007

A Shortcut to General Tree-level Scattering Amplitudes in $\mathcal{N} = 4$ SYM via Integrability

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Abstract

We combine recent applications of the two-dimensional quantum inverse scattering method to the scattering amplitude problem in four-dimensional $\mathcal{N} = 4$ Super Yang-Mills theory. Integrability allows us to obtain a general, explicit method for the derivation of the Yangian invariants relevant for tree-level scattering amplitudes in the $\mathcal{N} = 4$ model.

1 Introduction

For now many years there has been considerable interest in the integrable properties of planar gauge theories. Of special importance as the primary playground for testing ideas of integrability is the planar $\mathcal{N} = 4$ super Yang-Mills (SYM) model, the unique, maximally supersymmetric theory in four dimensions, for a comprehensive review see [1]. Integrability has already proven to be a good tool for calculating many observables for that model, including the set of anomalous dimensions of composite operators as well as certain families of structure constants. The former are believed to be understood at any finite value of the coupling constant, the most advanced method for calculating them taking the form of a "Quantum Spectral Curve" [2]. Recently, the attention of most workers in the field turned to the application of integrability methods to further quantities of interest, such as the expectation values of Wilson loops as well as scattering amplitudes. At strong coupling it already turned out to be very useful, allowing to recast the leading part of scattering amplitudes in terms of a Y-system [3]. There have also been impressive advances towards the exact computation of Wilson loops and of amplitudes at any coupling [4]. For example, the non-perturbative answer for expectation values of polygonal Wilson loops was reformulated as a sum over infinitely many particle contributions, which are, in principle, accessible via an asymptotic Bethe ansatz at any coupling. This provides a somehow orthogonal expansion compared to perturbation theory, giving rise to new predictions for the all-loop answers. There have also been important advances in understanding amplitudes using integrability at weak coupling. At tree-level the Yangian symmetry of amplitudes was proven in [5], which combined the invariance under explicit superconformal transformations of the model with its hidden counterpart – a second, dual superconformal symmetry [6]. This Yangian invariance, underlying a large class of rational two-dimensional integrable models, usually does not provide an immediate tool for calculations. However, it allows for a powerful approach termed the Quantum Inverse Scattering Method (QISM), where one constructs a family of operators in involution.

For the tree-level amplitudes, the first step towards the application of the QISM was done in [7], where the crucial notion of a spectral parameter was introduced to the scattering amplitude problem in $\mathcal{N} = 4$ SYM, see also [8]. This parameter found an interesting interpretation as a deformed, and thus in general unphysical, particle helicity. With the use of on-shell diagrams [9], providing a solution to the BCFW recursion relation [10], it allowed to deform tree-level amplitudes in such a way that they satisfy (generalized) Yang-Baxter equations. The latter are often interpreted as the quintessence of quantum integrable models. A full classification of Yangian invariants was also found there, based on a deep relation between on-shell diagrams and permutations due to Postnikov [11]. This relation to permutations was further studied in [12], and will be an essential ingredient in the construction proposed in this paper. The next steps in putting scattering amplitudes into the QISM framework were independently performed in [13,14], where it was proposed to study certain auxiliary spin-chain-like monodromies built from local Lax-operators. In this approach the amplitudes are found as "eigenstates" of these monodromies. The monodromies depend on an "auxiliary" spectral parameter, while the spectral parameters of [7,8] are encoded as inhomogeneities of the Lax operators. The amplitudes do not depend on the auxiliary spectral parameter, which is a key feature of the QISM. In [13] most details were given for a toy version of scattering amplitudes, where the complexified superconformal algebra $\mathfrak{gl}(4|4)$ was simplified to the $\mathfrak{gl}(2)$ case. It was then shown how to obtain the Yangian invariants from the monodromy matrix eigenproblem. Applying a modified version of the Algebraic Bethe Ansatz, this led to a system of Bethe equations for Yangian invariants. A different, more direct, and very powerful method, also based on the monodromy eigenproblem, was proposed in [14]. However, neither in [13] nor in [14] a systematic classification of Yangian invariants was provided. In this paper, we would like to fill this gap, and detail an integrability-based construction method for all Yangian invariants relevant to the tree-level scattering amplitudes in $\mathcal{N}=4$ SYM. As a byproduct, some interesting relations between the techniques in [13] and [14] as well as the observations in [12] will emerge.

The paper is organized as follows. In section 2 we begin by recapitulating some basic results of [13] and [14], and provide a link between the two approaches. The link helps to understand how to systematically generalize the powerful construction method of [14] to general Yangian invariants. This generalized construction is provided at the end of the section. As in [13], we will, for pedagogical reasons, mostly restrict the discussion in section 2 to $\mathfrak{gl}(2)$ or certain closely related compact representations of $\mathfrak{gl}(N|M)$. In section 3 we explain how the construction of the previous section generalizes, with small changes, to the problem of deformed $\mathfrak{psl}(4|4)$ invariant tree-level scattering amplitudes in $\mathcal{N}=4$ SYM. In section 4 we illustrate how our method works for particular examples, up to five external particles. We end with a summary and outlook.

2 Details of Construction

2.1 Introductory Remarks

The purpose of this paper is the systematic classification of Yangian invariants relevant for the tree-level scattering amplitudes of $\mathcal{N}=4$ SYM. Yangian invariance can be defined in a very compact form as a system of eigenvalue problems for the elements of a suitable monodromy matrix $M(u)$, cf. [13],

$$M_{ab}(u)|\Psi\rangle = \delta_{ab}\,|\Psi\rangle\,. \tag{1}$$

We are looking here for eigenvectors $|\Psi\rangle$ that are elements of the space $V=V_1\otimes\ldots\otimes V_n$ with V_i being a representation space of a particular representation of $\mathfrak{gl}(N|M)$. The representations we are interested in have the property that they can be built using a single family of harmonic oscillators transforming in the fundamental representation of $\mathfrak{gl}(N|M)$. In order to make our discussion more transparent, we focus first on the $\mathfrak{gl}(2)$ algebra and consider only compact representations. Later on we proceed to the general problem with emphasis on the case $N|M=4|4$, relevant for the $\mathcal{N}=4$ SYM amplitudes.

Following [13] we distinguish two oscillator realizations of the $\mathfrak{gl}(2)$ algebra,

$$J_{ab}=+\bar{\mathbf{a}}_a\mathbf{a}_b \qquad \text{with} \qquad [\mathbf{a}_a,\bar{\mathbf{a}}_b]=\delta_{ab}\,, \qquad \mathbf{a}_a|0\rangle=0\,, \tag{2}$$

$$\bar{J}_{ab}=-\bar{\mathbf{b}}_b\mathbf{b}_a \qquad \text{with} \qquad [\mathbf{b}_a,\bar{\mathbf{b}}_b]=\delta_{ab}\,, \qquad \mathbf{b}_a|\bar{0}\rangle=0\,, \tag{3}$$

where the fundamental indices a,b take the values $1,2$. We call (2) a *symmetric realization* and (3) a *dual realization*. The generators J_{ab} and $\bar{J}_{ab}$ act on the states obtained by applying the creation operators $\bar{\mathbf{a}}_a$ and $\bar{\mathbf{b}}_a$ to their respective Fock vacua $|0\rangle$ and $|\bar{0}\rangle$. The infinite-dimensional vector space spanned by these states decomposes into finite-dimensional representation spaces V_s and $\bar{V}_s$ of homogeneous polynomials of degree s in the creation operators. For each degree there is a highest weight state and the representation is labeled by the positive integer s which is an eigenvalue of one of the Cartan elements,

$$|\mathrm{hws}\rangle=(\bar{\mathbf{a}}_1)^s|0\rangle\,, \qquad J_{aa}|\mathrm{hws}\rangle = s\,\delta_{a1}|\mathrm{hws}\rangle\,, \tag{4}$$

$$\overline{|\mathrm{hws}\rangle}=(\bar{\mathbf{b}}_2)^s|\bar{0}\rangle\,, \qquad \bar{J}_{aa}\overline{|\mathrm{hws}\rangle} = -s\,\delta_{a2}\overline{|\mathrm{hws}\rangle}\,. \tag{5}$$

It is sometimes convenient to notationally hide the difference between the two types of oscillators $\mathbf{a}$ and $\mathbf{b}$, and to instead use only one type $\mathbf{w}$ satisfying

$$[\mathbf{w}_a,\overline{\mathbf{w}}_b]=\delta_{ab}\,. \tag{6}$$

This is just a relabeling, and it looks as follows:

$$\bar{\mathbf{a}}_a \leftrightarrow \overline{\mathbf{w}}_a\,, \qquad \mathbf{a}_a \leftrightarrow \mathbf{w}_a\,, \qquad \mathbf{w}_a|0\rangle = 0\,, \tag{7}$$

$$\bar{\mathbf{b}}_a \leftrightarrow -\mathbf{w}_a\,, \qquad \mathbf{b}_a \leftrightarrow \overline{\mathbf{w}}_a\,, \qquad \overline{\mathbf{w}}_a|\bar{0}\rangle = 0\,. \tag{8}$$

Note that we are not spelling out any conjugation properties of our oscillators, nor computing any norms, therefore there is no problem with negative norm states from (8).

Written in terms of these variables the generators and highest weight states of V_s and $\bar{V}_s$ become, respectively,

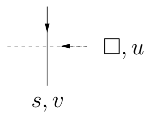
$$J_{ab} = \overline{\mathbf{w}}_a \mathbf{w}_b\,, \qquad |\mathrm{hws}\rangle = (+\overline{\mathbf{w}}_1)^s|0\rangle\,, \tag{9}$$

$$\bar{J}_{ab} = \mathbf{w}_b \overline{\mathbf{w}}_a\,, \qquad \overline{|\mathrm{hws}\rangle} = (-\mathbf{w}_2)^s|\bar{0}\rangle\,. \tag{10}$$

In the following we will use both notations, as is convenient.

The space $V = V_1 \otimes \ldots \otimes V_n$ of (1) can then be built out of factors V_i which are of the type $V_{s_i} := \mathrm{span}\{\overline{\mathbf{w}}^s|0\rangle\}$ or $\bar{V}_{s_i} := \mathrm{span}\{(-\mathbf{w})^s|\bar{0}\rangle\}$. We may think of V as the quantum space of a compact spin chain. The monodromy matrix $M(u)$ of this spin chain is defined on $V_\square \otimes V$, where $V_\square$ denotes an auxiliary space in the fundamental representation $\square$ of $\mathfrak{gl}(2)$. The monodromy matrix can be written with the help of Lax operators $L(u,v)$ and $\bar{L}(u,v)$ describing the "interaction" of the auxiliary space with, respectively, the spaces V_s and $\bar{V}_s$. We use similar Lax operators as in [13]. For the symmetric representations we take

$$L(u,v) = 1 + (u-v)^{-1} \sum_{a,b} e_{ab}\, \bar{\mathbf{a}}_b \mathbf{a}_a = \tag{11}$$



while for dual ones

$$\bar{L}(u,v) = 1 - (u-v-1)^{-1} \sum_{a,b} e_{ab}\, \bar{\mathbf{b}}_a \mathbf{b}_b\,. \tag{12}$$

In both cases the elementary matrices e_{ab} with matrix elements $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$ act on the auxiliary space. Compared to [13] we dropped a non-trivial normalization factor of the Lax operators and we introduced a shift of the parameter v in (12). This shift allows us to express both types of Lax operators in terms of

$$\mathcal{L}(u,v) = u - v + \sum_{a,b} e_{ab}\, \overline{\mathbf{w}}_b \mathbf{w}_a\,. \tag{13}$$

Using (7) and (8), respectively, we obtain

$$L(u,v) = (u-v)^{-1}\mathcal{L}(u,v)\,, \tag{14}$$

$$\bar{L}(u,v) = (u-v-1)^{-1}\mathcal{L}(u,v)\,. \tag{15}$$

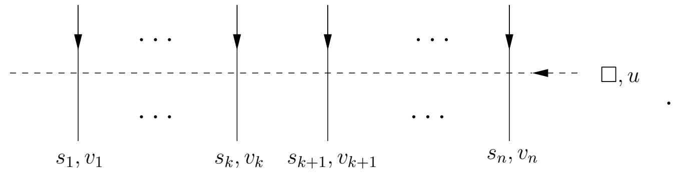
In order to render our discussion clearer, we take a spin chain with a very particular quantum space. The first k sites are represented with the use of the dual realization (3), and the last $n-k$ sites with the symmetric realization (2), i.e.

$$V = \bar{V}_{s_1} \otimes \cdots \otimes \bar{V}_{s_k} \otimes V_{s_{k+1}} \otimes \cdots \otimes V_{s_n}\,. \tag{16}$$

Finally, the monodromy matrix reads

$$M(u) = \bar{L}_1(u, v_1) \dots \bar{L}_k(u, v_k) L_{k+1}(u, v_{k+1}) \dots L_n(u, v_n) \tag{17}$$

$$= \tag{18}$$



Note that the monodromy $M(u)$ depends only on the spectral parameters u and v_i as well as on n and k, but not on the representation labels s_i. As we did already in (11), we have nevertheless attached these labels in the graphical depiction, to indicate the nature of the quantum space the monodromy is acting on. This monodromy provides a realization of the Yangian $Y(\mathfrak{gl}(2))$. Each Lax operator itself is an evaluation realization of the Yangian with evaluation parameter v_i, which is called an inhomogeneity in spin chain language.

2.2 Quantum Inverse Scattering Method for Yangian Invariants

It was shown in [13] that one can construct eigenvectors $|\Psi\rangle$ satisfying (1) using the Quantum Inverse Scattering Method (QISM), and in particular apply the Algebraic Bethe Ansatz technique. Most details of this construction were given for the simplest case of compact $\mathfrak{gl}(2)$ representations. In this case one writes the monodromy matrix (17) explicitly as a matrix acting in the fundamental auxiliary space,

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} . \tag{19}$$

The operators $A(u), B(u), C(u)$ and $D(u)$ act only on the quantum space V. The standard procedure is then to construct a "reference" state $|\Omega\rangle$ satisfying

$$C(u)|\Omega\rangle = 0 , \qquad A(u)|\Omega\rangle = \alpha(u)|\Omega\rangle , \qquad D(u)|\Omega\rangle = \delta(u)|\Omega\rangle , \tag{20}$$

where $\alpha(u)$ and $\delta(u)$ are scalar functions depending on the representations labels s_i and the inhomogeneities v_i of the monodromy. The reference state $|\Omega\rangle$ is realized as the tensor product of the highest weight states at each spin chain site. As already pointed out in (16), we focus on the case where the first k highest weight states are of the form (5) and the remaining $n-k$ highest weight states are as given in (4),

$$|\Omega\rangle = \bigotimes_{i=1}^{k} \overline{|\mathrm{hws}\rangle}_i \bigotimes_{i=k+1}^{n} |\mathrm{hws}\rangle_i = \prod_{i=1}^{k} (\bar{\mathbf{b}}_2^i)^{s_i} \prod_{i=k+1}^{n} (\bar{\mathbf{a}}_1^i)^{s_i} \, |\mathbf{0}\rangle \tag{21}$$

with

$$|\mathbf{0}\rangle = \underbrace{|\bar{0}\rangle \otimes \ldots \otimes |\bar{0}\rangle}_{k} \otimes \underbrace{|0\rangle \otimes \ldots \otimes |0\rangle}_{n-k} . \tag{22}$$

In order to construct Yangian invariants one proceeds to define the Bethe vectors

$$|\Psi\rangle_{n,k} = B(u_1) \dots B(u_K) |\Omega\rangle . \tag{23}$$

The Bethe roots u_j and the parameters s_i and v_i of the monodromy have to satisfy the first order Baxter equations

$$\frac{Q(u)}{Q(u+1)} = \delta(u) \qquad \text{with} \qquad Q(u) = \prod_{j=1}^{K}(u-u_j)\,, \tag{24}$$

$$\alpha(u)\delta(u-1) = 1\,. \tag{25}$$

These equations impose stronger conditions than the usual Bethe equations and they guarantee the Yangian invariance of the Bethe vector. For the monodromy (17) the functions $\alpha(u)$ and $\delta(u)$ can be worked out explicitly. This turns (24) into an equation[1] determining the Bethe roots u_j

$$\frac{Q(u)}{Q(u+1)} = \prod_{i=1}^{k} \frac{u-v_i-s_i-1}{u-v_i-1}\,. \tag{26}$$

In addition, (25) becomes

$$\prod_{i=1}^{k} \frac{u-v_i-s_i-2}{u-v_i-2} \prod_{i=k+1}^{n} \frac{u-v_i+s_i}{u-v_i} = 1\,, \tag{27}$$

which constrains the representation labels s_i and inhomogeneities v_i.

Equations (26) and (27) were solved explicitly in [13] for some sample invariants. The corresponding Bethe vectors were evaluated for small integer values of the representation labels s_i. Up to overall normalization factors this led to the invariants

$$|\Psi\rangle_{2,1} = (\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^2)^{s_2}|\mathbf{0}\rangle\,, \tag{28}$$

$$|\Psi\rangle_{3,1} = (\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^2)^{s_2}(\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^3)^{s_3}|\mathbf{0}\rangle\,, \tag{29}$$

$$|\Psi\rangle_{3,2} = (\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^3)^{s_1}(\bar{\mathbf{b}}^2 \cdot \bar{\mathbf{a}}^3)^{s_2}|\mathbf{0}\rangle\,, \tag{30}$$

$$\begin{aligned}|\Psi\rangle_{4,2} = \sum_{k=0}^{\infty} &\frac{1}{(s_1-k)!(s_2-k)!k!\Gamma(z-s_1+k+1)} \\ &\cdot (\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^3)^{s_1-k}(\bar{\mathbf{b}}^2 \cdot \bar{\mathbf{a}}^4)^{s_2-k}(\bar{\mathbf{b}}^2 \cdot \bar{\mathbf{a}}^3)^{k}(\bar{\mathbf{b}}^1 \cdot \bar{\mathbf{a}}^4)^{k}|\mathbf{0}\rangle\,,\end{aligned} \tag{31}$$

where $\bar{\mathbf{b}}^i \cdot \bar{\mathbf{a}}^i = \sum_a \bar{\mathbf{b}}^i_a \bar{\mathbf{a}}^i_a$. Surprisingly, perhaps, even though one acts in general with a large number of suitable operators $B(u_j)$ on the reference state $|\Omega\rangle$, the final result looks very simple for the first few invariants $|\Psi\rangle_{2,1}$, $|\Psi\rangle_{3,1}$ and $|\Psi\rangle_{3,2}$. As for the considerably more involved "harmonic R-matrix" $|\Psi\rangle_{4,2}$, note the following two features. Firstly, for finite dimensional representations, the sum in (31) is actually finite, of course. Secondly, a complex spectral parameter z appears.

The expressions (28)-(30) are reminiscent of the ones found in [14], where the authors studied a condition for Yangian invariance[2] similar to (1). Even though the method used there differs from the standard Algebraic Bethe Ansatz approach explained above, the idea bears many similarities. The construction in [14] is valid for any algebra $\mathfrak{gl}(N|M)$. Let us explain it here in a few steps, restricting to the $\mathfrak{gl}(2)$ case for simplicity. Now it is convenient to use (7) and (8) to express the $\mathbf{a}$, $\mathbf{b}$ oscillators in terms of the $\mathbf{w}$ oscillators. As opposed to the reference state $|\Omega\rangle$, the authors of [14] start from the Fock vacuum $|\mathbf{0}\rangle$ in the quantum space, defined in (22),

[1]Equations (26) and (27) are slightly different from the ones in [13]. This difference originates from the shift of the inhomogeneity in the dual Lax operator (12).

[2]Up to a very non-trivial normalization of the monodromy, which however will not play a role in this paper.

which corresponds to a trivial singlet representation. They then look for Yangian invariants of the form[3]

$$|\Psi\rangle = \mathcal{B}_{i_1 j_1}(\bar{u}_1)\dots\mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle\,, \tag{32}$$

with

$$\mathcal{B}_{ij}(u) = (-\overline{\mathbf{w}}^j\cdot\mathbf{w}^i)^u \tag{33}$$

and $i,j=1,\dots n$. As an example let us consider the invariants in (28), (29), (30) and notice that

$$|\Psi\rangle_{2,1} \propto \mathcal{B}_{12}(s_2)|\mathbf{0}\rangle\,, \tag{34}$$
$$|\Psi\rangle_{3,1} \propto \mathcal{B}_{12}(s_2)\mathcal{B}_{13}(s_3)|\mathbf{0}\rangle\,, \tag{35}$$
$$|\Psi\rangle_{3,2} \propto \mathcal{B}_{13}(s_1)\mathcal{B}_{23}(s_2)|\mathbf{0}\rangle\,. \tag{36}$$

At first, it seems impossible to also write down the four-point invariant in such a simple form since it is given in (31) as a complicated sum. In addition, it depends on the complex parameter z. However, using the formal algebraic commutation relations for the oscillators, one easily proves

$$|\Psi\rangle_{4,2} \propto \mathcal{B}_{12}(z)\mathcal{B}_{23}(s_1)\mathcal{B}_{12}(s_1-z)\mathcal{B}_{24}(s_2)|\mathbf{0}\rangle\,, \tag{37}$$

or more explicitly

$$|\Psi\rangle_{4,2} = \frac{1}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(z+1)}(\bar{\mathbf{b}}^1\cdot\mathbf{b}^2)^z(\bar{\mathbf{b}}^2\cdot\bar{\mathbf{a}}^3)^{s_1}(\bar{\mathbf{b}}^1\cdot\mathbf{b}^2)^{s_1-z}(\bar{\mathbf{b}}^2\cdot\bar{\mathbf{a}}^4)^{s_2}|\mathbf{0}\rangle\,. \tag{38}$$

The careful reader might be a bit puzzled here since in (37), (38) the spectral parameter z is a complex number. Let us therefore give some details on the purely algebraic derivation of (31) from (38). We interpret the action of the two rightmost operators as

$$(\bar{\mathbf{b}}^1\cdot\mathbf{b}^2)^{s_1-z}(\bar{\mathbf{b}}^2\cdot\bar{\mathbf{a}}^4)^{s_2}|\mathbf{0}\rangle = \frac{\Gamma(s_2+1)}{\Gamma(s_2-s_1-z+1)}(\bar{\mathbf{b}}^2\cdot\bar{\mathbf{a}}^4)^{s_2-s_1+z}(\bar{\mathbf{b}}^1\cdot\bar{\mathbf{a}}^4)^{s_1-z}|\mathbf{0}\rangle\,. \tag{39}$$

Then the forth operator $(\bar{\mathbf{b}}^1\cdot\mathbf{b}^2)^z$ produces the sum of terms in (31) by means of the generalized Leibniz rule. Finally, all complex powers of oscillators disappear and we obtain the invariant (31) for compact representations. This illustrates that we should not require in (33) that u is a non-negative integer; we take it as a general complex number $u\in\mathbb{C}$.

In [14] the authors studied a few more examples beyond (34), (35) and (36). However, no general construction for arbitrary n and k was proposed. In particular, no classification of such invariants was provided. In practical terms, we would like to find a prescription which tells us which (i_p, j_p) and $\bar{u}_p$ in (32) have to be taken in order to obtain the Yangian invariant $|\Psi\rangle_{n,k}$ for a given n and k. We will now fill this gap. It will turn out that there are many distinct solutions for fixed n and k, but that all of them are in one-to-one correspondence with permutations.

Before we proceed to this general construction let us come back for a moment to the first order Baxter equation (25) spelled out in (27). There exists a redefinition of the inhomogeneities which turns this equation into a much simpler form. Let us introduce

$$v_i' = \begin{cases} v_i + \frac{s_i}{2} + 2 & \text{for} \quad i = 1,\dots,k\,, \\ v_i - \frac{s_i}{2} & \text{for} \quad i = k+1,\dots,n\,. \end{cases} \tag{40}$$

In addition we define

$$\mathfrak{s}_i = \begin{cases} -s_i & \text{for} \quad i = 1,\dots,k\,, \\ s_i & \text{for} \quad i = k+1,\dots,n\,, \end{cases} \tag{41}$$

[3]We changed the notation from $R_{ij}(u)$ to $\mathcal{B}_{ij}(u)$ compared to [14], and furthermore changed the normalization.

and

$$v_i^{\pm} = v_i' \pm \tfrac{s_i}{2}\,. \tag{42}$$

In term of these variables (27) reads

$$\prod_{i=1}^{n}(u - v_i^{+}) = \prod_{i=1}^{n}(u - v_i^{-})\,. \tag{43}$$

Both sides of this equation are polynomials in u of degree n. Since the set of roots is unique for a given polynomial we conclude that for each v_i^+ there exists j such that $v_i^+ = v_j^-$. Altogether, it means that with each solution to (27) we can associate a permutation σ such that

$$v_{\sigma(i)}^{+} = v_i^{-}\,. \tag{44}$$

This provides a systematic construction of the solutions of (25), a problem proposed in [13]. We restrict our representations from now on and take generic values for the s_i, namely $s_i \neq 0$, then $\sigma(i) \neq i$, i.e. σ does not have fixed points.

Even though the above analysis was done for the case of $\mathfrak{gl}(2)$ we claim that it continues to be valid for any realization of the $\mathfrak{gl}(N|M)$ algebra in terms of a single family of oscillators. In particular this includes the $\mathfrak{gl}(4|4)$ case relevant for the $\mathcal{N} = 4$ SYM amplitudes. It is not an accident that on-shell diagrams are also parametrized by permutations. As was shown in [8] each on-shell diagram can be deformed by allowing non-physical particle helicities. While a generic deformation of the amplitude is not Yangian invariant, there exists a simple criterion enforcing it. It is sufficient to restrict to deformations such that all possible cluster transformations leave the appropriate integration measure invariant, see [8] for details. These constraints were studied in details in [12] and it was shown there that they can be naturally recast in the form (44). We will show that all such deformations can be obtained from (32) with an appropriate set of indices (i_p, j_p) and parameters $\bar{u}_p$, leading to a complete classification of Yangian invariants relevant for the tree-level amplitudes.

2.3 General Construction of Yangian Invariants

In this section we will construct a Yangian invariant $|\Psi\rangle_\sigma$ for each permutation σ using the operators $\mathcal{B}_{ij}(u)$ introduced in (33). The procedure is closely related to the combinatorics of permutations and scattering amplitudes in chapter 2 of [9].

Let σ be a permutation of n elements and k be the number of elements i with $\sigma(i) < i$. We decompose σ into transpositions $T_p = (i_p, j_p)$, which exchange the two elements $i_p < j_p$

$$\sigma = T_P \circ \ldots \circ T_2 \circ T_1 = (i_P, j_P) \cdots (i_2, j_2)(i_1, j_1)\,. \tag{45}$$

This (non-unique) decomposition is assumed to be *minimal*, meaning that there exists no other decomposition of σ into a smaller number of transpositions. In addition, the transpositions $T_p = (i_p, j_p)$ are required to be *adjacent*, c.f. [9], in the sense that

$$i_q, j_q \notin \{i_p + 1, \ldots, j_p - 1\} \qquad \text{for} \qquad q > p\,. \tag{46}$$

Note that this “generalized adjacency” of the indices i_p, j_p of the transposition T_p means that the condition $i_p + 1 = j_p$ is relaxed to $i_p + 1 < j_p$, iff all transpositions $T_q = (i_q, j_q)$ applied after T_p are restricted by (46). For practical purposes, the Mathematica program in [15] may be used to obtain such a decomposition for a given permutation σ.

We claim that a Yangian invariant is now constructible from the ansatz (32), which reads

$$|\Psi\rangle_\sigma = \mathcal{B}_{i_1 j_1}(\bar{u}_1)\cdots\mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle\,. \tag{47}$$

Here the indices of the operators are precisely the arguments of the transpositions in (45), and the number k defined above fixes the vacuum $|\mathbf{0}\rangle$ to be (22). The parameters $\bar{u}_p$, akin to Bethe roots in the Algebraic Bethe Ansatz, will be given below. Let us briefly outline the idea of how to show the Yangian invariance of this ansatz. The main tool is the "intertwining" relation

$$\mathcal{L}_i(u, y_i + \mathfrak{C}_i)\mathcal{L}_j(u, y_j + \mathfrak{C}_j)\mathcal{B}_{ij}(y_i - y_j) = \mathcal{B}_{ij}(y_i - y_j)\mathcal{L}_i(u, y_j + \mathfrak{C}_i)\mathcal{L}_j(u, y_i + \mathfrak{C}_j)\,, \tag{48}$$

with $\mathfrak{C}_i = \overline{\mathbf{w}}^i \cdot \mathbf{w}^i$ being the number operator of the oscillators at site i. This equation is easily verified by a direct computation, and may be depicted as in Figure 1. It is similar to, but distinct from a Yang-Baxter equation: Note that the indices on the Lax operators $\mathcal{L}_i$ and the number operators $\mathfrak{C}_i$ are not permuted, while the ones on the inhomogeneities y_i are. A closely related equation was introduced in [14]. One now expresses the monodromy $M(u)$ given in (17)

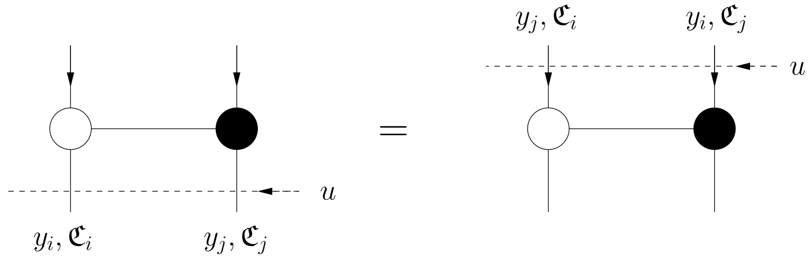


Figure 1: Fundamental relation (48) for the construction of Yangian invariants using the intertwining operators $\mathcal{B}_{ij}(y_i - y_j)$.

entirely in terms of Lax operators $\mathcal{L}_i(u, v_i)$ found in (13), and then acts with $M(u)$ on the ansatz (47). Then one uses (48) to commute all operators $\mathcal{B}_{i_p j_p}(\bar{u}_p)$ inside $|\Psi\rangle_\sigma$ to the left side of the monodromy $M(u)$. This constrains the parameters $\bar{u}_p$ in terms of the inhomogeneities v_i of the monodromy. It also modifies the distribution of inhomogeneities of the monodromy. However, one notices that this modified monodromy acts diagonally on the vacuum $|\mathbf{0}\rangle$. This shows that the ansatz (47) satisfies the invariance condition (1) up to a factor. The latter conveniently turns out to be unity in our conventions.

To implement this procedure we replace the inhomogeneities v_i of the monodromy by new variables[4] y_i defined in terms of the representation labels $\mathfrak{s}_i$, cf. (41)

$$v_i|\Psi\rangle_\sigma = (y_i + \mathfrak{C}_i)|\Psi\rangle_\sigma = \begin{cases} (y_i + \mathfrak{s}_i - N + M)|\Psi\rangle_\sigma & \text{for} \quad i = 1, \ldots, k\,, \\ (y_i + \mathfrak{s}_i)|\Psi\rangle_\sigma & \text{for} \quad i = k+1, \ldots, n\,. \end{cases} \tag{49}$$

For the convenience of the reader, here and in the formulas in the rest of this section we have already given the correct expressions for the algebra $\mathfrak{gl}(N|M)$; to specialize back to $\mathfrak{gl}(2)$ just put $N = 2$, $M = 0$. The action of the monodromy on the ansatz (47) in these variables reads

$$M(u)|\Psi\rangle_\sigma = \prod_{i=1}^{k}\frac{1}{u - v_i - 1}\prod_{i=k+1}^{n}\frac{1}{u - v_i}\mathcal{L}_1(u, y_1 + \mathfrak{C}_1)\cdots\mathcal{L}_n(u, y_n + \mathfrak{C}_n)|\Psi\rangle_\sigma\,. \tag{50}$$

[4]These variables are related to those in (40) by $y_i = v_i' - \frac{\mathfrak{s}_i}{2}$.

The arguments of the Lax operators are already in the form of (48). To apply (48) we also have to specify the variables $\bar{u}_p$ in (47) in terms of the y_i. For this purpose we introduced the permutations

$$\tau_p = \tau_{p-1} \circ (i_p, j_p) = (i_1, j_1) \cdots (i_p, j_p) \tag{51}$$

for $p = 1, \ldots, P$. As we will see shortly, the correct choice of the $\bar{u}_p$ is

$$\bar{u}_p = y_{\tau_p(j_p)} - y_{\tau_p(i_p)} \,. \tag{52}$$

Let us explain how to use (48) to commute the operators $\mathcal{B}_{i_p j_p}(\bar{u}_p)$ inside $|\Psi\rangle_\sigma$ to the left side of the monodromy $M(u)$. For a moment we disregard the arguments $\bar{u}_p$. If the indices of an operator $\mathcal{B}_{i_p j_p}$ are adjacent, $i_p + 1 = j_p$, we can directly apply (48) to move $\mathcal{B}_{i_p j_p}$ left of the Lax operators $\mathcal{L}_{i_p}\mathcal{L}_{i_p+1}$ and then on through the entire monodromy. However, if $i_p + 1 > j_p$, this is not immediately possible because in the monodromy there are some Lax operators in between $\mathcal{L}_{i_p}$ and $\mathcal{L}_{j_p}$. In this case the decomposition into adjacent transpositions guarantees, cf. (46), that all operators $\mathcal{B}_{i_q j_q}$ to the right of $\mathcal{B}_{i_p j_p}$ do not act on the spaces labeled $i_p + 1, \ldots, j_p - 1$. Hence, the corresponding Lax operators $\mathcal{L}_{i_p+1}, \ldots, \mathcal{L}_{j_p-1}$ in the monodromy act directly on the vacuum $|\mathbf{0}\rangle$. For any Lax operator acting on the vacuum we have

$$\mathcal{L}_i(u, y_j + \mathfrak{C}_i)|\mathbf{0}\rangle = \begin{cases} (u - y_j + N - M - 1)|\mathbf{0}\rangle & \text{for} \quad i = 1, \ldots, k\,, \\ (u - y_j)|\mathbf{0}\rangle & \text{for} \quad i = k+1, \ldots, n\,. \end{cases} \tag{53}$$

Consequently, the Lax operators between $\mathcal{L}_{i_p}$ and $\mathcal{L}_{j_p}$ effectively disappear from the monodromy. This means we can apply (48) to commute the operators $\mathcal{B}_{i_p j_p}$ past the monodromy also in the case $i_p + 1 > j_p$.

Finally, we give some details on the commutation of all operators $\mathcal{B}_{i_p j_p}(\bar{u}_p)$ inside $|\Psi\rangle_\sigma$ to the left of the monodromy $M(u)$. For this computation we introduce

$$\mathcal{M}(u; y_1, \ldots, y_n) = \mathcal{L}_1(u, y_1 + \mathfrak{C}_1)\,\mathcal{L}_2(u, y_2 + \mathfrak{C}_2) \cdots \mathcal{L}_n(u, y_n + \mathfrak{C}_n)\,. \tag{54}$$

Using (48), (52) and applying (53) once the whole monodromy hits the vacuum yields

$$M(u)|\Psi\rangle_\sigma = \prod_{i=1}^{k} \frac{1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{1}{u - v_i} \mathcal{M}(u; y_1, \ldots, y_n)\mathcal{B}_{i_1 j_1}(\bar{u}_1) \cdots \mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle \tag{55}$$

$$= \prod_{i=1}^{k} \frac{1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{1}{u - v_i} \mathcal{B}_{i_1 j_1}(\bar{u}_1)\mathcal{M}(u; \ldots, y_{j_1}, \ldots, y_{i_1}, \ldots)\mathcal{B}_{i_2 j_2}(\bar{u}_2) \cdots \mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle \tag{56}$$

$$= \prod_{i=1}^{k} \frac{1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{1}{u - v_i} \mathcal{B}_{i_1 j_1}(\bar{u}_1)\mathcal{M}(u; y_{\tau_1(1)}, \ldots, y_{\tau_1(n)})\mathcal{B}_{i_2 j_2}(\bar{u}_2) \cdots \mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle\,. \tag{57}$$

Continuing this calculation, we find

$$\prod_{i=1}^{k} \frac{1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{1}{u - v_i} \mathcal{B}_{i_1 j_1}(\bar{u}_1)\mathcal{B}_{i_2 j_2}(\bar{u}_2) \cdots \mathcal{B}_{i_P j_P}(\bar{u}_P)\mathcal{M}(u; y_{\tau_P(1)}, \ldots, y_{\tau_P(n)})|\mathbf{0}\rangle \tag{58}$$

$$= \prod_{i=1}^{k} \frac{u - y_{\tau_P(i)} + N - M - 1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{u - y_{\tau_P(i)}}{u - v_i} \mathcal{B}_{i_1 j_1}(\bar{u}_1)\mathcal{B}_{i_2 j_2}(\bar{u}_2) \cdots \mathcal{B}_{i_P j_P}(\bar{u}_P)|\mathbf{0}\rangle \tag{59}$$

$$= \prod_{i=1}^{k} \frac{u - y_{\tau_P(i)} + N - M - 1}{u - v_i - 1} \prod_{i=k+1}^{n} \frac{u - y_{\tau_P(i)}}{u - v_i} |\Psi\rangle_\sigma\,. \tag{60}$$

This proves that the ansatz for $|\Psi\rangle_\sigma$ given in (47) with $\bar{u}_p$ specified in (52) satisfies the Yangian invariance condition (1) up to a factor. Let us now argue that this factor is equal to 1. One may compute the representation labels $\mathfrak{s}_i$ of the vector $|\Psi\rangle_\sigma$, leading to

$$y_{\sigma(i)} + \mathfrak{s}_{\sigma(i)} = y_i \,. \tag{61}$$

Together with the variable redefinition (49) and $\tau_P = \sigma^{-1}$ this turns the scalar factor in (60) into 1, which shows the Yangian invariance of $|\Psi\rangle_\sigma$. Notice that defining $v_i^+ = y_i + \mathfrak{s}_i$ and $v_i^- = y_i$ turns (61) into (44).

3 Generalization to Superalgebras and Scattering Amplitudes

The general construction of Yangian invariants in the previous section easily generalizes to compact one-row representations of the superalgebra $\mathfrak{gl}(N|M)$. Moreover, we will argue that it also applies to the non-compact representations of $\mathfrak{gl}(4|4)$ relevant to tree-level scattering amplitudes of $\mathcal{N}=4$ SYM.

Let us first focus on the compact representations of $\mathfrak{gl}(N|M)$. We therefore replace the oscillators of section 2 by superoscillators. Generalizing (2) and (3), we obtain two families of realizations with generators

$$J_{ab} = +\bar{\mathbf{a}}_a \mathbf{a}_b \qquad \text{with} \qquad [\mathbf{a}_a, \bar{\mathbf{a}}_b\} = \delta_{ab} \,, \qquad \mathbf{a}_a|0\rangle = 0 \,, \tag{62}$$

$$\bar{J}_{ab} = -(-1)^{b+ab}\bar{\mathbf{b}}_b \mathbf{b}_a \qquad \text{with} \qquad [\mathbf{b}_a, \bar{\mathbf{b}}_b\} = \delta_{ab} \,, \qquad \mathbf{b}_a|\bar{0}\rangle = 0 \,, \tag{63}$$

where the oscillators labeled by $a,b = 1,\ldots,N$ are bosonic and those with $a,b = N+1,\ldots,N+M$ are fermionic, and the exponent of -1 is to be understood as the degree of the corresponding index. As in the purely bosonic case it is convenient to relabel the oscillators by introducing yet another oscillator family $\mathbf{w}$ as in (7) and (8),

$$\bar{\mathbf{a}}_a \leftrightarrow \overline{\mathbf{w}}_a \,, \qquad \mathbf{a}_a \leftrightarrow \mathbf{w}_a \,, \qquad \mathbf{w}_a|0\rangle = 0 \,, \tag{64}$$

$$\bar{\mathbf{b}}_a \leftrightarrow -(-1)^a\, \mathbf{w}_a \,, \qquad \mathbf{b}_a \leftrightarrow \overline{\mathbf{w}}_a \,, \qquad \overline{\mathbf{w}}_a|\bar{0}\rangle = 0 \,. \tag{65}$$

The Lax operator (13) then generalizes to a graded one,

$$\mathcal{L}(u,v) = u - v + \sum_{a,b}(-1)^b e_{ab}\, \overline{\mathbf{w}}_b \mathbf{w}_a \,, \tag{66}$$

whence the $\mathfrak{gl}(N|M)$ versions of $L(u,v)$ and $\bar{L}(u,v)$ may be deduced by imposing (14) and (15).

This is the setup we need in order to adapt the general construction method of section 2.3 to the superalgebra case. Importantly, the classification of the invariants using permutations and the decomposition of these permutations into transpositions does not make any reference to the specific symmetry algebra, let it be $\mathfrak{gl}(2)$ or $\mathfrak{gl}(N|M)$, nor the representation of interest. The form of the vacuum $|\mathbf{0}\rangle$ in (22), the operators $\mathcal{B}_{ij}(u)$ specified in (33), as well as the key relation (48) remain unchanged. Furthermore, all formulas in section 2.3 already contain $N-M$ at the appropriate places, and already pertain to the $\mathfrak{gl}(N|M)$ case. Hence the construction of Yangian invariants carries over to the case of compact $\mathfrak{gl}(N|M)$ representations, where the generators (62) and (63) are given in terms of a single family of superoscillators.

For the moment we stay with these compact representations, and discuss an integral realization of the operators $\mathcal{B}_{ij}(u)$. Soon this realization will be the basis for a formal transition to scattering amplitudes. As was observed in [14] (see also [13]), one may formally rewrite the intertwiner $\mathcal{B}_{ij}(u)$ in (33) for arbitrary complex numbers[5] $u \in \mathbb{C}$ as

$$\mathcal{B}_{ij}(u) = (-\overline{\mathbf{w}}^j \cdot \mathbf{w}^i)^u = -\frac{\Gamma(u+1)}{2\pi i} \int_{\mathcal{C}} \frac{d\alpha}{(-\alpha)^{1+u}} e^{\alpha\, \overline{\mathbf{w}}^j \cdot \mathbf{w}^i} \,, \tag{67}$$

where the Hankel contour $\mathcal{C}$ goes counterclockwise around the cut (for $u \notin \mathbb{Z}$) of the function $(-\alpha)^{1+u}$ defined to lie between its branch points at 0 and ∞. Now we make a further notational change, and realize the $\mathbf{w}$ oscillators in terms of "supertwistor" variables $\mathcal{W}$

$$\overline{\mathbf{w}}_a \leftrightarrow \mathcal{W}_a \,, \qquad \mathbf{w}_a \leftrightarrow \partial_{\mathcal{W}_a} \,, \quad |0\rangle \leftrightarrow 1 \,, \quad |\bar{0}\rangle \leftrightarrow \delta^{N|M}(\mathcal{W}) \,. \tag{68}$$

In this realization the vacuum state (22) in the construction of invariants becomes

$$|\mathbf{0}\rangle = \prod_{i=1}^{k} \delta^{N|M}(\mathcal{W}^i) \,. \tag{69}$$

The operators $\mathcal{B}_{ij}(u)$ in (67) then read

$$\mathcal{B}_{ij}(u) = -\frac{\Gamma(u+1)}{2\pi i} \int_{\mathcal{C}} \frac{d\alpha}{(-\alpha)^{1+u}} e^{\alpha\, \mathcal{W}^j \cdot \partial_{\mathcal{W}^i}} \,. \tag{70}$$

Note that for $u = s \in \mathbb{N}_0$ the cut in the complex α-plane disappears, and the only singularity inside $\mathcal{C}$ is a pole at $\alpha = 0$. In this case, the Hankel contour may be collapsed to a circular one encircling the pole counterclockwise, and (70) simplifies to

$$\mathcal{B}_{ij}(s) = \frac{(-1)^s\, s!}{2\pi i} \oint \frac{d\alpha}{\alpha^{1+s}} e^{\alpha\, \mathcal{W}^j \cdot \partial_{\mathcal{W}^i}} = (-\mathcal{W}^j \cdot \partial_{\mathcal{W}^i})^s \,, \tag{71}$$

which is essentially the expression, here in terms of twistor variables, given in (5.11) of [13] for compact representations. We may then express, as in (5.21) in [13], e.g. (28) in twistor variables

$$|\Psi\rangle_{2,1} \propto \oint \frac{d\alpha}{\alpha^{1+s_2}} \delta^{2|0}(\mathcal{W}^1 + \alpha\, \mathcal{W}^2) \,, \tag{72}$$

where we used the representation (69) of the vacuum. However, for more complicated compact invariants such as (37), where the intertwiners depend on complex parameters, we should use the integral representations (70) employing the Hankel contours.

Now, as already pointed out in [13], with a small further modification the compact formalism is easily modified to also apply to the non-compact $\mathfrak{gl}(4|4)$ Yangian invariants appearing in the $\mathcal{N} = 4$ tree-level scattering problem, which are expressed in terms of formal Graßmannian contour integrals [9]. As the procedure for building invariants is entirely algebraic, and reality conditions, conjugation properties of the operators, and the norms of the states are never considered, the construction immediately carries over to the amplitude problem. The price one pays is merely that the contour integrations and the delta functions, which are well-defined in the compact case (c.f. appendix A of [13]) become somewhat formal in the scattering problem, see again [9]. Let us now illustrate the method in some examples.

[5] For $u = -1, -2, \ldots$ one needs to take a limit, as the Gamma function diverges and the integral tends to zero since the integrand becomes completely analytic inside $\mathcal{C}$.

4 Sample Invariants

We will illustrate our construction in the first few cases. As we elaborated before, all the invariants are labelled by permutations. Here we restrict our discussion to the permutations $\sigma_{n,k}(i) = i + k \mod n$ that are relevant for the top cells of the positive Graßmannian $G(n,k)$. For the corresponding invariants we use the shorthand notation $|\Psi\rangle_{n,k} = |\Psi\rangle_{\sigma_{n,k}}$. Each invariant may be rewritten into integral form using (70). However, we will drop the reference to the Hankel contours, in order to stay general. There are at least three natural sets of variables in which the integral representation can be expressed. Directly using (70), we may write it using variables α_p, which are related to the decomposition into BCFW bridges (45), see also [9]. Another set of variables is given by the entries of the matrix $C = (c_{ij})_{j=1,\dots,n}^{i=1,\dots,k}$ making its appearance in the Graßmannian integral formulation studied in [16]. The last form is given in terms of the face variables f_i introduced in [11].

As was already mentioned in the previous section, the ensuing formulas are valid for both the Yangian invariants of compact $\mathfrak{gl}(N|M)$ algebras as well as for the amplitude problem of $\mathcal{N} = 4$ SYM with $\mathfrak{gl}(4|4)$ symmetry.

4.1 n=2, k=1

There is only one non-trivial two-point invariant, for which $k = 1$. The permutation reduces just to a single transposition

$$\sigma_{2,1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (12)\,, \tag{73}$$

The invariant (47) in that case is given by

$$|\Psi\rangle_{2,1} = \mathcal{B}_{12}(y_1 - y_2)|\mathbf{0}\rangle = \mathcal{B}_{12}(\mathfrak{s}_2)|\mathbf{0}\rangle \propto \int \frac{d\alpha_1}{\alpha_1^{1+\mathfrak{s}_2}} \delta^{N|M}(\mathcal{W}^1 + \alpha_1 \mathcal{W}^2)\,, \tag{74}$$

where we used the fact that $y_1 = y_2 + \mathfrak{s}_2$.

There are at least three distinguished graphical representations for each invariant. The first one, as proposed in [14], comes from the identification of $\mathcal{B}_{ij}$ with a BCFW bridge. Each such bridge can be depicted as a composition of one white and one black three-point vertex. Then the invariant can be drawn[6] as in Figure 2A. When we remove all dashed lines and replace all vertices with only two solid lines by a single solid line, we end up with a graphical representation analogous to the one in [13], as depicted in Figure 2B. It is now easy to translate the latter into on-shell diagram as in Figure 2C, which obviously is rather trivial in the case of the two-point invariant.

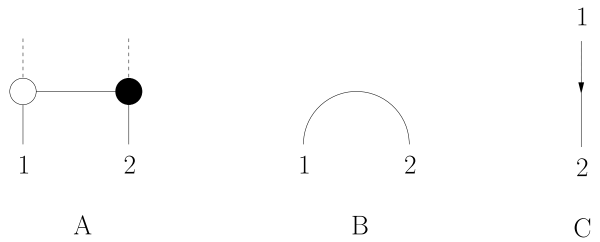


Figure 2: Two-point Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

[6] We thank Yvonne Geyer for bringing this construction to our attention in the case of undeformed amplitudes.

4.2 n=3, k=1

For the three-particle invariant there are two non-trivial values of k. Let us first take the permutation

$$\sigma_{3,1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (13)(12)\,, \tag{75}$$

for which $k = 1$. We specified here the decomposition of this permutation into transpositions. The invariant (47) is given by

$$|\Psi\rangle_{3,1} = \mathcal{B}_{12}(y_1 - y_2)\mathcal{B}_{13}(y_2 - y_3)|\mathbf{0}\rangle = \mathcal{B}_{12}(\mathfrak{s}_2)\mathcal{B}_{13}(\mathfrak{s}_3)|\mathbf{0}\rangle \tag{76}$$

$$\propto \int \frac{d\alpha_1 d\alpha_2}{\alpha_1^{1+\mathfrak{s}_2}\alpha_2^{1+\mathfrak{s}_3}} \delta^{N|M}(\mathcal{W}^1 + \alpha_1\mathcal{W}^2 + \alpha_2\mathcal{W}^3) \tag{77}$$

$$\propto \int \frac{dc_{12} dc_{13}}{c_{12}^{1+\mathfrak{s}_2} c_{13}^{1+\mathfrak{s}_3}} \delta^{N|M}(\mathcal{W}^1 + c_{12}\mathcal{W}^2 + c_{13}\mathcal{W}^3)\,. \tag{78}$$

This is exactly the deformed three-point $\overline{\text{MHV}}$ amplitude introduced in [7]. See Figure 3 for the graphical representation of $|\Psi\rangle_{3,1}$.

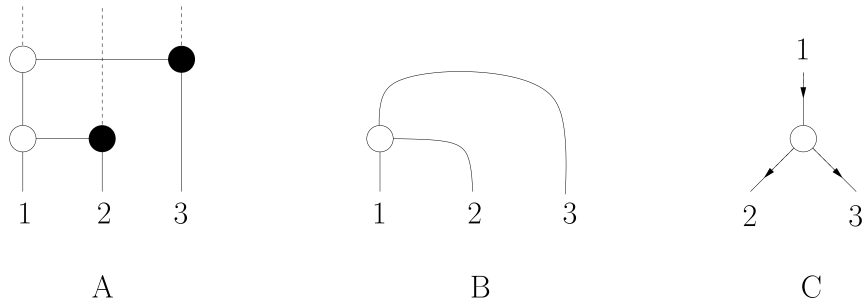


Figure 3: Three-point $\overline{\text{MHV}}$ Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

4.3 n=3, k=2

For the case of the three-particle invariant with $k = 2$ we have the permutation

$$\sigma_{3,2} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (23)(12)\,. \tag{79}$$

The invariant is given by

$$|\Psi\rangle_{3,2} = \mathcal{B}_{12}(y_1 - y_2)\mathcal{B}_{23}(y_1 - y_3)|\mathbf{0}\rangle = \mathcal{B}_{12}(-\mathfrak{s}_1)\mathcal{B}_{23}(\mathfrak{s}_3)|\mathbf{0}\rangle \tag{80}$$

$$\propto \int \frac{d\alpha_1 d\alpha_2}{\alpha_1^{1-\mathfrak{s}_1}\alpha_2^{1+\mathfrak{s}_3}} \delta^{N|M}(\mathcal{W}^1 + \alpha_1\mathcal{W}^2)\delta^{N|M}(\mathcal{W}^2 + \alpha_2\mathcal{W}^3) \tag{81}$$

$$\propto \int \frac{dc_{13} dc_{23}}{c_{13}^{1-\mathfrak{s}_1} c_{23}^{1-\mathfrak{s}_2}} \delta^{N|M}(\mathcal{W}^1 + c_{13}\mathcal{W}^3)\delta^{N|M}(\mathcal{W}^2 + c_{23}\mathcal{W}^3)\,. \tag{82}$$

This is again the deformed three-point MHV amplitude found in [7]. Together with (78) it is the building block for all deformations of on-shell diagrams, and subsequently all tree-level deformed amplitudes. For the graphical representation see Figure 4.

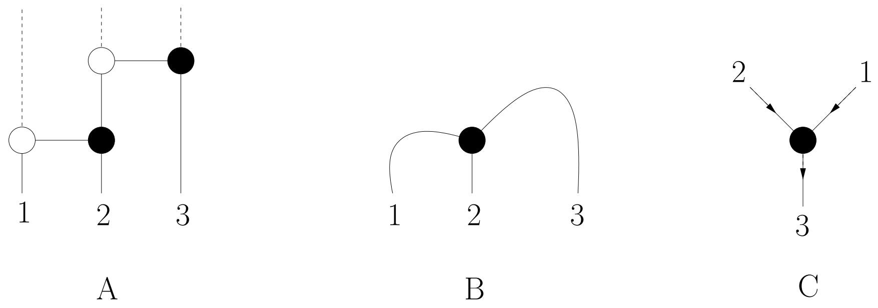


Figure 4: Three-point MHV Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

4.4 n=4, k=2

The four-point invariant with $k = 2$ is the first one which, interestingly, cannot be written solely by using representation labels. It corresponds to the deformation of the four-point tree amplitude obtained in [8] and depends on a spectral parameter z. Let us show how it arises in the context of this paper. The relevant permutation is

$$\sigma_{4,2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (24)(12)(23)(12)\,. \tag{83}$$

and the invariant is given by

$$\begin{aligned} |\Psi\rangle_{4,2} &= \mathcal{B}_{12}(y_1 - y_2)\mathcal{B}_{23}(y_1 - y_3)\mathcal{B}_{12}(y_2 - y_3)\mathcal{B}_{24}(y_2 - y_4)|\mathbf{0}\rangle & (84)\\ &= \mathcal{B}_{12}(z)\mathcal{B}_{23}(-\mathfrak{s}_1)\mathcal{B}_{12}(-z - \mathfrak{s}_1)\mathcal{B}_{24}(-\mathfrak{s}_2)|\mathbf{0}\rangle & (85)\\ &\propto \int \frac{df_1 df_2 df_3 df_4}{f_1^{1-\mathfrak{s}_1} f_2^{1-\mathfrak{s}_1-\mathfrak{s}_2} f_3^{1-z-\mathfrak{s}_1} f_4^{1-\mathfrak{s}_2}}\, \delta^{N|M}(\mathcal{W}^1 + f_1 f_2 \mathcal{W}^3 + (1 + f_3) f_1 f_2 f_4 \mathcal{W}^4) & (86)\\ &\qquad \delta^{N|M}(\mathcal{W}^2 + f_2\mathcal{W}^3 + f_2 f_4 \mathcal{W}^4)\,, & (87) \end{aligned}$$

where we defined $z = y_1 - y_2$. The form (85) of this invariant was already mentioned in (37). Its integral representation (86) exactly reproduces the four-point deformed amplitude in the form derived in [7], see also [8]. Note that a somewhat different looking form involving a hypergeometric function in the integrand of the harmonic R-matrix was given in [13].

4.5 n=5, k=2

In the five particle case the permutation

$$\sigma_{5,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} = (25)(12)(24)(12)(23)(12) \tag{88}$$

with $k = 2$ leads to the invariant

$$\begin{aligned} |\Psi\rangle_{5,2} &= \mathcal{B}_{12}(y_{12})\mathcal{B}_{23}(y_{13})\mathcal{B}_{12}(y_{23})\mathcal{B}_{24}(y_{24})\mathcal{B}_{12}(y_{34})\mathcal{B}_{25}(y_{35})|\mathbf{0}\rangle & (89)\\ &= \mathcal{B}_{12}(-\mathfrak{s}_1 - \mathfrak{s}_4)\mathcal{B}_{23}(\mathfrak{s}_3)\mathcal{B}_{12}(-\mathfrak{s}_2 - \mathfrak{s}_5)\mathcal{B}_{24}(\mathfrak{s}_4)\mathcal{B}_{12}(-\mathfrak{s}_1 - \mathfrak{s}_3)\mathcal{B}_{25}(\mathfrak{s}_5)|\mathbf{0}\rangle & (90)\\ &\propto \int \frac{df_1 df_2 df_3 df_4 df_5 df_6}{f_1^{1-\mathfrak{s}_1} f_2^{1-\mathfrak{s}_1-\mathfrak{s}_2} f_3^{1+\mathfrak{s}_4} f_4^{1+\mathfrak{s}_4+\mathfrak{s}_5} f_5^{1-\mathfrak{s}_1-\mathfrak{s}_3} f_6^{1+\mathfrak{s}_5}} & (91)\\ &\qquad \delta^{N|M}(\mathcal{W}^1 + f_1 f_2 \mathcal{W}^3 + (1 + f_3) f_1 f_2 f_4 \mathcal{W}^4 + (1 + f_3 + f_3 f_5) f_1 f_2 f_4 f_6 \mathcal{W}^5) & (92)\\ &\qquad \delta^{N|M}(\mathcal{W}^2 + f_2 \mathcal{W}^3 + f_2 f_4 \mathcal{W}^4 + f_2 f_4 f_6 \mathcal{W}^5)\,, & (93) \end{aligned}$$

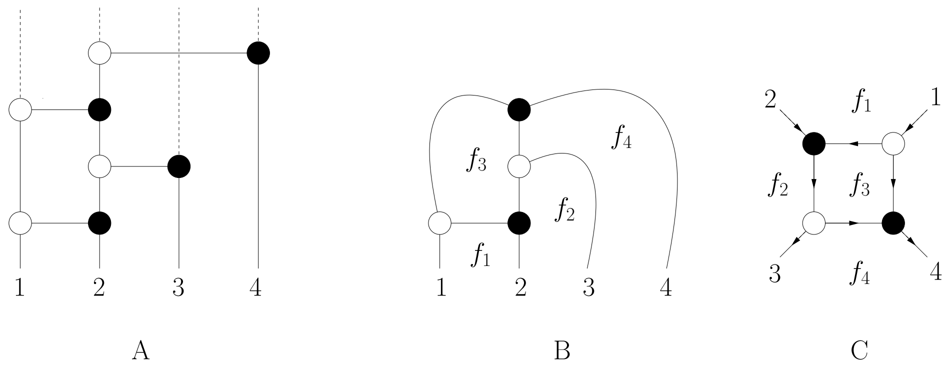


Figure 5: Four-point MHV Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

where we abbreviated $y_{ij} = y_i - y_j$. This provides a deformation of the five-point MHV amplitude. Notice that it is fully determined just using representation labels.

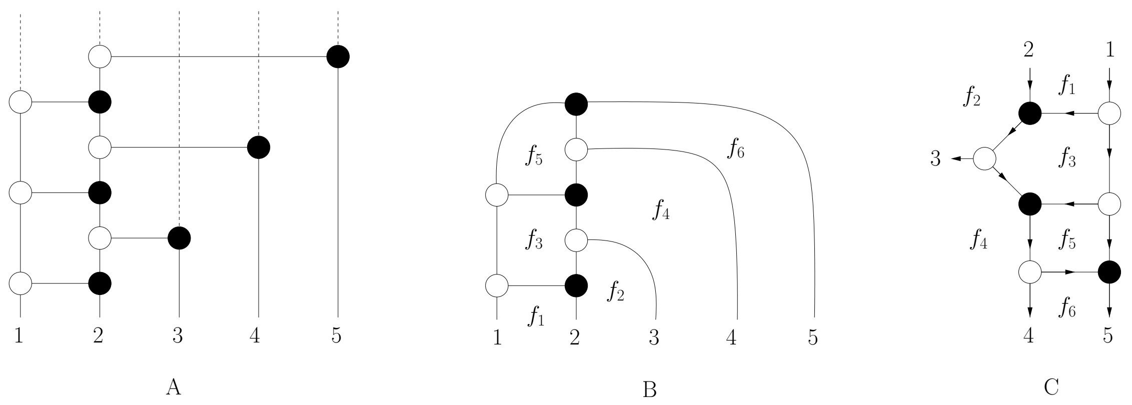


Figure 6: Five-point MHV Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

4.6 n=5, k=3

A five particle permutation with $k = 3$ is given by

$$\sigma_{5,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} = (35)(23)(34)(12)(23)(12)\,. \tag{94}$$

This yields the invariant

$$
\begin{aligned}
|\Psi\rangle_{5,3} &= \mathcal{B}_{12}(y_{12})\mathcal{B}_{23}(y_{13})\mathcal{B}_{12}(y_{23})\mathcal{B}_{34}(y_{14})\mathcal{B}_{23}(y_{24})\mathcal{B}_{35}(y_{25})|\mathbf{0}\rangle &(95)\\
&= \mathcal{B}_{12}(\mathfrak{s}_2+\mathfrak{s}_4)\mathcal{B}_{23}(-\mathfrak{s}_1)\mathcal{B}_{12}(\mathfrak{s}_3+\mathfrak{s}_5)\mathcal{B}_{34}(\mathfrak{s}_4)\mathcal{B}_{23}(-\mathfrak{s}_2)\mathcal{B}_{35}(\mathfrak{s}_5)|\mathbf{0}\rangle &(96)\\
&\propto \int \frac{df_1 df_2 df_3 df_4 df_5 df_6}{f_1^{1-\mathfrak{s}_1} f_2^{1-\mathfrak{s}_1-\mathfrak{s}_2} f_3^{1+\mathfrak{s}_3+\mathfrak{s}_5} f_4^{1+\mathfrak{s}_4+\mathfrak{s}_5} f_5^{1-\mathfrak{s}_2} f_6^{1+\mathfrak{s}_5}} &(97)\\
&\qquad \delta^{N|M}(\mathcal{W}^1 + f_1 f_2 f_4 \mathcal{W}^4 + f_1 f_2 f_4 f_6 (1+f_5+f_3 f_5)\mathcal{W}^5) &(98)\\
&\qquad \delta^{N|M}(\mathcal{W}^2 + f_2 f_4 \mathcal{W}^4 + f_2 f_4 f_6 (1+f_5)\mathcal{W}^5) &(99)\\
&\qquad \delta^{N|M}(\mathcal{W}^3 + f_4 \mathcal{W}^4 + f_4 f_6 \mathcal{W}^5)\,, &(100)
\end{aligned}
$$

which is a deformation of the five-point $\overline{\text{MHV}}$ amplitude.

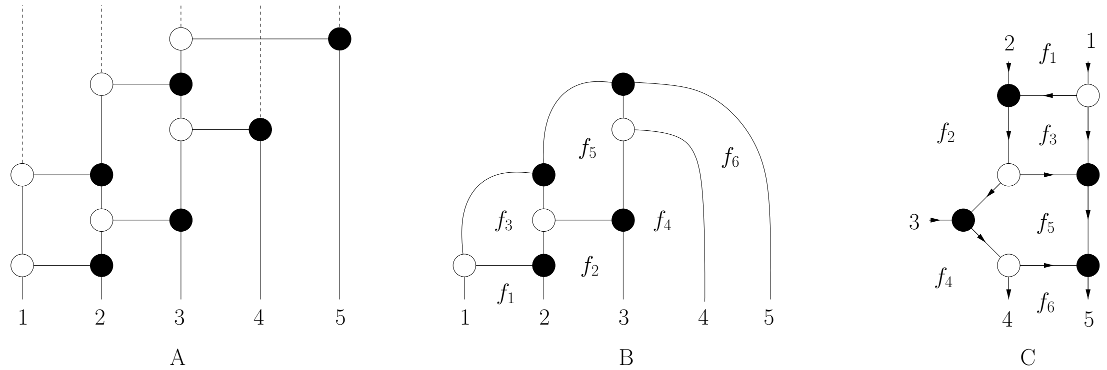


Figure 7: Five-point $\overline{\text{MHV}}$ Yangian invariant. A) Transposition decomposition; B) wilted on-shell diagram; C) on-shell diagram with a perfect orientation.

5 Summary and Outlook

In this paper we provided a full classification of Yangian invariants relevant to tree-level scattering amplitudes in $\mathcal{N}=4$ SYM. Our method combines the idea of deformation of on-shell diagrams [8] with the QISM as proposed in [13] and [14]. It gives a constructive way to build such invariants and provides a link to powerful Bethe Ansatz methods. It also introduces natural variables $v^{\pm}$, as studied in details by [12], which are very reminiscent of the Zhukovsky variables $x^{\pm}$ that play an important role in the all-loop solution of the spectral problem [17]. This leads to the hope that these variables will provide a good starting point for suitable generalizations of Yangian invariants to scattering amplitudes at higher loops.

Acknowledgments

We thank Livia Ferro, Rouven Frassek, Yvonne Geyer, Yumi Ko and Arthur Lipstein for useful discussions. This research is supported in part by the SFB 647 *"Raum-Zeit-Materie. Analytische und Geometrische Strukturen"* and the Marie Curie network GATIS (`gatis.desy.eu`) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA Grant Agreement No 317089. N.K. is supported by a *Promotionsstipendium* of the *Studienstiftung des Deutschen Volkes*, and receives partial support by the GK 1504 *"Masse, Spektrum, Symmetrie"*.

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